29. Mathematische Arbeitstagung

Bonn, 23. – 29. Juni 1990

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Programm der Mathematischen Arbeitstagung 1990 (I)

<u>Samstag, den 23.6.1990</u>

16.00 - 17.00 Uhr — M.F. ATIYAH: Topology and quantum field theory

<u>Sonntag, den 24.6.1990</u> *TAG DER KOMPLEXEN ANALYSIS*

10.15 - 11.15 Uhr — H. GRAUERT: Extension of quotient sheaves to edges

12.00 - 13.00 Uhr — T. OHSAWA: Geometric and analytic aspects of the L^2 theory in complex analysis

17.00 - 18.00 Uhr — S.K. DONALDSON: Differential topology and complex variables

<u>Montag, den 25.6.1990</u>

10.00 - 10.15 Uhr — Festlegung der nächsten Vorträge

10.15 - 11.15 Uhr — N. ELKIES: On Mordell-Weil lattices

12.00 - 13.00 Uhr — I. CHEREDNIK: Kac-Moody algebras and conformal field theory

17.00 - 18.00 Uhr — H. LENSTRA: On the 9^{th} Fermat number

<u>Dienstag, den 26.6.1990</u>

10.15 - 11.15 Uhr — CH. SOULÉ: Arithmetic Riemann-Roch theorems

Die Vorträge finden alle im *Großen Hörsaal*, Wegelerstraße 10, statt.
Erfrischungspausen mit Tee: Sonntag und Montag, 11.15-12.00 Uhr und 16.00-17.00 Uhr vor dem Großen Hörsaal.
Teilnehmerlisten und *Informationen* liegen vor dem Großen Hörsaal aus.
Alle Teilnehmer werden gebeten, sich in die Listen einzutragen.
Post liegt während der Teepausen aus.
Den *Tagungsbeitrag* bitte in den Teepausen vor dem Großen Hörsaal bezahlen.
Alle Tagungsteilnehmer mit ihren Damen oder Herren sind herzlich zum *Empfang des Rektors* eingeladen. Zeit: Montag, 25.6., 20.00 Uhr. Ort: Festsaal der Universität (Hauptgebäude); Eingang von der Straße "Am Hof" durch das Tor gegenüber Buchhandlung Röhrscheid.

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Dienstag, den 26.6.1990

13.00 Uhr	Schiffsausflug nach Linz. Abfahrt um 13.00 Uhr mit Motorschiff "Carmen Sylva", Ablegestelle Alter Zoll. Rückkehr ca. 19.30 Uhr.

Mittwoch, den 27.6.1990

10.15 - 11.15 Uhr	M. ZAIDENBERG: Acyclic affine curves and surfaces. Exotic algebraic structures on $\mathbb{C}^n$
12.00 - 13.00 Uhr	K. RUBIN: Euler systems and applications to elliptic curves and ideal class groups
17.00 - 18.00 Uhr	V. POENARU: New infinite processes in 3-dimensional topology

Donnerstag, den 28.6.1990

10.15 - 11.15 Uhr	U. HAMENSTÄDT: Rigidity of geodesic flows (on a conjecture of Katok)
12.00 - 13.00 Uhr	M. KRECK: Positive scalar curvature, $P_2(\mathbb{H})$, and elliptic homology
17.00 - 18.00 Uhr	H. BAUM: Killing spinors on Riemannian manifolds

Freitag, den 29.6.1990

10.15 - 11.15 Uhr	A. JUHL: Selberg ζ-functions for locally symmetric spaces of higher rank
12.00 - 13.00 Uhr	G. WÜSTHOLZ: Diophantine approximation (report on work of Vojta, Faltings and Bombieri)
16.30 - 17.30 Uhr (Kolloquium)	S. LANG: The non-Mordellic subset of an algebraic variety

Die Vorträge finden alle im *Großen Hörsaal* statt. *Erfrischungspausen mit Tee:* Mittwoch, Donnerstag und Freitag 11.15 - 12.00 Uhr, Mittwoch und Donnerstag 16.00 - 17.00 Uhr, Freitag 16.00 - 16.30 Uhr jeweils vor dem Großen Hörsaal. *Teilnehmerlisten* und *Informationen* liegen vor dem Großen Hörsaal aus. *Post* liegt während der Teepausen aus. Den *Tagungsbeitrag* bitte während der Teepausen vor dem Großen Hörsaal bezahlen.

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Titel: TOPOLOGY AND QUANTUM FIELD THEORY

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§1 Introduction

In recent years strong links have been developed between topology and quantum field theory. More specifically E. Witten, in a series of papers, has shown that many theories which produce topological invariants of manifolds can be interpreted as quantum field theories, with the invariants being expectation values or correlation functions. In particular Donaldson theory in dimension 4 and Jones theory in dimension 3 fit into this framework.

In theories such as Donaldson theory the quantum expectation values can be computed exactly in the semi-classical approximation by integrals over the moduli space of classical solutions. This suggests that similar integrals over other moduli spaces should also correspond to quantum theories.

Very recently there has been an explosive development by various groups of physicists [2] [3][4] of "2-dimensional gravity", based on matrix models. On the other hand Witten has developed a "topological gravity" theory [7] by working with the moduli spaces of Riemann surfaces. Witten [8] has provided strong evidence that his theory is equivalent to the matrix model theory. In this lecture I will outline all these ideas, following [8].

§2 Topological Gravity

Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces (algebraic curves) of genus g with n ordered marked points. This has a natural compactification $\overline{\mathcal{M}}_{g,n}$ given by "stable curves", where we allow double points on the curve but no coincidences of marked points.

On $\overline{\mathcal{M}}_{g,n}$ there are n line-bundles $\mathcal{L}_1, \dots \mathcal{L}_n$ given by the cotangent spaces at the marked points. Their Chern classes $c_1(\mathcal{L}_i)$ give cohomology classes and by forming monomials of the appropriate degree we get topological invariants

$$\left\langle \prod c_1(\mathcal{L}_i)^{d_i}, \overline{\mathcal{M}}_{g,n} \right\rangle \qquad \sum d_i = 3g-3+n .$$

These invariants are actually rational numbers (because $\overline{\mathcal{M}}_{g,n}$ has orbifold singularities). We denote them by $\langle \tau_0^{n_0} \tau_1^{n_1} \dots \rangle$ where the sequence $d_1, d_2, \dots$ contains 0 (n_0 times), 1 (n_1 times) etc. We then form the generating function

$$F(t_0, t_1, \dots) = \sum_{\{n_i\}} \prod_i \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \dots \rangle$$

summing over all sequences $\{n_i\}$ and putting $\langle 1 \rangle = -\frac{1}{12}$. This function F encodes all the invariants for all values of g and n.

The problem of calculating F would then be solved by the following:

THEOREM 1 $$\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i} \qquad \text{(String Equation)}$$

THEOREM 2 Put $U = \dfrac{\partial^2 F}{\partial t_0^2}$, then this obeys the KdV hierarchy

$$\frac{\partial U}{\partial t_n} = \frac{\partial R_{n+1}}{\partial t_0}(U, U', U'' \dots)$$

[Here the R_{n+1} are the polynomials in U and its t_0-derivatives which arise in the KdV theory - giving commuting flows]

At present Theorem 1 has been proved, and Theorem 2 has been verified for the contribution to F from $g \leq 3$. There is also a physical argument [6] claiming to establish Theorem 2.

§3 Triangulation of surfaces

Let T_n be the set of isomorphism classes of triangulations of a surface of genus g by n triangles, and put $V(g,n) = \sum_{T_n} \frac{1}{|\mathrm{Aut}\, T_n|}$. Then one can show that, for $n \to \infty$,

$$\text{(3.1)} \quad V(g,n) \sim e^{cn} n^{\gamma(2-2g)-1} b_g \left(1 + O\left(\tfrac{1}{n}\right)\right).$$

If we replace triangles by squares, pentagons etc we get a similar asymptotic formula with a different c but the same γ and the same b_g (up to a scaling factor t^{1-g}).

More generally consider using n squares ($n \to \infty$) and a few other shapes, i.e. u_2 2-gons, u_4 4-gons

u_6 6-gons etc. Then we get an asymptotic formula

(3.2) $W(g,n;u_2,u_4,\cdots) \sim e^{cn}\, n^{\gamma(2-2g)-1+\sum_{r=1}^{\infty}\delta_r u_{2r}} f_g(u_2,u_4,\cdots)$

(using matrix models, as in §4)

By very skilful argument it has been shown that an appropriate generating function made out of the $f_g(u_2,u_4,\cdots)$ satisfies Theorems 1 and 2 of §2. Witten's Conjecture (Theorem 2) is that the f_g are essentially the same as the $\langle \tau_0^{h_0}\tau_1^{h_1}\cdots\rangle$.

§4 Matrix Models

Let $W(g,n)$ be defined by squares as in §3 (put all $u_{2i}=0$) and define the generating function

$$F(N,\lambda) = \sum_{g=0}^{\infty}\sum_{n=0}^{\infty} W(g,n)\,\lambda^n N^{2-2g}$$

It is then a remarkable fact [1] that, by direct expansion, one can prove

(4.1) $\log F(N,-\lambda) = \int dM \exp - \mathrm{Tr}\left(\frac{M^2}{2} + \frac{\lambda M^4}{4N}\right)$

where we integrate over all $N\times N$ Hermitian matrices and dM is the Euclidean measure suitably normalized.

The asymptotic formulae of §3 are then obtained by a "double scaling limit" in which $N\to\infty$ and simultaneously $\lambda\to\lambda_c$, where $\lambda_c = -e^c$ and $N^2(\lambda-\lambda_c)^{-2\gamma}$ is kept fixed (c,γ as in (3.2)).

The integral in (4.1) is studied by diagonalizing M and then skilfully using orthogonal polynomials and Jacobi matrices.

One studies $W(g,n;u_2,u_4\cdots)$ by similar methods.

Autor: Michael Atiyah Seite: 5

Further Comments

(i) The above theory is called "pure gravity". It can be significantly generalized by "coupling to matter".

(ii) There is a close relation between these ideas and the computation of the Euler characteristic of M_g by Harer - Zagier [9] and Penner [5].

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① <u>Gap sheaves.</u> X = complex space, $\mathcal{S}$ = coherent analytic sheaf on X, $\mathcal{I} \subset \mathcal{S}$ coherent subsheaf, $\mathcal{R} = \mathcal{S}/\mathcal{I}$ quotient sheaf:

<u>Definition.</u> Take $q = 0, 1, 2, \dots$; denote $\mathcal{I}_q$ = maximal coherent subsheaf of $\mathcal{S}$ such that 1) $\mathcal{I} \subset \mathcal{I}_q$ 2) $\dim \operatorname{supp} (\mathcal{I}_q / \mathcal{I}) \leq q$ then $\mathcal{R}_q = \mathcal{S} / \mathcal{I}_q$ is the q-th gap sheaf of $\mathcal{R}$.

$\mathcal{R}_q$ is coherent; the bigger q the more $\mathcal{R}_q$ is filled.

<u>Example.</u> $A \subset X$ analytic set, $A = \bigcup A_\nu$ decomposition into irreducible components, $\mathcal{S} = \mathcal{H}_X$ structure sheaf, $\mathcal{I}$ = ideal sheaf of A, $B = \bigcup_{\dim A_\nu > q} A_\nu$, $\mathcal{H}_B$ = structure sheaf of B. Then $\mathcal{R}_q = \mathcal{H}_B$.

This will mean: extension of coherent quotient sheaves more general than analytic extension of analytic sets.

② <u>q-convexity.</u> $G \subset \mathbb{C}^n$ domain, $\sigma \in G$, φ real C^∞-function on G, $T_\sigma^{(\varphi)} = \{ \xi \in T_\sigma : \sum_{\nu=1}^{n} \xi_\nu \varphi_{z_\nu}(\sigma) = 0 \} \subset T_\sigma$ complex subvectorspace

= analytic tangent, where T_0 = tangent space in 0.

$L(\varphi) = \sum_{\nu,\mu=1}^{n} \varphi_{z_\nu \bar z_\mu}(0)\, dz_\nu\, d\bar z_\mu$ = Levi form in 0 = Hermitian form.

<u>Definition.</u> φ q-convex in 0 iff $L(\varphi) | T_0^{(a)}$ has $n-q$ positive eigenvalues, at least. φ = q-convex iff in every point of G.

X = complex space, φ real on X. The space X locally embeddable into domain $G \subset \mathbb{C}^N$. If φ always locally restriction of a q-convex function in G then φ q-convex on X.

Assume φ = q-convex on X, $c \in \mathbb{R}$. Then inner domain $X' = \{x \in X : \varphi(x) < c\}$ called q-convex, outer domain $X'' = \{x \in X : \varphi(x) > c\}$ q-concave. In this case:

<u>Theorem.</u> S coherent on X, $\mathcal{J} \subset S | X''$ coherent, $0 \in \partial X''$, $\hat q \geq q$. Then $R_{\hat q}$ has unique extension to 0.

<u>Proof</u>: [ST], p. 148.

<u>Example</u>: $Y \subset X = G \subset \mathbb{C}^n$ analytic set, $\dim Y < q$. Then $X'' = G - Y$. q-concave. Now $A \subset G - Y$ analytic set, $\dim A > q$ everywhere (i.e. $\dim A \geq \dim Y + 2$).

Then A extendable to G.

This is a special case of the Remmert-Stein theorem [RS]. If only $\dim A = q$ Remmert-Stein remains true, but for coherent $R = R_{q-1}$ it is false, in general.

③ Edges. $\varphi_1, \ldots, \varphi_\ell$ q-convex in X; $X'' = \bigcup \{\varphi_\nu > c_\nu\}$ with $c_\nu \in \mathbb{R}$. Then X'' = q-concave with edges.

In the edges $\partial X''$ very concave, but even analytic sets $A \subset X''$ with $\dim A > q$ every-where cannot be extended to edges. However, this is possible for $R = R_{2\hat{q}-1}$, $\hat{q} \geq q$ (smoothening theorem of M. Peternell).

Assume X = n-dimensional compact complex manifold, $Y \subset X$ complex submanifold of codimension q, normal bundle $N(Y)$ positive (in the most general sense), as always in the case $X = \mathbb{P}_n$. Then there exists a tube $U(Y)$ with a differentiable real φ in $U - Y$, q-convex and $\lim_{x \to Y} \varphi(x) = \infty$.

Problem. Can such a φ be extended to $X - Y$ as a differentiable function with edges, q-convex outside a closed set $A \subset X - Y$ with A nowhere dense, not locally separating X.

Solved $q = 1$. Then A = analytic. In case $q > 1$ the set A will not be analytic, in general. But theorems on extension and of other kind will follow. A brief construction principle is known.

Literature

[RS] R. Remmert and K. Stein: Über die wesentlichen Singularitäten analytischer Mengen. Math. Ann. 126 (1953).

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Titel: *Geometric and Analytic aspects of L^2 theory on complex manifolds*
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Let (X, ds^2) be a Hermitian manifold of dimension n, $C_0(X)$ the set of compactly supported differential forms on X, and let $d : C_0(X) \circlearrowleft$ be the exterior derivative. With respect to the L^2 norm, the minimal (resp. the maximal) closed extension of d is defined by $d_{min} = (d^*)^*$ (resp. $d_{max} = ((d^*|_{C_0(X)})_{min})^*$), where $*$ denotes the (Hilbert space) adjoint. The r-th L^2 de Rham cohomology of X w.r.t. ds^2 is defined as $H^r_{(2)}(X) = \mathrm{Ker}\, d_{max} \cap \overline{C^r_0(X)} / \mathrm{Im}\, d_{max} \cap \overline{C^r_0(X)}$, where $\overline{C^r_0(X)}$ denotes the completion of $C^r_0(X)$.

General Question: What is the relation between $H^r_{(2)}(X)$ $(r \geq 0)$ and the topology of the completion of X.

Cheeger-Goreski-MacPherson's conjecture (a specialized Q.)
Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a projective variety of dimension n, and let V' be the set of nonsingular points of V. Then, with respect to the Fubini-Study metric ds^2 (restricted to V'), $H^r_{(2)}(V') \cong IH^r(V)$, where IH^r denotes the r-th intersection cohomology with the middle perversity.

<u>Main Theorem</u> C-G-M conjecture is true if the singular points of V are isolated.

Sketch of Proof:

1. Known facts. Let $H^r(V')$ (resp. $H^r_0(V')$) denote the r-th de Rham cohomology group of V' (resp. that with compact support). Then the assertion is equivalent to the following.

$$(*)\qquad H^r_{(2)}(V') \cong \begin{cases} H^r(V') & \text{if } r<n \\ \mathrm{Im}(H^n_0(V')\to H^n(V')) & \text{if } r=n \\ H^r_0(V') & \text{if } r>n. \end{cases}$$

Lemma 1 $\lim\limits_{K\Subset V'} H^r_{(2)}(V'\setminus K)=0$ if $r>n$ (Publ. RIMS, '87)

There exists a similar L^2-vanishing result for the formal adjoint complex $\cdot\xrightarrow{\delta}\cdot\xrightarrow{\delta}\cdot$ to $\cdot\xrightarrow{d}\cdot\xrightarrow{d}\cdot$, which together with Lemma 1 establishes the following.

Theorem (*) is true if $r \neq n, n\pm 1$. Moreover $\dim H^{n\pm 1}_{(2)}(V') \le \dim IH^{n\pm 1}(V)$ and $\dim H^n_{(2)}(V') \ge \dim IH^n(V)$.

2. Saper's theorem (a stronger version in [ ]) There exists a complete Kähler metric of the shape $ds^2+\partial\bar\partial\psi$ for some C^∞-function $\psi: V'\to(-\infty,0]$ for which (*) holds. According to a computation based on the formula

$$[d, d\psi^*] + [(d^c)^*, d\psi^c] = [\sqrt{-1}\,dd^c\psi, \Lambda],$$

where $C=\sum \sqrt{-1}^{\,p-q}\,\Pi_{p,q}$, $\Pi_{p,q}$: projections to (p,q)-components, and Λ the adjoint of multiplication by the fundamental form of a prescribed Kähler metric, we have for all $f\in \overline{C^r_0(V')}_{ds^2+\partial\bar\partial\psi}\cap \mathrm{Dom}\,d\cap\mathrm{Dom}\,d^*$, $\|f\|\le C(\|f\|_K+\|df\|+\|d^*f\|)$. (**)

Here C and K do not depend on f.

By shifting the metric $ds^2 + \partial\bar{\partial}\psi$ to ds^2 by a path $ds^2 + \varepsilon\partial\bar{\partial}\psi$, $\varepsilon \in [0,1]$, we obtain the equality for the dimensions by establishing an L^2 estimate sharper than (**).

[O] A Nonexistence Theorem for L^2 Harmonic Forms on Stratified Riemannian Manifolds and its Application to Hodge Theory, to appear in Proc. of Taniguchi Symposium '89.

Titel: Differential Topology and Complex Variables
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Yang-Mills instantons define invariants which link the differential topology and geometry of a complex surface. In outline, the invariant are defined by evaluating certain "universal" cohomology classes on the fundamental homology class of ~~the~~ a moduli space M of stable holomorphic bundles on a surface X. The invariants actually depend on the differential topology of X, since we can interpret the stable holomorphic bundles as Yang-Mills instantons, and carry out the same definition for any Riemannian metric on X. To make the definition rigorous one needs also to discuss compactness properties of M, and to arrange some general position properties by, if necessary, deforming the problem to a nearby space M_ε; thought of as lying in the space $\mathcal{A}/\mathcal{G}$ of all equivalence classes of connections.

These invariant seem to be most useful for understanding the differential-topological meaning of the canonical class $c_1(K_X) \in H^2(X)$. In this first talk we focus on a problem raised by Hirzebruch:

Question What are the complex structures on the differentiable manifold $S^2 \times S^2$?

The standard examples are the (rational) Hirzebruch surfaces and question is essentially whether there is a surface X of general type diffeomorphic to $S^2 \times S^2$. Our approach to this problem is to try to find an invariant which distinguishes the manifolds differentiably. We can suppose ~~there~~ a diffeomorphism $f: S^2 \times S^2 \to X$ takes positive classes to positive classes, so in complex geometry the canonical classes are opposite:

$$K_X = \mathcal{O}(2,2) \quad , \quad K_{S^2 \times S^2} = \mathcal{O}(-2,-2)$$

using standard notation

There is a standard technique for describing rank 2 bundles over surfaces. (This technique has been used in work of other authors on the diff. topology of rational & irrational surfaces, including Friedman - Morgan, Okonek - Van de Ven, Kotschick, Mong and others.) Let $V \to X$ be a 2-bundle with a section s having isolated (non-degenerate) zeros. We fix an isomorphism $\Lambda^2 V \cong L^{\otimes 2}$ say, then we have data: $r_i = \Lambda^2 (ds)^{-1}_{x_i} \in (K \otimes L^2)^{-1}_{x_i}$. Sections σ of $K \otimes L^2$ give constraints on the data (r_i, x_i) which can arise.

One gets the following general picture: let us consider bundles E with $c_1(E) = 0$ and put $V = E \otimes L$. We have a decomposition

$$M = \bigcup_L M_L \quad , \quad \text{where } N_L \to M_L$$

and N_L is the moduli space of pairs $(E \otimes L, s)$. We construct a "secant bundle" R of all sets of data (r_i, x_i): so R maps to $s^{\ell}(X)$ with generic fibre $\mathbb{P}\left(\bigoplus (K \otimes L^2)_{x_i}\right)$. Over R there is a natural line bundle $\mathcal{U}$ and a section σ of $(K \otimes L^2) \to X$ induces a section $\tilde{\sigma}$ of $\mathcal{U} \to R$. A compactification $\overline{N}_L$ of N_L appears as the complete intersection:

$$\overline{N}_L = \bigcap_{\sigma} \text{Zeros of } \tilde{\sigma} \subset R.$$

One would like to make calculations in the cohomology of R. For example if $\dim R = 3\ell - 1 = \dim H^0(K \otimes L^2)$ then N_L is "generically" finite and $\nu = |N_L| = \langle c_1(\mathcal{U})^{\dim R}, [R] \rangle$. For $\ell = 1, 2$ this is ~~an~~ a standard calculation; e.g. for $\ell = 2$ if $K \otimes L^2$ gives an embedding in $\mathbb{P}^4$ ν is the number of double points.

For larger ℓ, Xu Ming-Wei has recently shown that formulae expressing ν in terms of standard invariants exist, and written them down explicitly in some cases.

In our case we consider the moduli space of bundles with $c_1 = 0$ & $c_2 = 2$. To describe the generic bundle one needs $l = 14$ points and this is too complicated to be practical at the moment. Instead we use a particularly convenient invariant. For any L the homology class of

$$\Delta = M_L \cup M_{K \otimes L^{-1}} \subset M$$

defines a smooth invariant (via an interpretation as the kernel of the Dirac operator). Choose $L = \mathcal{O}(2,0)$, so $K \otimes L^{-1} = \mathcal{O}(0,2)$ and in both cases $l = 2$. Δ is finite and the invariant is $\delta = |\Delta|$. One has to consider the double points of degree 24 surfaces in $\mathbb{P}^4$.

Theorem. (i) $\delta_{S^2 \times S^2} = 0$

(ii) If $H^0(X; \mathcal{O}(2,0)) = H^0(X, \mathcal{O}(0,2)) = 0$ then $\delta_X = 252$.

For the proof we have to compare deformations in R and $\mathcal{A}/\mathcal{G}$. Very likely the auxiliary hypothesis can be removed, thus answering Hirzebruch's problem.

Titel: *On Mordell-Weil lattices*

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OVERVIEW: A standard number-theoretical construction of lattices out of the sets of solutions to certain Diophantine equations turns out in some cases to be surprisingly relevant to the problem of packing congruent spheres in high-dimensional Euclidean spaces. We begin by introducing lattices and their associated sphere packings, and indicate how the lattices obtained below compare with previous constructions. We then outline the construction of a lattice from a general elliptic curve over a rational function field and exhibit the families of curves found independently by Shioda and the present author to give rise to good sphere packings. Finally we have a closer look at one of those families, including a report on recent work improving the estimates on their densities in high dimensions and the significance of these estimates for the arithmetic of the curves.

LATTICES: We essentially follow Chapter 1 of [C-S] here. GENERALITIES: A <u>lattice</u> L in n-dimensional Euclidean space is a discrete co-compact subgroup of the group of translations of the space; by picking an origin O in the space we identify L with the discrete subset $L(O)$ of the space. Thus L is a discrete copy of $\mathbb{Z}^n$ in $\mathbb{R}^n$, but $\mathbb{Z}^n$ regarded as an abstract group (not span of n distinguished generators), and likewise $\mathbb{R}^n$ has no distinguished coordinates but the Euclidean structure specifies an inner product $(v,v') = \frac{1}{2}[Q(v+v') - Q(v) - Q(v')]$ where Q is the positive-definite quadratic form $Q(v) = |v|^2$. Restricting Q to the lattice gives of course a pos.-def. quad. form on L; conversely, given such Q on $L \cong \mathbb{Z}^n$ we make L a lattice by extending Q to $L \otimes_{\mathbb{Z}} \mathbb{R}$, which thus becomes an n-dim. Euclidean space in which L is embedded as a lattice.

To do computations we can abandon this intrinsic approach, choosing generators $v_1, \ldots, v_n$ for L and orthonormal coordinates on $\mathbb{R}^n$ to write each v_i as a column vector $\overset{\downarrow}{v_i}$ and associate to $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ its $n \times n$ <u>generator</u> <u>matrix</u> $M = [\overset{\downarrow}{v_1}\, \overset{\downarrow}{v_2} \cdots \overset{\downarrow}{v_n}]$. Different choices of generators and coords multiply M on the right by $GL_n(\mathbb{Z})$ and on the left by $O_n(\mathbb{R})$ respectively, and $L = M \cdot {}^t\mathbb{Z}^n$. To avoid the choice of coordinates in $\mathbb{R}^n$, introduce the <u>Gram</u> <u>matrix</u> $A = {}^tMM = [(v_i, v_j)]_{i,j=1}^n$, well-defined up to $A \leftarrow {}^tgAg$ for $g \in GL_n(\mathbb{Z})\ (/\{\pm 1\})$. This is a symmetric $n \times n$ matrix; it is positive definite: if $v = \sum_i c_i v_i$ then $|v|^2 = |M\,{}^tc|^2 = cA\,{}^tc$. Now M is far from canonical, but $|\det M|$ is unchanged by mult. by $GL_n(\mathbb{Z})$ or $O_n(\mathbb{R})$, so must be an invariant of L itself; indeed it is the <u>covolume</u> or <u>sparsity</u> of L: $|\det M| = \mathrm{Vol}(\mathbb{R}^n/L)$. Also $\det A = \det^2 M$ is known as the <u>discriminant</u> of the lattice.

Further invariants: the <u>dual</u> <u>lattice</u> L^* of L is $L^* = \{u \in \mathbb{R}^n \mid (u,v) \in \mathbb{Z} \text{ for all } v \in L\}$. Generator and Gram matrices for L^* are ${}^tM^{-1}$ and A^{-1} respectively. L is <u>integral</u> if $(v,v') \in \mathbb{Z}$ for all

$v, v' \in L$ ($\Leftrightarrow L \subseteq L^* \Leftrightarrow A$ has integer entries), in which case its discriminant is $[L^* : L]$. In particular if $L = L^*$ ($\Leftrightarrow \det A = 1 \Leftrightarrow \det M = \pm 1$), L is said to be <u>unimodular</u>. A sufficient condition for integrality is is $Q(v) \in 2\mathbb{Z}$ for all $v \in L$; L is then called <u>even</u> ($\Leftrightarrow A$ integral with even diagonal entries); it is well-known that a lattice simultaneously unimodular and even can exist in $\mathbb{R}^n$ if and only if n is divisible by 8 (recall [Se]). Finally the <u>symmetry</u> (or <u>isometry</u>) <u>group</u> of L is the group of linear transformations of $\mathbb{R}^n$ that both permute L and preserve the inner product on $\mathbb{R}^n$; this is $\{g \in GL_n(\mathbb{Z}) \mid {}^t g A g = A\}$, and is a finite subgroup of $O_n(\mathbb{R})$.

[<u>Examples</u>: $n=1$ — any lattice is $\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}^*(/\{\pm 1\})$; here $M = [\alpha]$, $A = [\alpha^2]$, $L^* = \alpha^{-1}\mathbb{Z}$, L integral $\Leftrightarrow \alpha^2 \in \mathbb{Z}$, L even $\Leftrightarrow \frac{1}{2}\alpha^2 \in \mathbb{Z}$, L unimodular $\Leftrightarrow \alpha^2 = 1$, symmetry gp. $= \pm 1$.

$n=2$ — consider the <u>triangular lattice</u> scaled to minimal distance $\sqrt{2}$. (a chunk of L is represented by the points $\odot$ at right.) Choosing generators v_1, v_2 we may take $M = \begin{bmatrix} 2^{1/2} & 2^{-1/2} \\ 0 & (3/2)^{1/2} \end{bmatrix}$ and find $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, so L is even integral, of discriminant 3 with dual lattice $L^* = \mathbb{Z}v_1 + \mathbb{Z}\frac{v_1+v_2}{3}$ (points of $L^* - L$ marked by $\times$ at right), and symmetry gp. the 12-element dihedral gp. D_6. Already for $n=2$ we see that the choice of M or even A is in no way canonical ($A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ or even $\begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$ is reasonable) and that these coordinate-bound presentations of a lattice can obscure most of its symmetry.]

LATTICE PACKING OF SPHERES. The covolume tells us the asymptotic distribution of L: as $r \to \infty$, the ball $B(r)$ of radius r about O contains asymptotically $\frac{\mathrm{Vol}\, B_0(r)}{\mathrm{Covolume\ of\ } L}$ points of L (that's why "covolume" is also known as "sparsity"). A finer invariant is the distribution of L in spherical shells about O ("theta function"). Of particular interest is the smallest shell $|v| = r_0 > 0$ containing lattice pts. This r_0 is the <u>minimal distance</u> of L, and r_0^2 is by abuse of language the <u>minimal norm</u> N_0 (really minimal nonzero norm; least positive real represented by the form Q). Equivalently $\frac{r_0}{2}$ is the largest radius r such that the open balls $B_v(r)$: $v \in L$ do not overlap. (Think about the diagram of triangular L above.) [This also motivates calling $K = \#\{v \in L \mid |v| = r_0\}$ the <u>kissing number</u> of L, because $\{B_v(r) \mid |v| = r_0 = 2r\}$ is a configuration of K disjoint balls tangent to the central ball $B_0(r)$ of the same radius.] The <u>density</u> $d(L)$ of this packing of congruent balls in $\mathbb{R}^n$ is then $\mathrm{Vol}(B(r))/\mathrm{Vol}(\mathbb{R}^n/L) = (\mathrm{disc}\, L)^{-1/2}\gamma_n r^n = (\mathrm{disc}\, L)^{-1/2}\gamma_n (N_0/4)^{n/2}$ where γ_n, the volume of a unit ball in $\mathbb{R}^n$, is $\pi^{n/2}/\Gamma(\frac{n}{2}+1) \sim \left(\frac{2\pi e}{n}\right)^{n/2} \cdot$ (polynomial factor) by Stirling. Note that $d(L)$ is <u>scaling-invariant</u>: $d(\alpha L) = d(L)$, $\alpha \in \mathbb{R}^*$. So we ask, following Hilbert: <u>What is the largest possible value of $d(L)$ for a lattice $L \subset \mathbb{R}^n$?</u>

<u>Summary of known results</u>: there are basically 4 regimes, for different sizes of n:

<u>$1 \leq n \leq 8$</u>: It is known that $d(L)$ is maximized by the <u>root lattices</u> $\Lambda_n = \mathbb{Z}, A_2$ (above), $A_3, D_4, D_5, E_6, E_7, E_8$. This sequence may also be also be obtained by <u>lamination</u>: Λ_n is the union of parallel layers Λ_{n-1} stacked as close as possible, using the <u>holes</u> of Λ_{n-1} (points in $\mathbb{R}^n$ as distant as possible from the nearest point of Λ_{n-1}; thus the holes of $\alpha\mathbb{Z}$ are $\alpha(\mathbb{Z}+\frac{1}{2})$ and the holes of the triangular lattice are $L^* - L$): $\Lambda_8 = E_8$ is the unique even unimodular lattice in $\mathbb{R}^8$, and arises in mathematical fields as diverse as finite and Lie groups, theoretical physics, and coding/communication theory.

$8<n\le 24$: In no dimension $n>8$ is a lattice currently proved to maximize $d(L)$, but up to dim. 24 there are some very good candidates. In particular, continuing the lamination process past E_8 (and avoiding a misstep in dimension 12) we reach in dim. 24 a unique laminated lattice, the Leech lattice Λ_{24}, which is the unique even unimodular lattice in $\mathbb{R}^{24}$ of minimal norm 4, and like E_8 shows up in many different context. For each $n\le 24$, the best known lattice in $\mathbb{R}^n$ is a rank-n slice of Λ_{24}, and the orthogonal slice of Λ_{24} gives the record in $\mathbb{R}^{24-n}$. Particularly good among these records are the Coxeter-Todd lattice K_{12} (even; with disc $=3^6$, $N_0=4$; not a laminated lattice) and the Barnes-Wall lattice $BW_{16}\cong\Lambda_{16}$ (even, disc $=2^8$, $N_0=4$, laminated).

$24<n\lesssim 10^3$: Here the situation is less satisfactory; there are various competing constructions, often messy and coordinate-bound, but one does not expect that the last word has been said with the possible exceptions of $n=32$ and $n=48$ (lattices Q_{32} (Quebbeman), $P_{48p/q}$; even, with disc. 2^{16}, 1 respectively, and $N_0=6$).

$n\gtrsim 10^3$: Past $n\sim 10^3$ our state of knowledge is even worse: it's been known since Minkowski that "most" lattices have $d(L)>2^{-n}$; none of the known constructions improve on this, and it seems very hard to prove (as some "pessimists" expect) that 2^{-n} is in essence best possible.

[WARNINGS: — For $n>2$ there may be nonlattice sphere packings denser than any $d(L)$; indeed for $n=10,11,13$ there are nonlattice (though still periodic) packings that beat the best known lattices.
— in general $\kappa(L)$ and $d(L)$ may be maximized by different lattices L; there can even be local arrangements of κ disjoint spheres, tangent to and congruent to a central one, which come from no lattice at all, with $\kappa>\kappa(L)$ for any lattice L in the same dimension; this happens for $n=9$. However, in each of the five cases $n=1,2,3,8,24$ in which the kissing problem is solved, the maximum is attained by a lattice ($L=\mathbb{Z}, A_2, A_3, E_8, \Lambda_{24}$) which also gives the best-known sphere packing in $\mathbb{R}^n$.
— Given a matrix M or A, we determine for the corresponding lattice L the rank n by inspection, and the covolume easily, but as far as we know computing the minimal norm N_0 is computationally intractable for large n and general L. In particular
— We cannot even attain Minkowski's bound $d(L)\ge 2^{-n}$ by any explicit lattice for $n\ge 10^3$; even constructing infinite families with density provably $\ge 2^{-cn}$ as $n\to\infty$ for fixed $c<\infty$ is hard; $c<1.2$ has recently been attained by Quebbeman, Tsfasman et al., using quite beautiful ideas from number theory which however have little to do with the methods described below.]

We now indicate three closely related families of constructions which produce lattices in $\mathbb{R}^n$. All three constructions improve on the previous records in dimensions throughout the range $64\le n\lesssim 10^3$, producing truly coordinate-free lattices in these dimensions; two of the families also recover the known record lattices in smaller dimensions in a uniform way, and provide a new viewpoint on these lattices — for instance the Leech lattice will appear as simply the equation $y^2+y=x^3+t^{13}$.

ELLIPTIC CURVES. An <u>elliptic curve</u> over a field k is a curve$/k$ of genus 1 with a k-rational point. Embed the curve as a nonsingular cubic in $P_2(k)$ with affine equation (<u>Weierstrass form</u>) $E: y^2+a_1xy+a_3y=x^3+a_2x^2+a_4x+a_6$, $a_i \in k$, with the pt. at ∞ as the distinguished k-rational point. The study of such equations has been a mainstay of number theory from Fermat's age to the present; for an excellent modern introduction, containing all we'll use here and much more besides, see [Si]. It is well known that the set of k-rat'l points $E(k)$ [$=\{\infty\} \cup \{$solutions $(x,y)\in k^2$ to the Weierstrass equation$\}$] naturally forms an <u>abelian group</u> with neutral element ∞. Concerning the structure of this group, <u>Mordell</u> showed when $k=\mathbb{Q}$ that $E(k)$ is <u>finitely generated</u>, in effect generalizing Fermat's "method of descent", and <u>Weil</u> later generalized it further, proving $E(k)$ finitely generated whenever k is a number field or a <u>function field</u> such as $F(t)$, the field of rational functions in one indeterminate over a finite field F. [This is an example of the deep analogy between number and function fields in modern number theory.] Essentially these f.g. groups $E(k)$ (Mordell-Weil gp. of E/k) will be our lattices. Now it is not known how to construct elliptic curves over $\mathbb{Q}$ with large rank, but over $F(t)$ curves of arbitrarily high rank were constructed by Šafarevič and Tate a generation ago [Š-T]; we will use much the same construction, and henceforth restrict to the case $k=F(t)$.

The other ingredient we need to make a lattice is a quadratic form on $E(k)$. This is provided by the <u>Néron-Tate canonical height</u>, a function $\hat{h}: E(k)\to \mathbb{R}^{\geq 0}$ characterized by the properties: (i) $\hat{h}(\infty)=0$, $\hat{h}(x(t),y(t))=\deg x(t)+O(1)$ [so $\hat{h}$ measures the "complexity" of a solution to the Diophantine equation E]; (ii) $\hat{h}$ satisfies the parallelogram identity $\hat{h}(P+P')+\hat{h}(P-P')=2(\hat{h}(P)+\hat{h}(P'))$ for all $P,P'\in E(k)$ and is thus a quadratic form on $E(k)$; (iii) $\hat{h}$ is positive definite on the free abelian group $E(k)/$torsion. Thus $E(k)/$torsion is a lattice in $\mathbb{R}^n$ for $n=$rank of $E(k)$; this is the <u>Mordell-Weil lattice</u> of the elliptic curve E/k. That is Tate's algebraic approach; Néron proceeds more geometrically by considering the Weierstrass form of E as a single equation in the three variables x,y,t defining some <u>surface</u> $\mathcal{E}$, with the generic fiber of the map $t: \mathcal{E}\to\mathbb{P}^1(F)$ being the elliptic curve E; a point of $E(k)$ then becomes a <u>section</u> of that map, and the height is reinterpreted via intersection numbers on the surface $\mathcal{E}$. Either way, for the curves E that we will use it's easy to see that $\hat{h}$ actually takes values in $\mathbb{Q}$, and one naturally finds a subgroup $E_0(k)$ of finite index in $E(k)$, the <u>narrow</u> <u>M.-W. gp.</u> of E, in which $x(t)$ always has even degree (indeed all its poles have even order) and the correction $O(1)$ in $\hat{h}=\deg x+O(1)$ disappears; the corresponding <u>narrow</u> <u>M.-W. lattice</u> is then an even integral lattice, and when it is properly contained in the full M.-W. lattice the omitted points are candidate holes for the narrow lattice.

We now exhibit the three families of curves E, depending on the characteristic p of the field F:

(I) $p\equiv 5 \bmod 6$: $y^2=x^3+t^f-1$, f a factor of $p^\alpha+1$ for some odd integer α. These curves and the sphere-packing properties of their Mordell-Weil lattices were independently discovered by

Shioda [Sh] in the course of his work on the geometry of Fermat surfaces in finite characteristic. (note that the elliptic surface E here is a quotient of the $(p^\alpha+1)$st Fermat surface):
II $p=2$: $y^2+y=x^3+t^f$; III $p=3$: $y^2=x^3-x+t^f$; in both cases f divides $p^\alpha+1$ for some integer α. In all cases f should be either $p^\alpha+1$ itself or $\frac{p^\alpha+1}{c}$ for a small cofactor c, and (up to twist) F can be taken to be the finite field of $p^{2\alpha}$ elements.

Unlike the case of a lattice defined by matrices, we can here directly find the <u>minimal norm</u>, or at least an excellent lower bound for it: in all three cases <u>$\hat{h}(P)\geq\frac{1}{3}f$</u> for any $P\neq 0$ in $E(k)$. Computing the <u>rank</u> n can be tricky for general E, but for our families we can adapt the method of [Š-T] to obtain <u>$n\sim 2f$</u> (more precisely $n=2f-4$ in the family I, $n=2f-2$ in II and III). But curiously it is the <u>discriminant</u> that poses the greatest problem. There is a formula (Tate's analog [Ta] in the function-field case of the conjecture of Birch and Swinnerton-Dyer for elliptic curves/$\mathbb{Q}$, which in our cases is a theorem of Tate and Milne [M]): $[\mathrm{disc}(L)][|Ш|]=[\text{known quantity}]$, where Ш is a certain finite group (the <u>Tate-Šafarevič group</u> of $E(k)$) — in particular $|Ш|\geq 1$ — and the "known quantity" grows essentially as $(n/2)^{n/6}$ in all 3 families. Combining the bounds $N_0\geq\frac{f}{3}\approx\frac{n}{6}$ and $\mathrm{disc}(L)\lesssim\left(\frac{n}{2}\right)^{n/6}$ we find that the density is at least $\approx\left(\frac{n}{2}\right)^{-n/12}\left(\frac{\pi e}{12}\right)^{n/2}$. This bound is sharp if $|Ш|=1$; now Ш is hard to compute in general, but if Ш is trivial the fact can be verified with relative ease, and this has been done in all interesting cases of rank up to about 100 by combining calculations of Shioda, B. Gross and myself. It is clear that the bound $\left(\frac{n}{2}\right)^{-n/12}\left(\frac{\pi e}{12}\right)^{n/2}$ falls rapidly below Minkowski's 2^{-n} once n is sufficiently large — specifically $n\gtrsim 2\left(\frac{\pi e}{3}\right)^6$, which curiously is again approximately 10^3. But below $n\sim 10^3$ these lattices beat not only Minkowski but most of the previously best known constructions as well. Thus for family I Shioda and I independently observed that a new record is first obtained in dimension $n=80$ by taking $p=41$ and $f=42$.

For the family II even more can be said. Here even small values of f are of interest, reproducing record lattices in lower dimensions in a uniform manner: for $f=3,5,9,13$ the narrow Mordell-Weil lattice is isomorphic to $D_4, E_8, BW_{16}, \Lambda_{24}$ respectively, and in the cases $f=3,9$ the full M.-W. lattice gives holes that yield the best laminations D_5, E_8 and $\Lambda_{17},\Lambda_{18}$ of D_4 and BW_{16} respectively. [Since the record lattices in small dimensions all have discriminants containing only small prime factors it is clear that we could not expect to find them from I. The lattices E_6 (again) and K_{12} are produced by taking $f=4$, $f=7$ in III. The unimodular Λ_{24} requires an additional "miracle", as usual for the Leech lattice: the disc. of the M-W lattice is 2^{24}, but all the heights $\hat{h}(x,y)=\deg x$ can be shown to be <u>doubly</u> even; thus $\frac{1}{2}\hat{h}$ is a unimodular even lattice without norm-2 vectors (because $3\hat{h}\geq 13$), and so isomorphic to Leech.] Going further with $f=2^\alpha+1$ we find at $f=17$ a 32-dim. lattice with the same discriminant and minimal norm as (and likely isomorphic to) Quebbeman's record lattice of the same dimension, and for $f=33$ the first new record in dim. 64. Furthermore, for all $f=2^\alpha+1$ I can estimate the discriminant <u>by elementary means</u>, without appealing to the (disc)(|Ш|) theorem, by identifying a natural sublattice (spanned by all $(x(t),y(t))$ with $x(t)=x_0+x_1t+x_2t^2+x_4t^4+\dots+x_{2^{\alpha-1}}t^{2^{\alpha-1}}$ with $x_0,x_{2^i}\in F$) of the M.-W. lattice with a scaled copy of the Barnes-Wall lattice of the same dimension $2^{\alpha+1}$ (this works also with $t^{2^\alpha+1}$ replaced with more general poly(t)'s of the same degree as in [VV], and uses Gross' extension [G] of Thompson's work [Th] on invariant lattices in strongly irreducible representations of finite groups); knowing the discriminant of this sublattice and bounding from below its index in the full M-W group gives an upper bound on the discriminant of the full M-W lattice (and also other information which for instance explains the $4|\hat{h}$ "miracle" for $f=13$ via consideration of $f=65=5\cdot 13=2^6+1$). For the first few α this new bound coincides with the $|Ш|\geq 1$ bound, but for large f this bound is better, giving density $> \exp -O(n\sqrt{\log n}\log\log n)$ rather than $\gtrsim \exp(-\frac{n}{12}\log n)$. Thus Ш is large for $f\to\infty$, in sharp contrast to the behavior expected from the cases $n\leq 100$ with $|Ш|=1$: we have $|Ш|\geq 2^{12}$ for $\alpha=9$, and for $\alpha=28$ (the largest I could compute) Ш is of order $> 2^{5\cdot 10^8}$!

OPEN ENDS: All of this is very recent work, and almost certainly far from its final form. Promising further directions include: — determination of the exact density and other invariants (kissing number, symmetry group, ...) of the lattices obtained in (II); exploration of possible connections with coding theory suggested by the appearance of Barnes-Wall lattices and the hyperelliptic curves of [VV], both connected with the Reed-Muller codes. — Finding analogous approaches to the odd-characteristic families (I) and (III); in (I) Shioda already has (using very different methods) lower bounds on $|Ш|$ that are large as $\alpha\to\infty$ but probably not asymptotically sharp (note that for $y^2+y=x^3+t^{2^\alpha+1}$ we had $|Ш|=\exp[(1-o(1))\frac{n}{12}\log n]$ as $\alpha\to\infty$). — Using function fields k other than the rational one; Gross has already found alternative Mordell-Weil constructions of K_{12} and Λ_{24} with k of positive genus [G]. One can also contemplate exploiting intersection theory on surfaces other than elliptic ones, and perhaps even on varieties of higher dimension.

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Titel:
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Kac-Moody algebras and Conformal Field Theory (r-matrix Knizhnik-Zamolodchikov equations).

The system of differential equations for n-point functions from the two-dimensional conformal field theory was calculated by Knizhnik and Zamolodchikov in 1984 [1]. Now we realize that this one is very important for mathematics as well. It is precisely between the following main mathematical regions:

REPRESENTATION THEORY	TOPOLOGY (BRAIDS, ...)
KZ	
ANALYSIS (SPECIAL FUNCTIONS, V. NEUMANN FACTORS...)	ALGEBRAIC GEOMETRY, ARITHMETIC (CURVES, ...)

I'll define the so-called r-matrix KZ-equation [2] and formulate its key property. This equation is very convenient as some starting point to understand the generalized Knizhnik-Zamolodchikov equations and to appreciate the true possibilities of Kac-Moody algebras in mathematics and physics.

Let $\mathfrak{g}$ be a simple Lie algebra, $r(u)$ be a function of $u \in \mathbb{C}$ taking values in $\mathfrak{g} \otimes \mathfrak{g}$. We assume that $r(u) = t/u + \tilde{r}(u)$, where $t = \sum_\alpha I_\alpha \otimes I_\alpha$, $\{I_\alpha\}$ is an orthonormal basis with respect to the Killing form $(x,y)_K = Sp(ad_x\, ad_y)$, $x, y \in \mathfrak{g}$, $\tilde{r}(u)$ is an analytic function in some neighbourhood $U_0 \subset \mathbb{C}$ of $u=0$. Given a, b from the universal enveloping algebra $U(\mathfrak{g})$ (or from any of its quotients) let ${}^1a = a \otimes 1 \otimes 1 \otimes \ldots$, ${}^2a = 1 \otimes a \otimes 1 \otimes \ldots, \ldots,$ ${}^{ij}(a \otimes b) = {}^i a\, {}^j b$. For example, ${}^{13}(a \otimes b) = a \otimes 1 \otimes b$, ${}^{21}(a \otimes b) = b \otimes a$. We impose the condition ${}^{12}r(u) + {}^{21}r(-u) = 0$.

The matrix language. Given $\mathfrak{g}$-modules $V_1, \ldots, V_n$ we consider $\mathbf{V} = V_1 \otimes V_2 \otimes \ldots \otimes V_n$ as a $\mathfrak{g}^n$-module: $\mathfrak{g}^n = \mathfrak{g} \times \ldots \times \mathfrak{g}$ (n times), $(g_1 \times \ldots \times g_n)(v_1 \otimes \ldots \otimes v_n) = (g_1 v_1) \otimes v_2 \otimes \ldots + v_1 \otimes (g_2 v_2) \otimes v_3 \otimes \ldots + \ldots$. Then one has the homomorphism $U(\mathfrak{g})^{\otimes n} \to \mathrm{End}\, \mathbf{V}$. Let us denote by ${}^{ij}\bar{r}$ the image of ${}^{ij}r$ in $\mathrm{End}\, \mathbf{V}$. The r-matrix KZ-equation (KZ) is defined as follows:

$$\{ \varkappa\, \partial W / \partial u_i = \Big(\sum_{j \neq i} {}^{ij}\bar{r}(u_i - u_j)\Big) W,\ 1 \le i, j \le n,\ \varkappa \in \mathbb{C} \},$$

where the values of $W = W(u_1, \ldots, u_n)$ are from $\mathbf{V}$, $u_1, \ldots, u_n \in U_0$.

For $r=t/u$ one gets the KZ (and Kohno) system. Arbitrary r are expected to be connected with Wess-Zumino-Witten models with the asymmetrical chiral Lagrangian (a constant matrix between two currents). The most interesting generalization is to involve moduli of curves and $\mathfrak{g}$-bundles. One more way to generalize (KZ) is to consider other root systems in place of the set of the arguments $\{u_i-u_j\}$ (the latter is nothing else but the root system of type A_{n-1}). These equations (see [2,3]) are good candidates for WZW-models with boundary conditions.

<u>Loop Lie algebras:</u> ${}^i\mathbf{G}=\mathfrak{g}((\tilde{u}_i))=\{\sum_{k\geq p} g_k\tilde{u}_i^{\,k}, p\in\mathbb{Z}\}$, ${}^i\mathbf{G}_0=\mathfrak{g}[[\tilde{u}_i]]=\{\sum_{k\geq 0} g_k\tilde{u}_i^{\,k}\}$, where $1\leq i\leq n$, $\tilde{u}_i=u-u_i$ is the local parameter at $u=u_i$, $g_k\in\mathfrak{g}$. Let us introduce $\prod_{i=1}^{n}{}^iG=\mathbf{G}\supset\mathbf{G}_0=\prod_{i=1}^{n}{}^iG_0$.

<u>Theorem 1.</u> The following 3 assertions are equivalent:

(i) equations (KZ) are pairwise compatible (satisfy the cross-derivative integrability conditions);

(ii) $[{}^{13}r(u)+{}^{23}r(v), {}^{12}r(u-v)]=[{}^{13}r(u), {}^{23}r(v)]$ for any u,v;

(iii) Setting $\quad \mathbf{g}_r(u)=\sum \mathrm{Res}\,(r(u-v), g(v))_K\, dv$,
where $g\in G$, $(a\otimes b, c)_K \overset{\mathrm{def}}{=} (b,c)_K\, a$, the sum is over all components (and poles of g), $u\in U_0$, the values of g_r are in $\mathfrak{g}$, $g_r(u)$ is identified with the direct product of its expansions with respect to $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n$ at $u_1, \ldots, u_n$,

then the vector space $G_r = \{g_r, g \in G\} \subset G$ is a Lie subalgebra. ■

The equivalence of (ii) and (iii) is of great importance for the classical soliton theory. As for (i)$\Leftrightarrow$(ii) one can reduce the problem to the case $n=2$. Here $G_0 \subset G \supset G_r$ plays the role of the well-known triple: integer adeles $\subset$ adeles $\supset$ principal adeles. But we have the decomposition $G = G_0 \oplus G_r$, which is non-arithmetic. In fact, r-matrices from (ii) are in one-to-one correspondence with such Lie subalgebras $G_r \subset G$, that $G = G_0 \oplus G_r$. This statement can be made more precise, if one reformulate the unitary condition $({}^{12}r(u) + {}^{21}r(-u) = 0)$ and impose some other properties (see e.g. [4]). The general KZ-equation is connected with any discrete G_r $(\dim_{\mathbb{C}}(G_0 \cap G_r) < \infty)$ with compact quotient-spaces $(\dim_{\mathbb{C}}(G/(G_0 + G_r)) < \infty)$. These G_r correspond to the pairs {a curve + some principal g-bundle on it}.

Now we need the Kac-Moody algebra $\hat{G} = G \oplus \mathbb{C}c$:

$$[x + \xi c, y + \zeta c] = [x, y] + \sum \mathrm{Res}(dx, y)\, c, \quad dx = (\partial x/\partial u)\, du,$$

where $x = \prod_{i=1}^{n} x_i \subset G \supset \prod_{i=1}^{n} y_i = y$, $[x,y] = \prod_{i=1}^{n} [x_i, y_i]$, then the elements of $\deg = -1$ from the Virasoro algebra at $u_1, \dots, u_n$:

$${}^{i}L = \sum_{k \geq 0} \sum_{\alpha} (I_\alpha \tilde{u}_i^{-1-k})(I_\alpha \tilde{u}_i^{k}), \quad 1 \leq i \leq n,$$

the Verma module $M = \mathrm{Ind}_{\hat{G}_0}^{\hat{G}} V$, where $\hat{G}_0 = G_0 \oplus \mathbb{C}c$ acts on V with respect to the natural projection $G_0 \xrightarrow{\text{"the value"}} g^n$, $c \to \sigma \in \mathbb{C}$ (it is nothing else but the maximal $\hat{G}$-module generated by V as a $\hat{G}^0$-module), and the coinvariant: $\tau(m) \in V$, $m - \tau(m) \in G_r M$.

The last definition is correct ($\exists !$), since G_r is a Lie subalgebra in $\hat{G}$ (not only in G) and $\hat{G} = \hat{G}_0 \oplus G_r$. We have used the letter "τ", because the coinvariant is a direct generalization of τ from [4], generalizing in its turn the well-known τ-function by Date, Jimbo, Kashiwara, Miwa, which is in its turn closely connected with some Kac-Peterson construction.

Theorem 2. Given a $\mathbb{C}$-algebra F of functions in $u_1, \dots, u_n$ let $\tilde{M} = M \otimes_{\mathbb{C}} F \ni \tilde{m}$; $\partial/\partial u_i$ operate on F only. Then

$$-\tau((\varkappa \partial/\partial u_i + {}^iL)\tilde{m}) = (\varkappa \partial/\partial u_i - \sum_{j \neq i} {}^{ij}r(u_i - u_j))\tau(\tilde{m}) + \underline{nit},$$

where $\varkappa = \sigma + 1/2$, nit is a non-interesting term. ■

This theorem (proved in [5], p. 8-9) resembles the well-known identity $e^{\pi i} = -1$. It connects together 4 main objects of the modern representation theory ($\tau, L, r, \sigma + 1/2$). The action of ${}^iL$ on $\tilde{M}$ is "constant" ($\partial\, {}^iL/\partial u_j \overset{def}{\equiv} 0$). Hence, $W = \tau(\exp(-\sum_{i=1}^{n} (u_i/\varkappa)\, {}^iL)m)$, $m \in M$, is a generic solution of (KZ) in the sense of the RIMS papers on τ-functions.

This formula is, in fact, indefinite. Nevertheless it works well. It gives the Kac-Moody interpretation, the r-matrix generalization and the short proof ([5], p. 19-27) of the Schectman-Varchenko theorem [6], generalizing in its turn integral formulas for (KZ) by Dotsenko, Fateev, Aomoto, Christe, Flume, Date, Jimbo, Matsuo, Miwa, Lawrence. The formulas of this type should be connected with formulas, that can be obtained by means of [7] and other recent papers on the so-called "bosonization".

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Titel: THE NINTH FERMAT NUMBER

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In this lecture it is explained by which method the ninth Fermat number was factored. This was recently done by a team consisting of hundreds of individuals (and their machines), directed by A. K. LENSTRA (BELLCORE, New Jersey) and M. S. MANASSE (DEC SRC, California).

For $k \in \mathbb{Z}_{\geq 0}$, the k-th Fermat number F_k is defined by $F_k = 2^{2^k} + 1$. The number $F_9 = 2^{512} + 1$ has 155 decimal digits. A brief history:

- ≈ 1640: P. de Fermat expresses his strong belief that all F_k are prime. For $0 \leq k \leq 4$ he is right, but so far (1990) no other 'Fermat prime' has been found.
- 1738: Euler finds that $F_5 = 641 \cdot 6700417$.
- 1877: Pépin proves: F_k is prime $\Longleftrightarrow 3^{(F_k-1)/2} \equiv -1 \bmod F_k$ $(k>0)$.
- 1880: Landry & Le Lasseur factor F_6.
- 1903: A. E. Western finds that $2424833 = 37 \cdot 2^{16} + 1$ divides F_9.
- 1967: Brillhart proves that $F_9/2424833$ is composite.
- 1970: Morrison & Brillhart factor F_7.
- 1980: Brent & Pollard factor F_8. Now $F_9/2424833$ is the 'most wanted' number. It has 148 decimal digits.
- 1990: the New York Times, June 20, writes that $F_9/2424833$ has 7455602 8256478 8420833 7395736 2004549 1878336 6342657 as a prime factor of 49 digits, and that the cofactor, which has 99 digits, is also prime.

The factorization of F_9 was accomplished by means of the <u>number field sieve</u>, which is a new factoring algorithm that was proposed by J. M. POLLARD in 1988. It is more elementary than some other factoring algorithms, but it uses a new ingredient, namely algebraic number theory.

Like several other factoring algorithms, the number field sieve depends on the fact that, for a positive odd integer n, one has

$$\dim_{\mathbb{F}_2} \{x \in \mathbb{Z}/n\mathbb{Z} : x^2 = 1\} = \#\{p : p \text{ is prime}, p \mid n\}.$$

Hence if n is not a prime power, then some $x \in \mathbb{Z}/n\mathbb{Z}$ satisfies $x^2 = 1$, $x \neq \pm 1$. For any such x one has $(x+1)(x-1) = 0$, $x+1 \neq 0$, $x-1 \neq 0$, so that a non-trivial factorization of n is given by $n = \gcd(n, x-1) \cdot \gcd(n, x+1)$; the gcd's can be determined with Euclid's algorithm. The question remains how to solve $x^2 = 1$ in $\mathbb{Z}/n\mathbb{Z}$. In many factoring algorithms this is done by the following scheme:

- generate non-zero elements $a_i \in \mathbb{Z}/n\mathbb{Z}$, $i \in I$. These will usually belong to the unit group $(\mathbb{Z}/n\mathbb{Z})^*$ (and if not, a non-trivial factor of n is given by $\gcd(a_i, n)$). Let $\mathbb{Z}^I \longrightarrow (\mathbb{Z}/n\mathbb{Z})^*$ be the group homomorphism that sends the i-th standard basis vector to a_i. Often it is <u>onto</u>.
- generate relations <u>between</u> the a_i, i.e. elements of the kernel K of $\mathbb{Z}^I \rightarrow (\mathbb{Z}/n\mathbb{Z})^*$. Once one has a little more than $\# I$ relations, they generate often (almost) all of K.
- find $v \in \mathbb{Z}^I$ with $2v \in K$. This can be done by solving a system of $\# I$ linear equations in $\geq \# I$ unknowns over $\mathbb{F}_2$. If $v \mapsto x \in (\mathbb{Z}/n\mathbb{Z})^*$ then $x^2 = 1$. If $x \neq \pm 1$ then this splits n, and otherwise one tries a different v.

Different factoring algorithms have different ways to generate the a_i and the relations between them.

<u>Example</u>. Let the a_i be the elements $(p \bmod n)$, where p ranges over the prime numbers up to a suitably chosen bound B. To find relations, one can search for small integers b for which both b and $n+b$ are built up from those primes. (This can be done by means of a sieve.) Each such b gives a relation between the $p \bmod n$, by $b \equiv n+b \bmod n$. In this method, B must be chosen optimally: if B is chosen too

small, then not enough b can be found for which $b(n+b)$ is built up from the primes $\leq B$; and if B is chosen too large the linear system over $\mathbb{F}_2$ at the end becomes unwieldy. This method is not used in practice. Instead, one uses a variant (the <u>quadratic sieve</u>) in which the numbers that should be built up from small primes are not of order of magnitude n (such as the $b(n+b)$ above) but of order of magnitude $\sqrt{n}$. This clearly is an improvement, but for $F_9/2424833$ it is not good enough.

<u>The number field sieve</u>. In this method the numbers that should be built up from small primes are still smaller, since one uses higher degree number fields. I illustrate the method with $n = F_9/2424833$. The number field one uses is $\mathbb{Q}(\sqrt[5]{2})$. Its ring of integers is $\mathbb{Z}[\sqrt[5]{2}]$, which is a unique factorization domain. From $2^{512} \equiv -1 \bmod n$ one sees that $(2^{205})^5 = 2^{1025} \equiv 2 \bmod n$, so there is a ring homomorphism $\varphi: \mathbb{Z}[\sqrt[5]{2}] \longrightarrow \mathbb{Z}/n\mathbb{Z}$ sending $\sqrt[5]{2}$ to $2^{205} \pmod{n}$. The element $\alpha = \sqrt[5]{2}^3$ maps to -2^{103} which is quite 'small' (only 32 digits). To describe the α_i, $i \in I$, I let the set I consist of

- the first 99700 prime numbers p (up to $\approx 1.5 \cdot 10^6$);
- three elements that generate the unit group $\mathbb{Z}[\sqrt[5]{2}]^*$;
- one generator π for each of the 99500 smallest (measured by norm) prime ideals of $\mathbb{Z}[\sqrt[5]{2}]$ of the first degree (i.e., the residue class field is a <u>prime</u> field).

For $i \in I$, let $\alpha_i = \varphi(i)$. To find relations among the α_i, one searches for pairs a, b of coprime rational integers for which <u>both</u> $a - b\alpha$ is built up from the small prime ideals (exercise: each prime ideal dividing $a - b\alpha$ is of the first degree!) <u>and</u> $a + b2^{103}$ is built up from the small prime numbers. Factoring $a - b\alpha$ in $\mathbb{Z}[\sqrt[5]{2}]$, and $a + b\,2^{103}$ in $\mathbb{Z}$, and using that $\varphi(a - b\alpha) = \varphi(a + b\,2^{103})$, one sees that each such pair a, b gives a relation between the α_i.

For each b, the search for a can be done by means of a <u>sieving</u> technique. In practice, $2.2 \cdot 10^6$ values for b were used (all satisfying $1 \leq b \leq 2.5 \cdot 10^6$), and for a the interval $|a - b\sqrt[5]{8}| < 5 \cdot 10^7$ was searched. Then $|a + b \cdot 2^{103}| \lesssim 2.5 \cdot 10^{37}$, and the number $|a^5 - 8b^5|$ (which is built up from small prime numbers if and only if $a - b\alpha$ is built up from small prime ideals) is $\lesssim 3 \cdot 10^{38}$. Thus the numbers that one wants to be built up from small primes are much smaller than before.

On a modern workstation, one value for b can be dealt with in ≈ 7 minutes. Several hundreds of workstations were used, which communicated their relations by electronic mail to the headquarters. All values for b could be done in this way in less than two months. Actually only ≈ 50000 relations were found in this way, which is far too little, even if one also uses the 4944 'free' relations that come from totally splitting primes. 170000 additional relations were found by means of the 'large prime variation', which I do not describe here.

One now has a system of ≈ 200000 linear equations over $\mathbb{F}_2$ in ≈ 220000 unknowns to solve. By means of a 'structured' Gaussian elimination algorithm, depending on the sparsity of the matrix, this was reduced to a 72200×72500 system, which was solved on a large Connection Machine in Florida in three hours. The first solution gave $x = -1$, the second gave a non-trivial factorisation of n. The factors were proved prime by Andrew Odlyzko.

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Note.

If you are interested in assisting A. K. LENSTRA and M. S. MANASSE in their ongoing factorization efforts, please send electronic mail to

factor@src.dec.com

Your message should contain the (appropriately modified) lines

program#
path# yourlogin@yourhost.yourdomain

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Titel: ARIHMETIC RIEMANN-ROCH THEOREMS

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Joint work with H. GILLET

1) Let E be a lattice ($E \simeq \mathbb{Z}^n$) and h a positive definite quadratic form on $E_{\mathbb{R}} = E \underset{\mathbb{R}}{\otimes} \mathbb{Z}$.

let $\overline{E} = (E, h)$ and

$H^0(\mathbb{Z}, \overline{E}) = \{ s \in E \,/\, \|s\| \leq 1 \}$.

Choose a Haar measure on $E_{\mathbb{R}}$, let B_n be the unit ball in $E_{\mathbb{R}}$, and $\#$ the cardinality of a finite set. Let $\overline{E}^* = (E^*, h^*)$ be the dual of $\overline{E}$.

<u>Theorem 1</u>:

$$6^{-n} \mathrm{vol}(B_n) \leq \frac{\# H^0(\mathbb{Z}, \overline{E})}{\# H^0(\mathbb{Z}, \overline{E}^*)} \mathrm{vol}(E_{\mathbb{R}}/E) \leq 6^n \mathrm{vol}(B_n)^{-1}$$

2) Let X be a regular projective flat scheme over $\mathbb{Z}$. We endow $X(\mathbb{C})$ with a Kähler metric and let ω_0 be the Kähler form divided by 2π.

Let $\overline{E} = (E, h)$ be the pair formed by an algebraic vector bundle on X and an hermitian metric on the corresponding holomorphic vector bundle $E_{\mathbb{C}}$ on $X(\mathbb{C})$.

We assume that both the Kähler metric and the metric h are invariant under the complex conjugation F_∞ acting on $X(\mathbb{C})$.

Let $H^q(X, E)$ be the coherent cohomology of X with coefficients in E $(q \geq 0)$. This is a finitely generated $\mathbb{Z}$-module with torsion subgroup $H^q(X, E)_{tors}$. Its complex span $H^q(X, E) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ is isomorphic to

$$H^q(X(\mathbb{C}), E_{\mathbb{C}}) \simeq \operatorname{Ker} \Delta_q \quad ,$$

where $\underbrace{\Delta_q = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}}_{\text{is the Laplace operator}}$ on the space A^{0q} of forms of type $(0,q)$ on $X(\mathbb{C})$ with coefficients in $E_{\mathbb{C}}$. Given $\eta, \eta' \in A^{0q}$ we define their L^2-scalar product by the formula

$$\langle \eta, \eta' \rangle_{L^2} = \int_{X(\mathbb{C})} \langle \eta(x), \eta'(x) \rangle \frac{\omega_0^d}{d!}$$

where $d = \dim X(\mathbb{C})$, and the pointwise scalar product on $E_{\mathbb{C}}$ and $TX(\mathbb{C})$ has been used.

Let $\zeta_q(s) = \operatorname{Tr}(\Delta_q^{-s} \mid (\operatorname{Ker} \Delta_q)^{\perp})$, $\operatorname{Re}(s) > d$, be the zeta function of Δ_q. It extends meromorphically to $s \in \mathbb{C}$ and has no pole at $s = 0$.

Define a real number

$$\chi_Q(\bar{E}) = \sum_{q \geq 0} (-1)^q \left[\log \# H^q(X,E)_{\text{tors}} - \log \operatorname{covol}_{L^2} H^q(X,E) + \frac{1}{q} \zeta_q'(0) \right]$$

where $\operatorname{covol}_{L^2}$ is the covolume for the L^2-metric.

(This number is the arithmetic degree of the Knudsen–Mumford line bundle $\lambda(E)$, equipped with its Quillen metric).

When $Z = \sum_\alpha n_\alpha Z_\alpha$ is a codimension p cycle on X, we let δ_Z be the (p,p) current on $X(\mathbb{C})$ obtained by integrating forms on $Z(\mathbb{C})$:

$$\delta_Z(\eta) = \sum_\alpha n_\alpha \int_{Z_\alpha(\mathbb{C})} \eta .$$

Denote by $\widehat{CH}^p(X)$ the group generated by pairs (Z, g) where Z is a codimension p cycle, and g a $(p-1,p-1)$ real current such that $dd^c(g) + \delta_Z$ is smooth, $F_\infty^*(g) = (-1)^{p-1} g$ and modulo the relations

and $(0, \partial u + \bar\partial v) \sim 0$

$(\mathrm{div}(f), -\log|f|^2) \sim 0$ for any $f \in k(Y)^*$, Y irreducible of codimension $p-1$, $\mathrm{div}(f)$ being its divisor and

$$(\log|f|^2)(\eta) = \int_{Y(\mathbb{C})} \log|f|^2 \, \eta .$$

For instance $\widehat{CH}^1(\mathrm{Spec}\,\mathbb{Z}) = \mathbb{R}$.

There is a product structure

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$$

and a direct image

$$f_*: \widehat{CH}^{d+1}(X) \to \widehat{CH}^1(\mathrm{Spec}\,\mathbb{Z}) = \mathbb{R}.$$

Any hermitian bundle $\overline{E}$ has characteristic classes $\widehat{ch}(\overline{E}), \widehat{Td}(\overline{E}), \ldots$ in $\underset{p\geq 0}{\oplus} \widehat{CH}^p(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

Let $R(E_{\mathbb{C}})$ be the additive characteristic class in $H^*(X(\mathbb{C}), \mathbb{R})$ such that, when $L_{\mathbb{C}}$ is a line bundle,

$$R(L_{\mathbb{C}}) = \sum_{\substack{m \text{ odd} \\ m \geq 1}} \left[2\zeta'(-m) + \zeta(-m)\left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right)\right] \frac{c_1(L_{\mathbb{C}})^m}{m!},$$

where $\zeta(s)$ is Riemann zêta function.

Theorem 2:

$$\chi_{\mathbb{Q}}(\overline{E}) = f_*\left[\left(\widehat{ch}(E)\widehat{Td}(X)\right)^{(d+1)}\right] - \frac{1}{2}\int_{X(\mathbb{C})} ch(E_{\mathbb{C}})\, Td(TX_{\mathbb{C}})\, R(TX_{\mathbb{C}})$$

(the proof combines work of G.-S., Bismut-G.-S., Bismut, G.-S.-Zagier, and Bismut-Lebeau). Faltings has simplified this proof and generalized the result.

References:
1) Preprint IHES 1990
2) C.R.A.S. Paris 309, 1989, 929-932 and references therein.

Remark: $Th.1 \cap Th.2 = \emptyset$

$Th_1 \circ Th.2 = A.R.R.$

Titel: Affine acyclic curves and surfaces. Exotic algebraic structures on $\mathbb{C}^n$

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The talk is a very brief survey of some results in affine algebraic geometry and their mutual relations. Let us start with one result on plane affine algebraic curves.

__Theorem 1__ (S. Abhyankar, T. Moh and M. Suzuki (1973-1976) and also L. Rudolph in the smooth case; V. Lin, M. Zaidenberg in the general one (1982); see [Z] for references).

__Any simply connected algebraic curve Γ_0 in $\mathbb{C}^2$ is equivalent to a quasihomogeneous one with respect to the action of the group $\mathrm{Aut}\,\mathbb{C}^2$ of biregular automorphisms of $\mathbb{C}^2$.__

__Corollary 1.__ __Any simply connected irreducible curve Γ_0 in $\mathbb{C}^2$ can be given by exactly one of the equations__

$$x^k = y^l,$$

where k, l __are relatively prime positive integers, after a suitable invertible polynomial change of coordinates in $\mathbb{C}^2$.__

Corollary 2. Let $\Gamma_{k,l} = \{x^k = y^l\} \subset \mathbb{C}^2$, where $k, l \in \mathbb{N}$, $(k,l)=1$. Then all embeddings of the curve $\Gamma_{k,l}$ into $\mathbb{C}^2$ are equivalent via the action of the group $\mathrm{Aut}\,\mathbb{C}^2$ on $\mathbb{C}^2$.

Corollary 3. An irreducible simply connected algebraic curve in $\mathbb{C}^2$ cannot have more than one singular point.

Another proofs of Theorem 1 were obtained by M. Zaidenberg (1985) and in irreducible case by W. Neumann, L. Rudolph (1987-1988). W. Neumann (1989) also obtained the uniqueness theorems for some other classes of plane curves.

Let us now consider more general case of curves on affine acyclic surfaces.

Theorem 2 [Z]. Let X be a smooth acyclic algebraic surface. Then:

a) X contains a singular simply connected curve iff X is isomorphic to $\mathbb{C}^2$.

b) X contains a smooth curve Γ isomorphic to $\mathbb{C}$ iff either X is isomorphic to $\mathbb{C}^2$ or its logarithmic

Kodaira dimension $\bar{k}(X)$ *is equal to 1.*

Another proof was recently obtained by M. Miyanishi and T. Tsunoda. The complete list of all acyclic surfaces with $\bar{k}=1$ was obtained by R. Gurjar, M. Miyanishi (1987).

By the well known theorem of C. Ramanujam [R] the complex algebraic structure on $\mathbb{R}^4$ is unique. He also constructed an example of topologically contractible affine algebraic surface, nonisomorphic to $\mathbb{C}^2$. He suggested to use such surfaces for constructing exotic algebraic structures on $\mathbb{R}^{2n}$ $(n \geq 3)$.

Theorem 3. *For any* $n \geq 3$ *there exists a countable set of pairwise nonisomorphic complex affine algebraic structures on* $\mathbb{R}^{2n}$, *such that*

a) *the underlying analytic structures are pairwise biholomorphically nonequivalent too;*

b) *there are no regular injective mappings of* $\mathbb{C}^{n-1}$ *in these „exotic* $\mathbb{C}^n$".

The proof is based on the idea of C. Ramanujam and the following analytic version of the cancellation theorem.

Theorem 4. Let X_i $(i=1,2)$ be irreducible quasiprojective varieties with $\bar{k}(X_i) = \dim X_i$. Then for any biholomorphic mapping $\Phi : X_1 \times \mathbb{C}^k \longrightarrow X_2 \times \mathbb{C}^l$ must be $k=l$ and there exists an isomorphism $\varphi : X_1 \longrightarrow X_2$, such that the diagramm

$$\begin{array}{ccc} X_1 \times \mathbb{C}^k & \xrightarrow{\ \Phi\ } & X_2 \times \mathbb{C}^k \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\ \varphi\ } & X_2 \end{array}$$

commutes.

Recently there were obtained a number of examples of acyclic surfaces (R. Gurjar, M. Miyanishi, T. Sugie, T. tom Dieck, T. Petrie, M. Zaidenberg). Among them there exists a countable set $\{X_i\}_{i\in\mathbb{N}}$ of contractible pairwise nonisomorphic surfaces with logarithmic Kodaira dimension 2.

By the h-cobordism theorem the manifolds $X_i \times \mathbb{C}^{n-2}$ are diffeomorphic to $\mathbb{C}^n$ $(n \geq 3)$. By Theorem 4 they are analytically different. This leads to the proof of Theorem 3, a).

Proof of Theorem 3, b) is based on Theorem 2.

Literature

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Titel: Euler systems and applications

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This lecture is a survey of the methods of Kolyvagin and their applications. At the end we include some speculation on where one might look for additional applications.

Examples of Kolyvagin's Euler systems

I. Cyclotomic units. Fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and for $n \in \mathbb{Z}^+$ write $\zeta_n = e^{2\pi i/n}$. Fix an even integer m and set $F = \mathbb{Q}(\zeta_m)$. We have:

$$\begin{array}{ccc}
F(\zeta_n) & & n \in S = \{ \prod \ell_i : \ell_i \in \mathcal{L} \text{ distinct} \} \\
\vdots & & \\
\cdots\ F(\zeta_{\ell\ell'})\ \cdots & & \ell, \ell' \in \mathcal{L} \text{ distinct} \\
/ \quad \backslash \quad / \quad \backslash & & \\
\cdots\ F(\zeta_\ell)\ \cdots & & \ell \in \mathcal{L} = \{ \text{primes } \ell \nmid m \} \\
\backslash \quad | \quad / & & \\
F = \mathbb{Q}(\zeta_m) & & \\
| & & \\
\mathbb{Q} & &
\end{array}$$

Define E_F = global units of $F = \mathbb{Z}[\zeta_m]^\times$

$\cup$

C_F = cyclotomic units of $F = E_F \cap \langle 1 - \zeta_m^i : 1 \le i < m \rangle$

For every $n \in S$ write $\alpha_n = 1 - \zeta_{mn} \in F(\zeta_n)^\times$. These satisfy:

1. Distribution (Norm) Relation: $N_{F(\zeta_{n\ell})/F(\zeta_n)} \alpha_{n\ell} = \alpha_n^{1 - \mathrm{Fr}(\ell)^{-1}}$,

2. Congruence Relation: $\alpha_{n\ell} \equiv \alpha_n^{\mathrm{Fr}(\ell)^{-1}}$ modulo all primes above ℓ,

for every $n\ell \in S$, where $\mathrm{Fr}(\ell) \in \mathrm{Gal}(F(\zeta_n)/\mathbb{Q})$ is the Frobenius of ℓ.

Fix a prime $p \nmid [F:\mathbb{Q}]$ and write

$$A = \text{p-part of the ideal class group of } F$$
$$U = \text{p-part of } E_F/C_F.$$

Using the above family of cyclotomic units, Kolyvagin [1] proves:

<u>Theorem</u> (Mazur-Wiles 1980, Kolyvagin 1988) For every irreducible $\mathbb{Z}_p$-representation χ of $\mathrm{Gal}(F/\mathbb{Q})$ which is trivial on complex conjugation,

$$\# A^\chi = \# U^\chi.$$

(Here A^χ is the χ-component of A, $A = \bigoplus_\chi A^\chi$, etc.)

Idea of proof: use the $\{\alpha_n\}$ to construct relations in the ideal class group, and thereby bound $\# A^\chi$. The distribution and congruence conditions on the α_n are used to construct and identify these relations.

II Heegner points.

Let $E_{/\mathbb{Q}}$ be a modular elliptic curve of some level N. Fix an imaginary quadratic field $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ in which all primes dividing N split. We have:

$$\begin{tikzcd}[row sep=small]
\cdots & K[n] & \cdots \\
\cdots & K[\ell_1\ell_2] \arrow[u, dotted, dash] & \cdots \\
\cdots & K[\ell] \arrow[u, dash] & \cdots \\
 & K[1] \arrow[u, dash] & \\
 & K \arrow[u, dash] & \\
 & \mathbb{Q} \arrow[u, dash] &
\end{tikzcd}$$

$n \in S = \{\prod \ell_i : \ell_i \in \mathcal{L} \text{ distinct}\}$

$\ell_1, \ell_2 \in \mathcal{L}$ distinct

$\ell \in \mathcal{L} = \{\text{rational primes } \ell \nmid N D_K\}$

$D_K = \mathrm{discr}(K)$

Here $K[n]$ = ring class field of K modulo n, so $K[1]$ = Hilbert class field.

For every $n \in S$ one has a Heegner point $x_n \in E(K[n])$ and these points satisfy:

1. Distribution: $N_{K[n\ell]/K[n]} x_{n\ell} = -\mathrm{Fr}(\ell)^{-1}(1 - a_\ell \mathrm{Fr}(\ell) + \varepsilon_K(\ell)\,\mathrm{Fr}(\ell)^2)\, x_n$

where $\varepsilon_K(\ell) = \begin{cases} 1 & \ell \text{ splits in } K \\ -1 & \ell \text{ remains prime in } K, \end{cases}$

2. Congruence: $x_{n\ell} \equiv \mathrm{Fr}(\ell)^{-1} x_n$ modulo all primes above ℓ,

for every $n\ell \in S$. Using these Kolyvagin [1] proved:

<u>Theorem.</u> (Kolyvagin 1988) Write $x = N_{K[1]/K}\, x_1 \in E(K)$. If x has infinite order in $E(K)$ then $E(K)/\mathbb{Z}x$ is finite, $Ш_{E/K}$ is finite, and

$$\# Ш_{E/K} \;\Big|\; [E(K):\mathbb{Z}x]^2\, b_{E,K}$$

with an explicit constant $b_{E,K}$ only involving certain bad primes.

Combined with results of Gross-Zagier and Bump-Friedberg-Hoffstein/Murty-Murty, this proves:

<u>Theorem.</u> Suppose $E_{/\mathbb{Q}}$ is modular and $\mathrm{ord}_{s=1} L(E,s) = 0$ or 1. Then

$$\mathrm{rank}_{\mathbb{Z}}\, E(\mathbb{Q}) = \mathrm{ord}_{s=1} L(E,s)$$

and $Ш_{E/\mathbb{Q}}$ is finite.

Idea of proof of Kolyvagin's theorem: once again, use the $\{x_n\}$ to construct relations in the Selmer group of E.

<u>III</u> Elliptic units.

This is similar to the cyclotomic unit situation. Just replace

$\mathbb{Q} \longmapsto$ imaginary quadratic field K

$\mathbb{Q}(\zeta_n) \longmapsto$ ray class fields of K

cyclotomic units $\longmapsto$ elliptic units.

This system gives information on ideal class groups of abelian extensions of K in terms of elliptic units. This in turn gives information on the arithmetic of elliptic curves with complex multiplication. For details see [2].

Speculation: where to look for more Euler systems?

General philosophy: Euler systems come from special values of L-functions.

Examples:

① Cyclotomic units. Write $F = \mathbb{Q}(\zeta_m)^+$, and let χ be a Dirichlet character modulo m. Then

$$L'_m(0,\chi) = -\frac{1}{2}\sum_{\sigma\in \mathrm{Gal}(F/\mathbb{Q})} \chi(\sigma)\log|\varepsilon_m^\sigma|, \qquad \varepsilon_m = (1-\zeta_m)(1-\zeta_m^{-1})$$

where $L_m(s,\chi)$ is the L-function of χ with the Euler factors for primes dividing m removed.

Elliptic units satisfy similar formulas.

② Heegner points. Write $F = K[n]$, and let χ be a character of $\mathrm{Gal}(F/K)$. Then

$$L'_n(E,\chi,1) = * \,\hat{h}\Big(\sum_\sigma \chi(\sigma) x_n^\sigma\Big) \qquad \text{where } \hat{h} \text{ is the}$$

canonical height extended to $E(F)\otimes\mathbb{C}$. This was mostly proved by Gross + Zagier.

The distribution relations follow formally from these formulas.

"Theorem" Distribution relation $\Rightarrow$ weak (but mostly sufficient) form of congruence relation.

(see [3] for precise statement + details)

Thus, any family of formulas of this kind should give rise to an Euler system.

When should one expect a family of formulas of this kind?

One can hope for such formulas whenever one has a family of L-functions with simple zeros. For, if $L(s,?)$ has a simple zero at $s=k$, then conjecturally

$$L'(k,?) = * R$$

where R is a 1×1 regulator, that is, essentially an element, of something (a K-group, for example). A family of such formulas should give an Euler system.

The simplest case to work all this out in detail is the case of Stark's conjectures on units in abelian extensions of number fields. This is done in [3].

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Titel: New infinite processes in 3-dimensional topology.
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The new methods in question concern the homotopy spheres Σ^3 and the π_1^∞ of open 3-manifolds.

We will use the word "link" in two different ways in this talk: i) either we will mean a smooth pair (B^4, L) where $B^4 =$ the standard 4-ball and $L =$ a smooth submanifold, $L = \sum_1^k S_i^1 \times D_i^2 \subset \partial B^4$ (with null-framing); ii) or, as usual, we will mean a collection of disjoint curves in a 3-mfld, like for instance $\Gamma = \sum_1^k S_i^1 \times (\text{center of } D_i^2) \subset S^3 = \partial B^4$.

There is a canonical smooth noncompact 4-mfld V^4, with $\partial V^4 \neq \emptyset$, which one can attach to (B^4, L) as follows. Start by building the infinite connected sum (along the boundary, like it will always be the case in this talk), away from L

$$X^4 = B^4 \# (\infty \# (S^2 \times D^2))$$

and then define $V^4 = X^4 - (\partial X^4 - L)$. □

<u>A triviality criterion</u>. "If one can extend the
boundary of V^4 to be a copy of R^3, in such a way
that the link (R^3, Γ) is trivial, then (B^4, L) was
already trivial, to begin with." [Proof: This implies
that $\pi_1^\infty V^4$, which contains $\pi_1(S^3 - \Gamma)$ is locally free...]. □

Our general approach is to attach to Σ^3 a link
(B^4, L) with the following two properties (which very
easily imply that $\Sigma^3 = S^3$)

I) There is an $l \in Z^+$ and a diffeomorphism

$\{B^4 + (\text{the } k \text{ 2-handles defined by } L)\} \# (\ell \# (S^2 \times D^2)) =$
$= \{\Sigma^3 - (\text{the interiors of } k+\ell+1 \text{ disjoined 3-balls})\} \times [0,1]$.

II) The link (B^4, L) is trivial.

A very crucial intermediary step, the "COHERENCE THEOREM" (see [1] and/or [2] for a precise statement), automatically produces (B^4,L)'s satisfying I; with some work it also produces some (B^4,L)'s for which triviality can be eventually proved, by appealing to our criterion above. These (B^4,L)'s are the initial step of an INFINITE PROCESS which creates another smooth non-compact 4-manifold $W^4 \supset V^4$, with $\text{int } W^4 \underset{\text{DIFF}}{=} \text{int } V^4$ and $W^3 \overset{def}{=} \partial W^4 \supset \partial V^4 = L$.
[The hard work is to glue a W^3 with all the various desired properties to $\text{int } V^4$, extending the existing ∂V^4. On the other hand, although the statement of the COHERENCE THEOREM is purely combinatorial, its proof involves a similar type of extension, achieved by an appropriate infinite process.]

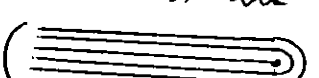
Our W^3 is an open connected and simply-connected manifold with a very large (for glueing purposes) $\pi_2 V$; it is WILD, in the sense that $\pi_1^\infty W^3 \neq 0$, but it comes equipped with some canonical structures, produced by the infinite process.

1°. One can "pull Γ to the infinity of W^3", in the sense that one can find a PROPER smooth embedding, extending Γ

$$(1) \quad \sum_1^k (D^2_i - h_i) \longrightarrow W^3,$$

with h_i = a Cantor set contained in $\text{int } D^2_i$.

IMPORTANT REMARK. Let X^3 be a simply-connected open 3-manifold and $\gamma \subset X^3$ a link which can be "pulled to the infinity" of X^3. If $\pi_1^\infty X^3 = 0$, then it follows, from Dehn's lemma that (X^3,γ) is trivial, but this is no longer true if $\pi_1^\infty X^3 \neq 0$.

2°. Our W^3 is also endowed with a LAMINATION $\mathcal{L}$, in the sense of Thurston (with some minor local singularities.) I will denote by $K = K(\mathcal{L})$ the closed set which consists of the leaves L^2 and of the singularities. Outside the singularities (W^3, K) has a cantorian transverse structure while the singularities themselves are a generalization of the 1-prongs () which occur for the measured foliations of surfaces with $\chi > 0$. We will also consider the 3-dimensional leaves, i.e. the connected components L^3 of $W^3 - K$. Here are some features of our lamination $\mathcal{L}$.

2°-1) The (non-singular) L^2's are planes.

2°-2) The L^3's are simply connected at infinity (but caution, the position of the L^2's at the infinity of L^3 can be WILD and this kind of non global singularities (i.e. the natural completions $\bar{L}^3$) accounts for the wildness of W^3)

2°-3) Most importantly, one has

(2) $$K \cap \Gamma = \emptyset .$$

Also K meets $\sum_1^k (D_i^2 - h_i)$ (see (1)) transversally and outside its singular locus. [In the infinite process which constructs (W^4, W^3) the set $K = K(\mathcal{L})$ occurs as the limit set of a certain sequence of 2-by-2 disjoint 2-disks, only FINITELY many of which touch Γ; this accounts for (2).]

The triviality of (W^3, Γ) follows now easily: property 2°-3 implies that Γ is contained in a finite collection of 3-dimensional leaves where one can pull it to infinity, by truncating (1). Then one applies the fact that $\pi_1^\infty L^3 = 0$ (see 2^o-2) and the IMPORTANT REMARK from 1^o, a.s.o. Once we know that (W^3, Γ) is trivial, we are in a good position for applying our triviality criterion and get $\overline{\text{II}}$, i.e. the triviality of (B^4, L).

Reference [1] is a slightly less sketchy overview of this approach. Complete details will be contained in six forthcoming papers of which [2] is the first one; the next two are being typed, while the last three exist, for the time being, only in hand written form (waiting to be typed...). □

SOME RESULTS ON π_1^∞ OF SIMPLY CONNECTED OPEN 3-MANIFOLDS (obtained by techniques inspired (from those described above) from [2], as well as from other parts of the work described

"Let V^3 be a simply-connected open 3-mfld such that $V^3 \times B^n$ (for some n) has NO HANDLES OF INDEX ONE. Then $\pi_1^\infty V^3 = 0$." So, for instance, one cannot kill stably the handles of index one of the Whitehead manifold. This result holds ONLY in dimension 3; complete proofs are given in [3].

The next idea is to start with a closed 3-mfld M^3, and using geometric assumptions à la Gromov (see [4]) on $\pi_1 M^3$ to get results about $\pi_1^\infty \tilde{M}^3$ (where $\tilde{M}^3$ is the universal covering space.)

"If $\pi_1 M^3$ is almost convex (in the sense of J. Cannon [5]) then $\pi_1^\infty \tilde{M}^3 = 0$." This theorem as well as the related result for **combable** groups, in the sense of Thurston (both proved in [6]) are also purely 3-dimensional: some of the manifolds of M. Davis [7] do indeed have almost convex fundamental groups. I should add that results of a similar nature (to [6]) have also been announced by A. Casson - J. Stallings.

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Titel: Rigidity of geodesic flows (on a conjecture of Katok)

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Let M be a compact Riemannian manifold of negative sectional curvature. The <u>geodesic flow</u> ϕ^t acts on the unit tangent bundle T^1M and preserves the <u>Lebesque-Liouville measure</u> λ . The <u>metric entropy</u> h_λ, i.e. the measure- entropy of the dynamical system (ϕ^t, λ), is not larger than the <u>topological entropy</u> h of ϕ^t. In 1982 Katok proposed the following conjecture:

<u>Conjecture</u> : If h and h_λ coincide then M is locally symmetric.

In the same paper ([K]) he gives some evidence for this conjecture by proving it for surfaces (dim M = 2).

The purpose of this lecture is to improve this result as follows:

<u>Theorem</u>: The conjecture is true provided that the dimension of M does not exceed 4.

The proof of this theorem consists of several steps which are described as follows:

Step 1 ([H]): If $h = h_\lambda$ then the mean curvature of the horospheres in the universal covering $\tilde{M}$ of M is constant.

This theorem is proved by reducing it to the following result of Ledrappier ([L]): The bottom of the L^2- spectrum of $\tilde{M}$ is not larger than $h^2/4$, with equality only if M is cmch (i.e. the mean curvature of the horosperes is constant).

Here for $v \in T^1M$ the second fundamental form of the stable horosphere determined by v is a self- adjoint automorphism U(v) of the orthogonal complement $v^{\perp}$ of v; its trace is the mean curvature of this horosphere.

Corollary ([H]): $h = h_\lambda$ and dim M = 2 , 3 implies that M has constant curvature.

The remaining steps for the proof of the 4-dim case are explained as follows:

Step 2: M is locally symmetric if and only if U(v) = U(-v) for all $v \in T^1M$.

Step 3: View U(v) as an automorphism of the tangent space at v of the standard sphere $S^{n-1} \sim T_p^1M$. For example, if M is a compact quotient of the complex hyperbolic plane, then U(v) is the operator whose eigenspace with respect to the eigenvalue 2 is the tangent bundle of the fibres of the Hopf-fibration $S^3 \to S^2$, the eigenspace with respect to the eigenvalue 1 is the orthogonal complement. Let $\mathcal{E}$ be the vector space of eigenfunctions of the Laplacian on S^{n-1} with respect to the smallest nonzero eigenvalue; then div (U(v) grad f) = (tr U(v)) f for all $f \in \mathcal{E}$ if M is cmch.

Step 4: Similar considerations for the Laplacian acting on 1-forms lead to the conclusion that for dim M = 4 we have U(v) = U(-v) for all $v \in T^1M$.

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Titel: Positive scalar curvature, P_2H and elliptic homology

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I dedicate this talk to Nico Kuiper (born June 28, 1920), whom I admire both as mathematician and as "guten Geist" of the IHES, and to my son Benjamin (born June 28, 1983).

This is a report about recent work of Stephan Stolz about the existence of manifolds with positive scalar curvature and about our joint work on elliptic homology.

In the study of smooth manifolds it is natural to ask for restriction in the topology of the manifold coming from prescribing curvature data. In particular one can ask under which conditions a manifold M admits a metric whose scalar curvature is positive at each point. Then we say M admits a psc-metric. If M is a closed Spin-manifold one has the following restriction coming from the Weitzenböck formula and the Atiyah-Singer Index Theorem.

Theorem (Lichnerowitz (1963), Hitchin (1974)): If a closed Spin manifold M^n admits a psc-metric, then $\alpha(M) \in KO_n(pt)$ vanishes.

Here

$$\alpha(M) = \begin{cases} \operatorname{ind}(\not{D}) & n = 0 \bmod 8 \\ \frac{1}{2}\operatorname{ind}(\not{D}) & n = 4 \bmod 8 \\ \dim H \bmod 2 & n = 1 \bmod 8 \\ \dim H^+ \bmod 2 & n = 2 \bmod 8 \end{cases},$$

$\not{D}$ the Dirac operator, H (H^+) the (positive) harmonic Spinors.

$n =$	0	1	2	3	4	5	6	7
$KO_n(pt) =$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0

On the other hand one has

<u>Theorem</u> (Gromov-Lawson / Schoen-Yau (1980)):

a) A closed 1-connected <u>non-Spin</u> manifold of dim ≥ 5 admits a psc-metric.

b) A closed 1-connected <u>Spin</u> manifold M of dim ≥ 5 admits a psc-metric if M is Spin-bordant to a manifold with psc-metric. Using appropriate generators of the Spin-bordism group $\Omega_*^{Spin} \otimes \mathbb{Q}$ this implies:

c) $n \geq 5$. $\alpha(M^n) = 0 \Rightarrow \exists k$ s.t. $\#_k M$ admits a psc-metric, where M is assumed 1-connected.

A more refined look at generators of $\Omega_*^{Spin} \otimes \mathbb{Z}[\frac{1}{2}]$ by Miyazaki shows that one can choose $k = 8$. But generators for Ω_*^{Spin} are not known. Nevertheless one has the

<u>Conjecture</u> (Gromov-Lawson): A 1-connected closed Spin-manifold M of dim ≥ 5 admits a psc-metric $\Leftrightarrow \alpha(M) = 0$

Theorem 1 (S. Stolz): The Gromov – Lawson conjecture holds.

Idea of proof: Let F be a closed manifold with p.s.c.-metric g and G a group of isometries. Then the total space of any smooth bundle with fibre F and structure group G has a psc-metric. Thus look for psc-manifold (F, g) and G such that $\alpha(M) = 0$ implies M is Spin-bordant to the total space of a bundle with fibre F. Note that $\alpha : \Omega_n^{spin} \longrightarrow KO_n(pt)$ is an isomorphism for $n \leq 7$ and has kernel $\mathbb{Z}$ generated by $P_2 H$ in dim 8. Thus a good candidate for (F, g) is $P_2\mathbb{H}$ with the standard metric and take $G = PSp(3) = Sp(3)/\pm 1$. The main step in Stolz's proof is the following very powerful

Proposition 1 (Stolz): M a closed Spin-manifold. Then $\alpha(M) = 0 \iff \exists\, k$ odd s.t. $\#_k M$ is Spin-bordant to the total space of an $P_2\mathbb{H}$-bundle with structure group $PSp(3)$.

Together with the information above this implies Theorem 1. The proof of Proposition 1 is rather difficult and based on a detailed analysis of the mod 2 Adams spectral sequence for $\pi_*(MSpin \wedge BSp(3)^+)$.

Now, we pass to elliptic homology.

Recall that a genus ϕ is a ring homomorphism from the oriented bordism ring Ω_*^{SO} to a commutative $\mathbb{Q}$-algebra Λ. ϕ is completely determined by its logarithm

$$\log_\phi(x) := \sum_{n\geq 0} \frac{1}{2n+1} \phi(\mathbb{CP}^{2n}) x^{2n+1} \in \Lambda[[x]].$$

According to Ochanine, ϕ is called elliptic if its log is an elliptic integral of the form

$$\log_\phi(x) = \int_0^x \frac{1}{\sqrt{1-2\delta t^2+\varepsilon t^4}}\, dt ,$$

where $\delta = \phi(\mathbb{CP}^2)$, $\varepsilon = \phi(P_2\mathbb{H})$. Every elliptic genus factors through the universal elliptic genus

$\phi_u : \Omega_*^{SO} \longrightarrow \mathbb{Q}[\delta,\varepsilon]$ and it turns out that $\mathrm{im}\, \phi_u \subset \mathbb{Z}[\frac{1}{2}, \delta, \varepsilon]$ making this into an Ω_*^{SO}-module.

Theorem (Landweber, Ravenel, Stong): The functor (LRS-functor) on finite CW-complexes

$$X \longmapsto \Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}} \mathbb{Z}[\tfrac{1}{2}, \delta, \varepsilon, \varepsilon^{-1}]$$ is a generalized homology theory called elliptic homology.

Problem: a) Give an intrinsic geometric description of this functor which b) is a homology theory without inverting 2.

For various reasons (Rigidity Theorem, integrality Theorem) the right category for elliptic genera should be

Spin-manifolds (note $\Omega_*^{SO}(X) \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \Omega_*^{Spin}(X) \otimes \mathbb{Z}[\tfrac{1}{2}]$).

The first important step towards a solution of the Problem was Ochanine's investigation of genus

$$\beta_q : \Omega_n^{Spin} \longrightarrow KO_n(pt)\,[[q]] \,.$$

It is a refinement of the universal elliptic genus ϕ_u and based on the Witten class. The values of $\beta_q(M)$ are q-expansions at the cusp ∞ of certain level 2 modular forms over $KO_n(pt)$. In analogy with the LRS-functor this suggests that the coefficients of elliptic homology should be $\operatorname{im} \beta_q\,[\varepsilon^{-1}]$.

<u>Definition:</u> $n \in \mathbb{Z}$ fixed. $\quad \mathcal{E}ll_n(X) := \sum_{k \in \mathbb{Z}} \Omega_{n+8k}^{Spin}(X)/\sim$,

where $\sim$ is the equivalence relation generated by identifying total spaces of $P_2\mathbb{H}$-bundles with the base space.

<u>Theorem 2 (S. Stolz + M.K.):</u> i) The functor $\mathcal{E}ll_*(X)$ on finite complexes is a generalized 8-periodic homology theory.

ii) The Ochanine genus β_q induces an isomorphism

$$\mathcal{E}ll_*(pt) \cong \operatorname{im} \beta_q\,[\varepsilon^{-1}]$$

iii) The mapping $[M^{n+8k}, f] \mapsto [M, f] \otimes \varepsilon^{-k}$ induces an isomorphism $\mathcal{E}ll_n(X) \otimes \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\cong} \left(\Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}} \mathbb{Z}[\tfrac{1}{2}, \delta, \varepsilon, \varepsilon^{-1}]\right)_n$,

the LRS-functor.

Titel: Killing spinors on Riemannian manifolds
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Let (M^n, g) be a Riemannian spin manifold. Let denote by S the spinor bundle of (M^n, g), by $\nabla^S : \Gamma(S) \to \Gamma(T^*M \otimes S)$ the spinor derivative and by $\mu : TM \otimes S \to S$ the Clifford multiplication. There are two natural operators acting on spinor fields, the Dirac and the twistor operator

$$\begin{tikzcd}
\Gamma(S) \arrow[r, "\nabla^S"] \arrow[rrr, bend left, "D = \text{Dirac operator}"] \arrow[rrrd, bend right, "\mathcal{D} = \text{Twistor operator}"'] & \Gamma(T^*M \otimes S) \arrow[r, phantom, "\cong"] & \Gamma(TM \otimes S) \arrow[r, "\mu"] \arrow[rd, "\mathrm{proj}_{\mathrm{Ker}\,\mu}"'] & \Gamma(S) \\
 & & & \Gamma(\mathrm{Ker}\,\mu)
\end{tikzcd}$$

Locally we have

$$\left.\begin{aligned} D\varphi &= \sum_{i=1}^{n} s_i \cdot \nabla^S_{s_i} \varphi \\ \mathcal{D}\varphi &= \sum_{i=1}^{n} s_i \otimes \left(\nabla^S_{s_i}\varphi + \tfrac{1}{n}\, s_i \cdot D\varphi\right) \end{aligned}\right\}$$ where $(s_1, \dots, s_n)$ is a local orthonormal basis.

A spinor field φ is called <u>twistor spinor</u>, if $\mathcal{D}\varphi = 0$.
A. Lichnerowicz ([8]) proposed to study the geometrical structure of manifolds admitting twistor spinors.
A special kind of twistor spinors are the so-called Killing spinors. A spinor field $\varphi \in \Gamma(S)$ is called <u>Killing spinor</u>, if there exists a complex number $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that for all vector fields X on M the differential

equation $\nabla^S_X \varphi = \lambda X \cdot \varphi$ is satisfied. λ is called the <u>Killing number</u> of φ. Killing spinors play a role in mathematical physics (for an outline see [4]). In differential geometry they appeared in connection with eigenvalue problems of the Dirac operator D which has on a compact manifold a discrete spectrum of real eigenvalues. For the first positive and negative eigenvalue $\lambda_1^\pm$ of D it is known:

<u>Theorem</u> ([6]): Let (M^n, g) be a compact spin manifold of positive scalar curvature R and $R_0 = \min_{x \in M} R(x)$. Then

$$|\lambda_1^\pm| \geq \frac{1}{2}\sqrt{\frac{R_0 n}{n-1}} .$$

Moreover, if φ is an eigenspinor to the smallest possible eigenvalue $\frac{1}{2}\sqrt{\frac{R_0 n}{n-1}}$ or $-\frac{1}{2}\sqrt{\frac{R_0 n}{n-1}}$, then φ is a Killing spinor to the Killing number $\mp \frac{1}{2}\sqrt{\frac{R_0}{n(n-1)}}$.

If (M^n, g) admits a non-zero Killing spinor to the Killing number λ, then it is an Einstein space of scalar curvature $R = 4n(n-1)\lambda^2$. Hence λ is either real or imaginary. Let (M^n, g) be complete. If φ is a real Killing spinor ($\lambda \in \mathbb{R}$), then M is a <u>compact</u> Einstein space of <u>positive</u> scalar curvature. If φ is an imaginary Killing spinor ($\lambda \in i\mathbb{R}$), then M is a <u>non-compact</u> Einstein space of <u>negative</u> scalar curvature.

There are several examples of manifolds with real Killing spinors. In dimension $n = 3, \dots, 8$ one has an almost complete classification of all compact manifolds with real Killing spinors (given by Friedrich / Kath ([5], [6]), Gruenewald ([7]), Hijazi ([8]).

Now, let us consider the non-compact case. We distinguish two types of imaginary Killing spinors. For a spinor field $\varphi \in \Gamma(S)$ we consider the distance between the subspace $V_\varphi(x) = \{ t \cdot \varphi(x) \mid t \in T_x M \}$ and the point $i\varphi(x)$ in the real vector space $(S_x, \operatorname{Re} \langle\, ,\rangle_{S_x})$ and denote by q_φ the function

$$q_\varphi(x) = \langle \varphi(x), \varphi(x) \rangle \cdot \operatorname{dist}^2 (V_\varphi(x), i\varphi(x)) .$$

If φ is an imaginary Killing spinor, then $q_\varphi \geq 0$ is constant.

<u>Theorem 1</u> ([1]): Let (M^n, g) be a complete, non-compact, connected spin manifold with a Killing spinor φ to the Killing number $i\mu$.

1) If $q_\varphi > 0$, then (M^n, g) is isometric to the hyperbolic space of constant sectional curvature $-4\mu^2$.
2) If $q_\varphi = 0$, then (M^n, g) is isometric to the warped product $(F^{n-1} \times \mathbb{R},\ e^{-4\mu t} h \oplus dt^2)$, where (F, h) is a complete spin manifold with parallel spinors.

Hence a non-compact complete spin manifold admits Killing spinors if and only if it is of the type described in Theorem 1.

Lichnerowicz proved in [9], that on a <u>compact</u> manifold the space of all twistor spinors is (up to a conformal deformation of the metric) the same as the space of all Killing spinors. For non-compact manifolds this is not the case. Lichnerowicz ([10]) and Rademacher ([11]) generalized Theorem 1 and described all non-compact manifolds, admitting twistor spinors satisfying the equation
$\nabla^S_X \varphi = f X \cdot \varphi$ for a smooth function f on M.

Killing spinors on compact manifolds are related to the <u>lowest</u> bound for the first eigenvalue of the Dirac operator. We can also prove an <u>upper</u> bound:

<u>Theorem 2</u> ([2]) Let (M^n, g), $n = 2m, 2m-1$, be a compact spin manifold of positive scalar curvature and let denote by K the sectional curvature and by r_{inj} the injectivity radius of (M, g). Then

$$|\lambda_1^{\pm}| \leq 2^{m-1} \sqrt{n} \max \left\{ \sqrt{K_{max}}, \frac{\pi}{r_{inj}} \right\}.$$

Other upper bounds were proved by Ch. Bär ([3]).

Autor: Helga Baum Seite: 5

The results on Killing and twistor spinors which were obtained in the last 5 years will be published in

H. Baum, Th. Friedrich, I. Kath, R. Grunewald:
"Twistor and Killing Spinors on Riemannian manifolds"
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Titel: Selberg zeta functions for locally symmetric spaces of higher rank.

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Let X be a compact Riemann surface of genus $g \geq 2$. Identify X with the space $\Gamma \backslash H^2$, where $H^2 \simeq PSL(2,\mathbb{R})/SO(2)$ is the upper half plane and $\Gamma \subset PSL(2,\mathbb{R})$ is a Fuchsian group acting properly discontinuous on H^2 by $z \mapsto \frac{az+b}{cz+d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. X inherits a hyperbolic metric of constant negative curvature from H^2. In [S] Selberg attached to X the (now well-known) zeta function

$$\zeta_S(z) = \prod_{N \geq 0} \prod_{\ell} \left(1 - e^{-(z+N)|\ell|}\right), \quad \operatorname{Re} z > 1,$$

where the inner product runs over the closed geodesics in X and $|\ell|$ denotes the length of the closed geodesic ℓ. The infinite product defining ζ_S converges for $\operatorname{Re} z > 1$ and defines a holomorphic function in this half-plane. It turns out that ζ_S can be continued holomorphically to the complex plane $\mathbb{C}$, its zeros are closely related to the spectrum of the Laplacian Δ_X on X and ζ_S satisfies a functional equation. The analytical properties of ζ_S are consequences of the Selberg trace formula. A corresponding theory exists also if we assume only $\operatorname{vol}(\Gamma \backslash H^2) < \infty$.

Gangolli, Warner, Wakayama and others extended the theory to locally symmetric spaces $X = \Gamma \backslash G/K$ of rank 1. Here G/K is a rank 1 Riemannian symmetric space of the non-compact type and $\Gamma \subset G$ is a uniform lattice (without torsion). The rank-condition means that the maximal dimension of a flat, totally geodesic submanifold of X is 1. Instead of a description of these (generalized) Selberg zeta functions we introduce a closely related zeta function, the definition of which is due to S. Smale and D. Ruelle, and which is natural from a dynamical point of view. Consider the geodesic flow Φ_t on the (unit)-sphere bundle SX of X. Then closed geodesics lift to periodic orbits of Φ_t and the length spectrum of X coincides with the spectrum of periods of the flow Φ_t.

Define the Ruelle zeta function by

$$\zeta_R(z) = \prod_{\ell \in \mathrm{Pr}'(\Phi_t, SX)} (1 - e^{-z|\ell|})^{-1}, \quad \mathrm{Re}\, z > h,$$

h = topological entropy of Φ_t, where the product runs over the primitive closed orbits of the flow Φ_t and $|\ell|$ is the (primitive) period of ℓ. ζ_R is holomorphic in the half plane $\mathrm{Re}\, z > h$ and extends to a meromorphic function on $\mathbb{C}$. If X is a compact Riemann surface of genus $g \geq 2$ then we have the obvious relation $\zeta_R(z) = \zeta_S(z+1)/\zeta_S(z)$. In general, for rank 1 spaces, the relation between Selberg type zeta functions and the Ruelle zeta function is more complicated. The singularities of ζ_R contain complete information on the spectrum of all the Laplacians on differential forms on X. The investigation of ζ_R along the usual way by using certain explicit Selberg trace formulas is technically very complicated and, to my opinion, not satisfying since the final results are not predictable. In particular, since the Selberg trace formula is group-theoretical in nature the contributions of the closed geodesics only come out after long chains of calculations.

So it is natural to look for a theory of ζ_R which is of dynamical nature. Observe that there is an analogy between the closed orbits of the flow Φ_t and the fixed points of a diffeomorphism $\varphi: M \to M$ of a compact manifold M. If the fixed points $\mathrm{Fix}(\varphi)$ of φ are non-degenerate in the sense that $\det(1 - d_x\varphi) \neq 0$, $x \in \mathrm{Fix}(\varphi)$ then we have the Lefschetz-fixed point formula $\sum_* (-1)^* \mathrm{Tr}(\varphi | H^*(M)) = \sum_{x \in \mathrm{Fix}(\varphi)} \mathrm{sgn} \det(1 - d_x\varphi)$ relating the local behaviour of φ at the fixed points of φ to global invariants of (M, φ). The analogy suggests to ask for a counterpart of the Lefschetz formula for the flow Φ_t again relating the local behaviour of closed geodesics to global information. It turns out that there is, in fact, such an analogue of the Lefschetz formula. Its description is based on the <u>Anosov</u>-property of Φ_t. In fact, the tangent bundle $T(SX)$

Φ_t - invariantly decomposes as

$$T(SX) = T^s(SX) \oplus T^0(SX) \oplus T^u(SX)$$

such that Φ_t (for $t>0$) contracts (expands) on T^s (T^u) exponentially. The bundles T^s and T^u integrate to the stable and the unstable foliations of SX. Now consider the complexes

$$\Lambda^{\cdot, q}: \quad 0 \to C^\infty(\Lambda^{0,q}(SX)) \xrightarrow{d_s} C^\infty(\Lambda^{1,q}(SX)) \xrightarrow{d_s} \cdots \xrightarrow{d_s} C^\infty(\Lambda^{top,q}(SX)) \to 0,$$

where $\Lambda^{p,q}(SX) = \Lambda^p(T_s^*(SX)) \otimes \Lambda^q(T_u^*(SX))$ are the stable-unstable forms of type (p,q) and d_s is the stable differential. The complexes $\Lambda^{\cdot,q}$ are Φ_t-equivariant, but, of course, <u>not elliptic</u>. Now the problem is to give a reasonable definition of the Φ_t-index of these complexes and what we would like to have is a formula which relates

$L(\Phi_t) = \sum_{p,q} (-1)^{p+q} \operatorname{Tr}(\Phi_t | H^{p,q}(SX))$ to the closed orbits of Φ_t; here $H^{p,q}(SX)$ denotes the cohomology of the complex $\Lambda^{\cdot,q}$. Of course, $L(\Phi_t)$ is <u>not well-defined</u>.

To obtain a better definition recall that 1. $L^2(\Gamma\backslash G)$ splits as $\sum_{\pi \in \hat{G}} N_\Gamma(\pi) V_\pi$, $N_\Gamma(\pi) \in \mathbb{Z}$, 2. $SX \simeq \Gamma\backslash G/M$, where $M \subset K$ is the centralizer of $\mathfrak{a}$ in K ($G = KAN$ Iwasawa decomposition) and the geodesic flow on SX can be identified with the action $\Gamma g M \to \Gamma g a M$ of A on $\Gamma\backslash G/M$, 3. the stable (unstable) leaf through $\Gamma g M$ is $\Gamma g N M$ ($\Gamma g \bar{N} M$, $\bar{N} = \theta N$, θ Cartan involution fixing K).

<u>Theorem 1</u> (Poisson summation formula).

$$\sum_{p,q} (-1)^{p+q} \Big\{ \sum_{\pi \in \hat{G}} N_\Gamma(\pi) \int_{A^+} \Theta_A(H^p(\bar{\mathfrak{n}}, V_{\pi,0}) \otimes_M \Lambda^q \mathfrak{n}^*)(a) \varphi(a) \alpha \Big\} =$$

$$= \epsilon \sum_{a \in \mathrm{Per}(A^+, \Gamma\backslash G/M)} \Big(\int_{X(a)} \alpha\Big) \varphi(a), \quad \varphi \in C_0^\infty(A^+).$$

Here A^+ corresponds to N, α is an A-invariant volume form on A (left hand side) which determines a 1-form α on $\Gamma\backslash G/M$ (right hand side), ϵ is a sign,

$X(a)$ is the space of a-periodic points and $H^p(\bar{n},\cdot)$ are the $\bar{n}$-cohomology groups of Harish-Chandra modules. The latter have been investigated by Casselman, Collingwood, Hecht, Schmid, Vogan, Wallach and others.

Theorem 1 can be used to give a description of the singularities of ζ_R. Now Theorem 1 admits far-reaching generalizations which, among other things, also give us a theory of Ruelle (and Selberg) zeta functions for locally symmetric spaces of arbitrary rank.

Let G be a semisimple linear connected Lie group with finite centre, $\Gamma \subset G$ a uniform lattice without torsion and K a maximal compact subgroup. Let $L \subset G$ be a θ-stable Cartan subgroup with decomposition $L = MA$, $M = L \cap K$. Let $\Delta_0 = \{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{l}_{\mathbb{C}}),\ \alpha|_{\mathfrak{a}} \not\equiv 0\}$ and choose compatible positive systems Δ_0^+ and Δ^+. Let $\mathfrak{a}^+ = \{X \in \mathfrak{a},\ \alpha(X) > 0,\ \alpha \in \Delta_0^+\}$ and $A^+ = \exp \mathfrak{a}^+$. Define $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$.

Theorem 2.
$$\sum_{p,q} (-1)^{p+q} \Big\{ \sum_{\pi \in \hat{G}} N_\Gamma(\pi) \int_{A^+} \Theta_A(H^p(\bar{\mathfrak{n}}, V_{\pi,0}) \otimes_M \Lambda^q \mathfrak{n}^*)(a)\, \varphi(a)\, \alpha \Big\} =$$
$$= \varepsilon \sum_{a \in \mathrm{Per}(A^+, \Gamma\backslash G/M)} \Big(\int_{X(a)} \alpha \wedge \mathrm{Eul}(T_{\mathbb{C}} X(a) \cap \mathcal{P}) \Big) \varphi(a), \quad \varphi \in C_0^\infty(A^+),$$
where $\mathcal{P}$ is the complex vector bundle $\Gamma\backslash G \times_M \mathfrak{n}$.

If $\dim A = 1$ then we define
$$R_{\Gamma,L}(\Lambda) = \prod_{a \in \mathrm{Per}'(A^+, \Gamma\backslash G/M)} \left(1 - e^{-\Lambda(\log a)}\right)^{h(a)}, \quad \Lambda \in \mathfrak{a}^*,\ \mathrm{Re}\,\Lambda > h,$$
where $h(a) = \alpha \wedge \mathrm{Eul}(T_{\mathbb{C}} X(a) \cap \mathcal{P})\,[X(a)] / \langle \alpha, \log a \rangle \in \mathbb{Z}$.

Theorem 3. $R_{\Gamma,L}$ can be continued to a meromorphic function in $\mathbb{C}$.

Moreover, theorem 2 yields a description of the singularities of $R_{\Gamma,L}$. Again, $R_{\Gamma,L}$ is related to zeta functions of Selberg's type.

Remarks. 1. The proofs depend on deep results in harmonic analysis. It would be interesting to eliminate harmonic analysis as much as possible.

2. Is there an analogue of theorem 1 for smooth contact Anosov systems?

3. $\| \log a \|$, $a \in \mathrm{Per}'(A^+, \Gamma\backslash G/M)$ coincides with the length of a closed geodesic in $\Gamma\backslash G/K$. Hence the theory has implications on the asymptotics of the length spectrum of closed geodesics.

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Titel: Diophantine Approximation (report on work of Vojta, Faltings and Bombieri

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1. Roth's Theorem

In 1842 Dirichlet proved his famous box principle: Given $\alpha \in \mathbb{R}$, $1 < Q \in \mathbb{R}$; then there exist p and q in $\mathbb{Z}$ such that

$$|\alpha q - p| < Q^{-1}, \quad 1 \leqq q < Q.$$

In 1844 Liouville obtained a result in the opposite direction: If α is real algebraic (and only this is the interesting case) of degree $d \geqq 2$ then

$$|\alpha q - p| \geqq c(\alpha)\, q^{-d+1}.$$

Finally in 1955 Roth succeeded to prove that the diophantine inequality

$$(*) \quad |\alpha q - p| < q^{-1-\varepsilon}$$

has only finitely many solutions p, q with $q > 0$. Before successively Thue, Siegel, Dyson and Schneider had improved Liouville's exponent $d-1$ by smaller ones.

The proof of Roth's theorem can be sketched as follows. One first defines the index of a polynomial. So let $P = P(x_1, \dots, x_m)$ be a polynomial with complex coefficients of degree $d_1, \dots, d_m \geq 1$ in $x_1, \dots, x_m$ respectively and let $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{C}^m$. Then the index $i(P,\xi)$ of P at ξ is the least value of

$$\frac{i_1}{d_1} + \dots + \frac{i_m}{d_m}$$

for which some $\left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{i_m} P(\xi) \neq 0$.

1st step: One fixes m sufficiently large and constructs a polynomial $0 \neq P(x_1, \dots, x_m)$ as above with integer coefficients such that for a given $\varepsilon > 0$

- $\deg_{x_i} P = d_i$
- $i(P, (\alpha, \dots, \alpha)) \geq \frac{m}{2}(1-\varepsilon)$
- the absolute value of the coefficients $\leq c^{d_1 + \dots + d_m}$

This is proved by writing down the conditions as a system of linear equations in the coefficients of the unknown P. For m large enough the number N of coefficients and the number M of linear equations satisfy $N > 2dM$ and one can apply Siegel's Lemma to obtain the desired P.

2nd step: Under the assumption that $(*)$ has infinitely many solutions one finds integers $1 \leq q_i, p_i$ such that

$$\left|\alpha - \frac{p_i}{q_i}\right| < q_i^{-2-\varepsilon}, \quad 1 \leq i \leq m,$$

of degree $d_1, \dots, d_m \geq 1$ in $x_1, \dots, x_m$ respectively and let $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{C}^m$. Then the index $i(P, \xi)$ of P at ξ is the least value of

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2nd step: Under the assumption that (*) has infinitely many solutions one finds integers $1 \leq q_i, p_i$ such that

$$\left|\alpha - \frac{p_i}{q_i}\right| < q_i^{-2-\varepsilon}, \quad 1 \leq i \leq m,$$

and such that the following growth estimates are satisfied:

$$d_1 \log q_1 \leq d_i \log q_i \leq (1+\varepsilon) d_1 \log q_1 .$$

Expanding $P(p_1/q_1, \dots, p_m/q_m)$ around $(\alpha, \dots, \alpha)$ and using the approximation above together with the index property one easily finds that the index at the point $(\frac{p_1}{q_1}, \dots, \frac{p_m}{q_m})$ is at least εm.

3rd step. This step is done by the so called Roth Lemma. It says that if for sufficiently small ω (depending only on m, ε) we have

- $\omega d_i \geq d_{i+1}$ $(1 \leq i < m)$
- $q_i^{\omega} \geq 2^{3m}$, $q_i^{d_i} \geq q_1^{d_1}$
- height of $P \leq q_1^{\omega d_1}$

then $i(P, (p_1/q_1, \dots, p_m/q_m)) \leq \varepsilon$.

4th step. The comparisation of the upper bound for the index at $(p_1/q_1, \dots, p_m/q_m)$ with the lower bound gives the desired contradiction.

2. Schmidt's Theorem

The work of Roth was generalized by Schmidt in a very sophisticated way to simultaneous approximation. We mention only the following result. Let $L_i(x_1, \dots, x_n)$, $1 \leq i \leq m$, be linear forms with real algebraic coefficients. Then this is called a Roth system if for all $\varepsilon > 0$ the system of inequalities

$$|L_i(x)| < \|x\|^{-(n-m)/m - \delta}$$

has only finitely many solutions. Here $x = (x_1, \ldots, x_n)$ and $\|x\|$ any norm of x.

Example: $n = 2$, $m = 1$, $L(x_1, x_2) = \alpha x_1 - x_2$ leads to Roth's Theorem.

Theorem (Schmidt 1971). Let $m < n$. Then $L_1, \ldots, L_m$ are a Roth system iff for all rational subspaces $W \subseteq \mathbb{R}^n$

$$\frac{\operatorname{rank}(L_1|_W, \ldots, L_m|_W)}{\dim W} \geqq \frac{m}{n}.$$

3. Heights

For $x \in \mathbb{P}^n(\mathbb{Q})$, $x = (x_0 : \cdots : x_n)$ we define the height as follows. Let $x_0, \ldots, x_n \in \mathbb{Z}$ and $1 = (x_0, \ldots, x_n)$. Then $h(x) = \max(\log |x_i|)$. This is the Logarithmic Weil-height. It can be suitably extended to points $x \in \mathbb{P}^n(\overline{\mathbb{Q}})$. If X is a projective variety and L a very line bundle over K we obtain an embedding $X \xrightarrow{\varphi_L} \mathbb{P}^n$ for some n. For $x \in X(\bar{\mathbb{Q}})$ we define $h_L(x) = h(\varphi_L(x))$. For an abelian variety A one can define a canonical height $\hat{h}_L$ starting from h_L. This is the socalled Néron-Tate height.

If C is a curve of genus $g \geq 2$ over a number field K then C can be embedded into its Jacobian $J(C)$. Let $L = \mathcal{O}(\Theta)$ where Θ is the Riemann theta divisor. Then

$$\langle x, y \rangle := \hat{h}_\Theta(x+y) - \hat{h}_\Theta(x) - \hat{h}_\Theta(y) : A(\bar{\mathbb{Q}}) \times A(\bar{\mathbb{Q}}) \to \mathbb{R}$$

is a symmetric bilinear form whose associated quadratic form $|x|^2$ is positive definite on $A(K)/_{tors}$.

The height attached to a line bundle can be extended to line bundles of the typ $\mathcal{L} \otimes \mathcal{M}^{-1}$ where $\mathcal{L}, \mathcal{M}$ are very ample: We embed X into $\mathbb{P}^n$ by $\mathcal{L}$ and into $\mathbb{P}^m$ by $\mathcal{M}$ and denote the homogeneous coordinates by $x_0, \ldots, x_n$ and $y_0, \ldots, y_m$. Then for $\xi \in X(\mathbb{Q})$ we put $(x_0', \ldots, x_n') = \varphi_{\mathcal{L}}(\xi)$, $(y_0', \ldots, y_m') = \varphi_{\mathcal{M}}(\xi)$ and put

$$h_{\mathcal{L} \otimes \mathcal{M}^{-1}}(\xi) = \min_i \max_j \log |x_i'/y_j'| \quad \text{if } K = \mathbb{Q}.$$

This extends easily to the general case of arbitrary $K \subseteq \bar{\mathbb{Q}}$. This height essentially replace the Arakelov theory in Bombieri's approach to be discussed later.

4. Vojta's proof of the Mordell conjecture.

Let K be a number field, $\mathcal{O}_K$ its ring of integers, C a curve of genus $g \geq 2$ over K and K_C its canonical divisor. Then we have various morphisms

$$C \xrightarrow{\Delta} C \times C \underset{p_2}{\overset{p_1}{\rightrightarrows}} C \quad , \quad \Delta = \text{diagonal}, \; p_i = \text{projection}.$$

We put

$$\Delta' = \Delta - \underbrace{p_1^* K_C / 2g-2}_{F_1} - \underbrace{p_2^* K_C / 2g-2}_{F_2} \in (\operatorname{Div} C \times C) \otimes \mathbb{Q}.$$

Vojta considers then divisors of the form $V = \Delta' + a_1 F_1 + a_2 F_2$ for rational a_1, a_2. Now Vojta's proof is as follows:

1. Choose $a_1 = a_1(r)$, $a_2 = a_2(r)$ as functions of a parameter r in a suitable way. Then it can be shown that $V = V_r$ is ample.
2. One extends then everything to $B = \operatorname{spec} \mathcal{O}_K$ to obtain an arithmetic surface $X \to B$ and replaces $C \times C$ by essentially the threefold $X \times_B X$ (more precisely by some modification $W \xrightarrow{q} B$ of it). Then one extends the Vojta divisor V to $\tilde{V} = \Delta' + a_1 F_1 + a_2 F_2 + bF$, b an integer > 0 and F a fibre of q.
3. By Gillet-Soulé's Riemann-Roch one finds for d sufficiently large a small section (see Soulé's talk).
4. One chooses rational points x_1, x_2 on C with heights h_1, h_2 such that

$$h_2 < rh_1 < (1+\varepsilon)h_2$$

and extend both to points of X. Then the intersection of the pull-backs to W of these points gives a curve E on W.

5. One calculates the intersection number $E.d\tilde{V}$ and shows that it becomes by the choice of the parameters negative. This gives then a lower bound for the index of the section σ constructed in 3. at (x_1, x_2). Here one uses a result of Mumford.

6. A zero estimate (Dyson's Lemma) gives an upper bound for the index of σ at (x_1, x_2).

7. The comparisation of the upper with the lower bound for the index of σ at (x_1, x_2) gives the contradiction

5. The work of Faltings.

Faltings considers an abelian variety over a number field K.

Theorem I: Let $X \subset A$ be a subvariety over K which does not contain a translate of a non-trivial abelian subvariety of K. Then $X(K)$ is finite

Corollary: Mordell's conjecture.

To state the next Theorem let $Y \subset A$ be a subvariety over K, v any place of K and $d_w(x, Y)$ a w-adic distance from $x \in A$ to Y. Fix a height function h on A.

Theorem II. Let $\varkappa > 0$ be arbitrary. Then

$$-\infty < \log d_W(x, Y) < -\varkappa h(x)$$

has only finitely many solutions $x \in A(K)$.

Corollary (Conjecture of S. Lang). Let $Y \subset A$ be an ample divisor. Then $A \setminus Y$ has only finitely many integral points.

The new ingredients which Faltings uses besides Vojta's work are

- a new definition of the height of a subvariety using Gillet–Soulé's Arakelov theory (see for another approach also P. Philippon's papers).
- a modification of Vojta's divisor in this context
- the product theorem
- a new version of Siegel's Lemma by a variant of Minkowski's theorem on the successive minima.

The Faltings bundle: Let $A \times A \xrightarrow{m} A$ be the multiplication and $p_i : A \times A \to A$ the projections, $i = 1, 2$. Then Faltings defines for a very ample symmetric line bundle L on A and for rational $\varepsilon, s_1, \ldots, s_m > 0$ the Faltings bundle (additive notation!)

$$L(-\varepsilon, s_1, \ldots, s_m) = -\varepsilon \sum_{i=1}^{m} s_i^2 \, pr_i^* L + \sum_{i=1}^{m-1} (s_i x_i - s_{i+1} x_{i+1})^* L .$$

This bundle replace the Vojta divisor in Faltings' work.

<u>The Product Theorem</u>. Let $P = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$, $L = \mathcal{O}(d_1, \dots, d_m)$ for positive integers $d_1, \dots, d_m$. If $f \in \Gamma(P, L)$ then the index $i(f, x)$ is defined using local trivialisations as in 1. For a fixed section f and $\sigma \in \mathbb{R}$ define the closed subscheme $Z_\sigma \subset P$ by $Z_\sigma : i(\cdot, x) \geq \sigma$.

<u>Theorem</u>. Given $\varepsilon > 0$ there exist $c = c(\varepsilon)$, $r = r(\varepsilon) > 0$ with the following property: If $d_i / d_{i+1} \geq r$ $(1 \leq i < m)$ and if Z is an irreducible component of $Z_{\sigma+\varepsilon}$ and Z_σ then

(i) $Z = Z_1 \times \dots \times Z_m$, $Z_i \subseteq \mathbb{P}^{n_i}$, $1 \leq i \leq m$,

(ii) $\deg Z_i \leq c(\varepsilon)$.

<u>Remark</u>. The Theorem has an extension which also gives bounds for the heights of Z_i.

6. Bombieri's approach to Vojta's work

The main ingredients in Bombieri's rather elementary proof of Mordell's conjecture based on Vojta's work are

- classical theory of heights (see 3.)
- Siegel's Lemma + classical Riemann-Roch for surfaces
- Roth's Lemma (see 1.).

The proof uses a modification of the Vojta divisor. Put $\Delta' = \Delta - P\times C - C\times P$ where $P\in C(K)$. Then Bombieri puts $V = d_1 P\times C + d_2 C\times P + d\Delta'$, d_1, d_2, d_3 positive integers. This Divisor is modified in order to calculate sections. Namely one can write

$$V = \delta_1 (N\cdot P)\times C + \delta_2 C\times(NP) - dB$$

$$B = ((s+1)P)\times C + C\times((s+1)P) - \Delta .$$

Then with $L = \mathcal{O}((NP)\times C + C\times(NP))$, $M = \mathcal{O}(B)$ one gets for N, s sufficiently large embeddings

$$\begin{array}{l} C\times C \xrightarrow{\varphi} \mathbb{P}^n \times \mathbb{P}^n \\ C\times C \xrightarrow[\psi]{} \mathbb{P}^m \end{array}, \quad \mathcal{O}(V) = \varphi^*\mathcal{O}(\delta_1,\delta_2)\otimes\psi^*\mathcal{O}(-d).$$

If σ is a section of $\mathcal{O}(V)$, s a section of $\mathcal{O}(dB)$ then $\sigma\cdot s$ is a section of $\mathcal{O}(\delta_1,\delta_2)|_{C\times C}$. Letting s run through linearly independent global sections of $\mathcal{O}(dB)$, say $s = y_0^d, \dots, y_m^d$ one obtains in this way homogeneous polynomials $F_i(x,x')$ of bidegree δ_1, δ_2 such that

$$s = (F_i(x,x')\cdot y_i^{-d})|_{C\times C}.$$

The conditions for homogeneous polynomials F_i $(0\leq i\leq m)$ as above to define a section are then given by

$$\left(F_i(x,x')y_i^{-d}\right)\Big|_{C\times C} = \left(F_j(x,x)y_j^{-d}\right)\Big|_{C\times C},\quad 0\le i,j\le m.$$

This leads to the description of sections for $\mathcal{O}(v)$ by homogeneous polynomials and one can apply the classical methods very much along the line of 1.

7. Final remarks

We had mentioned in 2. the work of Schmidt. By very new methods Faltings is able to extend these results to nonlinear varieties. It involves for instance the use of the Harder-Narasimhan filtration.

The results of Faltings described in 5. can be very likely be proved in a similar way as Bombieri did with Vojtas result.