THE CUT LOCUS ON NONCOMPACT MANIFOLDS

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It has been known for some time that the study of the cut locus of a compact Riemannian manifold leads to a map of the disc of the same dimension into the manifold which is a homeomorphism on the interior and takes the boundary to the cut locus. This mapping is useful in studying the topology of the manifold. In this paper it will be shown that such a mapping can also be defiend for a complete manifold, provided that both the manifold and the cut locus are compactified by the end compactification. This is of interest because of the further result that a manifold admits a complete metric with empty cut locus if and only if it is diffeomorphic to euclidean space. This, even though the manifold is contractible, there is a possibility that there will be interesting elements in the higher homotopy groups of the pair consisting of the compactified manifold and its compactified cut locus.

The precise statement regarding the existence and homotopy invariance of the mapping of the disc into the end compactification will be given in § 2, after the necessary

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definitions have been made. A brief discussion of topological consequences will be given in § 5. Throughout, we shall assume as the basic reference on the cut locus the paper of Kobayashi [2], and as the basic reference on ends of a manifold the paper of Siebenmann [3].

Throughout the paper M will denote a smooth, connected, paracompact, Hausdorff manifold of dimension m. Such a manifold admits a complete Riemannian metric and can be exhibited in many ways as the countable union of compact sets, each contained in the interior of the next.

1. The cut locus. Suppose M is given a complete Riemannian metric. Then given any point P and unit vector ξ at P, there is a semigeodesic $\gamma(t), 0 \leq t < \infty$, with $\gamma(0) = P$ and $\dot{\gamma}(0) = \xi$. (The parameter on a geodesic will always be taken to be the arc length.)

Definition 1.1. The cut point of P along γ is the point $\gamma(r)$, where r is the least upper bound of numbers s such that the segment $g(t)$, $0 \leq t \leq s$, is minimizing. If this set is unbounded, then γ is called a ray. The cut locus of P is the set consisting of all cut points of P along all semigeodesics starting at P. The cut locus will be denoted by C.

It is known that if $s < r$, then the segment from P to $\gamma(s)$ is uniquely minimizing, and a ray is uniquely minimizing for its entire length. Moreover, either $\gamma(r)$ is the first conjugate point of P along γ, or there exist at least two minimizing geodesics from P to $\gamma(r)$. In any case, the cut point comes before or at the first conjugate point.

Proposition 1.2. Given a manifild M and a point P of M, there is a complete Riemannian metric on M such that P has empty cut locus if and only if M is diffeomorphic to $\mathbb{R}^m$.

Proof. If M is diffeomorphic to $\mathbb{R}^m$, then the metric induced from the Euclidean metric by a diffeomorphism onto $\mathbb{R}^m$ has the required properties. If M is not diffeomorphic to $\mathbb{R}^m$. then for any complete Riemannian metric and any point P, the exponential map at P is not a diffeomorphism. Hence, either it is singular at some point, or it is not one-one. In the first case, some geodesic has a conjugate point and hence a cut point. In the second case, some point is reached by at least two geodesics, each of which must contain a cut point.

2. Rays and ends. First let us recall some facts about ends which will be needed. Let $\{K_j\}_{j=1}^{\infty}$ be a collection of compact subsets, each contained in the interior of the next, whose union is M. Let V_j be a connected component of $M-K_j$, and let $\{V_j\}_{j=1}^{\infty}$ be such that $V_{j+1} \subset V_j$. Then $\{V_j\}$ defines an end, and each V_j is a neighbourhood of that end. The set $E(M)$ of ends of M is topologized as a totally disconnected compact Hausdorff space, and $\hat{M} = M \cup E(M)$ is topologized as a Hausdorff compactification of M.

Definition 2.1. A ray approaches an end if given any neighbourhood of the end, all but a finite portion of the ray is contained in the neighbourhood.

Proposition 2.2. Each ray approaches some end, and each end is approached by some ray.

Proof. Suppose a ray γ starting at P is given, and consider a sequence $\{K_j\}$ of compact sets as above. Let

$$\sigma(K_j) = \sup \{ d(P,Q) \mid Q \in K_j \} .$$

Then for $t > \sigma(K_j)$, $\gamma(t) \notin K_j$, and since the rest of the ray beyond $\gamma(\sigma(K_j))$ is connected, it is contained in a single component $V_j(\gamma)$ of $M-K_j$. Since also $V_{j+1}(\gamma) \subset V_j(\gamma)$, the sequence $\{V_j(\gamma)\}$ defines an end which is approched by γ .

The proof of the second statement is an adaptation of the proof that any complete, noncompact manifold admits a ray (Gromoll and Meyer [1]). Let $\{V_j\}$ be a sequence defining an end as above, let $x_j \in V_j$, let γ_j be a minimal geodesic segment from P to x_j , and let ξ_j be the initial vector of γ_j . We may suppose $\{\xi_j\}$ approaches a limit ξ , and consider the semigeodesic γ from P with initial vector ξ. This semigeodesic is a ray, since the continuity of solutions of the geodesic differential equation with respect to the initial conditions implies that through any neighbourhood of any point of γ , there pass arbitrarily long minimal geodesic segments.

Moreover, this ray approaches the given end, so the proof is complete.

Let $\hat{C} = CUE(M)$. Let D be the unit disc in the tangent space at some point of M , let S be its boundary, and let $I = [0,1]$. The following theorem will be obtained as a corollary of a more general result proved in § 4.

Theorem 2.3. Given a point P_o of M and a smooth complete Riemannian metric g_o with cut locus C_o with respect to P_o , there is a continuous map

$$\varepsilon(0,\ ) : (D,S) \longrightarrow (\hat{M},\hat{C}_o)$$

which is equal to the exponential map near the origin and induces a homeomorphism of $D-S$ onto $M-C_o$. Given also $P_1, g_1, \varepsilon(1,\ )$ constructed as above, and a path $\gamma : I \longrightarrow M$ from P_o to P_1 , let $P_t = \gamma(t)$ and

$$g_t = (1-t)g_o + tg_1 .$$

Then g_t is a complete Riemannian metric with cut locus C_t and there is a continuous map

$$\varepsilon : I \times (D,S) \longrightarrow (I \times \hat{M}, V_t(\{t\} \times \hat{C}_t))$$

which commutes with projection onto I , satisfies the above conditions for each t , and such that $\varepsilon(0,)$ and $\varepsilon(1,)$ are the previously constructed maps.

Note that in the statement of the thereom, the product $I\times(D,S)$ is realized by using for $t\in I$ the unit disc in the targent space at P_t with respect to the metric g_t .

§ 3. Change of metric. Given two complete Riemannian metrics g_o and g_1 on M , let g_t be as in the statement of Theorem 2.3 and let $d(t,P,Q)$ be the distance from P to Q with respect to g_t . The proof of the theorem requires some properties of g_t , which will now be obtained.

Lemma 3.1. g_t is a smooth complete Riemannian metric.

Proof. Everything except the completeness is well-known. For any curve γ with parametrization $\gamma(S)$, the energy with respect to g_t is defined by

$$E_t(\gamma) = \int g_t(\dot\gamma,\dot\gamma)ds = (1-t)\int g_o(\dot\gamma,\dot\gamma)ds+t\int g_1(\dot\gamma,\dot\gamma)ds$$

so that $E_t(\gamma) = (1-t)E_O(\gamma)+tE_1(\gamma)$. Let t be fixed satisfying $0 < t < 1$. Then

$$d^2(0,P,Q) = \inf_\gamma E_o(\gamma) \leq \frac{1}{1-t} \inf_\gamma E_t(\gamma)$$
$$d^2(1,P,Q) = \inf_\gamma E_1(\gamma) \leq \frac{1}{t} \inf_\gamma E_t(\gamma) .$$

Hence, a sequence which is Cauchy for g_t is also Cauchy for g_o and g_1 , and conveges to the same limit for all three metrics. It follows that g_t is complete, as required.

Lemma 3.2. d is continuous on $I \times M \times M$.

Proof. Suppose d is discontinuous at (t,P,Q), and let γ be a minimal geodesic from P to Q with respect to g_t. Since the length of γ varies continuously with t, we know that for any given $\varepsilon > 0$, and for (t^1,P^1,Q^1) near enough to (t,P,Q), $d(t^1,P^1,Q^1) < d(t,P,Q) + \varepsilon$. Hence, there is an $\varepsilon > 0$ and a sequence $\{(t_i,P_i,Q_i)\}$ converging to (t,P,Q) so that $\{a_i = d(t_i,P_i,Q_i)\}$ converges to some number $a < d(t,P,Q) - \varepsilon$. If γ_i is a minimal geodesic from P_i to Q_i for the metric g_{t_i}, we may suppose that the initial unit vectors ξ_i converge to a unit vector ξ, necessarily based at P. Then the geodesic for g_t with initial vector ξ and length a joins P to Q, a contradiction. Here and later we need continiuty of solutions with respect to the parameter t as well as the initial conditions.

§ 4. Modified exponential mappings. The idea in constructing the maps required in Theorem 2.3 is to reparametrize each semigeodesic going out from a point so that either it reaches its cut point at time 1, or if it is a ray, it approaches its end as the time approaches 1. In order to do this in a continuous way, it is convenient to introduce a function $R(a,b,r)$ defined for

$$0 < a < b \leq \infty, \; a < \frac{3}{4}, \; 0 \leq r \leq 1$$

with values in $[0,\infty]$ and having the following properties;

i) R is continuous, and smooth on $R^{-1}([0,\infty))$.

ii) $R(a,b,\ )$ is equal to the identity on $[0,a]$, is a homeomorphism from $[0,1]$ to $[0,b]$, and is a diffeomorphism on $[0,1]$ (respectively $[0,1)$) if $b < \infty$ (respectively $b = \infty$) .

Such functions clearly exist.

Lemma 4.1. Given two functions R_o and R_1 with the above properties, then the function R_t defined by

$$R_t(a,b,r) = (1-t)R_o(a,b,r) + tR_1(a,b,r)$$

also has these properties.

Proof. This follows immediatly from the convexity of the set of positive real numbers.

If $T(M)$ is the tangent bundle of M , then a map of $I\times T(M)$ into $I\times M$ is obtained by using on each fiber of $\{t\}\times T(M)$ the exponential map of the metric g_t . If $D(M)$ and $S(M)$ are the unit disc and sphere bundles respectively, then $I\times D(M)$ and $I\times S(M)$ are constructed by using at (t,P) the metric g_t in the fiber over P. Let $\rho : I\times S(M) \longrightarrow [0,\infty]$ be the function such that $\rho(t,P,\xi)$ is the distance from P to the cut point on the g_t-semigeodesic with initial vector ξ. If this semigeodesic is a ray, then $\rho(t,P,\xi) = \infty$.

Lemma 4.2. ρ is continuous.

Proof. This is proved by Kobayashi for fixed t , and only minor modifications are required. Let $\pi : T(M) \longrightarrow M$ and suppose $\{(t_i,P_i,\xi_i)\}$ converges to (t,P,ξ) but

$\{\rho(t_i,P_i,\xi_i) = a_i\}$ converges to some number a different from $\rho(t,P,\xi)$. If $a>\rho(t,P,\xi)$, then

$$d(t,P,\exp a\xi) = \lim d(t_i,\pi(\xi_i),\exp(a_i\xi_i))$$
$$= \lim a_i = a$$

so that $\exp(t\xi)$ is minimizing out to a, a contradiction. If $a < \rho(t,P,\xi)$, then a_i and a are finite and strictly positive. Let $F: I\times T(M) \longrightarrow I\times M\times M$ be defined by

$$F(t,X) = (t,\pi(X), \exp(X))$$

Then F is a diffeomorphism on some neighbourhood V of $a\xi$. We may assume that all $a_k\xi_k$ belong to V, so $\exp(a_k\xi_k)$ is not a conjugate point, and therefore there is a second geodesic with $\exp(a_K\eta_k) = \exp(a_k\xi_k)$ and $a_k\eta_k$ not in V. We may assume that η_k converges to some η, where $a\eta$ is not in V. However, $\exp(a\eta) = \exp(a\xi)$, so we have 2 distinct geodesics to the same point which comes before the cut point, a contradiction.

Let $k(b) = \min\{\frac{1}{2},\frac{1}{2}b\}$. Unless $r=1$ and $\rho(t,P,\xi)=\infty$, we define $\varepsilon(t,P,r\xi)$ to be the exponential at P with respect to the metric g_t of the vector

$$R_t(k(\rho(t,P,\xi)),\ \rho(t,P,\xi),r)\ \xi\ .$$

In the remaining case, $\varepsilon(t,P,\xi)$ is defined to be the end approached by the ray with initial vector ξ.

Proposition 4.3. $\varepsilon: I\times D(M) \longrightarrow I\times\hat{M}$ is continuous, and for each fixed (t,P), it takes D onto $\{t\}\times\hat{M}$, S onto $\{t\}\times\hat{C}_t$, is equal to the exponential map on some neighbourhood of the origin, and induces a homeomorphism of $D-S$ onto $\{t\}\times(M-C_t)$.

Proof. The continuity follows from the lemmas, except in the case that $r=1$ and $\rho(t,P,\xi)=\infty$. However, continuity of solutions of the geodesic equation with respect to initial conditions and the parameter t implies that given any neighbourhood of the end $\varepsilon(t,P,\xi)$, any geodesic close enough to the g_t-ray with initial vector (P,ξ) is either a ray or has its cut point in the given neighbourhood. Thus, ε is continuous. The rest of the properties are clear.

Proof of theorem 2.3. Map I into $I\times M$ by taking t to $(t,\gamma(t))$. The pull-back of the bundle $I\times D(M)$ is a trivial disc bundle over I, and the function ε discussed in Proposition 4.3 gives rise to the mapping ε required in Theorem 2.3.

§ 5. Topological consequences. It follows that $\hat{C}$ is compact, arcwise connected, Hausdorff, and his a countable basis for its topology. Since it is arcwise connected, its fundamental group is defined, but it may not have a universal cover because the neighbourhoods of an end can be very complicated. Since $\hat{C}$ is compact, C is closed in M.

Just as in the compact case, $\pi_k(\hat{M},\hat{C})=0$ for $0\le k\le m-1$, and $\pi_m(\hat{M},\hat{C})$ maps onto $H_m(\hat{M},\hat{C})$. In addition, the homomorphisms

$$\varepsilon(t,P,\ )_{\#} : \pi_k(D,S) \longrightarrow \pi_k(\hat{M},\hat{C})$$

are defined for all $k\ge m$ and are independent of the choice of the metric, of the point with respect to which the cut locus is taken, and of the function R of § 4. These can also be viewed as homomorphisms

$$\pi_{k-1}(S^{m-1}) \longrightarrow \pi_k(\hat{M},\hat{C}) .$$

In particular,one would like to study these maps in the case of contractible open subsets of $\mathbb{R}^3$ and of smooth manifolds homeomorphic to $\mathbb{R}^4$, since in these cases the space itself has no homotopy-theoretic invariants. Since a contractible space has one end, in these cases $\hat{M}$ is just the one-point compactification.

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