The Kneser relation and the Hilbert pairing in multidimensional local field

Sergei Vostokov*

February 2002

Introduction

In the paper [Kn], M. Kneser obtained a relation (see [Kn], (12)), which allowed him to prove the Shafarevich reciprocity law (see [Sh]) directly from the definition of the Hilbert symbol. In the present paper, we generalize the Kneser relation to multidimensional local fields and obtain an explicit formula for the generalized Hilbert pairing dierctly from the definition of the pairing.

It is well known (see, e.g., [V1]) that the explicit formula for the Hilbert pairing is given by the residue of the product of two series, $\Phi(X)$ and $V(X)$, with integer coeffiecents. The first series is calculated by the arguments of the pairing. The second series is directly connected with primary elements. In fact, this series is the inverse of a series, which is substituted in the Artin–Hasse function to obtain a primary element. The Kneser relation, which we generalize to multidimensional local fields in section 2, gives, actually, the first series $\Phi(X)$ (see 2.6, (15) below). In section 3, we calculate the series $V(X)$. We find two congruences for $V(X)$ (see 3.3, (22) and 3.4, (24)). The first congruence follows from the explicit form of the primary elements. The second congruence follows from the fact that the Hilbert pairing is independent of the power series expansions of elements with respect to local parameters.

Solving the system of these two congruences, we obtain an explicit expression for $V(X)$ and an explicit formula for the Hilbert pairing. The author is grateful to Tanja Beljaeva and Alex.Zinoviev for several useful remarks.

*Author thanks the Max-Planck-Institut in Bonn for its valuable support. The paper was also supported by RFBR (grant no. 00-01-00140).

1 Notation and preparatory propositions

1.1

Let K be an *n-dimensional local field*, i.e. a finite sequence of fields $K^{(0)}$, $K^{(1)}, \ldots, K^{(n)} = K$, where $K^{(0)} = \mathbb{F}_q$ with $q = p^f$ is a finite field, and, for $1 \leqslant i \leqslant n$, the field $K^{(i)}$ is a complete discrete valuation field whose residue field is $K^{(i-1)}$.

The study of multidimensional local fields was initiated by A. Parshin und K. Kato in the seventies of the last century. Now these fields represent a well-developed subject of research (cf. [VZF]).

For an n-dimensional local field K, we have the following structural data:
1) A discrete valuation $v : K^* \longrightarrow \mathbb{Z}$ of rank 1
2) A discrete valuation $\bar{v} = (v_1, \ldots, v_{n-1}, v_n) : K^* \longrightarrow \mathbb{Z}^n$ of rank n
3) The chain of discrete valuations $\bar{v}_i = (v_{n-i+1}, v_{n-i+2}, \ldots, v_n) : K^* \longrightarrow \mathbb{Z}^i$
4) The chain $\mathfrak{m}_1 \subset \mathfrak{m}_2 \subset \ldots \subset \mathfrak{m}_n =: \mathfrak{m}$ of the valuation ideals which belong to the valuations $\bar{v}_i$.

We assume that all n-dimensional local fields considered in this paper are of characteristic zero.

Let $\mathfrak{o}_K$ denote the ring of integers for the valuation $\bar{v}$, and let $t_1, \ldots, t_{n-1}$, $t_n = \Pi$ be a system of local parameters for $\bar{v} = (v_1, \ldots, v_{n-1}, v_n = v)$. We denote by $\mathfrak{R}$ the Teichmüller set of representatives of $\mathbb{F}_q$ in K. Every α in K has a unique expansion of the form

$$\alpha = \sum_{\bar{r} \in I} \theta_{\bar{r}} t_1^{r_1} \ldots t_{n-1}^{r_{n-1}} \Pi^{r_n} \qquad \text{with} \quad \theta_{\bar{r}} \in \mathfrak{R},$$

where $\bar{r} = (r_1, \ldots, r_n)$ runs through a certain *admissible* subset $I = I(\alpha)$ of $\mathbb{Z}^n$. We call $I \subset \mathbb{Z}^n$ *admissible*, if for all $i_{l+1}, \ldots, i_n$ in $\mathbb{Z}, l \leqslant n$, there exists an $i \in \mathbb{Z}$ with the following property: If $\bar{r} = (r_1, \ldots, r_l, i_{l+1}, \ldots, i_n) \in I$, then $r_l \geqslant i$ (cf. [VZF]). We consider also expansions of the form

$$\alpha = \sum_{\bar{r} \in I} a_{\bar{r}} t_1^{r_1} \ldots t_{n-1}^{r_{n-1}} \Pi^{r_n}$$

with coefficients $a_{\bar{r}}$ in

$$\mathfrak{o} := \mathrm{W}(K^{(0)}),$$

the ring of Witt vectors over $K^{(0)}$; such expansions, however, are no longer unique. If α is contained in the maximal ideal $\mathfrak{m}_K$ of $\mathfrak{o}_K$, then

$$\alpha = \sum_{\substack{\bar{r} > \bar{0} \\ \bar{r} \in I}} a_{\bar{r}} t_1^{r_1} \ldots t_{n-1}^{r_{n-1}} \Pi^{r_n}.$$

1.2 The module of Cartier-curves of the multiplicative group of a multidimensional local field

Let K be an n-dimensional local field with last residue field $K^{(0)}$.We denote by $T = \mathrm{Quot}(\mathfrak{o})$ the quotient field of the ring of Witt vectors over $K^{(0)}$, and, as before $t_1, \dots, t_{n-1}, \Pi$ denotes a system of local parameters of K. Then every $\alpha \in K$ is representable in the form

$$\alpha = \sum_{\bar{r} \in I} a_{\bar{r}} t_1^{r_1} \dots t_{n-1}^{r_{n-1}} \Pi^{r_n} \qquad \text{with} \quad a_{\bar{r}} \in \mathfrak{o},$$

where $\bar{r} = (r_1, \dots, r_n)$ runs through a certain admissible subset of $\mathbb{Z}^n$ (see 1).

For algebraically independent variables $X_1, X_2, \dots, X_n$, we consider the $(n+1)$-dimensional local field

$$M = M(K) := T\{\{X_1\}\} \dots \{\{X_{n-1}\}\}((X_n))$$

with local parameters $X_1, \dots, X_{n-1}, p, X_n$. Introduce the following notation

$$\mathfrak{o}[[X]] = \Big\{ \sum_{\bar{r} \in I,\, \bar{r} \geqslant \bar{0}} a_{\bar{r}} X^{\bar{r}} \mid a_{\bar{r}} \in \mathfrak{o} \Big\},$$

where I is an admissible subset in $\mathbb{Z}^n$. Denote

$$X\mathfrak{o}[[X]] = \Big\{ \sum_{\bar{r} \in I,\, \bar{r} > \bar{0}} a_{\bar{r}} X^{\bar{r}} \in \mathfrak{o}[[X]] \mid a_{\bar{r}} \in \mathfrak{o} \Big\}.$$

Set

$$\begin{gathered} \mathfrak{o}_{(p)} := \mathfrak{o}\{\{X_1\}\} \dots \{\{X_{n-1}\}\}((X_n)), \\ U_m := 1 + X\mathfrak{o}[[X]], \\ \mathcal{H}_m := \langle X_1 \rangle \times \dots \times \langle X_n \rangle \times \mathfrak{o}^\times \times U_m \subset \mathfrak{o}_{(p)}^\times. \end{gathered}$$

We call $\mathcal{H}_m$ the *module of Cartier-curves for the multiplicative group of* K.

We have the following (noncanonical) surjective homomorphism:

$$\begin{gathered} \eta : \mathfrak{o}_{(p)} \longrightarrow K, \\ \alpha(X) \longmapsto \alpha(t_1, \dots, t_{n-1}, \Pi). \end{gathered}$$

By restriction, it induces a group homomorphism

$$\eta : \mathcal{H}_m \longrightarrow K^\times,$$

and the latter homomorphism is also surjective (see 1).

For each $\alpha \in K^\times$, we choose a preimage $\underline{\alpha}$ of α under η; it satisfies the conditions

$$\underline{\alpha} \in \mathcal{H}_m \quad \text{and} \quad \underline{\alpha}(t_1, \dots, t_{n-1}, \Pi) = \alpha. \tag{1}$$

We set $A := \mathfrak{o}\{\{X_1\}\}\dots\{\{X_{n-1}\}\}$.

Let $Eis(x) = p + a_1x + \ldots + a_ex^e + \ldots \in A[[x]]$ be an Eisenstein series for Π, i.e. such a series that $a_1, \ldots, a_{e-1}$ are divisible by p, a_e is an unit of the ring A, and

$$Eis(\Pi) = 0, \quad \text{i.e.,} \quad \eta(Eis(X_n)) = 0.$$

Lemma 1. *S Let $Eis(x)$ be an Eisenstein series for Π, and let $\rho(x)$ be a series of $A[[x]]$ such that $\rho(\Pi) = 0$ (i.e., $\eta(\rho(X_n)) = 0$). Then $\rho(x)$ is divisible by $Eis(x)$ in $A[[x]]$.*

Proof. By the Weierstrass Preparation Theorem, there exists an $\varepsilon(x) \in A[[x]]$ with invertible constant term such that

$$g(x) := Eis(x) \cdot \varepsilon(x)$$

is an Eisenstein polynomial of degree e over A . Without loss of generality, we may assume that the coefficients of $\rho(x)$ are relatively primer. Now, applying the Weierstrass Preparation Theorem to $\rho(x)$, we obtain a series $\varepsilon_1(x) \in A[[x]]$ with invertible constant term such that $\rho_* := \rho\varepsilon_1$ is a polynomial over A that admets Π as a root. It follows that $\rho_* = g \cdot \varphi$, where $\varphi \in A[x]$. Thus, we obtain $\rho = Eis(x) \cdot (\varphi \cdot \varepsilon \cdot \varepsilon_1^{-1})$, and the lemma is proved. □

It is easy to see that there exists an Eisenstein series for Π. By Lemma 1, such a series is uniquely determined up to units of $A[[x]]$.

2 Kneser's relation

2.1 The definition of the Hilbert symbol and primary elements

Let K be an n-dimensional local field containing all p^Nth roots of 1, and let Ω_N be the group of p^N-primary elements of the field K (an element $\omega \in K$ is called *primary* if the extension $K(\sqrt[p^N]{\omega})/K$ is *unramified*). We define the character χ on the group Ω_N as follows:

$$\chi(\omega) := \sqrt[p^N]{\omega}^{\varphi-1} \in \mu_N,$$

where φ is the canonical Frobenius automorphism of $K(\sqrt[p^N]{\omega})/K$, and μ_N is the group of all p^Nth roots of 1. Let

$$\Psi_K : K_n^{top}(K) \longrightarrow \mathrm{Gal}(K^{ab}/K)$$

be the reciprocity map à la Parshin-Kato from the topological Milnor group in the Galois group of the maximal abelian extension of the field K (cf. [Ka], [P1, 2]).

Definition 1. We define the multidimensional Hilbert symbol as follows:

$$(\cdot,\cdot) = (\cdot,\cdot)_{N,K} : K_n^{top}(K) \times K^\times \longrightarrow \mu_N \tag{2}$$
$$(\alpha,\beta)_{N,K} = \Psi_K(\alpha)(\tilde\beta)/\tilde\beta$$

where $\tilde\beta$ satisfies $(\tilde\beta)^{p^N} = \beta$.

Definition 2. By the Hilbert symbol, we mean a homomorphism

$$(,\ldots,)_{N,K} \ : \ K_{n+1}^{top}(K) \ \longrightarrow \ \mu_N,$$

normalized by the relation

$$(t_1,\ldots,t_n,\omega)_{N,K} = \chi(\omega)$$

for every set $t_1,\ldots,t_n$ of local parameters and every p^N-primary element ω in the field K.

Proposition 1. *The two definitions of the Hilbert symbol are equivalent.*

Proof. Let $(,)^{(1)}$ be the Hilbert symbol from the first definition, and $(,)^{(2)}$ be the symbol from the second one.

First, we must verify the relation

$$(t_1,\ldots,t_n,\beta)^{(1)} = (t_1,\ldots,t_n,\beta)^{(2)} \tag{3}$$

for every system of local parameters and all $\beta \in K^\times$.

Using arithmetic properties of K, we can represent β as follows:

$$\beta = t_1^{a_1}\ldots t_n^{a_n}\theta \prod_{\substack{\bar r>\bar 0\\ p\nmid \bar r}} (1-\theta_{\bar r} t_1^{r_1}\ldots t_n^{r_n})\,\omega_\beta,$$

where $\theta \in \mathfrak{R}^\times$, $\theta_{\bar r} \in \mathfrak{R}$, $\bar r = (r_1,\ldots,r_n)$ is a set of multi-indices, and ω_β is an p^N-primary element associated with β. Using the Steinberg relations, we obtain ([V2])

$$(t_1,\ldots,t_n,\beta)^{(1)} = (t_1,\ldots,t_n,\omega_\beta)^{(1)} = \chi(\omega_\beta)$$

and

$$(t_1,\ldots,t_n,\beta)^{(2)} = (t_1,\ldots,t_n,\omega_\beta)^{(2)} = \chi(\omega_\beta),$$

which gives relation (3).

The general case can be derived from (3) because K_{n+1}^{top} is topologically generated by the symbols of the form $\{t_1,\ldots,t_n,\beta\}$, where β is a principal unit. $\square$

In what follows, we mainly use the second definition.

2.2

Using the relation $(x, 1-x) = 1$ for $x \neq 0, 1$, and the fact that the Hilbert symbol is trivial if one of the components is a p^Nth power, Kneser (in [Kn]) obtained the following relation in a one-dimensional local field K:

$$(1-x, 1-y) = \prod_{(i,k)=1} (-x^{\iota} y^{\kappa}, 1 - x^i y^k), \quad x \equiv y \equiv 0 \mod \mathfrak{M},$$

where ι and κ are determined by the equation

$$\begin{vmatrix} i & \iota \\ k & \kappa \end{vmatrix} = 1$$

(here $\mathfrak{M}$ is the maximal ideal in the ring of integers of K). From this relation, Kneser obtained the following formula:

$$(\mathcal{E}(x), \mathcal{E}(y)) = \prod_{s=0}^{\infty} (-y^{-1}, \mathcal{E}(xy^{p^s})) \prod_{r=1}^{\infty} (-x, \mathcal{E}(x^{p^r} y)),$$

where

$$\mathcal{E}(x) = \exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \dots\right)$$

is the Artin-Hasse function.

Using the Kneser method in the case of an n-dimensional local field K, we obtain the following relation for principal units $\mathcal{E}(x)$ and $\mathcal{E}(y)$, $x \equiv y \equiv 0 \mod \mathfrak{M}$:

$$(\alpha_1, \dots, \alpha_{n-1}, \mathcal{E}(x), \mathcal{E}(y)) = \\ \prod_{s=0}^{\infty} (\alpha_1, \dots, \alpha_{n-1}, y^{-1}, \mathcal{E}(xy^{p^s})) \prod_{r=1}^{\infty} (\alpha_1, \dots, \alpha_{n-1}, x, \mathcal{E}(x^{p^r} y)), \quad (4)$$

where $\alpha_1, \dots, \alpha_{n-1} \in K^{\times}$

We suppose now that p is odd, then we can drop the minus signs at y^{-1} and x.

2.3

Let $x = \theta t_1^{i_1} \dots t_n^{i_n}$ and $y = \theta' t_1^{j_1} \dots t_n^{j_n}$, where $\theta, \theta' \in \mathfrak{R}$, be monomials in K; then, using the multiplicativity of the Hilbert symbol, we can represent (4) in the form

$$(\alpha_1,\dots,\alpha_{n-1},\mathcal{E}(x),\mathcal{E}(y)) = \prod_{k=1}^{n}\Big(\alpha_1,\dots,\alpha_{n-1},t_k,\prod_{s\geqslant 0}\mathcal{E}^{jk}(\theta\theta'^{p^s}t_1^{i_1+p^sj_1}\dots t_n^{i_n+p^sj_n})^{-1}\times \prod_{r\geqslant 1}\mathcal{E}^{ik}(\theta^{p^r}\theta' t_1^{p^ri_1+j_1}\dots t_n^{p^ri_n+j_n})\Big) \quad (5)$$

Further, we introduce the generalized Artin–Hasse function, which was first considered in [V1] and [V2]. Let $\triangle$ be the Frobenius operator over the ring $\mathfrak{o}_{(p)}$ that acts on the coefficients from $\mathfrak{o}$ as usual Frobenius operator and raises the parameters $X_1,\dots X_n$ to pth power, i.e.,

$$\triangle f = f^{\triangle} = \sum_{\bar r}\mathrm{Frob}(a_{\bar r})X_1^{pr_1}\cdots X_n^{pr_n}$$

We define the functions

$$l(\alpha) = \frac{1}{p}\log\frac{\alpha^p}{\alpha^{\triangle}}, \quad \alpha\in\mathcal{H}_m$$

and

$$E(\beta) = \exp\left(1+\frac{\triangle}{p}+\frac{\triangle^2}{p^2}+\dots\right)(\beta), \quad \beta\in X\mathfrak{o}[[X]].$$

Remark 1. Each element $\alpha\in\mathcal{H}_m$ can be represented in the form

$$\alpha = a_{\bar r_{01}}X_1^{r_{01}}\dots X_n^{r_{0n}}\varepsilon, \qquad a_{\bar r_0}\in\mathfrak{o}^{\times},$$

where $\varepsilon(X)\in U_m$(see [VZF]). We have

$$l(\alpha) = l(a_{\bar r_0}) + \left(1-\frac{\triangle}{p}\right)\log\varepsilon.$$

The function l is well defined for $\alpha\in\mathfrak{o}^{\times}$. This can be proved as follows: $\alpha\in\mathfrak{o}^{\times}$, then $\frac{a^p}{a^{\triangle}}\equiv 1 \mod p\mathfrak{o}$, i.e. $l(a)\in\mathfrak{o}$.

Remark 2. In general the functions $\mathcal{E}$ and E are different and they coincides only on the monomials $\theta X^{\bar r}$, where $\theta\in\mathfrak{R}$; i.e. $\mathcal{E}(\theta t_1^{r_1}\cdots t_n^{r_n}) = \eta(E(\theta X^{\bar r}))$. The function E depends on the choice of a system of local parameters of the field K, but the function $\mathcal{E}$ does not .

Proposition 2. *1. The function l induces the homomorphism*

$$l:\mathcal{H}_m\to X\mathfrak{o}[[X]],$$

and the function E induces the homomorphism

$$E: X\mathfrak{o}[[X]]\to U_m.$$

2. This homomorphisms are mutually inverse isomorphisms between U_m and $X\mathfrak{o}[[X]]$, i.e.

$$E(l(\varepsilon)) = \varepsilon, \quad l(E(\alpha)) = \alpha$$

for every $\varepsilon \in U_m$ and $\alpha \in X\mathfrak{o}[[X]]$.

3. If $\alpha \in \mathcal{H}_m$ and $\alpha = aX^{\bar{r}}\varepsilon(X)$, where $a = \theta a_1 \in \mathfrak{o}^{\times}$, $\theta \in \mathfrak{R}$, $a_1 \equiv 1 \mod p$, and $\varepsilon(X) \in U_m$, then

$$E(l(\alpha)) = a_1\varepsilon(X)$$

(for the proof, see [V1, 2]).

Remark 3. The function $\mathcal{E}$ is a homomorphism from the formal module $F(X\mathfrak{o}[[X]])$, where F is a Lubin–Tate formal group with an Artin–Hasse logarithm $x + x^p/p + x^{p^2}/p^2 + \ldots$, to the group U_m, i.e.,

$$\mathcal{E}(\alpha +_F \beta) = \mathcal{E}(\alpha)\mathcal{E}(\beta),$$

where $\alpha +_F \beta = F(\alpha, \beta)$

2.4

We replace the function $\mathcal{E}$ in (5) by the function E and obtain

$$(\alpha_1, \ldots, \alpha_{n-1}, \mathcal{E}(x), \mathcal{E}(y)) = (\alpha_1, \ldots, \alpha_{n-1}, \eta \circ E(\underline{x}), \eta \circ E(\underline{y})) = \prod_{k=1}^{n} (\alpha_1, \ldots, \alpha_{n-1}, t_k, \alpha_{n+1}), \quad (6)$$

with

$$\alpha_{n+1} = \eta \circ E\left(X_k \left\{\underline{x}\partial_k \left(1 + \frac{\triangle}{p} + \frac{\triangle^2}{p^2} + \ldots\right)(\underline{y}) - \underline{y}\partial_k \left(\frac{\triangle}{p} + \frac{\triangle^2}{p^2} + \ldots\right)(\underline{x})\right\}\right),$$

where $\partial_k = \frac{\partial}{\partial X_k}$, and $\underline{x}$ and $\underline{y}$ are the preimages of x and y respectively in $X\mathfrak{o}[[X]]$, i.e. $\underline{x} = \theta X_1^{i_1} \ldots X_n^{i_n}$, $\underline{y} = \theta' X_1^{j_1} \ldots X_n^{j_n}$.

Using the linearity of the function E, one can extend the coefficients θ and θ' from $\mathfrak{R}$ up to the coefficients from the ring $\mathfrak{o}$.

According to Proposition 2, one can write for any principal unit ε of the field K the following equality: $\varepsilon = \eta \circ E(l(\underline{\varepsilon}))$, where $\underline{\varepsilon}(X)$ is the preimage of ε in U_m (note that in general case $\varepsilon = a\varepsilon_1$, where $a \in \mathfrak{o}$, $a \equiv 1 \mod p$ and $\varepsilon_1 \equiv 1 \mod \mathfrak{M}$). From the equality (6) and the additivity of E we derive for principal units ε and ε' of K:

$$(\alpha_1, \ldots, \alpha_{n-1}, \varepsilon, \varepsilon') = (\alpha_1, \ldots, \alpha_{n-1}, \eta \circ E(l(\underline{\varepsilon})), \eta \circ E(l(\underline{\varepsilon}'))) = \prod_{k=1}^{n} (\alpha_1, \ldots, \alpha_{n-1}, t_k, \eta \circ E(X_k \Phi_k(\underline{\varepsilon}, \underline{\varepsilon}'))), \quad (7)$$

where $\Phi_k = l(\underline{\varepsilon})\partial_k \log \underline{\varepsilon}' - l(\underline{\varepsilon}')\partial_k \frac{\triangle}{p} \log \underline{\varepsilon}$.

Now, let α_n and α_{n+1} be an arbitrary elements of K^*.Then they can be written as $\alpha_n = \theta t_1^{a_1} \dots t_n^{a_n} \varepsilon$, $\alpha_{n+1} = \theta' t_1^{b_1} \dots t_n^{b_n} \varepsilon'$, where $\theta, \theta' \in \mathfrak{R}$ and ε and ε' are the principal units of the field K. From the multiplicativity of the Hilbert symbol, we obtain the following formula:

$$(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) = \prod_{k=1}^{n} \big(\alpha_1, \dots, \alpha_{n-1}, t_k, \eta \circ E(X_k \Phi_k(\underline{\varepsilon}, \underline{\varepsilon}')\underline{\varepsilon}^{-b_k} \underline{\varepsilon}'^{a_k})\big) \times \prod_{1 \leqslant k_1 \neq k_2 \leqslant n} (\alpha_1, \dots, \alpha_{n-1}, t_{k_1}, t_{k_2})^{\delta_{k_1,k_2}},$$

where

$$\delta_{k_1,k_2} = \begin{vmatrix} a_{k_1} & a_{k_2} \\ b_{k_1} & b_{k_2} \end{vmatrix}.$$

The second product disappears since the Hilbert symbol and the determinants δ_{k_1,k_2} are skew-symmetric. Now we calculate the argument of the function E in the first product by using the following easily verifiable relations:

$$\partial_k \log \underline{\varepsilon} + a_k X_k^{-1} = \underline{\alpha}_n^{-1} \partial_k \underline{\alpha}_n,$$
$$\partial_k \frac{\triangle}{p} \log \underline{\varepsilon}' + b_k X_k^{-1} = \frac{1}{p} \underline{\alpha}_{n+1}^{-\triangle} \partial_k \underline{\alpha}_{n+1}^{\triangle}.$$

Thus, we have

$$E(X_k \Phi_k(\underline{\varepsilon}, \underline{\varepsilon}')\underline{\varepsilon}^{-b_k} \underline{\varepsilon}'^{a_k}) = E(X_k(\Phi_k(\underline{\varepsilon}, \underline{\varepsilon}') + a_k l(\underline{\varepsilon}') X_k^{-1} - b_k l(\underline{\varepsilon}) X_k^{-1})) = E(X_k \Phi_k(\underline{\alpha}_n, \underline{\alpha}_{n+1})),$$

where

$$\Phi_k(\underline{\alpha}_n, \underline{\alpha}_{n+1}) = l(\underline{\alpha}_{n+1}) \underline{\alpha}_n^{-1} \partial_k \underline{\alpha}_n - l(\underline{\alpha}_n) \frac{1}{p} \underline{\alpha}_{n+1}^{-\triangle} \partial_k \underline{\alpha}_{n+1}^{\triangle}.$$

From this and (7), we obtain

$$(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) = \prod_{k=1}^{n} (\alpha_1, \dots, \alpha_{n-1}, t_k, \eta \circ E(X_k \Phi_k(\underline{\alpha}_n, \underline{\alpha}_{n+1}))). \qquad (8)$$

Remark 4. A similar relation is valid for each pair α_i, α_j.

Remark 5. Since the Hilbert symbol is uniquely determined, the valuations on the right-hand side of (8) are the same despite the fact that the preimages of the elements α_n, α_{n+1} in $\mathcal{H}_m$(see 1.2, (1)) are different.

2.5

We proceed to the next element α_{n-1} to make clear what might be the induction assumption.

Using Remark 4, we obtain

$$(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) = \prod_{1 \leqslant k_1 < k_2 \leqslant n} (\alpha_1, \dots, \alpha_{n-2}, t_{k_1}, t_{k_2}, \alpha'_{n+1})$$

with

$$\alpha'_{n+1} = \eta \circ E\left(X_{k_1}\Phi_{k_1}\left(\underline{\alpha}_{n-1}, E(X_{k_2}\Phi_{k_2}(\underline{\alpha}_n, \underline{\alpha}_{n+1}))\right) - X_{k_2}\Phi_{k_2}\left(\underline{\alpha}_{n-1}, E(X_{k_1}\Phi_{k_1}(\underline{\alpha}_n, \underline{\alpha}_{n+1}))\right)\right).$$

Now we calculate the argument of the function E (denoting this sum by S_{k_1,k_2}):

$$S_{k_1,k_2} = X_{k_1}X_{k_2}S_1 - l(\underline{\alpha}_{n-1})(X_{k_1}F - X_{k_2}G), \tag{9}$$

where

$$\begin{aligned} S_1 &= \Phi_{k_2}(\underline{\alpha}_n, \underline{\alpha}_{n+1})\underline{\alpha}_{n-1}^{-1}\partial_{k_1}\underline{\alpha}_{n-1} - \Phi_{k_1}(\underline{\alpha}_n, \underline{\alpha}_{n+1})\underline{\alpha}_{n-1}^{-1}\partial_{k_2}\underline{\alpha}_{n-1}, \\ F &= \partial_{k_1}\left(\frac{\triangle}{p} + \frac{\triangle^2}{p^2} + \dots\right)(X_{k_2}\Phi_{k_2}(\underline{\alpha}_n, \underline{\alpha}_{n+1})), \\ G &= \partial_{k_2}\left(\frac{\triangle}{p} + \frac{\triangle^2}{p^2} + \dots\right)(X_{k_1}\Phi_{k_1}(\underline{\alpha}_n, \underline{\alpha}_{n+1})). \end{aligned}$$

Using the form of the function Φ, we obtain the following relation for S_1:

$$S_1 = l(\underline{\alpha}_{n+1})D - l(\underline{\alpha}_n)D', \tag{10}$$

where

$$D = \begin{vmatrix} \underline{\alpha}_{n-1}^{-1}\partial_{k_1}\underline{\alpha}_{n-1} & \underline{\alpha}_{n-1}^{-1}\partial_{k_2}\underline{\alpha}_{n-1} \\ \underline{\alpha}_n^{-1}\partial_{k_1}\underline{\alpha}_n & \underline{\alpha}_n^{-1}\partial_{k_2}\underline{\alpha}_n \end{vmatrix},$$

$$D' = \begin{vmatrix} \underline{\alpha}_{n-1}^{-1}\partial_{k_1}\underline{\alpha}_{n-1} & \underline{\alpha}_{n-1}^{-1}\partial_{k_2}\underline{\alpha}_{n-1} \\ \frac{1}{p}\underline{\alpha}_{n+1}^{-\triangle}\partial_{k_1}\underline{\alpha}_{n+1}^{\triangle} & \frac{1}{p}\underline{\alpha}_{n+1}^{-\triangle}\partial_{k_2}\underline{\alpha}_{n+1}^{\triangle} \end{vmatrix}.$$

Then

$$F = \sum_{r \geqslant 1} \partial_{k_1}\frac{\triangle^r}{p^r}\left\{l(\underline{\alpha}_{n+1})(X_{k_2}\underline{\alpha}_n^{-1}\partial_{k_2}\underline{\alpha}_n) - l(\underline{\alpha}_n)\left(t_{k_2}\frac{1}{p}\underline{\alpha}_{n+1}^{-\triangle}\partial_{k_2}\underline{\alpha}_{n+1}^{\triangle}\right)\right\}.$$

Now, using the relation

$$(X_i\underline{\alpha}^{-1}\partial_i\underline{\alpha})^{\triangle^r} = X_i\frac{1}{p^r}\underline{\alpha}^{-\triangle^r}\partial_i\underline{\alpha}^{\triangle^r} \tag{11}$$

and differentiate F with respect to X_{k_1} we obtain

$$\begin{aligned} F = \sum_{r\geqslant 1} X_{k_2}\Bigg(\partial_{k_1}\frac{\triangle^r}{p^r}l(\underline{\alpha}_{n+1})\frac{1}{p^r}\underline{\alpha}_n^{-\triangle^r}\partial_{k_2}\underline{\alpha}_n^{\triangle^r} - \\ \partial_{k_1}\frac{\triangle^r}{p^r}l(\underline{\alpha}_n)\frac{1}{p^{r+1}}\underline{\alpha}_{n+1}^{-\triangle^{r+1}}\partial_{k_2}\underline{\alpha}_{n+1}^{\triangle^{r+1}}\Bigg) = \\ \sum_{r\geqslant 1} X_{k_2}\Bigg(\frac{\triangle^r}{p^r}l(\underline{\alpha}_{n+1})\partial_{k-1}\left(\frac{1}{p^r}\underline{\alpha}_n^{-\triangle^r}\partial_{k_2}\underline{\alpha}_n^{\triangle^r}\right) - \\ \frac{\triangle^r}{p^r}l(\underline{\alpha}_n)\partial_{k_1}\left(\frac{1}{p^{r+1}}\underline{\alpha}_{n+1}^{-\triangle^{r+1}}\partial_{k_2}\underline{\alpha}_{n+1}^{\triangle^{r+1}}\right)\Bigg). \end{aligned} \tag{12}$$

One may proceed the calculation for the series series G. Considering $X_{k_1}F - X_{k_2}G$ and taking into account that

$$\partial_i\left(\frac{1}{p^r}\underline{\alpha}^{-\triangle^r}\partial_j\underline{\alpha}^{\triangle^r}\right) = \partial_j\left(\frac{1}{p^r}\underline{\alpha}^{-\triangle^r}\partial_i\underline{\alpha}^{\triangle^r}\right), \quad r \geqslant 0,$$

we see that the second term in (12) disappears and so

$$X_{k_1}F - X_{k_2}G = X_{k_1}X_{k_2}\sum_{r\geqslant 1}(D_r + Dr^{'}),$$

where

$$D_r = \begin{vmatrix} \frac{1}{p^r}\underline{\alpha}_n^{-\triangle^r}\partial_{k_1}\underline{\alpha}_n^{\triangle^r} & \frac{1}{p^r}\underline{\alpha}_n^{-\triangle^r}\partial_{k_2}\underline{\alpha}_n^{\triangle^r} \\ \partial_{k_1}\frac{\triangle^r}{p^r}l(\underline{\alpha}_{n+1}) & \partial_{k_2}\frac{\triangle^r}{p^r}l(\underline{\alpha}_{n+1}) \end{vmatrix},$$

$$D_r^{'} = \begin{vmatrix} \partial_{k_1}\frac{\triangle^r}{p^r}l(\underline{\alpha}_n) & \partial_{k_2}\frac{\triangle^r}{p^r}l(\underline{\alpha}_n) \\ \frac{1}{p^{r+1}}\underline{\alpha}_{n+1}^{-\triangle^{r+1}}\partial_{k_1}\underline{\alpha}_{n+1}^{\triangle^{r+1}} & \frac{1}{p^{r+1}}\underline{\alpha}_{n+1}^{-\triangle^{r+1}}\partial_{k_2}\underline{\alpha}_{n+1}^{\triangle^{r+1}} \end{vmatrix}.$$

It is easy to see that

$$\partial_i\frac{\triangle^r}{p^r}l(\underline{\alpha}) = \frac{1}{p^r}\underline{\alpha}^{-\triangle^r}\partial_i\underline{\alpha}^{\triangle^r} - \frac{1}{p^{r+1}}\underline{\alpha}^{-\triangle^{r+1}}\partial_i\underline{\alpha}^{\triangle^{r+1}}$$

Using the latter relation for the second row of the determinant D_r and for the first row of D_r', we expand each of the determinants with respect to the corresponding row; then, summing $D_r + D_r'$ over r, we obtain

$$X_{k_1}F - X_{k_2}G = X_{k_1}X_{k_2}D'', \tag{13}$$

where
$$D'' = \begin{vmatrix} \frac{1}{p}\underline{\alpha}_n^{-\triangle}\partial_{k_1}\underline{\alpha}_n^{\triangle} & \frac{1}{p}\underline{\alpha}_n^{-\triangle}\partial_{k_2}\underline{\alpha}_n^{\triangle} \\ \frac{1}{p}\underline{\alpha}_{n+1}^{-\triangle}\partial_{k_1}\underline{\alpha}_{n+1}^{\triangle} & \frac{1}{p}\underline{\alpha}_{n+1}^{-\triangle}\partial_{k_2}\underline{\alpha}_{n+1}^{\triangle} \end{vmatrix}.$$

Thus, keeping in the mind the notation of (9), we have

$$(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) = \prod_{1 \leqslant k_1 < k_2 \leqslant n} (\alpha_1, \ldots, \alpha_{n-2}, t_{k_1}, t_{k_2}, \alpha'_{n+1})$$

with
$$\alpha'_{n+1} = \eta \circ E(X_{k_1} \ldots X_{k_{n-m}} \Phi_{k_1,k_2}(\underline{\alpha}_{n-1}, \underline{\alpha}_n, \underline{\alpha}_{n+1})),$$

where
$$\Phi_{k_1,k_2} = l(\underline{\alpha}_{n+1})D - l(\underline{\alpha}_n)D' + l(\underline{\alpha}_{n-1})D''$$

and D, D', and D'' are determined in (10) and (13), respectively.

2.6

Now it is clear how to state the induction assumption. We have

$$(\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_{n+1}) = \prod_{1 \leqslant k_1 < \cdots < k_{n-m} \leqslant n} (\alpha_1, \ldots, \alpha_m, t_{k_1}, \ldots, t_{k_{n-m}}, \alpha'_{n+1})$$

with
$$\alpha'_{n+1} = \eta \circ E(X_{k_1} \ldots X_{k_{n-m}} \Phi^{(m)}(\underline{\alpha}_{m+1}, \ldots, \underline{\alpha}_{n+1})),$$

where

$$\Phi^{(m)} = l(\underline{\alpha}_{n+1})D_{n+1}^{(n-m)} - l(\underline{\alpha}_n)D_n^{(n-m)} + \ldots + (-1)^{n-m+1} l(\underline{\alpha}_{m+1})D_{m+1}^{(n-m)}, \quad (14)$$

$$D_{n+1}^{(n-m)} = \begin{vmatrix} \underline{\alpha}_{m+1}^{-1}\partial_{k_1}\underline{\alpha}_{m+1} & \ldots & \underline{\alpha}_{m+1}^{-1}\partial_{k_{n-m}}\underline{\alpha}_{m+1} \\ \ldots & \ldots & \ldots \\ \underline{\alpha}_n^{-1}\partial_{k_1}\underline{\alpha}_n & \ldots & \underline{\alpha}_n^{-1}\partial_{k_{n-m}}\underline{\alpha}_n \end{vmatrix},$$

$$D_r^{(n-m)} = \begin{vmatrix} \underline{\alpha}_{m+1}^{-1}\partial_{k_1}\underline{\alpha}_{m+1} & \ldots & \ldots & \ldots & \ldots & \underline{\alpha}_{m+1}^{-1}\partial_{k_{n-m}}\underline{\alpha}_{m+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \underline{\alpha}_{r-1}^{-1}\partial_{k_1}\underline{\alpha}_{r-1} & \ldots & \ldots & \ldots & \ldots & \underline{\alpha}_{r-1}^{-1}\partial_{k_{n-m}}\underline{\alpha}_{r-1} \\ \frac{1}{p}\underline{\alpha}_{r+1}^{-\triangle}\partial_{k1}\underline{\alpha}_{r+1}^{\triangle} & \ldots & \ldots & \ldots & \ldots & \frac{1}{p}\underline{\alpha}_{r+1}^{-\triangle}\partial_{k_{n-m}}\underline{\alpha}_{r+1}^{\triangle} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{p}\underline{\alpha}_{n+1}^{-\triangle}\partial_{k1}\underline{\alpha}_{n+1}^{\triangle} & \ldots & \ldots & \ldots & \ldots & \frac{1}{p}\underline{\alpha}_{n+1}^{-\triangle}\partial_{k_{n-m}}\underline{\alpha}_{n+1}^{\triangle} \end{vmatrix},$$
$$m + 1 \leqslant r \leqslant n.$$

We leave to the reader the induction step itself, which can be performed in the same way as in 2.5 but with more cumbersome calculation.

Thus, using the generalized Kneser relation, we obtain the following equation:

$$(\alpha_1, \dots, \alpha_{n+1}) = (t_1, \dots, t_n, \eta \circ E(X\Phi(\alpha_1, \dots, \alpha_n))), \tag{15}$$

where

$$\Phi = \Phi^{(n)},$$

see (14).

3 Explicit Formulas

3.1 The series $V(X)$

Now we proceed to the most difficult part of the paper, i.e. to the calculation of the symbol $(t_1, \dots, t_n, \varepsilon)$ for the principal unit ε.

As before, let $X^{\bar{m}}$ denote the monomial $X_1^{m_1} \dots X_n^{m_n}$ for $\bar{m} = (m_1, \dots, m_n)$.

Lemma 2. *For every multiindex $\bar{m} > \bar{0}$ there exists an element $v_{\bar{m}}$ of $\mathfrak{o}$ such that, for all $\beta \in \mathfrak{o}$, the following equality holds:*

$$(t_1, \dots, t_n, \eta \circ E(\beta X^{\bar{m}})) = \zeta^{\operatorname{tr}(\beta v_{\bar{m}})},$$

where tr *is the trace operator from $\mathfrak{o}$ to $\mathbb{Z}_p$. The element $v_{\bar{m}}$ is uniquely determined* $\bmod p^N$. *Here we assume that if $p \nmid \bar{m}$ then $v_{\bar{m}} \equiv 0 \mod p^N$.*

Proof. 1) The existence of $v_{\bar{m}}$. We consider the dual bases $\alpha_1, \dots, \alpha_f$ and $\alpha_1', \dots, \alpha_f'$ of the ring $\mathfrak{o}$ over $\mathbb{Z}_p$ with respect to the trace operator. Then $\operatorname{tr} \alpha_i \alpha_j' = \delta_{ij}$. Let

$$(t_1, \dots, t_n, \eta \circ E(\alpha_i X^{\bar{m}})) = \zeta^{x_i}, \quad x_i \in \mathbb{Z}_p,$$

and let

$$v_{\bar{m}} = \sum_{i=1}^{f} x_i \alpha_i'.$$

If $\beta = y_1\alpha_1 + \dots + y_f\alpha_f$ is an arbitrary element of $\mathfrak{o}$, then

$$(t_1, \dots, t_n, \eta \circ E(\beta X^{\bar{m}})) = \zeta^{x_1 y_1 + \dots + x_f y_f}.$$

On the other hand, from the duality of the bases, we obtain

$$\operatorname{tr} \beta v_{\bar{m}} = \sum x_i y_j \ \operatorname{tr} \alpha_i \alpha_j' = \sum_{i=1}^{f} x_i y_i$$

2) Uniqueness $\mod p^N$. Let ω be another element of $\mathfrak{o}$ satisfying the same conditions. Then, for every $\beta \in \mathfrak{o}$, we have $\operatorname{tr}\beta(v_{\bar m} - \omega) \equiv 0 \mod p^N$; since the trace is nondegenerate, we obtain $v_{\bar m} \equiv \omega \mod p^N$.

3) Let $p \nmid \bar m$, and let $\theta \in \mathfrak{R}$. We have

$$\eta \circ E(\theta X^{\bar m}) = \mathcal{E}(\theta T^{\bar m}) = \prod_{(i,p)=1} \left(1 - (\theta T^{\bar m})^i\right)^{-\mu(i)/i},$$

where $T^{\bar m} = t_1^{m1} \dots t_n^{m_n}$ for the multiindex $\bar m = (m_1, \dots, m_n)$.

It follows from the condition $p \nmid \bar m$ that $p \nmid m_k$ for some k such that $1 \leqslant k \leqslant n$. We have

$$(t_1, \dots, t_n, 1 - (\theta t_1^{m_1} \dots t_n^{m_n})^i) =$$
$$(t_1, \dots, (\theta {t_1}^{m_1} \dots {t_n}^{m_n})^i, \dots, t_n, 1 - (\theta {t_1}^{m_1} \dots {t_n}^{m_n})^i)^{\frac{1}{m_k i}} = 1.$$

It follows from the last relation that, for for every $\beta \in \mathfrak{o}$, the following relation holds:

$$(t_1, \dots, t_n, \eta \circ E(\beta X^{\bar m})) = 1, \qquad p \nmid \bar m.$$

Then, from the uniqueness of $v_{\bar m} \mod p^N$, we obtain $v_{\bar m} \equiv 0 \mod p^N$. $\square$

We use the elements $v_{\bar m}$ to construct the series

$$V(X) = \sum_{\bar m > \bar 0} v_{\bar m} X^{-\bar m} \in M(K).$$

The property of the series $V(X)$ is given in the following statement, which immediately follows from Lemma 2 and the multiplicativity of the function E.

Proposition 3. *Let $\varphi(X) \in \mathfrak{o}[[X]]$. Then*

$$(t_1, \dots, t_n, \eta \circ E(X\varphi(X))) = \zeta^{\operatorname{tr}\operatorname{res}\varphi(X)V(X)},$$

where res *denotes the coefficient at ${X_1}^{-1} \dots {X_n}^{-1}$.*

From this Proposition and the relation (15), we obtain the following result:

Proposition 4. *For each element $\alpha = \{\alpha_1, \dots, \alpha_n\}$ of the K-group $K_n^{top}(K)$ and each $\beta \in K^*$, the Hilbert symbol satisfies the relation :*

$$(\alpha, \beta)_N = \zeta^{\operatorname{tr}\operatorname{res}\Phi(\alpha,\beta)V}$$

In particular,

$$(t_1, \dots, t_n, \varepsilon)_N = \zeta^{\operatorname{tr}\operatorname{res} X^{-1} l(\underline{\varepsilon}) V}, \tag{16}$$

where $\underline{\varepsilon}(X) \in 1 + X\mathfrak{o}[[X]]$.

3.2 The explicit form of primary elements

We need to know the explicit form of the primary elements of the field K to calculate the series $V(X)$. Let

$$s_m(X) := \underline{\zeta}^{p^m} - 1, \tag{17}$$

and

$$s(X) := s_N(X).$$

Proposition 5. *For every series $\varphi \in \mathfrak{o}[[X]]$, the element*

$$\omega(\varphi) = \eta \circ E(\varphi s) \tag{18}$$

is the p^N-primary and every p^N-primary element can be represented in the form (18) *with respect to* $\mod (k^\times)^{p^N}$. *Furthermore,*

$$(t_1, \dots, t_n, \omega(\varphi)) = \zeta^{\operatorname{tr}\varphi(0)} \tag{19}$$

(for the proof, see [V1, 2]).

3.3 The first congruence for the series $V(X)$

From relation (16) in Proposition 4 and relation (18) in Proposition 5, we obtain

$$(t_1, \dots, t_n, \omega(\varphi)) = \zeta^{\operatorname{tr}\operatorname{res} X^{-1}\varphi\, s\, V}, \tag{20}$$

since $l(\underline{\omega(\varphi)}) = l(E(\varphi s)) = \varphi s$.

Now we use the fact that φ can be chosen arbitrarily. First, let $\varphi(X) = a \in \mathfrak{o}$. Then from (19) and (20), we obtain $\operatorname{tr} a \equiv \operatorname{tr}\operatorname{res} aX^{-1}\; s\; V \bmod p^N$, and so

$$\operatorname{res} X^{-1} s(X) V(X) \equiv 1 \mod p^N \tag{21}$$

Now, we put $\varphi(X) = \beta X^{\bar{m}}$, $\bar{m} > \bar{0}$, $\beta \in \mathfrak{o}$. From (19) and (20), we obtain

$$\operatorname{tr}\beta a_{\bar{m}} \equiv 0 \bmod p^N, \qquad \beta \in \mathfrak{o},$$

where $a_{\bar{m}}$ is the coefficient of the series $s(X)V(X)$ at $X^{-\bar{m}}$. Then $a_{\bar{m}} \equiv 0 \bmod p^N$, and so the congruence $s(X)V(X) \equiv 0 \mod (p^N, \deg 0)$ is valid.

Thus, using the congruence (21), we obtain the first congruence for the series V:

$$V(X)s(X) \equiv 1 \mod (p^N, \deg 1)$$

which can be written as

$$V(X) = 1/s + g/s \mod p^N \tag{22}$$

for some $g \in X\mathfrak{o}[[X]]$.

Remark 6. From the congruence $s\ V \equiv 1 \mod (p^N, \deg 1)$, we obtain the following relation: $sV = 1 + g + p^N f$, where $g \in X\mathfrak{o}[[X]]$, $f \in M(K)$; the series $g(X)$ has integer coefficients since the series s and V have integer coefficients. Then $V = (1+g)/s + p^N f/s$, hence

$$V \equiv (1+g)/s \mod p^N.$$

3.4 The second congruence to the series V

To calculate the series $g(X)$ (and so, the series V), we use the following conclusion. For the element ε of K, its preimage $\underline{\varepsilon}(X)$ in $\mathcal{H}_m$ is not uniquely determined, and it may happen that $\underline{\varepsilon}(X) \neq 1$ but $\eta \circ \underline{\varepsilon} = 1$; in this case, for the valuation of the Hilbert symbol, we have, obviously,

$$(t_1, \ldots, t_n, \eta \circ \underline{\varepsilon}(X)) = 1.$$

On the other hand, by Proposition 2 and relation (16), we obtain

$$(t_1, \ldots, t_n, \eta \circ \underline{\varepsilon}(X)) = \zeta^{\operatorname{tr} \operatorname{res} X^{-1} l(\underline{\varepsilon}) V}$$

Thus,

$$\operatorname{tr} \operatorname{res} X^{-1} l(\varepsilon(X)) V(X) \equiv 0 \mod p^N \tag{23}$$

for all $\varepsilon(X) \in \mathcal{H}_m$ *suchthat* $\eta \circ \varepsilon(X) = 1$.

It is convenient to represent $\varepsilon(X)$ in the following equivalent form:

$$\varepsilon(X) = \mathcal{E}(\varphi(X))$$

for some $\varphi \in X\mathfrak{o}[[X]]$.

We can always do this, because, for $\varphi \in X\mathfrak{o}[[X]]$, we have $\mathcal{E}(\varphi) = 1 + H(\varphi)$, where $H(\varphi)$ is the composition of the series $\exp X - 1$ and $x + x^p/p + \ldots$. Thus, if $f(X) = \varepsilon(X) - 1$, then we can take $H^{-1}(f)$ as φ, where H^{-1} is the series inverse to H with respect to the composition.

It is obvious that

$$\eta \circ \varepsilon(X) = 1 \Leftrightarrow \eta \circ \varphi(X) = 0.$$

However, by Lemma 1, for $Eis(X) = u(X) := s/s_{N-1}$ we have

$$\eta \circ \varphi(X) = 0 \Leftrightarrow \varphi(X) = u(X)\psi(X),$$

where ψ is a series such that $\psi(X) \in X\mathfrak{o}[[X]]$. Then from congruence (23), we obtain the second congruence for V:

$$\operatorname{tr} \operatorname{res} X^{-1} l(\mathcal{E}(u\psi)) V \equiv 0 \mod p^N \tag{24}$$

for all series $\psi \in X\mathfrak{o}[[X]]$.

We use the following easily verifiable relation:

$$\operatorname{tr}\operatorname{res} X^{-1}h(X)^{\triangle} = \operatorname{tr}\operatorname{res} X^{-1}h(X), \tag{25}$$

which is true for all $h(X)$ in $M(K)$.

From relation (25) and the definition of the function l, we obtain

$$\operatorname{tr}\operatorname{res} X^{-1}\sum_{k\geqslant 0}\left(\frac{(u\psi)^{p^k}}{p^k}-\frac{(u\psi)^{p^k\triangle}}{p^{k+1}}\right)V = \\ \operatorname{tr}\operatorname{res} X^{-1}\sum_{k\geqslant 0}\left(\psi^{pk\triangle}\frac{u^{p^k\triangle}}{p^{k+1}}(1-p\triangle)V\right)\equiv 0 \mod p^N. \tag{26}$$

Lemma 3. *For all $m\geqslant 1$ and $p\neq 2$, the following congruence holds:*

$$\frac{u^{m\triangle}}{pm}(1-p\triangle)V\equiv\frac{p^{m-1}}{m}\frac{1}{s}(1-\triangle)g \mod (p^N,\deg 0).$$

Proof. By the definition of the series u(X), we have

$$u(X)=Q(s_{N-1}),$$

where $Q(X)=((1+x)^p-1)/x$. It is obvious that if $p\neq 2$, then

$$Q(-p^Nh)^m\equiv p^m \bmod mp^{N+1}, \tag{27}$$

where $h=(s-s_{N-1}^{\triangle})/p^N$. Further, for $f\in\mathfrak{o}[[X]]$, we have

$$\frac{u^{m\triangle}}{pm}f/s=\frac{Q(s_{N-1}^{\triangle})^m}{pm}f/s=\frac{Q(s-p^Nh)^m}{pm}f/s\equiv \\ \frac{Q(-p^Nh)^m}{p^m}f/s \mod (p^N,\deg 0).$$

Using (27), we obtain

$$\frac{u^{m\triangle}}{pm}f/s\equiv\frac{p^{m-1}}{m}\frac{f}{s} \mod (p^N,\deg 0). \tag{28}$$

From the definition of $u(X)$, we obtain

$$\frac{u^{m\triangle}}{m}f^{\triangle}/s^{\triangle}=\frac{u^{(m-1)\triangle}}{m}f^{\triangle}/s^{\triangle}N-1=\frac{Q(s_{N-1}^{\triangle})^{m-1}}{m}f^{\triangle}/s_{N-1}^{\triangle}= \\ \frac{(p+\binom{p}{2}s_{N-1}^{\triangle})^{m-1}}{m}f^{\triangle}/s_{N-1}^{\triangle}\equiv\frac{p^{m-1}}{m}f^{\triangle}/s_{N-1}^{\triangle} \mod \deg 0.$$

Since $p^{m-1}/m \in \mathbb{Z}_p$, $m \geqslant 1$, the latter congruence and the congruence

$$1/s \equiv 1/s_{N-1}^{\triangle} \mod p^N$$

(see section 3, [V1]) imply

$$\frac{u^{m\triangle}}{m} f^{\triangle}/s^{\triangle} \equiv \frac{p^{m-1}}{m} f^{\triangle}/s \mod (p^N, \deg 0). \tag{29}$$

Substituting f by $1+g$and subtracting congruence (29) from (28), we obtain the congruence of the lemma. □

As follows from the lemma, relation (26) takes the following form:

$$\operatorname{tr}\operatorname{res} X^{-1} \sum_{k \geqslant 0} \left(\beta^{p^k\triangle} \psi^{p^k\triangle} \frac{p^{p^k-1}}{p^k} 1/s(1-\triangle)(g) \right) \equiv 0 \mod p^N. \tag{30}$$

We note that the series ψ, $1/s$, g have integer coefficients and p^{p^k-p-1} is divisible by p^N when $k \geq N$, $p \neq 2$; therefore, congruence (30) takes the form

$$\operatorname{tr}\operatorname{res} X^{-1} \sum_{k=0}^{N-1} \psi^{p^k\triangle} p^{p^k-k-1} 1/s(1-\triangle)(g) \equiv 0 \mod p^N.$$

Using relation (25), we may write the latter congruence in the form

$$\operatorname{tr}\operatorname{res} X^{-1} \sum_{k=0}^{N-1} \psi^{p^k\triangle^{N-k+1}} p^{p^k-k-1}((1-\triangle)(g)1/s)^{\triangle^{N-k}} \equiv 0 \mod p^N. \tag{31}$$

Lemma 4. *Let $\gamma_0, \ldots, \gamma_M$ be elements of the ring $\mathfrak{o}$, and let the following congruence be true for all $\beta \in \mathfrak{o}$:*

$$\operatorname{tr}(\beta^{\triangle^M}\gamma_0 + \beta^{p\triangle^{M-1}}\gamma_1 + \cdots + \beta^{p^M}\gamma_M) \equiv 0 \mod p^N.$$

Then $\gamma_0 + \gamma_1 + \cdots + \gamma_M \equiv 0 \mod p^N$.

Proof. We prove the statement if we let β be the generator of the basis of the ring $\mathfrak{o}$ over $\mathbb{Z}_p$ formed by the set of the Teichmüller representatives, which is the lift of the basis of $K^{(0)} \cong \mathbb{F}_q$ over $\mathbb{F}_p$. □

Replacing the series $\psi(X)$ in (31) by $\beta\psi(X)$, $\beta \in \mathfrak{o}$ and using Lemma 4, we obtain the following congruence:

$$\operatorname{res} X^{-1} \sum_{k=0}^{N-1} \psi^{p^k\triangle^{N-k+1}} p^{p^k-k-1}((1-\triangle)(g)(1/s))^{\triangle^{N-k}} \equiv 0 \mod p^N.$$

Using the independence of the choice of ψ, we substitute the powers $X^{\bar{m}}, \bar{m} > \bar{0}$ for ψ. Then, as in 2.3, we obtain from the latter congruence:

$$\sum_{k=0}^{N-1} p^{p^k-k-1}((1-\triangle)g(1/s))^{\triangle^{N-k}} \equiv 0 \mod (p^N, \deg 0).$$

This means that

$$\sum_{k=0}^{N-1} p^{p^k-k-1}((1-\triangle)g(1/s))^{\triangle^{N-k}} = h + p^N f,$$

where $f \in M(K)$ and $h \in \mathfrak{o}[[X]]$, i.e. all series on the left-hand side have the coefficients in the ring $\mathfrak{o}$.

We multiply both sides of the latter congruence by $s^{\triangle^N}$ and obtain

$$\sum_{k=0}^{N-1} p^{p^k-k-1} s^{\triangle^N - \triangle^{N-k}} (1-\triangle) g^{\triangle^{N-k}} \equiv h\, s^{\triangle^N} \mod p^N, \tag{32}$$

where $h \in \mathfrak{o}[[X]]$.

From the congruence $s_m \equiv s_{m-1}^{\triangle} \mod p^m$ (see section 3, [V1] and §1, [V2]), it follows that

$$s^{\triangle^N} \equiv s_{2N} \mod p^N, \qquad s^{\triangle^{N-k}} \equiv s_{2N-k} \mod p^N.$$

By the definition of the series s_m (see 3.2, (17)), we have $s^{\triangle^N - \triangle^{N-k}} \equiv s_{2N}/s_{2N-k} =: f_k \mod p^N$, where $f_k \in \mathfrak{o}[[X]]$. Then, (32) transforms into the following congruence:

$$\mathcal{A}(\triangle)(1-\triangle)g \equiv h\, s^{\triangle^N} \mod p^N, \tag{33}$$

where

$$\mathcal{A}(\triangle) = \triangle^N + \sum_{k=1}^{N} p^{p^k-k-1} f k \triangle^{N-k}$$

is an operator of the ring $\mathfrak{o}[[X]][\triangle]$. We have

$$\mathcal{A}(\triangle)^{-1} = \triangle^{-N}\left(1 + \sum_{k=1}^{N} p^{p^k-k-1} f_k \triangle^{-k}\right)^{-1} \equiv$$

$$\triangle^{-N} + \sum_{k=1}^{N} a_k \triangle^{-N-k} \mod p^N, \qquad a_k \in p^k \mathfrak{o}[[X]].$$

We note that the following congruence is valid in the ring $\mathfrak{o}[[X^{1/p^m}]]$:

$$s^{\triangle^{-m}} \equiv s \mod p^N \mathfrak{o}[[X^{1/p^m}]]. \tag{34}$$

Indeed, the congruence

$$s \equiv s_{N+m} \equiv s^{\triangle^m} \mod p^N$$

means that

$$s = s^{\triangle^m} + p^N f, \qquad f \in \mathfrak{o}[[X]];$$

then

$$s^{\triangle^{-m}} = s + p^N f^{\triangle^{-m}}, \qquad f^{\triangle^{-m}} \in \mathfrak{o}[[X^{1/p^m}]].$$

We apply the operator $\mathcal{A}(\triangle)^{-1}$ to congruence (33) and use (34). Thus we obtain

$$(1 - \triangle)g \equiv \mathcal{A}(\triangle)^{-1}(hs^{\triangle^N}) \equiv h_1 s \mod p^N,$$

where $h_1 \in \mathfrak{o}[[X^{1/p^{2N}}]]$. It remains to apply the operator $(1-\triangle)^{-1} = 1 + \triangle + \triangle^2 + \dots$ to the both sides and obtain

$$g \equiv (1 + \triangle + \triangle^2 + \dots)(h_1 s) \equiv h_2 s \mod p^N, \quad h_2 \in \mathfrak{o}[[X^{1/p^{2N}}]].$$

We have

$$g/s \equiv h_2 \equiv 0 \mod (p^N, \deg 0),$$

and therefore

$$V \equiv 1/s \mod (p^N, \deg 0).$$

From this congruence and Proposition 4, we obtain the following explicit formula for the Hilbert symbol:

$$(\alpha, \beta)_N = \zeta^{\operatorname{tr} \operatorname{res} \Phi(\alpha,\beta)/s}$$

for every

$$\alpha = \{\alpha_1, \dots, \alpha_n\} \in K_n^{top}(K), \quad \beta \in K^{\times},$$

where the series Φ defined in 2.6, (15).

References

[Ka] K. Kato, A generalisation of local class field theory by using K-groups I, J. Fac. Sci. Univ. Tokyo, Sect. IA, **26** (1979), 303-376; II, J. Fac. Sci. Univ. Tokyo, Sect. IA, **27** (1980), 603-683

[Kn] M. Kneser, Zum explitizen Reziprozitätsgesetz von I. R. Shafarevich, Math. Nachr., **6** (1951/1952), 89-96

[P1] A. N. Parshin, Class field theory and algebraic K-theory, Uspekhi Mat. Nauk **30** (1975), 253-254

[P2] A. N. Parshin, Local class field theory, Trudy Mat. Inst. Steklov **165** (1984), 143-170; English transl. in Proc. Steklov Inst. Math. **3** (1985)

[Sh] I. R. Shafarevich, A general reciprocity law, Mat. Sbornik **26 (68)** (1950), 113-146; English transl. in Amer. Math. Soc. Transl. Ser. **2.4** (1956), 73-106

[V1] S. V. Vostokov, Explicit form for the reciprocity law, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 1287-1320; English transl. in Math. USSR Izv. **13** (1979), 557-588

[V2] S. V. Vostokov, An explicit construction of class field theory for multidimensional local fields, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), 283-308; English transl. in Math. USSR Izv. **26** (1986)

[VZF] S. V. Vostokov, I. B. Zhukov, and I. B. Fesenko, On the theory of multidimensional local fields. Methods and constructions, Algebra i Analis **2** (1990), No. 4, 91-118; English transl. in Leningrad Math. J. **2** (1991), No. 4, 775-800