Wolfgang Ziller

Report for the Arbeitstagung

"Curvature and symmetry of Milnor spaces"

In joint work with Karsten Grove, we examine the geometry and topology of cohomogeneity one manifolds, which are manifolds M^n with a Lie group action G such that $\dim(M/G) = 1$. On the geometric side we prove

Theorem A: Every cohomogeneity one manifold with two singular orbits of codimension two admits an invariant metric with non-negative sectional curvature.

As a first consequence one obtains:

Corollary: The four diffeomorphism types homotopy equivalent to $\mathbb{R}P^5$ admit infinitely many metrics with non-negative curvature.

On the topological side we prove

Theorem B: Every $L = SO(3)$ or $SO(4)$ principal bundle over S^4 admits a $L \times SO(3)$ cohomogeneity one metric with singular orbits of codimension two.

Theorem A and B imply:

Corollary: Every vector bundle and every sphere bundle over S^4 admits a metric with non-negative curvature.

In the case of vector bundles, this is the first progress in the 27-year-old Cheeger-Gromoll question, which asks which vector bundles over S^n admit a metric with non-negative curvature.

For sphere bundles, the interesting case are S^3 bundles over $S^4 : S^3 \to M_{k,\ell} \to S^4$ which are classified by a pair of integers (k, ℓ) in $\pi_3(SO(4)) = \mathbb{Z} + \mathbb{Z}$. They are homeomorphic to S^7 if $k + \ell = 1$, and their diffeomorphism classification is given by $\frac{k(k-1)}{2} \bmod 28$. Hence one obtains:

Corollary: 15 of the 27 exotic spheres in dimension 7 admit infinitely many metrics with non-negative curvature.

The $SO(3)$ actions in Theorem B which commute with the principal L action, induce (isometric) actions on the associated sphere bundles, which in most cases are almost free (i.e. have finite isotropy groups). This implies:

Corollary: For each $n \geq 1$, there exists an almost free action of $SO(3)$ on S^7 with isotropy groups $1, \mathbb{Z}_2, O_2, D_n, D_{n+1}, D_{n+2}, D_{n+3}$ which preserves the Hopf fibration $S^7 \to S^4$ and which does not extend to the disc D^8. Similarly, the 15 exotic Milnor spheres admit infinitely many almost free actions by $SO(3)$.