Motivic decomposition of anisotropic varieties of type F_4 into generalized Rost motives

S. Nikolenko, N. Semenov, K. Zainoulline *

Abstract

We prove that the Chow motive of an anisotropic projective homogeneous variety of type F_4 is isomorphic to the direct sum of twisted copies of a generalized Rost motive. In particular, we provide an explicit construction of a generalized Rost motive for a generically splitting variety for a symbol in $\mathrm{K}_3^M(k)/3$. We also establish a motivic isomorphism between two anisotropic non-isomorphic projective homogeneous varieties of type F_4. All our results hold for Chow motives with integral coefficients.

1 Introduction

The subject of the present paper begins with the celebrated result of M. Rost [Ro98] devoted to the motivic decomposition of a norm quadric. The existence of such a decomposition became one of the main ingredients in the proof of Milnor conjecture by V. Voevodsky. The generalization of this conjecture to other primes $p > 2$, known as the Bloch-Kato conjecture, was proven recently by M. Rost and V. Voevodsky. One of the ingredients of the proof is the fact that the motive with $\mathbb{Z}/p\mathbb{Z}$-coefficients of a splitting (norm) variety X contains as a direct summand a certain geometric motive M_{p-1} called *generalized Rost motive* [Vo03, Sect. 5]. This motive is indecomposable and splits as a direct sum of twisted Lefschetz motives over the separable closure of the base field.

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Note that Voevodsky's construction of M_{p-1} relies heavily on the language of triangulated category of motives. The main goal of the present paper is to provide an explicit and shortened construction of the motive M_2 ($p = 3$) working only within the classical category of Chow motives. More precisely, we provide such a construction for an exceptional projective homogeneous variety of type F_4 which splits the symbol (in $\mathrm{K}_3^M(k)/3$) given by the Rost-Serre invariant g_3 (see 4.5).

Note that if X is generically cellular and splits a pure symbol, it is expected that the motive of X is isomorphic to the direct sum of twisted copies of the motive M_{p-1}. The motivic decomposition that we obtain confirms these expectations. Namely, we prove the following

1.1 Theorem. *Let X be an anisotropic variety over a field k such that over a cubic field extension k' of k it becomes isomorphic to the projective homogeneous variety G/P, where G is a split simple group of type F_4 and P its maximal parabolic subgroup corresponding to the last or the first three vertices of the Dynkin diagram 4.1.*

Then the Chow motive of X (with integral coefficients) is isomorphic to the direct sum of twisted copies of an indecomposable motive R

$$\mathcal{M}(X) \cong \bigoplus_{i=0}^{7} R(i) \tag{1}$$

which has the property that over k' it becomes isomorphic to the direct sum of twisted Lefschetz motives $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$.

The next result provides the first known "purely exceptional" example of two different anisotropic varieties with isomorphic motives. Recall that the similar result for groups of type G_2 obtained in [Bo03] provides a motivic isomorphism between quadric and an exceptional Fano variety.

1.2 Theorem. *Let $\mathcal{G}$ be an anisotropic algebraic group over k such that over a cubic field extension k' of k it becomes isomorphic to a split simple group G of type F_4. Let X_1 and X_4 be two projective $\mathcal{G}$-homogeneous varieties which over k' become isomorphic to G/P_1 and G/P_4 respectively, where P_1 (resp. P_4) is the maximal parabolic subgroup corresponding to the last (resp. first) three vertices of the Dynkin diagram 4.1.*

Then the motives of X_1 and X_4 are isomorphic.

The main motivation for our work was the result of N. Karpenko where he gave a shortened construction of a Rost motive for a norm quadric [Ka98]. The key idea is to produce enough idempotents in the ring $\mathrm{CH}(X \times X)$ considered over the separable closure of k and then lift them to k using the Rost Nilpotence Theorem (see [CGM]). Contrary to the techniques used by Voevodsky, the proof of Theorem 1.1 is based on well-known and elementary facts about linear algebraic groups, projective homogeneous varieties, and Chow groups.

We expect that similar methods can be applied to other projective homogeneous varieties, thus providing analogous motivic decompositions. In particular, applying our arguments to a Pfister quadric one obtains the celebrated decomposition into Rost motives (see [KM02, Example 7.3]). For exceptional groups of type G_2 one immediately obtains the motivic decomposition of the Fano variety together with the motivic isomorphism constructed in [Bo03].

The paper is organized as follows. In Section 2 we provide background information on Chow motives and rational cycles. Section 3 is devoted to computational matters of Chow rings. Namely, we introduce Pieri and Giambelli formulae and discuss their relationships with Hasse diagrams. In Section 4 we apply the formulae introduced in Section 3 to projective homogeneous varieties X_1 and X_4 of type F_4. In Section 5 we prove Theorems 1.1 and 1.2. Finally, in the Appendix we explain the intermediate technical steps of computations used in the proofs.

2 Motives and rational cycles

In the present section we introduce the category of Chow motives over a field k following [Ma68] and [Ka01]. We remind the notion of a rational cycle and state the Rost Nilpotence Theorem for idempotents following [CGM].

2.1. Let k be a field and $\mathcal{V}ar_k$ be a category of smooth projective varieties over k. First, we define the category of *correspondences* (over k) denoted by $\mathcal{C}or_k$. Its objects are smooth projective varieties over k. For morphisms, called correspondences, we set $\mathrm{Mor}(X,Y) := \mathrm{CH}^{\dim X}(X \times Y)$. For any two correspondences $\alpha \in \mathrm{CH}(X \times Y)$ and $\beta \in \mathrm{CH}(Y \times Z)$ we define their composition $\beta \circ \alpha \in \mathrm{CH}(X \times Z)$ as

$$\beta \circ \alpha = pr_{13*}(pr_{12}^*(\alpha) \cdot pr_{23}^*(\beta)), \tag{2}$$

where pr_{ij} denotes the projection on the i-th and j-th factors of $X \times Y \times Z$ respectively and pr_{ij_*}, pr_{ij}^* denote the induced push-forwards and pull-backs for Chow groups.

The pseudo-abelian completion of $\mathcal{C}or_k$ is called the category of *Chow motives* and is denoted by $\mathcal{M}_k$. The objects of $\mathcal{M}_k$ are pairs (X, p), where X is a smooth projective variety and p is an idempotent, that is, $p \circ p = p$. The morphisms between two objects (X, p) and (Y, q) are the compositions $q \circ \mathrm{Mor}(X, Y) \circ p$. The motive (X, id) will be denoted by $\mathcal{M}(X)$.

2.2. By construction, $\mathcal{M}_k$ is a tensor additive category with self-duality, where the self-duality is given by the transposition of cycles $\alpha \mapsto \alpha^t$, and the tensor product is given by the usual fiber product $(X, p) \otimes (Y, q) = (X \times Y, p \times q)$. Moreover, the contravariant Chow functor $\mathrm{CH} : \mathcal{V}ar_k \to \mathbb{Z}\text{-}\mathcal{A}b$ (to the category of $\mathbb{Z}$-graded abelian groups) factors through $\mathcal{M}_k$, that is, one has a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{V}ar_k & \xrightarrow{\mathrm{CH}} & \mathbb{Z}\text{-}\mathcal{A}b \\ & {\scriptstyle\Gamma}\searrow \quad \nearrow{\scriptstyle CH} & \\ & \mathcal{M}_k & \end{array}$$

where $\Gamma : f \mapsto \Gamma_f$ is the covariant graph functor and $CH : (X, p) \mapsto \mathrm{Im}(p^*)$ is the realization.

2.3. Observe that the composition product $\circ$ induces the ring structure on the abelian group $\mathrm{CH}^{\dim X}(X \times X)$. The unit element of this ring is the class of the diagonal map Δ_X, which is defined by $\Delta_X \circ \alpha = \alpha \circ \Delta_X = \alpha$ for all $\alpha \in \mathrm{CH}^{\dim X}(X \times X)$.

2.4. Consider the morphism $(e, \mathrm{id}) : \{pt\} \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. Its image by means of the induced push-forward $(e, \mathrm{id})_*$ does not depend on the choice of the point $e : \{pt\} \to \mathbb{P}^1$ and defines the projector in $\mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ denoted by p_1. The motive $\mathbb{Z}(1) = (\mathbb{P}^1, p_1)$ is called *Lefschetz motive*. For a motive M and a nonnegative integer i we denote its twist by $M(i) = M \otimes \mathbb{Z}(1)^{\otimes i}$.

2.5. Let G be a split simple linear algebraic group over k. Let X be a projective G-homogeneous variety, that is, $X \simeq G/P$, where P is a parabolic subgroup of G. The abelian group structure of $\mathrm{CH}(X)$, as well as its ring structure, is well-known. Namely, X has a cellular filtration and the generators of Chow groups of the bases of this filtration correspond to the

free additive generators of $\mathrm{CH}(X)$. Note that the product of two projective homogeneous varieties $X \times Y$ has a cellular filtration as well, and $\mathrm{CH}^*(X \times Y) \cong \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(Y)$ as graded rings. The correspondence product of two cycles $\alpha = f_\alpha \times g_\alpha \in \mathrm{CH}(X \times Y)$ and $\beta = f_\beta \times g_\beta \in \mathrm{CH}(Y \times X)$ is given by (cf. [Bo03, Lem. 5])

$$(f_\beta \times g_\beta) \circ (f_\alpha \times g_\alpha) = \deg(g_\alpha \cdot f_\beta)(f_\alpha \times g_\beta), \tag{3}$$

where $\deg : \mathrm{CH}(Y) \to \mathrm{CH}(\{pt\}) = \mathbb{Z}$ is the degree map.

2.6. Let X be a projective variety of dimension n over a field k. Let k' be a field extension of k and $X' = X \times_k k'$. We say a cycle $J \in \mathrm{CH}(X')$ is *rational* if it lies in the image of the pull-back homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(X')$. For instance, there is an obvious rational cycle $\Delta_{X'}$ on $\mathrm{CH}^n(X' \times X')$ that is given by the diagonal class. Clearly, all linear combinations, intersections and correspondence products of rational cycles are rational.

2.7. Several techniques allow to produce rational cycles (cf. [Ka04, Prop. 3.3] for the case of quadrics). We shall use the following:

(i) Consider a variety Y and a morphism $X \to Y$ such that $X' = Y' \times_Y X$, where $Y' = Y \times_k k'$. Then any rational cycle on $\mathrm{CH}(Y')$ gives rise to a rational cycle on $\mathrm{CH}(X')$ by the induced pull-back $\mathrm{CH}(Y') \to \mathrm{CH}(X')$.

(ii) Consider a variety Y and a projective morphism $Y \to X$ such that $Y' = X' \times_X Y$. Then any rational cycle on $\mathrm{CH}(Y')$ gives rise to a rational cycle on $\mathrm{CH}(X')$ by the induced push-forward $\mathrm{CH}(Y') \to \mathrm{CH}(X')$.

(iii) Let X and Y be projective homogeneous varieties over k such that X splits completely (i.e., the respective group splits) over the function field $k(Y)$. Consider the following pull-back diagram

$$\begin{array}{ccc} \mathrm{CH}^i(X \times Y) & \xrightarrow{g} & \mathrm{CH}^i(X' \times Y') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathrm{CH}^i(X_{k(Y)}) & \xrightarrow{=} & \mathrm{CH}^i(X'_{k'(Y')}) \end{array}$$

where the vertical arrows are surjective by [IK00, §5]. Now take any cycle $\alpha \in \mathrm{CH}^i(X' \times Y')$, $i \leq \dim X$. Let $\beta = g(f^{-1}(f'(\alpha)))$. Then $f'(\beta) = f'(\alpha)$ and β is rational. Hence, $\beta = \alpha + J$, where $J \in \operatorname{Ker} f'$, and we conclude that $\alpha + J \in \mathrm{CH}^i(X' \times Y')$ is rational.

2.8 (Rost Nilpotence). Finally, we shall also use the following fact (see [CGM, Cor. 8.3]) that follows from Rost Nilpotence Theorem. Let p' be a non-trivial rational idempotent on $\mathrm{CH}^n(X' \times X')$, i.e., $p' \circ p' = p'$. Then there exists a non-trivial idempotent p on $\mathrm{CH}^n(X \times X)$ such that $p \times_k k' = p'$. Hence, the existence of a non-trivial rational idempotent p' on $\mathrm{CH}^n(X' \times X')$ gives rise to the decomposition of the Chow motive of X

$$\mathcal{M}(X) \cong (X, p) \oplus (X, \mathrm{id}_X - p).$$

3 Hasse diagrams and Chow rings

To each projective homogeneous variety X we may associate an oriented labeled graph $\mathcal{H}$ called Hasse diagram. It is known that the ring structure of $\mathrm{CH}(X)$ is determined by $\mathcal{H}$. In the present section we remind several facts concerning relations between Hasse diagrams and Chow rings. For detailed explanations of these relations see [De74], [Hi82a] and [Ko91].

3.1. Let G be a split simple algebraic group defined over a field k. We fix a maximal split torus T in G and a Borel subgroup B of G containing T and defined over k. We denote by Φ the root system of G, by Π the set of simple roots of Φ corresponding to B, by W the Weyl group, and by S the corresponding set of fundamental reflections.

Let $P = P_\Theta$ be a (standard) parabolic subgroup corresponding to a subset $\Theta \subset \Pi$, i.e., $P = BW_\Theta B$, where $W_\Theta = \langle s_\theta, \theta \in \Theta \rangle$. Denote

$$W^\Theta = \{w \in W \mid \forall\, s \in \Theta \quad l(ws) = l(w) + 1\},$$

where l is the length function. The pairing

$$W^\Theta \times W_\Theta \to W \qquad (w, v) \mapsto wv$$

is a bijection and $l(wv) = l(w) + l(v)$. It is easy to see that W^Θ consists of all representatives in the cosets W/W_Θ which have minimal length. Sometimes it is also convenient to consider the set of all representatives of maximal length. We shall denote this set as ${}^\Theta W$. Observe that there is a bijection $W^\Theta \to {}^\Theta W$ given by $v \mapsto vw_\theta$, where w_θ is the longest element of W_Θ. The longest element of W^Θ corresponds to the longest element w_0 of the Weyl group.

3.2. To a subset Θ of the finite set Π we associate an oriented labeled graph, which we call a Hasse diagram and denote by $\mathcal{H}_W(\Theta)$. This graph is constructed as follows. The vertices of this graph are the elements of W^Θ. There is an edge from a vertex w to a vertex w' labelled with i if and only if $l(w) < l(w')$ and $w' = ws_i$. A sample Hasse diagram is provided in 4.7. Observe that the diagram $\mathcal{H}_W(\emptyset)$ coincides with the Cayley graph associated to the pair (W, S).

3.3 Lemma. *The assignment $\mathcal{H}_W\colon \Theta \mapsto \mathcal{H}_W(\Theta)$ is a contravariant functor from the category of subsets of the finite set Π (with embeddings as morphisms) to the category of oriented graphs.*

Proof. It is enough to embed the diagram $\mathcal{H}_W(\Theta)$ to the diagram $\mathcal{H}_W(\emptyset)$. We do this as follows. We identify the vertices of $\mathcal{H}_W(\Theta)$ with the subset of vertices of $\mathcal{H}_W(\emptyset)$ by means of the bijection $W^\Theta \to {}^\Theta W$. Then the edge from w to w' of ${}^\Theta W \subset W$ has a label i if and only if $l(w) < l(w')$ and $w' = s_i w$ (as elements of W). Clearly, the obtained graph will coincide with $\mathcal{H}_W(\Theta)$. □

3.4. Now consider the Chow ring of a projective homogeneous variety G/P_Θ. It is well known that $\mathrm{CH}(G/P_\Theta)$ is a free abelian group with a basis given by varieties $[X_w]$ that correspond to the vertices w of the Hasse diagram $\mathcal{H}_W(\Theta)$. The degree of the basis element $[X_w]$ corresponds to the minimal number of edges needed to connect the respective vertex w with w_θ (which is the longest word). The multiplicative structure of $\mathrm{CH}(G/P_\Theta)$ depends only on the root system of G and the diagram $\mathcal{H}_W(\Theta)$.

By definition one immediately obtains

3.5 Lemma. *The contravariant functor $\mathrm{CH}\colon \Theta \mapsto \mathrm{CH}(G/P_\Theta)$ factors through the category of Hasse diagrams $\mathcal{H}_W$, i.e., the pull-back (ring inclusion)*

$$\mathrm{CH}(G/P_{\Theta'}) \hookrightarrow \mathrm{CH}(G/P_\Theta)$$

arising from the embedding $\Theta \subset \Theta'$ is induced by the embedding of the respective Hasse diagrams $\mathcal{H}_W(\Theta') \subset \mathcal{H}_W(\Theta)$.

3.6 Corollary. *Let B be a Borel subgroup of G and P its (standard) parabolic subgroup. Then $\mathrm{CH}(G/P)$ is a subring of $\mathrm{CH}(G/B)$. The generators of $\mathrm{CH}(G/P)$ are $[X_w]$, where $w \in {}^\Theta W \subset W$. The cycle $[X_w]$ in $\mathrm{CH}(G/P)$ has the codimension $l(w_0) - l(w)$.*

Proof. Apply the lemma to the case $B = P_\emptyset$ and $P = P_{\Theta'}$. □

Hence, in order to compute $\mathrm{CH}(G/P)$ it is enough to compute $\mathrm{CH}(X)$, where $X = G/B$ is the variety of complete flags. The following results provide tools to perform such computations.

3.7. In order to multiply two basis elements h and g of $\mathrm{CH}(G/P)$ such that $\deg h + \deg g = \dim G/P$ we use the following formula (see [Ko91, 1.4]):

$$[X_w] \cdot [X_{w'}] = \delta_{w,w_0w'w_\theta} \cdot [pt].$$

3.8 (Pieri formula). In order to multiply two basis elements of $\mathrm{CH}(X)$ one of which is of codimension 1 we use the following formula (see [De74, Cor. 2 of 4.4]):

$$[X_{w_0s_\alpha}][X_w] = \sum_{\beta\in\Phi^+,\, l(ws_\beta)=l(w)-1} \langle\beta^\vee, \overline{\omega}_\alpha\rangle [X_{ws_\beta}],$$

where the sum runs through the set of positive roots $\beta \in \Phi^+$, s_α denotes the simple reflection corresponding to α and $\bar\omega_\alpha$ is the fundamental weight corresponding to α. Here $[X_{w_0s_\alpha}]$ is the element of codimension 1.

3.9 (Giambelli formula). Let $P = P(\Phi)$ be the weight space. We denote as $\bar\omega_1, \ldots \bar\omega_l$ the basis of P consisting of fundamental weights. The symmetric algebra $S^*(P)$ is isomorphic to $\mathbb{Z}[\bar\omega_1, \ldots \bar\omega_l]$. The Weyl group W acts on P, hence, on $S^*(P)$. Namely, for a simple root α_i,

$$w_{\alpha_i}(\bar\omega_j) = \begin{cases} \bar\omega_i - \alpha_i, & i = j, \\ \bar\omega_j, & \text{otherwise.} \end{cases}$$

We define a linear map $c\colon S^*(P) \to \mathrm{CH}^*(G/B)$ as follows. For a homogeneous $u \in \mathbb{Z}[\bar\omega_1, \ldots, \bar\omega_l]$

$$c(u) = \sum_{w\in W,\, l(w)=\deg(u)} \Delta_w(u)[X_{w_0w}],$$

where for $w = w_{\alpha_1} \ldots w_{\alpha_k}$ we denote by Δ_w the composition of derivations $\Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_k}$ and the derivation $\Delta_{\alpha_i}\colon S^*(P) \to S^{*-1}(P)$ is defined by $\Delta_{\alpha_i}(u) = \frac{u - w_{\alpha_i}(u)}{\alpha_i}$. Then (see [Hi82a, ch. IV, 2.4])

$$[X_w] = c(\Delta_{w^{-1}}(\frac{d}{|W|})),$$

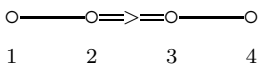
where d is the product of all positive roots in $S^*(P)$. In other words, the element $\Delta_{w^{-1}}(\frac{d}{|W|}) \in c^{-1}([X_w])$.

Hence, in order to multiply two basis elements $h, g \in \mathrm{CH}(X)$ one may take their preimages under the map c and multiply them in $S^*(P) = \mathbb{Z}[\bar{\omega}_1, \dots \bar{\omega}_l]$, finally applying c to the product.

4 Homogeneous varieties of type F_4

In the present section we remind several well-known facts concerning Albert algebras, groups of type F_4 and respective projective homogeneous varieties (see [PR94] and [Inv]). At the end we provide partial computations of Chow rings of these varieties.

4.1. From now on let X_i be an anisotropic projective homogeneous variety over a field k that over a cubic field extension k' of k becomes isomorphic to the projective homogeneous variety $X_i' = G/P_i$, where G is a split group of type F_4 and $P_i = P_{\theta_i}$ is a maximal parabolic subgroup of G corresponding to the subset $\theta_i = \{1, 2, 3, 4\} \setminus \{i\}$ of the Dynkin diagram



4.2 Remark. The variety X_i is a ${}_\xi G$-variety over k, where ${}_\xi G$ is an algebraic group over k that is the twisted form of a split group G of type F_4 by means of a 1-cocycle $\xi \in H^1(k, G(k'))$ (see [De77, Prop. 4]).

If the base field k has characteristic not 3 and $\mu_3 \subset k$, then all such groups ${}_\xi G$ are automorphism groups of Albert algebras coming from the first Tits construction.

The next two important properties of the varieties X_i will be extensively used in the sequel.

4.3 Lemma. *The Picard group* $\mathrm{Pic}(X_i')$ *is a free abelian group of rank 1 with a rational generator.*

Proof. Since P is maximal, $\mathrm{Pic}(X_i')$ is a free abelian group of rank 1. Consider the following exact sequence (see [Ar82] and [MT95, 2.3])

$$0 \longrightarrow \mathrm{Pic}\, X_i \longrightarrow (\mathrm{Pic}\, X_i')^\Gamma \xrightarrow{\alpha} \mathrm{Br}(k),$$

where $\Gamma = \mathrm{Gal}(k'/k)$ is the Galois group and $\mathrm{Br}(k)$ the Brauer group of k. The map α is explicitly described in [MT95] in terms of Tits classes. Since all groups of type F_4 are simply-connected and adjoint their Tits classes are trivial and so is α. Since Γ acts trivially on $\mathrm{Pic}(X_i')$ and the image of α is trivial, we have $\mathrm{Pic}(X_i) \simeq \mathrm{Pic}(X_i')$. □

4.4 Lemma. *For any $i, j \in \{1, 2, 3, 4\}$, the variety X_i splits completely over the function field $k(X_j)$.*

Proof. The following arguments belong to S. Garibaldi.

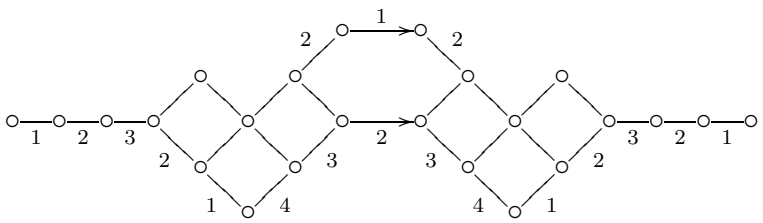
It is equivalent to show that the function field $K = k(X_j)$ splits the group $\widetilde{G} = {}_\xi G$. First, observe that if $\widetilde{G}$ is isotropic then it is split. Indeed, by Tits classification [Ti66, p. 60], the only other possibility is that $\widetilde{G}$ has a maximal parabolic P_4 defined over k, but no others. That is, the semisimple anisotropic kernel H of G is of type B_3. Since $\widetilde{G}$ is split by a cubic field extension, the same is true for H. But this is impossible for an anisotropic group of type B_3 by Springer's Theorem on quadratic forms and odd-degree extensions.

Now let k' be a cubic extension of k that splits $\widetilde{G}$. Since K is a regular extension of k, the tensor product $K' = K \otimes_k k'$ is a field and has dimension 1 or 3 over K. Hence, $\widetilde{G} \otimes_k K$ is isotropic and is split by the extension K'. By previous arguments $\widetilde{G} \otimes_k K$ is split. □

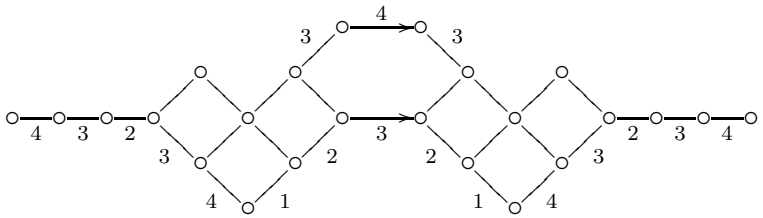
4.5 Remark. To any Albert algebra A one can associate a symbol $\{a_1, a_2, a_3\}$ in $\mathrm{K}_3^M(k)/3$ given by Rost-Serre's invariant g_3. If A is obtained by the first Tits construction, then $g_3 = 0$ iff A is split, and the variety X_j corresponding to A provides an example of a generically splitting variety for the symbol given by g_3 (see [Su05, Def. 1.8]). Observe that X_j is not exactly a norm (ν_n-)variety in the sense of [Vo03, Sect. 4], since it has the wrong dimension, but rather a "Pfister quadric" version of it.

4.6. From now on we consider only two anisotropic varieties X_1 and X_4. Recall that over a cubic field extension k' they become isomorphic to $X_1' = G/P_1$ and $X_4' = G/P_4$ respectively (see 4.1). Varieties X_1' and X_4' are not isomorphic (their Chow rings are different) and have the dimension 15. Observe that the variety X_4 is a twisted form of a hyperplane section of the Cayley plane $\mathbb{OP}^2$ which was extensively studied in [IM05].

4.7. The Hasse diagram (see 3.2) for X_1' looks as follows



and for X_4'



We draw the diagrams in such a way that the labels on opposite sides of a parallelogram are equal, and in that case we omit all labels but one. Recall that (see 3.4) the vertices of this graph correspond to the basis elements of the Chow group. The leftmost vertex is the unit class and the rightmost one is the class of a 0-cycle of degree 1.

4.8. We denote the basis elements of the respective Chow groups as follows

$$\mathrm{CH}^i(X_1') = \begin{cases} \langle h_1^i \rangle, & i = 0, \dots, 3, 12, \dots, 15, \\ \langle h_1^i, h_2^i \rangle, & i = 4, \dots, 11. \end{cases}$$

$$\mathrm{CH}^i(X_4') = \begin{cases} \langle g_1^i \rangle, & i = 0, \dots, 3, 12, \dots, 15, \\ \langle g_1^i, g_2^i \rangle, & i = 4, \dots, 11. \end{cases}$$

The generators with lower index 1 correspond to the lower vertices of the respective Hasse diagrams, and with index 2 to the upper ones (if there are two generators).

4.9. Applying 3.7 we immediately obtain the following partial multiplication table:

$$h_i^s h_j^{15-s} = \delta_{ij} h_1^{15}, \qquad g_i^s g_j^{15-s} = \delta_{ij} g_1^{15},$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

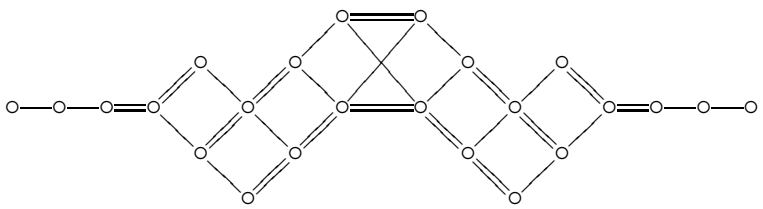
4.10. By Pieri formula 3.8 we obtain the following partial multiplication tables for $\mathrm{CH}(X_1')$:

$$
\begin{array}{llll}
h_1^1 h_1^1 = h_1^2, & h_1^1 h_1^2 = 2h_1^3, & h_1^1 h_1^3 = 2h_2^4 + h_1^4, & h_1 h_2^4 = h_2^5, \\
h_1^1 h_1^4 = 2h_2^5 + h_1^5, & h_1^1 h_2^5 = 2h_2^6 + h_1^6, & h_1^1 h_1^5 = 2h_1^6, & h_1^1 h_2^6 = h_2^7 + h_1^7, \\
h_1^1 h_1^6 = 2h_1^7, & h_1^1 h_2^7 = 2h_2^8 + h_1^8, & h_1^1 h_1^7 = h_2^8 + 2h_1^8, & h_1^1 h_2^8 = h_2^9, \\
h_1^1 h_1^8 = h_2^9 + 2h_1^9, & h_1^1 h_2^9 = 2h_2^{10}, & h_1^1 h_1^9 = h_2^{10} + 2h_1^{10}, & h_1^1 h_2^{10} = h_2^{11} + 2h_1^{11}, \\
h_1^1 h_1^{10} = h_1^{11}, & h_1^1 h_2^{11} = 2h_1^{12}, & h_1^1 h_1^{11} = h_1^{12}, & h_1^1 h_1^{12} = 2h_1^{13}, \\
h_1^1 h_1^{13} = h_1^{14}, & h_1^1 h_1^{14} = h_1^{15}. &&
\end{array}
$$

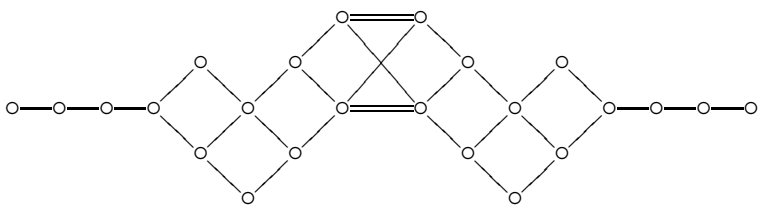
for $\mathrm{CH}(X_4')$:

$$
\begin{array}{llll}
g_1^1 g_1^1 = g_1^2, & g_1^1 g_1^2 = g_1^3, & g_1^1 g_1^3 = g_1^4 + g_2^4, & g_1^1 g_2^4 = g_2^5, \\
g_1^1 g_1^4 = g_1^5 + g_2^5, & g_1^1 g_2^5 = g_1^6 + g_2^6, & g_1^1 g_1^5 = g_1^6, & g_1^1 g_2^6 = g_1^7 + g_2^7, \\
g_1^1 g_1^6 = g_1^7, & g_1^1 g_1^7 = 2g_1^8 + g_2^8, & g_1^1 g_2^7 = g_1^8 + 2g_2^8, & g_1^1 g_2^8 = g_2^9, \\
g_1^1 g_1^8 = g_1^9 + g_2^9, & g_1^1 g_2^9 = g_2^{10}, & g_1^1 g_1^9 = g_1^{10} + g_2^{10}, & g_1^1 g_2^{10} = g_1^{11} + g_2^{11}, \\
g_1^1 g_1^{10} = g_1^{11}, & g_1^1 g_2^{11} = g_1^{12}, & g_1^1 g_1^{11} = g_1^{12}, & g_1^1 g_1^{12} = g_1^{13}, \\
g_1^1 g_1^{13} = g_1^{14}, & g_1^1 g_1^{14} = g_1^{15}. &&
\end{array}
$$

4.11. Observe that the multiplication tables 4.10 can be visualized by means of slightly modified Hasse diagrams. Namely, for the variety X_1' consider the following graph which is obtained from the respective Hasse diagram by adding a few more edges and erasing all the labels:



and for X_4':



Then the multiplication rules can be restored as follows: for a vertex u (that corresponds to a basis element of the Chow group) we set

$$H \cdot u = \sum_{u \to v} v,$$

where H denotes either h_1^1 or g_1^1 and the sum runs through all the edges going from u one step to the right (cf. [Hi82b, Cor. 3.3]).

4.12. Applying Giambelli formula 3.9 we obtain the following products (for details see Appendix):

$$h_1^4 h_1^4 = 8h_1^8 + 6h_2^8, \qquad g_1^4 g_1^4 = 4g_1^8 + 3g_2^8.$$

5 Construction of rational idempotents

The goal of the present section is to prove Theorems 1.1 and 1.2. The proof consists of several steps. First, using properties 4.3 and 4.4 we provide several important cycles ρ_i and prove their rationality. Multiplying and composing them, we obtain a set of pairwise orthogonal idempotents p_i' and q_i'. Then, using Rost Nilpotence Theorem (see 2.8) we obtain the desired motivic decomposition and, hence, finish the proof of 1.1. At the end we construct an explicit cycle which provides a motivic isomorphism of Theorem 1.2.

Proof of Theorem 1.1 Let X_1 and X_4 be the varieties corresponding to the last (resp. first) three roots of the Dynkin diagram (see 4.6). By the hypothesis of Theorem 1.1 the variety X is isomorphic either to X_1 or X_4 over k. As in 4.6 let k' denote the cubic field extension and let X_1' and X_4' be the respective base change. We start with the following obvious observation.

5.1. Since the variety X of Theorem 1.1 splits by a cubic field extension, transfer arguments show that any cycle of the kind $3z \in \mathrm{CH}(X')$ is rational. Hence, to prove that a cycle in $\mathrm{CH}(X')$ is rational it is enough to prove this modulo 3. We shall write $x \equiv y$ if $x - y = 3z$ for some cycle z.

5.2. The rational cycles to start with one obtains by Lemma 4.3. Namely, those are the classes of rational generators of the Picard groups h_1^1 and g_1^1 (see 4.8). Clearly, their powers $(h_1^1)^i$ and $(g_1^1)^i$, $i = 2, \ldots, 7$ are rational as well.

5.3. Apply the arguments of 2.7(iii) to $\mathrm{CH}^4(X_1' \times X_4')$ (this can be done because of Lemma 4.4). There exists a rational cycle $\alpha_1 \in \mathrm{CH}^4(X_1' \times X_4')$ such that $f'(\alpha_1) = h_1^4 \times 1$. This cycle must have the following form:

$$\alpha_1 = h_1^4 \times 1 + a_1 h_1^3 \times g_1^1 + a_2 h_1^2 \times g_1^2 + a_3 h_1^1 \times g_1^3 + a_4 1 \times g_2^4 + a' 1 \times g_1^4,$$

where $a_i, a' \in \{-1, 0, 1\}$. We may reduce α_1 by adding cycles that are known to be rational by 5.2 to

$$\alpha_1 = (h_1^4 \times 1) + a(1 \times g_1^4),$$

where $a \in \{-1, 0, 1\}$. Repeating the same procedure for a rational cycle $\alpha_2 \in \mathrm{CH}^4(X_1' \times X_4')$ such that $f'(\alpha_2) = 1 \times g_1^4$ and reducing it we obtain the rational cycle

$$\alpha_2 = b(h_1^4 \times 1) + (1 \times g_1^4),$$

where $b \in \{-1, 0, 1\}$. Hence, there is a rational cycle of the form

$$r = h_1^4 \times 1 + \varepsilon \cdot (1 \times g_1^4),$$

where the (indefinite) coefficient ε is either 1 or -1.

Now combining 5.2 and 5.3 together we obtain

5.4 Lemma. *For all $i = 0, \ldots, 7$ the cycles*

$$\rho_i = r^2 \cdot ((h_1^1)^i \times (g_1^1)^{7-i}) \in \mathrm{CH}^{15}(X_1' \times X_4')$$

are rational.

5.5. By the previous lemma all cycles

$$\rho_{7-i}^t \circ \rho_i \in \mathrm{CH}^{15}(X_1' \times X_1') \text{ and } \rho_i \circ \rho_{7-i}^t \in \mathrm{CH}^{15}(X_4' \times X_4')$$

where $i = 0, \ldots, 3$ are rational. Direct computations (see Appendix) show that these cycles are congruent modulo 3 to the following cycles in $\mathrm{CH}^{15}(X_1' \times X_1')$:

$$\begin{aligned}
&\rho_7^t \circ \rho_0 \equiv 1 \times h_1^{15} + h_1^4 \times (h_1^{11} + h_2^{11}) + h_1^8 \times (h_1^7 + h_2^7),\\
&\rho_6^t \circ \rho_1 \equiv h_1^1 \times h_1^{14} + (2h_2^5 + h_1^5) \times (h_2^{10} - h_1^{10}) + (h_2^9 - h_1^9) \times h_2^6,\\
&\rho_5^t \circ \rho_2 \equiv h_1^2 \times h_1^{13} + (h_1^6 + h_2^6) \times h_1^9 + (2h_1^{10} - h_2^{10}) \times (h_1^5 + h_2^5),\\
&\rho_4^t \circ \rho_3 \equiv h_1^3 \times h_1^{12} + h_2^7 \times (h_2^8 - h_1^8) + h_2^{11} \times (h_2^4 - h_1^4)
\end{aligned}$$

and in $\mathrm{CH}^{15}(X_4' \times X_4')$:

$$\begin{aligned}
\rho_0 \circ \rho_7^t &\equiv 1 \times g_1^{15} + g_1^4 \times (g_1^{11} - g_2^{11}) + g_1^8 \times (g_1^7 + g_2^7),\\
\rho_1 \circ \rho_6^t &\equiv g_1^1 \times g_1^{14} + (2g_1^5 - g_2^5) \times (g_1^{10} + g_2^{10}) + (g_1^9 + g_2^9) \times g_2^6,\\
\rho_2 \circ \rho_5^t &\equiv g_1^2 \times g_1^{13} + (g_1^6 - g_2^6) \times g_1^9 + (g_1^{10} + 2g_2^{10}) \times (g_2^5 - g_1^5),\\
\rho_3 \circ \rho_4^t &\equiv g_1^3 \times g_1^{12} + g_2^7 \times (g_2^8 - g_1^8) + g_2^{11} \times (g_1^4 + g_2^4)
\end{aligned}$$

respectively, which turn to be idempotents. Denote them by p_0', p_1', p_2', p_3' and q_0', q_1', q_2', q_4' respectively. Since they are congruent to rational cycles, they are rational. To complete the picture observe that the transposed cycles $(p_i')^t$ and $(q_i')^t$, $i = 0, \ldots, 3$, are rational idempotents as well. Hence, we have produced eight rational idempotents in each Chow ring.

5.6. Now easy computations show that these idempotents are orthogonal to each other, and their sum is equal to the respective diagonal cycle

$$\Delta_{X_1'} = \sum_{i,j,s} \delta_{ij} h_i^s \times h_j^{15-s} = \sum_{l=0}^{3} p_l' + (p_l')^t \in \mathrm{CH}^{15}(X_1' \times X_1'),$$

$$\Delta_{X_4'} = \sum_{i,j,s} \delta_{ij} g_i^s \times g_j^{15-s} = \sum_{l=0}^{3} q_l' + (q_l')^t \in \mathrm{CH}^{15}(X_4' \times X_4').$$

Hence, by 2.8 we obtain decompositions of the motives of X_1 and X_4

$$\mathcal{M}(X_1) = \bigoplus_{i=0}^{3} (X_1, p_i) \oplus (X_1, p_i^t),$$

$$\mathcal{M}(X_4) = \bigoplus_{i=0}^{3} (X_4, q_i) \oplus (X_4, q_i^t),$$

where $p_i \times_k k' = p_i'$ and $q_i \times_k k' = q_i'$ are pairwise orthogonal idempotents (over k).

5.7. By a straightforward computation using the definition of the idempotents p_i' and q_i' given in 5.5 we immediately obtain that

- there are isomorphisms of motives

$$\begin{aligned}
(X_1, p_i) &\simeq (X_1, p_0)(i), & (X_1, p_i^t) &\simeq (X_1, p_i)(7-i),\\
(X_4, q_i) &\simeq (X_4, q_0)(i), & (X_4, q_i^t) &\simeq (X_4, q_i)(7-i),
\end{aligned}$$

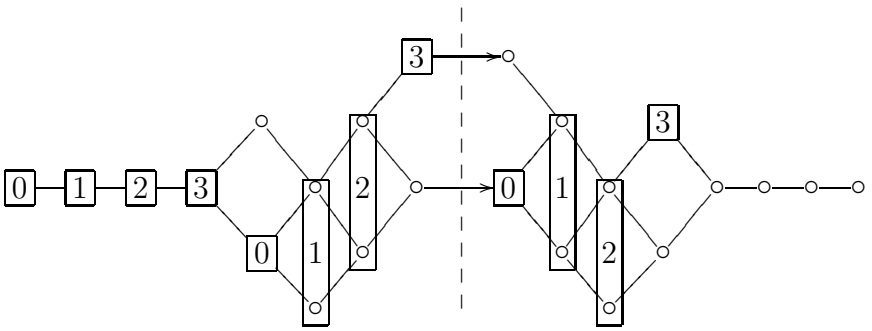
- over the cubic extension k' the motives (X_1, p_0) and (X_4, q_0) split as direct sums of Lefschetz motives $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$, where the shifts correspond to the codimensions of the first factors of p_0 and q_0.

5.8. To finish the proof of Theorem 1.1 we have to prove that the motives (X_1, p_0) and (X_4, q_0) are indecomposable. To see this observe that the group of endomorphisms $\mathrm{End}(X_1', p_0')$ is a free abelian group with the basis

$$\langle 1 \times h_1^{15}, h_1^4 \times (h_1^{11} + h_2^{11}), h_1^8 \times (h_1^7 + h_2^7) \rangle \tag{4}$$

Assume that (X_1, p_0) is decomposable, then the motive (X_1', p_0') is decomposable as well. The latter means that there exists a non-trivial rational idempotent in $\mathrm{End}(X_1', p_0')$. In this case easy computations show that one of the elements of the basis (4) must be rational. For instance, assume $1 \times h_1^{15}$ is rational. Then the cycle $(1 \times h_1^{15}) \cdot (h_1^{15} \times 1) = h_1^{15} \times h_1^{15}$ is rational and so is its image h_1^{15} by means of the push-forward $\mathrm{CH}_0(X_1' \times X_1') \to \mathrm{CH}_0(X_1')$ induced by a projection. Hence, we obtain a cycle of degree 1 on X_1, i.e., the variety X_1 must have a rational point [PR94, Cor. p. 205]. We have arrived to a contradiction. □

5.9 Remark. The following picture demonstrates how the realizations of motives (X_1', p_i'), $i = 0, 1, 2, 3$, are supported by the generators of the Hasse diagram (the numbers i drawn inside the rectangulars correspond to the motives (X_1', p_i')).



A similar picture for the motives (X_1', p_i^t) corresponding to transposed idempotents is obtained by the reflection along the vertical dashed line of the diagram.

Proof of Theorem 1.2. We use the notation of the proof of 1.1. Easy computations (see Appendix) show that

$$(\rho_0+\rho_1+\varepsilon\rho_2+\varepsilon\rho_3)+(\rho_0+\rho_1+\varepsilon\rho_2+\varepsilon\rho_3)^t\equiv\sum_{i,j,s}\pm\delta_{ij}h_i^s\times g_j^{15-s}.$$

Denote the right hand side by J. We have $J^t\circ J=\Delta_{X_1'}$ and $J\circ J^t=\Delta_{X_4'}$, i.e., J and J^t are two mutually inverse rational correspondences and the cycle J provides a rational motivic isomorphism between the motives of X_1' and X_4'. By Rost Nilpotence Theorem, J can be lifted to a motivic isomorphism between the motives of twisted forms X_1 and X_4. This completes the proof of Theorem 1.2. □

Appendix

Most of the computations of the present section were performed and checked using the Maple package by J. Stembridge [St04].

5.10. In the present paragraph we list the intermediate results of the computations of 5.5. First, using 4.12 we obtain

$$r^2\equiv -h_1^8\times 1-\varepsilon h_1^4\times g_1^4+1\times g_1^8.$$

Then, using 4.10 we obtain the following congruences for the cycles ρ_i

$$\begin{aligned}
\rho_0&\equiv -1\times g_1^{15}+\varepsilon h_1^4\times(g_1^{11}-g_2^{11})+h_1^8\times(g_1^7+g_2^7)\\
\rho_1&\equiv -h_1^1\times g_1^{14}+\varepsilon(h_2^5-h_1^5)\times(g_1^{10}+g_2^{10})+(h_2^9-h_1^9)\times g_2^6\\
\rho_2&\equiv -h_1^2\times g_1^{13}-\varepsilon(h_1^6+h_2^6)\times g_1^9+(h_1^{10}+h_2^{10})\times(g_2^5-g_1^5)\\
\rho_3&\equiv h_1^3\times g_1^{12}+\varepsilon h_2^7\times(g_2^8-g_1^8)-h_2^{11}\times(g_1^4+g_2^4)\\
\rho_4&\equiv (h_1^4-h_2^4)\times g_2^{11}+\varepsilon(h_2^8-h_1^8)\times g_2^7+h_1^{12}\times g_1^3\\
\rho_5&\equiv (h_1^5+h_2^5)\times(g_2^{10}-g_1^{10})+\varepsilon h_1^9\times(g_2^6-g_1^6)-h_1^{13}\times g_1^2\\
\rho_6&\equiv h_2^6\times(g_1^9+g_2^9)+\varepsilon(h_2^{10}-h_1^{10})\times(g_1^5+g_2^5)-h_1^{14}\times g_1^1\\
\rho_7&\equiv (h_1^7+h_2^7)\times g_1^8+\varepsilon(h_1^{11}+h_2^{11})\times g_1^4-h_1^{15}\times 1
\end{aligned}$$

5.11. The root enumeration follows Bourbaki [Bou]. To obtain the products of 4.12 we apply the Giambelli formula 3.9. Let $\bar\omega_i$, $i=1,\dots,4$, be the

fundamental weights. Then, the preimages of h_1^4 and g_1^4 in $S^*(P)$ can be expressed as polynomials in fundamental weights as follows

$$\begin{aligned} h_1^4 = c(&\frac{11}{6}\bar{\omega}_1^2\bar{\omega}_4^2 + \frac{3}{4}\bar{\omega}_1^2\bar{\omega}_2^2 - \frac{4}{3}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_3^2 + \frac{11}{6}\bar{\omega}_1^2\bar{\omega}_3^2 - \frac{2}{3}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4 + \frac{11}{12}\bar{\omega}_1^4 + \\ &\frac{1}{6}\bar{\omega}_2^4 - \frac{4}{3}\bar{\omega}_2\bar{\omega}_3^2\bar{\omega}_4 + \frac{4}{3}\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4^2 + \frac{2}{3}\bar{\omega}_2^2\bar{\omega}_3\bar{\omega}_4 + \frac{2}{3}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_4^2 - \frac{11}{6}\bar{\omega}_1^2\bar{\omega}_3\bar{\omega}_4 + \\ &2\bar{\omega}_1\bar{\omega}_3^2\bar{\omega}_4 - 2\bar{\omega}_1\bar{\omega}_3\bar{\omega}_4^2 - \frac{7}{12}\bar{\omega}_1^3\bar{\omega}_2 - \frac{11}{6}\bar{\omega}_1^2\bar{\omega}_2\bar{\omega}_3 + \frac{4}{3}\bar{\omega}_1\bar{\omega}_2^2\bar{\omega}_3 + \frac{2}{3}\bar{\omega}_2^2\bar{\omega}_3^2 - \\ &\frac{2}{3}\bar{\omega}_2^3\bar{\omega}_3 - \frac{1}{3}\bar{\omega}_1\bar{\omega}_2^3 - \frac{2}{3}\bar{\omega}_2^2\bar{\omega}_4^2), \end{aligned}$$

$$\begin{aligned} g_1^4 = c(&\frac{11}{6}\bar{\omega}_4^4 - \frac{7}{6}\bar{\omega}_3\bar{\omega}_4^3 + \frac{11}{12}\bar{\omega}_1^2\bar{\omega}_4^2 + \frac{3}{2}\bar{\omega}_3^2\bar{\omega}_4^2 - \frac{11}{6}\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4^2 + \frac{11}{12}\bar{\omega}_2^2\bar{\omega}_4^2 - \\ &\frac{11}{12}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_4^2 - \frac{2}{3}\bar{\omega}_3^3\bar{\omega}_4 - \frac{1}{2}\bar{\omega}_1^2\bar{\omega}_2\bar{\omega}_4 + \frac{1}{3}\bar{\omega}_1^2\bar{\omega}_3\bar{\omega}_4 + \frac{4}{3}\bar{\omega}_2\bar{\omega}_3^2\bar{\omega}_4 + \frac{1}{2}\bar{\omega}_1\bar{\omega}_2^2\bar{\omega}_4 - \\ &\frac{2}{3}\bar{\omega}_2^2\bar{\omega}_3\bar{\omega}_4 - \frac{1}{3}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4 + \frac{1}{3}\bar{\omega}_3^4 - \frac{1}{3}\bar{\omega}_1\bar{\omega}_2^2\bar{\omega}_3 + \frac{1}{3}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_3^2 - \frac{1}{3}\bar{\omega}_1^2\bar{\omega}_3^2 + \\ &\frac{1}{3}\bar{\omega}_1^2\bar{\omega}_2\bar{\omega}_3 + \frac{1}{3}\bar{\omega}_2^2\bar{\omega}_3^2 - \frac{2}{3}\bar{\omega}_2\bar{\omega}_3^3). \end{aligned}$$

Multiplying the respective polynomials and taking the c function, we find the products.

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Steklov Mathematical Institute, St. Petersburg, Russia

Fakultät für Mathematik, Universität Bielefeld, Germany

Max-Planck-Institut für Mathematik, Bonn, Germany
kirill@math.uni-bielefeld.de

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