Notes on Varieties of Codimension 3 in $\mathbb{P}^N$

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Introduction. This article contains a slightly expanded version of notes for an informal talk which I gave in Oberwolfach in August 1992. In this talk I tried to stimulate the investigation of smooth subvarieties of codimension 3 in $\mathbb{P}^N$ or other ambient spaces by collecting known results, pointing out some construction methods, and by raising some - hopefully interesting - problems. Afterwards several people suggested to write down and distribute the notes, which I do hereby.

In section 1 I review the construction of new smooth subvarieties from given ones by general linking. This is a classical construction method which has been applied very successfully in the codimension-2 case.

Here I want to stress the fact that linking allows to produce smooth subvarieties $X \subset P$ of any codimension provided the dimension of X is at most equal to 3. This fact is well known to the experts (I have learned it from A. Van de Ven) and appeared at least implicitly in the fundamental paper by Ch. Peskine and L. Szpiro [P/S], but - to the best of my knowledge - has not been used systematically for the construction of smooth subvarieties of codimension 3.

In section 2 I recall the construction of subvarieties $Y \subset P$ as zero-sets of general sections in smooth reflexive sheaves, and - closely related to this - as degeneracy loci of general vector bundle morphisms over P.

Both methods allow to construct smooth codimension-3 subvarieties $Y \subset P$ in projective N-folds P as long as $N \leq 7$.

This is a folklore fact for degeneracy loci; a useful filtered version has been proved by M.-C. Chang [C]. The idea to consider zero-sets of suitable reflexive sheaves is due to R. Hartshorne [H3], and has later been generalized by Ch. Okonek [O1], A. Hirschowitz/R. Marlin [H/M], and C. Bănică [B1].

The main novelty of this article is contained in section 3. Here I describe the construction of codimension-3 subvarieties $Z \subset P$ as Pfaffians of twisted skew-symmetric morphisms $f : E \to E^\vee \otimes L$ of vector bundles of odd ranks over P. A Bertini-type argument shows that this method yields smooth subvarieties if the dimension of P is strictly less than 10. The Grassmannian $G(1,4) \subset \mathbb{P}^9$ is the simplest example of this type.

Pfaffians occur naturally in many situations: As universal Pfaffians - like e.g. $G(1,4) \subset \mathbb{P}^9$ -, or as intersections of certain linked subvarieties of codimension 2; they can also be considered as natural generalizations of zero-sets of regular sections in rank-3 vector bundles. The latter point of view suggests an obvious question: Under which conditions is a submanifold $Z \subset P$ of codimension 3 a Pfaffian of a vector bundle morphisms $f : E \to E^\vee \otimes L, rkE \equiv 1(mod 2)$?

There is at least one necessary condition: Pfaffians are always subcanonical, i.e. their canonical sheaves are restrictions of line bundles on the ambient space P. It is not likely that this conditions is already sufficient; I have, however, not been able to say very much about this problem except translating it into a question about symplectic reflexive sheaves.
In section 4 I concentrate on the special case $P = \mathbb{P}^N$ and collect the known results about submanifolds $Z \subset \mathbb{P}^N$ of codimension 3. If $N \geq 10$, then we are in the Hartshorne range, so that all such submanifolds are expected to be complete intersections. It turns out that in the range $N = 8, 9$ all known smooth examples are actually Pfaffians. Since - as a consequence of the Barth-Lefschetz theorems - every smooth, codimension-3 subvariety in $\mathbb{P}^8$ and $\mathbb{P}^9$ is automatically subcanonical, it makes sense to ask, if all codimension-3 submanifolds in $\mathbb{P}^8$ and $\mathbb{P}^9$ are necessarily Pfaffians. If this were true, we had a second range $N = 8, 9$ below the Hartshorne range in which only one construction method would produce all smooth subvarieties. This is, of course, pure speculation at the moment.
For constructions in $\mathbb{P}^7$ reflexive sheaves and degeneracy loci of general vector bundle morphisms can be used. These degeneracy loci in $\mathbb{P}^7$ are simply-connected, again by Barth-Lefschetz. It would be interesting to know if there is a Barth-Lefschetz-type result for degeneracy loci of bundle morphisms over arbitrary base spaces P. In $\mathbb{P}^6$ we have - in addition to the previous methods - also general linking as a possible construction technique. I have written down some examples with small invariants. Finally - for the benefit of the reader - I have produced a list which contains the classification of all (families of) codimension-3 submanifolds in $\mathbb{P}^N, N = 6, 7, 8, 9$ of degree $d \leq 8$. This classification is due to F. Zak [Z], G. Scorza [S], T. Fujita [F], and P. Ionescu [I1], [I2], [I3]; the degree-9 and degree-10 classification is presently being worked out by M. Fania / L. Livorni [Fa/Li1], [Fa/Li2][1].

I like to thank F. Catanese, A. Van de Ven, and in particular F. Schreyer for very useful discussions about the subject of this article; in fact, originally it was planned as a joint paper with F. Schreyer, but it did not quite materialize in this way.
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1 Liaison

Let P be a complex-projective manifold of dimension N, $X \subset P$ a smooth subvariety of codimension c defined by the ideal I_X. Consider line bundles $L_i \in Pic(P)$ and sections $s_i \in H^0(P; I_X \otimes L_i), i = 1, \cdots, c$. The complete intersection $Z(\underline{s})$ has the form $Z(\underline{s}) = X \cup X'$ for a closed subvariety $X' \subset P$. X' is said to be linked to X via $\underline{s} = (s_1, \cdots, s_c)$ if X' is of codimension c and if X and X' have no common irreducible component.

[1] see 'Note added in proof'

Proposition. Let $X \subset P$ be a smooth subvariety of dimension n in a projective N-fold P. Suppose $L_i \in Pic(P)$ are line bundles such that $I_X \otimes L_i$ is globally generated for each $i = 1, \cdots, N-n$, and let X' be linked to X via a general section $\underline{s} \in H^0(P; \oplus_{i=1}^{N-n} I_X \otimes L_i)$. If $n < 4$, then X' is smooth.

Sketch of proof: Consider the blow-up $\sigma : \hat{P} \to P$ of P along X and denote the restriction of σ to the exceptional divisor $E \subset \hat{P}$ by π:

$$\begin{array}{ccc} E & \hookrightarrow & \hat{P} \\ \downarrow \pi & & \downarrow \sigma \\ X & \hookrightarrow & P \end{array}$$

The sections $s_i \in H^0(P; I_X \otimes L_i)$ correspond to sections $\hat{s}_i \in H^0(\hat{P}; \mathcal{O}_{\hat{P}}(-E) \otimes \sigma^* L_i)$; the line bundles $\mathcal{O}_{\hat{P}}(-E) \otimes \sigma^* L_i$ on $\hat{P}$ are globally generated as quotients of $\sigma^*(I_X \otimes L_i)$. Let $\hat{X} \subset \hat{P}$ be the complete intersection $\hat{X} = Z(\hat{\underline{s}})$ in $\hat{P}$ defined by $\hat{\underline{s}}$. The usual Bertini theorem on $\hat{P}$ shows $\hat{X}$ is smooth and intersects the exceptional divisor E transversally in $\hat{X} \cap E$.
Since $Z(\underline{s}) = X \cup X'$ with $X' = \sigma(\hat{X})$ we have to prove smoothness of $\sigma(\hat{X})$. But this is obvious in $P \backslash X$, and by transversality it suffices to show that the restriction $\pi | \hat{X} \cap E$ is one-to-one.
Since the fibers of $\pi | \hat{X} \cap E$ are linear subspaces of the fibers $E(x) = \pi^{-1}(x)$, we must guarantee that $\hat{X} \cap E$ contains no line of $E(x)$ for every point $x \in X$. Let therefore $\mathbb{G}_i := \{l \in \mathbb{G}_1(E) | L \subset Z(\hat{s}_i | E)\}$ be the subvariety of lines in E which are contained in the divisor $Z(\hat{s}_i | E)$. The codimension of $\mathbb{G}_i \subset \mathbb{G}_1(E)$ is 2, so that a dimension count shows $\bigcap_{i=1}^{N-n} \mathbb{G}_i = \emptyset$ if $2(N-n) > dim \mathbb{G}_1(E)$; the latter conditions is equivalent to $n < 4$.

Corollary. Suppose X' linked to X via a general section $\underline{s} \in H^0(P; \oplus_{i=1}^{N-n} I_X \otimes L_i), n < 4$. Then $X \cap X'$ is subcanonical in P.

Proof: $X \cap X' = \pi(\hat{X} \cap E)$, and by adjunction $\omega_{\hat{X} \cap E} \cong \omega_{\hat{P}} \otimes \mathcal{O}_{\hat{P}}(E) \otimes \otimes_{i=1}^{N-n} (\mathcal{O}_{\hat{P}}(-E) \otimes \sigma^* L_i) | \hat{X} \cap E$. Since $\omega_{\hat{P}} \cong \sigma^* \omega_P \otimes \mathcal{O}_{\hat{P}}(E)^{\otimes (N-n-1)}$, we find $\omega_{X \cap X'} \cong (\omega_P \otimes \otimes_{i=1}^{N-n} L_i) | X \cap X'$.

This corollary is useful also when $X \cap X'$ is not smooth, e.g. if $n \geq 4$. In this case it allows to construct explicitly projective small resolutions for singular subvarieties in P with "prescribed" canonical bundles [W].

Example. Let $X = \mathbb{P}^4 \subset \mathbb{P}^6$ be defined by $X_5 = X_6 = 0$. Choose sufficiently general forms $g_i, h_i \in \mathbb{C}[X_0, \cdots X_6]_{d_i}, i = 1, 2$, and let $X' \subset \mathbb{P}^6$ be linked to X via $\underline{f} := (f_1, f_2), f_i := X_5 g_i + X_6 h_i$.
The intersection $X \cap X'$ is the hypersurface $Z(\Delta | \mathbb{P}^4) \subset \mathbb{P}^4$ defined by

$$\Delta = \det \begin{pmatrix} g_1 & h_1 \\ g_2 & h_2 \end{pmatrix} \in \mathbb{C}[X_0, \cdots, X_6]_{d_1 + d_2}$$

$X \cap X'$ is a complete intersection with $(d_1 d_2)^2$ singular points, and the construction above yields a small resolution

$$\pi : \hat{X} \cap E \longrightarrow X \cap X'$$

with $\omega_{\hat{X}\cap E} \cong \sigma^* \mathcal{O}_{\mathbf{P}_6}(d_1 + d_2 - 5)|\hat{X} \cap E$, in particular $\omega_{\hat{X}\cap E} \cong \mathcal{O}_{\hat{X}\cap E}$ iff $d_1 + d_2 = 5$.

Remark. In order to calculate the numerical invariants of X' one could start with the formula $c_t(\hat{X}) \cdot c_t(N_{\hat{X}/\hat{P}}) = c_t(\hat{P})$. $c_t(N_{\hat{X}/\hat{P}})$ is determined by the Chern classes of $\mathcal{O}_{\hat{P}}(-E)$ and the $L_i's$; $c_t(\hat{P})$ is given by the formula $c_t(\hat{P}) = \sigma^* c_t(P) + j_*(\pi^* c_t(X) \cdot \alpha)$ for blowing-up Chern classes [Fu2, p.300].

The formula for the Segre class of $X \subset P$ [Fu2, p.75] allows to express the classes $\pi_*(c_1(\mathcal{O}_{\hat{P}}(E))^k)$ in terms of $c_t(X)$ and $c_t(P)$. The computations for $X \cap X'$ are similar. J. Spandaw has explicit formulas for 3-folds in $\mathbb{P}^5$ [Sp].

2 Smooth reflexive sheaves and degeneracy loci

Let P be a projective manifold of dimension $N, \mathcal{F}$ a reflexive sheaf of rank r over P with $hd\mathcal{F} \leq 1$. The singular set $Sing\mathcal{F}$ of $\mathcal{F}$ is in this case simply the support of the sheaf $\mathcal{E}xt^1_{\mathcal{O}_P}(\mathcal{F}, \mathcal{O}_P)$; the ideal Ann $(\mathcal{E}xt^1_{\mathcal{O}_P}(\mathcal{F}, \mathcal{O}_P))$ makes it a subvariety of P.

$\mathcal{F}$ is said to be smooth (a more generally, to have hypersurface singularities), if for every $x \in Sing(F)$ there exists a regular parameter system $(t_1, \cdots, t_N)$ of $\mathcal{O}_{P,x}$ such that

$$\mathcal{E}xt^1_{\mathcal{O}_P}(\mathcal{F}, \mathcal{O}_P)_x \cong \mathcal{O}_{P,x}/_{(t_1,\cdots,t_{r+1})}$$

$(\mathcal{E}xt^1_{\mathcal{O}_P}(\mathcal{F}, \mathcal{O}_P)_x \cong \mathcal{O}_{P,x}/_{(t_1,\cdots,t_r,\varphi)}$ where $(t_1, \cdots, t_r, \varphi)$ is a regular sequence in $\mathcal{O}_{P,x}$). This means that the singular set $Sing(\mathcal{F})$ is smooth of codimension $r + 1(Sing(\mathcal{F})$ is a hypersurface in a smooth subvariety of codimension r), and $\mathcal{E}xt^1_{\mathcal{O}_P}(\mathcal{F}, \mathcal{O}_P)$ is an invertible sheaf on $Sing(\mathcal{F})$. In [B2] C. Bănică shows:

Proposition. Let $\mathcal{F}$ be a globally generated reflexive sheaf of rank r over an N-fold P. If $\mathcal{F}$ is smooth or has at most isolated hypersurface singularities, and if $N \leq 2r+1$, then the zero-locus $Z(s)$ of a general section $s \in H^0(P; \mathcal{F})$ is smooth of codimension r (or empty) with $Sing(\mathcal{F}) \subset Z(s)$.

Sketch of proof for a smooth reflexive sheaf: On $P \backslash Sing(\mathcal{F})$ the usual Bertini theorem applies. Consider $x \in Sing(\mathcal{F})$: the zero-locus of a section $s \in H^0(P; \mathcal{F})$ is smooth around x if its value $s(x) \in \mathcal{F}(x)$ is non-zero in the vector space $\mathcal{F}(x)$. Since $ev(x) : H^0(P; \mathcal{F}) \to \mathcal{F}(x)$ is a surjection onto a vector space of dimension $r + 1$, its kernel $ev(x)^{-1}(0)$ is of codimension $r + 1$ in $H^0(P; \mathcal{F})$. There exists therefore a non-empty Zariski-open subset $U \subset H^0(P; \mathcal{F})$ of sections not vanishing in any point of $Sing(\mathcal{F})$ as long as $dimSing(\mathcal{F}) < r + 1$, i.e. for $N \leq 2r + 1$.

It is not completely trivial to relate the Chern classes of $\mathcal{F}$ to the numerical invariants of the zero-locus $Z(s)$ for a general section. For $r = 2, P = \mathbb{P}^N$ one can use Hilbert polynomials and identify the coefficients [O1]. Another method, which also produces formulas for $r = 3$, is to apply Riemann-Roch without denominators [Fu2, p.297] to the structure sheaf $\mathcal{O}_{Z(s)}$. Explicit formulas for the degree and sectional genus can be found in C. Bănică's paper [B1].

There is essentially only one way to construct smooth reflexive sheaves, namely as cokernels of suitable vector bundle morphisms.

Let F and E be vector bundles of ranks f and e respectively over the N-fold P, and let $u : F \to E$ be a bundle morphisms. For $0 \le k \le min(f,e)$ the k-th degeneracy locus $D_k(u) = \{x \in P | rku(x) \le k\}$ is the subspace defined by the vanishing of $\wedge^{k+1}u$; locally its ideal is defined by the $(k+1)$-minors of a matrix representing u. The codimension of $D_k(u) \subset P$ is at most $(e-k)(f-k)$ if $D_k(u) \ne \emptyset$. Set $D(u) = D_{f-1}(u)$ if $f < e$.

Proposition. Let F and E be vector bundles of ranks $f < e$ over a smooth projective N-fold P, such that $F^\vee \otimes E$ is globally generated. Suppose $N < 2(e-f+2)$. If $e - f \ge 2$, then the cokernel $coker(u)$ of a general morphism $u \in Hom(F,E)$ is a smooth reflexive sheaf of rank $e - f$ with singular set $D(u)$.

Proof: [B1, p.26 - 29].

C. Bănică also shows that for $N < 2(e-f+2)$ and $e - f = 1$ the cokernel of a general morphism $u \in Hom(F,E)$ has the form $coker(u) \cong I_Y \otimes detE \otimes (detF)^\vee$ for a smooth codimension-2 subvariety $Y \subset P$.

The previous two propositions allow to construct smooth codimension-r subvarieties in projective N-folds P if $N \le 2r+1$. The same range of dimensions can be reached by the closely related technique of general degeneracy loci.

Proposition. Let F and E be vector bundles of ranks $f < e$ over a projective manifold P of dimension N, such that $F^\vee \otimes E$ is globally generated. If $N < (f-k+1)(e-k+1)$, then for a general morphism $u \in Hom(F,E)$ the k-th degeneracy locus $D_k(u)$ is smooth of codimension $(f-k)(e-k)$ or empty.

Proof: This is well known for $k = f-1$, i.e. for $D(u)$, a "filtered version" is due to M.-C. Chang [C]. The proof in the general case $0 \le k < f$ can be found in [B1].

Remark. By taking for F a trivial bundle one recovers Kleiman's theorem on the smoothness of dependency loci of general sections in globally generated vector bundles [K].

Remark. In the borderline case $N = (f-k+1)(e-k+1)$ the degeneracy locus $D_k(u)$ of a general morphism is smooth of codimension $(f-k)(e-k)$ outside of finitely many points. The intersection of $D_k(u)$ with a smooth, transversally intersecting hypersurface not containing any of these singular points yields therefore a smooth subvariety of codimension $(f-k)(e-k)+1$.

Example. Let E and F be vector bundles of ranks e and $e-1$ over a projective 6-fold P, such that $F^\vee \otimes E$ is globally generated. Let $Y := D_{e-2}(u)$ for a general morphism u, so that I_Y has the resolution

$$0 \to F \xrightarrow{u} E \to I_Y \to 0.$$

If now $L \in Pic(P)$ is very ample with $h^1(P; E^\vee \otimes F \otimes L) = 0$, and if $s \in H^0(P; L)$ is sufficiently general, then $Z := Y \cap Z(s)$ is a smooth subvariety of codimension 3 in P whose ideal has a locally free resolution of the form

$$0 \to F \otimes L^\vee \to F \oplus E \otimes L^\vee \to E \oplus L^\vee \to I_Z \to 0.$$

The Eagon-Northcott complexes associated with u yield explicit locally free resolutions for the ideals of degeneracy loci [E/N].

Example. Let $D(u)$ be the degeneracy locus of a general morphism $u : F \longrightarrow E, f = e-2, F^\vee \otimes E$ globally generated. Then there is an exact sequence

$$0 \to \overset{e}{\wedge} E^\vee \otimes S^2 F \longrightarrow \overset{e-1}{\wedge} E^\vee \otimes F \longrightarrow \overset{e-2}{\wedge} E^\vee \longrightarrow I_{D(u)} \otimes det F^\vee \to 0.$$

The cohomology classes of degeneracy loci are given by the Thom - Porteous formula [Fu2]. An algorithm for calculating Chern numbers of $D_k(u)$ can be found in [P1], [P2] and [P/P]. To relate the invariants of degeneracy loci to the Chern classes of the bundles involved one can either use Riemann - Roch without denominators, or Hilbert polynomials if $P = \mathbb{P}^N$ [B1].
It may happen that a degeneracy locus $D_k(u)$ is empty even if the expected dimension $N-(f-k)(e-k)$ is non-negative. This cannot occur, however, if $F^\vee \otimes E$ is sufficiently positive. In [F/L2] W. Fulton / R. Lazarsfeld have shown that if $F^\vee \otimes E$ is ample, then $D_k(u)$ is non-empty for $N-(f-k)(e-k) \geq 0$, and connected when $N-(f-k)(e-k) \geq 1$. Furthermore, they remark [F/L2, p.277] that there is no obvious extension of this result to a Lefschetz-type vanishing theorem for higher relative homotopy groups.
Nevertheless one may ask for a vanishing result for general degeneracy loci - or zero-loci of general sections in smooth reflexive sheaves - which yields the Barth-Lefschetz theorem when applied to $P = \mathbb{P}^N$.

3 Pfaffians

Let P be a smooth projective variety of dimension $N, L \in Pic(P)$, and E a vector bundle of rank r over P. A vector bundle morphism $f : E \to E^\vee \otimes L$ is said to be skew-symmetric if $f^\vee \otimes id_L = (-1)\cdot f$. A skew-symmetric morphism f corresponds to an element $\hat{f} \in H^0(P; \wedge^2 E^\vee \otimes L)$; its determinant is a section $det f \in H^0(P; (det E)^{\otimes -2} \otimes L^{\otimes r})$. There are two essentially different cases:

If $r \equiv 0(mod2)$, then there exists a root $P(f) \in H^0(P; detE^\vee \otimes L^{\otimes \frac{r}{2}})$ of $detf$, the Pfaffian of the morphism f. $P(f)$ defines the degeneracy locus $D_{r-1}(f) \subset P$.

Example. Let V be a vector space of dimension $r \equiv 0(mod2), P := \mathbb{P}(\wedge^2 V^\vee)$ the projective space associated to $\wedge^2 V^\vee$.
There is a tautological morphism $f : V \otimes \mathcal{O}_P \to V^\vee \otimes \mathcal{O}_P(1)$ which corresponds to $id_{\wedge^2 V}$ under the identification

$$H^0(P; \wedge^2 V^\vee \otimes \mathcal{O}_P(1)) = \wedge^2 V^\vee \otimes \wedge^2 V.$$

The associated Pfaffian $P(f) \in H^0(P; detV^\vee \otimes \mathcal{O}_P(\frac{r}{2})$ defines a hypersurface of degree $\frac{r}{2}$ in a projective space of dimension $\binom{r}{2} - 1$, the universal Pfaffian hypersurface of rank r.
The simplest special case $r = 4$ yields the Grassmannian $G(1,3) \subset \mathbb{P}^5$ with the equation $X_0 X_5 - X_1 X_4 + X_2 X_3 = 0$.
The universal Pfaffian hypersurfaces are non-singular in codimensions ≤ 4.

If $f : E \to E^\vee \otimes L$ is a skew-symmetric morphism of a bundle of rank $r \equiv 1(mod2)$, then $detf \equiv 0$. In this case we consider the first non-trivial degeneracy locus $D(f) \subset P, D(f) := D_{r-2}(f) = D_{r-3}(f)$; the expected codimension of $D(f)$ in P is 3 [B/E].

Lemma. *Let* $f : E \to E^\vee \otimes L$ be a skew-symmetric morphism of a bundle E of odd rank r, given by a section $\hat{f} \in H^0(P; \wedge^2 E^\vee \otimes L)$. If $Pic(P)$ has no 2-torsion, and if $codim_P D(f) = 3$, then there exists an exact sequence

$$(*) 0 \to detE \otimes L^{\otimes -(\frac{r-1}{2})} \xrightarrow{g^\vee \otimes id_L} E \xrightarrow{f} E^\vee \otimes L \xrightarrow{g} I_{D(f)} \otimes detE^\vee \otimes L^{\otimes \frac{r+1}{2}} \to 0$$

Proof: Let $j : K \hookrightarrow E$ be the inclusion of the kernel of f into $E; K$ is reflexive of rank 1, hence invertible, and we obtain an exact sequence

$$0 \to K \xrightarrow{i} E \xrightarrow{f} E^\vee \otimes L \xrightarrow{j^\vee \otimes id_L} I_{Z(j)} \otimes K^\vee \otimes L \to 0,$$

where $Z(j)$ denotes the zero-locus of j. Since $detI_{Z(j)} \otimes K^\vee \otimes L \cong K^\vee \otimes L$, this exact sequence yields $K \otimes detE^\vee \otimes det(E^\vee \otimes L) \otimes K \otimes L^\vee \cong \mathcal{O}_P$, i.e. $K^{\otimes 2} \cong (detE \otimes L^{\otimes -(\frac{r-1}{2})})^{\otimes 2}$.

Remark. The section $g := j^\vee \otimes id_L \in H^0(P; E \otimes detE^\vee \otimes L^{\otimes \frac{r-1}{2}})$ corresponds to a global divided power $\hat{f}^{(\frac{r-1}{2})} \in H^0(P; \wedge^{r-1} E^\vee \otimes L^{\otimes \frac{r-1}{2}})$ under the natural identification $E \otimes detE^\vee = \wedge^{r-1} E^\vee$ [B/E].

Definition.A codimension-3 subvariety $Z \subset P$ in a smooth projective N-fold P is a Pfaffian subvariety if there exist bundles $L \in Pic(P), E$ of rank $r \cong 1(mod2)$ over P, and a skew-symmetric morphism $f : E \to E^\vee \otimes L$, such that $Z = D(f)$.

Example. Let V be a vector space of dimension $r \equiv 1(mod2)$, $P := \mathbb{P}(\wedge^2 V^\vee)$ the projective space associated to $\wedge^2 V^\vee$. The universal Pfaffian subvariety of P is the degeneracy locus $D(f) \subset P$ of the tautological morphism $f : V \otimes \mathcal{O}_P \to V^\vee \otimes \mathcal{O}_P(1)$. $D(f)$ is a codimension-3 subvariety of degree $\frac{1}{4}\binom{r+1}{3}$ which is non-singular in codimension ≤ 6 [B/E]. The simplest special case $r = 5$ yields the Grassmannian $G(1,4) \subset \mathbb{P}^9$ in its Plücker embedding.

Lemma. *Suppose $Z \subset P$ is a smooth Pfaffian subvariety associated to a morphism $f : E \to E^\vee \otimes L$.*
i) The ideal I_Z has the locally free resolution

$$(*_0)\ \ 0 \to L_0 \xrightarrow{g_0^\vee \otimes id_{L_0}} E_0 \xrightarrow{f_0} E_0^\vee \otimes L_0 \xrightarrow{g_0} I_Z \to 0$$

with $L_0 := (detE)^{\otimes 2} \otimes L^{\otimes -r}$ and $E_0 := E \otimes detE \otimes L^{\otimes -(\frac{r+1}{2})}$;
furthermore: $detE_0 = L_0^{\otimes \frac{r+1}{2}}$.
ii) $Z \subset P$ is subcanonical with $\omega_Z \cong L_0^\vee \otimes \omega_Z \otimes \mathcal{O}_Z$.

Proof: i) follows by a simple calculation from $(*)$;
ii) is a consequence of the local fundamental isomorphism.

Remark. The normalized resolution $(*_0)$ of a Pfaffian subvariety specializes to the Koszul complex of a regular section in a rank-3 vector bundle when $r = 3$.

Remark. If the ideal of a submanifold $Z \subset P$ of codimension c in a smooth projective N-fold P has a locally free resolution of the form

$$0 \to L \to E_{c-2} \to E_{c-3} \to .. \to E_0 \to I_Z \to 0,$$

then $\omega_Z \cong L^\vee \otimes \omega_Z \otimes \mathcal{O}_Z$, i.e. Z must be subcanonical.

At this point it is quite natural to ask for an analogue of the Serre-correspondence in codimension 3, i.e.: Given a submanifold Z of codimension 3 in a smooth projective N-fold P such that $\omega_Z \cong L^\vee \otimes \omega_P \otimes \mathcal{O}_Z$ for a line bundle $L \in Pic(P)$. Under which conditions is Z a Pfaffian of a suitable morphism $f : E \to E^\vee \otimes L$? I don't have a solution to this problem, but the following reformulation in terms of symplectic reflexive sheaves may be useful.

Lemma. *Suppose $Z \subset P$ is the Pfaffian subvariety associated with a skew-symmetric morphism $f_0 : E_0 \to E_0^\vee \otimes L_0$. Let $\mathcal{F}_0 := Imf_0$. Then:*
i) $\mathcal{F}_0$ is a reflexive sheaf with $hd(\mathcal{F}_0) \le 1$
ii) $\mathcal{E}xt^1_{\mathcal{O}_P}(\mathcal{F}_0^\vee, \mathcal{O}_P) \cong \mathcal{O}_Z$
iii) $\mathcal{F}_0$ is L_0-symplectic.

Proof: Property i) follows immediately from the normalized resolution $(*_0)$ of I_Z.
To prove ii) one double dualizes the sequence

$$0 \to \mathcal{F}_0 \xrightarrow{j} E_0^\vee \otimes L_0 \xrightarrow{g_0} I_Z \to 0$$

and compares it with the resulting exact sequence

$$0 \to \mathcal{F}_0 \xrightarrow{j^{\vee\vee}} E_0^\vee \otimes L_0 \xrightarrow{g_0^{\vee\vee}} \mathcal{O}_P \to \mathcal{E}xt^1_{\mathcal{O}_P}(\mathcal{F}_0^\vee, \mathcal{O}_P) \to 0.$$

Property iii) means there exists an isomorphism $\alpha : \mathcal{F}_0 \to \mathcal{F}_0^\vee \otimes L_0$ with $\alpha^\vee \otimes id_{L_0} = -\alpha$. To see this we dualize the first sequence again and twist it by L_0; this yields

$$0 \to L_0 \xrightarrow{g_0^\vee \otimes id_{L_0}} E_0 \xrightarrow{j^\vee \otimes id_{L_0}} \mathcal{F}_0^\vee \otimes L_0 \to 0.$$

The skew-symmetric morphism $f_0 : E_0 \to E_0^\vee \otimes L_0$ induces a map $f_0' : \mathcal{F}_0^\vee \otimes L_0 \to \mathcal{F}_0$ whose determinant $det f_0' : det\mathcal{F}_0^\vee \otimes L_0^{\otimes r-1} \to det\mathcal{F}_0$ corresponds to the identity under the identification $(det\mathcal{F}_0)^{\otimes -2} \otimes L_0^{\otimes r-1} \cong \mathcal{O}_P$; clearly $\alpha := (f_0')^{-1}$ satisfies $\alpha^\vee \otimes id_{L_0} = -\alpha$.

Remark. A reflexive sheaf $\mathcal{F}_0$ which admits an L_0-symplectic structure $\alpha : \mathcal{F}_0 \to \mathcal{F}_0^\vee \otimes L_0$ necessarily must have even rank s and $(det\mathcal{F}_0)^{\otimes 2} \cong L_0^{\otimes s}$. Every reflexive sheaf $\mathcal{F}_0$ of rank 2 has a natural $(det\mathcal{F}_0)$-symplectic structure induced by the second wedge-product.

The following is a partial converse of property iii).

Lemma. Let $\mathcal{F}_0$ be a reflexive sheaf with $hd(\mathcal{F}_0) \leq 1$ on a projective N-fold P, which admits an L_0-symplectic structure.
Suppose $\mathcal{E}xt^1_{\mathcal{O}_P}(\mathcal{F}_0^\vee, \mathcal{O}_P) \cong \mathcal{O}_Z$ for a smooth codimension-3 subvariety Z. If the identity section $id_Z \in H^0(P; \mathcal{O}_Z)$ lifts to a global element $\varepsilon \in Ext^1_{\mathcal{O}_P}(\mathcal{F}_0^\vee, \mathcal{O}_P)$, then $Sing(\mathcal{F}_0) = Z$ is a Pfaffian subvariety of P.

Proof: Choose a global extension class $\varepsilon \in Ext^1_{\mathcal{O}_P}(\mathcal{F}_0^\vee, \mathcal{O}_P)$ which localizes to $id_Z \in H^0(P; \mathcal{O}_Z) \cong H^0(P; \mathcal{E}xt^1_{\mathcal{O}_P}(\mathcal{F}_0^\vee, \mathcal{O}_P))$.
The corresponding exact sequence

$$(\varepsilon): \qquad 0 \to \mathcal{O}_P \xrightarrow{s} \mathcal{E} \xrightarrow{t} \mathcal{F}_0^\vee \to 0$$

has a locally free middle term $\mathcal{E}$ and dualizes to the exact sequence

$$0 \to \mathcal{F}_0 \xrightarrow{t^\vee} \mathcal{E}^\vee \xrightarrow{s^\vee} \mathcal{O}_P \to \mathcal{O}_Z \to 0$$

Setting $E_0 := \mathcal{E} \otimes L_0$ and using the L_0-symplectic structure $\alpha : \mathcal{F}_0 \to \mathcal{F}_0^\vee \otimes L_0$ one gets

$$\begin{array}{ccccccccc}
0 \longrightarrow & L_0 & \xrightarrow{s \otimes id_{L_0}} & E_0 & \xrightarrow{p \otimes id_{L_0}} & \mathcal{F}_0^\vee \otimes L_0 & \longrightarrow 0 & & \\
 & & & & & \cong \downarrow \alpha^{-1} & & & \\
 & & & 0 & \longrightarrow & \mathcal{F}_0 & \xrightarrow{p^\vee} E_0^\vee \otimes L_0 & \xrightarrow{s^\vee} I_Z & \longrightarrow 0;
\end{array}$$

the composition $f_0 := p^\vee \circ \alpha^{-1} \circ p \otimes id_{L_0}$ is the required skew-symmetric morphism which provides $Z = Sing(F_0)$ with the structure of a Pfaffian subvariety.

Corollary. *Let $Z \subset P$ be a disjoint union $Z = \coprod_i Z_i$ of smooth Pfaffian subvarieties Z_i associated to morphisms $f_i : E_i \to E_i^\vee \otimes L_i$. If the normalized line bundles $(L_i)_0$ coincide for all i, then Z is also a Pfaffian in P.*

Proof: Let $(\mathcal{F}_i)_0$ be the image of the normalized morphism $(f_i)_0 : (E_i)_0 \to (E_i)_0^\vee \otimes (L_i)_0$. Since $(L_i)_0 \cong L_0$ for fixed line bundle $L_0 \in Pic(P)$, every $(\mathcal{F}_i)_0$ is L_0-symplectic. The direct sum $\mathcal{F}_0 := \oplus_i(\mathcal{F}_i)_0$ is therefore also L_0-symplectic, and the identity section $id_Z \in H^0(P; \mathcal{O}_Z) \cong \oplus_i H^0(P; \mathcal{O}_{Z_i})$ lifts to a global extension class ε since every component id_{Z_i} has this property.

Example. Every subset $Z = \{z_1, \cdots, z_m\}$ of simple points in $\mathbb{P}^3$ is a Pfaffian subvariety. In general, however, it is not possible to express Z as zero-set of a regular section in a vector bundle of rank 3 [Kr].

There is one typical geometric situation in which Pfaffian subvarieties occur naturally, namely as divisors with induced adjoint line bundles in codimension-2 subvarieties. More precisely, suppose $V \subset P$ is a codimension-2 subvariety in P given as the degeneracy locus of a vector bundle morphism h

$$(**) \qquad 0 \to F \xrightarrow{h} E \to I_V \to 0.$$

Consider a divisor $Z \subset V$. If Z is a Pfaffian subvariety in P, then there exists a line bundle $L \in Pic(P)$ with $\omega_Z \cong L^\vee \otimes \omega_P \otimes \mathcal{O}_Z$, so that $\omega_V \otimes \mathcal{O}_V(Z) \otimes \mathcal{O}_Z \cong L^\vee \otimes \omega_P \otimes \mathcal{O}_Z$.
Assume now that already the adjoint line bundle $\omega_V \otimes \mathcal{O}_V(Z)$ of Z in V is induced by $L^\vee \otimes \omega_P$, i.e. $\omega_V \otimes \mathcal{O}_V(Z) \cong L^\vee \otimes \omega_P \otimes \mathcal{O}_V$. This equation can be rewritten as $I_{Z/V} \cong \omega_V \otimes L \otimes \omega_P^\vee$, so that by dualizing and twisting $(**)$ we obtain a locally free $\mathcal{O}_P$-resolution

$$0 \to L \to E^\vee \otimes L \to F^\vee \otimes L \to I_{Z/V} \to 0$$

for the ideal $I_{Z/V}$ of Z in V. The two exact sequences can be combined into the following diagram:

$$\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & 0 & \to & F & \xrightarrow{h} & E & \xrightarrow{s^\vee} & I_V & \to & 0 \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & I_Z & & \\
 & & & & & & & & \downarrow & & \\
0 & \to & L & \xrightarrow{s\otimes id_L} & E^\vee \otimes L & \xrightarrow{h^\vee\otimes id_L} & F^\vee \otimes L & \to & I_{Z/V} & \to & 0 \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & 0 & &
\end{array}$$

Under certain vanishing conditions it is possible to fill in a middle row into this diagram which is of the form

$$0 \to L \to F \oplus E^\vee \otimes L \xrightarrow{f} E \oplus F^\vee \otimes L \to I_Z \to 0,$$

and exhibits Z as the Pfaffian of a skew-symmetric morphism

$$f = \left[\begin{array}{c|c} 0 & h \\ \hline -h^\vee \otimes id_L & \psi \end{array}\right],$$

$\psi \in H^0(P; \wedge^2 E \otimes L^\vee)$.

An important special case in which the assumption $\omega_V \otimes \mathcal{O}_V(Z) \cong L^\vee \otimes \omega_P \otimes \mathcal{O}_V$ automatically holds is when $Z = V \cap V'$ is the intersection of generally linked subvarieties V, V' of codimension 2.

Example. Let $V \subset P$ be defined by

$$0 \to F \xrightarrow{h} E \to I_V \to 0,$$

and suppose V is linked to a subvariety V' whose ideal can be resolved by the cone of a morphism α

$$\begin{array}{ccccccccc} 0 & \to & L_1^\vee \otimes L_2^\vee & \xrightarrow{\binom{s_2}{-s_1}} & L_1^\vee \oplus L_2^\vee & \xrightarrow{(s_1^\vee, s_2^\vee)} & I_{V \cup V'} & \to & 0 \\ & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow & & \\ 0 & \to & F & \xrightarrow{h} & E & \to & I_V & \to & 0 \end{array}$$

i.e. $I_{V'}$ has a locally free resolution

$$0 \to E^\vee \oplus L \xrightarrow{(h^\vee \otimes id_{L'} \alpha^\vee \otimes id_L)} F^\vee \otimes L \oplus L_1^\vee \oplus L_2^\vee \to I_{V'} \to 0,$$

where $L := L_1^\vee \otimes L_2^\vee$.

Then $I_{V \cap V'} \cong I_V \oplus I_{V'} \big/ I_V \cap I_{V'}$, and the following diagram can be constructed:

$$\begin{array}{ccccccccc} 0 & \to & L_1^\vee \otimes L_2^\vee & \to & L_1^\vee \oplus L_2^\vee & \to & I_V \cap I_{V'} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F \oplus E^\vee \otimes L & \to & E \oplus F^\vee \otimes L \oplus L_1^\vee \oplus L_2^\vee & \to & I_V \oplus I_{V'} & \to & 0 \\ & & & \searrow_{f_0} & \downarrow & & \downarrow & & \\ & & & & E \oplus F^\vee \otimes L & \to & I_{V \cap V'} & \to & 0 \end{array}$$

Here f_0 is given by the block matrix

$$\left[\begin{array}{c|c} 0 & h \\ \hline -h^\vee \otimes id_L & \alpha \circ \varphi \circ \alpha^\vee \otimes id_L \end{array}\right],$$

and φ is the natural L-symplectic structure on $L_1^\vee \oplus L_2^\vee$.
Concrete special cases are divisors $Z(f_{a,b}) \subset V = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 = P$ defined by bihomogeneous polynomials $f_{a,b}$ of bidegrees $(a,b) = (-l-4, -l-3), l \leq -4$.

The next result is a Bertini-type theorem for Pfaffian subvarieties.

Proposition. Let P be a smooth projective N-fold, $L \in Pic(P)$, and E a vector bundle of rank $r \equiv 1(mod 2)$ over P. If $\wedge^2 E^\vee \otimes L$ is globally generated, and if $N \leq 9$, then the Pfaffian subvarieties associated with general sections $\hat{f} \in H^0(P; \wedge^2 E^\vee \otimes L)$ are smooth of codimension 3 in P, or empty.

Sketch of proof: Consider the subcones $D_{r-5}(\wedge^2 E^\vee \otimes L) \subset D_{r-3}(\wedge^2 E^\vee \otimes L) \subset \wedge^2 E^\vee \otimes L$ of skew-symmetric maps of ranks $\leq r-5(r-3)$. These subcones have codimensions 10 and 3 respectively in the total space of the bundle $\wedge^2 E^\vee \otimes L$. The evaluation map

$$ev : H^0(P; \wedge^2 E^\vee \otimes L) \times P \longrightarrow \wedge^2 E^\vee \otimes L$$

is a submersion, so that the preimages $\Delta := ev^{-1}(D_{r-5}(\wedge^2 E^\vee \otimes L))$ and $D := ev^{-1}(D_{r-3}(\wedge^2 E^\vee \otimes L))$ in $H^0(P; \wedge^2 E^\vee \otimes L) \times P$ are also subvarieties of codimensions 10 and 3; furthermore, $D \backslash \Delta$ is non singular. Let

$$\pi : D \to H^0(P; \wedge^2 E^\vee \otimes L)$$

be the restriction of the first projection to D. π is a proper map, and $\pi(\Delta) \neq H^0(P; \wedge^2 E^\vee \otimes L)$ if $dim P = N < 10$; $U := H^0(P; \wedge^2 E^\vee \otimes L) \backslash \pi(\Delta)$ is therefore a non-empty Zariski-open subset of $H^0(P; \wedge^2 E^\vee \otimes L)$ and its inverse image $\pi^{-1}(U) \subset D \backslash \Delta$ is smooth. Let $\hat{f} \in U$ be a regular value of $\pi|\pi^{-1}(U)$; the fiber over $\hat{f}$ is isomorphic to the degeneracy locus $D(f) = D_{r-3}(f) \subset P$ of the corresponding skew-symmetric morphism $f : E \to E^\vee \otimes L$, which is therefore smooth of codimension 3, or empty.

Remark. If $\wedge^2 E^\vee \otimes L$ is ample, and $dim P = N \geq 3$, then $D(f)$ is always non-empty [Fu2, p.216].

Example. Consider a projective manifold P of dimension $N \leq 9$ and a globally generated line bundle L over P. Fix a natural number $r \equiv 1(mod 2)$ and choose a general skew-symmetric $r \times r$-matrix of sections in L. The corresponding Pfaffian subvariety $Z \subset P$ is then smooth of codimension 3; its ideal I_Z has a locally free resolution

$$0 \to L^{\otimes -r} \to \oplus_{i=1}^{r} L^{\otimes -(\frac{r+1}{2})} \to \oplus_{i=1}^{r} L^{\otimes -(\frac{r-1}{2})} \to I_Z \to 0$$

The canonical sheaf of Z is $\omega_Z \cong L^{\otimes r} \otimes \omega_P \otimes \mathcal{O}_Z$.

Remarks. i) The previous proposition has an obvious analogue for symmetric morphisms which has been used to construct hypersurfaces with nodes in 3-folds [Ba]. The "second" degeneracy locus of a symmetric morphism allows the construction of codimension-3 subvarieties which are non-singular in codimensions ≤ 2. The Veronesean surface in $\mathbb{P}^5$ is a simple example of this type.

ii) *I* expect that there exists also a "filtered" version of the proposition above allowing some elements "near the diagonal" to be zero ; the numerics of such a version should "explain" the assumptions on the dimensions which guarantee smoothness of intersections of generally linked subvarieties of codimension 2.

iii) One could also ask for a Lefschetz-type result for degeneracy loci of skew-symmetric morphisms which gives the Barth-Lefschetz theorem when applied to $P = \mathbb{P}^N$.

4 Submanifolds of codimension 3 in $\mathbb{P}^N$

Let X be a smooth projective subvariety of codimension 3 in $\mathbb{P}^N$.

As usual, I denote the degree and the sectional genus of $X \subset \mathbb{P}^N$ by d and π respectively; clearly $d > 3$ if X is non-degenerate, and $\pi \leq \frac{(d-2)(d-3)}{6}$ [A/C/G/H].

The Barth-Lefschetz theorem - $\pi_i(\mathbb{P}^N, X) = 0$ for $i \leq N-5$ - yields necessary topological conditions: X must be simply-connected if $N \geq 7$, subcanonical for $N \geq 8$, and it must satisfy $\pi_3(X) = 0$ when $N \geq 9$ [B/L].

If $N \geq 10$, then we are in the Hartschorne range, i.e. every codimension-3 submanifold $X \subset \mathbb{P}^N, N \geq 10$ is conjectured to be a complete intersection [H1]. At present this has only been proven for arithmetically Gorenstein subvarieties [H1], but of course, there are no other examples known.

The most interesting range of dimension is therefore $N = 6,7,8,9$, and I will concentrate on this from now on.

By Zak's fundamental theorem we know that a smooth $X \subset \mathbb{P}^N$ of codimension 3 must be linearly normal if $N \geq 8$ [Z]. For $N = 6,7$ this does not hold, however, the exceptions have already been classified.

If $N = 7$ there is essentially only one exception, namely the Severi variety of degree 6 in $\mathbb{P}^7$.

It is a projection of a Segre embedded $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ from a general point $p \in \mathbb{P}^8 \setminus \mathbb{P}^2 \times \mathbb{P}^2$ [Z]. There are three different (families of) not linearly normal 3-folds in $\mathbb{P}^6$; they have degrees 6,7, and 8, and can be described as follows [F]:

$d = 6$: These 3-folds are general hyperplane sections of the Severi varieties in $\mathbb{P}^7$.
$d = 7$: A Veronese embedded $\mathbb{P}^3 \subset \mathbb{P}^9$ can be projected into $\mathbb{P}^8$ from an inner point $x_o \in \mathbb{P}^3$; the resulting 3-fold $\hat{\mathbb{P}}^3(x_o) \subset \mathbb{P}^8$ can be further projected from a general line $L \subset \mathbb{P}^8 \setminus \hat{\mathbb{P}}^3(x_o)$ into $\mathbb{P}^6$. The embedding $\hat{\mathbb{P}}^3(x_0) \subset \mathbb{P}^6$ is given by a 6-dimensional system of quadrics through x_0.
$d = 8$: These 3-folds are projections of a Veronese embedded $\mathbb{P}^3 \subset \mathbb{P}^9$ from a general plane $E \subset \mathbb{P}^9 \setminus \mathbb{P}^3$.
In addition to these general results the explicit classification of all linearly normal, non-degenerate submanifolds $X \subset \mathbb{P}^N$ up to degree 8 has been worked out [I1], [I2], [I3]. The basic ingredients used are standard formulas for the intersection numbers like e.g. doublepoint formulas, adjunction theoretic methods for the description of possible subvarieties, and mostly ad hoc constructions to prove their existence. I summarize the classification in the table below:
The varieties marked with a (*) are the not linearly normal exceptions described above.
1) $X \overset{2:1}{\to} \mathbb{P}^1 \times \mathbb{P}^2$ denotes a double covering of $\mathbb{P}^1 \times \mathbb{P}^2$ ramified along a smooth divisor of bidegree (2,2). The embedding is given by the pullback of $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$ to X.
2) $\mathbb{P}_C(E)$ is a scroll over an elliptic curve. There exists only one other 3-dimensional scroll over a curve in $\mathbb{P}^6$, namely $\mathbb{P}^1 \times \mathbb{P}^3 \cap \mathbb{P}^6$ [T].
3) $\mathbb{P}_{\mathbb{P}^2}(E)$ is the tautologically embedded projectivization of a rank-2 vector bundle E over $\mathbb{P}^2$. E is given as extension $0 \to \mathcal{O}_{\mathbb{P}^2} \to E \to I_Y(4) \to 0$, where $Y = \{p_1, \cdots, p_9\} \subset \mathbb{P}^2$ consists of 9 simple points [I2].
4) $Q \subset \mathbb{P}^7$ denotes a hyperquadric fibration; Q is a divisor in a $\mathbb{P}^4$-bundle $\mathbb{P}$ over $\mathbb{P}^1$, and the embedding is induced by a map $\mathbb{P} \to \mathbb{P}^7$ [I2].
5) $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(E)$ is the projective bundle associated to a rank-2 vector bundle E over a quadric $\mathbb{P}^1 \times \mathbb{P}^1$; the embedding is tautological. P. Ionescu used a slight modification of the Mumford/Fujita criterion to prove very ampleness [I3].

d	π	$\mathbb{P}^6$	$\mathbb{P}^7$	$\mathbb{P}^8$	$\mathbb{P}^9$	construction
4	0	$\mathbb{P}^1 \times \mathbb{P}^3 \cap \mathbb{P}^6$	$\mathbb{P}^1 \times \mathbb{P}^3$			Segre
5	1	$\mathbb{G}(1,4) \cap \mathbb{P}^6$	$\mathbb{G}(1,4) \cap \mathbb{P}^7$	$\mathbb{G}(1,4) \cap \mathbb{P}^8$	$\mathbb{G}(1,4)$	Plücker
6	1	$\pi_p(\mathbb{P}^2 \times \mathbb{P}^2) \cap \mathbb{P}^6$	$\pi_p(\mathbb{P}^2 \times \mathbb{P}^2)$			projection$^{(*)}$
6	2	$X \overset{2:1}{\rightarrow} \mathbb{P}^1 \times \mathbb{P}^2$				1)
7	1	$\mathbb{P}_C(E)$				2)
7	1	$\pi_L(\hat{\mathbb{P}}^3(x_0))$				projection$^{(*)}$
7	3	$\mathbb{P}_{\mathbb{P}^2}(E)$				3)
7	3	$Q \cap \mathbb{P}^6$	Q			4)
8	1	$\pi_E(\mathbb{P}^3)$				projection$^{(*)}$
8	4	$\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(E)$				5)
8	5	$V(2,2,2)$	$V(2,2,2)$	$V(2,2,2)$	$V(2,2,2)$	compl. int.

Now I will show that the linearly normal 3-folds $X \subset \mathbb{P}^6$ of degrees $d \leq 8$ - with the exception of the elliptic scroll and the hyperquadric fibration of degree 7 - can be constructed by general linking from simple known examples. To this end we have to be able to relate the degrees and sectional genera of linked 3-folds in $\mathbb{P}^6$. Suppose $X, X' \subset \mathbb{P}^6$ are smooth projective 3-folds of degrees d, d' with sectional genera π, π' respectively. If X and X' are linked via a complete intersection $Z(f_1, f_2, f_3)$ with equations f_i of degrees $deg f_i = d_i$, then the following formulas hold [Fu2, p.159]:

$$\begin{aligned} d + d' &= d_1 \cdot d_2 \cdot d_3 \\ \pi - \pi' &= \frac{1}{2}(d_1 + d_2 + d_3 - 5)(d' - d). \end{aligned}$$

Furthermore, the intersection $X \cap X'$ has degree

$$deg X \cap X' = (d_1 + d_2 + d_3 - 5)d - (2\pi - 2).$$

The following tables contain the degrees d' and the sectional genera π' of smooth 3-folds X' which are generally linked to "parent" 3-folds X via complete intersections $Z(\underline{f})$ of multidegrees $\underline{d}$.

Example 1: $X = \mathbb{P}^3$; $d = 1, \pi = 0$

$\underline{d}$	d'	π'
(2,2,2)	7	3
(2,2,3)	11	10
(2,2,4)	15	21
(2,3,3)	17	24

The 3-fold X' with $d' = 7, \pi' = 3$ is the projective bundle $\mathbb{P}_{\mathbb{P}^2}(E)$.

Example 2: $X =$ Quadric ; $d = 2, \pi = 0$

$\underline{d}$	d'	π'
(2,2,2)	6	2
(2,2,3)	10	8
(2,2,4)	14	18
(2,3,3)	16	21

The 3-fold with $d' = 6, \pi' = 2$ is a ramified double cover of $\mathbb{P}^1 \times \mathbb{P}^2$. The 3-fold X' with $d' = 10, \pi' = 8$ is a del Pezzo fibration over $\mathbb{P}^1$ [Fa/Li2].

Example 3: $X = \mathbb{P}^1 \times \mathbb{P}^2$; $d = 3, \pi = 0$

$\underline{d}$	d'	π'
(2,2,2)	5	1
(2,2,3)	9	6
(2,2,4)	13	15
(2,3,3)	15	18

The 3-fold with $d' = 5, \pi' = 1$ is $\mathbb{G}(1,4) \cap \mathbb{P}^6$.

The 3-fold X' with $d' = 9, \pi' = 6$ is quite interesting: X' is isomorphic to the blow-up of a point in a hyperquadric section of $\mathbb{G}(1,4) \cap \mathbb{P}^7$ [Fa/Li1]; the embedding of X' into $\mathbb{P}^6$ is given by the projection from this point. I know only one other 3-fold in $\mathbb{P}^6$ which is embedded in this way, the blow-up of a complete intersection of 4 general quadrics in $\mathbb{P}^7$.
Presumably there are only finitely many families of 3-folds in $\mathbb{P}^6$ which are such "inner projections" of 3-folds in $\mathbb{P}^7$.

Example 4: $X = \mathbb{P}^1 \times \mathbb{P}^3 \cap \mathbb{P}^6$; $d = 4, \pi = 0$
$\mathbb{P}^1 \times \mathbb{P}^3$ is the degeneracy locus of the tautological morphism on $\mathbb{P}^7 = \mathbb{P}(Hom(\mathbb{C}^2, \mathbb{C}^4))$:

$$u : \mathcal{O}_{\mathbb{P}^7}^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}^7}(1)^{\oplus 4}$$

The associated Eagon-Northcott complex yields a locally free resolution of $\mathcal{I}_{\mathbb{P}^1\times\mathbb{P}^3}$:

$$0 \to \mathcal{O}_{\mathbb{P}^7}(-4)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^7}(-3)^{\oplus 8} \to \mathcal{O}_{\mathbb{P}^7}(-2)^{\oplus 6} \to \mathcal{I}_{\mathbb{P}^1\times\mathbb{P}^3} \to 0$$

$\mathcal{I}_{\mathbb{P}^1\times\mathbb{P}^3\cap\mathbb{P}^6}(l)$ is therefore globally generated for $l \geq 2$, and general linking gives

$\underline{d}$	d'	π'
(2,2,2)	4	0
(2,2,3)	8	4
(2,2,4)	12	12
(2,3,3)	14	15

The 3-fold X' with $d' = 8, \pi' = 4$ is the $\mathbb{P}^1$-bundle $\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(E)$ over $\mathbb{P}^1\times\mathbb{P}^1$.

Example 5: $X = \mathbb{G}(1,4)\cap\mathbb{P}^6$; $d = 5, \pi = 1$
$\mathbb{G}(1,4) \subset \mathbb{P}^9$ is a universal Pfaffian, so that $\mathcal{I}_{\mathbb{G}(1,4)\cap\mathbb{P}^6}$ has the following resolution:

$$(*_0)\ 0 \to \mathcal{O}_{\mathbb{P}^6}(-5) \to \mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 5} \longrightarrow \mathcal{O}_{\mathbb{P}^6}(-2)^{\oplus 5} \to \mathcal{I}_{\mathbb{G}(1,4)\cap\mathbb{P}^6} \to 0$$

By general linking we obtain the following 3-folds:

$\underline{d}$	d'	π'
(2,2,2)	3	0
(2,2,3)	7	3
(2,2,4)	11	10
(2,3,3)	13	13

The 3-fold X' with $d' = 7, \pi' = 3$ is again $\mathbb{P}_{\mathbb{P}^2}(E)$.

In the following table I have listed the "new" 3-folds in $\mathbb{P}^6$ of degrees $d' \leq 15$ which I could construct via general linking from known examples X of degrees $d \leq 6$ (the symbol $Q\cap Q'\cap\mathbb{P}^5$ means that X is a general complete intersection of two hyperquadrics Q, Q' and a hyperplane).

d'	π'	X	$\underline{d}$
9	6	$\mathbb{P}^1 \times \mathbb{P}^2$	(2,2,3)
10	8	Quadric	(2,2,3)
11	10	$\mathbb{P}^3$	(2,2,3)
12	11	$X \overset{2:1}{\to} \mathbb{P}^1 \times \mathbb{P}^2$	(2,3,3)
12	12	$\mathbb{P}^1 \times \mathbb{P}^3 \cap \mathbb{P}^6$	(2,2,4)
12	13	$\emptyset$	(2,2,3)
13	13	$\mathbb{G}(1,4) \cap \mathbb{P}^6$	(2,3,3)
13	15	$\mathbb{P}^1 \times \mathbb{P}^2$	(2,2,4)
14	15	$\mathbb{P}^1 \times \mathbb{P}^3 \cap \mathbb{P}^6$	(2,2,3)
14	16	$Q \cap Q' \cap \mathbb{P}^5$	(2,3,3)
14	18	Quadric	(2,2,4)
15	18	$\mathbb{P}^1 \times \mathbb{P}^2$	(2,3,3)
15	21	$\mathbb{P}^3$	(2,2,4)

Remark. The 3-folds $X' \subset \mathbb{P}^6$ in this list with invariants $(d', \pi') \in \{(10,8), (12,11), (14,16), (14,18)\}$ are hyperquadric sections of cones over 3-folds in $\mathbb{P}^5$. The same construction applied to a 3-fold $\overline{X} \subset \mathbb{P}^5$ of degree 7 and sectional genus 4 yields another 3-fold of degree 14 in $\mathbb{P}^6$. This 3-fold has sectional genus 14, and can not be constructed via general linking from a 3-fold of smaller degree. A similar example is the general hypercubic section of the cone over a Castelnuovo 3-fold in $\mathbb{P}^5$; this gives a 3-fold of degree 15 and sectional genus 19.

Remark. The construction of codimension $c+1$ subvarieties in $\mathbb{P}^{N+1}$ as general hypersurface sections of cones over codimension c varieties in $\mathbb{P}^N$ is sometimes a specialization of the "borderline case" of section 2. If $\overline{X} \subset \mathbb{P}^N$ is smooth of degree $\overline{d}$ with sectional genus $\overline{\pi}$, and if $X \subset \mathbb{P}^{N+1}$ is the intersection of the cone over $\overline{X}$ with a general hypersurface of degree h, then the degree of X is $d = h\overline{d}$, and the sectional genus is given by $\pi = h(\overline{\pi}-1) + \binom{h}{2}\overline{d} + 1$ [B/N].

I leave it to the interested reader to construct smooth, codimension-3 subvarieties in $\mathbb{P}^7$ as zero-loci of sections in smooth reflexive sheaves, or as degeneracy loci of suitable vector bundle morphisms. There are no examples in degree 9 [Fa/Li1].

As far as submanifolds of codimension 3 in $\mathbb{P}^8$ and $\mathbb{P}^9$ are concerned, the only examples known-besides complete intersections- are $\mathbb{G}(1,4) \subset \mathbb{P}^9$, hyperplane sections, and pullbacks under finite maps thereof. I like to construct some new examples as Pfaffians of suitable vector bundles. The degree of a Pfaffian subvariety can be calculated as follows:

Lemma. Let $Z \subset \mathbb{P}^N$ be a Pfaffian submanifold defined by a skew-symmetric morphism $f : E \to E^\vee \otimes L$. The degree of Z in $\mathbb{P}^N$ is determined by the rank r and the Chern classes of E and L:

$$deg Z = c_3(E) - \frac{r-1}{2} c_2(E) \cdot c_1(L) + \frac{1}{4}\binom{r+1}{3} c_1(L)^3.$$

Proof: Tedious computation.

Example. Fix an odd integer $r > 0$, a natural number l and choose a general skew-symmetric $r \times r$-matrix $F_r(l)$ of forms of degree l on $\mathbb{P}^9$. Let $Z_r(l) \subset \mathbb{P}^9$ be the Pfaffian defined by the morphism

$$\mathcal{O}_{\mathbb{P}^9}^{\oplus r} \xrightarrow{F_r(l)} \mathcal{O}_{\mathbb{P}^9}(l)^{\oplus r}.$$

$Z_r(l)$ is a submanifold of codimension 3 in $\mathbb{P}^9$; its degree is $\frac{1}{4}\binom{r+1}{3} l^3$, its canonical sheaf is $\omega_{Z_r(l)} = \mathcal{O}_{Z_r(l)}(rl - 10)$. The Pfaffians with $rl < 10$ yield interesting Fano manifolds which I have listed below.

X	d	n	i	c
$Z_9(1)$	30	6	1	6
$Z_7(1)$	14	6	3	4
$Z_7(1) \cap \mathbb{P}^8$	14	5	2	4
$Z_7(1) \cap \mathbb{P}^7$	14	4	1	4
$Z_5(1)$	5	6	5	2
$Z_5(1) \cap \mathbb{P}^8$	5	5	4	2
$Z_5(1) \cap \mathbb{P}^7$	5	4	3	2
$Z_5(1) \cap \mathbb{P}^6$	5	3	2	2
$Z_5(1) \cap \mathbb{P}^5$	5	2	1	2

In addition to the degree d of X I have also noted the dimension n, the index i, and the coindex $c = n + 1 - i$ of X. The second Betti number $b_2(X)$ is always equal to 1 for $n \geq 3$. The Pfaffian $Z_7(1)$ can also be constructed from the tangent bundle of $\mathbb{P}^9$.

The list of manifolds with trivial canonical bundle which can be obtained as hyperplane sections of $Z_r(l)$'s is rather short

X	d	n
$Z_5(2)$	40	6
$Z_9(1) \cap \mathbb{P}^8$	30	5
$Z_7(1) \cap \mathbb{P}^6$	14	3
$Z_5(1) \cap \mathbb{P}^4$	5	1

As W. Decker has pointed out to me, it is, however, possible to construct further examples of Fano varieties or of manifolds with trivial canonical bundles by using skew-symmetric $r \times r$-matrices of forms whose degrees are not all equal.

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Note added in proof: After this paper was written, I received the new version [Fa/Li1] of the degree 9 classification of n-folds in projective spaces. There it is shown that one has only 2 families of 3-folds of degree 9 in $\mathbb{P}^6$: those mentioned in example 3, and the hypercubic sections of cones over $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. None of these 3-folds extend to a 4-fold in $\mathbb{P}^7$ [Fa/Li1].