THE MINIMAL NUMBER OF SINGULAR FIBERS OF A SEMISTABLE CURVE OVER $\mathbb{P}^1$

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1. Introduction

The purpose of this paper is to try to answer the following

Szpiro's Question. ([Sz], [B1]) Let $f : S \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ be a family of semistable curves of genus g, which is not trivial. Then, what is the minimal number of the singular fibers of f?

Beauville gives a lower bound for the number of singular fibers.

Beauville's Theorem. [B1] *With the notations as above, if $g \geq 1$, then*

1) *f admits at least 4 singular fibers.*

2) *If f admits 4 singular fibers, then S is algebraically simply connected with $p_g(S) = 0$, and the irreducible components of the 4 singular fibers are rational curves (may be singular), which generate a hyperplane of the $\mathbb{Q}$-vector space* $\mathrm{Pic}(S) \otimes \mathbb{Q}$.

Furthermore, Beauville [B1] gives an example of semistable elliptic fibration over $\mathbb{P}^1$ with 4 singular fibers, and one example of genus 3 with 5 singular fibers, and he gives also a series of such examples with 6 singular fibers for all $g > 1$. In fact, Beauville conjectured that for $g \geq 2$, there is no such fibrations with 4 singular fibers. In [B2], Beauville classified all semistable elliptic fibrations over $\mathbb{P}^1$ with 4 singular fibers.

Szpiro [Sz] considered that problem over a field with characteristic $p > 0$, and he proved that the minimal number of singular fibers is at least 3, and if the surface is of general type, then the number is at least 4.

The main result of this paper is

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Theorem 1 (Beauville's conjecture). *If $f : S \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ is a non-trivial semistable fibration of genus $g \geq 2$, then f admits at least 5 singular fibers.*

Theorem 1 is an immediate consequence of Beauville's Theorem and the following "strict canonical class inequality".

Theorem 2. *Let $f : S \longrightarrow C$ be a semistable fibration of genus $g \geq 2$ with s singular fibers. Then we have*

$$\deg f_*\omega_{S/C} < \frac{g}{2}(2g(C) - 2 + s).$$

The validity of these two theorems is heavily dependent on Miyaoka-Yau inequality. In Sect. 4, we shall give an example $f : S \longrightarrow \mathbb{P}^1$ of genus 2 with 5 singular fibers. Note that Beauville has also given such an example for $g = 3$.

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2. Preliminaries

2.1. Double coverings and ADE curve singularities. Let X be a normal surface, and Y a smooth surface, and let $\pi : X \longrightarrow Y$ be a double covering, branched along a curve $B \subset Y$. Then there exists a divisor δ on Y such that $B \equiv 2\delta$. Conversely, if B is an even divisor, i.e., $B \equiv 2\delta$ for some δ, then we can construct a double covering $\pi : X \longrightarrow Y$ such that π is branched along B. So π is determined by the data (Y, B, δ). The singularities of X come from those of B. Horikowa [Ho] gives a canonical resolution of the singularities of X, it goes as follows.

If p is a singular point of B, with multiplicity ν_p, then we blow up Y at p, $\sigma : Y_1 \longrightarrow Y$. Denote by E the exceptional curve of σ over p, then $\sigma^*(B) = \bar{B} + \nu_p E$. Let X_1 be the normalization of $X \times_Y Y_1$. Then it is a double covering of Y_1 with branch locus B_1,

$$B_1 = \bar{B} + (\nu_p - 2[\frac{\nu_p}{2}])E$$

Unless B_1 is nonsingular, we repeat this construction, and so on. It is not difficult to see that after a finite number of steps, the singularities of X can be resolved.

ADE curve singularities are defined as follows

$$\begin{aligned} A_n: &\quad x^2+y^{n+1}=0, \qquad n\geq 1,\\ D_n: &\quad y(x^2+y^{n-2})=0, \quad n\geq 4,\\ E_6: &\quad x^3+y^4=0,\\ E_7: &\quad x(x^2+y^3)=0,\\ E_8: &\quad x^3+y^5=0. \end{aligned}$$

Let $f:\Sigma\longrightarrow Y$ be a double covering determined by the data $(Y,\ B,\ \delta)$. If B has only ADE curve singularities, then Σ has only rational double points of the same type. Furthermore, we note that in this case, the canonical resolution of Σ is minimal.

2.2. Hyperelliptic fibrations. Let $f:S\longrightarrow C$ be a relatively minimal fibration of genus $g\geq 2$, i.e., S contains no (-1)-curves in a fiber of f. Let

$$\begin{aligned} \chi_f =&\chi(\mathcal{O}_S)-(g-1)(g(C)-1),\\ K_f^2 =&K_S^2-8(g-1)(g(C)-1),\\ e_f =&\chi_{\rm top}(S)-4(g-1)(g(C)-1). \end{aligned}$$

They are the basic invariants of f. If $K_{S/C}$ is the relative canonical divisor of f, then $K_f^2=K_{S/C}^2$ and $\chi_f=\deg f_*\omega_{S/C}$, where $\omega_{S/C}=\mathcal{O}(K_{S/C})$. If f is not locally trivial, by the well-known Arakelov-Parshine Theorem ([Ar], [Pa]), we know $\chi_f>0$ and $K_f^2>0$. Then we can define the slope of f as $\lambda_f=K_f^2/\chi_f$, which is an important invariant of f. In [X1], Xiao shows that if f is a locally non-trivial fibration of genus $g\geq 2$, then we have

$$\lambda_f\geq 4-\frac{4}{g}. \tag{1}$$

Furthermore, Konno [Ko] has recently shown that if the slope of f is $4-4/g$, then f is hyperelliptic, i.e., the general fibers of f are hyperelliptic curves. Cornalba and Harris [CH] have also obtained (1) and Konno's result for semistable fibrations.

Lemma 2.2. *If f is a (hyperelliptic) fibration with $\lambda_f=4-4/g$, then there is a geometrically ruled surface P over C and a double covering $\pi:\Sigma\longrightarrow P$, such that Σ has only rational double points as its singularities, and S is the canonical resolution of Σ. In fact, f is induced by the ruling of P over C. If $B\sim-(g+1)K_{P/C}+nF_0$ is the branch locus of π, then*

$$K_f^2=(2g-2)n, \qquad \chi_f=\frac{g}{2}n.$$

The proof of this Lemma can be found in [Pe] or [X2].

2.3. Miyaoka's inequality and Vojta's inequality. We refer to [Hi] for the details of the following Miyaoka's inequality.

Lemma 2.3. [Mi] *If S is a smooth surface of general type, and $E_1, \cdots, E_n$ are disjoint ADE curves on S, then we have*

$$\sum_{i=1}^{n} m(E_i) \leq 3c_2(S) - c_1^2(S),$$

where $m(E)$ is defined as follows,

$$\begin{aligned} m(A_r) &= 3(r+1) - \frac{3}{r+1}, \\ m(D_r) &= 3(r+1) - \frac{3}{4(r-2)}, \quad \textit{for } r \geq 4, \\ m(E_6) &= 21 - \frac{1}{8}, \\ m(E_7) &= 24 - \frac{1}{16}, \\ m(E_8) &= 27 - \frac{1}{40}. \end{aligned}$$

Finally, we should mention Vojta's interisting inequality for semistable fibrations f with s singular fibers, i.e., the "canonical class inequality"[Vo]:

$$K_f^2 \leq (2g-2)(2g(C) - 2 + s). \tag{2}$$

Our proof of Theorem 2 is based on these inequalities and Beauville's Theorem.

3. The proof of Theorem 2

In this section, we always assume that $f : S \longrightarrow C$ is a semistable fibration of genus $g \geq 2$ with s singular fibers $F_1, \cdots, F_s$. From (1) and (2), we have $\chi_f \leq \frac{g}{2}(2b - 2 + s)$, where b is the genus of C. In order to prove Theorem 2, we only need to derive a contradiction from the following assumption:

$$\chi_f = \frac{g}{2}(2b - 2 + s). \tag{3}$$

First we consider the base changes $\pi : \widetilde{C} \longrightarrow C$ of degree de, where d and e are natural numbers and π is ramified uniformly over the s critic points of f with ramification index e. By Kodaira-Parshin construction such a base change exists for all e if $b > 0$. Let $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{C}$ be the pullback fibration of f under π, then it is easy to know that $\widetilde{f}$ has ds singular fibers and the equality (3) also holds for $\widetilde{f}$. Note that if $b = 0$, then $s \geq 4$. By considering a base change totally ramified over the s critic points, we can assume that $b > 0$. Then by considering an étale base change of degree 2, we can assume also that s is even.

Combining (1) and (2) with (3), we have $\lambda_f = 4 - 4/g$.

By Lemma 2.2 we know that f is a hyperelliptic fibration and S can be constructed as in Lemma 2.2, i.e., it is from a double covering over a ruled surface $P \longrightarrow C$ and branched along a curve $B \sim -(g+1)K_{P/C} + nF_0$, where F_0 is a fiber of $P \longrightarrow C$. Since $K_f^2 = 2n(g-1)$ and $\chi_f = ng/2$, so we have

$$n = 2b - 2 + s. \tag{4}$$

Furthermore, Xiao [X2] has proved that $s > 0$ for locally non-trivial hyperelliptic fibrations.

In what follows, we shall consider the singularities of B so that f is semistable. First note that the fiber F_0 of P can not be contained in B. Otherwise, by Lemma 2.2, we know that the strict transform of F_0 in S is a curve with multiplicity 2, but not a (-1)-curve, this is impossible.

Lemma 3.1. *B has only double points as its singularities.*

Proof. Since f is semistable, so the connected components of the set of (-2)-curves in a fiber of f are of type A_n. Thus B has only double points as its singularities.

Q.E.D.

Lemma 3.2. *For any $p \in B \cap F_0$, the intersection number*

$$(B \cdot F_0)_p \leq 2.$$

Proof. Case I. (B, p) is nonsingular. We assume, on the contrary, that $n = (B \cdot F_0)_p \geq 3$. We shall claim that the fiber of f induced by F_0 has a singularity of type A_{n-1}, which is not an ordinary double point, a contradiction.

Indeed, in this case, there exists a local coordinate (x, y) at p such that

$$(F_0, p) = \{x = 0\}, \quad (B, p) = \{x + y^n = 0\}.$$

But the surface S is defined locally by $z^2 = x + y^n$, hence the fiber of f over F_0 is defined locally by

$$z^2 - y^n = 0,$$

which is a singular point of type A_{n-1}. This proves the claim.

Case II. (B, p) is a singular point. In this case, we consider the canonical resolution of the singularity. We denote by F the fiber of f corresponding to F_0. Let $\sigma : P_1 \longrightarrow P$ be the blowing-up of P at p, and let E be the exceptional curve of σ, we claim that $p_1 = \bar{F}_0 \cap E$ is not on $B_1 = \bar{B}$. Hence $(B \cdot F_0)_p = 2$.

Indeed, we assume, on the contrary, that $p_1 \in B_1$. If (B_1, p_1) is smooth, then the fiber F has a singular point over p_1, which is not a node, so F is not semistable, a contradiction. If (B_1, p_1) is singular, then it is a double point. Then we consider the next blowing-up σ_1 at p_1, it is easy to know that the strict inverse image of the exceptional curve of σ_1 is of multiplicity 2 in F, a contradiction. This proves the claim. **Q.E.D.**

From this lemma, we can divide the intersection points $p \in F_0 \cap B$ into the following three types.

A: $(F_0 \cdot B)_p = 1$,

B: $(F_0 \cdot B)_p = 2, (B, p)$ is smooth,

C: $(F_0 \cdot B)_p = 2, (B, p)$ is singular.

Now, we denote by F_{0i} the image of F_i in P. Let a_i (resp. b_i, c_i) be the number of points of type A (resp. B, C) on F_{0i}. Then, we have

$$a_i + 2b_i + 2c_i = 2g + 2, \qquad \text{for } i = 1, \cdots, 4. \tag{5}$$

Let $\mathcal{A} = \sum_{i=1}^{s} a_i$, $\mathcal{B} = \sum_{i=1}^{s} b_i$, $\mathcal{C} = \sum_{i=1}^{s} c_i$. Then, by (5), we have

$$\mathcal{B} + \mathcal{C} + \frac{1}{2}\mathcal{A} = sg + s. \tag{6}$$

In what follows, we denote by μ_p the Milnor number of the singular point (B, p), i.e., if (B, p) is of type A_n, then $\mu_p = n$. Let $\mu(B) = \sum_{p \in B} \mu_p$.

Lemma 3.3.

$$\mu(B) = (4n - s)g + 2n - s + \frac{1}{2}\mathcal{A}. \tag{7}$$

Proof. From (4), we have $2b - 2 = n - s$, hence $K_P \sim K_{P/C} + (n - s)F_0$, and

$$-2\chi(\mathcal{O}_B) = B^2 + K_P B = (g + 1)(6n - 2s) - 2n.$$

On the other hand,

$$\begin{aligned}
\chi_{\text{top}}(B) &= \chi_{\text{top}}\left(B - \sum_{i=1}^{s} F_{0i} \cap B\right) + \mathcal{A} + \mathcal{B} + \mathcal{C} \\
&= (2g + 2)(\chi_{\text{top}}(C) - s) + (g + 1)s + \frac{1}{2}\mathcal{A} \\
&= (g + 1)(s - 2n) + \frac{1}{2}\mathcal{A}.
\end{aligned}$$

Then by ([Ta], Lemma 1.1), we have

$$\mu(B) = \chi_{\text{top}}(B) - 2\chi(\mathcal{O}_B),$$

so we can obtain immediately the desired formula. **Q.E.D.**

Lemma 3.4.

$$g \leq 1 + \frac{3}{4s}\mathcal{B} + \sum_{\mu_p \geq 1} \frac{3}{4s(\mu_p + 1)}.$$

Proof. We shall prove this lemma by using Miyaoka's inequality. We consider a new curve $D = B + \sum_{i=1}^{s} F_{0i}$. By assumption s is even, so D is an even curve. Hence, we can construct a new double covering over P branched along D, $\pi_1 : S_1 \longrightarrow P$. If fact, we can see that D has only ADE singular points of the following types on F_{0i}:

$$a_i A_1 + b_i A_3 + \sum_{\mu_p \geq 1, p \in F_{0i}} D_{\mu_p + 3}. \tag{8}$$

From Lemma 2.2, we have

$$K_{f_1}^2 = (2g-2)(n+s), \quad \chi_{f_1} = \frac{g}{2}(n+s).$$

By Beauville's Theorem, if $b = 0$, then $n + s > 4$, which implies $K_{S_1}^2 > 0$ and $\chi(\mathcal{O}_{S_1}) \geq 2$. Hence we know that S_1 is of general type.

$$\begin{aligned} 3c_2(S_1) - c_1^2(S_1) &= 36\chi_{f_1} - 4K_{f_1}^2 + 2(g-1)(n-s). \\ &= (12n + 8s)g + 6n + 10s. \end{aligned} \tag{9}$$

Note that

$$m(A_1) = \frac{9}{2},\ m(A_3) = \frac{45}{4},\ m(D_{\mu_p+3}) = 3(\mu_p + 4) - \frac{3}{4(\mu_p + 1)}.$$

From (8), we have

$$\begin{aligned} \sum_p m(E_p) &= \frac{9}{2}\mathcal{A} + \frac{45}{4}\mathcal{B} + \sum_{\mu_p \geq 1} \left(3(\mu_p + 4) - \frac{3}{4(\mu_p + 1)}\right) \\ &= \frac{9}{2}\mathcal{A} + \frac{45}{4}\mathcal{B} + 3\mu(B) + 12\mathcal{C} - \sum_{\mu_p \geq 1} \frac{3}{4(\mu_p + 1)} \\ &= \frac{9}{2}\mathcal{A} + \frac{45}{4}\mathcal{B} + 3\left((4n - s)g + 2n - s + \frac{1}{2}\mathcal{A}\right) \\ &\quad + 12\left((g+1)s - \mathcal{B} - \frac{1}{2}\mathcal{A}\right) - \sum_{\mu_p \geq 1} \frac{3}{4(\mu_p + 1)} \\ &\qquad \text{by (6) and (7)} \\ &= (12n + 9s)g + 6n + 9s - \frac{3}{4}\mathcal{B} - \sum_{\mu_p \geq 1} \frac{3}{4(\mu_p + 1)}. \end{aligned}$$

Now by (9) and Miyaoka's inequality $3c_2(S_1) - c_1^2(S_1) \geq \sum_p m(E_p)$, we can obtain the inequality of the lemma. **Q.E.D.**

Lemma 3.5. *Under the assumptions above, we have $g \leq 1$.*

Proof. Note first that the inequality of Lemma 3.4 holds for all fibrations satisfying (3). Now we consider the pullback fibration $\widetilde{f}$ of f under a base change of degree de given at the begging of this section, where $e > 1$. In fact, $\widetilde{f}$ is determined by the data $(\widetilde{P}, \widetilde{B})$, where $\widetilde{P} \longrightarrow \widetilde{C}$ is the pullback ruled surface under the base change, and $\widetilde{B}$ is the inverse image of B in $\widetilde{P}$. If we denote by $\widetilde{\cdot}$ the corresponding objects of $\widetilde{B}$, then we have

$$\widetilde{s} = ds, \quad \widetilde{\mathcal{A}} = d\mathcal{A}, \quad \widetilde{\mathcal{B}} = 0, \quad \widetilde{\mathcal{C}} = d\mathcal{B} + \mathcal{C}, \quad \mu_{\widetilde{p}} = e\mu_p + e - 1.$$

Applying Lemma 3.4 to $\widetilde{f}$ we obtain

$$g \leq 1 + \frac{3}{4se}\mathcal{B} + \frac{3}{4se}\sum_{\mu_p \geq 1} \frac{1}{\mu_p + 1}.$$

Taking e large we have $g \leq 1$. **Q.E.D.**

This contradicts the assumption $g \geq 2$. Hence we have proved Theorem 2.

Remark 3.6. Using a similar method we can prove that the equality in Vojta's inequality (2) holds only if f is a smooth fibration.

4. An example of genus $g = 2$ with $s = 5$

In this section, we shall construct a semistable fibration $f : S \longrightarrow \mathbb{P}^1$ of genus 2 with 5 singular fibers.

Let ϕ and $\psi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be two morphisms with $\deg\phi + \deg\psi = 2g + 2$. We assume that there exists a subset $R = \{p_1, \cdots, p_5\} \subset \mathbb{P}^1$ satisfying

i) the branched points of ϕ and ψ are contained in R, and the ramification points of them are of index 2.

ii) $\phi^{-1}(p_i) \cap \psi^{-1}(p_i)$ consists of non-ramified points of ϕ and ψ, and if $p \notin R$, then $\phi^{-1}(p) \cap \psi^{-1}(p)$ is empty.

In $\mathbb{P}^1 \times \mathbb{P}^1$, we consider the divisors Γ_ϕ and Γ_ψ, graphes of ϕ and ψ respectively. Let $B = \Gamma_\phi + \Gamma_\psi$. Then B is an even divisor of type $(2g + 2, 2)$ satisfying the conditions of Lemma 3.2. Let $\pi : \Sigma \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a double cover branched along B, and let S be the canonical resolution of the singularities of Σ. Then the second projection $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ induces a semistable fibration of genus g with 5 singular fibers.

Now we give an example of genus 2 with 5 singular fibers. Let a and b be two nonzero complex numbers such that the discriminant of the polynomial

$$p(x) = x^3 + (2a - b^2)x^2 + (a^2 + 2ab^2)x - a^2b^2,$$

is zero. Hence $p(x)$ has (at most) two zeros x_1 and x_2. Let ϕ and $\psi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be two morphisms defined by

$$\phi(t) = t^2 + \frac{a^2}{t^2}, \qquad \psi(t) = -2a\frac{t^2 + b^2}{t^2 - b^2}.$$

Note that if $\phi(t) = \psi(t)$, then $p(t^2) = 0$. Take $R = \{\infty, 2a, -2a, x_1 + a^2/x_1, x_2 + a^2/x_2\}$. It is easy to check that ϕ and ψ satisfy i) and ii). This completes the construction.

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