CONTROL AND RELAXATION OVER THE CIRCLE

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Abstract. We formulate and proove a geometric version of the Fundamental Theorem of Algebraic K-Theory which relates the K-theory of the Laurent polynomial extension of a ring to the K-theory of the ring. The geometric version relates the higher simple homotopy theory of the product of a finite complex and a circle with that of the complex. By using methods of controlled topology, we also obtain a geometric version of the Fundamental Theorem of Lower Algebraic K-Theory. The main new innovation is a geometrically defined Nil space.

Contents

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1. Introduction

The Fundamental Theorem of Algebraic K–Theory relates the K–theory of the Laurent polynomial extension $R[t,t^{-1}]$ of a ring R to the K–theory of the ring (see [2]). For the functor K_1, the result is due to Bass, Heller and Swan [3]:

$$K_1(R[t,t^{-1}]) \cong K_1(R) \oplus K_0(R) \oplus \widetilde{\mathrm{Nil}}(R) \oplus \widetilde{\mathrm{Nil}}(R)$$

When R is the integral group ring $\mathbb{Z}\pi_1(X)$ of the fundamental group of a space X, then $R[t,t^{-1}] = \mathbb{Z}\pi_1(X \times S^1)$ and the Bass–Heller–Swan decomposition gives a calculation of the Whitehead group of $X \times S^1$:

$$\mathrm{Wh}(\pi_1(X \times S^1)) \cong \mathrm{Wh}(\pi_1(X)) \oplus \widetilde{K}_0(\mathbb{Z}\pi_1(X)) \oplus \widetilde{\mathrm{Nil}}(\mathbb{Z}\pi_1(X)) \oplus \widetilde{\mathrm{Nil}}(\mathbb{Z}\pi_1(X)).$$

For recent expositions of these results as well as for applications to topology, see Ranicki [42] and Rosenberg [43].

Whitehead groups measure the difference between homotopy equivalences and simple homotopy equivalences. That is, every homotopy equivalence $f : Y \to X$ between finite CW complexes determines an element $\tau(f) \in \mathrm{Wh}(\pi_1(X))$ which vanishes if and only if f is simple (see [13]). For a finite CW complex X all homotopy equivalences to X from other finite CW complexes can be organized into a moduli space, the Whitehead space $\mathcal{W}h(X)$, which is the domain of higher simple homotopy theory (see [23], [24], [27], [29]). One of the new results of this paper is a geometrically defined Nil space based on the earlier work of Prassidis [41].

Our main result is a moduli space version of the Bass-Heller-Swan decomposition where the moduli spaces involved in the decomposition are Whitehead spaces and Nil spaces. More specifically, we prove

Theorem (Main Theorem)**.** *If X is a finite CW-complex, then there is a homotopy equivalence*

$$\mathcal{W}h(X \times S^1) \simeq \mathcal{W}h(X) \times \Omega^{-1}\mathcal{W}h(X) \times \widetilde{\mathcal{N}il}(X) \times \widetilde{\mathcal{N}il}(X).$$

The classical Bass-Heller-Swan decomposition follows from this homotopy equivalence by considering the set of path components of the spaces involved. That is, there are isomorphisms

$$\pi_0\mathcal{W}h(X \times S^1) = \mathrm{Wh}(\pi_1(X \times S^1)),\ \pi_0\mathcal{W}h(X) = \mathrm{Wh}(\pi_1(X))$$
$$\pi_0\Omega^{-1}\mathcal{W}h(X) = \tilde{K}_0(\mathbb{Z}\pi_1(X)),\ \text{and}\ \pi_0\widetilde{\mathcal{N}il}(X) = \widetilde{\mathrm{Nil}}(\mathbb{Z}\pi_1(X)).$$

However, the present paper does not represent a new proof of the Bass-Heller-Swan decomposition.

The Fundamental Theorem of Algebraic K–Theory was extended to higher K–theory by Quillen (see [22], [45]). More recently, Klein, Vogell, Waldhausen and Williams have established a Fundamental Theorem in the A–theory of spaces [38]. Their decomposition descends under the linearization map to the decomposition in the higher K-theory of rings. It is widely believed (but as yet unpublished) that the fiber of the A–theory assembly map is the geometrically defined Whitehead space discussed above. One would like to compare the fundamental theorems of higher K-theory and A-theory with the decomposition which we obtain here. The problem with making such a comparison is that the techniques used in higher K-theory and A-theory are algebraic and rely on the description of $R[t,t^{-1}]$ as the ring obtained from the polynomial extension $R[t]$ by localizing at t. In fact, Bass, Heller and Swan [3] and Bass [2] also use this description in the original treatments. It seems that in order to understand those algebraic arguments from our geometric point of view, one needs a geometric model for the passage from a ring R to its polynomial extension $R[t]$ similar to the way crossing a space with the circle models the passage from R to $R[t, t^{-1}]$. This is an intriguing problem.

Ranicki [42] gives a proof of the original Bass–Heller–Swan decomposition which avoids $R[t]$. This is not surprising since Ranicki's algebra is often designed to match geometry. Our decomposition relies in part on finding parametric geometric arguments modeled on Ranicki's algebraic constructions.

Chapman-West theory allows us to formulate results involving simple homotopy theory and Whitehead groups in terms of Hilbert cube manifolds (see [6], [50]). The key result due to Chapman and West is that a homotopy equivalence $f : X \to Y$ between finite CW complexes is simple if and only if the map $f \times \mathrm{id}_Q : X \times Q \to Y \times Q$ between Hilbert cube manifolds is homotopic to a homeomorphism (here Q denotes the Hilbert cube). This allows $\mathcal{W}h(X)$ to be defined as the infinite dimensional analogue of the structure space from the surgery theory of finite dimensional manifolds.

We also obtain results related to Bass's Fundamental Theorem for lower algebraic K–theory [2]. The techniques come from controlled (or bounded) topology. The finite CW complex X is replaced by $X \times \mathbb{R}^n$ and homotopy equivalences to $X \times \mathbb{R}^n$ are required to be controlled in the $\mathbb{R}^n$-direction.

Topologists have long been interested in problems "over the circle S^1". Besides the classical Bass-Heller-Swan decomposition of the Whitehead group of the fundamental group of a product of a space with S^1 (or, more generally, of the total space of a bundle over S^1), there is the fibering problem (when is a map to S^1 homotopic to a fibration?) and the problem of determining the homotopy groups of the space of self-homeomorphisms on the product of a manifold with S^1 (see Farrell [17], Farrell and Hsiang [18], Siebenmann [19], Chapman [9], Burghelea [4], Waldhausen [47], Burghelea, Lashof and Rothenberg [5], Kinsey [37], Hughes, Taylor and Williams [32], and Prassidis [41].) While the literature is extensive, a synthesis still seems to be lacking. It is hoped that the current moduli space approach will lead to a more coherent theory.

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2. Statement of results

For a compact Hilbert cube manifold X the *Whitehead space of* X is a simplicial set $\mathcal{W}h(X)$ whose homotopy groups are the higher Whitehead groups of any finite CW-complex homotopy equivalent to X. For example, $\pi_0\mathcal{W}h(X) = \mathrm{Wh}(\pi_1(X))$, the classical algebraicly defined Whitehead group of the group $\pi_1(X)$, and $\Omega\mathcal{W}h(X)$ is homotopy equivalent to the space of pseudoisotopies on X (see [27], [29]). For an arbitrary Hilbert cube manifold and a proper map $p : X \to B$ the *controlled Whitehead space* is a simplicial set $\mathcal{W}h(p : X \to B)$ whose homotopy groups are the domain of higher Whitehead torsion with control in B.

The first result is a homotopy splitting of the controlled Whitehead group over a circle.

Theorem 2.1. *If X is a compact Hilbert and $X \times S^1 \to S^1$ is projection, then there is a homotopy equivalence*

$$\mathcal{W}h(X\times S^1 \to S^1) \simeq \mathcal{W}h(X) \times \Omega^{-1}\mathcal{W}h(X)$$

with $\Omega^{-1}\mathcal{W}h(X)$ a delooping of $\mathcal{W}h(X)$, that is, a (non–connected) simplicial set whose loop space is homotopy equivalent to $\mathcal{W}h(X)$, $\Omega\Omega^{-1}\mathcal{W}h(X) \simeq \mathcal{W}h(X)$.

We will obtain this splitting as follows. From Hughes, Taylor and Williams [32] there is a homotopy equivalence

$$\Psi : \mathcal{W}h(X\times S^1 \to S^1) \to \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R} \to \mathbb{R}))$$

(see §6.7). Evaluation at the basepoint of S^1 yields a fibration

$$\Omega\mathcal{W}h(X\times\mathbb{R} \to \mathbb{R}) \xrightarrow{I} \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R} \to \mathbb{R})) \xrightarrow{E} \mathcal{W}h(X\times\mathbb{R} \to \mathbb{R}).$$

In Chapter 6 we will define the *unwrapping* (or *infinite transfer*) map

$$u : \mathcal{W}h(X\times S^1 \to S^1) \to \mathcal{W}h(X\times\mathbb{R} \to \mathbb{R})$$

and the *wrapping up* map

$$w : \mathcal{W}h(X\times\mathbb{R} \to \mathbb{R}) \to \mathcal{W}h(X\times S^1 \to S^1)$$

such that

1. $uw \simeq 1_{\mathcal{W}h(X\times\mathbb{R}\to\mathbb{R})}$, and
2. $E\Psi \simeq u$.

In fact, all of the Whitehead spaces considered here carry the structure of *abelian monoid-like simplicial sets* (see §6.2) which induce abelian group structures on π_0 of these spaces and the usual group structures on higher homotopy groups. The natural simplicial group structures on

$$\Omega\mathcal{W}h(X\times\mathbb{R} \to \mathbb{R}) \quad \text{and} \quad \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R} \to \mathbb{R}))$$

induce abelian group structures on π_0 and the simplicial maps Ψ, I, E, u, and w induce group homomorphisms on homotopy groups including π_0.

If $i : \Omega\mathcal{W}h(X\times\mathbb{R} \to \mathbb{R}) \to \mathcal{W}h(X\times S^1 \to S^1)$ is any simplicial map such that $\Psi \circ i \simeq I$, then

$$i_* = (\Psi_*)^{-1}I_* : \pi_k\Omega\mathcal{W}h(X\times\mathbb{R} \to \mathbb{R}) \to \pi_k\mathcal{W}h(X\times S^1 \to S^1)$$

is a group homomorphism for each $k \geq 0$. Thus,

$$0 \to \pi_k\Omega\mathcal{W}h(X\times\mathbb{R} \to \mathbb{R}) \xrightarrow{i_*} \pi_k\mathcal{W}h(X\times S^1 \to S^1) \xrightarrow{u_*} \pi_k\mathcal{W}h(X\times\mathbb{R} \to \mathbb{R}) \to 0$$

is a split exact sequence of abelian groups for each $k \geq 0$ with

$$w_* : \pi_k \mathcal{W}h(X \times \mathbb{R} \to \mathbb{R}) \to \pi_k \mathcal{W}h(X \times S^1 \to S^1)$$

splitting u_*. It follows that

$$(i, w) : \Omega \mathcal{W}h(X \times \mathbb{R} \to \mathbb{R}) \times \mathcal{W}h(X \times \mathbb{R} \to \mathbb{R}) \to \mathcal{W}h(X \times S^1 \to S^1);\ (x, y) \mapsto i(x) + w(y)$$

induces isomorphisms between homotopy groups (including π_0) with $+$ denoting the abelian monoid-like sum in $\mathcal{W}h(X \times S^1 \to S^1)$. In particular, (i, w) is a homotopy equivalence.

To finish the proof of Theorem 2.1 we observe in §6.8 that the methods of Hughes [27], [29] imply that

$$\Omega \mathcal{W}h(X \times \mathbb{R} \to \mathbb{R}) \simeq \mathcal{W}h(X)$$

so that

$$\mathcal{W}h(X \times S^1 \to S^1) \simeq \mathcal{W}h(X) \times \mathcal{W}h(X \times \mathbb{R} \to \mathbb{R})$$

as desired.

It follows from Chapman [8] and Hughes [27] that

$$\pi_0 \mathcal{W}h(X) = \mathrm{Wh}(\pi_1(X)) \quad \text{and} \quad \pi_0 \mathcal{W}h(X \times \mathbb{R} \to \mathbb{R}) = \widetilde{\mathrm{K}}_0(\mathbb{Z}\pi_1(X))$$

so that

$$\pi_0 \mathcal{W}h(X \times S^1 \to S^1) = \mathrm{Wh}(\pi_1(X)) \oplus \widetilde{\mathrm{K}}_0(\mathbb{Z}\pi_1(X)).$$

For a compact Hilbert cube manifold X the *Nil space of* X is a simplicial set $\widetilde{\mathcal{N}il}(X)$ whose homotopy groups are the higher Nil groups of any finite CW complex homotopy equivalent to X. For example, $\pi_0 \widetilde{\mathcal{N}il}(X) = \widetilde{\mathrm{Nil}}(\mathbb{Z}\pi_1(X))$, the algebraically defined Nil group of Bass [2]. The simplicial set $\widetilde{\mathcal{N}il}(X)$ is defined using ideas from Prassidis [41] and carries the structure of an abelian monoid-like simplicial set. The second main result is a splitting of $\mathcal{W}h(X \times S^1)$ into a controlled part and an uncontrolled part. The uncontrolled part consists of two copies of $\widetilde{\mathcal{N}il}(X)$.

Theorem 2.2. *If X is a compact Hilbert cube manifold, then there is a homotopy equivalence*

$$\mathcal{W}h(X \times S^1) \simeq \mathcal{W}h(X \times S^1 \to S^1) \times \widetilde{\mathcal{N}il}(X) \times \widetilde{\mathcal{N}il}(X).$$

In order to prove Theorem 2.2 we define the *forget control* map

$$\varphi : \mathcal{W}h(X \times S^1 \to S^1) \to \mathcal{W}h(X \times S^1)$$

which induces homomorphisms between homotopy groups including π_0. By Hughes, Taylor and Williams [36] this map is homotopy split injective: there is a simplicial map

$$r : \mathcal{W}h(X \times S^1) \to \mathcal{W}h(X \times S^1 \to S^1)$$

which induces homomorphisms between homotopy groups including π_0 such that $r \circ \varphi \simeq 1_{\mathcal{W}h(X \times S^1 \to S^1)}$. The map r is called the *relaxation* map because it is related to the relaxation construction of Siebenmann [44] (see also Kinsey [37]). This is the source of the linguistic paradox: relaxing is the (left) inverse of forgetting control.

For each $k \geq 0$ we define a group homomorphism

$$J : \pi_k \widetilde{\mathcal{N}il}(X) \oplus \pi_k \widetilde{\mathcal{N}il}(X) \to \pi_k \mathcal{W}h(X \times S^1)$$

and a simplicial map

$$P : \mathcal{W}h(X \times S^1) \to \widetilde{\mathcal{N}il}(X) \times \widetilde{\mathcal{N}il}(X)$$

which induces homomorphisms between homotopy groups including π_0 such that $P_* \circ J = 1 : \pi_k \widetilde{\mathcal{N}il}(X) \oplus \pi_k \widetilde{\mathcal{N}il}(X) \to \pi_k \widetilde{\mathcal{N}il}(X) \oplus \pi_k \widetilde{\mathcal{N}il}(X)$ (see Proposition 8.21). Our goal is to show that for each $k \geq 0$ there is a split exact sequence of abelian groups

$$0 \to \pi_k \widetilde{\mathcal{N}il}(X) \oplus \pi_k \widetilde{\mathcal{N}il}(X) \xrightarrow{J} \pi_k \mathcal{W}h(X \times S^1) \xrightarrow{r_*} \pi_k \mathcal{W}h(X \times S^1 \to S^1) \to 0$$

with P_* splitting J and φ_* splitting r_*. Once we have established exactness, it will follow that

$$r \times P : \mathcal{W}h(X \times S^1) \to \mathcal{W}h(X \times S^1 \to S^1) \times \widetilde{\mathcal{N}il}(X) \times \widetilde{\mathcal{N}il}(X);\ x \mapsto (r(x), P(x))$$

induces isomorphisms on homotopy groups (including π_0) so that $r \times P$ is a homotopy equivalence, proving Theorem 2.2.

In order to prove exactness, we need to show that $r_* \circ J = 0$ and that $\mathrm{Ker}(r_*) \subseteq \mathrm{Im}(J)$. In Chapter 9 we will use transfer maps to establish the first property as follows. The standard s–fold cover ($s \geq 1$) of S^1 induces a simplicial map

$$\mathrm{tr}^s : \mathcal{W}h(X \times S^1) \to \mathcal{W}h(X \times S^1),$$

called the *s–fold transfer* map, such that

1. **(Transfer Additivity on Image of i)** the composition
$$\Omega \mathcal{W}h(X \times \mathbb{R} \to \mathbb{R}) \xrightarrow{i} \mathcal{W}h(X \times S^1 \to S^1) \xrightarrow{\varphi} \mathcal{W}h(X \times S^1) \xrightarrow{\mathrm{tr}^s} \mathcal{W}h(X \times S^1)$$
is homotopic to the map $x \mapsto \varphi i(x) + \cdots + \varphi i(x)$ (s times) with $+$ denoting the abelian monoid-like sum in $\mathcal{W}h(X \times S^1)$,
2. **(Transfer Invariance of Wrapping Up)** the composition
$$\mathcal{W}h(X \times \mathbb{R} \to \mathbb{R}) \xrightarrow{w} \mathcal{W}h(X \times S^1 \to S^1) \xrightarrow{\varphi} \mathcal{W}h(X \times S^1) \xrightarrow{\mathrm{tr}^s} \mathcal{W}h(X \times S^1)$$
is homotopic to $\varphi \circ w$,
3. **(Transfer Nilpotency on Image of J)** for each $x \in \pi_k \widetilde{\mathcal{N}il}(X) \oplus \pi_k \widetilde{\mathcal{N}il}(X)$ there exists $s \geq 1$ such that
$$\mathrm{tr}^S_*(J(x)) = 0 \in \pi_k \mathcal{W}h(X \times S^1) \text{ for each } S \geq s.$$

These transfer properties imply that

$$\mathrm{Im}(\varphi_* \circ i_*) \cap \mathrm{Im}(J) = 0 = \mathrm{Im}(\varphi_* \circ w_*) \cap \mathrm{Im}(J) \text{ in } \pi_k \mathcal{W}h(X \times S^1).$$

When these images are pushed into $\pi_n \mathcal{W}h(X \times S^1 \to S^1)$ by r_* we get

$$\mathrm{Im}(i_*) \cap \mathrm{Im}(r_* \circ J) = \mathrm{Im}(r_* \circ \varphi_* \circ i_*) \cap \mathrm{Im}(r_* \circ J) = 0$$

and

$$\mathrm{Im}(w_*) \cap \mathrm{Im}(r_* \circ J) = \mathrm{Im}(r_* \circ \varphi_* \circ w_*) \cap \mathrm{Im}(r_* \circ J) = 0.$$

Since $\pi_k \mathcal{W}h(X \times S^1 \to S^1) = \mathrm{Im}(i_*) \oplus \mathrm{Im}(w_*)$ (from the proof of Theorem 2.1 above), it follows that $\mathrm{Im}(r_* \circ J) = 0$, i.e. $r_* \circ J = 0$.

Finally, the proof that $\mathrm{Ker}(r_*) \subseteq \mathrm{Im}(J)$ is given in Chapter 10 using a complicated argument involving mapping tori (together with some technical results from controlled topology given in an appendix, Chapter 12). It is this part of our argument which is modeled on the Ranicki's algebraic pentagon [42].

The following result follows immediately from Theorems 2.1 and 2.2. It is a restatement of the Main Result of the Introduction using Hilbert cube manifold theory to interchange finite CW complexes with compact Hilbert cube manifolds.

Theorem 2.3. *If X is a compact Hilbert cube manifold, then there is a homotopy equivalence of simplicial sets*

$$\mathcal{W}h(X\times S^1) \simeq \mathcal{W}h(X) \times \Omega^{-1}\mathcal{W}h(X) \times \widetilde{\mathcal{N}il}(X) \times \widetilde{\mathcal{N}il}(X).$$

A *pseudoisotopy* of a space X is a self-homeomorphism on $X\times[0,1]$ which restricts to the identity on $X\times\{0\}$. Let $\mathcal{P}(X)$ denote the simplicial set of pseudoisotopies on X. If X is a compact Hilbert cube manifold, then $\mathcal{P}(X) \simeq \Omega\mathcal{W}h(X)$ (see Proposition 6.12 and [29]).

Corollary 2.4. *If X is a compact Hilbert cube manifold, then there is a homotopy equivalence of simplicial sets*

$$\mathcal{P}(X \times S^1) \simeq \mathcal{P}(X) \times \mathcal{W}h(X) \times \Omega\widetilde{\mathcal{N}il}(X) \times \Omega\widetilde{\mathcal{N}il}(X).$$

This result can be stated in finite dimensional language: Chapman [7] proved that if M is a PL manifold, then $\mathcal{P}(M\times Q)$ is homotopy equivalent to the simplicial set of *stable* pseudoisotopies on M (i.e., the direct limit of $\mathcal{P}(M) \to \mathcal{P}(M \times I) \to \mathcal{P}(M \times I^2) \to \cdots$). Hatcher [23] also obtained a decomposition of $\mathcal{P}(M \times S^1)$ with $\mathcal{P}(M) \times \mathcal{W}h(M)$ as a factor, but did not have a splitting of the Nil part.

We actually prove our results in more generality than stated above. The Whitehead and Nil spaces of X are generalized to Whitehead and Nil spaces of $X \times \mathbb{R}^n$ with control in $\mathbb{R}^n$ (so that the first results correspond to $n = 0$).

Theorem 2.5. *If X is a compact Hilbert cube manifold, then there is a homotopy equivalence of simplicial sets*

$$\begin{aligned}\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to \mathbb{R}^n) \simeq{}& \mathcal{W}h(X\times S^1\times\mathbb{R}^n \to \mathbb{R}^n) \times \mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \times\\ & \widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n) \times \widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n).\end{aligned}$$

By considering the set of path components of the spaces in the Theorem above, one obtains the statement of the Fundamental Theorem of Lower Algebraic K-Theory (see [43]). This is because of the following calculations:

$$\begin{aligned}\pi_0\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to \mathbb{R}^n) &= \begin{cases} \widetilde{K}_0(\mathbb{Z}\pi_1(X\times S^1)) & \text{if } n=1\\ K_{1-n}(\mathbb{Z}\pi_1(X\times S^1)) & \text{if } n>1\end{cases}\\ \pi_0\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) &= K_{-n}(\mathbb{Z}\pi_1(X)) \qquad \text{if } n\geq 1\\ \pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n) &= \mathrm{NK}_{1-n}(\mathbb{Z}\pi_1(X)).\end{aligned}$$

The first two are due to Chapman [7] and the third is carried out in Chapter 8.

The Fundamental Theorems of Lower Algebraic K-theory on which the results are modeled are (see e.g. [43]):

$$\widetilde{K}_0(R[t,t^{-1}]) \cong \widetilde{K}_0(R)\oplus K_{-1}(R) \oplus \mathrm{NK}_0(R) \oplus \mathrm{NK}_0(R),$$

and

$$K_{1-n}(R[t,t^{-1}]) \cong K_{1-n}(R) \oplus K_{-n}(R) \oplus \mathrm{NK}_{1-n}(R) \oplus \mathrm{NK}_{1-n}(R), \quad n>1.$$

There is also an analogue of Corollary 2.4 for *bounded pseudoisotopies* on $X\times\mathbb{R}^n$. Let $\mathcal{P}_b(X\times\mathbb{R}^n)$ be the simplicial set of bounded pseudoisotopies on $X\times\mathbb{R}^n$. In §6.8, it is shown (using [27], [29]) that, for $n\geq 1$,

$$\begin{aligned}\mathcal{P}_b(X\times\mathbb{R}^n) &\simeq \Omega\mathcal{W}h(X\times\mathbb{R}^n \to \mathbb{R}^n)\\ &\simeq \mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}).\end{aligned}$$

The following result is a consequence of Theorem 2.5.

Corollary 2.6. *If X is a compact Hilbert cube manifold, then there is a homotopy equivalence of simplicial sets*

$$\begin{aligned}\mathcal{P}_b(X\times S^1\times\mathbb{R}^n) \simeq \mathcal{P}_b(X\times\mathbb{R}^n)\times\mathcal{W}h(X\times\mathbb{R}^n \to \mathbb{R}^n)\times \\ \Omega\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)\times\Omega\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n).\end{aligned}$$

As mentioned above, controlled topology plays a major role in our results. One important part of controlled topology is the theory of manifold approximate fibrations. Manifold approximate fibrations arise from the controlled Whitehead spaces as follows: if $f : M \to X\times B$ is a vertex of $\mathcal{W}h(X\times B \to B)$, then the composition $\mathrm{proj}\circ f : M \to B$ is a manifold approximate fibration. In fact, this construction induces a map between moduli spaces. Many of our constructions are carried out on the level of spaces of manifold approximate fibrations and then getting results about Whitehead spaces as corollaries. One consequence of this approach is that the manifold approximate fibration results hold equally well for finite dimensional manifolds (of dimension greater than four) as for Hilbert cube manifolds. However, we offer no analogue for finite dimensional manifolds of our splitting of the uncontrolled part of $\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to \mathbb{R}^n)$ into two Nil spaces.

In addition to giving parametric treatments of the wrapping up, unwrapping and relaxation constructions for moduli spaces of manifold approximate fibrations, another innovation is to give interpretations of these as maps between classifying spaces. The spaces of manifold approximate fibrations and the Whitehead spaces were shown to be homotopy equivalent to spaces of cross sections of certain bundles of classifying spaces in [30] and [32]. We obtain simple descriptions of our geometric constructions as maps between these spaces of cross sections. These descriptions should allow easier application of the complicated geometry in the future.

There are two more new ideas in this paper which should be mentioned here. The first is a clarification of the boundedness properties of the infinite cyclic cover of a space with a map to S^1. As pointed out in [31], the infinite cyclic cover is finitely dominated if and only if the natural map to $\mathbb{R}$ is a bounded fibration, in which case the space is called a *band*. Here we encounter spaces with maps to $S^1\times\mathbb{R}^n$ and the situation is complicated by the noncompactness of $\mathbb{R}^n$.

The second is a parametric version of the sum theorem for Whitehead torsion. Our proof uses Hilbert cube manifold machinery.

For background on Hilbert cube manifolds including the notion of Z-sets, consult the books by Chapman [6] and van Mill [39]. The Hilbert cube Q is the countable infinite product of closed intervals. A *Hilbert cube manifold*, or *Q-manifold*, is a separable metric space which is locally homeomorphic to open subsets of Q. We will also need basic results from Q-manifold theory parametrized by fiber bundle projections, including the notion of *sliced* Z-sets (sometimes called *fibered* Z-sets). The standard references for the parametrized theory are Chapman and Wong [11], Chapman and Ferry [12], Ferry [20], and Toruńczyk and West [46]. For a recent survey describing the connection between Hilbert cube manifold theory and finite dimensional manifolds, see Weiss and Williams [49].

3. Moduli spaces of manifolds and maps

Throughout this section let B denote a locally compact, separable metric space with a fixed metric and let $m \in \{1, 2, 3, \ldots, \infty\}$ be fixed. When $m = \infty$ an m-manifold means a Hilbert cube manifold. The Hilbert cube is denoted by Q and when $m = \infty$ we also write $m = Q$.

3.1. **The simplicial set of manifolds.** A k–simplex of the simplicial set $\mathrm{Man}^m(B)$ of *m–manifolds over B* consists of a subspace $M \subseteq \ell_2 \times B \times \Delta^k$ of small capacity such that

(i) the projection $\rho : M \to \Delta^k$ is a fibre bundle projection with fibres m–manifolds without boundary, and
(ii) the projection $p : M \to B \times \Delta^k$ is a proper map.

See [32] for the notion of small capacity of subspaces of $\ell_2 \times B \times \Delta^k$. We will usually ignore this embedding property; the reader can easily supply the missing details. For us the important information is the map $p : M \to B \times \Delta^k$; the embedding $M \subseteq \ell_2 \times B \times \Delta^k$ just allows us to avoid defining k–simplices of $\mathrm{Man}^m(B)$ as equivalence classes of such maps. The "small capacity" condition is a technical one which allows easy manipulation of the embedding (this is similar to Z–embeddings in Hilbert cube manifold theory). In other words, we often specify a k–simplex of $\mathrm{Man}^m(B)$ by giving a proper map $p : M \to B \times \Delta^k$ such that the composition

$$\rho : M \to B \times \Delta^k \to \Delta^k$$

is a fibre bundle projection with fibres m–manifolds without boundary. This should be thought of as a k–parameter family of manifolds mapping to B.

Proposition 3.1. *Two vertices $p : M \to B$ and $q : N \to B$ are in the same component of $\mathrm{Man}^m(B)$ if and only if there exists a homeomorphism $h : M \to N$ such that p is properly homotopic to qh.*

Proof. See Hughes–Taylor–Williams [35], Proposition 1.6. $\square$

3.2. **The simplicial set of manifold approximate fibrations.** Recall the following definition.

Definition 3.2. If $\mathcal{U}$ is an open cover of B, then a map $p\colon M \to B$ is a *$\mathcal{U}$–fibration* if for every commuting diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\times 0"'] & M \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "F"] & B \end{tikzcd}$$

there is a map $\tilde{F}\colon X \times [0,1] \to M$ such that $\tilde{F}|X \times 0 = f$ and $p\tilde{F}$ is $\mathcal{U}$–close to F.

If $c > 0$, then we also use c to denote the open cover of B by c–balls. Thus, we speak of *c–fibrations*. If $p\colon M \to B$ is a c–fibration for some $c > 0$, then we say p is a *bounded fibration*.

We assume that the reader is familiar with the basic properties of *approximate fibrations* as discussed in [32], Appendix. In particular, if M and B are ANRs (as they always will be in this paper), then a map $p\colon M \to B$ is a c–fibration for every $c > 0$ if and only if p is an approximate fibration.

A *manifold approximate fibration* is a proper approximate fibration between manifolds without boundary. Important special cases to keep in mind are fibrations (i.e., maps with the homotopy lifting property for all spaces) and projection maps of locally trivial fibre bundles, as long as these maps are proper and have manifolds without boundary as domain and range.

Assume now that B is a manifold without boundary with a fixed metric. The simplicial set $\mathrm{MAF}^m(B)$ of *manifold approximate fibrations over* B was defined in [32]. A k–simplex consists of a subspace $M \subseteq \ell_2 \times B \times \Delta^k$ of small capacity such that

(i) the projection $\rho : M \to \Delta^k$ is a fibre bundle projection with fibres m–manifolds without boundary, and
(ii) the projection p: $M \to B \times \Delta^k$ has the property that for each $t \in \Delta^k$,

$$p| : p^{-1}(B \times \{t\}) \to B \times \{t\}$$

is a manifold approximate fibration.

We will usually ignore the ℓ_2 embedding property in the definition, as mentioned above in the case of the simplicial set of manifolds over B.

By analogy with Proposition 3.1 we have the following characterization of components of the simplicial set of manifold approximate fibrations. Let B be a fixed manifold with $\dim B < \infty$ and assume that B is either a closed manifold or a product $Y \times \mathbb{R}^n$ of a closed manifold Y and $\mathbb{R}^n$ (with the standard metric) and that B has the product metric. The reason that we make this assumption here and elsewhere is so that we can use epsilonics rather than open covers of B in making estimates.[1]

Proposition 3.3. *If $m \geq 5$, then there exists $\epsilon_0 > 0$ so that if $p_i : M_i \to B$, $i = 1, 2$, are two vertices of $\mathrm{MAF}^m(B)$, then the following are equivalent :*

(i) *p_1 and p_2 are in the same component of $\mathrm{MAF}^m(B)$.*
(ii) *There exists a homeomorphism $h : M_1 \to M_2$ such that $p_2 h$ is ϵ_0–homotopic to p_1.*
(iii) *For every $\epsilon > 0$ there exists a homeomorphism $h_\epsilon : M_1 \to M_2$ such that $p_2 h_\epsilon$ is ϵ–homotopic to p_1.*

Proof. See Hughes–Taylor–Williams [32], Corollary 7.12. □

3.3. **The simplicial set of manifold bands.** We continue to let B denote a manifold without boundary with a fixed metric. The standard exponential map is denoted by $\mathrm{e} : \mathbb{R} \to S^1$ and is an infinite cyclic covering projection. We also generically use e to denote the product of e with identity maps, for example $\mathrm{e} = \mathrm{id} \times \mathrm{e} \times \mathrm{id}$, so that e is always an infinite cyclic covering projection.

A k–simplex of the simplicial set $\mathrm{ManBan}^m(S^1 \times B)$ is a k–simplex of $\mathrm{Man}^m(S^1 \times B)$ with projection

$$p : M \to S^1 \times B \times \Delta^k$$

[1] More generally, B could be a non-compact manifold with a sufficiently homogeneous metric, for example, hyperbolic space or any cover of a closed manifold.

so that the infinite cyclic cover

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{p}} & \mathbb{R} \times B \times \Delta^k \\ \downarrow & & \downarrow \mathrm{e} \\ M & \xrightarrow{p} & S^1 \times B \times \Delta^k \end{array}$$

has the property that for every $\epsilon > 0$ there exists $c > 0$ such that for every $t \in \Delta^k$ the restriction

$$\bar{p}| : \bar{p}^{-1}(\mathbb{R} \times B \times \{t\}) \to \mathbb{R} \times B \times \{t\}$$

is a $(c \times \epsilon)$–fibration where $(c \times \epsilon)$ denotes the open cover of $\mathbb{R} \times B$ consisting of products of c–balls in $\mathbb{R}$ with ϵ–balls in B.

A k–simplex of $\mathrm{ManBan}^m(S^1 \times B)$ is called a *k–parameter manifold band over B*, and a 0–parameter manifold band over B is just a *manifold band over B*.

In the case $B = \{\text{point}\}$, the property defining a manifold band over B is that there exists $c > 0$ such that each restriction

$$\bar{p}| : \bar{p}^{-1}(\mathbb{R} \times \{t\}) \to \mathbb{R} \times \{t\}$$

is a c-fibration. By Hughes-Ranicki ([31], Proposition 17.14), this is equivalent to saying each $\bar{p}^{-1}(\mathbb{R} \times \{t\})$ is finitely dominated. This explains the terminology because a *manifold band* is a manifold M with a map $M \to S^1$ such that the infinite cyclic cover $\bar{M}$ is finitely dominated.

In Chapter 6 we will encounter homotopy equivalences to manifolds of the form $X \times S^1 \times B$ in the definition of Whitehead spaces. When the manifolds are compact, passing to infinite cyclic covers produces manifold bands over B as Proposition 3.5 below shows. After that result, we will discuss the noncompact case. We begin with a simple lemma which illustrates how infinite cyclic covers induce boundedness. This lemma is established in [35], Lemma 3.2, in a more general situation when S^1 is replaced by any closed manifold of nonpositive curvature.

Lemma 3.4 (Bounding Lifts)**.** *Let Z be a compact space with a homotopy $K : Z \times [0,1] \to S^1$. There exists $c > 0$ such that if $\omega : [0,1] \to \mathbb{R}$ is any path covering a track of K with respect to* e *(i.e., there exists $z \in Z$ such that $\mathrm{e}\,\omega(t) = K(z,t)$ for each $t \in [0,1]$), then $\mathrm{diam}(\omega) < c$.*

Proposition 3.5. *Suppose M and X are compact manifolds, B is a closed manifold, and $f : M \to X \times S^1 \times B$ is a homotopy equivalence such that the composition*

$$p_B \circ f : M \xrightarrow{f} X \times S^1 \times B \xrightarrow{p_B} B$$

is a vertex of $\mathrm{MAF}^m(B)$. Then the composition

$$p_{S^1 \times B} \circ f : M \xrightarrow{f} X \times S^1 \times B \xrightarrow{p_{S^1 \times B}} S^1 \times B$$

is a vertex of $\mathrm{ManBan}^m(S^1 \times B)$.

Proof. Form the pull-back diagram

$$\begin{array}{ccccc} \bar{M} & \xrightarrow{\bar{f}} & X \times \mathbb{R} \times B & \xrightarrow{p_{\mathbb{R} \times B}} & \mathbb{R} \times B \\ \bar{\mathrm{e}} \downarrow & & \downarrow \mathrm{e} & & \downarrow \mathrm{e} \\ M & \xrightarrow{f} & X \times S^1 \times B & \xrightarrow{p_{S^1 \times B}} & S^1 \times B \end{array}$$

It is known that the conditions on f imply that for every $\epsilon > 0$, there is a $p_B^{-1}(\epsilon)$-homotopy inverse for f; that is, there exist a map $g : X \times S^1 \times B \to M$ and homotopies $F : \mathrm{id}_{X\times S^1\times B} \simeq fg$ and $G : \mathrm{id}_M \simeq gf$ such that $p_B \circ F$ and $p_B \circ f \circ G$ are ϵ-homotopies. Similarly, it suffices to show that given $\epsilon > 0$ there exist $c > 0$ and a $p_{\mathbb{R}\times B}^{-1}(c \times \epsilon)$-homotopy inverse $\bar{g}$ for $\bar{f}$ (see [27]). So let $\epsilon > 0$ be given and let $g : X \times S^1 \times B \to M$ be a $p_B^{-1}(\epsilon)$-homotopy inverse for f with homotopies F, G as above. The lifting problem

$$\begin{array}{ccc} X \times \mathbb{R} \times B & \xrightarrow{\mathrm{id}_{X\times\mathbb{R}\times B}} & X \times \mathbb{R} \times B \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{e}} \\ X \times \mathbb{R} \times B \times [0,1] & \xrightarrow{F(\mathrm{e}\times \mathrm{id}_{[0,1]})} & X \times S^1 \times B \end{array}$$

has a solution $\tilde{F} : X \times \mathbb{R} \times B \times [0,1] \to X \times \mathbb{R} \times B$ so that $\tilde{F}_0 = \mathrm{id}_{X\times\mathbb{R}\times B}$ and $\mathrm{e}\,\tilde{F}_t = F_t\,\mathrm{e}$ for each $t \in [0,1]$. Since the tracks of $p_{\mathbb{R}}\tilde{F}$ cover tracks of $p_{S^1}F$ with respect to e, it follows from Lemma 3.4 that there exists $c_1 > 0$ such that $p_{\mathbb{R}}\tilde{F}$ is a c_1-homotopy. Moreover, each track of $p_B\tilde{F}$ is a track of p_BF, so $\tilde{F}$ is a $p_B^{-1}(\epsilon)$-homotopy. Thus, $\tilde{F}$ is a $p_{\mathbb{R}\times B}^{-1}(c_1 \times \epsilon)$-homotopy.

Define $\bar{g} : X \times \mathbb{R} \times B \to \bar{M}$ by $\bar{g}(x) = (g\,\mathrm{e}(x), \tilde{F}_1(x)) \in \bar{M} \subseteq M \times (X \times \mathbb{R} \times B)$. Note that $\bar{\mathrm{e}}\bar{g} = g\,\mathrm{e}$ and $\tilde{F}_1 = \bar{f}\bar{g}$ so that $\tilde{F} : \mathrm{id}_{X\times\mathbb{R}\times B} \simeq \bar{f}\bar{g}$ is a $p_{\mathbb{R}\times B}^{-1}(c_1 \times \epsilon)$-homotopy.

It remains to show that $\bar{g}$ is a left homotopy equivalence for $\bar{f}$ (with the correct control in $\mathbb{R} \times B$). Define $K : \bar{M} \times [0,1] \to M$ by $K(x,t) = G(\bar{\mathrm{e}}(x), 1-t)$. The lifting problem

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{g}\bar{f}} & \bar{M} \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle \bar{\mathrm{e}}} \\ \bar{M} \times I & \xrightarrow{K} & M \end{array}$$

has a solution $\tilde{K} : \bar{M} \times [0,1] \to \bar{M}$ so that $\tilde{K}_0 = \bar{g}\bar{f}$ and $\bar{\mathrm{e}}\tilde{K}_1 = \bar{\mathrm{e}}$. Thus, $\tilde{K}_1 : \bar{M} \to \bar{M}$ is a covering translation and, letting $\gamma = \tilde{K}_1^{-1}$, it follows that $\gamma\bar{g}$ is a left homotopy inverse for $\bar{f}$.

Since tracks of $\tilde{F}$ cover tracks of F with respect to e, it follows that tracks of $\gamma\bar{g}\tilde{F}$ cover tracks of gF with respect to $\bar{\mathrm{e}}$, and tracks of $p_{\mathbb{R}}\bar{f}\gamma\bar{g}\tilde{F}$ cover tracks of $p_{S^1}fgF$ with respect to e. It follows from Lemma 3.4 that $\gamma\bar{g}\tilde{F}$ is a $(p_{\mathbb{R}}\bar{f})^{-1}(c_2)$-homotopy for some $c_2 > 0$. Thus, $\gamma\bar{g}\tilde{F}\bar{f} : \gamma\bar{g}\bar{f} \simeq \gamma\bar{g}\bar{f}\bar{g}\bar{f}$ is a $(p_{\mathbb{R}}\bar{f})^{-1}(c_2)$-homotopy. Moreover, $p_B\bar{f}\gamma\bar{g}\tilde{F}$ is a 3ϵ-homotopy.

Tracks of the homotopy $\gamma\tilde{K}$ cover tracks of G with respect to $\bar{\mathrm{e}}$. Thus, tracks of $p_{\mathbb{R}}\bar{f}\gamma\tilde{K}$ cover tracks of $p_{S^1}fG$ with respect to e. It follows from Lemma 3.4 that $\gamma\tilde{K} : \gamma\bar{g}\bar{f} \simeq \mathrm{id}_{\bar{M}}$ is a $(p_{\mathbb{R}}\bar{f})^{-1}(c_3)$-homotopy for some $c_3 > 0$. Moreover, $p_B\bar{\gamma}\tilde{K}$ is an ϵ-homotopy.

It follows that there is a $(p_{\mathbb{R}\times B}\bar{f})^{-1}((c_3 + 2c_2) \times 7\epsilon)$-homotopy

$$\mathrm{id}_{\bar{M}} \simeq \gamma\bar{g}\bar{f} \simeq \gamma\bar{g}(\bar{f}\bar{g})\bar{f} = (\gamma\bar{g}\bar{f})\bar{g}\bar{f} \simeq \bar{g}\bar{f}$$

completing the proof. □

When B is not compact, passing to an infinite cyclic cover may not produce boundedness. We next give one general situation where boundedness is produced, and then give an example when it is not. For notation, we use

$$\mathrm{e}^n = \mathrm{e} \times \mathrm{e} \times \cdots \times \mathrm{e} : \mathbb{R}^n \to T^n$$

for the standard covering projection as well as $\mathrm{e}^n = \mathrm{id} \times \mathrm{e}^n \times \mathrm{id}$.

Proposition 3.6. *Suppose M and X are compact manifolds, Y is a closed manifold, and $f : M \to X \times S^1 \times Y \times T^n$ is a homotopy equivalence such that*

$$p_{Y\times T^n} \circ f : M \xrightarrow{f} X \times S^1 \times Y \times T^n \xrightarrow{p_{Y\times T^n}} Y \times T^n$$

is a vertex of $\mathrm{MAF}^m(Y \times T^n)$*. Then the map*

$$\bar{f} : \bar{M} \to X \times S^1 \times Y \times \mathbb{R}^n$$

induced by pulling back along the cover $\mathrm{e}^n : X \times S^1 \times Y \times \mathbb{R}^n \to X \times S^1 \times Y \times T^n$ *is a vertex of* $\mathrm{ManBan}(S^1 \times Y \times \mathbb{R}^n)$*.*

Proof. Consider the diagram of pull-back squares:

$$\begin{array}{ccc}
\bar{\bar{M}} & \xrightarrow{\bar{\bar{f}}} & X \times \mathbb{R} \times Y \times \mathbb{R}^n \\
\big\downarrow & & \big\downarrow \mathrm{e} \\
\bar{M} & \xrightarrow{\bar{f}} & X \times S^1 \times Y \times \mathbb{R}^n \\
\big\downarrow & & \big\downarrow \mathrm{e}^n \\
M & \xrightarrow{f} & X \times S^1 \times Y \times T^n
\end{array}$$

Given $\epsilon > 0$ it needs to be shown that there exists $c > 0$ such that $p_{\mathbb{R}\times Y\times\mathbb{R}^n} \circ \bar{\bar{f}} : \bar{\bar{M}} \to \mathbb{R} \times Y \times \mathbb{R}^n$ is a $(c \times \epsilon)$-fibration (where $\mathbb{R}\times Y\times\mathbb{R}^n = \mathbb{R}\times(Y\times\mathbb{R}^n)$). Note that $\bar{\bar{f}}$ also fits into the diagram of pull-back squares

$$\begin{array}{ccc}
\bar{\bar{M}} & \xrightarrow{\bar{\bar{f}}} & X \times \mathbb{R} \times Y \times \mathbb{R}^n \\
\big\downarrow & & \big\downarrow \mathrm{e}^n \\
M' & \xrightarrow{f'} & X \times \mathbb{R} \times Y \times T^n \\
\big\downarrow & & \big\downarrow \mathrm{e} \\
M & \xrightarrow{f} & X \times S^1 \times Y \times T^n.
\end{array}$$

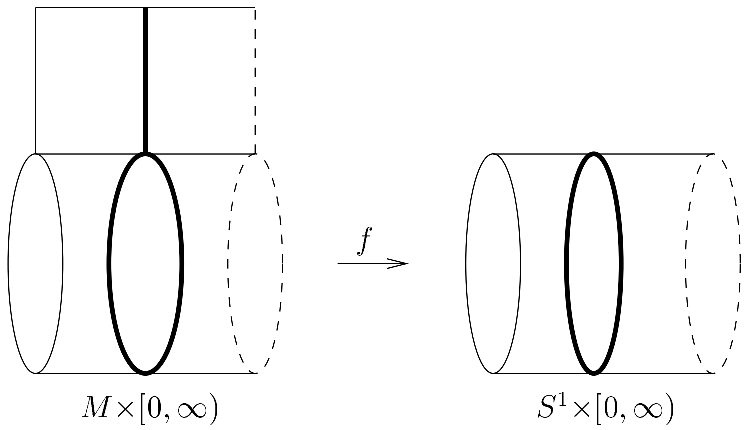
It follows from Proposition 3.5 that there exists $c > 0$ such that $p_{\mathbb{R}\times Y\times T^n} \circ f' : M' \to \mathbb{R} \times Y \times T^n$ is a $(c \times \epsilon)$-fibration. Now the diagram

$$\begin{array}{ccc}
\bar{\bar{M}} & \xrightarrow{p_{\mathbb{R}\times B}\circ\bar{\bar{f}}} & \mathbb{R} \times Y \times \mathbb{R}^n \\
\big\downarrow & & \big\downarrow \mathrm{e}^n \\
M' & \xrightarrow{p_{\mathbb{R}\times Y\times T^n}\circ f'} & \mathbb{R} \times Y \times T^n
\end{array}$$

immediately implies the result (provided ϵ is small enough). $\square$

Example 3.7. This example illustrates the failure of Proposition 3.5 in the case that $B = [0, \infty)$. For notation let $S^1 = \{[t] \mid t \in \mathbb{R}\}$ where $[t] = [t']$ if and only if $t - t' \in \mathbb{Z}$. Thus, the exponential map $\mathrm{e} : \mathbb{R} \to S^1$ is given by $\mathrm{e}(t) = [t]$. Let

$M = [0,1] \vee S^1 = \{(s,[t]) \in [0,1] \times S^1 \mid s = 0 \text{ or } t \in \mathbb{Z}\}$, the circle with a sticker attached. Define a homotopy equivalence $f : M \times [0,\infty) \to S^1 \times [0,\infty)$ by $(s,[t],u) \mapsto ([t+su],u)$ (see figure below).



Let $\bar{M} = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y \in \mathbb{Z} \text{ and } 0 \leq x \leq 1\}$. There is a pull-back diagram

$$\begin{array}{ccc} \bar{M} \times [0,\infty) & \xrightarrow{\bar{f}} & \mathbb{R} \times [0,\infty) \\ {\scriptstyle \bar{\mathrm{e}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{e}} \\ M \times [0,\infty) & \xrightarrow{f} & S^1 \times [0,\infty) \end{array}$$

where $\bar{f}(x,y,u) = (y + xu, u)$ and $\bar{\mathrm{e}}(x,y,u) = (x,[y],u)$. Since $p_{[0,\infty)} \circ f : M \times [0,\infty) \to [0,\infty)$ is projection, it is certainly an approximate fibration. However, the composition $p_{\mathbb{R}\times[0,\infty)} \circ \bar{f} : \bar{M} \times [0,\infty) \to \mathbb{R} \times [0,\infty)$ is not a $(c \times 1)$-fibration for any $c > 0$. Of course, M is not a manifold, but manifold examples can be constructed by taking a regular neighborhood of M or crossing with the Hilbert cube (to get a Hilbert cube manifold).

Because of this example, we introduce another simplicial set, $\mathrm{ManBan}^m_c(S^1 \times B)$, which will contain homotopy equivalences to $B \times S^1$ even though B is not compact. A k–simplex of $\mathrm{ManBan}^m_c(S^1 \times B)$ is a k–simplex of $\mathrm{Man}^m(S^1 \times B)$ with projection

$$p : M \to S^1 \times B \times \Delta^k$$

so that the infinite cyclic cover

$$\bar{p} : \bar{M} \to \mathbb{R} \times B \times \Delta^k$$

has the property that for every compact subspace $K \subseteq B$ and for every $\epsilon > 0$ there exists $c > 0$ such that for every $t \in \Delta^k$ the restriction

$$\bar{p}| : \bar{p}^{-1}(\mathbb{R} \times B \times \{t\}) \to \mathbb{R} \times B \times \{t\}$$

is a $(c \times \epsilon)$–fibration over $\mathbb{R} \times K \times \{t\}$. Thus, there is an inclusion map

$$\mathrm{ManBan}^m(S^1 \times B) \to \mathrm{ManBan}^m_c(S^1 \times B)$$

which is the identity if B is compact. The subscript c stands for "compact subsets," and $\mathrm{ManBan}^m_c(S^1 \times B)$ is called *the simplicial set of manifold bands with respect to compact subsets*.

We need the following analogue of Proposition 3.4. Its proof is omitted since it follows that of Proposition 3.4 almost word for word.

Proposition 3.8. *Suppose X is a compact manifold, M and B are manifolds without boundary, and $f : M \to X \times S^1 \times B$ is a proper homotopy equivalence such that the composition*

$$p_B \circ f : M \xrightarrow{f} X \times S^1 \times B \xrightarrow{p_B} B$$

is a vertex of $\mathrm{MAF}^m(B)$. *Then the composition*

$$p_{S^1\times B} \circ f : M \xrightarrow{f} X \times S^1 \times B \xrightarrow{p_{S^1\times B}} B$$

is a vertex of $\mathrm{ManBan}^m_c(S^1 \times B)$.

In Proposition 3.10 below it will be shown that the inclusion

$$\mathrm{ManBan}^m(S^1 \times B) \to \mathrm{ManBan}^m_c(S^1 \times B)$$

is a homotopy equivalence in the case $B = Y \times \mathbb{R}^n$ with Y a closed manifold.

3.4. **The forget control map.** If B is a manifold, then

$$\varphi\colon\ \mathrm{MAF}^m(B) \to \mathrm{Man}^m(B)$$

will denote the inclusion and we call φ the *forget control map*. Note that the forget control map on $\mathrm{MAF}^m(S^1 \times B)$ actually has image in the simplicial set of manifold bands,

$$\varphi : \mathrm{MAF}^m(S^1 \times B) \to \mathrm{ManBan}^m(S^1 \times B).$$

This is because if $p : M \to S^1 \times B$ is a manifold approximate fibration, then so is $\bar{p} : \bar{M} \to \mathbb{R} \times B$.

There is also a natural map

$$q : \mathrm{ManBan}^m(S^1 \times B) \to \mathrm{MAF}^m(B)$$

defined by taking a k–simplex $p : M \to S^1 \times B \times \Delta^k$ of $\mathrm{ManBan}^m(S^1 \times B)$ to the composition

$$M \xrightarrow{p} S^1 \times B \times \Delta^k \xrightarrow{\mathrm{proj}} B \times \Delta^k.$$

Of course, one needs to verify that this composition is a k-simplex of $\mathrm{MAF}^m(B)$. We give the proof of this fact for vertices ($k = 0$), the more general case being entirely analogous.

Lemma 3.9. *If $p : M \to S^1 \times B$ is a vertex of* $\mathrm{ManBan}^m(S^1 \times B)$, *then the composition $p_B p : M \to B$ is a vertex of* $\mathrm{MAF}^m(B)$.

Proof. Consider the commuting diagram

$$\begin{tikzcd} \bar{M} \arrow[r, "\bar{p}"] \arrow[d, "\bar{e}"'] & \mathbb{R} \times B \arrow[r, "p_B"] \arrow[d, "e"'] & B \arrow[d, "\mathrm{id}_B"] \\ M \arrow[r, "p"] & S^1 \times B \arrow[r, "p_B"] & B \end{tikzcd}$$

where the square on the left is a pull-back. By assumption, for every $\epsilon > 0$ there exists a $c > 0$ such that $\bar{p}$ is a $(c \times \epsilon)$-fibration. It follows easily that for every $\epsilon > 0$,

$p_B\bar{p}$ is an ϵ-fibration. We need to show the same for $p_Bp : M \to B$. To this end let $\epsilon > 0$ be given and consider the a lifting problem:

$$\begin{array}{ccc} Z & \xrightarrow{f} & M \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle p_B p} \\ Z \times I & \xrightarrow{F} & B. \end{array}$$

Form the pull-back

$$\begin{array}{ccc} \bar{Z} & \xrightarrow{\bar{f}} & \bar{M} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \bar{e}} \\ Z & \xrightarrow{f} & M \end{array}$$

and define $\bar{F} : \bar{Z} \times [0,1] \to B$ by $\bar{F}(\bar{z},t) = F(g(\bar{z}),t)$. This gives a lifting problem

$$\begin{array}{ccc} \bar{Z} & \xrightarrow{\bar{f}} & \bar{M} \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle p_B \bar{p}} \\ \bar{Z} \times I & \xrightarrow{\bar{F}} & B. \end{array}$$

Let $F^\sharp : \bar{Z} \times [0,1] \to \bar{M}$ be an ϵ'-solution where ϵ' is as small as we like (and depends on ϵ).

If $\zeta_{\bar{M}} : \bar{M} \to \bar{M}$ and $\zeta_{\bar{Z}} : \bar{Z} \to \bar{Z}$ denote the covering translations induced from the $+1$ covering translation on $\mathbb{R}$, then it follows that $\zeta_{\bar{M}}\bar{f} = \bar{f}\zeta_{\bar{Z}}$, $g\zeta_{\bar{Z}} = g$, and $\bar{F} \circ (\zeta_{\bar{Z}} \times \mathrm{id}_{[0,1]}) = \bar{F}$.

Let $\bar{Z}_i = (p_{\mathbb{R}}\bar{p}\bar{f})^{-1}(i)$ for $i = 0,1$ and $\bar{Z}_{[0,1]} = (p_{\mathbb{R}}\bar{p}\bar{f})^{-1}([0,1])$. Define $\hat{F}^\sharp : (\bar{Z}_0 \cup \bar{Z}_1) \times I \cup \bar{Z} \times \{0\} \to \bar{M}$ by

$$\hat{F}^\sharp(\bar{z},t) = \begin{cases} F^\sharp(\bar{z},t), & \text{if } \bar{z} \in \bar{Z}_0 \text{ or } t = 0 \\ \zeta_{\bar{M}} F^\sharp(\zeta_{\bar{Z}}^{-1}(\bar{z}),t), & \text{if } \bar{z} \in \bar{Z}_1. \end{cases}$$

Note that this is an ϵ'-lift of $\bar{F}|$. Hence, if ϵ' is chosen small enough, $\hat{F}^\sharp$ extends to a map $F^! : \bar{Z}_{[0,1]} \times [0,1] \to \bar{M}$ which is an ϵ-lift of $\bar{F}|$. Finally, define an ϵ-solution $\hat{F} : Z \times I \to M$ of the original problem by $\hat{F}(z,t) = \bar{e}F^!((g|)^{-1}(z),t)$ where $g| : \bar{Z}_{[0,1]} \to Z$. $\square$

The proof of Lemma 3.9 shows that the map $q : \mathrm{ManBan}^m(S^1 \times B) \to \mathrm{MAF}^m(B)$ extends to a map $q_c : \mathrm{ManBan}^m_c(S^1 \times B) \to \mathrm{MAF}^m(B)$ defined by the formula: a k-simplex $p : M \to S^1 \times B \times \Delta^k$ of $\mathrm{ManBan}^m_c(S^1 \times B)$ is sent to the composition

$$M \xrightarrow{p} S^1 \times B \times \Delta^k \xrightarrow{\mathrm{proj}} B \times \Delta^k.$$

This extension exists because the property of a map to B being an approximate fibration is a local property (see [16], [14]).

The composition of these two maps gives yet another simplicial map, also denoted φ and called the *forget control map*,

$$\varphi : \mathrm{MAF}^m(S^1 \times B) \to \mathrm{MAF}^m(B).$$

Note that this map is really just a partial forgetting of control: it forgets control in the S^1 direction, but remembers the control in the B direction.

We end with a result showing that there is essentially no difference between ManBan and ManBan_c in the cases we will need.

Proposition 3.10. *If Y is a closed manifold, then the inclusion*

$$\mathrm{ManBan}^m(S^1 \times Y \times \mathbb{R}^n) \to \mathrm{ManBan}_c^m(S^1 \times Y \times \mathbb{R}^n)$$

is a homotopy equivalence.

Proof. It suffices to show that the inclusion induces an isomorphism on homotopy groups. To this end let $f : M \to S^1 \times Y \times \mathbb{R}^n \times \Delta^k$ be a k-simplex of $\mathrm{ManBan}_c^m(S^1 \times Y \times \mathbb{R}^n)$ which is a union of $(k-1)$-simplices of $\mathrm{ManBan}^m(S^1 \times Y \times \mathbb{R}^n)$ over $S^1 \times Y \times \mathbb{R}^n \times \partial\Delta^k$. By a deformation rel $\partial\Delta^k$ we may assume that f has the ManBan property over a collar neighborhood C of $\partial\Delta^k$ in Δ^k. Let U be an open subset of $\mathbb{R}^n \times \Delta^k$ such that $U \cap (\mathbb{R}^n \times \{t\})$ is the ball of radius 1 about the origin for $t \notin C$, $U \cap (\mathbb{R}^n \times \partial\Delta^k) = \mathbb{R}^n \times \partial\Delta^k$, and $U \cap (\mathbb{R}^n \times \{t\})$ is a ball of radius $r(t) > 1$ for $t \in C \setminus \partial\Delta^k$ such that $r(t) \to \infty$ as $t \to \partial\Delta^k$. Let $h : U \to \mathbb{R}^n \times \Delta^k$ be a radial homeomorphism rel $\partial\Delta^k$ and let $M_U = f^{-1}(S^1 \times Y \times U)$. Then $f_U = (\mathrm{id}_{S^1 \times Y} \times h^{-1}) \circ f : M_U \to S^1 \times Y \times \mathbb{R}^n \times \Delta^k$ is a k-simplex of $\mathrm{ManBan}^m(S^1 \times Y \times \mathbb{R}^n)$ which is homotopic in $\mathrm{ManBan}_c^m(S^1 \times Y \times \mathbb{R}^n)$ rel $\partial\Delta^k$ to f. The key ideas needed to verify this are in [30], §6. That f_U is in ManBan rather than just ManBan_c follows from the fact that $U \cap (\mathbb{R}^n \times (\Delta^k \setminus C))$ has compact closure. □

4. Wrapping-up and unwrapping as simplicial maps

Fix $m \in \{5, 6, \ldots, \infty\}$ and suppress m from the notation for the simplicial sets defined in Chapter 3. Let B denote a fixed manifold with $\dim B < \infty$ and assume that B is either a closed manifold or a product $Y \times \mathbb{R}^n$ of a closed manifold Y and $\mathbb{R}^n$ (with the standard metric) and that B has the product metric.

In this section we define two simplicial maps, *wrapping-up*

$$w : \mathrm{MAF}(\mathbb{R}\times B) \to \mathrm{MAF}(S^1 \times B)$$

and *unwrapping*

$$u : \mathrm{ManBan}(S^1 \times B) \to \mathrm{MAF}(\mathbb{R}\times B)$$

such that $uw \simeq 1_{\mathrm{MAF}(\mathbb{R}\times B)}$. More precisely, we show that the composition

$$\mathrm{MAF}(\mathbb{R}\times B) \xrightarrow{w} \mathrm{MAF}(S^1 \times B) \xrightarrow{\varphi} \mathrm{ManBan}(S^1 \times B) \xrightarrow{u} \mathrm{MAF}(\mathbb{R}\times B)$$

is homotopic to the identity (where φ is the forget control map defined in §3.4).

4.1. **Unwrapping.** The idea behind the map $u : \mathrm{ManBan}(S^1 \times B) \to \mathrm{MAF}(\mathbb{R}\times B)$ is simple. Just pull-back along the infinite cyclic cover $\mathbb{R}\times B \to S^1 \times B$. The problem is that the result of the pull–back construction is only a $(c\times\epsilon)$–fibration rather than an approximate fibration (this is because of the very definition of $\mathrm{ManBan}(S^1 \times B)$). Thus the Sucking Principle for manifold approximate fibrations must be applied (hence the dimension restriction). This principle originated in Chapman's papers [8], [9] and was developed parametrically by Hughes [26], [28] and by Hughes, Taylor and Williams [32]. In order to get a simplicial map the construction is carried out inductively over the skeleta and the following lemma allows us to carry out the induction.

Lemma 4.1 (Sucking to get MAFs)**.** *For every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, m, k, B) > 0$ so that if $p : M \to \mathbb{R}\times B \times \Delta^k$ is a proper map such that*

1. *the composition $\rho : M \to \mathbb{R} \times B \times \Delta^k \to \Delta^k$ is a fibre bundle projection with fibres m–manifolds without boundary, and*
2. *there exists $b > 0$ such that for all $t \in \Delta^k$ the restriction*

$$p| : \rho^{-1}(t) \to \mathbb{R}\times B \times \{t\}$$

is a $(b \times \delta)$–fibration.

Then there exist $c > 0$ and a f.p. $(c \times \epsilon)$–homotopy $h : p \simeq p'$ with the property that for each $t \in \Delta^k$ the restriction

$$p'| : \rho^{-1}(t) \to \mathbb{R}\times B \times \{t\}$$

is a manifold approximate fibration.

The property of p' above can be restated as "p' is a f.p. manifold approximate fibration".

Addendum 4.2.
1. *If $p'| : \rho^{-1}(\partial\Delta^k) \to \mathbb{R}\times B\times\partial\Delta^k$ is already a f.p. approximate fibration, then the homotopy $h : p \simeq p'$ can be chosen rel $\rho^{-1}(\partial\Delta^k)$.*
2. *If $m = \infty$, $X \subseteq M$ is a sliced Z-set Hilbert cube manifold and $p| : X \to \mathbb{R}\times B\times\Delta^k$ is a fibration, then the homotopy $h : p \simeq p'$ can be taken rel X (in addition to the other properties above).*

The proof of 4.1 and 4.2 can be obtained from [26].

For $n \geq 0$ consider the

Inductive Statement S_n: For each k, $0 \leq k \leq n$, and each k-simplex

$$p : M \to S^1 \times B \times \Delta^k$$

of $\mathrm{ManBan}(S^1 \times B)$ and each $\epsilon > 0$, there exists a k-simplex

$$\bar{M} \xrightarrow{u(p)} \mathbb{R} \times B \times \Delta^k$$

of $\mathrm{MAF}(\mathbb{R} \times B)$ and $c > 0$ together with a homotopy $H_p : \bar{M} \times [0,1] \to \mathbb{R} \times B \times \Delta^k$ such that

1. $\bar{M}$ comes from forming the pull-back diagram

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{p}} & \mathbb{R} \times B \times \Delta^k \\ \downarrow & & \downarrow \\ M & \xrightarrow{p} & S^1 \times B \times \Delta^k, \end{array}$$

2. H_p is a f.p. $(c \times \epsilon)$–homotopy from $u(p)$ to $\bar{p}$,
3. for each i, $0 \leq i \leq k$,
$$H_{\partial_i p} = H_p|$$
as a homotopy from $u(\partial_i p)$ to $\overline{\partial_i p} = \bar{p}|$,
4. for each i, $0 \leq i \leq k$, $u(\partial_i p) = \partial_i(u(p))$.

Proof of S_0. Given a vertex $M \to S^1 \times B$ of $\mathrm{ManBan}(S^1 \times B)$ and $\epsilon > 0$, form the pull-back

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{p}} & \mathbb{R} \times B \\ \downarrow & & \downarrow \\ M & \xrightarrow{p} & S^1 \times B \end{array}$$

and let $\delta > 0$ be given by Lemma 4.1. By the definition of $\mathrm{ManBan}(S^1 \times B)$, there exists $b > 0$ such that $\bar{p}$ is a $(b \times \delta)$–fibration. By Lemma 4.1 there exist $c > 0$ and a map $u(p) : \bar{M} \to \mathbb{R} \times B$ which is a vertex in $\mathrm{MAF}(\mathbb{R} \times B)$ together with a $(c \times \epsilon)$–homotopy $H_p : u(p) \simeq \bar{p}$. □

Proof that S_{n-1} implies S_n for $n > 0$. It suffices to consider an n-simplex

$$p : M \to S^1 \times B \times \Delta^n$$

of $\mathrm{ManBan}(S^1 \times B)$. Form the pull-back

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{p}} & \mathbb{R} \times B \times \Delta^n \\ \downarrow & & \downarrow \\ M & \xrightarrow{p} & S^1 \times B \times \Delta^n \end{array}$$

Let $\bar{\rho} : \bar{M} \to \Delta^n$ denote projection; that is, the composition

$$\bar{\rho} : \bar{M} \xrightarrow{\bar{p}} \mathbb{R} \times B \times \Delta^n \to \Delta^n.$$

For each i, $0 \leq i \leq n$, we have constructed

$$u(\partial_i p) : \bar{\rho}^{-1}(\partial_i \Delta^n) \to \mathbb{R} \times B \times \partial_i \Delta^n$$

by our inductive statement S_{n-1}. These fit together to give a map

$$u(\partial p) : \bar{\rho}^{-1}(\partial\Delta^n) \ \to\ \mathbb{R}\times B \times \partial\Delta^n.$$

We also have a f.p. $(b \times \delta)$–homotopy

$$H_{\partial p} : u(\partial p) \ \simeq\ \bar{p}|\bar{\rho}^{-1}(\partial\Delta^n)$$

for some $b, \delta > 0$ where b is possibly large, but δ is as small as we desire. Using this homotopy and a exterior collar $\partial\Delta^n \times [0,1]$ of $\partial\Delta^n$ on Δ^n so that $\partial\Delta^n \times \{1\} = \partial\Delta^n$, we can define a new map

$$p^* : \bar{M} \cup \bar{\rho}^{-1}(\partial\Delta^n)\times[0,1] \ \to\ \mathbb{R}\times B\times(\Delta^n\cup(\partial\Delta^n\times[0,1]))$$

such that

1. $p^* = u(\partial p)$ on $\bar{\rho}^{-1}(\partial\Delta^n) \times \{0\}$,
2. $p^* = H_{\partial p}$ on $\bar{\rho}^{-1}(\partial\Delta^n) \times [0,1]$, and
3. $p^* = \bar{p}$ on $\bar{M}$.

The point is that p^* is an f.p. $(b \times \delta)$–fibration, so Lemma 4.1 can be used to deform p^* rel $\partial\Delta^n\times\{0\}$ to a map in which we can find $u(p)$ and H_p. That is, we get a map

$$F : \bar{M}\times[0,1] \ \to\ \mathbb{R}\times B \times \Delta^n\times[0,1]$$

which is f.p. over $\Delta^n\times[0,1]$ such that

$$F|\bar{M}\times\{1\} = \bar{p} \quad \text{and} \quad F|\bar{\rho}^{-1}(\partial\Delta^n)\times[0,1] = H_{\partial p}.$$

Then we take $H_p = F$ and $u(p) = F|\bar{M}\times\{0\}$. □

It follows that we can define the simplicial map

$$u : \mathrm{ManBan}(S^1\times B) \to \mathrm{MAF}(\mathbb{R}\times B)$$

by letting $u(p)$ be given by the construction above.

Remark 4.3 (The Infinite Transfer). The composition

$$\mathrm{MAF}(S^1\times B) \ \xrightarrow{\varphi}\ \mathrm{ManBan}(S^1\times B) \ \xrightarrow{u}\ \mathrm{MAF}(\mathbb{R}\times B)$$

is homotopic to the simplicial map given by the pull–back construction (no sucking needed). This composition is also called the *unwrapping*, or *infinite transfer*, map and denoted

$$u : \mathrm{MAF}(S^1\times B) \ \to\ \mathrm{MAF}(\mathbb{R}\times B).$$

Finally, we note that the unwrapping map can be extended to a simplicial map

$$u : \mathrm{ManBan}_c(S^1\times B) \ \to\ \mathrm{MAF}(\mathbb{R}\times B).$$

The idea is to write B as an ascending union of compact subsets, $B = \cup_{i=1}^{\infty} B_i$, and, after pulling back, to inductively shrink in the $\mathbb{R}$–coordinates of $\mathbb{R} \times (B_{i+1} \setminus B_i)$.

4.2. **Wrapping up.** The construction of the map $w : \mathrm{MAF}(\mathbb{R}\times B) \to \mathrm{MAF}(S^1\times B)$ is based on the existence of certain engulfing isotopies which we get from the Approximate, or Controlled, Isotopy Covering Principle for manifold approximate fibrations. This principle originated in [25] and [28] and has been exposited in [32], [34] and [31]. We only need a special case which we now develop.

The initial construction of w is rather technical because the isotopies mentioned above have to be constructed inductively on the skeleta of $\mathrm{MAF}(\mathbb{R}\times B)$. However, once we have this concrete definition of wrapping-up in hand, we will be able to use the classification results of [32] to give w a simple description (see Theorems 4.11 and 4.13 below). As a corollary we will be able to show that when $B = \mathbb{R}^n$ the wrapping-up $w : \mathrm{MAF}(\mathbb{R}^{n+1}) \to \mathrm{MAF}(S^1\times\mathbb{R}^n)$ is independent (up to homotopy) of which $\mathbb{R}$–factor is used to wrap-up.

Recall that an isotopy on X is a homeomorphism $h : X\times[0,1] \to X\times[0,1]$ which is f.p. over $[0,1]$ and $h_0 = \mathrm{id}_X$.

Let $g : \mathbb{R}\times[0,1] \to \mathbb{R}\times[0,1]$ be the PL isotopy such that

1. g is supported on $[-1,3]\times[0,1]$, and
2. for each $s \in [0,1]$, g_s takes $[-1,-0.5]$ linearly onto $[-1, s-0.5]$, takes $[-0.5, 1.5]$ linearly onto $[s-0.5, s+1.5]$ and takes $[1.5, 3]$ linearly onto $[s+1.5, 3]$.

In particular, $g_1|[-0.5,1.5]$ has the form $t \mapsto t+1$.

Lemma 4.4 (Controlled Isotopy Covering)**.** *Given a k–simplex*

$$p : M \to \mathbb{R}\times B\times\Delta^k$$

of $\mathrm{MAF}(\mathbb{R}\times B)$ *and $\epsilon_0 > 0$, there exists a "continuous family of isotopies"*

$$G(p) : M\times[0,1]\times[0,\infty) \to M\times[0,1]$$

such that

1. *$G(p)$ is a map which is f.p. over $[0,1]$ and Δ^k,*
2. *for each $t \in [0,\infty)$, $G(p)_t : M\times[0,1] \to M\times[0,1]$ is an isotopy supported on $p^{-1}([-2,4]\times B\times\Delta^k)$,*
3. *$(p\times\mathrm{id}_{[0,1]})\circ G(p)_t : M\times[0,1] \to \mathbb{R}\times[0,1]\times B\times\Delta^k$ converges uniformly to $(g\times\mathrm{id}_{B\times\Delta^k})\circ(p\times\mathrm{id}_{[0,1]})$ as t goes to ∞, and*
4. *for each $t \in [0,\infty)$, $(p\times\mathrm{id}_{[0,1]})\circ G(p)_t$ is ϵ_0–close to $(g\times\mathrm{id}_{B\times\Delta^k})\circ(p\times\mathrm{id}_{[0,1]})$.*

Addendum 4.5. 1. *If $\rho : M \to \Delta^k$ is the projection and*

$$G(\partial p) : \rho^{-1}(\partial\Delta^k)\times[0,1]\times[0,\infty) \to \rho^{-1}(\partial\Delta^k)\times[0,1]$$

is already such a family, then $G(p)$ can be chosen so that

$$G(p)|\rho^{-1}(\partial\Delta^k)\times[0,1]\times[0,\infty) = G(\partial p).$$

2. *If $m = \infty$, X is a compact Hilbert cube manifold, $X\times\mathbb{R}\times B\times\Delta^k$ is a sliced Z–set in M, and $p| : X\times\mathbb{R}\times B\times\Delta^k \to \mathbb{R}\times B\times\Delta^k$ is projection, then $G(p)$ can be chosen so that for each $t\in[0,\infty)$*

$$G(p)_t| : X\times\mathbb{R}\times B\times\Delta^k \to X\times\mathbb{R}\times B\times\Delta^k$$

is $g\times\mathrm{id}_{X\times B\times\Delta^k}$ (in addition to the other properties above).

Proof of 4.4, 4.5. See [28], [27], [34], [31]. □

We will also need the following version of the Sucking Principle for manifold approximate fibrations (cf. Lemma 4.1).

Lemma 4.6 (Sucking to get MAFs)**.** *For every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, m, k, B) > 0$ such that if $p : M \to S^1 \times B \times \Delta^k$ is a proper map satisfying*

1. *the composition*
$$\rho : M \xrightarrow{p} S^1 \times B \times \Delta^k \to \Delta^k$$
is a fibre bundle projection with fibres m–manifolds without boundary,
2. *for each $t \in \Delta^k$,*
$$p| : \rho^{-1}(t) \to S^1 \times B \times \{t\}$$
is a δ–fibration, and
3. *for each $t \in \partial\Delta^k$,*
$$p| : \rho^{-1}(t) \to S^1 \times B \times \{t\}$$
is a manifold approximate fibration,

then there is a homotopy $p \simeq p'$ which is f.p. over Δ^k and is rel $\rho^{-1}(\partial\Delta^k)$ such that for each $t \in \Delta^k$
$$p'|\rho^{-1}(t) \to S^1 \times B \times \{t\}$$
is a manifold approximate fibration.

Addendum 4.7. *If $m = \infty$, X is a compact Hilbert cube manifold, $X \times S^1 \times B \times \Delta^k$ is a sliced Z–set in M, and $p| : X \times S^1 \times B \times \Delta^k \to S^1 \times B \times \Delta^k$ is projection, then the homotopy $p \simeq p'$ can be rel $X \times S^1 \times B \times \Delta^k$ (in addition to the other properties above).*

Proof of 4.6, 4.7. See [26], [28], [32]. □

One more result on manifold approximate fibrations is required.

Lemma 4.8 (Local Connectivity for MAFs)**.** *For every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, m, k, B) > 0$ such that if $p_0, p_1 : M \to S^1 \times B \times \Delta^k$ are k–simplices of* $\mathrm{MAF}(S^1 \times B)$ *satisfying*

1. *$p_0 = p_1$ over $S^1 \times B \times \partial\Delta^k$,*
2. *p_0 and p_1 are δ–close,*

then there is a map
$$p : M \to S^1 \times B \times \Delta^k \times [0, 1]$$
such that

1. *$p = p_i$ over $S^1 \times B \times \Delta^k \times \{i\}$ for $i = 0, 1$,*
2. *$p| = p_0| \times \mathrm{id} = p_1| \times \mathrm{id}$ over $S^1 \times B \times \partial\Delta^k \times [0, 1]$,*
3. *the composition*
$$\rho : M \xrightarrow{p} S^1 \times B \times \Delta^k \times [0, 1] \to \Delta^k \times [0, 1]$$
is a fibre bundle projection with fibres m–manifolds without boundary, and
4. *for each $t \in \Delta^k \times [0, 1]$,*
$$p| : \rho^{-1}(t) \to S^1 \times B \times \{t\}$$
is a manifold approximate fibration.

Addendum 4.9. *If $m = \infty$, X is a compact Hilbert cube manifold, $X \times S^1 \times B \times \Delta^k$ is a sliced Z–set in M, and both $p_0|, p_1| : X \times S^1 \times B \times \Delta^k \to S^1 \times B \times \Delta^k$ are projection, then the map p can be chosen so that $p| : X \times S^1 \times B \times \Delta^k \times [0, 1] \to S^1 \times B \times \Delta^k \times [0, 1]$ is projection.*

Proof of 4.8, 4.9. See [26], [28]. □

4.2.1. *Definition of* w. The definition of w is given inductively over the skeleta of $\mathrm{MAF}(\mathbb{R}\times B)$. To begin the construction let $p : M \to \mathbb{R}\times B$ be a vertex of $\mathrm{MAF}(\mathbb{R}\times B)$. Let

$$G(p) : M\times[0,1]\times[0,\infty) \to M\times[0,1]$$

be given by Controlled Isotopy Covering Lemma 4.4. Choose $t_0 > 0$ so large that if $t \geq t_0$ then

$$p^{-1}((-\infty,0)\times B)\subseteq G(p)_{(1,t)}(p^{-1}((-\infty,0]\times B))$$

where $G(p)_{(1,t)} : M \to M$ is

$$G(p)| : M\times\{1\} \times \{t\} \to M\times\{1\}.$$

For such t let

$$Y_t = G(p)_{(1,t)}(p^{-1}((-\infty,0]\times B))\backslash p^{-1}((-\infty,0)\times B)$$

and $\hat{M}_t = Y_t/\sim$ where $\sim$ is generated by

$$x \sim G(p)_{(1,t)}(x) \text{ if } p(x) \in \{0\}\times B.$$

For each $t \geq t_0$ there exists a map $q_t : Y_t \to [0,1]\times B$ constructed by first considering

$$p| : Y_t \to [0,2]\times B.$$

The constant map $[1,2] \to 1$ induces maps

$$c : [0,2]\times B \to [0,1]\times B, \quad cp| : Y_t \to [0,1]\times B.$$

Now $cp|$ is homotopic rel $p^{-1}((-\infty,0.5]\times B)\cap Y_t$ to a map $q_t : Y_t \to [0,1]\times B$ such that $q_t\circ G(p)_{(1,t)}(p^{-1}(\{0\}\times B)) = \{1\}\times B$ and the diagram

$$\begin{array}{ccc} p^{-1}(\{0\}\times B) & \xrightarrow{G(p)_{(1,t)}} & q_t^{-1}(\{1\}\times B) \\ {\scriptstyle p=q_t}\downarrow & & \downarrow{\scriptstyle q_t} \\ \{0\}\times B & \xrightarrow{+1} & \{1\}\times B \end{array}$$

commutes.

By identifying S^1 with $[0,1]/\{0=1\}$, q_t induces a map $\hat{p}_t : \hat{M}_t \to S^1\times B$. Moreover, $\hat{p}_t$ is a δ_t–fibration where $\delta_t \to 0$ as $t \to \infty$.

The construction of q_t from $cp|$ involves the use of the Homotopy Extension Property, so is not completely canonical. However, any two choices will yield two maps $\hat{p}_t$ and $\hat{p}'_t$ which can be assumed δ_t–close with δ_t as above. It follows from Lemmas 4.6 and 4.8 that there exists a manifold approximate fibration $\hat{p} : \hat{M} \to S^1\times B$ such that $[\hat{p}] \in \pi_0\,\mathrm{MAF}(S^1\times B)$ is well–defined by the construction above.

It is easy to see how to continue this construction inductively to construct

$$w : \mathrm{MAF}(\mathbb{R}\times B) \to \mathrm{MAF}(S^1\times B).$$

Theorem 4.10. *The composition*

$$\mathrm{MAF}(\mathbb{R}\times B) \xrightarrow{w} \mathrm{MAF}(S^1\times B) \xrightarrow{\varphi} \mathrm{ManBan}(S^1\times B) \xrightarrow{u} \mathrm{MAF}(\mathbb{R}\times B)$$

is homotopic to the identity.

Proof. We need to recall a bit of germ theory from [32]. Let $\mathrm{GMAF}(\mathbb{R}\times B)$ be the simplicial set whose k–simplices are equivalence classes of k–simplices of $\mathrm{MAF}(\mathbb{R}\times B)$; two such being equivalent if they are equal over some neighborhood of $\{0\}\times B\times\Delta^k$ in $\mathbb{R}\times B\times\Delta^k$. This process of taking germs induces a homotopy equivalence Γ : $\mathrm{MAF}(\mathbb{R}\times B) \to \mathrm{GMAF}(\mathbb{R}\times B)$. Now one only needs to observe that since $u\varphi w$ does not affect a neighborhood of $\{0\}\times B\times\Delta^k$ it follows that $\Gamma u\varphi w \simeq \Gamma$. □

4.3. **Classifying space interpretation.** In the special case $B=\mathbb{R}^n$ for $n\geq 0$, w has an especially pleasant description in terms of the classification results of [32]. Since $S^1\times\mathbb{R}^n$ is parallelizable and $S^1\times\mathbb{R}^n\simeq S^1$, according to [32] (Theorem 2.2.2 and Example 4.8) there is a homotopy equivalence

$$\Psi : \mathrm{MAF}(S^1\times\mathbb{R}^n) \to \mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1}))$$

where $\mathrm{Map}(X,Y)$ denotes the simplicial set of maps from X to Y.

Theorem 4.11. *The composition*

$$\mathrm{MAF}(\mathbb{R}\times\mathbb{R}^n) \xrightarrow{w} \mathrm{MAF}(S^1\times\mathbb{R}^n) \xrightarrow{\Psi} \mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1}))$$

is homotopic to the map which takes a k–simplex

$$p : M \to \mathbb{R}\times\mathbb{R}^n\times\Delta^k = \mathbb{R}^{n+1}\times\Delta^k$$

of $\mathrm{MAF}(\mathbb{R}\times\mathbb{R}^n)$ *to the constant map in* $\mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1}))$ *with image the k–simplex p.*

Before proving Theorem 4.11 we need to make five comments about the homotopy equivalence Ψ.

Comment 4.12. The first comment involves an explicit model for the simplicial set $\mathrm{Map}(S^1,\mathrm{MAF}(B))$. A k–simplex consists of a subspace $M\subseteq\ell_2\times S^1\times B\times\Delta^k$ of small capacity such that

1. the projection $\rho : M \to S^1\times\Delta^k$ is a fibre bundle projection with fibres m–manifolds without boundary,
2. the projection $p : M \to S^1\times B\times\Delta^k$ has the property that for each $(s,t)\in S^1\times\Delta^k$,
$$p| : p^{-1}(\{s\}\times B\times\{t\}) \to \{s\}\times B\times\{t\}$$
is a manifold approximate fibration.

In this model constant maps are described as follows. If $p : M \to B\times\Delta^k$ is a k–simplex of $\mathrm{MAF}(B)$, then the constant map in $\mathrm{Map}(S^1,\mathrm{MAF}(B))$ with image the k–simplex p is simply the k–simplex $\mathrm{id}_{S^1}\times p : S^1\times M \to S^1\times B\times\Delta^k$. Thus, Theorem 4.11 is equivalent to Theorem 4.13.

Theorem 4.13. $\Psi\circ w$ *is homotopic to the map which takes a k–simplex $p : M \to \mathbb{R}^{n+1}\times\Delta^k$ of* $\mathrm{MAF}(\mathbb{R}^{n+1})$ *to the k–simplex of* $\mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1}))$ *given by*

$$\mathrm{id}_{S^1}\times p : S^1\times M \to S^1\times\mathbb{R}^{n+1}\times\Delta^k.$$

Comment 4.14. The second comment involves recalling the simple geometric description of the homotopy equivalence Ψ. This is a distillation of the differential of [32].[2] Fix a small number $a>0$ ($a<\pi=\mathrm{diam}(S^1)$) and for $x\in S^1$ let U_x denote the open interval about x of radius a. Let $\eta : S^1\times\mathbb{R} \to S^1\times S^1$ be an open embedding such that for each $x\in S^1$, $\eta(x,\cdot) : \{x\}\times\mathbb{R} \to \{x\}\times S^1$ takes $\mathbb{R}$ homeomorphically

[2]It should be emphasized that the differential in [32] is based on the differential in [30] which in turn is inspired by the differential of immersion theory.

onto U_x with $\eta(x,0) = (x,x)$. (Thus, η is just a trivialization of the topological tangent microbundle of S^1.) Then given a k–simplex

$$p : M \to S^1 \times B \times \Delta^k$$

of $\mathrm{MAF}(S^1 \times B)$, form

$$\mathrm{id}_{S^1} \times p : S^1 \times M \to S^1 \times S^1 \times B \times \Delta^k$$

and let

$$M' = (\mathrm{id} \times p)^{-1}(\eta(S^1 \times \mathbb{R}) \times B \times \Delta^k).$$

Then

$$\Psi(p) : M' \xrightarrow{(\mathrm{id} \times p)|} \eta(S^1 \times \mathbb{R}) \times B \times \Delta^k \xrightarrow{\eta^{-1} \times \mathrm{id}_{B \times \Delta^k}} S^1 \times \mathbb{R} \times B \times \Delta^k$$

is a k–simplex of $\mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R} \times B))$. In other words, there is a pull-back diagram

$$\begin{array}{ccc} M' & \longrightarrow & S^1 \times M \\ {\scriptstyle \Psi(p)}\downarrow & & \downarrow {\scriptstyle \mathrm{id}_{S^1} \times p} \\ S^1 \times \mathbb{R} \times B \times \Delta^k & \xrightarrow{\eta \times \mathrm{id}_{B \times \Delta^k}} & S^1 \times S^1 \times B \times \Delta^k. \end{array}$$

In the case that $B = \mathbb{R}^n$, it follows from [32] that this construction defines a homotopy equivalence

$$\Psi : \mathrm{MAF}(S^1 \times \mathbb{R}^n) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})).$$

Comment 4.15. The third comment is to describe another interpretation of the map Ψ which is reminiscent of the constructions in [35].[3] For this construction represent S^1 as the quotient $\mathbb{R}/\sim$ where the quotient map is the exponential map $\mathrm{e} : \mathbb{R} \to S^1$; $t \mapsto [t]$. There is another version of the exponential map , namely

$$\mathrm{e}' : S^1 \times \mathbb{R} \to S^1 \times S^1;\ ([s], t) \mapsto ([s], [s+t]).$$

We may assume that the trivialization $\eta : S^1 \times \mathbb{R} \to S^1 \times S^1$ mentioned above is defined so that it agrees with e' over a neighborhood of the diagonal in $S^1 \times S^1$. Form the pull-back

$$\begin{array}{ccc} N & \longrightarrow & S^1 \times M \\ {\scriptstyle p'}\downarrow & & \downarrow {\scriptstyle \mathrm{id}_{S^1} \times p} \\ S^1 \times \mathbb{R} \times B \times \Delta^k & \xrightarrow{\mathrm{e}' \times \mathrm{id}} & S^1 \times S^1 \times B \times \Delta^k. \end{array}$$

Define $\Psi' : \mathrm{MAF}(S^1 \times B) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R} \times B))$ by $\Psi'(p) = p'$.

Lemma 4.16. $\Psi \simeq \Psi' : \mathrm{MAF}(S^1 \times B) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R} \times B))$.

Proof. Note that $\Psi(p)$ and $\Psi'(p)$ agree over a neighborhood of $S^1 \times \{0\} \times B \times \Delta^k$ in $S^1 \times \mathbb{R} \times B \times \Delta^k$. The result then follows from the uniqueness of germs principle for manifold approximate fibrations (cf. [32], Proposition 3.2). $\square$

[3] We are invoking the fact that S^1 has nonpositive curvature.

Comment 4.17. The fourth comment concerns the construction of yet another map

$$\Psi'' : \mathrm{MAF}(S^1 \times B) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R} \times B))$$

which we will show is homotopic to Ψ (for arbitrary B). Given a k–simplex $p : M \to S^1 \times B \times \Delta^k$ form the pull-back

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{p}} & \mathbb{R} \times B \times \Delta^k \\ \Big\downarrow & & \Big\downarrow{\scriptstyle e \times \mathrm{id}} \\ M & \xrightarrow{p} & S^1 \times B \times \Delta^k. \end{array}$$

Let $\zeta : \bar{M} \to \bar{M}$ be the $+1$ generating covering translation, let $\gamma : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ be the isotopy defined by $\gamma(x,t) = (x+t, t)$, let $\gamma^B = \gamma \times \mathrm{id}_{B \times \Delta^k} : \mathbb{R} \times B \times \Delta^k \times [0,1] \to \mathbb{R} \times B \times \Delta^k \times [0,1]$. Let $T(\zeta^{-1})$ be the mapping torus of ζ^{-1} (thus $T(\zeta^{-1}) = \bar{M} \times [0,1]/\sim$ where $(x,1) \sim (\zeta^{-1}(x), 0)$, Definition 8.1). Represent S^1 as above and define

$$\Psi''(p) : T(\zeta^{-1}) \to S^1 \times \mathbb{R} \times B \times \Delta^k$$

by $[x,t] \mapsto ([t], (\gamma^B_t)^{-1} \bar{p}(x))$.

Lemma 4.18. $\Psi' \simeq \Psi'' : \mathrm{MAF}(S^1 \times B) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R} \times B))$.

Proof. Using the notation above define a homeomorphism $\gamma : T(\zeta^{-1}) \to N$ as follows. First define $\alpha : T(\zeta^{-1}) \to S^1 \times \mathbb{R} \times B \times \Delta^k$ by representing a point of $T(\zeta^{-1})$ as $[x,t,y,s]$ where $s \in [0,1]$ and $(x,t,y) \in \bar{M} \subseteq M \times \mathbb{R} \times B \times \Delta^k$ with $x \in M$, $t \in \mathbb{R}$, $y \in B \times \Delta^k$ so that $p(x) = ([t], y)$. Then let $\alpha([x,t,y,s]) = ([s], t-s, y)$.

Now define $\beta : T(\zeta^{-1}) \to S^1 \times M$ by $\beta([x,t,y,s]) = ([s], x)$. Since one can check that $(e' \times \mathrm{id})\alpha = (\mathrm{id} \times p)\beta$ and N is the pull–back of $e' \times \mathrm{id}$ and $\mathrm{id} \times p$ it follows that α, β uniquely determine a map $\gamma : T(\zeta^{-1}) \to N$. Moreover, $p' \circ \gamma = \alpha$. Since $\alpha = \Psi''(p)$ and $p' = \Psi(p)$ we are done. □

Comment 4.19. The fifth and final comment is about yet another map

$$\Phi : \mathrm{MAF}(\mathbb{R} \times B) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R} \times B)).$$

The definition will be illustrated only for vertices; the remaining details are similar to those in the constructions above. Given a vertex $p : M \to \mathbb{R} \times B$ in $\mathrm{MAF}(\mathbb{R} \times B)$ we have associated to p an isotopy $G : M \times [0,1] \to M \times [0,1]$ in the definition of w above (using Lemma 4.6). In the notation above $G = G(p)_t$ for some sufficiently large t. If $T(G_1^{-1})$ denotes the mapping torus of $G_1^{-1} : M \to M$, define

$$p' : T(G_1^{-1}) \to S^1 \times \mathbb{R} \times B$$

by $[x,t] \mapsto ([t], pG_t^{-1}(x))$. Then $\Phi(p) = p'$ is the map which we are interested in for the following reason.

Lemma 4.20. $\Phi : \mathrm{MAF}(\mathbb{R} \times B) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R} \times B))$ *is homotopic to the map* $p \mapsto \mathrm{id}_{S^1} \times p$.

Proof. The mapping torus $T(G^{-1})$ of G^{-1} provides a one–simplex from $\mathrm{id}_{S^1} \times p$ to $\Phi(p)$. □

Proof of Theorem 4.13. For a vertex $p : M \to \mathbb{R} \times B$ of $\mathrm{MAF}(\mathbb{R} \times B)$ we will construct a one simplex from $\Psi'' w(p)$ to $\Phi(p)$. This construction can then be generalized to provide a homotopy $\Psi'' \circ w \simeq \Phi$. By letting $B = \mathbb{R}^n$, the proof of Theorem 4.13 then follows from Lemmas 4.16, 4.18 and 4.20. Let $G : M \times [0,1] \to$

$M\times[0,1]$ be the isotopy used in the definition of $\Phi(p)$ above. In particular, pG_t is close to $g_t p$ for $0\le t\le 1$. Let

$$Y = G_1 p^{-1}((-\infty,0]\times B)\backslash p^{-1}((-\infty,0])$$

and

$$\bar{Y} = \bigcup_{n=-\infty}^{n=\infty} G_1^n(Y) \subseteq M.$$

The wrapping–up map w yields a manifold approximate fibration $w(p) = q : \hat{M} \to S^1\times B$ where $\hat{M} = Y/\sim$, $\bar{Y}$ is the infinite cyclic cover of $\hat{M}$, and q induces $\bar{q} : \bar{Y} \to \mathbb{R}\times B$. The $+1$ generating covering translation $\zeta : \bar{Y} \to \bar{Y}$ is such that $\zeta = G_1|\bar{Y}$. The key observation now is that the map $\Psi'' w(p) = \Psi''(q)$ is given by

$$T(\zeta^{-1}) \to S^1\times\mathbb{R}\times B; \quad [x,t] \mapsto ([t], (\gamma_t^B)^{-1}\bar{q}(x)).$$

Over $[-0.5, 1.5]\times B$, p is close to $\bar{q}$ and $\gamma_t = g_t$. Thus, over $[-0.5, 1.5]\times B$ we have $(\gamma_t^B)^{-1}\bar{q}$ close to pG_t^{-1}. It follows that the composition

$$T(\zeta^{-1}) \hookrightarrow T(G_1^{-1}) \xrightarrow{p'} \mathbb{R}\times B\times S^1$$

is close to

$$\Psi''(q) : T(\zeta^{-1}) \to \mathbb{R}\times B\times S^1$$

over a neighborhood of $S^1\times\{0\}\times B$ in $S^1\times\mathbb{R}\times B$. Moreover, these two maps are equal over a neighborhood of $\{0\}\times B\times\{1\}$ in $\mathbb{R}\times B\times\{1\}$ (because the homotopy $p \simeq q$ is rel such a neighborhood). Now once again use the fact that taking germs about 0 induces a homotopy equivalence together with local connectivity (Lemma 4.8) to show that $\Phi(p) = p'$ and $\Psi''(q)$ are in the same component of $\mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}\times B))$. ☐

We now will interpret the unwrapping (or infinite transfer) map $u : \mathrm{MAF}(S^1\times\mathbb{R}^n) \to \mathrm{MAF}(\mathbb{R}^{n+1})$ in terms of the classifying map Ψ. Evaluation at the basepoint of S^1 yields a fibration

$$\Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) \xrightarrow{I} \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})) \xrightarrow{E} \mathrm{MAF}(\mathbb{R}^{n+1}).$$

Theorem 4.21. *The composition*

$$\mathrm{MAF}(S^1\times\mathbb{R}^n) \xrightarrow{\Psi} \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})) \xrightarrow{E} \mathrm{MAF}(\mathbb{R}^{n+1})$$

is homotopic to the unwrapping map u.

Proof. According to Lemma 4.16 it suffices to show that $u \simeq E\Psi'$ where Ψ' is constructed in Comment 4.15. If $p : M \to S^1\times\mathbb{R}^n\times\Delta^k$ is a k–simplex of $\mathrm{MAF}(S^1\times\mathbb{R}^n)$, then $\Psi'(p) = p'$ is formed by pulling–back along $e'\times\mathrm{id}_{\mathbb{R}^n\times\Delta^k}$ to obtain $p' : N \to S^1\times\mathbb{R}^{n+1}\times\Delta^k$. Then $E\Psi'(p) = E(p')$ is formed by restricting p' over $\{1\}\times\mathbb{R}^{n+1}\times\Delta^k$ to obtain

$$E\Psi'(p) = p'| : (p')^{-1}(\{1\}\times\mathbb{R}^{n+1}\times\Delta^k) \to \{1\}\times\mathbb{R}^{n+1}\times\Delta^k.$$

Since $e' : \{1\}\times\mathbb{R} \to \{1\}\times S^1$ is the standard exponential cover, $E\Psi'(p)$ is obtained from p by the pull–back construction and the result now follows from Remark 4.3. ☐

4.4. **Independence of covering isotopy.** As a consequence of Theorem 4.13 note that the homotopy class of the wrapping up $w(p)$ is independent of the approximate covering isotopy which is used to perform the wrapping up construction. This is because the map $p \mapsto \mathrm{id}_{S^1} \times p$ has nothing whatsoever to do with a covering isotopy. The following result is an application of this observation.

Corollary 4.22. *If $p : M \to \mathbb{R}$ is a vertex of $\mathrm{MAF}(\mathbb{R})$ and $\hat{p} : \hat{M} \to S^1$ is its image in $\mathrm{MAF}(S^1)$ under the wrapping-up map, then $M \times S^1$ is homeomorphic to $\hat{M} \times \mathbb{R}$.*

Proof. Let $\rho : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ be the isotopy which rotates $\mathbb{R}^2$ through an angle of $\frac{\pi}{2}$ in the clockwise direction; that is,

$$\rho(x,y,t) = \begin{pmatrix} \cos\frac{\pi t}{2} & \sin\frac{\pi t}{2} \\ -\sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $g_s : \mathbb{R} \to \mathbb{R}$, $0 \le s \le 1$, be the isotopy used in the construction of w, and let $G_s : M \to M$, $0 \le s \le 1$, be an isotopy which approximately covers g_s (that is, $g_s p$ is close to pG_s).

Consider the vertex $p \times \mathrm{id}_{\mathbb{R}} : M \times \mathbb{R} \to \mathbb{R}^2$ of $\mathrm{MAF}(\mathbb{R}^2)$ and note that the component $[w(p \times \mathrm{id}_{\mathbb{R}})]$ in $\mathrm{MAF}(S^1 \times \mathbb{R})$ determined by the image of $p \times \mathrm{id}_{\mathbb{R}}$ under the wrapping up map w is the component $[\hat{p} \times \mathrm{id}_{\mathbb{R}}]$ determined by $\hat{p} \times \mathrm{id}_{\mathbb{R}} : \hat{M} \times \mathbb{R} \to \mathbb{R}^2$. This is because $G_s \times \mathrm{id}_{\mathbb{R}}$ is an isotopy on $M \times \mathbb{R}$ which approximately covers $g_s \times \mathrm{id}_{\mathbb{R}}$ and hence can be used to wrap-up $p \times \mathrm{id}_{\mathbb{R}}$; the result of this wrapping up is $\hat{p} \times \mathrm{id}_{\mathbb{R}}$.

On the other hand, $\rho_1(p \times \mathrm{id}_{\mathbb{R}}) : M \times \mathbb{R} \to \mathbb{R}^2$ is in the same component of $\mathrm{MAF}(\mathbb{R}^2)$ as $p \times \mathrm{id}_{\mathbb{R}}$ (because $\rho_t(p \times \mathrm{id}_{\mathbb{R}})$, $0 \le t \le 1$ provides a path). Note that the diagram

$$\begin{array}{ccc} M \times \mathbb{R} & \xrightarrow{\mathrm{id}_M \times g_s} & M \times \mathbb{R} \\ {\scriptstyle \rho_1(p \times \mathrm{id}_{\mathbb{R}})} \downarrow & & \downarrow {\scriptstyle \rho_1(p \times \mathrm{id}_{\mathbb{R}})} \\ \mathbb{R}^2 & \xrightarrow{g_s \times \mathrm{id}_{\mathbb{R}}} & \mathbb{R}^2 \end{array}$$

commutes. This means that $\mathrm{id}_M \times g_s$ is an isotopy *exactly* covering $g_s \times \mathrm{id}_{\mathbb{R}}$ with respect to $\rho_1(p \times \mathrm{id}_{\mathbb{R}})$ and so can be used to wrap-up $\rho_1(p \times \mathrm{id}_{\mathbb{R}})$. The result is

$$\rho_1'(p \times \mathrm{id}_{S^1}) : M \times S^1 \to S^1 \times \mathbb{R}$$

where $\rho_1' : \mathbb{R} \times S^1 \to S^1 \times \mathbb{R}$ is the map $(x,y) \mapsto (y,-x)$.

Thus,

$$\rho_1'(p \times \mathrm{id}_{S^1}) : M \times S^1 \to S^1 \times \mathbb{R} \text{ and } \hat{p} \times \mathrm{id}_{\mathbb{R}} : \hat{M} \times \mathbb{R} \to S^1 \times \mathbb{R}$$

are in the same component of $\mathrm{MAF}(S^1 \times \mathbb{R})$. In particular, $\hat{M} \times \mathbb{R}$ and $M \times S^1$ are homeomorphic. □

Note that the wrapping up $w : \mathrm{MAF}(\mathbb{R}^{n+1}) \to \mathrm{MAF}(S^1 \times \mathbb{R}^n)$ as constructed above appears to depend on the fact that the first $\mathbb{R}$ factor was the one which was wrapped-up to get the S^1 factor. However, the map in Theorems 4.11 and 4.13 which is homotopic to w does not depend on which factor is used. We next make this observation precise.

Let σ be a permutation of $\{1, 2, \dots, n+1\}$. Then σ induces a homeomorphism

$$\sigma : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}; \quad (x_1, \dots, x_{n+1}) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n+1)})$$

and therefore a simplicial isomorphism

$$\sigma : \mathrm{MAF}(\mathbb{R}^{n+1}) \to \mathrm{MAF}(\mathbb{R}^{n+1}); \quad (p : M \to \mathbb{R}^{n+1} \times \Delta^k) \mapsto (\sigma \times \mathrm{id}_{\Delta^k}) \circ p.$$

This in turn induces a simplicial isomorphism

$$\sigma : \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1}))$$

and thus we can define (up to homotopy) a simplicial isomorphism on $\mathrm{MAF}(S^1 \times \mathbb{R}^n)$ so that the following diagram homotopy commutes :

$$\begin{array}{ccc}
\mathrm{MAF}(S^1 \times \mathbb{R}^n) & \xrightarrow{\sigma} & \mathrm{MAF}(S^1 \times \mathbb{R}^n) \\
\Psi \downarrow & & \downarrow \Psi \\
\mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})) & \xrightarrow{\sigma} & \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})).
\end{array}$$

Corollary 4.23. *The following diagram homotopy commutes:*

$$\begin{array}{ccc}
\mathrm{MAF}(\mathbb{R}^{n+1}) & \xrightarrow{\sigma} & \mathrm{MAF}(\mathbb{R}^{n+1}) \\
w \downarrow & & \downarrow w \\
\mathrm{MAF}(S^1 \times \mathbb{R}^n) & \xrightarrow{\sigma} & \mathrm{MAF}(S^1 \times \mathbb{R}^n).
\end{array}$$

Proof. Theorem 4.13 reduces this to observing that the following diagram commutes

$$\begin{array}{ccc}
\mathrm{MAF}(\mathbb{R}^{n+1}) & \xrightarrow{\sigma} & \mathrm{MAF}(\mathbb{R}^{n+1}) \\
\downarrow & & \downarrow \\
\mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})) & \xrightarrow{\sigma} & \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1}))
\end{array}$$

where the vertical maps are both given by $p \mapsto \mathrm{id}_{S^1} \times p$. $\square$

5. Relaxation as a simplicial map

In this section we define a simplicial map

$$r : \mathrm{ManBan}(S^1 \times B) \to \mathrm{MAF}(S^1 \times B),$$

called the *relaxation map*, such that $r\varphi \simeq \mathrm{id}_{\mathrm{MAF}(S^1\times B)}$. In particular, the forget control map $\varphi : \mathrm{MAF}(S^1 \times B) \to \mathrm{ManBan}(S^1 \times B)$ is homotopy split injective.

The definition of r is given inductively over the skeleta of $\mathrm{ManBan}(S^1 \times B)$. To begin the construction let $p : M \to S^1 \times B$ be a vertex of $\mathrm{ManBan}(S^1 \times B)$. Form the pull-back

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{p}} & \mathbb{R}\times B \\ \big\downarrow & & \big\downarrow \mathrm{e} \\ M & \xrightarrow{p} & S^1 \times B. \end{array}$$

Let $\zeta : \bar{M} \to \bar{M}$ be the $+1$ generating covering translation. The unwrapping map of §4.1 yields

$$p' = u(\bar{p}) : \bar{M} \to \mathbb{R}\times B$$

which is a vertex of $\mathrm{MAF}(\mathbb{R}\times B)$. We now follow closely the definition of the wrapping up map in §4.2 closely with p' taking the place of p and ζ playing a role. Let

$$G(p') : \bar{M}\times[0,1]\times[0,\infty) \to \bar{M}\times[0,1]$$

be given by Controlled Isotopy Covering Lemma 4.4. Choose $t_0 > 0$ so large that if $t \geq t_0$ then

$$(p')^{-1}((-\infty,0)\times B) \subseteq \zeta G(p')_{(1,t)}((p')^{-1}((-\infty,0]\times B))$$

where $G(p')_{(1,t)} : \bar{M} \to \bar{M}$ is

$$G(p')| : M\times\{1\}\times\{t\} \to M\times\{1\}.$$

For such t let

$$Y_t = \zeta G(p')_{(1,t)}((p')^{-1}((-\infty,0]\times B)) \setminus (p')^{-1}((-\infty,0)\times B)$$

and $\hat{M}_t = Y_t/\sim$ where $\sim$ is generated by

$$x \sim G(p')_{(1,t)}(x), \ \text{ if } \ p'(x)\in\{0\}\times B.$$

Equivalently, $\hat{M}_t$ may be described as a quotient space of $\bar{M}$:

$$\hat{M}_t = \bar{M}/\zeta G(p')_{(1,t)}.$$

As in the construction of w, it is possible to construct a continuous family of maps $\hat{p}_t : \hat{M}_t \to S^1\times B$ such that $\hat{p}_t$ is a δ_t–fibration with $\delta_t \to 0$ as $t \to \infty$. Then Lemmas 4.6 and 4.8 imply that there exists a canonical manifold approximate fibration $r(p) = \hat{p} : \hat{M} \to S^1\times B$ associated to the family $\{\hat{p}_t\}$. This construction can be continued inductively to construct $r : \mathrm{ManBan}(S^1\times B) \to \mathrm{MAF}(S^1\times B)$.

Given a k–simplex $p : M \to S^1\times B\times\Delta^k$ of $\mathrm{ManBan}(S^1\times B)$, we will need to describe the infinite cyclic cover $\bar{\hat{p}} : \bar{\hat{M}} \to \mathbb{R}\times B\times\Delta^k$ of the relaxation $r(p)$. To this end note from the construction above that there is a fiber preserving isotopy $G : \bar{M}\times[0,1] \to \bar{M}\times[0,1]$ such that $G = G(p')_t$ for sufficiently large t so that $r(p) = \hat{p} : \hat{M} = \hat{M}_t \to S^1\times B\times\Delta^k$. If $\zeta : \bar{M} \to \bar{M}$ is the $+1$ generating covering translation of the pull-back $\bar{p} : \bar{M} \to \mathbb{R}\times B\times\Delta^k$ of p, then it follows easily that

$\hat{\zeta} = \zeta \circ G_1 : \bar{M} \to \bar{M}$ is a covering translation of an infinite cyclic cover of $\bar{M}/\zeta G_1 = \bar{M}/\hat{\zeta} = \hat{M}$. We summarize this as follows.

Proposition 5.1. *The relaxation $\hat{p} : \hat{M} \to S^1 \times B \times \Delta^k$ of $p : M \to S^1 \times B \times \Delta^k$ is such that M and $\hat{M}$ have the same infinite cyclic cover $\bar{M}$ so that $M = \bar{M}/\zeta$ and $\hat{M} = \bar{M}/\hat{\zeta}$ with $\zeta, \hat{\zeta} : \bar{M} \to \bar{M}$ the $+1$ generating covering translations. Moreover, there exists a fibre preserving isotopy $G : \bar{M} \times [0,1] \to \bar{M} \times [0,1]$ supported on $\bar{p}^{-1}([-2,4] \times B \times \Delta^k)$ so that $\hat{\zeta} = \zeta \circ G_1$.*

We now specialize to the case $B = \mathbb{R}^n$.

Theorem 5.2. *The composition*

$$\mathrm{MAF}(S^1 \times \mathbb{R}^n) \xrightarrow{\varphi} \mathrm{ManBan}(S^1 \times \mathbb{R}^n) \xrightarrow{r} \mathrm{MAF}(S^1 \times \mathbb{R}^n)$$

is homotopic to the identity.

Proof. We will show that $\Psi' r \varphi \simeq \Psi'$ where

$$\Psi' : \mathrm{MAF}(S^1 \times \mathbb{R}^n) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1}))$$

is the homotopy of Comment 4.15. It suffices to illustrate the homotopy on vertices, so let $p : M \to S^1 \times \mathbb{R}^n$ be a vertex of $\mathrm{MAF}(S^1 \times \mathbb{R}^n)$. Since the pull-back $\bar{p} : \bar{M} \to \mathbb{R}^{n+1}$ is a manifold approximate fibration, we may take $p' = \bar{p}$ in the construction above (no sucking is needed). Let $G : \bar{M} \times [0,1] \to \bar{M} \times [0,1]$ be an isotopy such that $G = G(\bar{p})_t$ for some sufficiently large t so that $r(p) = \hat{p} : \hat{M} = \hat{M}_t \to S^1 \times \mathbb{R}^n$. As above the infinite cyclic cover of $\hat{M}$ is $\bar{M}$ with $+1$ covering translation $\hat{\zeta} = \zeta G_1 : \bar{M} \to \bar{M}$. The construction also shows that $\bar{p}, \bar{\hat{p}} : \bar{M} \to \mathbb{R} \times B \times \Delta^k$ are boundedly homotopic. It then follows that $\Psi'(p)$ are $\Psi'(r(p))$ homotopic. $\square$

The relaxation map agrees with Siebenmann's relaxation construction [44] as is shown in [31] (Chapter 18) and with the splitting of the forget control map of Hughes-Taylor-Williams [35], [36].

Finally, we note that the relaxation map extends to a simplicial map

$$r : \mathrm{ManBan}_c(S^1 \times B) \to \mathrm{MAF}(S^1 \times B).$$

This extension is accomplished just as for the unwrapping map in §4.1.

6. The Whitehead spaces

In this section we recall the definition of the controlled Whitehead space as given in Hughes [27] and define analogues of the wrapping, unwrapping, relaxation and forget control maps.

6.1. **The Controlled Whitehead space.** Let B be a finite dimensional manifold without boundary with a fixed metric and let X be a Hilbert cube manifold with a fixed closed embedding $X \subseteq \ell_2$ of small capacity. Let $p : X \to B$ be a manifold approximate fibration. Consider X as a subset of $\ell_2 \times B$ of small capacity via the embedding $x \mapsto (x, p(x))$. A k–simplex of the simplicial set $\mathcal{W}h(p : X \to B)$ consists of a subspace $M \subseteq \ell_2 \times B \times \Delta^k$ of small capacity such that

1. the projection $\rho : M \to \Delta^k$ is a fibre bundle projection with Q–manifold fibres,
2. $X \times \Delta^k$ is a sliced Z–set in M (in particular, $\rho | X \times \Delta^k$ is projection),
3. there is a fibre preserving strong deformation retraction $f : M \to X \times \Delta^k$,
4. the map f is actually a proper $(p \times \mathrm{id}_{\Delta^k})^{-1}(\epsilon)$-sdr for every $\epsilon > 0$.

Condition (4) above is equivalent (in the presence of the other conditions) to

4′. the composition $(p \times id_{\Delta^k}) \circ f : M \to B \times \Delta^k$ is a k–simplex of $MAF^Q(B)$ (see [27]).

As a consequence note that this defines a simplicial map

$$\mathcal{W}h(p : X \to B) \to \mathrm{MAF}^Q(B).$$

As in Chapter 3 we will usually ignore the small capacity embedding in ℓ_2 and just denote a k–simplex of $\mathcal{W}h(p : X \to B)$ by a map $f : M \to X \times \Delta^k$ satisfying the properties above.

When $B = \{\text{point}\}$ we denote $\mathcal{W}h(p : X \to B)$ by $\mathcal{W}h(X)$. This is the classical Whitehead space of X with $\pi_0 \mathcal{W}h(X) = \mathrm{Wh}(\mathbb{Z}\pi_1(X))$ and $\Omega \mathcal{W}h(X)$ homotopy equivalent to the space of pseudoisotopies on X (see [29]).

It is easy to see that $\mathcal{W}h(p : X \to B)$ is a Kan complex (see [32]). In order to describe a basepoint of $\mathcal{W}h(p : X \to B)$, fix an embedding $X \times [0,1] \subseteq \ell_2$ of small capacity such that $X \times \{0\} = X \subseteq \ell_2$. The base vertex e_0 is the projection $X \times [0,1] \to X$. (The identity $X \times \Delta^k \to X \times \Delta^k$ can not be used because of the Z–set condition.) The degenerate k–simplex e_k on e_0 is the projection $X \times [0,1] \times \Delta^k \to X \times \Delta^k$. The subcomplex $e = \{e_k\}$ of $\mathcal{W}h(p : X \to B)$ is the basepoint at which homotopy groups are based unless otherwise mentioned.

By analogy with Propositions 3.1 and 3.3 we have the following characterization of the components of the controlled Whitehead spaces. The version for higher simplices is also explicitly stated here. Let B be a fixed manifold with $\dim B < \infty$ and assume that B is either a closed manifold or a product $Y \times \mathbb{R}^n$ of a closed manifold Y and $\mathbb{R}^n$ (with the standard metric) and that B has the product metric.

Proposition 6.1. *Suppose X is a Hilbert cube manifold, and $p : X \to B$ is a manifold approximate fibration.*

(1) *There exists $\epsilon_0 > 0$ so that if $f_i : M_i \to X$, $i = 1, 2$, are two vertices of $\mathcal{W}h(p : X \to B)$, then the following are equivalent:*
   (i) *f_1 and f_2 are in the same component of $\mathcal{W}h(p : X \to B)$.*
   (ii) *There exists a homeomorphism $h : M_1 \to M_2$ such that $h|X$ is the inclusion and pf_2h is ϵ_0–homotopic to pf_1 rel X.*

(iii) *For every* $\epsilon > 0$ *there exists a homeomorphism* $h_\epsilon : M_1 \to M_2$ *such that* $h_\epsilon | X$ *is the inclusion and* $pf_2 h_\epsilon$ *is* ϵ*–homotopic to* pf_1 *rel* X.
(iv) *For every* $\epsilon > 0$ *there exists a homeomorphism* $h_\epsilon : M_1 \to M_2$ *such that* $pf_2 h_\epsilon$ *is* ϵ*–homotopic to* pf_1.

(2) *For each* $k \geq 1$ *there exists* $\epsilon_k > 0$ *so that if* $f_i : M_i \to X \times \Delta^k$, $i = 1, 2$, *are two* k*–simplices of* $\mathcal{W}h(p : X \to B)$ *which represent classes* $[f_1], [f_2] \in \pi_k(\mathcal{W}h(p : X \to B), e)$, *then the following are equivalent:*
(i) $[f_1] = [f_2] \in \pi_k(\mathcal{W}h(p : X \to B), e)$.
(ii) *There exists a homeomorphism* $h : M_1 \to M_2$ *such that*

$$h|(X \times \Delta^k) \cup (X \times [0,1] \times \partial \Delta^k) = \textit{inclusion}$$

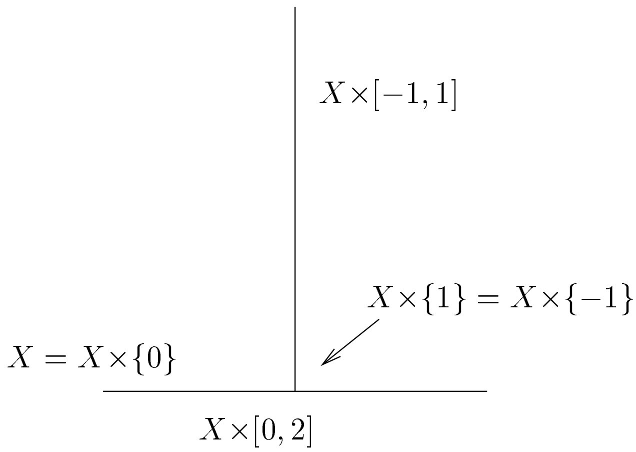
and $pf_2 h$ *is* ϵ_0*–homotopic to* pf_1 *rel* $(X \times \Delta^k) \cup (X \times [0,1] \times \partial \Delta^k)$.
(iii) *For every* $\epsilon > 0$ *there exists a homeomorphism* $h_\epsilon : M_1 \to M_2$ *such that*

$$h_\epsilon|(X \times \Delta^k) \cup (X \times [0,1] \times \partial \Delta^k) = \textit{inclusion}$$

and $pf_2 h_\epsilon$ *is* ϵ*–homotopic to* pf_1 *rel* $(X \times \Delta^k) \cup (X \times [0,1] \times \partial \Delta^k)$.
(iv) *For every* $\epsilon > 0$ *there exists a homeomorphism* $h_\epsilon : M_1 \to M_2$ *such that* $h_\epsilon|(X \times [0,1] \times \partial \Delta^k)$ *is the inclusion and* $pf_2 h_\epsilon$ *is* ϵ*–homotopic to* pf_1 *rel* $X \times [0,1] \times \partial \Delta^k$.

Proof. See [29], Theorems 3.2, 3.3. □

6.2. **Abelian monoid-like structures.** We will now define a binary operation which gives $\mathcal{W}h(p : X \to B)$ what might be called the structure of an *abelian monoid-like simplicial set*. However, the operation is not basepoint preserving. We begin by describing a new basepoint of $\mathcal{W}h(p : X \to B)$ which will be the product of the standard basepoint with itself. Let $Y = (X \times [0,2]) \amalg (X \times [-1,-2]) / \sim$ where $\sim$ is the equivalence relation generated by $(x,1) \sim (x,-1)$ for each $x \in X$. Then Y is a Hilbert cube manifold homeomorphic to X [6]. The key features of Y are that it contains a natural copy of X as a Z-set and that it corresponds to what should be the product of the basepoint with itself (see figure below).



Fix a closed embedding $Y \subseteq \ell_2$ of small capacity which agrees with the embedding $X \times [0,1] \subseteq \ell_2$. Let $e_0 \vee e_0$ denote the vertex $Y \to X$; $[(x,t)] \mapsto x$, of $\mathcal{W}h(p : X \to B)$. The degenerate k–simplex on $e_0 \vee e_0$ is denoted $e_k \vee e_k$ (essentially the product

$e_0 \vee e_0 \times \Delta^k$), and $e \vee e = \{e_k \vee e_k\}$ is a basepoint of $\mathcal{W}h(p : X \to B)$. Consider the simplicial maps:

$$i_1 : \mathcal{W}h(p : X \to B) \to \mathcal{W}h(p : X \to B) \times \mathcal{W}h(p : X \to B);\ x \mapsto (x, e),$$
$$i_2 : \mathcal{W}h(p : X \to B) \to \mathcal{W}h(p : X \to B) \times \mathcal{W}h(p : X \to B);\ x \mapsto (e, x),$$
$$\Delta : \mathcal{W}h(p : X \to B) \to \mathcal{W}h(p : X \to B) \times \mathcal{W}h(p : X \to B);\ x \mapsto (x, x).$$

Let B be a fixed manifold with $\dim B < \infty$ and assume that B is either a closed manifold or a product $Y \times \mathbb{R}^n$ of a closed manifold Y and $\mathbb{R}^n$ (with the standard metric) and that B has the product metric.

Proposition 6.2. *Suppose X is a Hilbert cube manifold, and $p : X \to B$ is a manifold approximate fibration.*

(1) *There exists a simplicial map*

$$\mu : \mathcal{W}h(p : X \to B) \times \mathcal{W}h(p : X \to B) \to \mathcal{W}h(p : X \to B); \quad \mu(x, y) = x + y$$

satisfying the following properties:

(i) $\mu \circ i_1 \simeq \mathrm{id} \simeq \mu \circ i_2$,

(ii) *the two maps $\mathcal{W}h(p : X \to B) \times \mathcal{W}h(p : X \to B) \times \mathcal{W}h(p : X \to B) \to \mathcal{W}h(p : X \to B)$; $(x, y, z) \mapsto \mu(\mu(x, y), z)$ and $(x, y, z) \mapsto \mu(x, \mu(y, z))$ are homotopic,*

(iii) *the two maps $\mathcal{W}h(p : X \to B) \times \mathcal{W}h(p : X \to B) \to \mathcal{W}h(p : X \to B)$; $(x, y) \mapsto \mu(x, y)$ and $(x, y) \mapsto \mu(y, x)$ are homotopic,*

(iv) $\mu(e, e) = e \vee e$.

(2) *For each $k \geq 1$ there exists an isomorphism $\nu_k : \pi_k(\mathcal{W}h(p : X \to B), e \vee e) \to \pi_k(\mathcal{W}h(p : X \to B), e)$ such that the operation induced by the composition*

$$\pi_k(\mathcal{W}h(p : X \to B), e) \times \pi_k(\mathcal{W}h(p : X \to B), e) \xrightarrow{\mu_*} \pi_k(\mathcal{W}h(p : X \to B), e \vee e)$$
$$\xrightarrow{\nu_k} \pi_k(\mathcal{W}h(p : X \to B), e)$$

is commutative and agrees with the standard homotopy group operation.

(3) *μ_* induces an abelian group structure on $\pi_0 \mathcal{W}h(p : X \to B)$.*

Proof. (1) The idea behind the definition of the operation $\mu : \mathcal{W}h(p : X \to B) \times \mathcal{W}h(p : X \to B) \to \mathcal{W}h(p : X \to B)$ is simple : if $f_i : M_i \to X \times \Delta^k$ are k–simplices of $\mathcal{W}h(p : X \to B)$ for $i = 1, 2$, define $M = M_1 \cup_{X \times \Delta^k} M_2$ and $f : M \to X \times \Delta^k$ by $f | M_i = f_i$. Then $\mu(f_1, f_2)$ should be f. However, there are two technical problems with this. First, some attention must be paid to the required embedding into $\ell_2 \times B \times \Delta^k$. Second, $X \times \Delta^k$ is not a sliced Z–set in M. To deal with the embedding problem, let $\Lambda_i : \ell_2 \to \ell_2$, $i = 1, 2$, be two embeddings of small capacity such that $\Lambda_i| : X \times [0, 1] \to \ell_2$ is the inclusion and $\Lambda_1(\ell_2) \cap \Lambda_2(\ell_2) = X \times [0, 1]$. Let $\Lambda'_i = \Lambda_i \times 1_{B \times \Delta^k} : \ell_2 \times B \times \Delta^k \to \ell_2 \times B \times \Delta^k$. Given k–simplices $f_i : M_i \to X \times \Delta^k$ are k–simplices of $\mathcal{W}h(p : X \to B)$ for $i = 1, 2$, the embeddings $M_i \to \ell_2 \times B \times \Delta^k$, $x \mapsto \Lambda'_i(x)$, determines an embedding $M = M_1 \cup_{X \times \Delta^k} M_2 \to \ell_2 \times B \times \Delta^k$ of small capacity. Other aspects of the ℓ_2 embeddings are handled in a standard way ([32], §15 is useful here) and no further mention of this problem will be made. The second technical problem is dealt with as follows. Define

$$M' = M \coprod (X \times [0, 1] \times \Delta^k) / (X \times \Delta^k = X \times \{1\} \times \Delta^k).$$

Then $X \times \{0\} \times \Delta^k$ is a sliced Z–set in M' and the composition

$$f' : M' \xrightarrow{f \cup 1} X \times [0, 1] \times \Delta^k \xrightarrow{\mathrm{proj}} X \times \Delta^k$$

is a k–simplex of $\mathcal{W}h(p : X \to B)$ defining $\mu(f_1, f_2)$. The proofs of properties (i)–(iv) are straightforward.
(2) The natural retraction $Y \to X\times[0,1]$ is cell–like, so it can be approximated by a homeomorphism $h : Y \to X\times[0,1]$. The mapping cylinder $M(h)$ is a 1–simplex in $\mathcal{W}h(p : X \to B)$ from e_0 to $e_0 \vee e_0$. In the usual way, the 1–simplex $M(h)$ (and degenerate simplices on it) define the isomorphism ν_k. More explicitly, if $f : M \to X\times\Delta^k$ represents a class $[f]\in\pi_k(\mathcal{W}h(p : X \to B), e\vee e)$, then $f^{-1}(X\times\partial\Delta^k) = Y \times \partial\Delta^k$. Now f induces a map $\tilde{f} : \tilde{M} = M\cup_{Y\times\partial\Delta^k}(M(h)\times\partial\Delta^k) \to X\times\Delta^k$ representing a class $[\tilde{f}]\in\pi_k(\mathcal{W}h(p : X \to B), e)$. Set $\nu_k([f]) = [\tilde{f}]$. The proof that $\nu_k\mu_*$ is the standard homotopy group operation follows the usual proof for H–spaces in homotopy theory, aided by the fact that there is a fibre preserving homeomorphism $M(h) \to X\times[0,1]\times\Delta^1$ which is the identity on $(X\times[0,1]\times\partial_0\Delta^1)\cup(X\times\{0\}\times\Delta^1)$ and is h from Y to $X\times[0,1]\times\partial_1\Delta^1$. That the operation is commutative follows from the characterization of 6.2.
(3) Existence of inverses (in π_0) follows from the usual geometric construction ([13], p. 21) with control ([8], p.320) (cf. [27], §4). This can even be done inductively on the dimension of the simplices of $\mathcal{W}h(p : X \to B)$ to define a simplicial map. As with the higher homotopy groups, commutativity follows from the characterization of 6.2. □

6.3. **The various simplicial maps.** In order to define forget control, unwrapping, wrapping up and relaxation maps, we consider only controlled Whitehead spaces in special situations. Thus, for the remainder of this chapter X denotes a compact Hilbert cube manifold, and Y and B denote finite dimensional manifolds with Y closed and B without boundary and with a fixed metric. We then have projections

$$p = p_{Y\times B} : X\times Y\times B \to Y\times B \text{ and } p = p_B : X\times Y\times B \to B,$$

and consider the controlled Whitehead spaces

$$\mathcal{W}h(X\times Y\times B \to Y \times B) \text{ and } \mathcal{W}h(X\times Y\times B \to B)$$

with unlabeled arrows out of cartesian products of spaces always denoting projection maps. As in the previous sections we will be interested only in the special cases $Y = S^1$ or $Y = \{\text{point}\}$ and $B = \mathbb{R}^n$ or $B = \mathbb{R}^{n+1}$ with the standard metric. We will define the following simplicial maps:

(i) the *forget control* map

$$\varphi : \mathcal{W}h(X\times S^1\times B \to S^1 \times B) \to \mathcal{W}h(X\times S^1\times B \to B),$$

(ii) the *unwrapping* map

$$u : \mathcal{W}h(X\times S^1\times B \to B) \to \mathcal{W}h(X\times\mathbb{R}\times B \to \mathbb{R} \times B),$$

(iii) the *wrapping up* map

$$w : \mathcal{W}h(X\times\mathbb{R}\times B \to \mathbb{R} \times B) \to \mathcal{W}h(X\times S^1\times B \to S^1 \times B),$$

(iv) the *relaxation* map

$$r : \mathcal{W}h(X\times S^1\times B \to B) \to \mathcal{W}h(X\times S^1\times B \to S^1 \times B).$$

6.4. **The forget control map.** A k–simplex

$$f : M \to X\times S^1\times B\times\Delta^k$$

of $\mathcal{W}h(X\times S^1\times B \to S^1 \times B)$ has the property that the composition

$$M \xrightarrow{f} X\times S^1\times B\times\Delta^k \to S^1\times B\times\Delta^k$$

is a k–simplex of $\mathrm{MAF}^Q(S^1\times B)$. Hence, the composition

$$M \xrightarrow{f} X\times S^1\times B\times\Delta^k \to B\times\Delta^k$$

is a k–simplex of $\mathrm{MAF}^Q(B)$. In this way the forget control map is defined

$$\varphi : \mathcal{W}h(X\times S^1\times B \to S^1\times B) \to \mathcal{W}h(X\times S^1\times B \to B).$$

This is compatible with the forget control map of Chapter 3 so that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{W}h(X\times S^1\times B \to S^1\times B) \arrow[r, "\varphi"] \arrow[d] & \mathcal{W}h(X\times S^1\times B \to B) \arrow[d] \\
\mathrm{MAF}^Q(S^1\times B) \arrow[r, "\varphi"] & \mathrm{MAF}^Q(B)
\end{tikzcd}$$

The proof of the following proposition is obvious.

Proposition 6.3. *The forget control map* $\varphi : \mathcal{W}h(X\times S^1\times B \to S^1\times B) \to \mathcal{W}h(X\times S^1\times B \to B)$ *induces homomorphisms between homotopy groups* (*based at* e) *including* π_0.

6.5. **The unwrapping map.** The construction is similar to the construction of the unwrapping map in §4.1 so we will not give all of the details. Given a k–simplex $f : M \to X\times S^1\times B\times\Delta^k$ of $\mathcal{W}h(X\times S^1\times B \to B)$, the composition

$$p : M \xrightarrow{f} X\times S^1\times B\times\Delta^k \to S^1\times B\times\Delta^k$$

is a k–simplex of $\mathrm{ManBan}_c^Q(S^1\times B)$ (see §3.3). Pulling back along $\mathbb{R} \to S^1$ gives maps

$$\overline{f} : \overline{M} \to X\times\mathbb{R}\times B\times\Delta^k$$

and

$$\overline{p} : \overline{M} \xrightarrow{\overline{f}} X\times\mathbb{R}\times B\times\Delta^k \to \mathbb{R}\times B\times\Delta^k.$$

As in §4.1, $\overline{p}\simeq u(p)$ with $u(p)$ a k–simplex of $\mathrm{MAF}^Q(\mathbb{R}\times B)$. Using the Addenda 4.2, the homotopy can be taken rel $X\times\mathbb{R}\times B\times\Delta^k$. Lifting this homotopy gives a homotopy $\overline{f}\simeq u(f)$ rel$X\times\mathbb{R}\times B\times\Delta^k$ so that $u(p)$ is the composition

$$u(p) : \overline{M} \xrightarrow{u(f)} X\times\mathbb{R}\times B\times\Delta^k \to \mathbb{R}\times B\times\Delta^k.$$

Thus, $u(f)$ is a k–simplex of $\mathcal{W}h(X\times\mathbb{R}\times B \to \mathbb{R}\times B)$ and this defines the map u so that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{W}h(X\times S^1\times B \to B) \arrow[r, "u"] \arrow[d] & \mathcal{W}h(X\times\mathbb{R}\times B \to \mathbb{R}\times B) \arrow[d] \\
\mathrm{ManBan}_c^Q(S^1\times B) \arrow[r, "u"] & \mathrm{MAF}^Q(\mathbb{R}\times B).
\end{tikzcd}$$

Proposition 6.4. *The unwrapping map*

$$u : \mathcal{W}h(X \times S^1 \times B \to S^1 \times B) \to \mathcal{W}h(X \times \mathbb{R} \times B \to \mathbb{R} \times B)$$

induces homomorphisms between homotopy groups (based at e) including π_0.

Proof. Taking pull-backs is compatible with the sum operation. After taking pull-backs, the remaining modifications are rel $X \times \mathbb{R} \times B \times \Delta^k$. □

Remark 6.5 (The Infinite Transfer). As in Remark 4.3 the composition

$$\mathcal{W}h(X \times S^1 \times B \to S^1 \times B) \xrightarrow{\varphi} \mathcal{W}h(X \times S^1 \times B \to B) \xrightarrow{u} \mathcal{W}h(X \times \mathbb{R} \times B \to \mathbb{R} \times B)$$

is homotopic to the simplicial map given by the pull-back construction. This composition is also called the *unwrapping*, or *infinite transfer* map and denoted

$$u : \mathcal{W}h(X \times S^1 \times B \to S^1 \times B) \to \mathcal{W}h(X \times \mathbb{R} \times B \to \mathbb{R} \times B).$$

6.6. **The wrapping up map.** As with the unwrapping map we will just note that minor modifications can be made to the wrapping up map $w : \mathrm{MAF}(\mathbb{R} \times B) \to \mathrm{MAF}(S^1 \times B)$ of Chapter 4. Thus, given a vertex $f : M \to X \times \mathbb{R} \times B$ of $\mathcal{W}h(X \times \mathbb{R} \times B \to \mathbb{R} \times B)$, the composition

$$p : M \xrightarrow{f} X \times \mathbb{R} \times B \to \mathbb{R} \times B$$

is a vertex of $\mathrm{MAF}^Q(\mathbb{R} \times B)$. Adopting the notation of Chapter 4 and using the full strength of the addenda 4.2, 4.5 and 4.7, it is possible to construct a lift $\hat{f} : \hat{M} \to X \times S^1 \times B$ of the map $\hat{p} : \hat{M} \to S^1 \times B$ such that $\hat{f}$ is a homotopy equivalence and $\hat{f}|X \times S^1 \times B$ is the identity so that $w(f) = \hat{f}$ is a vertex of $\mathcal{W}h(X \times S^1 \times B \to S^1 \times B)$. The construction is continued inductively to define the map w so that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{W}h(X \times \mathbb{R} \times B \to \mathbb{R} \times B) \arrow[r, "w"] \arrow[d] & \mathcal{W}h(X \times S^1 \times B \to S^1 \times B) \arrow[d] \\
\mathrm{MAF}^Q(\mathbb{R} \times B) \arrow[r, "w"] & \mathrm{MAF}^Q(S^1 \times B)
\end{tikzcd}$$

Proposition 6.6. *The wrapping up map*

$$w : \mathcal{W}h(X \times \mathbb{R} \times B \to \mathbb{R} \times B) \to \mathcal{W}h(X \times S^1 \times B \to S^1 \times B)$$

induces homomorphisms between homotopy groups (based at e) including π_0.

Proof. Since the addenda 4.5, 4.7 and 4.9 are used, all the constructions are compatible along $X \times \mathbb{R} \times B \times \Delta^k$. □

By analogy with Theorem 4.10 we have

Theorem 6.7. *The composition*

$$\begin{aligned}\mathcal{W}h(X \times \mathbb{R} \times B \to \mathbb{R} \times B) &\xrightarrow{w} \mathcal{W}h(X \times S^1 \times B \to S^1 \times B) \\ &\xrightarrow{\varphi} \mathcal{W}h(X \times S^1 \times B \to B) \xrightarrow{u} \mathcal{W}h(X \times \mathbb{R} \times B \to \mathbb{R} \times B)\end{aligned}$$

is homotopic to the identity.

Proof. This follows from germ theory as in Theorem 4.10. See [30] for the analogue of GMAF in this case. □

6.7. **Classifying space interpretation.** There are also analogues of Theorems 4.10 and 4.11 in the special case $B=\mathbb{R}^n$ for $n\geq 0$, in which w and u have descriptions in terms of the classification results of [30] and [27]. According to [30] and [32] there is a homotopy equivalence

$$\Psi : \mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n) \to \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}))$$

making the following diagram commute:

$$\begin{tikzcd}
\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n) \arrow[r, "\Psi"] \arrow[d] & \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})) \arrow[d] \\
\mathrm{MAF}^Q(S^1\times\mathbb{R}^n) \arrow[r, "\Psi"] & \mathrm{Map}(S^1, \mathrm{MAF}^Q(\mathbb{R}^{n+1})).
\end{tikzcd}$$

Theorem 6.8. *The composition*

$$\begin{aligned}\mathcal{W}h(X\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}\times\mathbb{R}^n) &\xrightarrow{w} \mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n)\\ &\xrightarrow{\Psi} \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}))\end{aligned}$$

is homotopic to the map which takes a k–simplex

$$f : M \to X\times\mathbb{R}\times\mathbb{R}^n\times\Delta^k = X\times\mathbb{R}^{n+1}\times\Delta^k$$

of $\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})$ *to the constant map in* $\mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}))$ *with image the k–simplex f.*

Proof. One just needs to make the relative additions to the proof of 4.11. □

Evaluation at the basepoint of S^1 yields a fibration compatible with the one in §4.3 so that we get a commuting diagram of fibrations:

$$\begin{tikzcd}
\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) \arrow[r] \arrow[d, "I"'] & \Omega\,\mathrm{MAF}^Q(\mathbb{R}^{n+1}) \arrow[d, "I"] \\
\mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})) \arrow[r] \arrow[d, "E"'] & \mathrm{Map}(S^1, \mathrm{MAF}^Q(\mathbb{R}^{n+1})) \arrow[d, "E"] \\
\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) \arrow[r] & \mathrm{MAF}^Q(\mathbb{R}^{n+1}).
\end{tikzcd}$$

The proof of the following result follows the proof of 4.21.

Theorem 6.9. *The composition*

$$\begin{aligned}\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n) &\xrightarrow{\Psi} \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}))\\ &\xrightarrow{E} \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})\end{aligned}$$

is homotopic to the unwrapping (or infinite transfer) map u.

Proposition 6.10. *Ψ, I and E induce homomorphisms between homotopy groups (based at e) including π_0.*

Proof. This is obvious for I and E. The result for Ψ is obvious as well if one is armed with the geometric description in Comment 4.14 of Ψ and the relative modifications needed to define Ψ on Whitehead spaces. □

Let $i : \Omega\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) \to \mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n)$ be any simplicial map such that $\Psi\circ i\simeq I$.

Proposition 6.11. *The simplicial map*

$$(i,w) : \Omega\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \times \mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \to$$
$$\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n),\ (x,y)\mapsto i(x)+w(y)$$

is a homotopy equivalence.

Proof. Note that

$$i_* = (\Psi_*)^{-1}I_* : \pi_k\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \to \pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n)$$

is a group homomorphism for each $k \geq 0$, and the fibration above implies that there is a long exact sequence

$$\cdots \to \pi_k\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \xrightarrow{i_*} \pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n)$$
$$\xrightarrow{u_*} \pi_k\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \to \pi_{k-1}\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \xrightarrow{i_*} \cdots$$

Since the wrapping up map w induces a splitting of u_* by Theorem 6.7, the long exact sequence reduces to split short exact sequences:

$$0 \to \pi_k\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \xrightarrow{i_*} \pi_k\mathcal{W}h(X\times\mathbb{R}^n \to \mathbb{R}^n)$$
$$\xrightarrow{u_*} \pi_k\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \to 0.$$

Thus, (w,i) induces isomorphisms on homotopy groups and, hence, is a homotopy equivalence. ☐

6.8. **Delooping the Whitehead space.** We show how results already in the literature can be used to provide a delooping of the controlled Whitehead space (Corollary 6.14 below). The proof uses pseudoisotopy theory.

The *simplicial set* $\mathcal{P}(X)$ *of pseudoisotopies on* X has k-simplices of the form

$$h : X\times[0,1]\times\Delta^k \to X\times[0,1]\times\Delta^k$$

where h is a homeomorphism which is fibre preserving over Δ^k and $h| : X\times\{0\}\times\Delta^k \to X\times\{0\}\times\Delta^k$ is the identity.

The *simplicial set* $\mathcal{P}(X\times\mathbb{R}^n \to \mathbb{R}^n)$ *of controlled pseudoisotopies* is defined in [29]. Essentially, a k-simplex is a parametrized family

$$h_t : X\times[0,1]\times\Delta^k \to X\times[0,1]\times\Delta^k,\ t\geq 0,$$

of k-simplices of $\mathcal{P}(X\times\mathbb{R}^n)$ such that $\mathrm{proj}\circ h_t : X\times\mathbb{R}^n\times[0,1]\times\Delta^k \to \mathbb{R}^n$ converges uniformly to the projection as $t\to\infty$.

Proposition 6.12. *If X is a compact Hilbert cube manifold, then there is a homotopy equivalence*

$$\Omega\mathcal{W}h(X\times\mathbb{R}^n \to \mathbb{R}^n) \simeq \mathcal{P}(X\times\mathbb{R}^n \to \mathbb{R}^n).$$

Proof. The result is explicitly stated in [29] for control measured in a compact base space, but the proof works equally well for the base space $\mathbb{R}^n$ with the standard metric (cf. the footnote in §3.2). ☐

Proposition 6.13. *If X is a compact Hilbert cube manifold, then there is a homotopy equivalence*

$$\mathcal{P}(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \simeq \mathcal{W}h(X\times\mathbb{R}^n \to \mathbb{R}^n).$$

Proof. It is not hard to adapt the techniques of [27], §6, to give a proof of this result. ☐

The following result follows immediately from the previous two propositions.

Corollary 6.14. *If X is a compact Hilbert cube manifold, then there is a homotopy equivalence*

$$\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})\simeq\mathcal{W}h(X\times\mathbb{R}^n\to\mathbb{R}^n).$$

Corollary 6.15. *If X is a compact Hilbert cube manifold, then there is a homotopy equivalence*

$$\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n)\simeq\mathcal{W}h(X\times\mathbb{R}^n\to\mathbb{R}^n)\times\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}).$$

Proof. Apply Proposition 6.11 and Corollary 6.14. □

6.9. **The relaxation map.** The construction is similar to the construction of the relaxation map in Chapter 5. Moreover, the necessary modifications are so similar to the modifications needed to define the wrapping up map that we omit further details. Note that if $f:M\to X\times S^1\times B\times\Delta^k$ is a k–simplex of $\mathcal{W}h(X\times S^1\times B\to B)$, then the composition

$$p:M\xrightarrow{f}X\times S^1\times B\times\Delta^k\to S^1\times B\times\Delta^k$$

is a k–simplex of $\mathrm{Man}^Q(S^1\times B)$. Moreover, it is easily seen that p is actually a k–simplex of $\mathrm{ManBan}^Q_c(S^1\times B)$. Thus, there is a simplicial map $\mathcal{W}h(X\times S^1\times B\to B)\to\mathrm{ManBan}^Q_c(S^1\times B)$ and the following diagram commutes:

$$\begin{array}{ccc}\mathcal{W}h(X\times S^1\times B\to B) & \xrightarrow{\ r\ } & \mathcal{W}h(X\times S^1\times B\to S^1\times B)\\ \downarrow & & \downarrow\\ \mathrm{ManBan}^Q_c(S^1\times B) & \xrightarrow{\ r\ } & \mathrm{MAF}^Q(S^1\times B).\end{array}$$

Proposition 6.16. *The relaxation map*

$$r:\mathcal{W}h(X\times S^1\times B\to B)\to\mathcal{W}h(X\times S^1\times B\to S^1\times B)$$

induces homomorphisms between homotopy groups (based at e) including π_0.

By analogy with Theorem 5.2 we have

Theorem 6.17. *The composition*

$$\begin{aligned}\mathcal{W}h(X\times S^1\times B\to S^1\times B) &\xrightarrow{\varphi}\mathcal{W}h(X\times S^1\times B\to B)\\ &\xrightarrow{r}\mathcal{W}h(X\times S^1\times B\to S^1\times B)\end{aligned}$$

is homotopic to the identity.

We make some remarks about the relaxation construction. Given a k–simplex $f:M\to X\times S^1\times B\times\Delta^k$ of $\mathcal{W}h(X\times S^1\times B\to B)$, let $\bar f:\bar M\to X\times\mathbb{R}\times B\times\Delta^k$ be the pull-back so and $\zeta:\bar M\to\bar M$ the $+1$ generating covering translation. There is a fibre preserving isotopy $G:\bar M\times[0,1]\to\bar M\times[0,1]$ supported on $\bar f^{-1}(X\times[-2,4]\times B\times\Delta^k)$ such that $\hat\zeta=\zeta\circ G_1$ is a covering translation of the infinite cyclic cover of the relaxation $\bar M/\hat\zeta=\hat M$. Moreover, $G_1|:X\times[0,1]\times B\times\Delta^k\to X\times\mathbb{R}\times B\times\Delta^k$ has the form $(x,s,y,t)\mapsto(x,s+1,y,t)$. Note that there is a natural copy of $X\times S^1\times B\times\Delta^k$ in $\hat M$ obtained from $X\times[0,2]\times B\times\Delta^k\subseteq\bar M$. Note that $\hat\zeta(x,0,y,t)=(x,2,y,t)$ for each $(x,y,t)\in X\times B\times\Delta^k$. The map $\hat f:\hat M\to X\times S^1\times B\times\Delta^k$ is the identity on this copy of $X\times S^1\times B\times\Delta^k$. More precisely, on the S^1–coordinates the map has the form

$$S^1=[0,2]/0=2\to S^1=[0,1]/0=1;\ [s]\mapsto[s/2].$$

6.10. **The boundedness condition on the infinite cyclic cover.** Proposition 3.5 allows the construction of a map

$$\mathcal{W}h(X\times S^1 \to B) \to \mathrm{ManBan}^Q(S^1\times B)$$

if B is a closed manifold. On the other hand, it follows from Proposition 3.8 that there is a map

$$\alpha : \mathcal{W}h(X\times S^1\times B \to B) \to \mathrm{ManBan}_c^Q(S^1\times B)$$

if B is any manifold without boundary. Let $\mathcal{W}h_{mb}(X\times S^1 \to B)$ denote the inverse image $\alpha^{-1}(\mathrm{ManBan}^Q(S^1\times B))$. The subscript "mb" stands for "manifold band" and indicates a boundedness condition after passing to the infinite cyclic cover. Proposition 3.6 shows that the image of

$$u^n : \mathcal{W}h(X\times S^1\times Y\times T^n \to Y\times T^n) \to \mathcal{W}h(X\times S^1\times Y\times \mathbb{R}^n \to Y\times\mathbb{R}^n)$$

lies in $\mathcal{W}h_{mb}(X\times S^1\times Y\times\mathbb{R}^n \to Y\times\mathbb{R}^n)$, where Y is a closed manifold. Therefore there is a map

$$u^n_{mb} : \mathcal{W}h(X\times S^1\times Y\times T^n \to Y\times T^n) \to \mathcal{W}h_{mb}(X\times S^1\times Y\times\mathbb{R}^n \to Y\times\mathbb{R}^n)$$

given by pulling back. Moreover, the composition

$$\begin{aligned}\mathcal{W}h(X\times S^1\times Y\times T^n \to Y\times T^n) &\xrightarrow{u^n_{mb}} \mathcal{W}h_b(X\times S^1\times Y\times\mathbb{R}^n \to Y\times\mathbb{R}^n)\\ &\xrightarrow{\text{inclusion}} \mathcal{W}h(X\times S^1\times Y\times\mathbb{R}^n \to Y\times\mathbb{R}^n)\end{aligned}$$

is equal to u^n, the previously defined n-fold unwrapping (see Remark 4.3). Hence, since $u^n\circ w^n$ is homotopic to the identity (Theorem 6.7), the composition

$$\begin{aligned}&\mathcal{W}h(X\times S^1\times Y\times\mathbb{R}^n \to Y\times\mathbb{R}^n) \xrightarrow{w^n} \mathcal{W}h(X\times S^1\times Y\times T^n \to Y\times T^n) \xrightarrow{u^n_{mb}}\\ &\mathcal{W}h_{mb}(X\times S^1\times Y\times\mathbb{R}^n \to Y\times\mathbb{R}^n) \xrightarrow{\text{inclusion}} \mathcal{W}h(X\times S^1\times Y\times\mathbb{R}^n \to Y\times\mathbb{R}^n),\end{aligned}$$

where w^n is the n-fold wrapping map, is homotopic to the identity.

In analogy with Proposition 3.10 we have the following result.

Proposition 6.18. *If Y is a closed manifold, then the inclusion*

$$\mathcal{W}h_{mb}(X\times S^1\times Y\times\mathbb{R}^n \to Y\times\mathbb{R}^n) \to \mathcal{W}h(X\times S^1\times Y\times\mathbb{R}^n \to Y\times\mathbb{R}^n)$$

is a homotopy equivalence.

6.11. **Bounded Whitehead and Pseudoisotopy Spaces.** We recall the definitions of the bounded Whitehead and pseudoisotopy spaces. The bounded pseudoisotopy space was used in Chapter 2 in the formulations of some of the main results. The bounded Whitehead space will be used for various technical reasons in Chapter 8 in the discussion of the Nil space. The consequence of Proposition 6.19 below is that there is no essential difference between the bounded and controlled theories. This principle is exploited in the series of papers [34], [35], and [36].

The *bounded Whitehead space* $\mathcal{W}h_b(X\times\mathbb{R}^n \to \mathbb{R}^n)$ is defined similarly to the controlled Whitehead space in §6.1. The difference is that a k-simplex $f : M \to X\times\mathbb{R}^n\times\Delta^k$ is required to be a proper $(p\times \mathrm{id}_{\Delta^k})^{-1}(c)$-sdr for some $c>0$ (rather than for every $c>0$).

A *bounded pseudoisotopy* on $X\times\mathbb{R}^n$ is a pseudoisotopy $h : X\times\mathbb{R}^n\times[0,1] \to X\times\mathbb{R}^n\times[0,1]$ such that there exists a $c>0$ such that $\mathrm{proj}\circ h : X\times\mathbb{R}^n\times[0,1]\to\mathbb{R}^n$ is c-close to the projection. Let $\mathcal{P}_b(X\times\mathbb{R}^n\to\mathbb{R}^n)$ be the simplicial set of bounded pseudoisotopies on $X\times\mathbb{R}^n$.

Proposition 6.19. *If X is a compact Hilbert cube manifold, then there are homotopy equivalences*

$$\mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n) \simeq \mathcal{W}h_b(X \times \mathbb{R}^n \to \mathbb{R}^n)$$

and

$$\mathcal{P}(X \times \mathbb{R}^n \to \mathbb{R}^n) \simeq \mathcal{P}_b(X \times \mathbb{R}^n \to \mathbb{R}^n).$$

Proof. This follows from standard results about manifold approximate fibrations. See [27]. □

The following result follows immediately from the previous proposition and Corollary 6.14. In fact, the proofs of 6.11–6.15 use the bounded approach of [27].

Corollary 6.20. *If X is a compact Hilbert cube manifold, then there are homotopy equivalences*

$$\Omega\mathcal{W}h_b(X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \simeq \mathcal{W}h_b(X \times \mathbb{R}^n \to \mathbb{R}^n)$$

and

$$\Omega\mathcal{P}_b(X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \simeq \mathcal{P}_b(X \times \mathbb{R}^n \to \mathbb{R}^n).$$

7. Torsion and a higher sum theorem

Let X be a Hilbert cube manifold with a fixed closed embedding $X \subseteq \ell_2$ of small capacity. Let $p : X \to B$ be a manifold approximate fibration with B a finite dimensional manifold without boundary with a fixed metric. In this section we show how certain homotopy equivalences to $X \times \Delta^k$ represent elements in $\pi_k \mathcal{W}h(p : X \to B)$. This is analogous to taking the torsion of a homotopy equivalence in classical simple homotopy theory (cf. [13]). The higher controlled version is a straightforward generalization of the construction in [27], §3. We also prove a sum formula for elements in $\pi_k \mathcal{W}h(p : X \to B)$. For $k = 0$ and $B = \{\text{point}\}$ this formula reduces to the classical sum formula of simple homotopy theory (cf. [13]). Chapman established a sum formula for controlled simple homotopy theory ([10]).

It is understood that the basepoint of the Whitehead space $\mathcal{W}h(p : X \to B)$ is generated by the vertex e_0 which is the projection $X \times [0,1] \to X$.

Let (ϵ, ρ, f, h) denote the following data :

(i) a constant $\epsilon > 0$,
(ii) a bundle projection $\rho : M \to \Delta^k$ with Hilbert cube manifold fibres,
(iii) a fibre preserving $(p \times \mathrm{id}_{\Delta^k})^{-1}(\epsilon)$–homotopy equivalence $f : M \to X \times \Delta^k$ so that there exists a fibre preserving homeomorphism $h : X \times [0,1] \times \partial\Delta^k \to \rho^{-1}(\partial\Delta^k)$ such that

$$fh = \text{ projection} : X \times [0,1] \times \partial\Delta^k \to X \times \partial\Delta^k.$$

In this chapter, let B be a fixed manifold with $\dim B < \infty$ and assume that B is either a closed manifold or a product $Y \times \mathbb{R}^n$ of a closed manifold Y and $\mathbb{R}^n$ (with the standard metric) and that B has the product metric.

Theorem/Definition 7.1. (1) *If $\epsilon > 0$ is sufficiently small (or, in the case $B = \mathbb{R}^n$, for every $\epsilon > 0$), then data (ϵ, ρ, f, h) determines a well-defined element*

$$\tau(f) \in \pi_k \mathcal{W}h(p : X \to B),$$

called the **torsion** *of f.*

(2) *If $f : M \to X \times \Delta^k$ is a k–simplex of $\mathcal{W}h(p : X \to B)$ which represents a class $[f] \in \pi_k \mathcal{W}h(p : X \to B)$, then there is data $(\epsilon, \rho, f, \mathrm{id})$ for every $\epsilon > 0$ and $\tau(f) = [f]$.*

(3) *There exists $\epsilon_0 > 0$ (or, in the case $B = \mathbb{R}^n$, for every $\epsilon_0 > 0$) such that if (ϵ, ρ, f, h) and $(\epsilon', \rho', f', h')$ are two sets of data associated to $f : M \to X \times \Delta^k$ and $f' : M' \to X \times \Delta^k$, respectively, for which the torsions are defined and $\epsilon, \epsilon' \leq \epsilon_0$, then $\tau(f) = \tau(f') \in \pi_k \mathcal{W}h(p : X \to B)$ if and only if there exists a fibre preserving homeomorphism $H : M \to M'$ such that the composition*

$$X \times [0,1] \times \partial\Delta^k \xrightarrow{h} \rho^{-1}(\partial\Delta^k) \xrightarrow{H|} (\rho')^{-1}(\partial\Delta^k) \xrightarrow{(h')^{-1}} X \times [0,1] \times \Delta^k$$

is the identity, and $f'H$ is fibre preserving $(p \times \mathrm{id}_{\Delta^k})^{-1}(\epsilon_0)$–homotopic to f rel $\rho^{-1}(\partial\Delta^k)$.

Proof. We will review the definition of torsion and refer to [27] for the remaining details. Given a $(p \times \mathrm{id}_{\Delta^k})^{-1}(\epsilon)$–homotopy equivalence $f : M \to X \times \Delta^k$ as in the data, choose an inverse $g : X \times \Delta^k \to M$ which is fibre preserving over Δ^k. We may assume that g is a sliced Z–embedding and $g| : X \times \partial\Delta^k \to \rho^{-1}(\partial\Delta^k)$ is given by $g(x,t) = h(x,0,t)$ for each $(x,t) \in X \times \partial\Delta^k$. If $X \times \Delta^k$ is identified with the image of g, then f can be slightly deformed rel $\rho^{-1}(\partial\Delta^k)$ to a map $f' : M \to X \times \Delta^k$

which is a $(p\times \mathrm{id}_{\Delta^k})^{-1}(\epsilon')$–sdr where the size of $\epsilon' > 0$ depends on ϵ. The the sucking principle (Lemma 4.1) can be used to deform f' to a map $\tilde{f} : M \to X\times\Delta^k$ representing a class $\tau(f) = [\tilde{f}]\in\pi_k\mathcal{W}h(p : X \to B)$. □

Remark 7.1 (Hypothesis for the Higher Sum Theorem 7.2). Suppose $f : M \to X\times\Delta^k$ is a k–simplex of $\mathcal{W}h(p : X \to B)$ and M is a union of closed subspaces $M = M_1\cup M_2$ with $M_0 = M_1\cap M_2$ and M_0 a sliced Z–set in both M_1 and M_2 so that $f_i = f|M_i : M_i \to X\times\Delta^k$, $i = 0, 1, 2$ are k–simplices of $\mathcal{W}h(p : X \to B)$. If $k > 0$, suppose there exists a fibre preserving homeomorphism $h : X\times[0, 1]\times[-1, 1]\times\partial\Delta^k \to f^{-1}(X\times\partial\Delta^k)$ such that

(a) $fh = \mathrm{proj} : X\times[0, 1]\times[-1, 1]\times\partial\Delta^k \to X\times\partial\Delta^k$,
(b) $h^{-1}(M_1\cap\rho^{-1}(\partial\Delta^k)) = X\times[0, 1]\times[-1, 0]\times\partial\Delta^k$,
(c) $h^{-1}(M_2\cap\rho^{-1}(\partial\Delta^k)) = X\times[0, 1]\times[0, 1] \times \partial\Delta^k$,
(d) $h| : X\times\{0\}\times\{0\}\times\partial\Delta^k \to M$ agrees with the given Z–embeddings $X\times\Delta^k \to M_i$, $i = 0, 1, 2$, when restricted to $X\times\partial\Delta^k$.

Even though M, M_0, M_1, M_2 are k–simplices of $\mathcal{W}h(p : X \to B)$ they are not based at the basepoint e (with the possible exception of M_0). We now correct this by adjoining canonical collars to get new k–simplices which are based at e. These canonical collars are mapping cylinders of certain maps. To this end let

$$c : X\times[0, 1]\times[-1, 1]\times\partial\Delta^k \to X\times[0, 1]\times\partial\Delta^k$$

be the projection. This map restricts to projections

1. $c_1 : X\times[0, 1]\times[-1, 0]\times\partial\Delta^k \to X\times[0, 1]\times\partial\Delta^k$
2. $c_2 : X\times[0, 1]\times[0, 1]\times\partial\Delta^k \to X\times[0, 1]\times\partial\Delta^k$
3. $c_0 : X\times[0, 1]\times\{0\}\times\partial\Delta^k \to X\times[0, 1]\times\partial\Delta^k$

The mapping cylinders are such that

$$M(c) = M(c_1)\cup M(c_2),\ M(c_1)\cap M(c_2) = M(c_0) = X\times[0, 1]\times[0, 1]\times\partial\Delta^k.$$

Consider $\tilde{M} = M\cup M(c)$ and $\tilde{M}_i = M_i\cup M(c_i)$, $i = 0, 1, 2$, where the mapping cylinders are adjoined to $\tilde{M}$ and $\tilde{M}_i$ at their tops. The map $f : M \to X\times\Delta^k$ induces a map $\tilde{f} : \tilde{M} \to X\times\Delta^k$ by using a collar of $\partial\Delta^k$ in Δ^k, and $\tilde{f}$ restricts to maps $\tilde{f}_i : \tilde{M}_i \to X\times\Delta^k$ so that $\tilde{f}, \tilde{f}_i$ are k–simplices of $\mathcal{W}h(p : X \to B)$ which are based at e and, hence, define classes in $\pi_k\mathcal{W}h(p : X \to B)$.

Theorem 7.2 (Higher Sum Theorem). (i) *If $k = 0$, then*

$$[f] = [f_1] + [f_2] - [f_0]\in\pi_0\mathcal{W}h(p : X \to B).$$

(ii) *If $k > 0$, then*

$$[\tilde{f}] = [\tilde{f}_1] + [\tilde{f}_2] - [\tilde{f}_0]\in\pi_k\mathcal{W}h(p : X \to B).$$

Proof. Let

$$N_1 = M\cup_{X\times\Delta^k}M_0 = (M\coprod M_0)/(X\times\Delta^k = X\times\Delta^k)$$

and

$$N_2 = M_1\cup_{X\times\Delta^k}M_2 = (M_1\coprod M_2)/(X\times\Delta^k = X\times\Delta^k).$$

There are maps $g_1 = f\cup f_0 : N_1 \to X\times\Delta^k$ and $g_2 = f_1\cup f_2 : N_2 \to X\times\Delta^k$. Now g_1 and g_2 are not k–simplices of $\mathcal{W}h(p : X \to B)$ because $X\times\Delta^k$ is not a sliced Z–set in N_1 or N_2. We will show how to construct a homeomorphism $H : N_1 \to N_2$

which is fibre preserving over Δ^k and then show further below how to use this construction to prove the result. For notational simplicity we will assume that

$$f^{-1}(X\times\partial\Delta^k) = X\times[0,1]\times[-1,1]\times\partial\Delta^k$$

and that h is the identity. Since M_0 is a sliced Z–set in both M_1 and M_2, M_0 is fibre preserving collared in both M_1 and M_2. Thus, let $M_0\times[-1,1]$ be a fibre preserving bicollaring of M_0 in M with

$$M_0\times[-1,0]\subseteq M_1 \text{ and } M_0\times[0,1]\subseteq M_2$$

so that M_0 is identified with $M_0\times\{0\}$. In the case $k>0$, over $\partial\Delta^k$ we have already identified M_0 with $X\times[0,1]\times\{0\}\times\partial\Delta^k$ and we may assume that the bicollaring coordinates $[-1,1]$ of M_0 in M agree with the standard bicollaring coordinates $[-1,1]$ of $X\times[0,1]\times\{0\}\times\partial\Delta^k$ in $X\times[0,1]\times[-1,1]\times\partial\Delta^k$. Let

$$A = (M_0\times[-1,1]\coprod M_0)/(X\times\{0\}\times\Delta^k = X\times\Delta^k)\subseteq N_1$$

and

$$B = ((M_0\times[-1,0])\coprod(M_0\times[0,1]))/(X\times\{0\}\times\Delta^k = X\times\{0\}\times\Delta^k)\subseteq N_2.$$

Let $\alpha : M_0\times[-1,1]\to M_0\times[-1,0]$ be the homeomorphism $\alpha(x,t)=(x,t/2-1/2)$. Then

$$X\times\{0\}\times\Delta^k\hookrightarrow M_0\times[-1,1]\xrightarrow{\alpha} M_0\times[-1,0]$$

is f.p. homotopic to the inclusion $X\times\{0\}\times\Delta^k\hookrightarrow M_0\times[-1,0]$ so that fibred Z–set unknotting implies that α is f.p. isotopic to a homeomorphism $\alpha' : M_0\times[-1,1]\to M_0\times[-1,0]$ such that $\alpha'| : X\times\{0\}\times\Delta^k\to M_0\times[-1,0]$ is the inclusion. Let $\beta : M_0\to M_0\times[0,1]$ be a f.p. homeomorphism such that β^{-1} is close to the projection $M_0\times[0,1]\to M_0$ (at least close enough to imply that they are homotopic). Then

$$X\times\Delta^k\hookrightarrow M_0\xrightarrow{\beta} M_0\times[0,1]$$

is f.p. homotopic to the inclusion $X\times\Delta^k = X\times\{0\}\times\Delta^k\hookrightarrow M_0\times[0,1]$ so that fibred Z–set unknotting implies that β is f.p. isotopic to a homeomorphism $\beta' : M_0\to M_0\times[0,1]$ such that $\beta'| : X\times\Delta^k = X\times\{0\}\times\Delta^k\to M_0\times[0,1]$ is the inclusion. Thus, α' and β' glue together to define a f.p. homeomorphism $\gamma : A\to B$ such that $\gamma| : X\times\{0\}\times\Delta^k\to B$ is the inclusion. We now modify γ by a f.p. isotopy to get a homeomorphism $\gamma' : A\to B$ such that $\gamma'| : M_0\times\{-1,1\}\to B$ is the inclusion. To this end note that

$$M_0\times\{-1,1\}\hookrightarrow A\xrightarrow{\gamma} B$$

is the composition

$$M_0\times\{-1,1\}\hookrightarrow M_0\times[-1,1]\xrightarrow{\alpha'} M_0\times[-1,0]\hookrightarrow B$$

which is f.p. homotopic to

$$M_0\times\{-1,1\}\hookrightarrow M_0\times[-1,1]\xrightarrow{\alpha} M_0\times[-1,0]\hookrightarrow B$$

which is the map

$$(x,-1)\mapsto(x,-1),\ (x,1)\mapsto(x,0).$$

Since $X\times\Delta^k\hookrightarrow M_0$ is a f.p. homotopy equivalence, there exists a f.p. strong deformation retraction of M_0 to $X\times\Delta^k$. Thus, $M_0\times\{0\}\hookrightarrow M_0\times[-1,0]\hookrightarrow B$ is f.p. homotopic to $M_0\times\{0\}\hookrightarrow M_0\times[0,1]\hookrightarrow B$. It follows that $M_0\times\{-1,1\}\hookrightarrow A\xrightarrow{\gamma} B$ is f.p. homotopic to the inclusion, and fibred Z–set unknotting implies that γ is f.p. isotopic to a homeomorphism γ' as above. Now γ' extends via the identity to a f.p. homeomorphism $H : N_1 \to N_2$.

One property of the construction is that if $\epsilon > 0$ is given, then H can be constructed so that $g_2 H$ is fibre preserving $(p\times \mathrm{id}_{\Delta^k})^{-1}(\epsilon)$–homotopic to g_1. To see that this is the case, first note that we may assume that the bicollaring intervals $\{x\}\times[-1,1]$, $x\in M_0$, are small when mapped to $X\times\Delta^k$ by f. Using this fact and the full strength of the estimates available in Z–set unknotting, it follows that γ can be constructed so that $g_2\gamma$ is $(p\times \mathrm{id}_{\Delta^k})^{-1}(\epsilon)$–homotopic to $g_1|A$ where $\epsilon > 0$ is as small as we want. Then in the modification of γ to γ', one must use the fact that the homotopy equivalence $X\times\Delta^k \hookrightarrow M_0$ has good control when measured in $X\times\Delta^k$.

Now when $k = 0$, note that $g_1 : N_1 \to X$ and $g_2 : N_2 \to X$ determine torsions and 7.1(3) implies that $\tau(g_1) = \tau(g_2)\in\pi_0\mathcal{W}h(p : X \to B)$. On the other hand, it is clear from the definitions and Z–set unknotting that

$$\mu_*(f, f_0) = \tau(g_1) \text{ and } \mu_*(f_1, f_2) = \tau(g_2) \text{ in } \pi_0\mathcal{W}h(p : X \to B).$$

It then follows from 6.2 that

$$[f] + [f_0] = [f_1] + [f_2]\in\pi_0\mathcal{W}h(p : X \to B).$$

Now when $k > 0$, according to Chapter 6, it suffices to show that $\nu_k\mu_*([\tilde f], [\tilde f_0]) = \nu_k\mu_*([\tilde f_1], [\tilde f_2])$ for then $[\tilde f]+[\tilde f_0] = [\tilde f]+[\tilde f_0]$. Note that $\mu(f, f_0) = N_1\cup(X\times[0,1]\times\Delta^k)$ and $\mu(f_1, f_2) = N_2\cup(X\times[0,1]\times\Delta^k)$. Since the composition $X \times \Delta^k \hookrightarrow N_1 \xrightarrow{H} N_2$ is f.p. homotopic to the inclusion, with good control in $B \times \Delta^k$, Z–set unknotting can be used to induce a f.p. homeomorphism $H' : \mu(f, f_0) \to \mu(f_1, f_2)$ with $\mu(f, f_0) \xrightarrow{H'} \mu(f_1, f_2) \to X\times\Delta^k$ f.p. homotopic to $\mu(f, f_0) \to X\times\Delta^k$, with good control in $B\times\Delta^k$. Now observe that H' induces a homeomorphism $\tilde H : \mu(\tilde f, \tilde f_0) \to \mu(\tilde f_1, \tilde f_2)$. The construction of $\tilde H$ involves suitable modifications of α', β' and γ' over $\partial\Delta^k$. For example, the use of Z–set unknotting to modify α to α' implies that the exists a family of homeomorphism

$$\partial\alpha_t : X\times[0,1]\times[-1,1]\times\partial\Delta^k \to X\times[0,1]\times[-1,0]\times\partial\Delta^k, 0 < t\leq 1,$$

such that $\partial\alpha_1 = \alpha'|$, $\partial\alpha'|X\times\{0\}\times\{0\}\times\partial\Delta^k$ is the inclusion and $\lim_{t\to 1} c_1\circ\partial\alpha_t = c$, so that there is an induced map between mapping cylinders. Similar considerations apply to β' and γ'. Finally use 6.1(2) to conclude that $\nu_k\mu_*([\tilde f], [\tilde f_0]) = \nu_k\mu_*([\tilde f_1], [\tilde f_2])$.

□

8. Nil as a geometrically defined simplicial set

In this section, we will give the definition of the Nil-space of a compact Q-manifold X equipped with a map to a manifold B, $\widetilde{\mathcal{N}il}(p : X \to B)$. It is going to be a simplicial set such that $\pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$ has a natural group structure which makes it isomorphic to the lower reduced K-theory Nil-group of the integral group ring of X, for a compact Q-manifold X. The definition generalizes the geometric definition of Nil-groups given in [41].

8.1. **Preliminaries.**

Definition 8.1. Let $f : Y \to Y$ be a map.

1. The *mapping torus* of a map $f : Y \to Y$ is the identification space
$$T(f) = Y\times[0,1]/\{(x,1) = (f(x),0)|\, x\in Y\}.$$
2. The *reversed mapping torus* of f is given by
$$T'(f) = Y\times[0,1]/\{(f(x),1) = (x,0)|\, x\in Y\}.$$

We also define the canonical infinite cyclic covers of the mapping tori:

1. The *canonical infinite cyclic cover* $\overline{T}(f)$ of $T(f)$ is the identification space
$$\overline{T}(f) = \left(\coprod\nolimits_{j\in\mathbb{Z}}(Y\times[0,1]\times\{j\})\right)/\{(x,1,j) = (f(x),0,j+1)|\, x\in Y,\ j\in\mathbb{Z}\}.$$
2. For the inverse mapping torus we have a similar construction
$$\overline{T}'(f) = \left(\coprod\nolimits_{j\in\mathbb{Z}}(Y\times[0,1]\times\{j\})\right)/\{(f(x),1,j) = (x,0,j+1)|\, x\in Y,\ j\in\mathbb{Z}\}.$$

There are maps $q : T(f) \to S^1$, $[x,t]\mapsto[t]$, ($q' : T'(f) \to S^1$, $[x,t]\mapsto[t]$) and $\overline{q} : \overline{T}(f) \to \mathbb{R}$, $[x,t,j]\mapsto t+j$ ($\overline{q}' : \overline{T}'(f) \to \mathbb{R}$, $[x,t,j]\mapsto t+j$). For $a\in\mathbb{R}$,
$$\overline{T}^{+}_{a}(f) = \overline{q}^{-1}[0,\infty),\quad \overline{T}^{-}_{a}(f) = \overline{q}^{-1}(-\infty,0].$$
We use similar notation for the reversed constructions.

Let B be a finite dimensional manifold without boundary with a fixed metric. In most of our calculations B will be a product $Y\times\mathbb{R}^n$, with Y a finite dimensional closed manifold, with the standard metric on $\mathbb{R}^n$ and the product metric on B. We give $B\times\Delta^k$ the standard metric denoted d.

Definition 8.2. Suppose Z is a compact space equipped with a proper map $p : Z \to B$. Let M be a space containing $Z\times\Delta^k$ with inclusion $i : Z\times\Delta^k \to M$ and $p' = p\times\mathrm{id}_{\Delta^k} : Z\times\Delta^k \to B\times\Delta^k$.

1. A map $f : M \to M$ is *boundedly close* to a map $g : M \to Z\times\Delta^k$ if there exists a positive number b such that $d(p'gf(x), p'g(x)) < b$ for each $x\in M$.
2. A homotopy $h : M\times[0,1] \to M$ with $h_1(M)\subseteq Z\times\Delta^k$ is *bounded* if there exists a positive number b such that $d(p'h_1h(x,t), p'h_1(x)) < b$ for each $(x,t)\in M\times[0,1]$.
3. A map $f : M \to M$ is *boundedly homotopic* to a map $g : M \to Z\times\Delta^k$ if there is a bounded homotopy $h : f\simeq ig$.

8.2. **The simplicial set of nil simplices.** Let X be a Hilbert cube manifold with a fixed embedding $X \subseteq \ell_2$ of small capacity, and let $p : X \to B$ be a manifold approximate fibration.

Definition 8.3. (1) A *nil k-simplex over X* consists of a pair (M, f), with M a Q-manifold, such that

(i) $M \subseteq \ell_2 \times B \times \Delta^k$ is a subspace of small capacity,
(ii) the projection $\rho : M \to \Delta^k$ is a fiber bundle projection with Q–manifold fibers (in what follows, 'fiber preserving' refers to fiber preserving with respect to the projection ρ),
(iii) $X \times \Delta^k$ is a sliced Z–set in M (in particular, $\rho | X \times \Delta^k$ is the projection),
(iv) $f : M \to M$ is a fiber preserving map with $f | X \times \Delta^k$ equal to the inclusion $i : X \times \Delta^k \to M$,
(v) there exist a positive integer s and a fiber preserving retraction $r : M \to X \times \Delta^k$ such that
   1. the composition $M \xrightarrow{r} X \times \Delta^k \xrightarrow{p \times \mathrm{id}_{\Delta^k}} B \times \Delta^k$ is a proper map.
   2. f is fiber preserving boundedly close to r, and f^s is boundedly homotopic to r rel $X \times \Delta^k$.

(2) Two nil k-simplices over X, (M_α, f_α), $\alpha = 1, 2$, are *related*, written $(M_1, f_1) \sim (M_2, f_2)$, if they are germ related in the following sense:

   there is a nil k-simplex (N, g) over X with N containing both M_1 and M_2 and inclusions denoted $j_\alpha : M_\alpha \to N$ such that $g j_\alpha = j_\alpha f_\alpha$ $(\alpha = 1, 2)$ and $g(N) \subseteq h_1(M_1) \cap h_2(M_2)$.

(3) Let $\simeq$ denote the equivalence relation on the set of nil k-simplices over X generated by the relation $\sim$. The equivalence class of a nil simplex (M, f) is denoted $[M, f]$.
(4) The simplicial set $\widetilde{\mathcal{N}il}(p : X \to B)$ *of nil simplices over* X is the simplicial set whose k–simplices are equivalence classes of a nil k–simplices over X. Face and degeneracy operations are induced from the standard ones on Δ^k.

For $B = \{\text{point}\}$, we use the notation $\widetilde{\mathcal{N}il}(X) = \widetilde{\mathcal{N}il}(p : X \to \text{point})$.

Remark 8.4. If the nil k-simplex (N, g) provides the relation $(M_1, f_1) \sim (M_2, f_2)$ between two nil k-simplices, then so does the nil k-simplex $(N \times [0, 1], g')$, where

$$g' : N \times [0,1] \xrightarrow{\text{proj}} N \xrightarrow{g} N \xrightarrow{\times 0} N \times [0,1].$$

Let (N, g) provide the germ relation between (M_α, f_α), $(\alpha = 1, 2)$. By replacing (N, g) by $(N \times [0, 1], g')$, if necessary, we may assume without loss of generality that $j_\alpha : M_\alpha \to N$, $\alpha = 1, 2$, are fiber preserving Z-embeddings.

Lemma 8.5. *The simplicial set $\widetilde{\mathcal{N}il}(p : X \to B)$ is a Kan complex.*

Proof. We will show the extension condition for 1-simplices. The general extension condition follows inductively. Let (M_α, f_α), $\alpha = 1, 2$, represent two simplices of $\widetilde{\mathcal{N}il}(p : X \to B)$ such that $\partial_0[M_1, f_1] = \partial_1[M_2, f_2]$. Since the projections $M_\alpha \to \Delta^1 = [0, 1]$ are trivial fiber bundles, there are Q-manifolds N_α and fibre preserving homeomorphisms $h_\alpha : M_\alpha \to N_\alpha \times [0, 1]$, $\alpha = 1, 2$. Then $f'_\alpha = h_\alpha f_\alpha h_\alpha^{-1} : N_\alpha \times [0, 1] \to N_\alpha \times [0, 1]$ is a fiber preserving map over $[0, 1]$. By a slight abuse of notation we use h_α to replace M_α with $N_\alpha\times[0,1]$ and assume that

$$(N_1\times\{1\}, f_1'|) \simeq (N_2\times\{0\}, f_2'|).$$

Therefore there are nil 0-simplices (A_j, a_j), $j = 1, 2, \dots m$, over X such that:

$$(N_1\times\{1\}, f_1'|) = (A_0, a_0) \sim (A_1, a_1) \sim \cdots \sim (A_{m+1}, a_{m+1}) = (N_2\times\{0\}, f_2'|).$$

Let (L_i, ℓ_i), $i = 0, 1, \dots m$, be the nil 0-simplex that provides the relation between (A_i, a_i) and (A_{i+1}, a_{i+1}) $(i = 0, 1, \dots m)$ and let

$$\iota_i : A_i \to L_i, \quad i = 0, 1, \dots m, \quad \kappa_j : A_j \to L_{j-1}, \quad j = 1, 2, \dots m + 1$$

be the inclusions which, according to Remark 8.4, can be assumed to be fibre preserving Z–embeddings. Thus $\ell_i\iota_i = \iota_i a_i$ $(i = 0, 1, \dots m)$ and $\ell_{j-1}\kappa_j = \kappa_j a_j$ $(j = 0, 1, \dots m + 1)$. Use the homotopy extension property to extend the map

$$(\iota_0 \times \mathrm{id}_{[0,1]})\circ f_1' : N_1\times[0,1] \to \iota_0(N_1)\times[0,1]$$

to a fibre preserving map

$$\lambda^0 : L_0\times[0,1] \to \iota_0(N_1)\times[0,1] \subseteq L_0$$

such that $\lambda^0_1 = \ell_0$. Thus $(N_1\times[0,1], f_1') \sim (L_0\times[0,1], \lambda^0)$. The adjunction space $L_0\cup_{A_1} L_1$ is a Q-manifold since ι_0 and κ_1 are Z-embeddings. By construction,

$$(\ell_0\cup\ell_1)(L_0\cup_{A_1} L_1) \subset \kappa_1(A_1) \subset L_0.$$

To this end, if $\ell_0^{s_0}$ is boundedly homotopic to a retraction to X, rel X,

$$(\ell_0\cup\ell_1)^{s_0+1} = (\ell_0\cup\ell_1)^{s_0}(\ell_0\cup\ell_1) = \ell_0^{s_0}(\ell_0\cup\ell_1)$$

which is boundedly homotopic to a retraction to X, rel X. Thus the pair $(L_0\cup_{A_1} L_1, \ell_0\cup\ell_1)$ is also a nil 0-simplex. Using the homotopy extension property again we construct an extension of λ^0 to a map λ^1 defined on $(L_0\cup_{A_1} L_1)\times[0,1]$, such that $\lambda^1_1 = \ell_0\cup\ell_1$. Repeating the same argument as before, we show that

$$(L_0\times[0,1], \lambda^0) \sim ((L_0\cup_{A_1} L_1)\times[0,1], \lambda^1)$$

Set $L = L_0\cup_{A_1} L_1 \dots \cup_{A_m} L_m$. Continuing this process, we construct a sequence of relations

$$(N_1\times[0,1], f_1') \sim (L_0\times[0,1], \lambda^0) \sim \dots (L\times[0,1], \lambda^m)$$

where $L = L_0\cup_{A_1} L_1\cup_{A_2} \dots \cup_{A_{m+1}} L_m$. Thus $(N_1\times[0,1], f_1') \simeq (L\times[0,1], \lambda^m)$ with $\lambda^m_1 = \ell_0\cup\ell_1 \dots \cup\ell_m$. Repeating the same process 'backwards' starting from N_2 we conclude that

$$(N_2\times[0,1], f_2') \simeq (L\times[0,1], \mu^m)$$

with $\mu^m_0 = \ell_0\cup\ell_1 \dots \cup\ell_m$. Since $\lambda^m_1 = \mu^m_0$, we glue the simplices $(L\times[0,1], \lambda^m)$ and $(L\times[0,1], \mu^m)$ to form an nil 1-simplex $(L\times[0,1], \nu)$ by

$$\nu(x,t) = \begin{cases} \lambda^m(x, \frac{t}{2}) & 0 \le t \le \frac{1}{2} \\ \mu^m(x, 2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Set

$$\nu' : L\times[0,1]\times[0,1] \to L\times[0,1]\times[0,1]; \quad \nu'(x,t,t') = (\nu(x,t), t').$$

Let $h : [0,1]\times[0,1] \to \Delta^2$ be a homeomorphism that maps $[0,1/2]\times\{0\}$ and $[1/2,1]\times\{1\}$ to the horn Λ_0 by mapping each space to a face of Λ_0. Define a nil 2-simplex by $(L\times\Delta^2, h\circ\nu'\circ h^{-1})$. Then $\rho^{-1}(\Lambda_0)$ of the last simplex consists of two faces that are equal to the simplices (M_1, f_1), (M_2, f_2) respectively $(\rho : L\times\Delta^2 \to \Delta^2$ is the projection map). □

8.2.1. *Characterization of nil simplices.* We give a characterization of nil k-simplices over X analogous to the one given in [41], Proposition 2.6. Let $p : X \to B$ be as before and $p' = p\times\mathrm{id}_{\Delta^k} : X\times\Delta^k \to B\times\Delta^k$. As always, $i : X\times\Delta^k \to M$ denotes the inclusion map of $X\times\Delta^k$ as a sliced Z–subset. Let $f : M \to M$ be a fibre preserving map such that $f|X\times\Delta^k = \mathrm{id}_{X\times\Delta^k}$. Let $\iota : X\times\Delta^k\times S^1 \to T(f)$ denote the induced inclusion map and $\rho' : T(f) \to \Delta^k$ be the map defined by $\rho'(x,t) = \rho(x)$. By replacing f by a sliced Z–embedding if necessary, we can assume that ρ' is a fiber bundle projection with Q-manifold fibers. We say that there is a f.p. bounded strong deformation retraction of $T(f)$ to $X\times S^1\times\Delta^k$ if there exists a f.p. bounded homotopy $h : T(f)\times[0,1] \to T(f)$ rel $X\times S^1\times\Delta^k$ with $h_0 = \mathrm{id}_{T(f)}$ and $h_1(T(f))\subseteq X\times S^1\times\Delta^k$. The homotopy h induces a f.p. (over Δ^k) strong deformation retraction $\bar{f} : \overline{T}(f) \to \overline{T}(f)$ of $\overline{T}(f)$ to $X\times\mathbb{R}\times\Delta^k$ which is bounded over $B\times\Delta^k$.

Proposition 8.6. (i) *If (M, f) is a nil k-simplex over X, then $X\times S^1\times\Delta^k$ is a f.p. bounded strong deformation retraction of $T(f)$.*

(ii) *Let (M, f) be a pair satisfying the conditions (i)–(iv) of Definition 8.3. Let $h : T(f) \to T(f)$ be a f.p. strong deformation retraction of $T(f)$ to $X\times S^1\times\Delta^k$ such that $\bar{h}$ is f.p., bounded over $B\times\mathbb{R}\times\Delta^k$ strong deformation retraction. Then (M, f) is a nil k-simplex over X.*

Proof. (i) Since (M, f) is a nil k-simplex over X, there exist $s \in \mathbb{Z}_+$ and a f.p., bounded homotopy $H : M\times[0,1] \to M$ rel $X\times\Delta^k$ such that $H_0 = f^s$ and $H_1 = ir$ for some retraction $r : M \to X \times \Delta^k$. Let $c > 0$ be a constant such that $p'rf$ is c–close to $p'r$ (by Definition 8.3(1)(v)) and which is a bound for the diameters of the tracks of $p'rH : M\times[0,1] \to B \times \Delta^k$. Define $H' : M\times[0,1]\times[0,1] \to M$ by

$$H'(x,t,u) = \begin{cases} f^{s+1}H(x,(1-2u)t), & \text{if } 0 \le u \le \frac{1}{2} \\ f^s H(f(x),(2u-1)t), & \text{if } \frac{1}{2} \le u \le 1. \end{cases}$$

Observe that H' is bounded in the sense that $p'rH'(x,t,u)$ is $(s+2)c$–close to $p'r(x)$ for every $(x,t,u) \in M\times[0,1]\times[0,1]$. To see this note first that since $p'r$ is c–close to $p'rf$, it follows that $p'r$ is sc–close to $p'rf^s$ (and $p'r$ is $(s+1)c$-close to $p'rf^{s+1}$). Thus, $p'rH'(x,t,u)$ is

$$\begin{cases} (s+1)c\text{–close to } p'rH(x,(1-2u)t), & \text{if } 0 \le u \le \frac{1}{2} \\ sc\text{–close to } p'rH(f(x),(2u-1)t), & \text{if } \frac{1}{2} \le u \le 1. \end{cases}$$

Now H is c–bounded, so $p'rH(x,v)$ is c–close to $p'rH_1(x) = p'r(x)$ for every $(x,v) \in M\times[0,1]$ and the $(s+2)c$–bound on H' follows. The map H' has the form $f^s\circ L$, where $L : M\times[0,1]\times[0,1] \to M$. Define a homotopy

$$h : M\times[0,1]\times[0,1]\times[0,1] \to M; \quad h(x,t,u,v) = H(L(x,t,u),v).$$

Then

$$\begin{aligned} h(x,t,u,0) &= H(L(x,t,u),0) &&= f^s\circ L(x,t,u) &&= H'(x,t,u) \\ h(x,t,u,1) &= H(L(x,t,u),1) &&= r\circ L(x,t,u). \end{aligned}$$

Since the distance from f^s to r is bounded, h is a bounded f.p. homotopy from H' to a retraction to $X\times\Delta^k$. Furthermore, the homotopy h has the following properties:

$$\begin{aligned} h(x,1,0,v) &= H(L(x,1,0),v) &&= H(fH(x,1),v) &&= ir(x) \\ h(x,1,1,v) &= H(L(x,1,1),v) &&= H(H(f(x),1),v) &&= irf(x). \end{aligned}$$

Thus h is a homotopy relative to $M\times\{1\}\times\{0,1\}\cup X\times\Delta^k\times[0,1]\times[0,1]$. We write h^- for the homotopy, $h^-(x,t,u,v)=h(x,t,u,1-v)$. Define the restrictions

$$h' = h^-|(M\times[0,1]\times\{0,1\}\cup M\times\{0\}\times[0,1])\times[0,1]$$
$$h'' = h|(M\times\{1\}\times[0,1])\times[0,1].$$

On the intersection $M\times\{1\}\times\{0,1\}\times[0,1]$ the two maps agree because

$$\begin{array}{rcccccc} h'(x,1,0,v) & = & h(x,1,0,1-v) & = & ir(x) & = & h''(x,1,0,v) \\ h'(x,1,1,v) & = & h(x,1,1,1-v) & = & irf(x) & = & h''(x,1,1,v). \end{array}$$

Using the estimated homotopy extension property, there is a f.p. bounded homotopy

$$G : M\times[0,1]\times[0,1]\times[0,1] \to M$$

extending $h'\cup h''$, rel $X\times\Delta^k\times[0,1]\times[0,1]$. Define $\kappa : T(f)\times[0,1] \to T(f)$ by $\kappa([x,t],u)=[G(x,u,t,1),t]$. Then

1. κ is well-defined: On $M\times\{1\}\times[0,1]$,

$$\begin{array}{rcccl} \kappa([x,1],u) & = & [G(x,u,1,1),1] & = & [h'(x,u,1,1),1] = \\ [h(x,u,1,0),1] & = & [H(L(x,u,1),0),1] & = & [f^sH(f(x),u),1]. \end{array}$$

On $f(M)\times\{0\}\times[0,1]$,

$$\begin{array}{rcccl} \kappa([f(x),0],u) & = & [G(f(x),u,0,1),0] & = & [h'(f(x),u,0,1),0] = \\ [h(f(x),u,0,0),0] & = & [H(L(f(x),u,0),0),0] & = & [f^{s+1}H(f(x),u),0] \end{array}$$

which are equal in $T(f)$.

2. κ is a homotopy between

$$\begin{array}{rcccl} \kappa([x,t],0) & = & [G(x,0,t,1),t] & = & [h'(x,0,t,1),t] = \\ [h(x,0,t,0),t] & = & [H(L(x,0,t),0),t] & = & [f^{2s+1}(x),t] \end{array}$$

and the map

$$\begin{array}{rcccl} \kappa([x,t],1) & = & [G(x,1,t,1),t] & = & [h''(x,1,t,1),t] = \\ [h(x,1,t,1),t] & = & [H(L(x,1,t),1),t] & = & [irL(x,1,t),t)]. \end{array}$$

Thus $\kappa(T(f)\times\{1\})\subseteq X\times S^1\times\Delta^k$. Also, by Mather's trick, $\kappa|T(f)\times\{0\}$ is f.p. bounded homotopic to the identity. Combining the two homotopies, we complete the proof of (i).

(ii) Let $h : T(f) \to T(f)$ be the strong deformation retraction given and $\bar{h} : \overline{T}(f) \to \overline{T}(f)$ the infinite cyclic cover of h which is bounded over $B\times\mathbb{R}$. Identify M with $M\times\{0\}\times\{0\}\subset\overline{T}(f)$. Since $\bar{h}$ is bounded over $\mathbb{R}$, there is a number $b'\geq 0$ such that

$$\bar{h}(M\times[0,1])\subset\bar{h}_1^{-1}(X\times(-\infty,b']\times\Delta^k)\subset\overline{T}_b^-(f).$$

for some positive integer b. Also, there is a map induced by collapse maps

$$c_b : \overline{T}_b^-(f) \to M\times\{1\}\times\{b\}, \quad c_b([x,t,j])=[f^{b-j}(x),1,b].$$

Define a homotopy

$$c_b\circ\bar{h}| : M\times[0,1] \to M = M\times\{1\}\times\{b\}$$

which is f.p. over Δ^k and bounded over $B\times\Delta^k$. Then $c_b\circ\bar{h}(x,0)=f^b(x)$ $(x\in M)$, and $c_b\circ\bar{h}(-,1)$ is a retraction to $X\times\Delta^k$, rel $X\times\Delta^k$. Thus (M,f) is a nil k-simplex over X. $\square$

Remark 8.7. The lift of the homotopy κ, constructed in the first part of the proof, to $\overline{T}(f)$ is given by:

$$\bar{\kappa} : \overline{T}(f)\times[0,1] \to \overline{T}(f); \quad \bar{\kappa}([x,t,j],u) = [G(x,u,t,1),t,j].$$

Thus the retraction of $\overline{T}(f)$ to $X\times\mathbb{R}\times B\times\Delta^k$ is given by:

$$\bar{r} = \bar{\kappa}_1 : \overline{T}(f) \to X\times\mathbb{R}\times\Delta^k.$$

which implies that

$$\bar{r}^{-1}(X\times[0,\infty)\times\Delta^k) = T_0^+(f).$$

Let $q : X\times\mathbb{R}\times\Delta^k \to \mathbb{R}$ denote the projection. Let $[x,t,j]\in\overline{T}(f)$. Then, for $u\in[0,1]$,

$$q\bar{r}\bar{\kappa}([x,t,j],u) = q\bar{r}[G(x,u,t,1),t,j] = q[G(G(x,u,t,1),1,t,1),t,j] = t+j$$

which implies that $p\bar{r}\bar{\kappa}(\{[x,t,j]\}\times[0,1]) = \{t+j\}$. Thus $\bar{\kappa}$ is a bounded homotopy over $\mathbb{R}$.

Similar calculations apply to the reversed mapping torus $T'(f)$.

8.2.2. *Abelian monoid-like structures.* We will now define a binary operation which induces a structure of an abelian monoid-like simplicial set on $\widetilde{\mathcal{N}il}(p : X \to B)$ We modify the construction given in §6.2. We begin by describing a new basepoint of $\widetilde{\mathcal{N}il}(p : X \to B)$. Let $Y = (X\times[0,2])\sqcup(X\times[-1,-2])/\sim$ where $\sim$ is the equivalence relation generated by $(x,1)\sim(x,-1)$ for each $x\in X$ (see §6.2). Let $e_0\vee e_0$ denote the vertex $Y \to Y$; $[(x,t)]\mapsto[x,0]$, of $\widetilde{\mathcal{N}il}(p : X \to B)$. The degenerate k–simplex on $e_0\vee e_0$ is denoted $e_k\vee e_k$, and $e\vee e = \{e_k\vee e_k\}$ is a basepoint of $\widetilde{\mathcal{N}il}(p : X \to B)$. Consider the simplicial maps:

$$i_1 : \widetilde{\mathcal{N}il}(p : X \to B) \to \widetilde{\mathcal{N}il}(p : X \to B)\times\widetilde{\mathcal{N}il}(p : X \to B);\ x\mapsto(x,e),$$

$$i_2 : \widetilde{\mathcal{N}il}(p : X \to B) \to \widetilde{\mathcal{N}il}(p : X \to B)\times\widetilde{\mathcal{N}il}(p : X \to B);\ x\mapsto(e,x),$$

$$\Delta : \widetilde{\mathcal{N}il}(p : X \to B) \to \widetilde{\mathcal{N}il}(p : X \to B)\times\widetilde{\mathcal{N}il}(p : X \to B);\ x\mapsto(x,x).$$

Proposition 8.8. *Suppose X is a Hilbert cube manifold, and $p : X \to B$ is a manifold approximate fibration.*

(1) *There exists a simplicial map*

$$\mu : \widetilde{\mathcal{N}il}(p : X \to B)\times\widetilde{\mathcal{N}il}(p : X \to B) \to \widetilde{\mathcal{N}il}(p : X \to B); \quad \mu(x,y) = x+y$$

satisfying the following properties:

  (i) $\mu\circ i_1 \simeq \mathrm{id} \simeq \mu\circ i_2$,
  (ii) *the two maps* $\widetilde{\mathcal{N}il}(p : X \to B)\times\widetilde{\mathcal{N}il}(p : X \to B)\times\widetilde{\mathcal{N}il}(p : X \to B) \to \widetilde{\mathcal{N}il}(p : X \to B)$; $(x,y,z)\mapsto\mu(\mu(x,y),z)$ *and* $(x,y,z)\mapsto\mu(x,\mu(y,z))$ *are homotopic,*
  (iii) *the two maps* $\widetilde{\mathcal{N}il}(p : X \to B)\times\widetilde{\mathcal{N}il}(p : X \to B) \to \widetilde{\mathcal{N}il}(p : X \to B)$; $(x,y)\mapsto\mu(x,y)$ *and* $(x,y)\mapsto\mu(y,x)$ *are homotopic,*
  (iv) $\mu(e,e) = e\vee e$.

(2) *For each $k\geq1$ there exists an isomorphism $\nu_k : \pi_k(\widetilde{\mathcal{N}il}(p : X \to B), e\vee e) \to \pi_k(\widetilde{\mathcal{N}il}(p : X \to B), e)$ such that the operation induced by the composition*

$$\begin{aligned}\pi_k(\widetilde{\mathcal{N}il}(p : X \to B),e) \times \pi_k(\widetilde{\mathcal{N}il}(p : X \to B),e) &\xrightarrow{\mu_*} \pi_k(\widetilde{\mathcal{N}il}(p : X \to B),e\vee e)\\ &\xrightarrow{\nu_k} \pi_k(\widetilde{\mathcal{N}il}(p : X \to B),e)\end{aligned}$$

is commutative and agrees with the standard homotopy group operation.

Proof. The operation μ is defined as a push-out construction. The exact definition and the properties of μ are given in Proposition 6.2. $\square$

Remark 8.9. In Proposition 6.2, there is a construction of inverses for $\mathcal{W}h(p : X \to B)$. We are going to prove the existence of inverses for $\widetilde{\mathcal{N}il}(p : X \times \mathbb{R}^n \to \mathbb{R}^n)$, with X a closed Q-manifold and p the projection map, later in this chapter.

8.3. **The simplicial maps** $p_\pm : \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n) \to \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$**.** We assume that the control space $B = \mathbb{R}^n$, X a compact Q-manifold with control map proj : $X \times \mathbb{R}^n \to \mathbb{R}^n$. The main technical reason for the restriction is that in this case, the Whitehead spaces defined using controlled and bounded strong deformation retractions are homotopy equivalent (Proposition 6.19). Let $f : M \to X \times \mathbb{R}^n \times S^1 \times \Delta^k$ represent a k-simplex in $\mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$. We assume that f represents an element in $\mathcal{W}h_{mb}(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$ because the two Whitehead spaces are homotopy equivalent (Proposition 6.18). Form the pull-back:

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{f}} & X \times \mathbb{R}^n \times \mathbb{R} \times \Delta^k \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \times \mathbb{R}^n \times S^1 \times \Delta^k. \end{array}$$

Set $U_+ = \bar{f}^{-1}(X \times \mathbb{R}^n \times [0,\infty) \times \Delta^k)$ and $U_- = \bar{f}^{-1}(X \times \mathbb{R}^n \times (-\infty, 0] \times \Delta^k)$. By a small homotopy of f, we may assume that $\rho : U_- \cap U_+ \to \Delta^k$ and $\rho : U_\pm \to \Delta^k$ are fibre bundle projections with Q–manifold fibres and that $U_- \cap U_+$ is a sliced Z–set in $U_\pm$. Let $\bar{p} : \bar{M} \to \mathbb{R}$ be the composition

$$\bar{p} : \bar{M} \xrightarrow{\bar{f}} X \times \mathbb{R}^n \times \mathbb{R} \times \Delta^k \xrightarrow{\text{proj}} \mathbb{R}.$$

Lemma 8.10. *There exist closed Q–submanifolds $K_\pm \subseteq U_\pm$ of $\bar{M}$ such that*

1. $\bar{p}| : K_\pm \to \mathbb{R}$ *is proper,*
2. $\rho : K_\pm \to \Delta^k$ *is a fibre bundle projection with Q–manifold fibres,*
3. $K_\pm \cap (X \times \mathbb{R}^n \times \mathbb{R} \times \Delta^k)$ *is a sliced Z–set in* $K_\pm$,
4. $K_\pm$ *dominates* $U_\pm$ *as follows: there exist homotopies* $h^\pm : U_\pm \times [0,1] \to U_\pm$ *rel* $\bar{f}^{-1}(X \times \mathbb{R}^n \times [-1,1] \times \Delta^k) \cup (K_\pm \cap (X \times \mathbb{R}^n \times \mathbb{R} \times \Delta^k))$ *such that*
   (i) $h_0^\pm = \mathrm{id}_{U_\pm}$.
   (ii) *There are* $\gamma_\pm \in \mathbb{R}$ *such that*
   $$\begin{array}{rcl} h_1^+|\bar{f}^{-1}(X \times \mathbb{R}^n \times [\gamma_+, \infty) \times \Delta^k) & = & r_1 \circ \bar{f}| \\ h_1^-|\bar{f}^{-1}(X \times \mathbb{R}^n \times (-\infty, \gamma_-] \times \Delta^k) & = & r_2 \circ \bar{f}|, \end{array}$$
   where r_1, r_2 denote the retractions
   $$\begin{array}{rcl} r_1 : X \times \mathbb{R}^n \times [\gamma_+, \infty) \times \Delta^k & \to & X \times \mathbb{R}^n \times \{\gamma_+\} \times \Delta^k \\ r_2 : X \times \mathbb{R}^n \times (-\infty, \gamma_-] \times \Delta^k & \to & X \times \mathbb{R}^n \times \{\gamma_-\} \times \Delta^k \end{array}$$
   Also, $h_1^\pm(U_\pm) \subseteq K_\pm$, and $h_t^\pm$ is a fibre preserving bounded homotopy.
5. *Set* $\partial U_+ = \bar{f}^{-1}(X \times \mathbb{R}^n \times [0,\infty) \times \partial\Delta^k)$ *and* $\partial U_- = \bar{f}^{-1}(X \times \mathbb{R}^n \times (-\infty, 0] \times \partial\Delta^k)$. *Suppose $\partial K_\pm \subseteq \partial U_\pm$ are closed Q–submanifolds of $\bar{M}$ such that the analogues of (1)–(4) hold over $\partial\Delta^k$. Then $K_\pm$ can be chosen such that $\partial K_\pm \subseteq K_\pm$.*

Proof. We will show how to construct K_+, the construction of K_- being analogous. We use the construction presented in [10], p. 42. Let $\bar{F} : \bar{M} \to \bar{M}$ be the homotopy from the identity to $\bar{f}$ and let c be the bound of $\bar{F}$ over $\mathbb{R}$. Set $K_+ = \bar{f}^{-1}(X\times\mathbb{R}^n\times[0,3c]\times\Delta^k)$ Define h as follows

$$h(x,t) = \begin{cases} x, & \text{on } \bar{f}^{-1}(X\times\mathbb{R}^n\times[0,c]\times\Delta^k) \\ \bar{F}(x,t(\alpha-1)), & \text{on } \bar{f}^{-1}(X\times\mathbb{R}^n\times\{\alpha c\}\times\Delta^k),\ 1\leq\alpha\leq 2 \\ \bar{F}(x,t), & \text{elsewhere} \end{cases}$$

Then $h_1(U_+)\subseteq K_+\cup(X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k)$ and h satisfies all the assumptions of (4) except (ii). Let ℓ be a f.p. bounded homotopy on $K_+\cup(X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k)$ starting from the identity, rel $K_+\cap(X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k)$, such that

$$\ell_1(K_+\cup(X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k))\subseteq K_+$$

(the homotopy ℓ is induced by a homotopy on $[\gamma_+,\infty)$ from the identity to $\{\gamma_+\}$, rel $\{\gamma_+\}$). Using the estimated homotopy extension property, we can extend ℓ to a f.p. bounded homotopy ℓ' on U_+ starting from the identity. Then $h_1\circ\ell'$ is a homotopy on U_+ such that

$$h_1\circ\ell' : h_1\simeq h_1\circ\ell'_1 = h_1\circ\ell_1.$$

Then the homotopy h^+, defined as the composition of the homotopies h and $h_1\ell'$, satisfies all the requirements. $\square$

Let $\zeta:\bar{M}\to\bar{M}$ be the generator of the group of covering transformations such that $\zeta(U_+)\subseteq U_+$. Since U_+ is dominated (in the sense of Lemma 8.10) by K_+, there are map

$$K_+\xrightarrow{i}U_+\xrightarrow{d}K_+$$

(with i the inclusion and $d=h_1^+$) such that $i\circ d$ is f.p., bounded homotopic to id_{U_+} Let $V_+=U_+/\sim$ where "$\sim$" is the relation generated by $(x,s,\delta)\sim(x,0,\delta)$ where $x\in X\times\mathbb{R}^n$, $s\in\mathbb{R}$, $\delta\in\Delta^k$. Let $\sigma:U_+\to V_+$ be the projection map and $L_+=\sigma(K_+)$. Since $(X\times\mathbb{R}^n\times\mathbb{R}\times\Delta^k)\cap K_+$ is a Z–subset of K_+ and σ is a CE-map, L_+ is a Q-manifold. Denote by f_+ the map induced by the composition

$$f_+ : L_+\xrightarrow{i'}V_+\xrightarrow{z}V_+\xrightarrow{d'}L_+$$

to L_+, where i', z, d' are induced by i, ζ, d respectively.

Lemma 8.11. *The pair (L_+,f_+) is a nil k-simplex over $X\times\mathbb{R}^n$.*

Proof. We will show that there is $s\in\mathbb{N}$ such that f_+^s is f.p. bounded homotopic to a retraction to $X\times\mathbb{R}^n\times\Delta^k$, rel $X\times\mathbb{R}^n\times\Delta^k$. Let $h_t^+ : U_+\to U_+$ be a f.p. bounded homotopy constructed in Lemma 8.10. Let γ_+ be as in Lemma 8.10. Set $A=\bar{f}^{-1}(X\times\mathbb{R}^n\times[0,\gamma_+]\times\Delta^k)$. Again from Lemma 8.10, $h_1^+|U_+\backslash A=\bar{f}|U_+\backslash A$. Choose $s\in\mathbb{N}$ so large that $U_+\backslash A$ contains $\zeta^s(U_+)$. Then $h_t^+\circ\zeta^s : U_+\to U_+$ is f.p. homotopy, which restricts to a homotopy on $X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k$, from ζ^s to $h_1^+\zeta^s$. But

$$h_1^+\zeta^s(U_+)\subseteq h_1^+(U_+\setminus A)=\bar{f}(U_+\setminus A)\subseteq X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k$$

Then $h_t=h_1^+(h_t^+\circ\zeta^s)i : K_+\to K_+$ is a f.p. bounded homotopy, restricting to a homotopy on $K_+\cap X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k$, from $h_1^+\zeta^s i$ to a map whose image is contained in $X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k$. But $(h_1^+\zeta i)^s\simeq h_1^+\zeta^s i$ through a f.p. bounded homotopy. Combining the two homotopies, we construct a f.p. bounded homotopy $H_t^+ : K_+\to K_+$ from $(h_1^+\zeta i)^s$ to a map whose image is contained in $X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k$. The map $h_1^+\zeta i$ induces the map f_+ on L_+. Thus H_t^+ induces a f.p. bounded homotopy $H'_t : L_+ \to L_+$ from f^s_+ to a retraction to $X\times\mathbb{R}^n\times\Delta^k$, rel $X\times\mathbb{R}^n\times\Delta^k$. Therefore (L_+, f_+) is a nil k-simplex over $X\times\mathbb{R}^n$. □

We can now define a map of simplicial sets

$$p_+ : \mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n) \to \widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n).$$

The construction of p_+ will be done inductively on skeleta.

Let $f : M \to X\times\mathbb{R}^n\times S^1$ represent a 0-simplex in $\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$. Then we define $p_+([f]) = [L_+, f_+]$, where (L_+, f_+) is the nil 0-simplex constructed in Lemma 8.11.

We assume that p_+ has been defined on the $(k-1)$-skeleton of $\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$. Let $f : M \to X\times\mathbb{R}^n\times S^1\times\Delta^k$ represent a k-simplex of $\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$. Using Lemma 8.10(5) we can construct $[L_+, f_+]$, as in Lemma 8.11, extending the construction already done over $\partial\Delta^k$.

Let $f : M \to X\times\mathbb{R}^n\times S^1\times\Delta^k$ represent an element in $\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$. The construction p_+ outlined above gives an element in the group $\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$. We will check if the resulting element is independent of certain choices made in the construction of p_+. Let $a\in\mathbb{R}$ and $U'_+ = \bar{f}^{-1}(X\times\mathbb{R}^n\times[a,\infty)\times\Delta^k)$. As in Lemma 8.10, we can construct a Q-submanifold K'_+ and a domination g^+ satisfying properties analogous to the ones listed in Lemma 8.10.

Lemma 8.12. *Let $[L'_+, f'_+]$ be the nil k-simplex constructed using the data (K'_+, g^+) and $[L_+, f_+]$ be the nil k-simplex constructed using the data (K_+, h^+). Then*

$$[L_+, f_+] = [L'_+, f'_+] \in \pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n).$$

Proof. We will split the proof of the Lemma into three cases.

Case 1. We first consider the case $U_+ = U'_+ = \bar{f}^{-1}(X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k)$. Then

$$f_+ : L_+ \xrightarrow{i} V_+ \xrightarrow{z} V_+ \xrightarrow{h} L_+$$
$$f'_+ : L'_+ \xrightarrow{i'} V_+ \xrightarrow{z} V_+ \xrightarrow{g} L'_+$$

where i,, i' are the inclusion maps, z is the map induced by the translation ζ, h and g are induced by the maps h^+_1, g^+_1, respectively. Let $M(gi)$ be the mapping cylinder of the map gi. Set $M = M(gi)/\!\sim$ where $\sim$ is the relation generated by $(x,y,t,\delta)\sim(x,y,0,\delta)$ for $(x,y,t,\delta) \in X\times\mathbb{R}^n\times[0,1]\times\Delta^k$. By approximating gi by a Z–embedding, if necessary, we assume that M is a Q-manifold and the map ρ induces a bundle map to Δ^k. The space M contains X as a Z–subset. We denote by $c : M \to L'_+$ the collapse map and by $j : L_+ \to M$, $j' : L'_+ \to M$ the inclusion maps. Define

$$i'' = i'{\circ}c : M \to V_+; \quad \kappa = j'g : V_+ \to M; \quad m = \kappa z i'' : M \to M.$$

The composition $\kappa{\circ}i''$ is f.p. bounded homotopic to the identity on V_+ (rel $X\times\mathbb{R}^n\times\Delta^k$) because

$$\kappa{\circ}i'' = j'gi'c \simeq j'c \simeq \mathrm{id}_{V_+}.$$

Thus (M,m) is a nil k-simplex and it represents an element in $\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$. By construction,

$$mj' = j'gzi'cj' = j'gzi' = j'f'_+$$

which implies that the following diagram commutes

$$\begin{array}{ccc} L'_+ & \xrightarrow{j'} & M \\ {\scriptstyle f'_+}\downarrow & & \downarrow{\scriptstyle m} \\ L'_+ & \xrightarrow{j'} & M \end{array}$$

Also by construction, $m(M) \subset j'(L'_+)$. By the germ relation on the Nil-spaces, $[M,m] = [L'_+, f'_+]$ in $\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$.

Now we will compare $[L_+, f_+]$ to $[M,m]$. By construction,

$$i''j = i'cj = i'gi \simeq i, \quad \mathrm{rel}\, X\times\mathbb{R}^n\times\Delta^k.$$

Thus

$$[L_+, f_+] = [L_+, hzi] = [L_+, hzi''j]; \quad \text{in } \pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n).$$

A similar calculation can be made for m:

$$jh \simeq j'cjh = j'gih \simeq j'g = \kappa, \quad \mathrm{rel}\, X\times\mathbb{R}^n\times\Delta^k.$$

which implies

$$[M,m] = [M, \kappa zi''] = [M, jhzi'']; \quad \text{in } \pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n).$$

There is a commutative diagram

$$\begin{array}{ccc} L_+ & \xrightarrow{j} & M \\ {\scriptstyle hzi''j}\downarrow & & \downarrow{\scriptstyle jhzi''} \\ L_+ & \xrightarrow{j} & M \end{array}$$

and $jhzi''(M) \subset L_+$. Thus $[M, jhzi''] = [L_+, hzi''j]$ in $\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$. Therefore, in $\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$,

$$[L'_+, f'_+] = [M,m] = [M, jhzi''] = [L_+, hzi''j] = [L_+, f_+].$$

Case 2. Let $|a| < 1$. For simplicity we will assume that $a > 0$, the other case follows similarly. Let K_+ dominates U_+ and K'_+ dominates U'_+. Then the space $K''_+ = K'_+ \cup \bar{f}^{-1}(X\times\mathbb{R}^n\times[0,a]\times\Delta^k)$ dominates also U_+ (just choose the homotopy to be the identity on the second set). Let $[L''_+, f''_+]$ the element constructed using K''_+. By the result in Case 1,

$$[L'_+, f'_+] = [L''_+, f''_+] \in \pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n).$$

Let $i : L'_+ \to L''_+$ be the inclusion map. Then the diagram commutes

$$\begin{array}{ccc} L'_+ & \xrightarrow{i} & L''_+ \\ {\scriptstyle f'_+}\downarrow & & \downarrow{\scriptstyle f''_+} \\ L'_+ & \xrightarrow{i} & L''_+ \end{array}$$

and $f''_+(L''_+) \subset L'_+$ because $a < 1$ and the map f''_+ is induced by the translation. Thus $[L'_+, f'_+] = [L''_+, f''_+]$ in $\widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$ and the result follows. The proof also works in the following case

General Case. We are going to show the proof when $a > 0$. The other case follows similarly. Let $b \in \mathbb{N}$ such that $0 < a - b < 1$. For $c = a - b, a - b + 1, \ldots, a$ we write

$$U_c = \bar{f}^{-1}(X \times \mathbb{R}^n \times [c, \infty) \times \Delta^k), \quad K_c = K'_+ \cup \bar{f}^{-1}(X \times \mathbb{R}^n \times [0, a] \times \Delta^k)$$

and (K_c, f_c) as the nil k-simplex constructed using K_c. By repeated applications of the methods in case 2

$$[L_+, f_+] = [L_{a-b}, f_{a-b}] = [L_{a-b+1}, f_{a-b+1}] = \cdots = [L'_+, f'_+]$$

which completes the proof. □

8.4. **The homomorphism P_* is a split epimorphism.**

Definition 8.13. Define a simplicial map

$$P = (p_+, p_-) : \mathcal{W}h(X \times \times S^1 \to \mathbb{R}^n) \to \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \times \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n).$$

Since $\widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$ satisfies the Kan condition (Lemma 8.5), it makes sense to talk about the homotopy groups of $\widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$. Following the ideas in [41] we will show that the maps $p_\pm$ induce split epimorphisms on the homotopy groups. A k-simplex (M, f) of $\widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$ such that $(\rho^{-1}(\partial_i \Delta^k), f|)$ is equivalent to the "base $(k-1)$-simplex" of $\widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$ for each $i = 0, 1, \ldots, k$ represents an element of $\pi_k \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$.

We start with some useful properties of elements of $\pi_k \mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n)$. For the rest of the chapter we will identify $\mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n)$ with the bounded Whitehead space $\mathcal{W}h_b(X \times \mathbb{R}^n \to \mathbb{R}^n)$ (Proposition 6.19).

Lemma 8.14. *Let $g : N \to X \times \Delta^k$ be a strong deformation retraction that represents an element in $\pi_k \mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n)$ with $\rho : N \to \Delta^k$ a fiber bundle surjection with Q-manifold fibers. Let $g' : N \to X \times \mathbb{R}^n \times \Delta^k$ be a f.p. retraction which is bounded distance apart from id_N, when distances are measured in $\mathbb{R}^n \times \Delta^k$ such that $\rho^{-1}(\partial_i \Delta^k)$ is in the same component as the "base $(k-1)$-simplex" of $\mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n)$. Then g' represents an element in $\pi_k \mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n)$ such that $[g] = [g']$.*

Proof. Let $F : N \times [0, 1] \to N$ be a f.p. bounded homotopy between id_N and g. Then $g' \circ F$ is a bounded homotopy, rel $X \times \mathbb{R}^n \times \Delta^k$, from g to g'. Thus $[g] = [g'] \in \pi_k \mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n)$. □

Lemma 8.15. *Let M be a Hilbert cube manifolds, containing $X \times \mathbb{R}^n \times \Delta^k$, equipped with bundle maps to Δ^k and $f_i : M \to M$, $i = 1, 2$, be f.p. maps, which are the identity on $X \times \mathbb{R}^n \times \Delta^k$, such that there are strong deformation retractions*

$$r_i : T(f_i) \to X \times \mathbb{R}^n \times S^1 \times \Delta^k$$

representing elements in $\pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$. If there is a f.p. bounded homotopy between f_1 and f_2, then $[r_1] = [r_2]$ in $\pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$.

Proof. By Proposition 12.2, there is a f.p. bounded homotopy equivalence $f : T(f_1) \to T(f_2)$ which can be chosen to be rel $X \times \mathbb{R}^n \times S^1 \times \Delta^k$. The map f can be written as a composition of f.p. CE-maps and their inverses. Thus $[r_1 \circ f] = [r_2]$ and by Lemma 8.14, $[r_1] = [r_1 \circ f]$. The result follows. $\square$

Corollary 8.16. *Let M be a Q-manifold equipped with a bundle map to Δ^k and $f : M \to M$ be a f.p. map which is f.p. bounded (over $\mathbb{R}^n$) homotopic rel $X \times \mathbb{R}^n \times \Delta^k$, to a retraction $r : M \to X \times \mathbb{R}^n \times \Delta^k$. If there is a deformation retraction*

$$r' : T(f) \to X \times \mathbb{R}^n \times S^1 \times \Delta^k$$

representing an element in $\pi_k \mathcal{W}h(p : X \to \mathbb{R}^n)$ then $[r'] = 0$.

Proof. Let $i : X \times \mathbb{R}^n \times \Delta^k \to M$ be the inclusion map. Then there is a sequence of f.p. bounded homotopy equivalences (Proposition 12.2)

$$T(f) \to T(ir) \to T(ri) = X \times \mathbb{R}^n \times S^1 \times \Delta^k$$

and each one can be written as a composition of bounded CE-maps and their inverses. The result follows from Lemma 8.14. $\square$

8.4.1. *Definition of the maps $j_\pm : \pi_* \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \to \pi_* \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$.* We will define Bass-Heller-Swan type injections from the homotopy groups of the Nil-spaces to the homotopy groups of the Whitehead spaces.

We start by defining

$$j_+ : \pi_k \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \to \pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n).$$

Actually the image of j_+ will be in $\pi_k \mathcal{W}h_{mb}(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$ and it will be represented by a bounded (over $\mathbb{R}^n$) strong deformation retraction. Let $[M, f]$ represent an element in $\pi_k \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$. We assume that the map f is a sliced Z–embedding. Define

$$j_+([M, f]) = (r : T'(f) \to X \times S^1 \times \Delta^k)$$

where r is the retraction given in Proposition 8.6.

1. j_+ is well-defined: Let (M, f) and (N, g) be two nil k-simplices that represent elements in $\pi_k \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$ such that $M \subset N$, $g(N) \subseteq M$, $g|M = f$. We write $i : M \to N$ for the inclusion map and $g : N \to M$ for the map induced by g. Then $T'(ig) \cong T'(gi) = T'(f)$ and $j_+([M, f]) = j_+([N, g])$. Thus j_+ maps elements that are germ equivalent to the same element. Using Lemma 8.15, it is also easy to see that j_+ respects the homotopy relation in $\pi_k \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$.
2. j_+ is a group homomorphism if $k > 1$ and a monoid homomorphism for $k = 0$: It follows from the geometric definition of addition.

Similarly we define

$$j_- : \pi_k \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \to \pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n).$$

by $j_-([M, f]) = (r : T(f) \to X \times \mathbb{R}^n \times S^1 \times \Delta^k)$. Define

$$J = j_+ + j_- : \pi_k \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \times \pi_k \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \to \pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n).$$

Proposition 8.17. *The map*

$$P : \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n) \to \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \times \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$$

induces a split monoid epimorphism on the monoid of path components and a split epimorphism on the higher homotopy groups with splitting induced by J. Furthermore, Im(j_+) and Im(j_-) are two orthogonal summands of $\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$.

Proof. We will show that $p_+j_+ = \mathrm{id}, p_+j_- = 0$. The equalities $p_-j_- = \mathrm{id}, p_-j_+ = 0$ follow similarly.

Let $[M,f]\in\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$. Then $j_+([M,f])$ is represented by the reversed mapping torus of f. Form the pull back

$$\begin{array}{ccc} \overline{T}'(f) & \xrightarrow{\bar{r}} & X\times\mathbb{R}^n\times\mathbb{R}\times\Delta^k \\ \downarrow & & \downarrow \\ T'(f) & \xrightarrow{r} & X\times\mathbb{R}^n\times S^1\times\Delta^k \end{array}$$

By Remark 8.7, $\bar{r}^{-1}(X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k) = \overline{T}_0'^{+}(f)$ which is homotopy equivalent to the "base" $M\times\{0\}\times\{0\}$ via the collapse map. So we can choose M to correspond to the space K_+ of Lemma 8.10. Thus $p_+j_+([M,f]) = [M,c']$, where

$$c' : M \xrightarrow{i} \overline{T}_0'^{+}(f) \xrightarrow{\zeta} \overline{T}_0'^{+}(f) \xrightarrow{c} M$$

which is equal to f.

For the other composition, let $[M,f]$ be an element as before with $f^s \simeq ir$ rel $X\times\mathbb{R}^n\times\Delta^k$. Again by Remark 8.7, the infinite cyclic cover of $T(f)$ is $\overline{T}(f)$ and $\bar{r}^{-1}(X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k) = T_0^+(f)$. But there is a sequence of f.p. bounded homotopy equivalences rel $X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k$ ([21], Corollary 2.5, [41], Proposition 3.3)

$$T_0^+(f) \to T_0^+(f^s) \to T_0^+(ir) \to X\times\mathbb{R}^n\times[0,\infty)\times\Delta^k.$$

Thus for the definition of p_+ we can choose $L_+ = X\times\mathbb{R}^n\times[0,1]\times\Delta^k$ and the map f_+ to be the projection. Thus $p_+j_-([M,f]) = 0$. $\square$

8.5. **Homotopy inverses for $\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$.** For constructing homotopy inverses, we will use the geometric construction of inverses of nil elements presented in [41], Lemma 2.11. Let $[M,f]$ represent an element in $\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$ and $T' = T(f)/\sim$, where the relation $\sim$ is generated by

$$(x,y,t,\delta) \sim (x,y,0,\delta);\ (x,y,t,\delta)\in X\times\mathbb{R}^n\times S^1\times\Delta^k.$$

Mather's trick implies that the map f induces a map ϕ on T' which is homotopic to the identity. By replacing the natural inclusion $M \to T(f)$ by a sliced Z–embedding we can assume that $T'\cup_M T'$ is a Q-manifold equipped with a bundle surjection to Δ^k. Then the pair $(T'\cup_M T', \phi\cup\phi)$ represents a nil k-simplex.

We will first construct inverses in $\pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$

Lemma 8.18. *The map μ induces an abelian group structure on the set of path components $\pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$.*

Proof. We will show that $(T'\cup_M T', \phi\cup\phi)$, constructed above, is the inverse of $[M,f]$. Since j_+ induces a monomorphism on the monoid of path components, it is enough to show that

$$j_+[M,f] + j_+[T'\cup_M T', \phi\cup\phi] = 0 \in \pi_0\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n).$$

The assumptions of Remark 7.1 are satisfied in the splitting $T'\cup_M T'$ and the Higher Sum Theorem (Theorem 7.2) implies

$$j_+[T'\cup_M T', \phi\cup\phi] = j_+[T', \phi] + j_+[T', \phi] - j_+[M, f] = -j_+[M, f]$$

which completes the proof of the claim. □

Corollary 8.19. *Let x be any simplex in the component of the identity of $\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$. Then the map*

$$\mu(-,x) : \pi_k(\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n), e) \to \pi_k(\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n), x)$$

is an isomorphism.

Proof. Let y be the inverse of x in $\pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$. Then the inverse of $\mu(-,x)$ is given by $\sigma_*\mu(-,y)$ where σ_* is the isomorphism induced by a path σ from $\mu(x,y)$ to the base point e. □

Proposition 8.20. *The map μ induces an abelian group-like structure on the simplicial space $\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$.*

Proof. It is enough to show that the map

$$\alpha : \widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)\times\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n) \to \widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)\times\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$$

given by $\alpha(x,y) = (x, x+y)$ is a homotopy equivalence ([48], p. 119). Actually it is enough to show that α induces isomorphism on homotopy groups. By Lemma 8.18, α induces an isomorphism on the group of path components. Using Corollary 8.19, it is enough to show that α induces an isomorphism on the higher homotopy groups of the component based on e. There is a similar map α' defined on $\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$ which is a homotopy equivalence, since $\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$ is an abelian group-like space. Since J and $(p_+)_*$ are monoid homomorphisms,

$$\alpha'\circ(J,J) = (J,J)\circ\alpha, \quad ((p_+)_*, (p_+)_*)\circ\alpha' = \alpha\circ((p_+)_*, (p_+)_*).$$

Thus α is an isomorphism and the result follows. □

Combining the result of Proposition 8.20 with the result of Proposition 8.17 we show

Proposition 8.21. *The map*

$$P_* : \pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n) \to \pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)\times\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$$

is a split group epimorphism for all $k \geq 0$. with splitting induced by J. Furthermore, $Im(j_+)$ and $Im(j_-)$ are two orthogonal summands of $\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$ for all $k \geq 0$.

9. Transfers

Fix $m \in \{5, 6, \ldots, \infty\}$ but suppress it from the notation as usual. Let B be a fixed manifold with $\dim B < \infty$ and assume that B is either a closed manifold or a product $Y \times \mathbb{R}^n$ of a closed manifold Y and $\mathbb{R}^n$ (with the standard metric) and that B has the product metric. Let X denote a fixed compact Hilbert cube manifold.

9.1. **Transfers for manifold approximate fibrations.** For each nonzero integer s define the standard s–fold covering projection $s : S^1 \to S^1$ by $e^{2\pi i t} \mapsto e^{2\pi i s t}$. There are naturally induced s–fold transfer maps

- $\mathrm{tr}^s : \mathrm{MAF}(S^1 \times B) \to \mathrm{MAF}(S^1 \times B)$ and
- $\mathrm{tr}^s : \mathrm{Map}(S^1, \mathrm{MAF}(B)) \to \mathrm{Map}(S^1, \mathrm{MAF}(B))$.

First $\mathrm{tr}^s : \mathrm{MAF}(S^1 \times B) \to \mathrm{MAF}(S^1 \times B)$ is defined as follows. If $p : M \to S^1 \times B \times \Delta^k$ is a k–simplex of $\mathrm{MAF}(S^1 \times B)$, then $\mathrm{tr}^s(p)$ is defined by the pull-back diagram:

$$\begin{array}{ccc} \bar{M} & \longrightarrow & M \\ {\scriptstyle \mathrm{tr}^s(p)}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 \times B \times \Delta^k & \xrightarrow{s \times \mathrm{id}_{B \times \Delta^k}} & S^1 \times B \times \Delta^k \end{array}$$

Second, transfers on the mapping spaces are defined by precomposition with the covering projection s. For example, if $f : S^1 \times \Delta^k \to \mathrm{MAF}(B)$ is a k–simplex of $\mathrm{Map}(S^1, \mathrm{MAF}(B))$, then $\mathrm{tr}^s(f)$ is the composition

$$\mathrm{tr}^s(f) : S^1 \times \Delta^k \xrightarrow{s \times \mathrm{id}_{\Delta^k}} S^1 \times \Delta^k \xrightarrow{f} \mathrm{MAF}(B).$$

Recall from §4.3 the classifying homotopy equivalence

$$\Psi : \mathrm{MAF}(S^1 \times \mathbb{R}^n) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})).$$

Lemma 9.1. *For each nonzero integer s the following diagram homotopy commutes:*

$$\begin{array}{ccc} \mathrm{MAF}(S^1 \times \mathbb{R}^n) & \xrightarrow{\mathrm{tr}^s} & \mathrm{MAF}(S^1 \times \mathbb{R}^n) \\ {\scriptstyle \Psi}\downarrow & & \downarrow{\scriptstyle \Psi} \\ \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})) & \xrightarrow{\mathrm{tr}^s} & \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})) \end{array}$$

Proof. We first interpret $\mathrm{tr}^s : \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})) \to \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1}))$ in the model of 4.12 in which a k–simplex of $\mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1}))$ is given by a projection $p : M \to S^1 \times \mathbb{R}^{n+1} \times \Delta^k$ having certain properties. In this model, $\mathrm{tr}^s(p)$ is given by the pull–back construction:

$$\begin{array}{ccc} \bar{M} & \longrightarrow & M \\ {\scriptstyle \mathrm{tr}^s(p)}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 \times \mathbb{R}^{n+1} \times \Delta^k & \xrightarrow{s \times \mathrm{id}_{\mathbb{R}^{n+1} \times \Delta^k}} & S^1 \times \mathbb{R}^{n+1} \times \Delta^k \end{array}$$

With this observation we can check that the diagram commutes after replacing Ψ by the homotopic map Ψ' of Comment 4.15. For suppose $p : M \to S^1 \times \mathbb{R}^n \times \Delta^k$ is a k–simplex of $\mathrm{MAF}(S^1 \times \mathbb{R}^n)$. Then $\Psi'(p)$ is given by a pull-back construction, and then another pull-back construction is formed to give $\mathrm{tr}^s(\Psi'(p))$. These two pull-back diagrams have the form

$$\begin{array}{ccc} \bar{N} & \xrightarrow{\mathrm{tr}^s(\Psi'(p))} & S^1 \times \mathbb{R}^{n+1} \times \Delta^k \\ \big\downarrow & & \big\downarrow{\scriptstyle s \times \mathrm{id}_{\mathbb{R}^{n+1} \times \Delta^k}} \\ N & \xrightarrow{\Psi'(p)} & S^1 \times \mathbb{R} \times \mathbb{R}^n \times \Delta^k \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{e}' \times \mathrm{id}_{\mathbb{R}^n \times \Delta^k}} \\ S^1 \times M & \longrightarrow & S^1 \times S^1 \times \mathbb{R}^n \times \Delta^k. \end{array}$$

On the other hand, $\mathrm{tr}^s(p)$ is given by a pull-back diagram

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{s}} & M \\ {\scriptstyle \mathrm{tr}^s(p)}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ S^1 \times \mathbb{R}^n \times \Delta^k & \xrightarrow{s \times \mathrm{id}_{\mathbb{R}^n \times \Delta^k}} & S^1 \times \mathbb{R}^n \times \Delta^k. \end{array}$$

Note that by crossing the spaces in this diagram with S^1, crossing the horizontal maps by $s : S^1 \to S^1$ and by crossing the vertical maps by id_{S^1}, we obtain another pull-back diagram:

$$\begin{array}{ccc} S^1 \times \bar{M} & \xrightarrow{s \times \bar{s}} & M \\ {\scriptstyle \mathrm{id}_{S^1} \times \mathrm{tr}^s(p)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{S^1} \times p} \\ S^1 \times S^1 \times \mathbb{R}^n \times \Delta^k & \xrightarrow{s \times s \times \mathrm{id}_{\mathbb{R}^n \times \Delta^k}} & S^1 \times S^1 \times \mathbb{R}^n \times \Delta^k. \end{array}$$

It follows that $\Psi'(\mathrm{tr}^s(p))$ appears in the following pull-back diagram:

$$\begin{array}{ccc} N' & \xrightarrow{\Psi'(\mathrm{tr}^s(p))} & S^1 \times \mathbb{R} \times \mathbb{R}^n \times \Delta^k \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{e}' \times \mathrm{id}_{\mathbb{R}^n \times \Delta^k}} \\ S^1 \times \bar{M} & \xrightarrow{\mathrm{id}_{S^1} \times \mathrm{tr}^s(p)} & S^1 \times S^1 \times \mathbb{R}^n \times \Delta^k \\ {\scriptstyle s \times \bar{s}}\big\downarrow & & \big\downarrow{\scriptstyle s \times s \times \mathrm{id}_{\mathbb{R}^n \times \Delta^k}} \\ M & \xrightarrow{\mathrm{id}_{S^1} \times p} & S^1 \times S^1 \times \mathbb{R}^n \times \Delta^k. \end{array}$$

To see that $\Psi'(\mathrm{tr}^s(p)) = \mathrm{tr}^s(\Psi'(p))$ one just has to check that the outer rectangles in the two double pull-back diagrams above are the same. This follows from the fact that the compositions of the maps along the bottoms of the two diagrams are equal; i.e.,

$$(s \times s)\,\mathrm{e}' = \mathrm{e}'(s \times \mathrm{id}_{\mathbb{R}}) : S^1 \times \mathbb{R} \to S^1 \times S^1.$$

$\square$

Recall from §4.3 that there is a fibration sequence

$$\Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) \xrightarrow{I} \mathrm{Map}(S^1, \mathrm{MAF}(\mathbb{R}^{n+1})) \xrightarrow{E} \mathrm{MAF}(\mathbb{R}^{n+1})$$

where E is evaluation at the basepoint of S^1. Let

$$i : \Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) \to \mathrm{MAF}(S^1 \times \mathbb{R}^n)$$

be a simplicial map such that $\Psi i \simeq I$ (thus, i is well-defined up to homotopy). Theorem 4.21 implies that there is a homotopy commuting diagram

$$\begin{array}{ccccc} \Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) & \xrightarrow{i} & \mathrm{MAF}(S^1\times\mathbb{R}^n) & \xrightarrow{u} & \mathrm{MAF}(\mathbb{R}^{n+1}) \\ {\scriptstyle =}\downarrow & & {\scriptstyle\Psi}\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle =} \\ \Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) & \xrightarrow{I} & \mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1})) & \xrightarrow{E} & \mathrm{MAF}(\mathbb{R}^{n+1}) \end{array}$$

where u is the unwrapping map. Moreover, the fibration

$$E : \mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1})) \to \mathrm{MAF}(\mathbb{R}^{n+1})$$

has a section $c : \mathrm{MAF}(\mathbb{R}^{n+1}) \to \mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1}))$ which takes a k–simplex p to the constant map with image p. According to Theorem 4.11 this section is compatible with the homotopy splitting of u, the wrapping-up map w, so that there is a homotopy commuting diagram

$$\begin{array}{ccc} \mathrm{MAF}(\mathbb{R}^{n+1}) & \xrightarrow{w} & \mathrm{MAF}(S^1\times\mathbb{R}^n) \\ {\scriptstyle =}\downarrow & & {\scriptstyle\simeq}\downarrow{\scriptstyle\Psi} \\ \mathrm{MAF}(\mathbb{R}^{n+1}) & \xrightarrow{c} & \mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1})). \end{array}$$

With these preliminaries the effect of the s–fold transfer on $\mathrm{MAF}(S^1\times\mathbb{R}^n)$ can now be described. In what follows the s–fold transfer on loop spaces is defined by precomposing a loop with the s-fold cover of S^1. In particular,

$$\mathrm{tr}^s : \Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) \to \Omega\,\mathrm{MAF}(\mathbb{R}^{n+1})$$

is defined by considering $\Omega\,\mathrm{MAF}(\mathbb{R}^{n+1})$ as a subspace of $\mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1}))$.

Theorem 9.2. 1. *The following diagram homotopy commutes:*

$$\begin{array}{ccc} \Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) & \xrightarrow{i} & \mathrm{MAF}(S^1\times\mathbb{R}^n) \\ {\scriptstyle \mathrm{tr}^s}\downarrow & & \downarrow{\scriptstyle\mathrm{tr}^s} \\ \Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) & \xrightarrow{i} & \mathrm{MAF}(S^1\times\mathbb{R}^n). \end{array}$$

2. **(Transfer Additivity on Image of i)** *On homotopy groups*

$$\mathrm{tr}^s_*\, i_* : \pi_k\Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) \to \pi_k\,\mathrm{MAF}(S^1\times\mathbb{R}^n)$$

is given by $[p] \mapsto i_*(s\cdot[p])$ *for* $k \geq 0$.

3. **(Transfer invariance of Wrapping Up)** *The composition*

$$\mathrm{MAF}(\mathbb{R}^{n+1}) \xrightarrow{w} \mathrm{MAF}(S^1\times\mathbb{R}^n) \xrightarrow{\mathrm{tr}^s} \mathrm{MAF}(S^1\times\mathbb{R}^n)$$

is homotopic to w.

Proof. (1) Since $\Psi\circ i \simeq I$ by definition and $\Psi\circ\mathrm{tr}^s \simeq \mathrm{tr}^s\circ\Psi$ by Lemma 9.1, it suffices to observe that

$$\begin{array}{ccc} \Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) & \xrightarrow{I} & \mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1})) \\ {\scriptstyle \mathrm{tr}^s}\downarrow & & \downarrow{\scriptstyle\mathrm{tr}^s} \\ \Omega\,\mathrm{MAF}(\mathbb{R}^{n+1}) & \xrightarrow{i} & \mathrm{Map}(S^1,\mathrm{MAF}(\mathbb{R}^{n+1})) \end{array}$$

commutes.

(2) From (1) it follows that $\mathrm{tr}^s_* i_*([p]) = i_* \mathrm{tr}^s_*([p])$. The very definition of the transfer on loop spaces implies that $\mathrm{tr}^s_*([p]) = s \cdot [p] \in \pi_k \Omega \,\mathrm{MAF}(\mathbb{R}^{n+1})$. This is valid even for $k = 0$ when π_0 of a loop space is given its usual group structure.

(3) From Lemma 9.1 and the fact that $\Psi \circ w = c$ (which follows from Theorem 4.11 as noted above) it follows that $\Psi \circ \mathrm{tr}^s \circ w \simeq \mathrm{tr}^s \circ \Psi \circ w \simeq \mathrm{tr}^s \circ c$. Obviously $\mathrm{tr}^s \circ c = c$ so $\Psi \circ \mathrm{tr}^s \circ w \simeq c \simeq \Psi \circ w$. Since Ψ is a homotopy equivalence, $\mathrm{tr}^s \circ w \simeq w$ as required.

$\square$

9.2. **Transfers for Whitehead spaces.** The s–fold covering projection $s : S^1 \to S^1$ also induces s–fold transfer maps for Whitehead spaces

- $\mathrm{tr}^s : \mathcal{W}h(X \times S^1 \times B \to B) \to \mathcal{W}h(X \times S^1 \times B \to B)$,
- $\mathrm{tr}^s : \mathcal{W}h(X \times S^1 \times B \to S^1 \times B) \to \mathcal{W}h(X \times S^1 \times B \to S^1 \times B)$, and
- $\mathrm{tr}^s : \mathrm{Map}(S^1, \mathcal{W}h(X \times B \to B)) \to \mathrm{Map}(S^1, \mathcal{W}h(X \times B \to B))$

in analogy with the transfer maps defined for simplicial sets of manifold approximate fibrations. To define the transfer map on the Whitehead space, let

$$f : M \to X \times S^1 \times B \times \Delta^k$$

be a k-simplex in $\mathcal{W}h(X \times S^1 \times B \to B)$ and form the pull-back

$$\begin{tikzcd} \bar{M} \arrow[r] \arrow[d, "\mathrm{tr}^s(f)"'] & M \arrow[d, "f"] \\ X \times S^1 \times B \times \Delta^k \arrow[r, "\mathrm{id}_X \times s \times \mathrm{id}_{\mathbb{R}^n \times \Delta^k}"] & X \times S^1 \times B \times \Delta^k. \end{tikzcd}$$

The same construction defines the transfer on $\mathcal{W}h(X \times S^1 \times B \to S^1 \times B)$. This is compatible with the forget control map φ of §6.4. so that there is a commuting diagram

$$\begin{tikzcd} \mathcal{W}h(X \times S^1 \times B \to S^1 \times B) \arrow[r, "\mathrm{tr}^s"] \arrow[d, "\varphi"'] & \mathcal{W}h(X \times S^1 \times B \to S^1 \times B) \arrow[d, "\varphi"] \\ \mathcal{W}h(X \times S^1 \times B \to B) \arrow[r, "\mathrm{tr}^s"] & \mathcal{W}h(X \times S^1 \times B \to B). \end{tikzcd}$$

The transfer on the mapping space $\mathrm{Map}(S^1, \mathcal{W}h(X \times B \to B))$ is defined by pre-composition with s and the transfer on the loop space $\Omega \mathcal{W}h(X \times B \to B)$ is defined by considering $\Omega \mathcal{W}h(X \times B \to B)$ as a subspace of $\mathrm{Map}(S^1, \mathcal{W}h(X \times B \to B))$.

Recall from §6.7 the classifying homotopy equivalence

$$\Psi : \mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to S^1 \times \mathbb{R}^n) \to \mathrm{Map}(S^1, \mathcal{W}h(X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1})).$$

Lemma 9.3. *For each nonzero integer s the following diagram homotopy commutes:*

$$\begin{tikzcd} \mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to S^1 \times \mathbb{R}^n) \arrow[r, "\mathrm{tr}^s"] \arrow[d, "\Psi"'] & \mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to S^1 \times \mathbb{R}^n) \arrow[d, "\Psi"] \\ \mathrm{Map}(S^1, \mathcal{W}h(X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1})) \arrow[r, "\mathrm{tr}^s"] & \mathrm{Map}(S^1, \mathcal{W}h(X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1})) \end{tikzcd}$$

Proof. From §6.7 there is a commuting diagram

$$\begin{array}{ccc}
\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n) & \longrightarrow & \mathrm{MAF}^Q(S^1\times\mathbb{R}^n) \\
\Psi\big\downarrow & & \big\downarrow\Psi \\
\mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})) & \longrightarrow & \mathrm{Map}(S^1, \mathrm{MAF}^Q(\mathbb{R}^{n+1})).
\end{array}$$

Since the transfers for Whitehead spaces and manifold approximate fibrations are defined by pull-back constructions, there is a commuting diagram

$$\begin{array}{ccc}
\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n) & \xrightarrow{\mathrm{tr}^s} & \mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n) \\
\big\downarrow & & \big\downarrow \\
\mathrm{MAF}^Q(S^1\times\mathbb{R}^n) & \xrightarrow{\mathrm{tr}^s} & \mathrm{MAF}^Q(S^1\times\mathbb{R}^n).
\end{array}$$

Likewise there is a commuting diagram

$$\begin{array}{ccc}
\mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})) & \xrightarrow{\mathrm{tr}^s} & \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Map}(S^1, \mathrm{MAF}^Q(\mathbb{R}^{n+1})) & \xrightarrow{\mathrm{tr}^s} & \mathrm{Map}(S^1, \mathrm{MAF}^Q(\mathbb{R}^{n+1})).
\end{array}$$

The lemma follows by combining these three diagrams with the homotopy commuting diagram of Lemma 9.1. □

Recall from §6.7 that there is a fibration sequence

$$\begin{array}{rl}
\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) \xrightarrow{I} & \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})) \\
\xrightarrow{E} & \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})
\end{array}$$

where E is evaluation at the basepoint of S^1. Let $i : \Omega\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) \to \mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n)$ be a simplicial map such that $\Psi i \simeq I$ (thus, i is well-defined up to homotopy). Theorem 6.9 implies that there is a homotopy commuting diagram

$$\begin{array}{ccc}
\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) & \xrightarrow{=} & \Omega\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) \\
i\big\downarrow & & \big\downarrow I \\
\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n) & \underset{\simeq}{\xrightarrow{\Psi}} & \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})) \\
u\big\downarrow & & \big\downarrow E \\
\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) & \xrightarrow{=} & \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})
\end{array}$$

where u is the unwrapping map. Moreover, the fibration

$$E : \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})) \to \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})$$

has a section

$$c : \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) \to \mathrm{Map}(S^1, \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}))$$

which takes a k–simplex p to the constant map with image p. According to Theorem 6.8 this section is compatible with the homotopy splitting of u, the wrapping-up map w, so that there is a homotopy commuting diagram

$$\begin{array}{ccc} \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) & \xrightarrow{w} & \mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n) \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \simeq\ \Psi} \\ \mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) & \xrightarrow{c} & \mathrm{Map}(S^1,\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})). \end{array}$$

With these preliminaries the effect of the s–fold transfer on $\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to\mathbb{R}^n)$ can now be described.

Theorem 9.4. 1. *The following diagram homotopy commutes:*

$$\begin{array}{ccc} \Omega\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) & \xrightarrow{i} & \mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n) \\ {\scriptstyle \mathrm{tr}^s}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{tr}^s} \\ \Omega\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) & \xrightarrow{i} & \mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n). \end{array}$$

2. **(Transfer Additivity on Image of i)** *On homotopy groups*

$$\mathrm{tr}^s_*\, i_* : \pi_k\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1})\to\pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n)$$

is given by $[p]\mapsto i_*(s\cdot[p]) = s\cdot i_*([p])$ *for* $k\geq 0$.

3. **(Transfer Invariance of Wrapping Up)** *The composition*

$$\begin{aligned}\mathcal{W}h(X\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}) &\xrightarrow{w} \mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n)\\ &\xrightarrow{\mathrm{tr}^s} \mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n)\end{aligned}$$

is homotopic to w.

Proof. The proof is analogous to the proof of Theorem 9.2. There is just one additional point which needs to be made. In (2) the equation $i_*(s\cdot[p]) = s\cdot i_*([p])$ is to have the following interpretation when $k=0$. On the left-hand side the multiplication s times $[p]$ takes place with $\pi_0\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n)$ given the usual group structure of π_0 of a loop space (i.e., induced by loop concatenation). On the right-hand side $s\cdot i_*([p])$ is the multiplication in $\pi_0\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to S^1\times\mathbb{R}^n)$ discussed in §6.2. Equality follows from Proposition 6.10. □

We will now explain how the transfer acts nilpotently on image of the nil elements in the homotopy groups of the Whitehead space. Recall from §8.4 that the simplicial map

$$P:\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to\mathbb{R}^n)\to\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\to\mathbb{R}^n)\times\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\to\mathbb{R}^n)$$

is a split epimorphism on homotopy groups. This is shown by defining homomorphisms

$$j_\pm:\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\to\mathbb{R}^n)\to\pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to\mathbb{R}^n)$$

whose images are orthogonal summands of $\mathcal{W}h(X\times\mathbb{R}^n\to\mathbb{R}^n)$. Then the map

$$J=j_+\oplus j_-:\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\to\mathbb{R}^n)\oplus\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\to\mathbb{R}^n)\to\pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to\mathbb{R}^n)$$

is a right inverse of P_*, $P_*J=1$. In this section we will show that transfer maps act nilpotently on the image of J.

Theorem 9.5. 1. *For each* $x\in\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\to\mathbb{R}^n)$, *there exists* $s'\in\mathbb{Z}$ *such that*

$$\mathrm{tr}^s(j_+(x)) = \mathrm{tr}^s(j_-(x)) = 0\in\pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to\mathbb{R}^n),\ \textit{for all } s>s'.$$

2. **(Transfer Nilpotency on Image of J)** *For each*

$$(x,y)\in\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\to\mathbb{R}^n)\oplus\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\to\mathbb{R}^n),$$

there exists $s'\in\mathbb{Z}$ such that $\mathrm{tr}^s(J(x,y))=0\in\pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to\mathbb{R}^n)$ for all $s>s'$.

Proof. (1) Let (M,f) be a nil k-simplex representing the element $x\in\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\to\mathbb{R}^n)$. We may assume that $f:M\to M$ is a f.p. Z–embedding. Then the image of $j_+(x)$ is given by

$$j_+(x)=(r:T'(f)\to X\times\mathbb{R}^n\times S^1\times\Delta^k)$$

where r is the deformation retraction constructed in Proposition 8.6. For each $s\in\mathbb{Z}$, the transfer $\mathrm{tr}^s(j_+(x))$ is given by the pull-back diagram

$$\begin{array}{ccc} T'_s(f) & \xrightarrow{\bar{r}} & X\times\mathbb{R}^n\times S^1\times\Delta^k \\ \downarrow & & \downarrow \\ T'(f) & \xrightarrow{r} & X\times\mathbb{R}^n\times S^1\times\Delta^k \end{array}$$

By a repeated application of Corollary 2.5 in [21] (also [10], p. 35, [41], Lemma 1.10), there is an ANR M, equipped with a map to Δ^k, and f.p. CE-maps

$$T'_s(f)\xleftarrow{\alpha}M\xrightarrow{\beta}T'(f^s)$$

such that composition

$$T'_s(f)\hookrightarrow M\xrightarrow{\beta}T'(f^s)$$

is a f.p. bounded (over $\mathbb{R}^n$) homotopy equivalence. Because (M,f) is a nil k-simplex, there is a positive integer s' such that $f^{s'}$ is f.p. bounded homotopic to a retraction σ to $X\times\mathbb{R}^n\times\Delta^k$, rel $X\times\mathbb{R}^n\times\Delta^k$. Thus for each $s>s'$, f^s has the same properties. Proposition 8.6 and Corollary 8.16 imply that there is a strong deformation retraction

$$r':T'(f^s)\to X\times\mathbb{R}^n\times S^1\times\Delta^k$$

such that $[r']=0$ in $\pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to\mathbb{R}^n)$. By Lemma 8.14, the composition

$$r'':T'_s(f)\hookrightarrow M\xrightarrow{\beta}T'_s(f)\xrightarrow{r}X\times\mathbb{R}^n\times S^1\times\Delta^k$$

represents an element in $\pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to\mathbb{R}^n)$ such that $[r'']=[\bar{r}]$. At the same time, $[r'']=[r']=0$. Thus

$$[\bar{r}]=[r']=0\in\pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to\mathbb{R}^n).$$

Part (2) follows from (1) $\square$

Remark 9.6. Let (M,f) be a k-nil simplex over $X\times\mathbb{R}^n$ representing an element x in $\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\to\mathbb{R}^n)$ and

$$r:T'(f)\to X\times S^1\times\mathbb{R}^n\times\Delta^k$$

be the strong deformation retraction constructed in Proposition 8.6. The methods used in the proof of Theorem 9.5 show that $\mathrm{tr}^s(j_+(x))$ can be represented

1. by a retraction $r^+:T'(f^s)\to X\times S^1\times\mathbb{R}^n\times\Delta^k$, if $s>0$,
2. by a retraction $r^-:T(f^s)\to X\times S^1\times\mathbb{R}^n\times\Delta^k$, if $s<0$.

Corollary 9.7. *The images of J, w_* and i_* are orthogonal direct summands of $\pi_k\mathcal{W}h(X\times S^1\times\mathbb{R}^n\to\mathbb{R}^n)$.*

Proof. Let $x \in \mathrm{Im}(w_*) \cap \mathrm{Im}(J)$. Then by Theorem 9.4(3), $\mathrm{tr}^s(x) = x$ for all $s \in \mathbb{Z}$ and by Theorem 9.5(2), there is $s \in \mathbb{Z}$ such that $\mathrm{tr}^s(x) = 0$. Thus $x = 0$ and $\mathrm{Im}(w_*)$ is orthogonal to $\mathrm{Im}(J)$.

Let $x \in \mathrm{Im}(i_*) \cap \mathrm{Im}(J)$. Then by Theorem 9.5(2), there is $s' \in \mathbb{Z}$ such that for all $s > s'$, $\mathrm{tr}^s(x) = 0$. For such s, $\mathrm{tr}^s(x) = \mathrm{tr}^{s+1}(x) = 0$. By Theorem 9.4(2),

$$sx = \mathrm{tr}^s(x) = \mathrm{tr}^{s+1}(x) = (s+1)x \Rightarrow x = 0.$$

A similar calculation shows that $\mathrm{Im}(i_*) \cap \mathrm{Im}(w_*) = \{0\}$. $\square$

Form the composition

$$\begin{aligned}
\hat{r} : &\mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to \mathbb{R}^n) \xrightarrow{r} \mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to S^1 \times \mathbb{R}^n) \xrightarrow{\Psi} \\
&\mathrm{Map}(S^1, \mathcal{W}h(X \times S^1 \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1})) \xrightarrow{\simeq} \\
&\Omega\mathcal{W}h(X \times S^1 \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \times \mathcal{W}h(X \times S^1 \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \xrightarrow{\mathrm{proj}} \\
&\Omega\mathcal{W}h(X \times S^1 \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \xrightarrow{\simeq} \mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n)
\end{aligned}$$

where the last equivalence follows from Theorem 6.12. We summarize the splitting results obtained so far.

Theorem 9.8. 1. *There is a map between abelian group-like simplicial sets*

$$(\hat{r}, u, p_+, p_-) : \mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to \mathbb{R}^n) \to$$
$$\mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n) \times \mathcal{W}h(X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \times \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \times \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$$

which induces a split epimorphism on homotopy groups. In particular, the homotopy groups of the product are direct summands of the homotopy groups of $\mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to \mathbb{R}^n)$.

2. *There is a map between abelian group-like simplicial sets*

$$(\phi, p_+, p_-) : \mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to \mathbb{R}^n) \to$$
$$\mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to S^1 \times \mathbb{R}^n) \times \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \times \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n)$$

which induces a split epimorphism on homotopy groups. In particular, the homotopy groups of the product are direct summands of the homotopy groups of $\mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to \mathbb{R}^n)$.

Proof. Part (2) follows from Part (1) from the decomposition

$$\mathcal{W}h(X \times S^1 \times \mathbb{R}^n \to S^1 \times \mathbb{R}^n) \simeq \mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n) \times \mathcal{W}h(X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}).$$

For Part (1), notice that the map (i_*, w_*, j_+, j_-) is a right inverse of the map induced on the homotopy groups by $(\hat{r}, u, p_+, p_-)_*$. The result follows from Corollary 9.7. $\square$

10. Completion of the proof

We have developed the necessary machinery for completing the proof of the main result. For the rest of the section, X is a compact Q-manifold.

Theorem 10.1 (Main Theorem). 1. *The map*

$$(\hat{r}, u, p_+, p_-) : \mathcal{W}h(X\times S^1\times\mathbb{R}^n \to \mathbb{R}^n) \to$$
$$\mathcal{W}h(X\times\mathbb{R}^n \to \mathbb{R}^n)\times\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1})\times\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)\times\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$$

is a homotopy equivalence of abelian group-like simplicial sets.

2. *The map*

$$(\phi, p_+, p_-) : \mathcal{W}h(X\times S^1\times\mathbb{R}^n \to \mathbb{R}^n) \to$$
$$\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n)\times\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)\times\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$$

is a homotopy equivalence of abelian group-like simplicial sets.

Parts (1) and (2) are equivalent because of the splitting of $\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to S^1\times\mathbb{R}^n)$. We will prove Part(2). Theorem 9.8 implies that the map (ϕ, p_+, p_-) induces a split epimorphism on homotopy groups. We will show that the map induces an isomorphism.

Actually we will show that the sequence of simplicial sets

$$\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n\times S^1) \xrightarrow{\phi} \mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$$
$$\xrightarrow{P} \widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)\times\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$$

is homotopy fibration with the map ϕ admitting a homotopy splitting. Equivalently, it is enough to show that the sequence

$$0 \to \pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n\times S^1) \xrightarrow{\phi_*} \pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$$
$$\xrightarrow{P_*} \pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)\times\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n) \to 0$$

is a split short exact sequence. Because of Theorem 9.8, the only remaining ingredient that needs to be proved is that the sequence is exact in the middle term.

10.1. **Preliminaries and notation.** Let $f : M \to X\times\mathbb{R}^n\times S^1\times\Delta^k$ be a strong deformation retraction representing an element in $\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$ such that $P([f]) = 0$. We will work with the image of $[f]$ in $\pi_k\mathcal{W}h_{mb}(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$, which we still call $[f]$. Form the pull-back diagram

$$\begin{tikzcd}
\bar{M} \arrow[r, "\bar{f}"] \arrow[d, "\pi"'] & X\times\mathbb{R}^n\times\mathbb{R}\times\Delta^k \arrow[d] \\
M \arrow[r, "f"] & X\times\mathbb{R}^n\times S^1\times\Delta^k
\end{tikzcd}$$

The retraction $\bar{f}$ is a f.p. bounded (over $\mathbb{R}^n\times\mathbb{R}\times\Delta^k$) strong deformation retraction. Let $\zeta : \bar{M} \to \bar{M}$ be the $+1$ generating covering translation. Then ζ is f.p. bounded (over $\mathbb{R}^n\times\mathbb{R}\times\Delta^k$) homotopic to $\mathrm{id}_{\bar{M}}$ because

$$\bar{f}\zeta = (+1)\bar{f} \simeq \bar{f} \Rightarrow \zeta \simeq \mathrm{id}_{\bar{M}} \simeq \bar{f}$$

where $(+1)$ is the map on $X\times\mathbb{R}^n\times\mathbb{R}\times\Delta^k$ that is $+1$ on the $\mathbb{R}$-coordinate and the identity on $X\times\mathbb{R}^n\times\Delta^k$. A similar argument implies that ζ^{-1} is f.p. bounded homotopic $\mathrm{id}_{\bar{M}}$.

Let c, a positive integer, be the bound (over $\mathbb{R}$) of the strong deformation retraction $\bar{f}$. We write

$$\bar{p}:\bar{M}\ \xrightarrow{\bar{f}}\ X\times\mathbb{R}^n\times\mathbb{R}\times\Delta^k\ \xrightarrow{\text{proj}}\ \mathbb{R}.$$

Let $\alpha>0$ be a positive integer, $\alpha>3c$, and $\beta=\alpha+3c$. By [29] (also Chapter 6) there is an f.p. isotopy $G_s:\bar{M}\ \to\ \bar{M}$, bounded over $\mathbb{R}^n\times\Delta^k$, such that

(i) $G_0=\mathrm{id}_{\bar{M}}$,
(ii) $G_1(\bar{p}^{-1}([\alpha,\infty)))\subset\bar{p}^{-1}((\beta+1,\infty))$
(iii) G_s is supported in $\bar{p}^{-1}([1,\gamma])$ for some positive integer γ.
(iv) G_s restricts to an isotopy of $X\times\mathbb{R}^n\times\mathbb{R}\times\Delta^k$ to itself.

By Lemma 8.10, there is a domination h^+ of $\bar{p}^{-1}([0,\infty))$ by $\bar{p}^{-1}([0,3c])$, rel $\bar{p}^{-1}(\{0\})$, which restricts to a retraction of $p^{-1}[3c,\infty)$ to $X\times\mathbb{R}^n\times\{3c\}\times\Delta^k$. We use h^+ to define dominations

$$h(+)=\zeta^{\alpha}(h^+\cup\mathrm{id})(\zeta^{-\alpha}\times\mathrm{id}_{[0,1]}),\ h=\zeta^{\gamma}(h^+\cup\mathrm{id})(\zeta^{-\gamma}\times\mathrm{id}_{[0,1]})$$

such that

(i) $h(+)_0$, h_0 are the identity maps.
(ii) $h(+)_s|\bar{p}^{-1}([0,\alpha])$, $h_s|\bar{p}^{-1}([0,\gamma])$ are the identity maps.
(iii) $h(+)_1(\bar{p}^{-1}([0,\infty)))\subset\bar{p}^{-1}([0,\beta])$
$h_1(\bar{p}^{-1}([0,\infty)))\subset\bar{p}^{-1}([0,\gamma+3c])$.
(iv) $h(+)_s|\bar{p}^{-1}([\beta,\infty))$ is a retraction to $X\times\times\mathbb{R}^n\{\beta\}\times\Delta^k$
$h_s|\bar{p}^{-1}([\gamma+3c,\infty))$ is a retraction to $X\times\mathbb{R}^n\{\gamma+3c\}\times\Delta^k$.

Let K_- be a compact Q-submanifold of $\bar{M}$ dominating $p^{-1}((-\infty,0])$. We can assume that K_- contains $\bar{p}^{-1}([-1,0])$ and that the domination is relative $\bar{p}^{-1}([-1,0])$ (just choose $h(-)=\zeta^{-1}(h^-\cup\mathrm{id})(\zeta\times\mathrm{id}_{[0,1]})$, where h^- is a homotopy on $\bar{p}^{-1}((-\infty,0])$ having properties analogous to h^+). We choose K_+ as a compact Q-submanifold of $\bar{M}$ so that

$$h(+)_s(\bar{p}^{-1}([\alpha,\gamma+3c]))\subset K_+\cap\bar{p}^{-1}([\alpha,\infty)),\quad\text{for all } s\in[0,1].\qquad(*)$$

Then $K=K_-\cup K_+$ is also a Q-submanifold of $\bar{M}$ and $H=h(-)\cup h$ is a homotopy of the identity on $\bar{M}$ such that $H_1(\bar{M})\subset K$. Write $i:K\ \to\ \bar{M}$ for the inclusion map.

Let $\bar{N}=\bar{M}/{\sim}$ where $(x,0)\sim(x,t)$ for all $(x,t)\in(X\times\mathbb{R}^n)\times\mathbb{R}$ and $\sigma:\bar{M}\ \to\ \bar{N}$ denote the projection map. The projection ρ from M to Δ^k induces a map (not necessarily a fiber bundle projection) $\bar{N}\ \to\ \Delta^k$. The strong deformation retraction f induces a f.p. bounded over $\mathbb{R}^n\times\mathbb{R}\times\Delta^k$, strong deformation retraction $f':\bar{N}\ \to\ X\times\mathbb{R}^n\times\mathbb{R}\times\Delta^k$. The translation ζ^{-1} induces a homeomorphism on $\bar{N}$, denoted z^{-1}. Let $L=\sigma(K)$, $L_\pm=\sigma(K_\pm)$. Then $H_1\zeta^{-1}i$ induces a map $w'=H_1'z^{-1}i':L\ \to\ L$. For a subset $A\subset\mathbb{R}$, set $\bar{N}_A=\sigma(\bar{p}^{-1}(A))$.

We review the definition of coequalizer of two maps $f,g:U\ \to\ V$ ([31]). It is the space defined by

$$\mathcal{W}(f,g)=U\times[0,1]\bigsqcup V/\{(u,0)\sim f(u),\ (u,1)\sim g(u),\ u\in U\}$$

We use the description of the mapping torus of a self map $f:U\ \to\ U$ (denoted $T(f)$) as the coequalizer $\mathcal{W}(f_0,\iota_1)$ where

$$f_0:U\ \to\ U\times[0,1],\ f_0(u)=(f(u),0),\quad \iota_1:U\times\{1\}\ \to\ U\times[0,1],\ \iota_1(u)=(u,1)$$

The reversed mapping torus of f (denoted $T'(f)$) is the coequalizer $\mathcal{W}(\iota_0, f_1)$ with notation similar as above. There are CE-maps from the mapping tori defined above to the classical ones ([31]).

Since w' is a fiber preserving map over Δ^k, there is a natural map $T(w') \to \Delta^k$ which is not necessarily a fiber bundle projection. Actually, $T(w')$ is not necessarily a Q-manifold. We use the properties of coequalizers ([31], Proposition 13.18, Proposition 14.2) to analyze the space $T(w')$. By definition $w' = H_1' z^{-1} i'$. Since z^{-1} is f.p. bounded (over $\mathbb{R}^n \times \mathbb{R} \times \Delta^k$) homotopic to the identity on $\bar{N}$, w' is f.p. bounded (over $\mathbb{R}^n \times \Delta^k$) homotopic to a retraction to $X \times \Delta^k$. If ι is the inclusion map, $\iota : X \times \mathbb{R}^n \times \Delta^k \hookrightarrow L$ then $T(w')$ and $X \times \mathbb{R}^n \times S^1 \times \Delta^k$ are f.p. bounded over $\mathbb{R}^n \times \Delta^k$, homotopy equivalent (Proposition 12.2) through a sequence of homotopy equivalences

$$\begin{aligned} \nu : & T(w') = T(H_1' z^{-1} i') \to T(H_1' f' i') = T(\iota f' i') = T(\iota (f'|)) \to \\ & T((f'|)\iota) = X \times \mathbb{R}^n \times S^1 \times \Delta^k \end{aligned}$$

where each one can be written as a composition of CE-maps and their inverses.

10.2. **Description of finite structures on mapping tori.** We will construct a f.p. map $\omega : L \to L$ close to w' such that

$$\omega | L \cap (X \times \mathbb{R}^n \times \Delta^k) \cup \bar{N}_{[-1,\gamma]} = w'| = z^{-1} i'|,$$

and $\rho : T(\omega) \to \Delta^k$ is a fiber bundle projection with Q-manifold fibres. To this end let

$$L'_- = \bar{N}_{(-\infty,-1]} \cap L_-, \quad L'_+ = \bar{N}_{[\gamma,\infty)} \cap L_+.$$

Approximate $w'| : L'_- \to L_-$ by a sliced Z–embedding ω_- rel $(L'_- \cap (X \times \mathbb{R}^n \times \Delta^k)) \cup \bar{N}_{-1}$. Likewise approximate $w'| : L'_+ \to L_+$ by a sliced Z–embedding $\omega_+ : L'_+ \to L_+$ rel $(L'_+ \cap (X \times \mathbb{R}^n \times \Delta^k)) \cup \bar{N}_\gamma$. Then set $\omega | L'_\pm = \omega_\pm$ and $\omega | \sigma(\bar{p}^{-1}([-1,\gamma])) = w'| = z^{-1} i'|$. By Proposition 12.2, there is an ANR T equipped with a map to Δ^k and f.p. CE-maps

$$\chi' : T \to T(\omega), \quad \chi'' : T \to T(w').$$

Define

$$\begin{aligned} \chi_0 : T(\omega) & \hookrightarrow T \xrightarrow{\chi''} T(w') \\ \chi_1 : T(w') & \hookrightarrow T \xrightarrow{\chi'} T(\omega). \end{aligned}$$

The map χ_0 induces a f.p., bounded over $\mathbb{R}^n \times \Delta^k$, homotopy equivalence

$$f_0 : T(\omega) \xrightarrow{\chi_0} T(w') \xrightarrow{\nu} X \times S^1 \times \Delta^k$$

rel $X \times S^1 \times \Delta^k$. The part of L that lies over $\partial \Delta^k$ is

$$X' = X \times \mathbb{R}^n \times [0,1] \times I \times \partial \Delta^k / \sim, \quad (x,t,\delta) \sim (x,0,\delta),$$

for $(x,t) \in (X \times \mathbb{R}^n) \times [0,1]$, $\delta \in I \times \partial \Delta^k$ (I an interval). The part of $T(\omega)$ that lies over $\partial \Delta^k$ is the mapping torus of the identity map on X'. But the element $X' \times S^1 \to X \times \mathbb{R}^n \times S^1$ of $\mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$ lies in the same component as the base simplex. Thus f_0 represents a class in $\pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$.

Lemma 10.2. $[f_0] = 0 \in \pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$.

Proof. The result follows from Corollary 8.16. ∎

10.3. **Definition of an embedding** $M \to T(\omega)$**.** Let $D = \bar{p}^{-1}([-1,0]) \subset \bar{M}$. Then $M = D/\zeta^{-1}$ (D is a fundamental domain for the action of ζ on $\bar{M}$). Let $M_{-1} = \bar{p}^{-1}\{-1\}$ and

$$\theta : M_{-1}\times[0,1] \to D, \quad \theta|M_{-1}\times\{0\} = \mathrm{id}$$

be a collar of M_{-1} in D.

First we define a map $\mu : \bar{M} \to [0,1]$

$$\mu(m) = \begin{cases} -\bar{p}(m) & \text{if } -1 \le \bar{p}(m) \le 0 \\ 0 & \text{if } \bar{p}(m) \ge 0 \\ 1 & \text{if } \bar{p}(m) \le -1. \end{cases}$$

Then we define an embedding

$$\eta' : D \to T(\zeta^{-1}) = \bar{M}\times[0,1]\bigsqcup\bar{M}\times[0,1]/\{(m,0)_1 \sim (\zeta^{-1}(m),0)_2, (m,1)_1 \sim (m,1)_2\}$$

as follows

$$\eta'(m) = \begin{cases} (m', 2t)_2 & \text{if } m = \theta(m',t),\ t\in[0,\frac{1}{2}] \\ (\theta(m',2t-1), \mu\theta(m',2t-1))_1 & \text{if } m = \theta(m',t),\ t\in[\frac{1}{2},1] \\ (m,\mu(m))_1 & \text{if } m \in D - \theta\big(M_{-1}\times[0,1)\big) \end{cases}$$

where the subscript indicates the copy of $\bar{M}\times[0,1]$. The map η' induces the embedding $\eta_0 : M = D/\zeta \to T(z^{-1})$. But the image of η_0 is contained in $T(w')$. Thus η_0 induces an embedding $M \to T(w')$. Actually η_0 induces an embedding $\eta : M \to T(\omega)$ (Figure 1). From now on, we identify M with $\mathrm{im}(\eta)$.

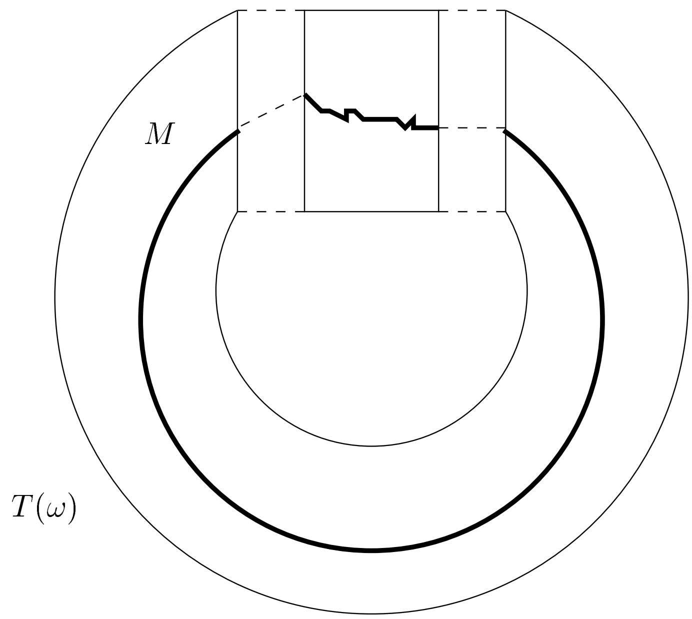


FIGURE 1

Corollary 10.3. $[f_0|M] = [f] \in \pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$.

Proof. $f_0|M$ is a f.p. retraction to $X\times\mathbb{R}^n\times\Delta^k$ whose distance from id_M is bounded, (since f_0 is bounded homotopic to $\mathrm{id}_{T(\omega)}$). The condition over $\partial\Delta^k$ is satisfied because it is satisfied for f_0. Thus we can apply Lemma 8.14 to get $[f_0|M]=[f]\in\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1\to\mathbb{R}^n)$. □

10.4. **Definition of the relaxation.** Let $\tilde{\zeta}=\zeta^{-1}G_1$. We need a variation of the construction of the relaxation presented in Chapters 5 and 6. Set

$$Y=\tilde{\zeta}\bar{p}^{-1}((-\infty,\alpha])\setminus\bar{p}^{-1}((-\infty,\alpha))\subset\bar{M}$$

and $\tilde{M}=Y/\tilde{\zeta}$. Also there is a map $\tilde{f}:\tilde{M}\to X\times\mathbb{R}^n\times S^1\times\Delta^k$ representing an element of $\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1\to\mathbb{R}^n)$. The methods of §6.9 show that the element $\tilde{f}$ is in the image of the forget control map ϕ.

The infinite cyclic cover of $\tilde{M}$ is $\bar{M}$, the covering translation is equal to $\tilde{\zeta}$ and the fundamental domain of the action is Y. Furthermore the map $\tilde{\zeta}$ induces a homeomorphism $\tilde{z}$ on $\bar{N}$. We write $\tilde{w'}:L\to L$ for the map induced by $H_1\tilde{\zeta}i$.

10.4.1. *Description of finite structures on $T(\tilde{w'})$.* We will give a finite structure on $T(\tilde{w'})$ as we did with $T(w')$. We will approximate $\tilde{w'}$ by a fiber preserving map $\tilde{\omega}:L\to L$ such that $\tilde{\omega}|(L\cap(X\times\mathbb{R}^n\times\mathbb{R}\times\Delta^k))\cup\bar{N}_{[-1,\gamma]}=\tilde{w'}|=\tilde{z}i'|$, and $\rho:T(\tilde{\omega})\to\Delta^k$ is a fibre bundle projection with Q-manifold fibres. By Proposition 12.2, there is an ANR $\tilde{T}$, equipped with a map to Δ^k, and f.p. CE-maps

$$\tilde{\chi}':\tilde{T}\to T(\tilde{\omega}),\quad \tilde{\chi}'':\tilde{T}\to T(\tilde{w'}).$$

Define

$$\begin{aligned}\tilde{\chi}_0:T(\tilde{\omega})\ &\hookrightarrow\ \tilde{T}\xrightarrow{\tilde{\chi}''}T(\tilde{w'})\\ \tilde{\chi}_1:T(w')\ &\hookrightarrow\ \tilde{T}\xrightarrow{\tilde{\chi}'}T(\tilde{\omega}).\end{aligned}$$

The map $\tilde{\chi}_0$ induces a f.p., bounded over $\mathbb{R}^n\times\Delta^k$, strong deformation retraction

$$\tilde{f}_0:T(\tilde{\omega})\to X\times\mathbb{R}^n\times S^1\times\Delta^k$$

$\mathrm{rel}X\times\mathbb{R}^n\times S^1\times\Delta^k$, such that $[\tilde{f}_0]=0$ in $\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1\to\mathbb{R}^n)$.

10.4.2. *Definition of an embedding of $\tilde{M}$ into $T(\tilde{w'})$.* We work as in §10.3. Set $M_\alpha=\bar{p}^{-1}\{\alpha\}$. Let

$$\tilde{\theta}:M_\alpha\times[0,1]\to Y,\quad \tilde{\theta}|M_\alpha\times\{0\}=\mathrm{id}$$

be a collar of M_α in Y. First we define a Urysohn map $\tilde{\mu}:\bar{M}\to[0,1]$, so that

(i) $\tilde{\mu}^{-1}(1)=\bar{p}^{-1}(\{\alpha\})$.
(ii) $\tilde{\mu}^{-1}(0)=\tilde{\zeta}\bar{p}^{-1}(\{\alpha\})$.
(iii) $\tilde{\mu}|:X\times\mathbb{R}^n\times[\alpha,\beta+2]\to[0,1]$ is decomposed as $X\times\mathbb{R}^n\times[\alpha,\beta+2]\to[\alpha,\beta+2]\to[0,1]$ where the first map is the projection and the second is the order reversing linear map.

As before, we define a map $\tilde{\eta}':Y\to T(\zeta)$

$$\tilde{\eta}'(y)=\begin{cases}(y',2t)_2 & \text{if } y=\tilde{\theta}(y',t),\ t\in[0,\frac{1}{2}]\\ (\tilde{\theta}(y',2t-1),\tilde{\mu}\tilde{\theta}(y',2t-1))_1 & \text{if } y=\tilde{\theta}(y',t),\ t\in[\frac{1}{2},1]\\ (y,\tilde{\mu}(y))_1 & \text{if } y\in Y-\tilde{\theta}\big(M_a\times[0,1)\big)\end{cases}$$

where the subscript indicates the copy of $\bar{M}\times[0,1]$. The map $\tilde{\eta}'$ induces the embedding $\tilde{\eta}_0:\tilde{M}=Y/\tilde{\zeta}\to T(\tilde{z})$. But the image of $\tilde{\eta}_0$ is contained in $T(\tilde{w}')$. Finally $\tilde{\eta}_0$

induces an embedding $\tilde{\eta} : \tilde{M} \to T(\tilde{\omega})$. We identify $\tilde{M}$ with $\mathrm{im}(\tilde{\eta})$ and $[\tilde{f}_0|\tilde{M}] = [\tilde{f}]$ in $\pi_k \mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$ as in Corollary 10.3.

10.5. **Splittings.** In this section we describe a splitting of $T(\omega)$ into a union of spaces corresponding to the Bass-Heller-Swan splitting. The argument is a geometrization of "Ranicki's pentagon" ([42], p. 95).

10.5.1. *First Splitting.* The embedding

$$\eta'| : D - \theta\big(M_{-1}\times[0,\frac{1}{2})\big) \to \bar{M}\times[0,1]$$

induces an embedding

$$\eta'' : D - \theta\big(M_{-1}\times[0,\frac{1}{2})\big) \to \bar{M}\times[0,1] \to \bar{N}\times[0,1] \to T(z^{-1})$$

into the first copy of $\bar{N}\times[0,1]$ in $T(z^{-1})$. The image of η'' decomposes $\bar{N}\times[0,1]$ into $Z_+\cup_{im(\eta'')}Z_-$ where

$$Z_+ = (\sigma\times\mathrm{id})(\{(m,t) : t \geq \mu(m)\}), \quad Z_- = (\sigma\times\mathrm{id})(\{(m,t) : t \leq \mu(m)\}).$$

If we restrict ω to L_-, we get a decomposition of $T(\omega) = \mathcal{T}(-)\cup_M\mathcal{T}'$. Set $L_{1-} = L_- \setminus \bar{N}_{(-1,0]}$. The space $\mathcal{T}(-)$ is given by

$$\mathcal{T}(-) = Z_-\bigsqcup L_{1-}\times[0,1]/\sim'$$

where we identify

$$x \sim' (\omega(x),0) \text{ for } x\in Z_-\cap(L\times\{0\}), \quad x \sim' (x,1) \text{ for } x\in Z_-\cap(L\times\{1\}).$$

There is a CE-map from $\mathcal{T}(-)$ to $T(\omega|L_-)$ constructed in three steps:

Step 1. There is a CE-map

$$s_1 : L_-\times[0,1] \to L_-\times\{1\}\cup L_{1-}\times[0,1] = L_{2-}$$

which is the identity on L_{2-} (Figure 2).

Proof We will construct a CE-map

$$s_1' : \bar{N}_{[-1,0]}\times[0,1] \to \bar{N}_{[-1,0]}\times\{1\}\cup\bar{N}_{-1}\times[0,1],\ \mathrm{rel}\bar{N}_{[-1,0]}\times\{1\}\cup\bar{N}_{-1}\times[0,1].$$

Since $\bar{p}^{-1}(\{-1\})$ is collared in $\bar{p}^{-1}([-1,0])$, $\bar{N}_{-1}$ is collared in $\bar{N}_{[-1,0]}$. Thus

$$\bar{N}_{[-1,0]}\times\{1\}\cup\bar{N}_{-1}\times[0,1] \cong \bar{N}_{[-1,0]}\times\{1\}$$

($\bar{N}_{-1}\times[0,1]$ can be considered as an extension of the collar of $\bar{N}_{-1}$ in $\bar{N}_{[-1,0]}$). Thus there are two homotopic Z–embeddings i_1 and i_2 of $\bar{N}_{[-1,0]}\times\{1\}$ into $\bar{N}_{[-1,0]}\times[0,1]$ with images

$$i_1(\bar{N}_{[-1,0]}\times\{1\}) = \bar{N}_{[-1,0]}\times\{1\}, \quad i_2(\bar{N}_{[-1,0]}\times\{1\}) = \bar{N}_{[-1,0]}\times\{1\}\cup\bar{N}_{-1}\times[0,1].$$

Actually, the homotopy between the two embeddings can be chosen to be f.p. and bounded (over $\mathbb{R}^n\times\Delta^k$). By the Z–set unknotting theorem there is a f.p., bounded isotopy $\iota : \bar{N}_{[-1,0]}\times[0,1] \to \bar{N}_{[-1,0]}$ from the identity such that $\iota_1\circ i_2 = i_1$. Then the map $s_1' = \iota_1^{-1}\circ\mathrm{proj}\circ\iota_1$ satisfies the conditions required ($\mathrm{proj} : M\times[0,1] \to M\times\{1\}$ is the projection map).

Step 2. There is a CE-map

$$s_2 : L_-\times[0,1] \to Z_-.$$

Proof. The embedding $\eta'|$ is the graph of the function μ. Thus there is a CE-map

$$K_-\times[0,1] \to \{(m,t) : t \leq \mu(m)\}$$

which maps

$$(x,t) \mapsto \begin{cases} (x,\mu(x)), & \text{for } (x,t)\in Z_+ \\ (x,t), & \text{otherwise.} \end{cases}$$

Actually the map is the identity on $\{(m,t) : t \leq \mu(m)\}$. The above map induces a CE-map s_2 on the quotients.

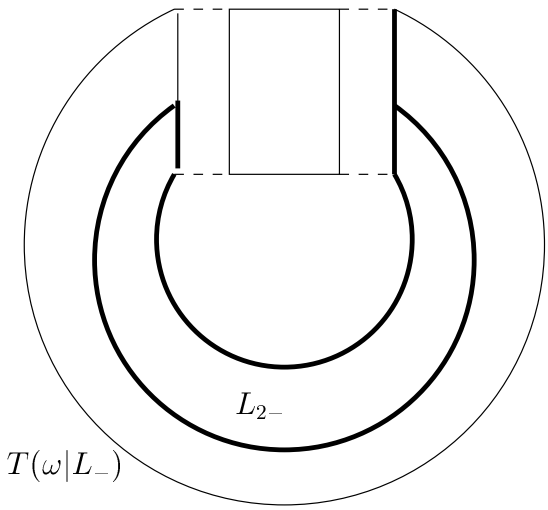


FIGURE 2

Step 3. The two maps constructed in the first two steps are combined to form a CE-map $s : T(\omega|L_-) \to \mathcal{T}(-)$ (Figure 3).

Proof. By definition the space $T(\omega|L_-)$ has the form

$$T(\omega|L_-) = L_-\times[0,1] \bigsqcup L_-\times[0,1]/\sim'', \quad (x,0)_1\sim''(\omega(x),0)_2,\ (x,1)_1\sim''(x,1)_2.$$

The map s is defined as the composition

$$s : T(\omega|L_-) = L_-\times[0,1] \bigsqcup L_-\times[0,1]/\sim'' \xrightarrow{\mathrm{id}\sqcup s_1} L_-\times[0,1] \bigsqcup L_{2-} \xrightarrow{s_2\sqcup s_2'} \mathcal{T}(-).$$

where s_2' is defined to be the restriction of s_2 on $L_-\times\{1\}$ and the identity elsewhere:

$$s_2'(x,t) = \begin{cases} s_2(x,1), & \text{for } x\in L_-,\ t=1 \\ (x,t), & \text{otehrwise.} \end{cases}$$

Notice that $s_2(x,1)\in Z_-$. We use the notation $(x,t)'$ for a pair that is an element of Z_-. Then

(i) $s((x,0)_1) = s_2(x,0) = (x,0)' = (\omega(x),0) = s((\omega(x),0)_2)$.
(ii) For the element $((x,1)_1)$ we consider two cases:
  (a) $x\in\bar{N}_{[-1,0]}$. Then, $s((x,1)_1) = s_2(x,1) = (x,\mu(x))' = s_2'(x,1) = s((x,1)_2)$.
  (b) $x\in L_-\setminus\bar{N}_{[-1,0]}$. In this case, $s((x,1)_1) = (x,1)' = (x,1) = s((x,1)_2)$.

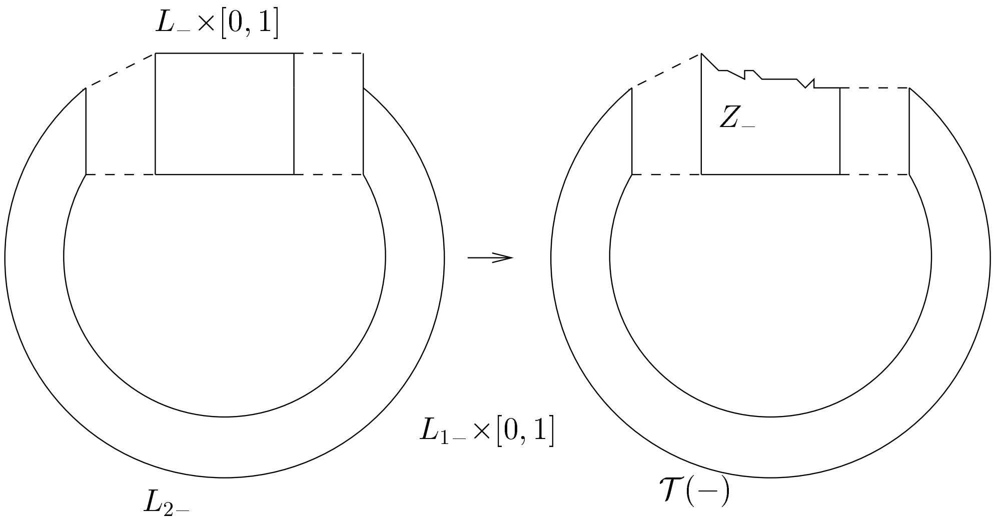


FIGURE 3

Therefore s is a well-defined f.p. CE-map.

Lemma 10.4. *$j_-p_-([f]) = [f_0|\mathcal{T}(-)]$ in $\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$.*

Proof. There is a strong deformation retraction $r : T(\omega|L_-) \to X\times\mathbb{R}^n\times S^1\times\Delta^k$ representing $j_-p_-([f])$. Then the retraction

$$r' : T(\omega|L_-) \hookrightarrow \mathcal{T}(-) \xrightarrow{f_0|} X\times\mathbb{R}^n\times S^1\times\Delta^k$$

is a f.p. retraction which satisfies the conditions of Lemma 8.14. Thus, in the group $\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1 \to \mathbb{R}^n)$,

$$j_-p_-([f]) = [r] = [r'] = [f_0|\mathcal{T}(-)].$$

$\square$

10.5.2. *First description of the space $\mathcal{T}'$.* The space $\mathcal{T}'$ admits a description as the space $\mathcal{T}(-)$ above. Set $L_{1+} = L_+\cup\bar{N}_{[-1,0]}$. Then

$$\mathcal{T}' = Z_+\bigsqcup L_{1+}\times[0,1]/\sim'$$

where we identify

$$x\sim'(\omega(x),0) \text{ for } x\in Z_+\cap(L\times\{0\}), \quad x\sim'(x,1) \text{ for } x\in Z_+\cap(L\times\{1\}).$$

Then $\mathcal{T}'$ is homotopy equivalent to the space $\mathcal{T}''$, defined as

$$\mathcal{T}'' = Z_+\bigsqcup L_{1+}\times[0,1]/\sim'$$

where we identify

$$x\sim'(\tilde{\omega}(x),0) \text{ for } x\in Z_+\cap(L\times\{0\}), \quad x\sim'(x,1) \text{ for } x\in Z_+\cap(L\times\{1\}).$$

The homotopy equivalence is constructed as follows:

First of all $Z_+ \cap (L \times \{0\}) = L_+$ and $Z_+ \cap (L \times \{1\}) = L_{1+}$. The identification is constructed using the map

$$\omega_1 = ((\omega|) \times \{0\}) \sqcup (\mathrm{id} \times \{1\}) : L_+ \sqcup L_{1+} \to L_{1+} \times [0,1].$$

The homotopy between ζ and ζ^{-1} induces a homotopy between

$$\omega_1 \simeq \tilde{\omega}_1 = ((\tilde{\omega}|) \times \{0\}) \sqcup (\mathrm{id} \times \{1\}).$$

The space $\mathcal{T}''$ is formed as $\mathcal{T}'$ using the map $\tilde{\omega}_1$ for the identification. Then there is a space $\tilde{T}'$ and cell-like maps from $\tilde{T}'$ to $\mathcal{T}'$ and $\mathcal{T}''$ (Lemma 12.1, [31], Proposition 13.18). The space $\mathcal{T}''$ is a subset of $T(\tilde{\omega})$. Furthermore the embedding $\tilde{\eta}$ in §10.4.2 induces an embedding of $\tilde{M}$ to $\mathcal{T}''$.

10.6. **Splitting of $\mathcal{T}''$. Second splitting of $T(\omega)$.** The embedding $\tilde{\eta}$ induces a decomposition $\mathcal{T}'' = \mathcal{T}_1 \cup_{\tilde{M}} \mathcal{T}_2$.

10.6.1. *Description of $\mathcal{T}_1$.* As in §10.5.1, the embedding

$$\tilde{\eta}'| : Y - \tilde{\theta}\big(M_\alpha \times [0, \frac{1}{2})\big) \to \bar{M} \times [0,1]$$

induces an embedding

$$\tilde{\eta}'' : Y - \tilde{\theta}\big(M_\alpha \times [0, \frac{1}{2})\big) \to \bar{M} \times [0,1] \to \bar{N} \times [0,1] \to T(\tilde{z}^{-1})$$

into the first copy of $\bar{N} \times [0,1]$ in $T(\tilde{z}^{-1})$. The image of $\tilde{\eta}''$ is contained in Z_+. It decomposes $Z_+ = W_+ \cup_{im(\tilde{\eta}''|)} W_-$ where

$$W_+ = (\sigma \times \mathrm{id})(\{(m,t) : t \geq \tilde{\mu}(m)\}), \quad W_- = (\sigma \times \mathrm{id})(\{(m,t) : t \leq \tilde{\mu}(m)\})$$

(Figure 4).

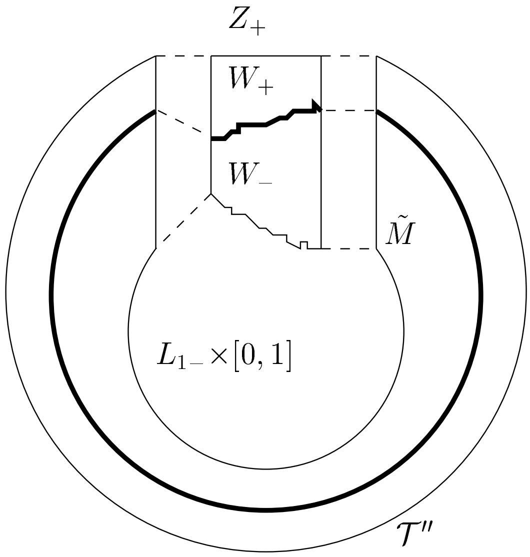


FIGURE 4

Thus $\mathcal{T}_1$ admits a descriptions, $\mathcal{T}_1 = W_- \sqcup L_1' \times [0,1]/\sim'$, where $L_1' = \tilde{\omega}(\bar{N}_{[0,\alpha]})$. The identifications are done using the map

$$((\tilde{\omega}|)\times\{0\}) \sqcup (\mathrm{id}\times\{1\})) : W_-\cap(L\times\{0\}) \bigsqcup W_-\cap(L\times\{1\}) \to L_1'\times[0,1]$$

(notice that $W_-\cap(L\times\{0\}) = \bar{N}_{[0,\alpha]}$ and $W_-\cap(L\times\{1\}) = L_1'$). By construction, the restriction

$$\tilde{\omega}|\bar{N}_{[0,\alpha]} = \tilde{w'}|\bar{N}_{[0,\alpha]} = H_1'\tilde{z}i'|\bar{N}_{[0,\alpha]}.$$

which is equal to the map induced on $\bar{p}^{-1}([0,\alpha])$ by $H_1\tilde{\zeta}i$. But

$$\tilde{\zeta}i|\bar{p}^{-1}([0,\alpha]) = \zeta^{-1}G_1(\bar{p}^{-1}([0,\alpha]) \subset \zeta^{-1}(\bar{p}^{-1}([0,\gamma])) \subset \bar{p}^{-1}([-1,\gamma])$$

because the support of G lies in $\bar{p}^{-1}([1,\gamma])$. By construction, the homotopy H is the identity on $\bar{p}^{-1}([-1,\gamma])$. Thus

$$\tilde{\omega}|\bar{p}^{-1}([0,\alpha]) = H_1\tilde{\zeta}i|\bar{p}^{-1}([0,\alpha]) = \tilde{\zeta}|\bar{p}^{-1}([0,\alpha]).$$

Therefore $\tilde{\omega}|\bar{N}_{[0,\alpha]} = \tilde{z}|\bar{N}_{[0,\alpha]}$ which is a homeomorphism to its image. Set $T_1' = \mathcal{W}(j_0', \tilde{\omega}^{-1})$ where

$$j_0' : L_1' \hookrightarrow L\times\{1\}\cap W_- \hookrightarrow W_-, \quad \tilde{\omega}^{-1} : L_1' \to L\times\{1\}\cap W_- \hookrightarrow W_-.$$

There is a homeomorphism $\psi : \mathcal{T}_1 \to T_1'$ defined by

$$\begin{array}{lcll} \psi(x,t) & = & (x, 1-t), & x\in L_1' \\ \psi(y) & = & y, & y\in W_-. \end{array}$$

(i) For $(x,0)\in L_1'\times\{0\}$, $(\tilde{\omega}(x),0)$ is identified with $x\in W_-\cap L\times\{0\}$ in $\mathcal{T}_1$. Then

$$\psi(\tilde{\omega}(x),0) = (\tilde{\omega}(x),1) = \tilde{\omega}^{-1}(\tilde{\omega}(x)) = x = \psi(x).$$

(ii) For $(x,0)\in L_1'\times\{0\}$, $(x,1)$ is identified with $x\in W_-\cap L\times\{1\}$ in $\mathcal{T}_1$. Then

$$\psi(x,1) = (x,0) = x = \psi(x).$$

Let

$$\lambda_s = h(+)_s G_s^{-1} : \bar{p}^{-1}([0,\infty)) \to \bar{p}^{-1}([0,\infty)).$$

The restriction

$$\lambda_1\zeta|\tilde{\zeta}(\bar{p}^{-1}([0,\alpha])) = \tilde{\zeta}^{-1}|\tilde{\zeta}(\bar{p}^{-1}([0,\alpha])) \qquad (**)$$

(that is because for $x\in\bar{p}^{-1}([0,\alpha])$,

$$\lambda_1\zeta(\tilde{\zeta}(x)) = h(+)_1G_1^{-1}\zeta\zeta^{-1}G_1(x) = h(+)_1(x) = x$$

and the last equality follows from the fact that $h(+)$ is the identity on $\bar{p}^{-1}([0,\alpha])$ by construction). The homotopy λ_s induces a homotopy λ_s' on L_+. Since $\tilde{\omega}|\bar{N}_{[0,\alpha]} = \tilde{z}|\bar{N}_{[0,\alpha]}$, (**) implies that for $x\in\bar{N}_{[0,\alpha]}$,

$$\lambda_1'z(\tilde{\omega}(x)) = \lambda_1'z\tilde{z}(x) = x$$

Therefore $\lambda_1'z|L_1' = \tilde{\omega}^{-1}|L_1'$ which implies that $T_1' = \mathcal{W}(j_0'', \lambda_1'z|)$ with

$$j_0'' : L_1' \hookrightarrow L\times\{1\}\cap W_- \hookrightarrow W_-, \quad \lambda_1'z| : L_1' \to L\times\{0\}\cap W_- \hookrightarrow W_-$$

As in §10.5.1, there is a CE-map from $T'(\lambda_1'z|L_1')$ to T_1':
We first construct a CE-map $\tau : L_1'\times[0,1] \to W_-$ which is the identity on W_-. The map τ is constructed as the composition

$$\tau : L_1'\times[0,1] \to W_-\cup(Z_-\cap\bar{N}_{[-1,0]}) \to W_-$$

where the two maps are constructed as in Step 2 of §10.5.1. We write τ_- for the CE-map

$$\tau_- : L'_1 \times [0,1] \to W_-, \quad \tau_-(x,t) = \tau(x, 1-t).$$

The CE-map from $T'(\lambda'_1 z | L'_1)$ to T'_1 is constructed as follows (Figure 5)

$$\tau' : T'(\lambda'_1 z|L'_1) = L'_1 \times [0,1] \bigsqcup L'_1 \times [0,1]/\sim \xrightarrow{\mathrm{id} \cup \tau_-} L'_1 \times [0,1] \bigsqcup W_-/\sim = T'_1$$

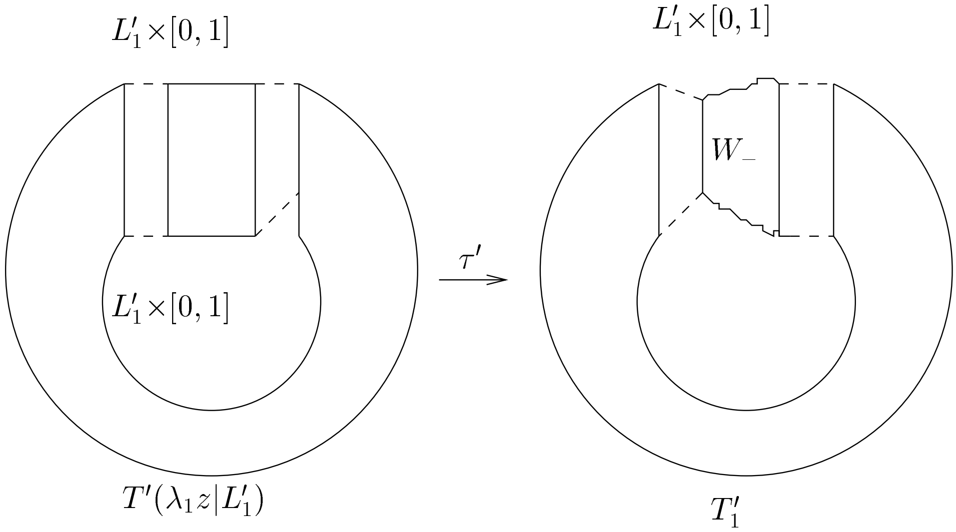


FIGURE 5

(i) $(x,0)_1$ is identified with $(x,0)_2$ in $T'(\lambda'_1 z|L'_1)$. But

$$\tau'((x,0)_2) = \tau(x,1) = (x,1) = (x,0) = \tau'((x,0)_1)$$

where the second equality follows because $(x,1) \in W_-$ and τ is the identity on W_-, and the third equality follows from the relations in T'_1, namely $(x,0) \in L'_1 \times \{0\}$ is identified with its image in $L \times \{1\} \cap W_-$.

(ii) The second relation in $T'(\lambda'_1 z|L'_1)$ is given by identifying $(x,1)_1$ with $(\lambda'_1 z(x), 1)_2$. Then

$$\tau'((\lambda'_1 z(x),1)_2) = \tau(\lambda'_1 z(x), 0) = (\lambda'_1 z(x), 0) = (x,1) = \tau'((x,1)_1)$$

where the second equality follows because $\lambda'_1 z(x) \in W_-$ and the third equality follows from the defining relations for T'_1.

We use the homotopy λ' to define an element of $\pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$ as in the construction of j_+. By Lemma 8.12, there is a strong deformation retraction $f_1 : T'(\lambda'_1 z|L'_1) \to X \times \mathbb{R}^n \times S^1 \times \Delta^k$ such that $[f_1] = j_+ p_+([f])$ in $\pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$. From Lemma 10.4, we derive

$$[f_1] = [f_1 | T'_1] = [(f_1|T'_1) \circ \psi] = [f_0 | \mathcal{T}_1]$$

in $\pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$. Thus

Lemma 10.5. $[f_0|\mathcal{T}_1] = [f_1] = j_+ p_+([f])$ *in* $\pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n)$.

10.6.2. *Description of the space $\mathcal{T}_2$.* By construction,

$$\begin{aligned}\tilde{w}(\bar{p}^{-1}([\alpha,\infty))) &= H_1\tilde{\zeta}i(\bar{p}^{-1}([\alpha,\infty))) = H_1\zeta^{-1}G_1(\bar{p}^{-1}([\alpha,\infty)))\\ &\subset H_1\zeta^{-1}(\bar{p}^{-1}((\beta+1,\infty)))\\ &\subset H_1(\bar{p}^{-1}((\beta,\infty))) = h_1(\bar{p}^{-1}((\beta,\infty)))\\ &\subset \bar{p}^{-1}((\beta,\infty))\end{aligned}$$

the last inclusion holds because h is the identity on $\bar{p}^{-1}([0,\gamma])$ and $\beta<\gamma$. Similarly, $\tilde{w}(K_+)\subset K_+$. Thus

$$\tilde{w}(K_+\cap\bar{p}^{-1}([\alpha,\infty)))\subset K_+\cap\bar{p}^{-1}([\beta,\infty)).$$

Let $L_2 = L_+\cap\bar{N}_{[\alpha,\infty)}$ which is a subset of L_+. Then

$$\tilde{\omega}(L_2) = \tilde{w'}(L_2)\subset L_+\cap\bar{N}_{[\beta,\infty)}.$$

By construction,

$$\mathcal{T}_2 = W_+\bigsqcup L_2\times[0,1]/\sim''$$

where

$$x\sim''(\tilde{\omega}(x),0) \text{ for } x\in W_+\cap(L\times\{0\}),\quad x\sim''(x,1) \text{ for } x\in W_+\cap(L\times\{1\}).$$

As in §10.5.1, we construct a CE-map from $\mathcal{T}_2$ to the mapping torus $T(\tilde{\omega}|L_2)$. Condition (*) implies that the homotopy $h(+)$ induces a homotopy $h'(+)$ on $\tilde{\omega}(L_2)$ starting from the identity and ending to a map that it is a retraction of $\tilde{\omega}(L_2)$ to $X\times\mathbb{R}^n\times\Delta^k$. Thus

$$h'(+)\tilde{\omega}|L_2 : \tilde{\omega}\simeq r'$$

where r' is a retraction. Thus there is a CE-map from $T(\tilde{\omega}|L_2)$ to the mapping torus of a retraction. Therefore $[f_0|\mathcal{T}_2]=0$ in $\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1\ \to\ \mathbb{R}^n)$ (Corollary 8.16).

10.7. **Completion of the proof.** .

We now use the Higher Sum Theorem to express the element $[f_0]$ as a sum of elements of $\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1\ \to\ \mathbb{R}^n)$. The assumptions of Remark 7.1 of the Higher Sum Theorem (Theorem 7.2) are satisfied for the splitting of $T(\omega)$. Thus

$$0=[f_0]=[f_0|\mathcal{T}(-)]+[f_0|\mathcal{T}']-[f_0|M]\qquad(1)$$

The first summand is equal to $j_-p_-([f])$ (Lemma 10.4) which is zero by assumption and the last term is equal to $[f]$ (Corollary 10.3). Thus, again by the Higher Sum Theorem,

$$[f]=[f_0|\mathcal{T}']=[f_0|\mathcal{T}_1]+[f_0|\mathcal{T}_2]-[f_0|\tilde{M}]\qquad(2)$$

By Lemma 10.5, $[f_0|\mathcal{T}_1]=j_+p_+([f])$ which is zero by our assumption. $[f_0|\mathcal{T}_1]$ is also zero by §10.6.2. Thus

$$[f]=-[f_0|\tilde{M}]=-[\tilde{f}]\in\operatorname{im}(\phi_*).$$

The element $[f]$ belongs to the image of ϕ_* and the sequence is exact.

Remark 10.6. We summarize the structure of the construction presented in Chapter 10. Theorem 9.8 implies that there is a spit injection

$$\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1\ \to\ \mathbb{R}^n\times S^1)\times\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\ \to\ \mathbb{R}^n)\times\pi_k\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n\ \to\ \mathbb{R}^n)$$
$$\xrightarrow{(\phi_*,j_+,j_-)}\pi_k\mathcal{W}h(X\times\mathbb{R}^n\times S^1\ \to\ \mathbb{R}^n)$$

with splitting given by $(r_*, (p_+)_*, (p_-)_*)$. In Chapter 10, we show that the split injection (ϕ_*, j_+, j_-) is an epimorphism. More specifically, equations (1) and (2) imply that

$$[f] = [f_0|M] = [f_0|\mathcal{T}(-)] + [f_0|\mathcal{T}'] = [f_0|\mathcal{T}(-)] + [f_0|\mathcal{T}_1] + [f_0|\mathcal{T}_2] - [f_0|\hat{M}].$$

In Chapter 10, we have shown that the summands satisfy the following:

$$\begin{aligned}
[f_0|\mathcal{T}(-)] &= j_-(p_-)_*([f]) \\
[f_0|\mathcal{T}_1] &= j_+(p_+)_*([f]) \\
[f_0|\mathcal{T}_2] &= 0 \\
[f_0|\hat{M}] &\in \mathrm{im}(\phi_*).
\end{aligned}$$

Thus, we can rewrite the decomposition of $[f]$ as

$$[f] = j_-(p_-)_*([f]) + j_+(p_+)_*([f]) - \phi_*([f'])$$

for some $[f'] \in \pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n \times S^1)$. That shows that the injection (ϕ_*, j_+, j_-) is a surjection and therefore an isomorphism. The last equation also implies that

$$[f'] = -r_*([f]) \in \pi_k \mathcal{W}h(X \times \mathbb{R}^n \times S^1 \to \mathbb{R}^n \times S^1).$$

11. Comparison with the lower algebraic nil groups

In this section, we assume that X is a compact Q-manifold and $p : X\times\mathbb{R}^n \to \mathbb{R}^n$ is the projection map. For simplicity we will assume that the space X is path connected. The general case follows by taking the direct sum over the set of path-components of X. We will compare $\pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$ to the algebraically defined exotic lower nil group $\mathrm{NK}_{1-n}(\mathbb{Z}\pi_1(X))$ of [2] (also [42], [43]). In particular,

$$\pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n) \cong \mathrm{NK}_1(\mathbb{Z}\pi_1(X)) \cong \widetilde{\mathrm{Nil}}(\mathbb{Z}\pi_1(X)).$$

In lower K-theory, the Nil-groups appearing in the splitting theorems for the reduced K-groups coincide with the Nil-groups that appear in the splitting of the non-reduced K-groups. In this section we will work with the Nil-groups that are summands of the non-reduced K-groups.

11.1. **Preliminaries on the algebraic lower Nil-groups.** We start by reviewing the algebraic construction given in [42]. For an additive category $\mathcal{A}$, let $\mathbb{C}_n(\mathcal{A})$ denote the graded category over $\mathbb{Z}^n$ (or equivalently over $\mathbb{R}^n$). The objects of $\mathbb{C}_n(\mathcal{A})$ are the graded objects of $\mathcal{A}$ over $\mathbb{Z}^n$. Morphisms are bounded morphisms between them. We write $\mathbb{P}_n(\mathcal{A})$ for the idempotent completion of $\mathbb{C}_n(\mathcal{A})$. The category $\mathcal{A}[z]$ denotes the polynomial extension of $\mathcal{A}$, and $\mathcal{A}[z,z^{-1}]$ the Laurent extension category of $\mathcal{A}$ ([42], §7). If R is a ring and $\mathcal{A} = \mathbf{B}^f(R)$ is the category of finitely generated free based left R-modules we write $\mathbb{C}_n(R)$ for the graded category, over $\mathbb{Z}^n$ or $\mathbb{R}^n$.

The lower Nil-groups that are of interest are the ones appearing in the Bass-Heller-Swan splitting ($n \geq -1$)

$$K_{-n}(R[z,z^{-1}]) \cong K_{-n}(R)\oplus K_{-n-1}(R) \oplus \mathrm{NK}_{-n}(R) \oplus \mathrm{NK}_{-n}(R).$$

Actually, the lower Nil groups are expressed as the cokernels of the homomorphism induced by the inclusion

$$\begin{aligned}\mathrm{NK}_{-n}(R) = &\operatorname{coker}(i : K_{-n}(R) \to K_{-n}(R[z])) = \\ &\operatorname{coker}(i : K_1(\mathbb{C}_{1+n}(R)) \to K_1(\mathbb{C}_{1+n}(R[z])))\end{aligned}$$

where the last identification follows from [42], §11. In the proof of Proposition 11.2 in [42], the cokernel of i is shown to be isomorphic to $\widetilde{\mathrm{Nil}}_0(\mathbb{C}_{1+n}(R))$, the Grothendieck group of the category of Nil objects in the additive category $\mathbb{C}_{1+n}(R)$. Therefore

$$\mathrm{NK}_{-n}(R) = \widetilde{\mathrm{Nil}}_0(\mathbb{C}_{1+n}(R)).$$

The reduced Nil-groups of $\mathcal{A}$ are defined in two equivalent ways. One, denoted $\widetilde{\mathrm{Nil}}_0(\mathcal{A})$, is defined using the pairs (A,ν) of objects of $\mathcal{A}$ and nilpotent endomorphisms. The second definition involves pairs (C_*,ν_*) of finite dimensional chain complexes over $\mathcal{A}$ and chain homotopy nilpotent self-chain maps ν_* and it is denoted $\widetilde{\mathrm{Nil}}_0^{\infty}(\mathcal{A})$ ([42], §9). The two definitions produce naturally isomorphic groups ([42], Proposition 9.3). We are going to use both definitions.

11.2. **The definition of the homomorphism between algebraic and geometric Nil-groups.** Following the ideas in [41] §4, we define a group homomorphism:

$$\alpha : \pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n) \to \mathrm{NK}_{1-n}(\mathbb{Z}\pi_1(X))$$

Let $\pi = \pi_1(X)$. Let $[M, f]$ represent a 0-simplex in $\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$. Then there is a positive integer s such that f^s is boundedly homotopic rel $X \times \mathbb{R}^n$ to a proper retraction $r : M \to X\times\mathbb{R}^n$. We choose a finite complex K such that $X \cong K\times Q$ and a locally finite complex L, containing K, such that $M \cong L\times Q$. The maps f and r induce maps

$$f' : L \to L, \quad r' : L \to K\times\mathbb{R}^n$$

such that ${f'}^s$ is boundedly homotopic to r', rel $K\times\mathbb{R}^n$. We choose a triangulation of $\mathbb{R}^n$ which is bounded in the sense that all the cells have diameter less than or equal to a fixed positive number. The product cell structure on $K\times\mathbb{R}^n$ is a bounded cell structure over $\mathbb{R}^n$. Form the pull back diagram

$$\begin{array}{ccc} \bar{L} & \xrightarrow{\overline{r'}} & \tilde{K}\times\mathbb{R}^n \\ \downarrow & & \downarrow \\ L & \xrightarrow{r'} & K\times\mathbb{R}^n \end{array}$$

where $\tilde{K}$ is the universal cover of K. Since K is finite complex and r' is proper, the chain complex $C_*(\bar{L}, \tilde{K}\times\mathbb{R}^n)$ is a complex in the additive category $\mathbb{C}_n(\mathbb{Z}\pi)$ ([40], §4, [42], p. 14). Let $\bar{f'} : \bar{L} \to \bar{L}$ be the pull-back of the map f' (the domain of $\bar{f'}$ is $\bar{L}$ because $f'r' = r'$ and so the pull-back using $f'r'$ is equal to the pull-back using r'). Since the map ${f'}^s$ is boundedly homotopic to a retraction, rel $K\times\mathbb{R}^n$, the chain map $\bar{f'}^s_*$ induced by $\bar{f'}^s$ is chain homotopic to the zero map in the category $\mathbb{C}_n(\mathbb{Z}\pi)$. Therefore the pair $(C_*(\bar{L}, \tilde{K}\times\mathbb{R}^n), \bar{f'}_*)$ represents an element in $\widetilde{\mathrm{Nil}}_0^{(\infty)}(\mathbb{C}_n(\mathbb{Z}\pi))$ which is canonically isomorphic to $\widetilde{Nil}_0(\mathbb{C}_n(\mathbb{Z}\pi))$. Define

$$\alpha(M, f) = (C_*(\bar{L}, \tilde{K}\times\mathbb{R}^n), \bar{f'}_*)$$

The construction of α does not depend on the choice of the retraction r because any two retractions homotopic to some power of f are boundedly homotopic rel $K\times\mathbb{R}^n$. The construction is independent of the choices of the complexes K and L because of the chain homotopy condition in $\widetilde{\mathrm{Nil}}_0(\mathbb{C}_n(\mathbb{Z}\pi))$.

Proposition 11.1. *The map*

$$\alpha : \pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n) \to \widetilde{Nil}_0(\mathbb{C}_n(\mathbb{Z}\pi)) \cong NK_{1-n}(\mathbb{Z}\pi)$$

is a group isomorphism.

Proof. First we will show that α is well-defined. Let (M_i, f_i), $i = 1, 2$, be two equivalent nil 0-simplices. We will assume that they are germ equivalent. The general case follows from repeated applications of the germ equivalence. Let f_i^s $(s > 0)$ be boundedly homotopic to a retraction r_i, $i = 1, 2$. Then there is a nil 0-simplex (N, g) and embeddings $h_i : M_i \to N$ such that $gh_i = h_i f_i$, $i = 1, 2$, and $g(N) \subseteq h_1(M_1)\cap h_2(M_2)$. We will assume that M_i is a subspace of N and h_i is the inclusion map $(i = 1, 2)$. By construction

$$g^{s+1} = g^s \circ g = f_i^s \circ g \simeq r_i g, \quad i = 1, 2.$$

In particular, $r_1 g \simeq r_2 g$ and the homotopy is bounded over $\mathbb{R}^n$. We choose complexes L_i, $i = 1, 2$, and T such that

1. K is a subcomplex of L_i, $i = 1, 2$, and T.
2. $M_i \cong L_i\times Q$, $N \cong T\times Q$.
3. $L_1\cup L_2$ is a subcomplex of T.

We denoted by f_i', r_i' ($i = 1, 2$) and g' the corresponding maps induced on the complexes. Thus there is a Mayer-Vietoris exact sequence of chain complexes and homotopy nilpotent maps, for $i = 1, 2$

$$0 \to (C_*(\bar{L}_i, \tilde{K}\times\mathbb{R}^n), \bar{f'}_{i*}) \to (C_*(\bar{T}_i, \tilde{K}\times\mathbb{R}^n), \bar{g'}_{i*}) \to (C_*(\bar{T}_i, \bar{L}_i), \bar{g'}_{i*}) \to 0$$

where $\bar{T}_i$ is the pull-back of $\tilde{K} \times \mathbb{R}^n$ using the retraction $r_i'g'$ and $\bar{g'}_i$ is the lifting of g'. But $\bar{g'}_{i*}$ is the zero map because the image of $\bar{g}'$ is contained in $\bar{L}_i$ ($i = 1, 2$). Therefore $(C_*(\bar{T}_i, \bar{L}_i), \bar{g'}_{i*}) = 0$ in $\widetilde{\mathrm{Nil}}_0(\mathbb{C}_n(\mathbb{Z}\pi))$. The exact sequence relation in the algebraic Nil-group implies

$$\alpha(M_1, f_1) = (C_*(\bar{L}_1, \tilde{K}\times\mathbb{R}^n), \bar{f'}_{1*}) = (C_*(\bar{T}_1, \tilde{K}\times\mathbb{R}^n), \bar{g'}_{1*}) =$$
$$(C_*(\bar{T}_2, \tilde{K}\times\mathbb{R}^n), \bar{g'}_{2*}) = (C_*(\bar{L}_2, \tilde{K}\times\mathbb{R}^n), \bar{f'}_{2*}) = \alpha(M_2, f_2)$$

(the third equality holds because g_1' is boundedly homotopic rel $K\times\mathbb{R}^n$ to g_2'.) The chain homotopy relation in $\widetilde{\mathrm{Nil}}_0^{(\infty)}(\mathbb{C}_n(\mathbb{Z}\pi))$ implies that α induces a map on $\pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n)$ and the exact sequence relation implies that α is a group homomorphism.

Claim: α is an epimorphism.
We represent an element of $\mathrm{NK}_{1-n}(\mathbb{Z}\pi)$ by a pair (F, ν) where F is an object in $\mathbb{C}_n(\mathbb{Z}\pi)$ and ν is a nilpotent endomorphism of F of nilpontency s. Then F is graded over $\mathbb{Z}^n$ i.e. $F = \oplus_{t\in\mathbb{Z}^n}F_t$, where F_t is a finitely generated based free $\mathbb{Z}\pi$-module of rank k_t for all t and ν is a bounded nilpotent endomorphism. Let L be the space obtained from $K\times\mathbb{R}^n$ ($K\times Q \cong X$) by attaching the wedge of k_t 2-spheres at $K\times\{t\}$ for each $t\in\mathbb{Z}^n$. The morphism ν induces a map $f' : L \to L$ extending the identity on $K\times\mathbb{R}^n$. The nilpontency condition on ν implies that that the map f'^s is boundedly homotopic (rel $K\times\mathbb{R}^n$) to a retraction. Set $M = L\times Q$ and $f = f'\times\mathrm{id}_Q$. Then (M, f) is a 0-nil simplex over X and $\alpha(M, f) = (F, \nu)$.
Claim: α is a monomorphism.
The following diagram commutes ([41], Proposition 4.3)

$$\begin{array}{ccccc}
\pi_0\widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n) & \xrightarrow{j_+} & \pi_0\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to \mathbb{R}^n) & \xrightarrow{\tau} & \widetilde{K}_{1-n}(\mathbb{Z}\pi_1(X\times S^1)) \\
{\scriptstyle\alpha}\Big\downarrow & & & & \Big\downarrow{\scriptstyle\cong} \\
\widetilde{\mathrm{Nil}}_0(\mathbb{C}_n(\mathbb{Z}\pi_1(X))) & \xrightarrow{\bar{N}_+'} & K_1(\mathbb{C}_n(\mathbb{Z}\pi_1(X)[z, z^{-1}])) & \xrightarrow{\beta} & K_1(\mathbb{C}_n(\mathbb{Z}\pi_1(X\times S^1)))
\end{array}$$

($\widetilde{K}_{1-n}$ denotes the Whitehead group for $n = 0$, the reduced K_0-group if $n = 1$ and the lower K-groups, K_{1-n}, if $n > 1$) The map τ is the torsion of an element in $\pi_0\mathcal{W}h(X\times S^1\times\mathbb{R}^n \to \mathbb{R}^n)$ and it is an isomorphism ([27]), $\bar{N}_+'$ is the geometrically significant injection map in [42], §10, and β is the isomorphism in [42], §11. The commutativity of the diagram implies that α is injective. Thus α is an isomorphism. $\square$

11.3. **Delooping of Nil-spaces.** We will construct a homotopy equivalence

$$d_{\mathcal{N}} : \widetilde{\mathcal{N}il}(X\times\mathbb{R}^n \to \mathbb{R}^n) \to \Omega\widetilde{\mathcal{N}il}(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}).$$

The definition of $d_{\mathcal{N}}$ is along the ideas developed in [1], [23] and [27].

We start by defining explicitly the delooping for Whitehead spaces as explained in §6.8. We will define an isomorphism

$$(d_{\mathcal{W}})_* : \pi_k\mathcal{W}(X\times\mathbb{R}^n \to \mathbb{R}^n) \to \pi_k\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1})$$

such that the following diagram commutes

$$\begin{array}{ccc} \pi_k\mathcal{P}(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) & \xrightarrow{\beta} & \pi_k\mathcal{W}(X\times\mathbb{R}^n \to \mathbb{R}^n) \\ {\scriptstyle (d_{\mathcal{P}})_*}\big\downarrow & & {\scriptstyle (d_{\mathcal{W}})_*}\big\downarrow \\ \pi_k\Omega\mathcal{P}(X\times\mathbb{R}^{n+2} \to \mathbb{R}^{n+2}) & \xrightarrow{\beta} & \pi_k\Omega\mathcal{W}h(X\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \end{array} \qquad (*)$$

where β is the isomorphism defined in [27], §6 and $d_{\mathcal{P}}$ is the homotopy equivalence defined in [23] and [1].

For the definition of $(d_{\mathcal{W}})_*$ we use an isomorphism

$$\kappa : \mathcal{W}h(X\times\mathbb{R}^n \to \mathbb{R}^n) \to \mathcal{W}h(X\times[0,1]\times\mathbb{R}^n \to \mathbb{R}^n)$$

induced by a homeomorphism $X\times[0,1] \to X$. Let $f : M \to X\times[0,1]\times\mathbb{R}^n\times\Delta^k$ be strong deformation retraction representing an element in $\pi_k\mathcal{W}h(X\times[0,1]\times\mathbb{R}^n \to \mathbb{R}^n)$. As in [27], §5, there is a strong deformation retraction

$$f' : X\times[0,1]\times[0,1]\times\mathbb{R}^n\times\Delta^k \to X\times[0,1]\times\mathbb{R}^n\times\Delta^k$$

such that $[f] = [f']$ in $\pi_k\mathcal{W}h(X\times[0,1]\times\mathbb{R}^n \to \mathbb{R}^n)$. Let

$$g : X\times\{0,1\}\times[0,1]\times\mathbb{R}^n\times\Delta^k \to X\times[0,1]\times\mathbb{R}^n\times\Delta^k$$

be a f.p. Z–set embedding approximating $f'|$. Using the estimated homotopy extension property we construct a strong deformation retraction

$$f'' : X\times[0,1]\times[0,1]\times\mathbb{R}^n\times\Delta^k \to X\times[0,1]\times\mathbb{R}^n\times\Delta^k$$

such that $[f'] = [f'']$ in $\pi_k\mathcal{W}h(X\times[0,1]\times\mathbb{R}^n \to \mathbb{R}^n)$ and

$$f''|X\times\{0,1\}\times[0,1]\times\mathbb{R}^n\times\Delta^k = g.$$

On the other hand, let

$$p : X\times\{0,1\}\times[0,1]\times\mathbb{R}^n\times\Delta^k \to X\times\{0,1\}\times\mathbb{R}^n\times\Delta^k,\ p(x,s,t,r,\delta) = (x,s,r,\delta)$$

and p' a f.p. Z–set embedding approximating p. We will further assume that

$$p'(X\times\{i\}\times[0,1]\times\mathbb{R}^n\times\Delta^k) \subset X\times\{i\}\times\mathbb{R}^n\times\Delta^k,\ i = 0,1.$$

By applying the f.p. Z–set unknotting theorem to the embeddings p' and g, we construct a strong deformation retraction

$$f''' : X\times[0,1]\times[0,1]\times\mathbb{R}^n\times\Delta^k \to X\times[0,1]\times\mathbb{R}^n\times\Delta^k$$

such that $[f''] = [f''']$ in $\pi_k\mathcal{W}h(X\times[0,1]\times\mathbb{R}^n \to \mathbb{R}^n)$ and

$$f'''|X\times\{0,1\}\times[0,1]\times\mathbb{R}^n\times\Delta^k = p'.$$

Now we can apply the usual construction for delooping. Let $\sigma : (0,1) \to \mathbb{R}$ be an increasing homeomorphism. Also, let

$$p'' : X\times\big((-\infty,0]\cup[1,\infty)\big)\times[0,1]\times\mathbb{R}^n\times\Delta^k \to X\times\big((-\infty,0]\cup[1,\infty)\big)\times\mathbb{R}^n\times\Delta^k$$

be a f.p. Z–set embedding approximating the projection map and extending p'. Define

$$\bar{f} : X\times\mathbb{R}\times[0,1]\times\mathbb{R}^n\times\Delta^k \to X\times\mathbb{R}\times\mathbb{R}^n\times\Delta^k$$

to be

$$\bar{f}(x,y,t,r,\delta) = \begin{cases} f'''(x,y,t,r,\delta), & \text{if } y\in(0,1) \\ p''(x,y,t,r,\delta), & \text{otherwise.} \end{cases}$$

We define $(d_{\mathcal{W}})_*([f])$ to be the class of $[\tilde{f}]$ where

$$\tilde{f} : X \times \mathbb{R} \times [0,1] \times \mathbb{R}^n \times \Delta^k \times [0,1] \to X \times \mathbb{R}^n \times \Delta^k \times [0,1]$$

is given by

$$\tilde{f}(x,y,t,r,\delta,u) = \begin{cases} (T^{-1}_{\sigma(u)} \bar{f} T_{\sigma(u)}(x,y,t,r,\delta),u) & \text{if } u \in (0,1) \\ p''(x,y,t,r,\delta,u) & \text{otherwise,} \end{cases}$$

where T_y denotes the homeomorphism induced on the corresponding space by translation, on the $\mathbb{R}$-direction, by y.

The proof that the diagram (*) commutes follows from the proof of Proposition 6.6 in [27]. Thus if we define

$$\begin{aligned} d_{\mathcal{W}} : & \mathcal{W}h(X \times \mathbb{R}^n \to \mathbb{R}^n) \xrightarrow{\simeq} \mathcal{W}h(X \times [0,1] \times \mathbb{R}^n \to \mathbb{R}^n) \xrightarrow{\simeq} \\ & \mathcal{P}(X \times [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \xrightarrow{d_{\mathcal{P}}} \Omega \mathcal{P}(X \times \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}) \xrightarrow{\simeq} \\ & \Omega \mathcal{W}h(X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}) \end{aligned}$$

then $d_{\mathcal{W}}$ is a homotopy equivalence inducing the homomorphism $(d_{\mathcal{W}})_*$ on the homotopy groups.

A similar construction defines a homomorphism

$$(d_{\mathcal{N}})_* : \pi_k \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \to \pi_k \Omega \widetilde{\mathcal{N}il}(X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1})$$

such that $(d_{\mathcal{N}})_* P_* = P_*(d_{\mathcal{W}})_*$, $\quad (d_{\mathcal{W}})_* J = J(d_{\mathcal{N}})_*$. Thus $(d_{\mathcal{N}})_*$ is an isomorphism and it is induced by a map

$$d_{\mathcal{N}} : \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \to \Omega \widetilde{\mathcal{N}il}(X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1})$$

which factors through the corresponding Whitehead spaces. Therefore $d_{\mathcal{N}}$ is a homotopy equivalence. Thus we have proved

Proposition 11.2. *There is a homotopy equivalence*

$$d_{\mathcal{N}} : \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \to \Omega^m \widetilde{\mathcal{N}il}(X \times \mathbb{R}^{n+m} \to \mathbb{R}^{n+m})$$

The following is the analogue for Nil spaces of Corollary 1 in [27].

Corollary 11.3. *There is an isomorphism*

$$\alpha : \pi_k \widetilde{\mathcal{N}il}(X \times \mathbb{R}^n \to \mathbb{R}^n) \to NK_{1-n+k}(\mathbb{Z}\pi_1(X)), \ \textit{if}\ 0 \le k \le n.$$

Proof. The proof follows by combining Proposition 11.2 and Proposition 11.1. ☐

12. Appendix: Controlled homotopies on mapping tori

We construct homotopy equivalences between adjunction spaces when the gluing maps are homotopic and we show that the homotopy equivalence is small when the homotopy is small for appropriate control maps. We use the above homotopies in the construction of small homotopy equivalences between mapping tori.

Let X and Y be Hilbert cube manifolds equipped with maps ρ and ρ' over Δ^k. In practice the maps ρ and ρ' are locally trivial bundles with Q-manifold fibers. Let $f_j : X\times\{0,1\} \to Y$, $j = 0, 1$, be f.p. proper maps. Form the adjunction spaces $X\times[0,1]\cup_{f_j} Y$, $j = 0, 1$ which are ANR's equipped with maps to Δ^k. Let $p : X\times[0,1]\cup_{f_1} Y \to B$ be a map to a metric space such that there is a positive number b,

$$\operatorname{diam}\{p(x,s) :\ s\in[0,1]\} < b, \text{ for all } x\in X.$$

Let $F : X\times\{0,1\}\times[0,1] \to Y$ be a proper f.p. bounded (in B) homotopy from f_0 to f_1 (with control the restriction $p|Y$).

Lemma 12.1. *There is a f.p. bounded homotopy equivalence from $X\times[0,1]\cup_{f_0} Y$ to $X\times[0,1]\cup_{f_1} Y$. Actually, there is an ANR T and CE-maps*

$$\tau^j : T \to X\times[0,1]\cup_{f_j} Y.$$

Proof. Let $F : X\times\{0,1\}\times[0,1] \to Y$ be the homotopy between f_0 and f_1 as above. We form the adjunction space $T = X\times[0,1]\times[0,1]\cup_F Y$. The maps ρ and ρ' induce a map on T over Δ^k. We will show that both adjunction spaces are strong deformation retracts of T. Choose strong deformation retractions

$$s^j : [0,1]\times[0,1] \to \{0,1\}\times[0,1]\cup[0,1]\times\{j\}, \quad j = 0, 1.$$

We choose the homotopies ϕ^j of the deformation retractions so that the tracks of the homotopies in $\{0,1\}\times[0,1]\cup[0,1]\times\{j\}$ are points i.e. $s^j\phi^j(t_1,t_2,t') = s^j(t_1,t_2)$ for all $t'\in[0,1]$. Crossing with X we get f.p. strong deformation retractions

$$X\times[0,1]\times[0,1] \to X\times\{0,1\}\times[0,1]\cup X\times[0,1]\times\{j\}, \quad j = 0, 1.$$

In terms, they induce strong deformation retractions

$$\sigma^j : T \to (X\times\{0,1\}\times[0,1]\cup X\times[0,1]\times\{j\})\cup_F Y, \quad j = 0, 1.$$

with the strong deformation retraction homotopies given by H^j_t $(j = 0, 1)$. By the construction of H^j, $\sigma^j H^j(x,t) = \sigma^j(x)$ for all $x\in T$. The retractions from T to $X\cup_{f_j} Y$ are induced by the homeomorphisms

$$h_j : (X\times\{0,1\}\times[0,1]\cup X\times[0,1]\times\{j\})\cup_F Y \to X\times[0,1]\cup_{f_j} Y, \quad j = 0, 1$$

given by

$$\begin{aligned} h_j(x,t,t') &= \begin{cases} (x,t,t') & \text{if } (x,t,t')\in X\times[0,1]\times\{j\} \\ F(x,t,t') & \text{if } (x,t,t')\in X\times\{0,1\}\times[0,1] \end{cases} \\ h_j(y) &= \quad y, \qquad\qquad \text{if } y\in Y \end{aligned}$$

We define the homotopy equivalence between the adjunction spaces as follows:

$$f : X\times[0,1]\cup_{f_0} Y \xrightarrow{h_1\sigma^1\iota_0 h_0^{-1}} X\times[0,1]\cup_{f_1} Y$$

with inverse $h_0\sigma^0\iota_1 h_1^{-1}$ where ι_j, $j = 0, 1$, are the inclusion maps. Thus the composite

$$(h_0\sigma^0\iota_1 h_1^{-1})(h_1\sigma^1\iota_0 h_0^{-1}) = h_0\sigma^0\iota_1\sigma^1\iota_0 h_0^{-1} : X\times[0,1]\cup_{f_0} Y \to X\times[0,1]\cup_{f_1} Y$$

is f.p. homotopic to the identity with homotopy $h_0\sigma^0 H_t^1\iota_0 h_0^{-1}$. Notice that the homotopy is the identity on Y. Composing with the control map we get

$$(ph_1\sigma^1\iota_0 h_0^{-1})(h_0\sigma^0 H_t^1\iota_0 h_0^{-1}) = ph_1\sigma^1\iota_0\sigma^0 H_t^1\iota_0 h_0^{-1}.$$

Using the properties of the maps we get for $(x,s)\in X\times[0,1]$,

$$\begin{array}{rcll} \iota_0 h_0^{-1}(x,s) & = & (x,s,0) & \Rightarrow \\ H_t^1\iota_0 h_0^{-1}(x,s) & = & (x,s_1',s_2') & \Rightarrow \\ \iota_0\sigma^0 H_t^1\iota_0 h_0^{-1}(x,s) & = & (x,s_1'',s_2'') \Rightarrow & \\ ph_1\sigma^1\iota_0\sigma^0 H_t^1\iota_0 h_0^{-1}(x,s) & = & ph_1(x,s_1'',s_2'') & . \end{array}$$

If $(x,s)\notin X\times\{0,1\}$ then $(x,s_1')\notin X\times\{0,1\}$ and $(x,s_1'')\notin X\times\{0,1\}$. Then

$$\iota_0\sigma^0 H_t^1\iota_0 h_0^{-1}(x,s) = (x,s_1'',1)$$

and $ph_1(x,s_1'',s_2'') = p(x,s_1'')$. Then

$$\mathrm{diam}\{ph_1(x,s_1'',s_2'') :\ s_1'',s_2''\in[0,1]\} < b.$$

If $(x,s)\in X\times\{0,1\}$ then $(x,s_1'')\in X\times\{0,1\}$ and

$$h_1(x,s_1'',s_2'') = F(x,s_1'',s_2'').$$

Let $b' > 0$ be the bound of F. Then

$$\mathrm{diam}\{pF(x,s_1'',s_2'') :\ s_2''\in[0,1]\} < b'$$

which implies that

$$\mathrm{diam}\{ph_1(x,s_1'',s_2'') :\ s_2''\in[0,1]\} < b'.$$

and

$$\mathrm{diam}\{ph_1(x,s_1'',s_2'') :\ s_1''\in\{0,1\},\ s_2''\in[0,1]\} < 2b'.$$

Therefore the homotopy $h_0\sigma^0 H_t^1\iota_0 h_0^{-1}$ is a f.p. bounded homotopy. Similarly, we can show that the other composition

$$h_0\sigma^0\iota_1 h_1^{-1} : X\times[0,1]\cup_{f_1} Y \to X\times[0,1]\cup_{f_1} Y$$

is f.p. boundedly homotopic to the identity. Therefore $X\times[0,1]\cup_{f_0} Y$ is f.p. boundedly homotopy equivalent to $X\times[0,1]\cup_{f_1} Y$.

The maps $\tau^j = h_j\circ\sigma^j$, $j=0,1$, are CE-maps by construction. □

We now apply Lemma 12.1 in constructing homotopy equivalences between mapping tori of homotopic maps. We use the definition of the mapping torus as a coequalizer (Chapter 10).

Proposition 12.2. (i) *Let X be an ANR equipped with a map to Δ^k and a control map $p : X \to B$ to a metric space B. Let $f,g : X \to X$ be two f.p. boundedly homotopic (over B) maps. Then there is an ANR T and f.p. CE-maps*

$$T(f) \xleftarrow{\alpha} T \xrightarrow{\beta} T(g)$$

which induce a f.p. bounded homotopy equivalence.

(ii) *Let X and Y be ANR's as in Part(i) and*

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

be proper maps. Then there is an ANR T' and f.p, CE-maps

$$T(gf) \xleftarrow{\alpha'} T' \xrightarrow{\beta'} T(fg).$$

Proof. Part (i) follows immediately from Lemma 12.1. The homotopy equivalence for Part (ii) is constructed as in [42], Proposition 13.18, Proposition 14.2. The fact that the homotopy equivalences constructed are bounded follows from Lemma 12.1. □

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