Distribution of the components of a real Enriques surface

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INTRODUCTION

A *real Enriques surface* is a complex Enriques surface equipped with an anti-holomorphic involution (called *complex conjugation*). With the only possible exception when the fixed point set of this involution, the *real part* of the surface, is empty, the involution can be lifted to the covering $K3$-surface. Thus the study of real Enriques surfaces with non-empty real part is equivalent to the study of real $K3$-surfaces equipped with a holomorphic fixed point free involution which commutes with the real structure.

A systematic study of the topological properties of real Enriques surfaces was started by V. Nikulin. It is his preprint [N2] that stimulated our investigation. In our preceding paper [KhD] we have completed the classification of real Enriques surfaces by the topological types of their real part.

This classification has a natural refinement (also first studied by V. Nikulin): the real part $E_{\mathbb{R}}$ of a real Enriques surface admits a natural decomposition in two halves $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)}$, each half being a union of components of $E_{\mathbb{R}}$. This splitting is due to the fact that the real structure lifts to the covering K3 surface in two different ways: each half is covered by the fixed point set of one of the two liftings (see 1.3). This gives rise to the following problem: to classify the triads $(E_{\mathbb{R}}; E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ up to homeomorphism.

For a large number of topological types an arbitrary splitting is realizable. For some other types the splittings are determined by the only restriction: the orientation double covering of a half must either consist of two topological tori or have at most one nonspherical component. The surfaces constructed in [KhD] show the existence of such splittings in many cases. On the other hand, as it was discovered by Nikulin, there are topological types whose distributions must satisfy to certain restrictions.

It is the distribution of the components between the two halves that is the principal subject of the present paper. Our results and the methods which we use are different from those by V. Nikulin: using a more topological approach we obtain some prohibitions which apply as well to other classes of surfaces with non simply connected complexification. More precisely, in this paper we treat what we call *generalized Enriques surfaces*: quotients of a nonsingular compact complex surface X with $H_1(X;\mathbb{Z}/2) = 0$ and $w_2(X)=0$ by a fixed

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point free holomorphic involution (see 1.2 and Appendix D). The prohibitions obtained (see 2.1, 2.2, and Appendix D) are a combination of the inequality-type and congruence-type prohibitions. To an extent they may be regarded as some kind of refinement of the Smith-Thom inequality and extension of the Arnold-Rokhlin congruences. (It is worth mentioning that the prohibitions obtained for the generalized Enriques surfaces are an example, probably the first one, which shows that the topology of surfaces with non simply connected complexification contains some elements which have no precise analogues in the simply connected case.)

We apply these results to the classical Enriques surfaces and complete the classification of the distributions of their components (see 2.3.2).

Another by-product is some prohibitions on the topology of a generalized Enriques surface, see 2.1, which contain some results on the classical case (see [KhD, 3.7–3.10]) as a direct consequence, and provide them with a new proof.

Note that there are 'quite classical' examples of generalized Enriques surfaces: in Horikawa's construction (see Section 8.1) bi-degree $(4,4)$ can be replaced with $(4k,4k)$, $k\in\mathbb{Z}_+$. Thus, our results also provide some prohibitions on the topology of symmetric real curves on quadrics.

The key rôle in our present study is played by so called Kalinin's spectral sequence and Viro homomorphisms, used in combination with more traditional tools of topology of real algebraic varieties. The spectral sequence in question is derived from the Borel-Serre spectral sequence: it is some sort of its stabilization with only one grading. It converges to the homology of the fixed point set, and the corresponding filtration and identification with the limit term are given by the Viro homomorphisms, which have an explicit geometrical definition (see Section 5 for the details).

The paper consists of eight sections and four appendices. In Section 1 we introduce the main objects, such as a generalized $K3$-surface (which, from our point of view, is just a Spin-surface X with $H_1(X;\mathbb{Z}/2)=0$) and a generalized Enriques surface, give some definitions and fix the principal notation. In Section 2 we formulate the main results and apply them to the classical Enriques surfaces. In Section 3 we expose some auxiliary results on the arithmetic of involutions. Section 4 is devoted to the study of the basic topological properties of generalized Enriques surfaces. In Section 5 we introduce Kalinin's spectral sequences and Viro homomorphisms and examine their general properties which we need in subsequent proofs; these results are then applied to generalized Enriques surfaces in Section 6. Finally, in Section 7 we prove the main results announced in Section 2, and in Section 8 we construct some surfaces to extend the list of distributions found in [KhD] and thus complete the classification for the case of classical Enriques surfaces.

In Appendices A–C we discuss some properties of Kalinin's spectral sequence, which complete and develop the content of Section 5. Certainly, these properties must be well known to the specialists on transformation groups, but we could not find them in the literature.

Appendix A is intended for those who prefer the Smith exact sequence: we show how Kalinin's spectral sequence and Viro homomorphisms can be extracted from the Smith sequence. In Appendix B we study the multiplicative structure of Kalinin's spectral sequence and, in the case of an involution on a closed manifold, give a formula relating the intersection pairings on the manifold and on the fixed point set. In Appendix C we study the relation between the Steenrod squares acting in Kalinin's spectral sequence and those acting in the cohomology of the fixed point set.

In Appendix D we introduce Spin generalized Enriques surfaces and extend to them the main results of Section 2.

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1. Preliminary definitions and notation

1.1. Notation. We agree that, unless specified explicitly, the coefficients of all the homology and cohomology groups are $\mathbb{Z}/2$. When this does not lead to a confusion, both the cohomology characteristic classes of a closed smooth manifold and their dual homology classes are denoted by w_i. Throughout the paper we use the following notation:

- b_r and β_r stand for the Betti numbers with the integral and $\mathbb{Z}/2$-coefficients respectively: $b_r(\,\cdot\,) = \operatorname{rk} H_r(\,\cdot\,;\mathbb{Z})$ and $\beta_r(\,\cdot\,) = \dim H_r(\,\cdot\,)$;
- β_* is the total Betti number: $\beta_*(\,\cdot\,) = \sum_{r\geqslant 0} \beta_r(\,\cdot\,)$;
- $\chi(X)$ is the Euler characteristic of a topological space X;
- $\sigma(M)$ is the signature of an orientable manifold M;
- $\operatorname{Tors}_2 G$ is the 2-primary component of an abelian group G.

1.2. Generalized Enriques surfaces. A nonsingular compact complex surface X will be called a *generalized K3-surface* if $H_1(X;\mathbb{Z}/2) = 0$ and $w_2(X) = 0$. A *generalized Enriques surface* is a complex surface E which (1) has $w_2(E) \neq 0$, and (2) can be obtained as the orbit space X/τ of a generalized $K3$-surface by a fixed point free holomorphic involution $\tau\colon X \to X$; the latter is called the *Enriques involution*.

As it follows, for example, from the Ghysin exact sequence, $H_1(E;\mathbb{Z}/2) = \mathbb{Z}/2$ (cf. 4.2.1). Thus, X is the only double covering space of E, and τ is its deck translation. Hence, they can both be uniquely recovered from E.

Remark. Orbit spaces of generalized $K3$-surfaces with $w_2(E) = 0$ are considered in Appendix D.

1.3. Decomposition of the real part. As usually, by a *real structure* on a nonsingular complex surface we mean an anti-holomorphic involution. When not empty, the fixed point set of such an involution is a real 2-manifold.

Let E be a generalized Enriques surface, and let $\operatorname{conj}\colon E \to E$ be the real structure on E. Denote by $E_{\mathbb{R}}$ the real part, $E_{\mathbb{R}} = \operatorname{Fix}\operatorname{conj}$.

1.3.1. Lemma. *If $E_{\mathbb{R}} \neq \varnothing$, then there are two and only two liftings $t^{(1)}, t^{(2)}\colon X \to X$ of conj to X. Both the liftings are involutions. They are anti-holomorphic, commute with each other, and their composition is τ.*

Both the real parts $X^{(i)}_{\mathbb{R}} = \operatorname{Fix} t^{(i)}$, $i = 1, 2$, and their images $E^{(1)}_{\mathbb{R}}$, $E^{(2)}_{\mathbb{R}}$ in E are disjoint, and $E^{(1)}_{\mathbb{R}} \cup E^{(2)}_{\mathbb{R}} = E_{\mathbb{R}}$.

The proof is obvious as soon as the points of X are represented by homotopy classes of paths in E starting at a point of $E_{\mathbb{R}}$: two such classes define the same point in X if and only if they differ by a loop homologous to zero in $H_1(E; \mathbb{Z}/2)$. □

Due to the above lemma, $E_{\mathbb{R}}$ canonically splits into two disjoint parts, which we will refer to as the *halves* of $E_{\mathbb{R}}$. Note that both $E^{(1)}_{\mathbb{R}}$ and $E^{(2)}_{\mathbb{R}}$ consist of whole components of $E_{\mathbb{R}}$, and that $X^{(1)}_{\mathbb{R}}$ and $X^{(2)}_{\mathbb{R}}$ are unramified double coverings of $E^{(1)}_{\mathbb{R}}$ and $E^{(2)}_{\mathbb{R}}$ respectively. In most cases these coverings are determined by $E_{\mathbb{R}}$ intrinsically:

1.3.2. Lemma. *$X_{\mathbb{R}}$ is orientable. The restriction of the projection $X \to E$ to the real parts $X_{\mathbb{R}} = X^{(1)}_{\mathbb{R}} \cup X^{(2)}_{\mathbb{R}} \to E_{\mathbb{R}}$ is the orientation double covering unless $\sigma(X) \equiv 0 \pmod{32}$, one of the halves is empty, and the nonempty half is orientable.*

Proof. The orientability of $X_{\mathbb{R}}$ is well known (see [E], [S], or [K]). For the rest, one can repeat, almost literally, the proof of Theorem A.2 from [KhD]. The assumption $\sigma(X) \equiv 16 \pmod{32}$ in [KhD] is used to prove the following two assertions: E is not a Spin-manifold, and if one of the halves (say, $X^{(2)}_{\mathbb{R}}$) is empty, then the quotient $X/t^{(2)}$ is not a Spin-manifold either. The first assertion is a part of our definition of generalized Enriques surfaces now. As to the second one, we have to replace it by the following: if $E^{(2)}_{\mathbb{R}} = \varnothing$, then τ either preserves or reverses the canonical orientation of all the components of $X_{\mathbb{R}}$ simultaneously. For proof just note that the Spin-structure on X defines a canonical *pair* of opposite orientations on $X_{\mathbb{R}}$, and it is this structure that is preserved by Spin-diffeomorphisms of X. □

Since E is a compact surface, each component C of $E_{\mathbb{R}}$ is a closed manifold. By the first part of 1.3.2, C may be of one of the following three types:

- S_g – a trivially covered orientable surface of genus $g \geqslant 0$;
- V_g – a nonorientable surface of genus $g > 0$, $V_g \cong \#_g\mathbb{R}\mathrm{p}^2$, covered by an orientable component $S_{2g-2} \subset X_{\mathbb{R}}$;
- T_g – a nontrivially covered orientable surface of genus $g > 0$.

When denoting the topological types we use any of $S = S_0 = V_0$ for the 2-sphere S^2. To describe the decomposition of $E_{\mathbb{R}}$ into the two halves, we write $E_{\mathbb{R}} = \{\text{half } E_{\mathbb{R}}^{(1)}\} \sqcup \{\text{half } E_{\mathbb{R}}^{(2)}\}$.

Remark. The empty set has nothing to be distributed, and in what follows we never consider the case $E_{\mathbb{R}} = \varnothing$.

Remark. According to Lemma 1.3.2, the type T_g is a very special one: $E_{\mathbb{R}}$ may have a component of type T_g only if $\sigma(X) \equiv 0 \pmod{32}$ (or, equivalently, $\sigma(E) \equiv 0 \pmod{16}$), and in this case one of the halves of $E_{\mathbb{R}}$ must be empty and the other one must be orientable. In particular, this type never occurs in the case of the classical Enriques surfaces.

Remark. Lemma 1.3.2 gives rise to the following problem: Let X be a closed complex surface with $H_1(X) = 0$ and $w_2(X) = 0$, and let τ and conj be two commuting fixed point free involutions on X, holomorphic and anti-holomorphic respectively. If X/τ is not Spin, can X/conj be Spin?

1.4. Types of the real part. Let Y be a nonsingular compact complex surface with a real structure. Then, since $Y_{\mathbb{R}}$ is a closed (real 2-dimensional) manifold, it has a well defined $\mathbb{Z}/2$-homology fundamental class $[Y_{\mathbb{R}}]$. We say that $Y_{\mathbb{R}}$ is of type $\mathrm{I}_{\mathrm{abs}}$ if $Y_{\mathbb{R}}$ is homologous to zero in $H_2(Y)$ and of type $\mathrm{I}_{\mathrm{rel}}$ if $Y_{\mathbb{R}}$ is homologous to $w_2(Y)$. The surface is said to be of type I if it is of type $\mathrm{I}_{\mathrm{abs}}$ or $\mathrm{I}_{\mathrm{rel}}$; otherwise it is said to be of type II.

In the case of a generalized Enriques surface E and its double covering X the notion of type obviously extends to the halves $E_{\mathbb{R}}^{(i)}$ and $X_{\mathbb{R}}^{(i)}$. For the covering and its halves the types $\mathrm{I}_{\mathrm{abs}}$ and $\mathrm{I}_{\mathrm{rel}}$ coincide.

1.5. $(M-d)$-surfaces. According to the Smith-Thom inequality, for any complex surface Y with a real structure one has $\beta_*(Y_{\mathbb{R}}) \leqslant \beta_*(Y)$, and the difference $\beta_*(Y) - \beta_*(Y_{\mathbb{R}})$ is even. By definition, Y is called an $(M-d)$-*surface* if the above difference is $2d$.

2. Main results

From now on we fix a generalized real Enriques surface E with $E_{\mathbb{R}} \neq \varnothing$ and follow the notation introduced in Section 1: $\text{conj}\colon E \to E$ is the real structure on E, X is the double covering of E with the Enriques involution $\tau\colon X \to X$, and $t^{(1)}, t^{(2)}$ are the two real structures on X determined by conj (see Lemma 1.3.1).

2.1. Prohibitions on the topological type.

2.1.1. Theorem. *Suppose that $X_{\mathbb{R}}^{(1)}$ is of type I and both the halves are nonempty. Then*

(1) *$E_{\mathbb{R}}$ has no nonorientable components of odd genus (i.e., V_{2g+1});*

(2) *at most one of the two halves $E_{\mathbb{R}}^{(1)}$, $E_{\mathbb{R}}^{(2)}$ may have a nonorientable component.*

2.1.2. Theorem. *Suppose that $E_{\mathbb{R}}$ is orientable. Then E is an $(M-d)$-surface with $d \geqslant 2$, and*

(1) *if $d=2$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pmod{16}$ and $E_{\mathbb{R}}$ is of type* I;
(2) *if $d=3$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 2 \pmod{16}$;*
(3) *if $d=4$ and $\chi(E_{\mathbb{R}}) \equiv \sigma(E)+8 \pmod{16}$, then $E_{\mathbb{R}}$ is of type* I.

If, in addition, all the components of $E_{\mathbb{R}}$ are spheres, then $d \geqslant 3$.

Remark. The last assertion of Theorem 2.1.2 follows, in fact, from Comessatti-Severi inequality: $\chi(E_{\mathbb{R}}) \leqslant h^{1,1}(E)$ (see [Co]). If E is a generalized Enriques surface and $E_{\mathbb{R}} = kS$, this inequality transforms into $d \geqslant 3+h^{2,0}(E)$. Thus, an $(M-d)$-surface with only spherical components and $d \leqslant 2$ cannot exist, and an $(M-3)$-surface with only spherical components may exist only if $H_2(E;\mathbb{Z})$ is a hyperbolic lattice. Note that this is the case for classical Enriques surfaces.

2.2. Prohibitions on the distribution of components.

2.2.1. Theorem. *Suppose that $E_{\mathbb{R}}$ consists of a single half and does not have nonorientable components of odd genus (i.e., V_{2g+1}). Then E is an $(M-d)$-surface with $d \geqslant 2$, and*

(1) *if $d=2$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pmod{16}$ and $E_{\mathbb{R}}$ is of type* I;
(2) *if $d=3$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 2 \pmod{16}$;*
(3) *if $d=4$ and $\chi(E_{\mathbb{R}}) \equiv \sigma(E)+8 \pmod{16}$, then $E_{\mathbb{R}}$ is of type* I.

2.2.2. Theorem. *Let E be an $(M-3)$-surface with $E_{\mathbb{R}} = kS$. Then $E_{\mathbb{R}} = \{4pS\} \sqcup \{(4q+1)S\}$, both the halves being nonempty unless $k \equiv 1 \pmod 8$.*

2.2.3. Theorem. *Let $E_{\mathbb{R}} = V_{2g} \sqcup kS$, $g>0$. Suppose that E is an $(M-d)$-surface and $\chi(E_{\mathbb{R}}) \equiv \sigma(E)+2\delta \pmod{16}$. Then for all the values of (d,δ) listed in Table 1 one has $E_{\mathbb{R}} = \{V_{2g} \sqcup k^{(1)}S\} \sqcup \{k^{(2)}S\}$, where $k^{(2)} \pmod 4$ may take only the values given in the table and $k^{(2)} \neq 0$ with the possible exception of the case $d=2$, $\delta=0$, $E_{\mathbb{R}}$ is of type* I. *Besides, there are the following additional prohibitions:*

(1) *if $d=0$, then $E_{\mathbb{R}}^{(1)}$ is of type $\mathrm{I}_{\mathrm{abs}}$ and $E_{\mathbb{R}}^{(2)}$ is of type $\mathrm{I}_{\mathrm{rel}}$;*
(2) *if $d=0$, then $k^{(1)} \neq 0$ unless $k \equiv 0 \pmod 8$;*
(3) *if $d=1$ and $k^{(1)}=0$, then either $k \equiv \delta \pmod 8$, or $k \equiv 0 \pmod 4$ and $E_{\mathbb{R}}^{(2)}$ is of type $\mathrm{I}_{\mathrm{rel}}$.*

Remark. Note that in the case $d=3$ the last theorem only states that, if $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 6 \pmod{16}$, then both the halves are not empty. This follows also from Theorem 2.2.1.

2.3. Classical Enriques surfaces. The topological types realizable by the real part of a classical Enriques surface were enumerated in [KhD]. In that

TABLE 1

d	δ	$k^{(2)}$ (mod 4)
0	0	0
1	1	0, 1
	−1	0, 3
2	0	0, 2 (if $E_{\mathbb{R}}$ is of type I); 0, 1, 3 (if $E_{\mathbb{R}}$ is of type II)
	2	0, 1, 2
	−2	0, 2, 3
	4	0, 2
3	±3	0, 1, 2, 3

paper we treated separately three types, $6S$, $S_1 \sqcup 5S$, and $3V_2$, and one series, $S_1 \sqcup V_1 \sqcup \ldots$, which were not prohibited by the standard inequalities and congruences known in topology of real algebraic varieties. The prohibition of these types is now an immediate consequence of the general results of the previous section: the first two are prohibited by Theorem 2.1.2, the others—by Theorem 2.1.1. To apply Theorem 2.1.1 one should note that, if the real part of a real $K3$-surface contains two components S_1, then this real part is of type I and it cannot have any other component, see [Kh].

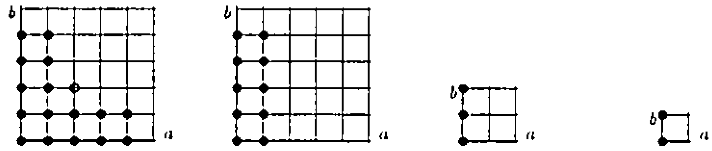
Consider now the decomposition $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)}$. The following obvious observation can be found, e.g., in [KhD]:

2.3.1. *Each half of a classical real Enriques surface may only be of one of the following three types:*

(1) $\alpha V_g \sqcup a V_1 \sqcup bS$, $g > 1$, $a \geqslant 0$, $b \geqslant 0$, $\alpha = 0, 1$;
(2) $2V_2$;
(3) S_1.

In [KhD] and in Section 8 we construct a number of different realizations of Enriques surfaces which is sufficient to show that, with a few exceptions, any distribution satisfying 2.3.1 is realizable. The exceptional topological types are listed in Figure 1: the distributions marked by the black nodes are realized, e.g., in [KhD]; the white node represents the distributions $\{2S\} \sqcup \{2S\}$ and $\{V_2 \sqcup 2S\} \sqcup \{2S\}$ constructed in [N2]. Theorems 2.2.2 and 2.2.3 imply that this list is complete:

2.3.2. Theorem. *With the exception of the types kS and $V_{2r} \sqcup kS$ any distribution of the components of a real Enriques surface satisfying 2.3.1 is realizable. The exceptional topological types admit only the distributions listed in Figure 1.*



$\{aS\}\sqcup\{bS\}$, $\{V_2\sqcup aS\}\sqcup\{bS\}$ — $\{V_4\sqcup aS\}\sqcup\{bS\}$ — $\{V_6\sqcup aS\}\sqcup\{bS\}$ — $\{V_8\sqcup aS\}\sqcup\{bS\}$, $\{V_{10}\sqcup aS\}\sqcup\{bS\}$

FIGURE 1. Exceptional topological types

Remark. There are four distributions, $\{2S\}\sqcup\{2S\}$, $\{V_2\sqcup 2S\}\sqcup\{2S\}$, $\{V_2\sqcup 2S\}\sqcup\{V_2\sqcup 2S\}$, and $\{V_2\sqcup 4S\}\sqcup\{V_2\}$, which are not constructed in [KhD] or Section 8. Existence of these distributions is announced in [N2]. The first two distributions *cannot* be obtained by our construction, i.e., the covering $K3$-surface is not a double of a symmetric quadric. (Proof will be published elsewhere.)

3. Involutions on modules

In this section we expose some elementary facts on the Galois cohomology of modules with involution and on the discriminant forms of integral lattices with involution. Most of the results of this section appear, explicitly or implicitly, in [N1]. We give the proofs when it is easier than to find a precise reference or when the direct proof is simpler.

3.1. Galois cohomology of $\mathbb{Z}/2$-vector spaces with involution. The zero-dimensional cohomology group of a $\mathbb{Z}/2$-vector space V with an involution c is $H^0(V)=\operatorname{Ker}(1+c)$. All the other cohomology groups are isomorphic to $\operatorname{Ker}(1+c)/\operatorname{Im}(1+c)$; to be short and in accordance with the notation commonly used in the literature we denote them by $\hat H^0(V)$.

3.1.1. Lemma. *Let V and V' be finite dimensional vector spaces over $\mathbb{Z}/2$ with involution. If they are connected by one of the following two short exact sequences of spaces with involution*

$$0\to\mathbb{Z}/2\to V\to V'\to 0 \qquad \text{or} \qquad 0\to V'\to V\to\mathbb{Z}/2\to 0,$$

then $\dim\hat H^0(V)-\dim\hat H^0(V')=\pm1$. In the former case the difference is -1 if and only if the generator of the subgroup $\mathbb{Z}/2$ vanishes in $\hat H^0(V)$. In the latter case it is -1 if and only if the generator of the quotient group $\mathbb{Z}/2$ does not lift to $\hat H^0(V)$, i.e., does not belong to the image of $\operatorname{Ker}(1+c)\subset V$.

Proof. Denote by c, c', and c_0 the involutions on V, V', and $\mathbb{Z}/2$ respectively. Then $\operatorname{Ker}(1+c_0)=\operatorname{Coker}(1+c_0)=\mathbb{Z}/2$, and the result follows immediately from the additivity of dimension and the Ker-Coker exact sequences (see, e.g., [CE], Lemma V.10.1)

$$0 \to \operatorname{Ker}(1+c_0) \to \operatorname{Ker}(1+c) \to \operatorname{Ker}(1+c') \to \\ \to \operatorname{Coker}(1+c_0) \to \operatorname{Coker}(1+c) \quad \text{and}$$

$$\operatorname{Ker}(1+c) \to \operatorname{Ker}(1+c_0) \to \\ \to \operatorname{Coker}(1+c') \to \operatorname{Coker}(1+c) \to \operatorname{Coker}(1+c_0) \to 0. \quad \square$$

Suppose now that V is equipped with a c-equivariant symmetric bilinear form $\circ\colon V \otimes V \to \mathbb{Z}/2$. Then $\circ$ induces, in a natural way, a symmetric bilinear form on $\hat{H}^0(V)$.

3.1.2. Lemma. *If $\circ\colon V \otimes V \to \mathbb{Z}/2$ is nondegenerate, then so is the induced form $\circ\colon \hat{H}^0(V) \otimes \hat{H}^0(V) \to \mathbb{Z}/2$.*

Proof. Since $\hat{H}^0(V) = \operatorname{Ker}(1+c)/\operatorname{Im}(1+c)$, the result follows from the additivity of dimension and the existence of the induced form. $\square$

3.2. Free abelian groups with involution. Let L be a finitely generated free abelian group with involution c. Consider its eigensubgroups

$$L^+ = \{x \in L \mid cx = x\}, \quad L^- = \{x \in L \mid cx = -x\}$$

and the cohomology group of the associated $\mathbb{Z}/2$-vector space $L/2L = L \otimes \mathbb{Z}/2$:

$$\hat{H}(L) = \hat{H}^0(L/2L).$$

Obviously, both $L^\pm$ are primitive in L (i.e., the quotients $L/L^\pm$ are torsion free), and $L^+ \cap L^- = 0$.

3.2.1. Lemma. *One has*

$$\begin{aligned} \operatorname{Ker}[(1+c)\colon L/2L \to L/2L] &= (L^+/2L) + (L^-/2L), \\ \operatorname{Im}[(1+c)\colon L/2L \to L/2L] &= (L^+/2L) \cap (L^-/2L), \\ \dim \hat{H}(L) &= \dim L - 2\dim[(L^+/2L) \cap (L^-/2L)]. \end{aligned}$$

Proof. In $L \otimes \mathbb{Q}$ each element x is represented as $x = x^+ + x^-$, where $x^+ = \frac{1}{2}(x + cx)$ and $x^- = \frac{1}{2}(x - cx)$. The first statement follows from the fact that, given an $x \in L$, the elements $\frac{1}{2}(x + cx)$ and $\frac{1}{2}(x - cx)$ belong to L if and only if $x \equiv cx \pmod{2L}$. To prove the second statement just notice that $(1+c)y \equiv (1-c)y \pmod{2L}$ for any $y \in L$, and that whenever $x^+ \in L^+$ and $x^- \in L^-$ are such that $x^+ \equiv x^- \pmod{2L}$, one has $x^+ = y + cy$, where $y = \frac{1}{2}(x^+ + x^-) \in L$.

The last statement is an immediate consequence of the first two. $\square$

3.3. Integral lattices. Suppose now that L is a unimodular integral even lattice, i.e., L is supplied with a symmetric bilinear pairing $\circ\colon L\otimes L\to\mathbb{Z}$ so that (1) the *correlation* $\varphi\colon L\to L^* = \operatorname{Hom}(L,\mathbb{Z})$, $\varphi x(y) = x\circ y$, is an isomorphism (L is unimodular), and (2) $x\circ x\in 2\mathbb{Z}$ for any $x\in L$ (L is even). Assume also that L is supplied with an involution $c\colon L\to L$ which is a lattice morphism, i.e., $cx\circ cy = x\circ y$ for any $x,y\in L$.

Under these assumptions each of the sublattices $L^\pm$ is the orthogonal complement of the other one, and they are both nondegenerated, i.e., their correlations are injective. Thus, one can define two finite $\mathbb{Z}/4$-quadratic spaces $\mathcal{D}^\pm$, which are called the *discriminant spaces* of $L^\pm$, in the following way:

The underlying finite groups, called the *discriminant groups*, are $\mathcal{D}^\pm = (L^\pm)^*/L^\pm$; here each $(L^\pm)^*$ is considered, via the correlation, as an extensions of the corresponding lattice $L^\pm$ in $L^\pm\otimes\mathbb{Q}$. The discriminant quadratic functions $q\colon\mathcal{D}^\pm\to\mathbb{Q}/2\mathbb{Z}$ are induced from the bilinear form extended from $L^\pm$ to $L^\pm\otimes\mathbb{Q}$: given $x\in(L^\pm)^*\subset L^\pm\otimes\mathbb{Q}$, define $q(x) = x\circ x \pmod 2$.

3.3.1. Lemma (see [N1]). *The quadratic spaces $(\mathcal{D}^\pm, q)$ are anti-isometric, i.e., there exists a group isomorphism $\alpha\colon\mathcal{D}^+\to\mathcal{D}^-$ such that $q(\alpha x) = -q(x)$ for any $x\in\mathcal{D}^+$.*

At the group level this statement has the following consequence:

3.3.2. Lemma. *One has $2(L^\pm)^*\subset L$, and the quotient*

$$\alpha^\pm\colon\mathcal{D}^\pm = (L^\pm)^*/L^\pm\to L/2L$$

of the multiplication by 2 establishes an isomorphism between $\mathcal{D}^\pm$ and the intersection $(L^+/2L)\cap(L^-/2L)\subset L/2L$. In particular, $\mathcal{D}^\pm$ are 2-periodic groups and $\dim\hat H(L) = \operatorname{rk} L - 2\dim\mathcal{D}^\pm$.

Proof. Let $x\in(L^+)^*$, i.e., let $x\in L^+\otimes\mathbb{Q}$ be an element such that $x\circ L^+\in\mathbb{Z}$. Then for any $y\in L$ one has $2x\circ y = 2x\circ(y^+ + y^-) = 2x\circ y^+ = x\circ(y+cy)\in\mathbb{Z}$. Hence, $2x\in L^* = L$ and $2(L^+)^*\subset L$. Since $2L^+\subset 2L$, the multiplication by 2 has a well defined quotient $\alpha^+\colon\mathcal{D}^+ = (L^+)^*/L^+\to L/2L$.

Let $x\in\operatorname{Ker}\alpha^+$, i.e., $2x\in 2L$. Then $x\in L\cap(L^+\otimes\mathbb{Q}) = L^+$, i.e., $x=0$ in $\mathcal{D}^+$. Thus, $\operatorname{Ker}\alpha^+ = 0$ and $\mathcal{D}^+$ is a 2-periodic group.

Given $2x = (1+c)y\in(L^+/2L)\cap(L^-/2L)$ (see Lemma 3.2.1), for any $z\in L^+$ one has $x\circ z = \frac{1}{2}(y\circ z + cy\circ cz)\in\mathbb{Z}$, i.e., $x\in(L^+)^*$. This proves that $\operatorname{Im}\alpha^+\supset(L^+/2L)\cap(L^-/2L)$.

Since $\mathcal{D}^+$ is a 2-periodic group, $2x\in L^+$ for any $x\in(L^+)^*$. Hence $\operatorname{Im}\alpha^+\subset L^+/2L$. Since L^+ is primitive in the unimodular lattice L, the map $L = L^*\to(L^+)^*$ induced by the inclusion $L^+\subset L$ is onto, and, given $x\in(L^+)^*$, there is some $y\in L$ so that $(x-y)\circ L^+ = 0$. Then $z = 2x-2y\in L^- = (L^+)^\perp$ and $2x\equiv z \pmod{2L}$. Hence $\operatorname{Im}\alpha^+\subset L^-/2L$. This completes the proof for α^+; the other isomorphism is constructed similarly. □

3.3.3. Corollary. *An $x \in L^+$ vanishes in $\hat{H}(L)$ if and only if $x \circ L^+ \in 2\mathbb{Z}$.*

Proof. According to Lemmas 3.2.1 and 3.3.2, x vanishes in $\hat{H}(L)$ if and only if $x \bmod 2L \in \operatorname{Im}\alpha^+$, i.e., $\frac{1}{2}x \in (L^+)^*$. □

Remark. The result follows as well from Lemmas 3.2.1 and 3.1.2, which gives a more direct proof.

3.3.4. To formulate the next statement, remind that, given a (not necessary unimodular) nondegenerate lattice M and a nondegenerate primitive sublattice $M' \subset M$, one can define subgroups $\Gamma' \subset \operatorname{discr} M'$ and $\Gamma'' \subset \operatorname{discr} M'^{\perp}$ and an anti-isometry $\alpha \colon \Gamma' \to \Gamma''$ so that M is the pull back of the graph Γ of α under the projection $(M')^* \oplus (M'^{\perp})^* \to \operatorname{discr} M \oplus \operatorname{discr} M'^{\perp}$ and $\operatorname{discr} M = \Gamma^{\perp}/\Gamma$. (Details can be found in Nikulin [N1].)

3.3.5. Lemma. *Suppose that M' is a primitive nondegenerate sublattice of L^+ and M is the primitive hull of $M' \oplus L^-$ in L. Let $x \in M' \subset L^+$ be an element with $x \circ M' \in 2\mathbb{Z}$, so that $\frac{1}{2}x$ defines an element in $\operatorname{discr} M'$. If this element belongs to the subgroup Γ' defined above, then x vanishes in $\hat{H}(L)$.*

Proof. According to Nikulin's construction, if the element defined by $\frac{1}{2}x$ in $\operatorname{discr} M'$ belongs to Γ', there are some $y \in L^-$ and $z \in M$ such that $z = \frac{1}{2}x + \frac{1}{2}y$. Then $x = 2z - y$ and $x \circ L^+ \in 2\mathbb{Z}$ (since $y \circ L^+ = 0$). The statement follows now from Corollary 3.3.3. □

4. Basic topological properties of generalized Enriques surfaces

4.1. General facts. First, let us consider an arbitrary algebraic surface Y equipped with a real structure $\operatorname{conj}\colon Y \to Y$. Denote $L = H_2(Y;\mathbb{Z})/\operatorname{Tors}$ and $\mathcal{D}^{\pm} = \operatorname{discr} L^{\pm}$, where $L^{\pm}$ are the subgroups of conj_*-invariant and conj_*-skew-invariant elements of L.

4.1.1. Lemma. *The fundamental class $[Y_{\mathbb{R}}] \in H_2(Y)$ and the Stiefel-Whitney class $w_2(Y)$ are integral, i.e., they belong to the image of $H_2(Y;\mathbb{Z})$ in $H_2(Y)$.*

Proof. As it is known (see [HH]), $w_2(Y)$ is integral for any closed orientable 4-dimensional manifold.[1]

According to [Ar], Lemma 3[2], $[Y_{\mathbb{R}}]$ is the characteristic class of the *twisted intersection form* $(x, y) \mapsto x \circ \operatorname{conj}_* y$. In particular, it is orthogonal to the image of $\operatorname{Tors} H_2(Y;\mathbb{Z})$ in $H_2(Y)$, which, by Poincaré duality, is the orthogonal complement of the image of $H_2(Y;\mathbb{Z})$. □

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[1] For complex manifolds this assertion is completely obvious as $w_2(Y) = c_1(Y) \bmod 2$.

[2] Arnol'd formulates and proves this assertion only for orientable $Y_{\mathbb{R}}$; the proof in the general case is literally the same.

Thus, the projections of $[Y_{\mathbb{R}}]$ and $w_2(Y)$ to $L/2L$ are well defined, and since both these classes are conj_*-invariant, they further descend to $\hat{H}(L)$.

4.1.2. Lemma. *The projections of $[Y_{\mathbb{R}}]$ and $w_2(Y)$ in $\hat{H}(L)$ coincide.*

Proof. Since $\hat{H}(L)$ consists of only conj_*-invariant classes, the twisted intersection form on it coincides with the standard intersection form. It remains to note that $w_2(Y)$ is the characteristic class of the standard intersection form, $[Y_{\mathbb{R}}]$ is the characteristic class of the twisted intersection form (Arnold's Lemma, *loc. cit.*), and the characteristic class is unique (Lemma 3.1.2). □

4.1.3. Lemma. *If Y is an $(M-d)$-surface, then*

(1) $\chi(Y_{\mathbb{R}}) \equiv \sigma(Y) + 2\operatorname{Br}\mathcal{D}^-\pmod{16}$;
(2) $\dim \mathcal{D}^- \equiv d \pmod 2$;

Proof. Hirzebruch's signature theorem gives $\nu(Y_{\mathbb{R}}) = \sigma(L^+) - \sigma(L^-)$. The left hand side here is the normal Euler number of $Y_{\mathbb{R}}$ in Y and is equal to $-\chi(Y_{\mathbb{R}})$; the right hand side is $-\sigma(Y) + 2\sigma(L^+) \equiv -\sigma(Y) - 2\operatorname{Br}\mathcal{D}^- \pmod{16}$. This proves (1).

Since Y is an algebraic surface, $\sigma(Y) \equiv b_2(Y) + 2 \equiv \beta_*(Y) \pmod 4$. By definition, $\beta_*(Y) = \beta_*(Y_{\mathbb{R}}) + 2d$. Substituting this into (1) and replacing $\chi(Y_{\mathbb{R}})$ with $\beta_*(Y_{\mathbb{R}}) \equiv \chi(Y_{\mathbb{R}}) \pmod 4$ and $\operatorname{Br}\mathcal{D}^-$ with $\dim\mathcal{D}^- \equiv \operatorname{Br}\mathcal{D}^- \pmod 2$ gives (2). □

4.1.4. Lemma. *The quadratic space $\mathcal{D}^-$ is even (i.e., $q(\hat{x}) \in \mathbb{Z}/2\mathbb{Z}$ for any $\hat{x} \in \mathcal{D}^-$) if and only if $[Y_{\mathbb{R}}] - w_2(Y)$ belongs to the image of $\operatorname{Tors} H_2(Y;\mathbb{Z})$ in $H_2(Y)$.*

Proof. $[Y_{\mathbb{R}}]$ and $w_2(Y)$ are the characteristic classes of the (respectively, twisted and standard) intersection forms. In particular, they are both orthogonal to the image of $\operatorname{Tors} H_2(Y;\mathbb{Z})$ in $H_2(Y)$. In addition, they are both integral (see Lemma 4.1.1). Thus, the condition that $[Y_{\mathbb{R}}] - w_2(Y)$ belongs to the image of $\operatorname{Tors} H_2(Y;\mathbb{Z})$ in $H_2(Y)$ is equivalent to the condition that this difference annihilates all the integral classes, which is equivalent to the congruence $x^2 \equiv x \circ \operatorname{conj}_* x \pmod 2$ for any $x \in L$.

Let $x^\pm = \frac{1}{2}(x \pm \operatorname{conj}_* x) \in L^\pm \otimes \mathbb{Q}$. Then $x = x^+ + x^-$ and $x^2 - x \circ \operatorname{conj}_* x \equiv 2(x^-)^2 \pmod{2\mathbb{Z}}$. Since $x^- \circ L^- = x \circ L^-$ takes integral values, x^- belongs to $(L^-)^*$ and, hence, represents an element in $\mathcal{D}^-$. Moreover, each element in $\mathcal{D}^-$ admits such a representative. Thus, $(x^-)^2 \in \mathbb{Z}$ for any $x \in L$ if and only if $\mathcal{D}^-$ is even. □

4.1.5. Corollary. *Suppose that the 2-primary component $\operatorname{Tors}_2 H_2(Y;\mathbb{Z})$ is generated by $w_2(Y)$. (This is the case for generalized Enriques surfaces; see Lemma 4.2.3 below.) Then $Y_{\mathbb{R}}$ is of type I if and only if $\mathcal{D}^-$ is even.*

All the preceding statements, except Lemma 4.1.3[3], extend, word by word, to any (not necessary anti-holomorphic) orientation preserving involution conj on any (not necessary complex) oriented 4-manifold Y. In this extended version Lemma 4.1.4 has the following corollary:

4.1.6. Corollary. *Let conj be a fixed point free orientation preserving involution on an oriented 4-manifold Y. Then the quadratic spaces $\mathcal{D}^\pm$ are even if and only if so is $H_2(Y;\mathbb{Z})/\operatorname{Tors}$.*

4.2. Homology of a generalized Enriques surface. We now consider a generalized Enriques surface E covered by a generalized $K3$-surface X with Enriques involution τ. We denote by $\operatorname{pr}\colon X \to E$ the projection and by $\operatorname{tr}\colon H_*(E;R) \to H_*(X;R)$ the transfer (with the coefficients in a group R).

Note that $H_1(X)=0$ implies $\operatorname{Tors}_2 H_2(X;\mathbb{Z})=0$.

4.2.1. Lemma. *There are isomorphisms $\operatorname{Tors}_2 H_1(E;\mathbb{Z}) = H_1(E) = \mathbb{Z}/2$ and an exact sequence*

$$0 \to \operatorname{Tors}_2 H_2(E;\mathbb{Z}) \to H_2(E) \xrightarrow{\operatorname{tr}} H_2(X),$$

where $\operatorname{Tors}_2 H_2(E;\mathbb{Z}) = \mathbb{Z}/2$ is generated by $w_2(E)$.

Proof. From the Smith-Ghysin exact sequence

$$\begin{array}{ccccccccc} H_1(X) & \xrightarrow{\operatorname{pr}_*=0} & H_1(E) & \longrightarrow & H_0(E) & \xrightarrow{\operatorname{tr}} & H_0(X) & \xrightarrow[\cong]{\operatorname{pr}_*} & H_0(E) \\ \| & & & & & & & & \\ 0 & & & & & & & & \end{array}$$

it follows that $H_1(E) = H_0(E) = \mathbb{Z}/2$ and, hence, $\operatorname{Tors}_2 H_1(E;\mathbb{Z})$ is a cyclic group. It cannot be larger than $\mathbb{Z}/2$ since otherwise X would have a nontrivial double covering. From Poincaré duality and universal coefficient formula it now follows that $H_3(E;\mathbb{Z})=0$, $H_3(E)=\mathbb{Z}/2$, and $\operatorname{Tors}_2 H_2(E;\mathbb{Z}) = \mathbb{Z}/2$, and another portion of the Smith-Ghysin exact sequence,

$$\begin{array}{ccccc} H_3(E) & \longrightarrow & H_2(E) & \xrightarrow{\operatorname{tr}} & H_2(X), \\ \| & & & & \\ \mathbb{Z}/2 & & & & \end{array}$$

shows that $\operatorname{Ker}[\operatorname{tr}_2\colon H_2(E) \to H_2(X)]$ is at most $\mathbb{Z}/2$. On the other hand, since $H_2(X;\mathbb{Z})$ does not have 2-torsion, $\operatorname{Tors}_2 H(E;\mathbb{Z})$ is contained in $\operatorname{Ker}\operatorname{tr}_2$. Thus, $\operatorname{Ker}\operatorname{tr}_2$ is $\mathbb{Z}/2$ and, since $w_2(E)\neq 0$ and $\operatorname{tr} w_2(E) = w_2(X) = 0$, its only nontrivial element is $w_2(E)$. □

[3]Lemma 4.1.3 extends to any anti-holomorphic involution on any quasi-complex variety, cf. [Wi].

4.2.2. Lemma. *For any $p = 1, 2, 3$ there is a short exact sequence*

$$0 \to \operatorname{Tors}_2 H_p(E;\mathbb{Z}) \to H_p(E;\mathbb{Z}) \xrightarrow{\operatorname{tr}_p} H_p^{+\tau}(X;\mathbb{Z}) \to 0,$$

where $H_p^{+\tau}(X;\mathbb{Z})$ denotes the subgroup of τ_*-invariant elements.

4.2.3. Lemma. *Let $\bar L = H_2(X;\mathbb{Z})/\operatorname{Tors}$ and let $\bar L^{\pm\tau}$ be the sublattices of τ_*-invariant and τ_*-skew-invariant elements of $\bar L$. Then $H_2(E;\mathbb{Z})/\operatorname{Tors}$ is an even lattice isometric via* tr *to $\bar L^{+\tau}(\frac12)$, which is $\bar L^{+\tau}$ with the modified pairing $(x,y) \mapsto \frac12(x \circ y)$.*

Proof of Lemmas 4.2.2 and 4.2.3. The transfer $H_*(E;R) \to H_*^{+\tau}(X;R)$ for $R = \mathbb{Q}$ and $R = \mathbb{Z}/q$, q odd, is an isomorphism (see, e.g., [Br]). Thus, in the integral homology we have $\operatorname{Ker}\operatorname{tr}_p = \operatorname{Tors}_2 H_p(E;\mathbb{Z})$, and, to complete the proof of 4.2.2, it only remains to show that tr_2 reduced modulo torsion maps $H_2(E;\mathbb{Z})/\operatorname{Tors}$ onto $\bar L^{+\tau}$.

Denote $L = H_2(E;\mathbb{Z})/\operatorname{Tors}$ and $L' = \bar{\operatorname{tr}} L \subset \bar L$, where $\bar{\operatorname{tr}}$ is the integral transfer reduced modulo torsion. Then $L' \subset \bar L^{+\tau}$ is a subgroup of finite index. The identity $\operatorname{tr} x \circ \operatorname{tr} y = 2(x \circ y)$ implies that $L = L'(\frac12)$ as a lattice and, since L is unimodular, the discriminant group of L' is 2-periodic of dimension equal to $\operatorname{rk} L = \operatorname{rk} L'$. Since, due to Lemma 4.2.1, the index of L' in $\bar L$ and, hence, in $\bar L^{+\tau}$ is odd ($\bar{\operatorname{tr}} \otimes \mathbb{Z}/2$ is a monomorphism) and $\operatorname{discr} \bar L^{+\tau}$ is also 2-periodic (Lemma 3.3.2), these two subgroups coincide.

Thus $\bar{\operatorname{tr}}_2$ provides an isometry between the lattices $H_2(E;\mathbb{Z})/\operatorname{Tors}$ and $\bar L^{+\tau}(\frac12)$ and an isomorphism between the groups $H_2(E;\mathbb{Z})/\operatorname{Tors}$ and $\bar L^{+\tau}$. The lattice $\bar L^{+\tau}(\frac12)$ is even due to Corollary 4.1.6. □

4.3. Eigenspaces of conj_*. Let now E be a generalized Enriques surface equipped with a real structure $\operatorname{conj}\colon E \to E$. The following fact is well known and follows immediately from Lefschetz fixed point theorem (part (1)) and Hirzebruch signature theorem (part (2)). Note that Statement (2) applies, in fact, to any real algebraic surface, and Statement (1) applies to any surface E with $H_1(E;\mathbb{Q}) = 0$.

4.3.1 Lemma. *Let $L = H_2(E;\mathbb{Z})/\operatorname{Tors}$ and let $L^\pm$ be the subgroups of conj_*-invariant and conj_*-skew-invariant elements of L. Then*

$$(1)\quad \operatorname{rk} L^+ = \frac12(b_2(E) + \chi(E_{\mathbb{R}})) - 1, \qquad \operatorname{rk} L^- = \frac12(b_2(E) - \chi(E_{\mathbb{R}})) + 1;$$

$$(2)\quad \sigma(L^+) = \frac12(\sigma(E) - \chi(E_{\mathbb{R}})), \qquad \sigma(L^-) = \frac12(\sigma(E) + \chi(E_{\mathbb{R}})).$$

5. Kalinin's spectral sequence and Viro homomorphisms

In this section we summarize some auxiliary results from algebraic topology of involutions. The constructions below are presented in both the cohomology and homology settings. They require, in principle, a cautious choice of the cohomology and homology theories used, as well as certain appropriate conditions on the underlying topological spaces. One possibility is to use the sheaf theories and suppose that the topological spaces involved are locally compact and finite dimensional. Fortunately, in this paper we do not need any definite choice and can use any homology theory (or even several theories). The reason is the fact that all the results are applied to the best topological spaces one can possibly expect—smooth compact manifolds.

Throughout this section Y is a good (see the paragraph above) topological space with involution $c\colon Y \to Y$.

5.1. Kalinin's homology spectral sequence.

5.1.1. *There exist a filtration*

$$0 = \mathcal{F}^{n+1} \subset \mathcal{F}^n \subset \dots \subset \mathcal{F}^0 = H_*(\operatorname{Fix} c),$$

a $\mathbb{Z}$-graded spectral sequence (H^r_*, d^r_*), where

$$d^r_q\colon H^r_q \to H^r_{q+r-1}, \quad d^r_{q+r-1} \circ d^r_q = 0,$$

$$(H^0_*, d^0_*) \text{ is the chain complex of } Y, \text{ and } H^{r+1}_q = \operatorname{Ker} d^r_q / \operatorname{Im} d^r_{q-r+1},$$

and homomorphisms $\operatorname{bv}_r\colon \mathcal{F}^r \to H^\infty_r$ such that

1. $H^1_* = H_*(Y)$ *and* $d^1_* = 1 + c_*$;
2. *a cycle $x_p \in H^0_p$ survives to H^r_p if and only if there are some chains $y_p = x_p, y_{p+1}, \dots, y_{p+r-1}$ in Y so that $\partial y_{i+1} = (1 + c_*)y_i$. In this case $d^r_p x_p = (1 + c_*)y_{p+r-1}$;*
3. *bv_q annihilates $\mathcal{F}^{q+1}$ and maps $\mathcal{F}^q/\mathcal{F}^{q+1}$ isomorphically onto H^∞_q;*
4. *the filtration, spectral sequence, and homomorphisms are all natural with respect to equivariant mappings.*

When necessary, we will use the notation $H^r_q = H^r_q(Y)$ and $\mathcal{F}^q = \mathcal{F}^q(Y)$ to indicate the original space Y.

The original construction of this spectral sequence is due to I. Kalinin [Ka][4], who derived it from the Borel-Serre spectral sequence and related results by Borel (see [Bo]). This construction is briefly outlined in Appendix B. Property (2) is proven in [D]. In Appendix A we give an alternative description of Kalinin's spectral sequence, which is based upon the Smith exact sequence.

The following results are straightforward consequences of 5.1.1.

[4] He presented the result only in its cohomological setting (see 5.3 below), but the construction is literally translated to the homology language.

5.1.2. Corollary. *If Y is connected and $\operatorname{Fix} c \neq \varnothing$, then*

(1) $H_0(Y) = H_0^2(Y) = H_0^\infty(Y) = \mathbb{Z}/2$;
(2) *each nonzero element of $H_1^2(Y)$ which survives to $H_1^\infty(Y)$ is nonzero in $H_1^\infty(Y)$.*

5.1.3. Corollary-definition. *If a cycle admits a representation by an equivariant chain, it survives to $H_*^\infty(Y)$. Thus, in particular, there are tautological homomorphisms $H_p(\operatorname{Fix} c) \to H_p^\infty(Y)$; with certain abuse of terminology we will call them the* inclusion homomorphisms.

5.1.4. Corollary. *One has $H_2^2(Y) = \hat{H}^0(H_2(Y))$.*

5.2. Viro homomorphisms. The homomorphisms bv_* appearing in Kalinin's spectral sequence were discovered, in an equivalent form, by O. Viro before Kalinin's work. That is why we call them *Viro homomorphisms*. The following geometrical description of Viro homomorphisms, given in terms of Kalinin's spectral sequence, is close to their original form due to Viro (cf. [VZ]).

5.2.1. *Suppose that $\operatorname{Fix} c \neq \varnothing$. Then*

(1) $\mathrm{bv}_0\colon H_*(\operatorname{Fix} c) \to H_0^\infty(Y)$ *is zero on $H_{\geqslant 1}(\operatorname{Fix} c)$; its restriction to $H_0(\operatorname{Fix} c) \to H_0^\infty(Y) = H_0(Y)$ coincides with the inclusion homomorphism (cf 5.1.2 and 5.1.3);*
(2) *a (nonhomogeneous) element $x \in H_*(\operatorname{Fix} c)$ represented by a cycle $\sum x_i$ belongs to $\mathcal{F}_p = \operatorname{Ker} \mathrm{bv}_{p-1}$ (see 5.1.1) if and only if there exist some chains y_i, $1 \leqslant i \leqslant p$, so that $\partial y_1 = x_0$ and $\partial y_{i+1} = x_i + (1+c_*)y_i$ for $i \geqslant 1$; the class of $x_p + (1+c_*)y_p$ in $H_p^\infty(Y)$ represents then $\mathrm{bv}_p\, x$.*

This result is proven in [D].

5.2.2. Evident Corollary. *For any p the Viro homomorphism bv_p is zero on $H_{>p}(\operatorname{Fix} c)$ and coincides with the inclusion homomorphism (see 5.1.3) when restricted to $H_p(\operatorname{Fix} c) \to H_p^\infty(Y)$.*

5.3. Kalinin's cohomology spectral sequence. Though in applications the homology groups are more transparent and easier to manipulate with, in a number of intermediate considerations it is the cohomology language that is more convenient and gives the results. In particular, the cohomology spectral sequence has the advantage that it carries a canonical multiplicative structure; we will use this structure to introduce in a formal way and to evaluate (in the general case, see Appendix B) the intersection pairing in the homology spectral sequence.

5.3.1. *There exist a filtration*

$$H^*(\operatorname{Fix} c) = \mathcal{F}_n \supset \mathcal{F}_{n-1} \supset \cdots \supset \mathcal{F}_{-1} = 0,$$

a $\mathbb{Z}$-graded spectral sequence (H^*_r, d^*_r), where

$$d^q_r : H^q_r \to H^{q-r+1}_r, \quad d^{q-r+1}_r \circ d^q_r = 0,$$

$$(H^*_0, d^*_0) \text{ is the cochain complex of } Y, \text{ and } H^q_{r+1} = \operatorname{Ker} d^q_r / \operatorname{Im} d^{q+r-1}_r,$$

and homomorphisms $\operatorname{bv}^r : H^r_\infty \to H^*(\operatorname{Fix} c)/\mathcal{F}_{r-1}$ *such that*

(1) bv^q *maps* H^q_∞ *isomorphically onto* $\mathcal{F}_q/\mathcal{F}_{q-1}$;
(2) *the spectral sequence, homomorphisms, and filtration are all natural with respect to equivariant mappings;*
(3) *the spectral sequence is multiplicative, the multiplication being induced by the cup-product in* H^*_0; *the filtration and homomorphisms* bv^q *preserve the multiplication;*
(4) $H^r_*(Y)$ *is a graded differential module over* H^*_r *(via the cap-product); the homology filtration and homomorphisms* bv_q *preserve the module structure.*

This spectral sequence is dual to that of 5.1.1 in the following sense: $H^q_r = \operatorname{Hom}(H^r_q; \mathbb{Z}/2)$, $\mathcal{F}_{r-1} = \operatorname{Ker}[H^*(\operatorname{Fix} c) \to \operatorname{Hom}(\mathcal{F}^r; \mathbb{Z}/2)]$, *and* d^q_r *and* bv^q *are dual to* d^r_{q-r+1} *and* bv_q *respectively.*

The cohomology part of this statement is proved in [Ka]; the rest is the standard relation between dual cohomology and homology objects.

5.3.2. Corollary. *If Y is a closed n-dimensional manifold and* $\operatorname{Fix} c \neq \varnothing$, *then for any r,* $1 \leqslant r \leqslant +\infty$, *one has* $H^n_r \cong \mathbb{Z}/2$, *and the product map* $H^p_r \otimes H^{n-p}_r \to H^n_r$ *is a nondegenerate pairing.*

To our knowledge, the only publication where this result is stated explicitly is [Ka]. It is a straightforward consequence of the Poincaré duality and a simple lemma on spectral sequences which states that, given a multiplicative spectral sequence of $\mathbb{Z}/2$-algebras, if for some $r = r_0$ one has $H^n_r = H^n_\infty = \mathbb{Z}/2$, and the product paring $H^p_r \otimes H^{n-p}_r \to H^n_r$ is nondegenerate for $r = r_0$, then so it is for all $r = r_0, \dots, \infty$ (cf. Lemma 3.1.2).

5.3.3. Corollary (the dual version of 5.3.2). *If Y is a closed n-dimensional manifold and* $\operatorname{Fix} c \neq \varnothing$, *then the intersection pairing in* $H_*(Y)$ *descends to a nondegenerate pairing* $H^\infty_p \otimes H^\infty_{n-p} \to \mathbb{Z}/2$.

Poincaré duality between cohomology and homology translates 5.3.2 into 5.3.3, along with the above proof.

The pairing $H^\infty_p \otimes H^\infty_{n-p} \to \mathbb{Z}/2$ introduced in 5.3.3 is called below the *intersection form.*

5.4. Application to a real structure of a complex surface. Let Y be a compact nonsingular complex surface with a real structure $c: Y \to Y$. Then the $\mathbb{Z}/2$-homology fundamental class $[Y_\mathbb{R}]$ of $Y_\mathbb{R} = \operatorname{Fix} c$ is well defined.

5.4.1. Lemma. *The Stiefel-Whitney class $w_2(Y)$ survives to $H_2^\infty(Y)$. The projection of $w_2(Y)$ in $H_2^\infty(Y)$ coincides with $\operatorname{bv}_2[Y_\mathbb{R}]$.*

Proof. As any Chern or Stiefel-Whitney class, $w_2(Y)$ is realized by the fundamental class of a c-invariant divisor. (The earliest reference which we could find in the literature is [BH]; the statement is based on the simple observation that Schubert cycles are defined over $\mathbb{R}$ and even over $\mathbb{Z}$.) Thus, w_2 survives to $H_2^\infty(Y)$. The other part of the lemma follows from 5.3.3, 5.1.4, and the fact that the image of $[Y_\mathbb{R}]$ in $H_2(Y)$ coincides with the characteristic class of the twisted intersection form (cf. the proof of Lemmas 4.1.1 and 4.1.2). ▢

Let $C_1, C_2, \dots, C_k$ be the components of $Y_\mathbb{R}$. Denote by $\langle C_i\rangle \in H_0(\operatorname{Fix} c)$ and $[C_i] \in H_2(\operatorname{Fix} c)$ the classes represented by a component C_i, and consider the following values of Viro homomorphisms:

- $\operatorname{bv}_0\langle C_i\rangle$ in $H_0^\infty(Y)$;
- $\operatorname{bv}_1 \alpha$ and $\operatorname{bv}_1\langle C_i - C_j\rangle$ in $H_1^\infty(Y)$ (where α is an element of $H_1(Y_\mathbb{R})$, and $\langle C_i - C_j\rangle = \langle C_i\rangle - \langle C_j\rangle$);
- $\operatorname{bv}_2[C_i]$, $\operatorname{bv}_2 \alpha$, $\operatorname{bv}_2\langle C_i - C_j\rangle$, and $\operatorname{bv}_2(\alpha + \langle C_i - C_j\rangle)$ in $H_2^\infty(Y)$.

From 5.2.1 and 5.2.2 it immediately follows that:

- all the above classes but the last three are always well defined;
- $\operatorname{bv}_2 \alpha$ is defined if and only if $\operatorname{bv}_1 \alpha = 0$, i.e., if $\operatorname{in}_* \alpha = (1 + c_*)y_1$, where y_1 is a cycle in Y. In the latter case $\operatorname{bv}_2 \alpha$ is represented by the cycle $(1+c_*)y_2$, where y_2 is any 2-chain in Y such that $\partial y_2 + (1+c_*)y_1$ belongs to $Y_\mathbb{R}$ and represents α;
- $\operatorname{bv}_2\langle C_i - C_j\rangle$ is defined if and only if $\operatorname{bv}_1\langle C_i - C_j\rangle = 0$. The latter class is represented by the equivariant circle $(1 + c_*)y_1$, where y_1 is a segment in Y joining a point in C_i and a point in C_j; it vanishes if $(1 + c_*)y_1$ bounds in Y (for some appropriate choice of y_1), and in this case $\operatorname{bv}_2\langle C_i - C_j\rangle$ is represented by $(1 + c_*)y_2$, where y_2 is any 2-chain in Y with $\partial y_2 = (1 + c_*)y_1$;
- $\operatorname{bv}_2(\alpha + \langle C_i - C_j\rangle)$ is defined if and only if $\operatorname{bv}_1 \alpha = \operatorname{bv}_1\langle C_i - C_j\rangle$; in this case it is represented by the cycle $(1 + c_*)y_2$, where y_2 is a 2-chain such that ∂y_2 consists of an equivariant circle representing $\operatorname{bv}_1\langle C_i - C_j\rangle$ and a cycle representing α in $Y_\mathbb{R}$.

One can smoothen all the chains above. For our purpose it is sufficient to smoothen them in a tubular neighborhood W of $Y_\mathbb{R}$ and thus to represent the last four classes near $Y_\mathbb{R}$ by smooth equivariant 2-submanifolds of Y.

Remark. If $c_* = \mathrm{id}$ on $H_1(Y)$, then one can ignore the term $(1 + c_*)y_1$ in the above description of bv. This is the case, e.g., if Y is a generalized Enriques surface.

5.4.2. Intersection matrix. *The intersection form on $H_2^\infty(Y) = \operatorname{Im} \operatorname{bv}_2$ is that defined by Table 2, where $C_i, \dots, C_l$ are some connected components* *of $Y_{\mathbb{R}}$, and α, β are some 1-dimensional homology classes in $Y_{\mathbb{R}}$. The intersection $\alpha\circ\beta$ is regarded as an element of $H_0(Y_{\mathbb{R}})$, and $(\alpha\circ\beta)[Y_{\mathbb{R}}^{(1)}]$ and $(\alpha\circ\beta)[C_i]$ are, respectively, the total intersection number and its part which falls into C_i. δ_{ij} stands for the Kronecker symbol: $\delta_{ii}=1$ and $\delta_{ij}=0$ if $i\neq j$. The intersection form extends linearly to the classes of the form $\operatorname{bv}_2(\alpha+(C_i-C_j))$, as if $\operatorname{bv}_2\alpha$ and $\operatorname{bv}_2(C_i-C_j)$ were well defined.*

TABLE 2

	$\operatorname{bv}_2(C_i-C_j)$	$\operatorname{bv}_2\alpha$	$\operatorname{bv}_2[C_i]$
$\operatorname{bv}_2(C_k-C_l)$	0	0	$\delta_{ik}+\delta_{il}$
$\operatorname{bv}_2\beta$	0	$(\alpha\circ\beta)[Y_{\mathbb{R}}^{(1)}]$	$(\beta\circ\beta)[C_i]$
$\operatorname{bv}_2[C_k]$	$\delta_{ik}+\delta_{jk}$	$(\alpha\circ\alpha)[C_k]$	$\delta_{ik}\chi(C_i)$

Proof. Pick some smooth equivariant representatives (see above) of two elements of the table. By 5.3.3, their intersection number in $H_2(Y)$ is equal to the intersection number of the corresponding classes in $H_2^{\sim}$. By a small equivariant perturbation put the representatives in a position without common points in $W\setminus\operatorname{Fix}c$ (see above). Since the intersection numbers are considered modulo 2, one can ignore all the imaginary intersection points, which appear in pairs (cf., for example [Kh2], Lemma 2.3), and counting the intersection points in W of some (not necessary equivariant now) perturbations of the cycles gives the desired result. When counting $\operatorname{bv}_2[C_i]\circ\operatorname{bv}_2\beta$ and $\operatorname{bv}_2[C_i]\circ\operatorname{bv}_2[C_i]$, one should take into consideration the fact that the normal bundle of $\operatorname{Fix}c$ in Y is (anti-)isomorphic to its tangent bundle. □

Remark. One can avoid the geometrical arguments in the proof and to generalize the result to higher dimensions, see Appendix B.

6. VIRO HOMOMORPHISMS IN GENERALIZED ENRIQUES SURFACES

Recall that we denote by E a generalized real Enriques surface, which is supposed to have nonempty real part: $E_{\mathbb{R}}\neq\varnothing$.

The main goal of this section is to prove Propositions 6.1 and 6.2 below. In the proofs we use Kalinin's homology spectral sequence H_*^r; we denote $\dim H_p^r$ by β_p^r.

6.1. Dimension of the discriminant space. *Let E be an $(M-d)$-surface, and let $\mathcal{D}^-$ be the discriminant space of the sublattice of conj-skew-invariant vectors in $H_2(E;\mathbb{Z})/\operatorname{Tors}$. Then:*

$d-\dim\mathcal{D}^-=0$ if either

(1) $E_{\mathbb{R}}$ has a component V_{2g+1} (i.e., $w_2(E_{\mathbb{R}})\neq 0$), or

(2) $E_{\mathbb{R}}$ is nonorientable and both the halves are nonempty;

$d - \dim \mathcal{D}^- = 2$ *if either*

(1) $E_{\mathbb{R}}$ *is nonorientable,* $w_2(E_{\mathbb{R}}) = 0$, *and one of the halves is empty, or*
(2) $E_{\mathbb{R}}$ *is orientable and both the halves are nonempty;*

$d - \dim \mathcal{D}^-$ *may be 2 or 4 if* $E_{\mathbb{R}}$ *is orientable and one of the halves is empty.*

6.2. Relations between real components. *There is at least one and at most two relations between the elements of* $H_2^{\omega}(E)/w_2(E)$ *realized by the fundamental classes of the components of* $E_{\mathbb{R}}$. *One relation is* $\operatorname{bv}_2[E_{\mathbb{R}}] = w_2(E)$; *the only other possible relation is* $\operatorname{bv}_2[E_{\mathbb{R}}^{(1)}] \equiv \operatorname{bv}_2[E_{\mathbb{R}}^{(2)}] \equiv 0 \pmod{w_2(E)}$.

6.3. Proof of Proposition 6.1.

6.3.1. Lemma. *Let* C_1, C_2 *be two components of* $E_{\mathbb{R}}$. *Then* $\operatorname{bv}_1\langle C_1 - C_2\rangle = 0$ *if and only if these two components belong to the same half of* $E_{\mathbb{R}}$.

Proof. Pick two points $c_i \in C_i$ and connect them with a path γ in E. By 5.1.2, $\operatorname{bv}_1\langle C_1 - C_2\rangle = 0$ if and only if the loop $\delta = (\operatorname{conj}\gamma)^{-1}\cdot\gamma$ is homologous to zero in $H_1(E)$. Thus $\operatorname{bv}_1\langle C_1 - C_2\rangle = 0$ if and only if δ lifts to a loop in X. Suppose that $C_1 \in E_{\mathbb{R}}^{(1)}$ and lift γ to a path $\tilde\gamma$ with the endpoints $\tilde c_1, \tilde c_2$. Then $\tilde\delta = \tilde\gamma\cdot(t^{(1)}\tilde\gamma)^{-1}$ is a lift of δ which connects $t^{(1)}\tilde c_2$ and $\tilde c_2$. It is a loop if and only if $t^{(1)}\tilde c_2 = \tilde c_2$, i.e., $c_2 \in E_{\mathbb{R}}^{(1)}$. ☐

6.3.2. Lemma. *Let* α *be an element of* $H_1(E_{\mathbb{R}})$. *Then* $\operatorname{bv}_1\alpha \neq 0$ *if and only if* $\omega \circ \alpha = 1$, *where* $\omega \in H_1(E_{\mathbb{R}})$ *is the characteristic element of the covering* $X_{\mathbb{R}} \to E_{\mathbb{R}}$. *Moreover,* $\operatorname{bv}_1\alpha \neq 0$ *whenever* $\alpha^2 = 1$.

Proof. Since $H_1(E) = \mathbb{Z}/2$, from 5.1.2 it follows that $\operatorname{bv}_1\alpha = 0$ if and only if $\operatorname{in}_*\alpha \in H_1(E)$ is zero, or, equivalently, if $\omega\circ\alpha = 0$. The last assertion follows from Lemma 1.3.2: if $w_1(E_{\mathbb{R}}) \neq 0$, then $\omega = w_1(E_{\mathbb{R}})$. ☐

6.3.3. Lemma. *The Stiefel-Whitney class* $w_2(E)$ *(which, due to 5.4.1, always survives to* $H_2^{\omega}(E)$*) represents a nonzero element in* $H_2^{\omega}(E)$ *if and only if either*

(1) $E_{\mathbb{R}}$ *has a component* V_{2g+1} *(i.e.,* $w_2(E_{\mathbb{R}}) \neq 0$*), or*
(2) $E_{\mathbb{R}}$ *is nonorientable and both the halves are nonempty.*

Proof. By 5.3.3 and since $w_2(E)$ is a characteristic element of the intersection form, $w_2(E) \neq 0$ in $H_2^{\omega}(E)$ if and only if there is an element $x \in H_*(E_{\mathbb{R}})$ with $(\operatorname{bv}_2 x)^2 \neq 0$. According to 5.4.2 such an x can be found in one of the following three forms: (i) $x = [C_1]$, where $C_1 \subset E_{\mathbb{R}}$ is a component of odd Euler characteristic; (ii) $x = \alpha + \langle C_1 - C_2\rangle$, where $\alpha \in H_1(E_{\mathbb{R}})$ is an element with $\alpha^2 = 1$ and $\operatorname{bv}_1\alpha \neq 0$; (iii) $x = \alpha \in H_1(E_{\mathbb{R}})$ with $\alpha^2 = 1$ and $\operatorname{bv}_1\alpha = 0$. In (i) we have case (1) of the lemma. In (ii), according to 6.3.1, we have case (2). Finally, (iii) contradicts to 6.3.2. ☐

6.3.4. Lemma. *$H_1^\infty(E) \neq 0$ if and only if either*

(1) *$E_{\mathbb{R}}$ is nonorientable, or*
(2) *$E_{\mathbb{R}}$ has a component T_g, or*
(3) *both the halves of $E_{\mathbb{R}}$ are nonempty.*

If $H_1^\infty(E) \neq 0$, then the spectral sequence collapses at H_*^2; in particular, $\beta_2^2 - \beta_2^\infty = 0$. If $H_1^\infty(E) = 0$, then $\beta_2^2 - \beta_2^\infty = 0$ or 2 and $\beta_1^\infty = \beta_3^\infty = 0$.

Proof. By 5.2.1, $H_1^\infty(E) = \operatorname{bv}_1 H_{\leqslant 1}(\operatorname{Fix} c)$. According to 6.3.1 and 6.3.2, a homogeneous element $x \in H_*(E_{\mathbb{R}})$ with $\operatorname{bv}_1 x \neq 0$ is either $\alpha \in H_1(E_{\mathbb{R}})$ with $\omega \circ \alpha = 1$ (cases (1) and (2) of the lemma, see 1.3.2) or $(C_1 - C_2)$, where $C_i \subset E_{\mathbb{R}}^{(i)}$ are two components from different halves of $E_{\mathbb{R}}$ (case (3) of the lemma).

The last statement of the lemma is a straightforward consequence from the relations $\beta_0^2 = \beta_0^\infty = 1$ and $\beta_1^2 = 1 \geqslant \beta_1^\infty$ and from the existence of the nondegenerate pairing in the spectral sequence. In the case $H_1^\infty \neq 0$ one has $\beta_2^2 - \beta_2^\infty = 0$ if $H_1^2(E)$ is killed by d^3, and $\beta_2^2 - \beta_2^\infty = 2$ if it is killed by d^2. □

6.3.5. End of the proof.

By definition, $2d = \beta_*(E) - \beta_*^\infty$. According to Lemma 4.3.1, we have $2\dim \mathcal{D}^- = b_2(E) - b_2^2$, where $b_2^2 = \dim \hat{H}(\operatorname{conj}_*, H_2(E;\mathbb{Z})/\operatorname{Tors})$. Therefore,

$$2(d - \dim \mathcal{D}^-) = \left[(2 - \beta_1^\infty - \beta_3^\infty) + (\beta_2^2 - \beta_2^\infty)\right] + \left[2 - (\beta_2^2 - b_2^2)\right].$$

The first term of this expression is zero if $H_1^\infty(E) \neq 0$ and 2 or 4 otherwise, see 6.3.4. Applying Lemma 3.1.1 to the exact sequences

$$0 \to \operatorname{Tors}_2 H_2(E;\mathbb{Z}) \to H_2(E;\mathbb{Z}) \otimes \mathbb{Z}/2 \to (H_2(E;\mathbb{Z})/\operatorname{Tors}) \otimes \mathbb{Z}/2 \to 0,$$
$$0 \to H_2(E;\mathbb{Z}) \otimes \mathbb{Z}/2 \to H_2(E) \to \mathbb{Z}/2 \to 0$$

gives that $\beta_2^2 - b_2^2$ is equal to 2 if $w_2(E) \neq 0$ in $H_2^2(E)$, and it is equal to 0 or -2 otherwise. The combination $\beta_2^2 - b_2^2 = 0$ and $w_2(E) \neq 0$ in $H_2^2(E)$ is excluded by an additional argument: the intersection form on $H_2^2(E)$ is nondegenerate, hence, $w_2(E)$, which generates $\operatorname{Tors}_2 H_2(E;\mathbb{Z}/2) \subset H_2(E)$, and an arbitrary element, which generates the quotient $H_2(E)/(H_2(E;\mathbb{Z}) \otimes \mathbb{Z}/2)$ and thus has a nonzero intersection with $w_2(E)$, must either both survive to $H_2^2(E)$ or both disappear.

Now the lemma follows from Lemmas 6.3.3 and 6.3.4 and the $(\operatorname{mod} 2)$-congruence given by Lemma 5.1.2(2). □

6.4. Proof of Proposition 6.2.

The relation $bv_2[E_{\mathbb{R}}] = w_2(E)$ is given by Lemma 5.4.1.

Suppose that $\operatorname{bv}_2([C_1] + \dots + [C_r]) = k w_2(E)$, $k \in \mathbb{Z}/2$, is a relation other than $\operatorname{bv}_2[E_{\mathbb{R}}^{(1)}] \equiv 0 \pmod{w_2(E)}$ or $\operatorname{bv}_2[E_{\mathbb{R}}^{(2)}] \equiv 0 \pmod{w_2(E)}$. This means that one of the components C_i involved in the relation, say C_1, belongs to $E_{\mathbb{R}}^{(1)}$, and there is another component of $E_{\mathbb{R}}^{(1)}$, say D, which does not belong to the relation. Then $\operatorname{bv}_2(C_1 - D)$ is well defined, and, according to 5.4.2, $\operatorname{bv}_2(C_1 - D) \circ \operatorname{bv}_2([C_1] + \dots + [C_r]) = 1$ and $(\operatorname{bv}_2(C_1 - D))^2 = 0$. On the other hand, $w_2(E)$ survives to $H_2^{\sim}(E)$, and, since $w_2(E)$ is the characteristic class, one has $\operatorname{bv}_2(C_1 - D) \circ w_2(E) = (\operatorname{bv}_2(C_1 - D))^2 = 0$. This contradicts to $\operatorname{bv}_2([C_1] + \dots + [C_r]) = k w_2(E)$ and $\operatorname{bv}_2(C_1 - D) \circ \operatorname{bv}_2([C_1] + \dots + [C_r]) = 1$.

7. Proof of the main results

Below, as in Section 2, E is a generalized real Enriques surface with nonempty real part, $\operatorname{conj}: E \to E$ is the real structure on E, and X is the double covering of E with the Enriques involution $\tau: X \to X$ and two real structures $t^{(1)}$, $t^{(2)}$ determined by conj.

7.1. Proof of Theorem 2.1.1. By the hypothesis, the fundamental class of $X_{\mathbb{R}}^{(1)}$ vanishes in $H_2(X)$. On the other hand, it is equal to the image of the fundamental class of $E_{\mathbb{R}}^{(1)}$ under the transfer homomorphism $\operatorname{tr}: H_2(E) \to H_2(X)$, whose kernel is generated by $w_2(E)$ (see Lemma 4.2.1). Thus, the half $E_{\mathbb{R}}^{(1)}$ realizes either 0 or $w_2(E)$ in $H_2(E)$. Since, according to Lemma 5.4.1, the union $E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)}$ realizes $w_2(E)$ in $H_2^{\sim}(E)$, the half $E_{\mathbb{R}}^{(2)}$ realizes either $w_2(E)$ or 0. In any case at least one of the two halves realizes zero in $H_2^{\sim}(E)$.

Suppose that $C_1 \subset E_{\mathbb{R}}^{(1)}$ is a component of type V_{2k+1} and that $E_{\mathbb{R}}^{(2)}$ has at least one component, say, C_2. Then, by 5.4.2, $\operatorname{bv}_2[E_{\mathbb{R}}^{(1)}] \circ \operatorname{bv}_2[C_1] = 1$ and $\operatorname{bv}_2[E_{\mathbb{R}}^{(2)}] \circ \operatorname{bv}_2(w_1(C_1) + (C_1 - C_2)) = 1$, i.e., both the halves realize nontrivial classes in $H_2^{\sim}(E)$. This contradiction proves the first assertion.

Suppose now that each of the two halves contains a nonorientable component $C_i \subset E_{\mathbb{R}}^{(i)}$ (which, due to the first statement, must be of an even genus). Pick some classes $\alpha_i \in H_1(C_i)$ with $\operatorname{bv}_1 \alpha_i \neq 0$. Then for both $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$ one has $\operatorname{bv}_2(\alpha_j + (C_1 - C_2)) \circ \operatorname{bv}_2[E_{\mathbb{R}}^{(i)}] = 1$, which is also a contradiction. □

7.2. Proof of Theorems 2.1.2 and 2.2.1. Let $\mathcal{D}^-$ be the discriminant form of the sublattice of conj_*-skew-invariant vectors in $H_2(E;\mathbb{Z})/\operatorname{Tors}$. From Lemma 6.1 it follows that, under the hypotheses, $d - \dim \mathcal{D}^- = 2$ or 4. Since the dimension is nonnegative, $d \geqslant 2$.

All the congruences are derived from $\chi(E_{\mathbb{R}}) \equiv \sigma(E) + 2 \operatorname{Br} \mathcal{D}^- \pmod{16}$ given by Lemma 4.1.3(1) (just like the other congruences known in topology of real algebraic manifolds, cf. [Kh3], [M], and [N1]).

If $d = 2$, then $\mathcal{D}^- = 0$ and $\operatorname{Br} \mathcal{D}^- = 0$. This gives the congruence. The fact that $E_{\mathbb{R}}$ is of type I follows from Corollary 4.1.5.

If $d = 3$, then $\dim \mathcal{D}^- = 1$. Hence $\mathcal{D}^- = \langle \pm\frac{1}{2} \rangle$ and $\operatorname{Br} \mathcal{D}^- = \pm 1$.

If $d = 4$ and $\chi(E_{\mathbb{R}}) \equiv \sigma(E)+8 \pmod{16}$, then $\operatorname{Br}\mathcal{D}^- = 4$ and $\dim\mathcal{D}^- = 2$. The only such form is the one given by the (2×2)-matrix $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. This form is even and Corollary 4.1.5 applies to prove that $E_{\mathbb{R}}$ is of type I. $\square$

7.3. Proof of Theorems 2.2.2 and 2.2.3. In addition to the lattice $L = H_2(E;\mathbb{Z})/\operatorname{Tors}$ with involution conj_*, the eigenlattices $L^{\pm}$ of conj_*, and their discriminant forms $\mathcal{D}^{\pm}$, let us consider the sublattice M' of L^+ generated by the classes $s_1, \dots, s_k \in L$ realized by the spherical components of $E_{\mathbb{R}}$ (with some orientations), and denote by N the orthogonal complement of M' in L^+. Recall that L and all its sublattices are even, see 4.2.3.

7.3.1. Lemma. *If M' is not primitive in L^+, then either $E_{\mathbb{R}}$ has a half $\{lS\}$ of type* I *with $l \equiv 0 \pmod 4$, or $E_{\mathbb{R}} = kS$, it is of type* I, *and $k \equiv 0 \pmod 4$. If all the k spherical components constitute one half of $E_{\mathbb{R}}$ and, besides, $\mathcal{D}^- = 0$ and $\operatorname{rk} N = k-2$, then $k \equiv 0 \pmod 8$.*

Proof. Since $s_i \circ s_j = -2\delta_{ij}$, nonprimitiveness of M' means that there is an $x \in L$ such that $2x = s_1 + \dots + s_l$, $l > 0$. (We simplify the notation and assume that the relation involves the first l components.) Pick such a relation with the smallest possible number l of components. Then, due to 6.2 and 6.3.4, either the first l spherical components form a half $\{lS\}$ of $E_{\mathbb{R}}$ of type I, or $\{lS\} = E_{\mathbb{R}}$ and $E_{\mathbb{R}}$ is of type I. Since $l = -2x^2$, the first part of the lemma follows from the fact that L^+ is an even lattice[5].

Suppose that all the spherical components form together one half of $E_{\mathbb{R}}$. As it follows from the first part of the proof, no partial sum of $s_1, \dots, s_k$ is divisible by 2 (as otherwise the corresponding components would form a half), and the primitive hull M'' of M' in L^+ is generated by M' and an $x \in L$ such that $2x = s_1 + \dots + s_k$. Thus, the discriminant form of M'' is the nondegenerate part of the restriction of $-\frac{1}{2}(\theta_1^2 + \dots + \theta_k^2)$, $\theta_j \in \mathbb{Z}/2$, to $\theta_1 + \dots + \theta_k = 0$. In particular, $\dim\operatorname{discr} M'' = k-2$ and $\operatorname{discr} M''$ is an even form. If $\mathcal{D}^- = 0$, then $\mathcal{D}^+ = 0$ and L^+ is unimodular. If, in addition, $\operatorname{rk} N = k-2$, then, since $\dim\operatorname{discr} N = \dim\operatorname{discr} M'' = k-2$, the lattice $\frac{1}{2}N$ is integral and unimodular. Besides, it is even, since so are $\operatorname{discr} M''$ and L^+. Hence, $k = -\sigma(M') = \sigma(\frac{1}{2}N) - \sigma(L^+) \equiv 0 \pmod 8$. $\square$

7.3.2. Lemma. *If M' is primitive in L^+ and $\dim\operatorname{discr} M' + \dim\mathcal{D}^- > \dim\operatorname{discr} N$, then either $E_{\mathbb{R}}$ has a half $\{lS\}$, or $E_{\mathbb{R}} = lS$, where $l \neq 0$ and $l \equiv 2q(y) \pmod 4$ for some non trivial element $y \in \mathcal{D}^-$. If, in addition, $l = k$, $\dim\mathcal{D}^- = 1$, and $\operatorname{rk} N = k-1$, then $k \equiv \operatorname{Br}\mathcal{D}^- \pmod 8$.*

Remark. If $\dim\mathcal{D}^- = 1$, then $\mathcal{D}^-$ contains only one nontrivial element, and

[5] As it follows from the existence of equivariant representatives of the Chern classes, cf. 5.4.1, L^+ is even for any compact complex (and even quasicomplex) surface with a real structure.

$2q(y) = \operatorname{Br} \mathcal{D}^-$ (mod 8). In all cases $y = \frac{1}{2}y_-$ (mod L^-) for some element $y_- \in L^-$, and $2q(y) \equiv \frac{1}{2}y_-^2$ (mod 4).

Proof. Denote by M the primitive hull of $L^- \oplus M'$ in L. Since M and N are the orthogonal complements of each other in the unimodular even lattice L, their discriminant forms are anti-isometric. On the other hand, $\dim \operatorname{discr} M' + \dim \mathcal{D}^- > \dim \operatorname{discr} N = \dim \operatorname{discr} M$ by the hypotheses, and, hence, $L^- \oplus M'$ is not primitive in L and the subgroup $\Gamma' \subset \operatorname{discr} M'$ (see 3.3.4) is nontrivial: for some $l > 0$ there exists an element $y_- \in L^-$ which represents a nonzero element $y \in \operatorname{discr} M'$ so that the class $s = \frac{1}{2}(y_- + s_1 + \cdots + s_l)$ belongs to L. Then $s_1 + \cdots + s_l = s + \operatorname{conj}_* s$. Thus $s_1 + \cdots + s_l$ vanishes in $\hat{H}(L)$ and therefore the element realized by the corresponding l spherical components of $E_{\mathbb{R}}$ in $\hat{H}(H_2(E, \mathbb{Z}))$ is either 0 or w_2.

Due to 6.2 and 6.3.4, either these components form a half of $E_{\mathbb{R}}$, or $E_{\mathbb{R}} = lS$ and $l = k$. Furthermore, $2q(y) \equiv \frac{1}{2}y_-^2 \equiv \frac{1}{2}(s_1 + \cdots + s_l)^2 \equiv l$ (mod 2).

If the additional assumptions hold, then $\operatorname{discr} M$ is an even discriminant form of dimension $(k-1)$. Therefore, as in 7.3.1, $\frac{1}{2}N$ is an integral even unimodular lattice and $k - \operatorname{Br} \mathcal{D}^- \equiv \sigma(\frac{1}{2}N) \equiv 0$ (mod 8). □

Now, in order to complete the proof of theorems 2.2.2 and 2.2.3, consider separately the different cases.

7.3.3. The case $E_{\mathbb{R}} = kS$ (Theorem 2.2.2). Comessatti-Severi inequality $\chi(E_{\mathbb{R}}) \leqslant h^{1,1}(E)$ gives $d \geqslant 3 + h^{2,0}(E)$. Hence $d \geqslant 3$ and, if $d = 3$, then $\sigma(E) = 2 - b_2(E)$. In the latter case a calculation using Lemma 4.3.1 shows that L^- is a positive definite lattice of rank 1 and L^+ is a negative definite lattice of rank $2k-1$. In particular, $\dim \mathcal{D}^- = 1$ and $\operatorname{Br} \mathcal{D}^- = 1$. By 4.1.3, this implies that $k \equiv 1$ (mod 4). This congruence excludes, in particular, the second choice $E_{\mathbb{R}} = lS$, $l \equiv 0$ (mod 4) in Lemma 7.3.1. The theorem follows now from 7.3.1 and 7.3.2, which cover the two possibilities for M' and both give the same decomposition $\{4pS\} \sqcup \{(4q+1)S\}$ (with $l = 4q+1$ in the latter case). □

7.5. The case $E_{\mathbb{R}} = V_{2g} \sqcup kS$ (Theorem 2.2.3). From Lemma 4.3.1 (1) it follows that $\operatorname{rk} L^+ = 2k + d - 2$ and, hence, $\dim \operatorname{discr} N \leqslant \operatorname{rk} N = k + d - 2$. If $d = 0$, then L^+ is a unimodular lattice and $\dim \operatorname{discr} M' > \dim \operatorname{discr} N$. Hence M' cannot be primitive and 7.3.1 applies. Corollary 4.1.5 gives the missing information: $E_{\mathbb{R}}$ is of type I. If $d = 1$, then $\dim \mathcal{D}^- = 1$ and $\dim \operatorname{discr} N \leqslant k - 1$, and the statement follows from 7.3.1 and 7.3.2. The possibility "$k \equiv 0$ (mod 4), $E_{\mathbb{R}}^{(2)}$ is of type I" for $k^{(1)} = 0$ arises from the case when M' is not primitive: then $k = k^{(2)}$ must be divisible by 4. If $d = 2$, then $\mathcal{D}^-$ is one of the forms given in Table 3. $\mathcal{D}^- = 0$ is the exceptional case of Theorem 2.2.3 when $k^{(2)}$ may be trivial. (In fact, $k^{(2)}$ is trivial in this case since $\dim \mathcal{D}^- = d - 2$ and, according to Lemma 6.1, $E_{\mathbb{R}}$ must consist of a single half.) In all the other cases 7.3.1 and 7.3.2 give all the values of $k^{(2)}$ (mod 4) listed in Table 1.

TABLE 3

Odd forms

$\mathcal{D}^-$	$\operatorname{Br}\mathcal{D}^-$
0	0
$\langle\frac{1}{2}\rangle\oplus\langle\frac{1}{2}\rangle$	2
$\langle\frac{1}{2}\rangle\oplus\langle-\frac{1}{2}\rangle$	0
$\langle-\frac{1}{2}\rangle\oplus\langle-\frac{1}{2}\rangle$	-2

Even forms

$\mathcal{D}^-$	$\operatorname{Br}\mathcal{D}^-$
$\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}$	0
$\begin{pmatrix}1 & 1/2\\ 1/2 & 1\end{pmatrix}$	4

The remaining case $d = 3$, $\delta = \pm 3$ follows from Theorem 2.2.1, see the remark in 2.2. (Though, due to 6.6 and 4.1.3, in this case $\dim\mathcal{D}^- = 3$, and one can also apply 7.3.2.)

Finally, to decide whether type I in (1) and (3) is $\mathrm{I}_{\mathrm{abs}}$ or $\mathrm{I}_{\mathrm{rel}}$ it suffices to notice that, under the hypotheses, $w_2(E)$ represents a nontrivial element in $H_2^\infty(E)$ (see Lemma 6.3.3) and, hence, a half is of type $\mathrm{I}_{\mathrm{abs}}$ if and only if its fundamental class vanishes in $H_2^\infty(E)$, i.e., if it belongs to the kernel of the intersection form. Using 5.4.2 one can easily see that the spherical half realizes $w_2(E)$; hence, it is of type $\mathrm{I}_{\mathrm{rel}}$. □

8. CONSTRUCTION

8.1. General idea (see [KhD] for details). Let X be the $K3$-surface obtained as the double covering of $Y = \mathbb{C}\mathrm{p}^1\times\mathbb{C}\mathrm{p}^1$ branched over a non-singular curve $C\subset Y$ of bi-degree $(4,4)$. Denote by $s\colon Y\to Y$ the Cartesian product of the nontrivial involutions $(u:v)\mapsto(-u:v)$ of the factors. If C is s-symmetric, s lifts to two different involutions on X, which commute with the deck translation d of $X\to Y$. If, besides, C does not pass through the fixed points of s, then exactly one of these two involutions, which we denote by τ, is fixed point free (see, e.g., [H] or [BPV]), and, hence, the orbit space $E = X/\tau$ is an Enriques surface.

Suppose now that Y is equipped with a real structure conj_Y which commutes with s, and C is a real curve. Then $s\circ\operatorname{conj}_Y$ is another real structure on Y and C. We denote the real point sets of these two structures by $Y_{\mathbb{R}}^{(i)}$ and $C_{\mathbb{R}}^{(i)}$, $i = 1,2$ ($i = 1$ corresponding to conj_Y) and call them the *halves* of Y and C respectively. The involutions conj_Y and $s\circ\operatorname{conj}_Y$ lift to four different commuting real structures ($t^{(1)}$, $t^{(2)} = \tau\circ t^{(1)}$, $d\circ t^{(1)}$, and $d\circ t^{(2)}$) on X, which, in turn, descend to two real structures on E; we call them the *expositions* of E. A choice of an exposition is determined by a choice of one of the two liftings $t^{(1)}$, $t^{(2)}$ of conj_Y to X.

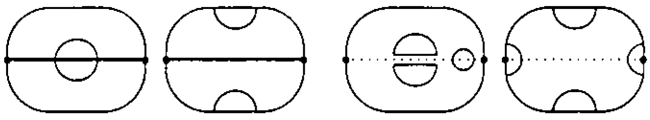
We use for Y a quadric in $\mathbb{C}\mathrm{p}^3$ real in respect to the standard complex conjugation involution and invariant in respect to the real symmetry $s\colon\mathbb{C}\mathrm{p}^3\to$

$\mathbb{C}p^3$, $(x_0\colon x_1\colon x_2\colon x_3) \mapsto (x_0\colon x_1\colon -x_2\colon -x_3)$. Since the bi-degree of C is even, $C_{\mathbb{R}}^{(i)}$ separates $Y_{\mathbb{R}}^{(i)}$ into two parts, which have $C_{\mathbb{R}}^{(i)}$ as their common boundary (at least one of the two parts is non-empty unless $Y_{\mathbb{R}}^{(i)}$ is empty). The fixed point set $X_{\mathbb{R}}^{(i)}$ of $t^{(i)}$ is the pull-back of one of the parts. Thus, a choice of $t^{(1)}$ is equivalent to a choice of one of the two parts of $Y_{\mathbb{R}}^{(1)}$, and, since $t^{(2)} = \tau o t^{(1)}$, the latter determines as well the part of $Y_{\mathbb{R}}^{(2)}$ whose pull-back is $\operatorname{Fix} t^{(2)}$. This correlation is easily controlled due to the fact that $X_{\mathbb{R}}^{(1)}$ and $X_{\mathbb{R}}^{(2)}$ are disjoint and, hence, the pull-back of a point of $Y_{\mathbb{R}}^{(1)} \cap Y_{\mathbb{R}}^{(2)}$ is contained in exactly one of the sets $X_{\mathbb{R}}^{(1)}$, $X_{\mathbb{R}}^{(2)}$. (Note that in all the examples we use here the above intersection is not empty.)

To construct the branch curve $C \in Y$ we start with a singular s-symmetric curve $\tilde{C} \in Y$, given by an equation $f = 0$, and perturb it to the curve C given by $f + \epsilon h = 0$; here f and h are homogeneous real bi-degree $(4,4)$ polynomials either both s-symmetric or both s-skew-symmetric and ϵ is a small real parameter. All the facts necessary to construct a perturbation and to control its topology can be found in [KhD, Sect. 4].

8.2. The distributions of $2V_1 \sqcup kS$. It suffices to construct the distributions $\{aS\} \sqcup \{2V_1 \sqcup bS\}$ and $\{V_1 \sqcup aS\} \sqcup \{V_1 \sqcup bS\}$ with $(a,b) = (1,3)$, $(2,2)$, or $(3,1)$; the rest is constructed in [KhD]. We start with the ellipsoid Y given by $x_0^2 = x_1^2 + x_2^2 + x_3^2$ and the singular curve $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$, where $\tilde{C}_1$ and $\tilde{C}_2$ are cut on Y by $x_3^2 = 0$ and $2(x_2^2 - x_3^2) = x_0^2$ respectively (see Fig. 2 (a), which represents the two halves of Y, which are both topological spheres, and $\tilde{C}$. The two black dots in each figure are the fixed points of the restriction of s to the corresponding half.) To perturb $\tilde{C}$ we take for h the equation of a bi-degree $(4,4)$ s-symmetric real curve which intersects the two real halves of $\tilde{C}_1$ at eight points (the *ramification points*); all these points must be outside of the ovals of $\tilde{C}_2$ and different from the fixed points of s. Then, under a proper choice of the sign of ϵ, the portions of the real part of $\tilde{C}_1$ which are either inside the ovals of $\tilde{C}_2$ or between pairs of the ramification points double, and the rest of $\tilde{C}_1$ disappears (see, e.g., Fig. 2 (b), which corresponds to the distribution $\{3S\} \sqcup \{2V_1 \sqcup S\}$; to obtain the other distributions note that one or both the ovals surrounding the fixed points can be moved to the 'left hand' half, and the pair of small ovals can be moved to the 'right hand' half). If the exposition is chosen so that $X_{\mathbb{R}}^{(2)}$ covers the interior of the two ovals surrounding the fixed points of s, then these two ovals produce the V_1 components of $E_{\mathbb{R}}$; the other pairs of symmetric ovals produce spheres.

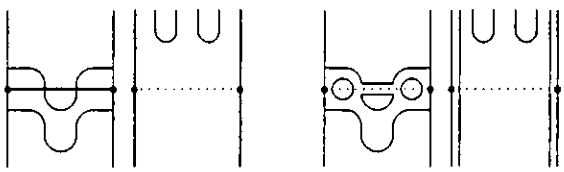
8.3. The distributions of $2V_2 \sqcup kS$. The distributions constructed here are $\{V_2 \sqcup aS\} \sqcup \{V_2 \sqcup bS\}$ for all (a,b) except $(0,0)$, $(4,0)$, $(2,2)$, and $(0,4)$. (The first exception is found in [KhD], the others, in [N2], see the remark at the end of 2.3.) Let Y be the hyperboloid $x_0^2 = x_1^2 + x_2^2 - x_3^2$, and let $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$, where



(a) (b)

FIGURE 2

$\widetilde{C}_1$ and $\widetilde{C}_2$ are given, respectively, by $x_3^2 = 0$ and $(2x_3 - x_2)^2 = \epsilon(x_0^2 + x_3^2)$ for some small real $\epsilon > 0$ (see Fig. 3 (a), which represents the projections of the affine part $x_0 \neq 0$ of $Y_{\mathbb{R}}^{(1)}$ and $Y_{\mathbb{R}}^{(2)}$ to the planes $(x_0 : x_1 : x_3)$ and $(x_0 : x_1 : ix_2)$ respectively; note that the right half of $\widetilde{C}_1$ coincides with the visible counter of $Y_{\mathbb{R}}^{(2)}$). The s-symmetric real perturbative term is chosen so that its zero set does not intersect the right half of $\widetilde{C}_1$ and intersects its left half at $4(a-1)$ points, $a = 1, 2, 3$, located close to the fixed points of s. Under a proper choice of the sign of the perturbation, the right half of $\widetilde{C}_1$ doubles and the ramification points generate $2(a-1)$ ovals which do not contain the fixed points of s (Fig. 3 (b)). The exposition is chosen so that the two strips containing the fixed points of s in the right half $Y_{\mathbb{R}}^{(2)}$ of Y are covered by $X_{\mathbb{R}}^{(2)}$; these two strips produce the components V_2 of $E_{\mathbb{R}}$. Thus we obtain the distributions $\{V_2 \sqcup aS\} \sqcup \{V_2 \sqcup bS\}$ with $a = 1, 2, 3$ and $b = 1$. To construct surfaces with $b = 0$, we replace $\widetilde{C}_2$ with the curve given by $(2x_3 - x_2)^2 = \epsilon(x_0^2 - x_3^2)$ for some small real $\epsilon > 0$; this makes the two ovals at the top of the front side (and the bottom of the back side) of Fig. 3 (a) and (b) disappear.



(a) (b)

FIGURE 3

8.4. The distributions of $V_3 \sqcup V_1 \sqcup kS$. It suffices to construct the distributions $\{V_3 \sqcup V_1 \sqcup aS\} \sqcup \{bS\}$ and $\{V_3 \sqcup aS\} \sqcup \{V_1 \sqcup bS\}$ with $2 \leqslant a + b \leqslant 4$ and $a \geqslant 1$; the rest is found in [KhD]. (In fact, here we also cover the case $1 \leqslant a + b \leqslant 3$.) Let us start with a quartic $Q \subset \mathbb{R}\mathrm{p}^2$ with $(k+1)$ real components, $1 \leqslant k \leqslant 3$, obtained by perturbing the union of two conics (see Figure 4; in order to reduce the number of real components one should change the perturbation so that two or three upper ovals form a single oval). Pick an oval O (the lowest one in Figure 4) and denote by L the double tangent to O and by L_a, $0 \leqslant a \leqslant k$, another tangent, which together with L separates in the real projective plane O from a other ovals of Q.

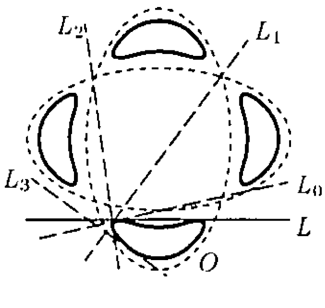


FIGURE 4

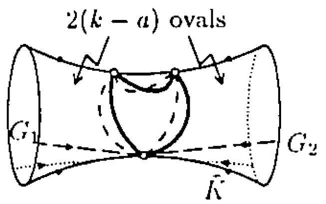


FIGURE 5

We make use of the following technical result, whose proof we postpone to the end of this section.

8.4.1. Lemma. *The union $L \cup L_a$ can be perturbed to an irreducible conic K which is still tangent to O at three points, has no other real intersection points with Q, and such that O is in the outer part of the oval of K.*

Let K be the conic given by the lemma. Consider the double cover $\bar Y$ of the projective plane branched over K. Denote by $\bar s$ the deck translation involution, by $\bar K$ its fixed point set (which projects to K), and by $\bar Q$ the pull-back of Q. Due to $\bar s$ (cf. 8.1), each of $\bar Y$, $\bar Q$ and $\bar K$ has two real halves. $\bar Y^{(1)}_{\mathbb R}$ is the hyperboloid shown in Figure 5: $\bar Q^{(1)}_{\mathbb R}$ has a component (the pull-back of O) with three nondegenerate double points in $\bar K^{(1)}_{\mathbb R}$ and $(k-a)$ pairs of symmetric ovals. The other half $\bar Y^{(2)}_{\mathbb R}$ is an ellipsoid in which $\bar Q^{(2)}_{\mathbb R}$ has a pairs of ovals disjoint from $\bar K^{(2)}_{\mathbb R}$. Now (Y, s) is obtained from $(\bar Y, \bar s)$ by the following real $\bar s$-symmetric birational transformation: we blow up the three singular points of $\bar Q$ and then blow down the proper transforms of $\bar K$ and the two generatrices G_1, G_2 of $\bar Y$ through one of the singular points (more precisely, through the singular point whose image in $\mathbb{R}p^2$ is close to the tangency point of L_a and O). Let $\tilde C$ be the transform of $\bar Q$. It is easy to see that $\tilde C^{(1)}_{\mathbb R}$ consists of a large oval $\tilde O$ (the transform of the singular component) surrounding $(k-a)$ pairs of symmetric ovals and three isolated double points: one (the image of $\bar K$) is fixed under s, and the two others (the images of G_1, G_2) are symmetric. The other half $\tilde C^{(2)}_{\mathbb R}$ consists of a pairs of ovals and an isolated double point (the image of $\bar K$). All the ovals but $\tilde O$ are not nested and do not surround the singular points of $\tilde C$. Finally, we perturb $\tilde C$ to a nonsingular symmetric curve C (see 4.3.1 in [KhD]); the fixed double point, which produces the V_1 component of the resulting Enriques surface, can be made to pop up in either side, and the two symmetric double points may either form a pair of symmetric ovals or disappear. Thus, the distributions obtained are $\{V_3 \sqcup V_1 \sqcup (k-a+\varepsilon)S\} \sqcup \{aS\}$ and $\{V_3 \sqcup (k-a+\varepsilon)S\} \sqcup \{V_1 \sqcup aS\}$, where $\varepsilon = 0, 1$.

Proof of Lemma 8.4.1. Given an imaginary point $u \in Q$, define an involution ρ_u of a Zariski open subset of the symmetric power S^3Q in the following way: for a generic triple $(x_1, x_2, x_3) \in S^3Q$ there is a unique conic through $u, \bar{u}, x_1, x_2, x_3$; it intersects Q at three more points y_1, y_2, y_3, and we let $\rho_u(x_1, x_2, x_3) = (y_1, y_2, y_3)$. Clearly, the above conic is tangent to Q at x_1, x_2, x_3 if and only if (x_1, x_2, x_3) is a fixed point of ρ_u.

Denote by a_1, a_2, a_3 the three tangency points of $L \cup L_a$ and Q, and by v one of the two imaginary intersection points of L_a and Q. Then the graph Γ_v of ρ_v intersects the diagonal $\Delta \subset S^3Q \times S^3Q$ at $a = (a_1, a_2, a_3) \times (a_1, a_2, a_3)$ transversally. (Note that S^3Q is smooth at this point.) Indeed, let p_1, p_2 be the two projections $S^3Q \times S^3Q \to S^3Q$, and let e_i be some real generators of the tangent spaces $T_{a_i}Q$, which we regard as basis vectors of $T_{(a_1,a_2,a_3)}S^3Q$. Then $T_a\Delta$ is spanned by $p_1^*e_i + p_2^*e_i$, $i = 1, 2, 3$, and $T_a\Gamma_v$ is spanned by $p_1^*e_i + \alpha_i p_2^*e_i$, $i = 1, 2, 3$, with some real $\alpha_i < 0$. (To see that, one can move one point at a time; then the conic is still reducible, and it is easy to estimate the tangent vectors.) Thus, for any other point v' close to v the graph of $\rho_{v'}$ also has a unique (and hence real) intersection point with Δ close to a, i.e., there is a real conic K through v' tangent to Q at three real points close to a_1, a_2, a_3. If the line $(v'\bar{v}')$ is not tangent to Q, this conic is irreducible. Finally, to control the topology (actually, to choose one of the two possible real directions of the perturbation), just note that K has no real intersection points with $(v'\bar{v}')$; hence, this line lies outside of the oval of K, and if v' is chosen so that $(v'\bar{v}')$ intersects O at two real points, then O is also outside. □

Remark. The involution utilized in the proof is similar to that from [GH, Sect. 7], where it is used for a similar purpose. It also seems possible to apply Shustin's approach [Sh].

Appendix A.
Viro homomorphisms and Smith exact sequence

A.1. Viro homomorphisms and differentials. Recall that the *Smith exact sequence* of a $\mathbb{Z}/2$-space (Y, c) is the exact sequence

$$\to H_{p+1}(Y', \operatorname{Fix} c) \xrightarrow{\Delta} H_p(\operatorname{Fix} c) \oplus H_p(Y', \operatorname{Fix} c) \to$$
$$\xrightarrow{\text{in}_* + \text{tr}} H_p(Y) \xrightarrow{\text{rel} \bullet \text{pr}_*} H_p(Y', \operatorname{Fix} c) \to,$$

where $Y' = Y/c$ is the orbit space, in: $\operatorname{Fix} c \to Y$ is the inclusion, pr: $Y \to Y'$ is the projection, tr: $H_p(Y', \operatorname{Fix} c) \to H_p(Y)$ is the transfer map, and rel: $H_p(Y') \to H_p(Y', \operatorname{Fix} c)$ is the relativization map. The connecting homomorphism Δ is defined as follows. Given a relative cycle y' in $(Y', \operatorname{Fix} c)$, lift it to a chain y in Y. Then $\partial y = \partial y' + \operatorname{tr} z$ for some cycle z in $(Y', \operatorname{Fix} c)$, and we let $\Delta y' = \partial y' \oplus z$.

Viro homomorphisms and the differentials of Kalinin's spectral sequence are incorporated in the Smith exact sequence. To extract them, one should regard both d_p^r and bv_p as additive relations (i.e., partial many-valued homomorphisms) $H_p(Y) \to H_{p+r-1}(Y)$ and $H_*(\operatorname{Fix} c) \to H_p(Y)$, and consider the relation $\Delta^{-1}\colon H_p(\operatorname{Fix} c) \oplus H_p(Y', \operatorname{Fix} c) \to H_{p+1}(Y', \operatorname{Fix} c)$ inverse to Δ (see, e.g., [McL] for the notion of additive relation and its properties).

A.1.1. Proposition. *The differentials of Kalinin's spectral sequence and Viro homomorphisms, regarded as additive relations $d_p^r\colon H_p(Y) \to H_{p+r-1}(Y)$ and $\operatorname{bv}_p\colon H_*(\operatorname{Fix} c) \to H_p(Y)$ respectively, are given by*

$$d_p^r = \operatorname{tr}_{p+r-1} \circ (\Delta^{-1})^{r-1} \circ \operatorname{pr}_p \quad \textit{and}$$

$$\operatorname{bv}_p\Bigl(\sum\nolimits_{i\leqslant p} x_i\Bigr) = \operatorname{in}_p x_p + \operatorname{tr}_p y'_p,$$

where, in the latter equation, $x_i \in H_i(\operatorname{Fix} c)$ and $y'_p \in H_p(Y', \operatorname{Fix} c)$ is defined recursively via $y'_0 = 0$, $y'_{i+1} = \Delta^{-1}(x_i \oplus y'_i) \in H_{i+1}(Y', \operatorname{Fix} c)$.

Proof. To prove the first assertion, pick a cycle x_p in Y and consider some cycles y'_i in $(Y', \operatorname{Fix} c)$, $p \leqslant i \leqslant p+r-1$, representing the iterated pull-backs $(\Delta^{-1})^{i-p}(0 \oplus \operatorname{pr} x_p)$. By the definition of Δ, for $i > p$ one gets $\partial y'_i = 0$ and there are some chains y_i in Y such that $\operatorname{pr} y_i = y'_i$ and $\partial y_i = \operatorname{tr} y_{i-1}$ for $i > p$. Now, replacing each y'_i with y_i and using the fact that $\operatorname{tr} \circ \operatorname{pr} = 1 + c_*$, one obtains the definition of d_p^r given in 5.1.1.

The second assertion is proved similarly: we start with a sequence of cycles x_i in $\operatorname{Fix} c$, $0 \leqslant i \leqslant p$, choose relative cycles y'_i in $(Y', \operatorname{Fix} c)$ so that $\Delta y'_{i+1} = x_i \oplus y'_i$, and lift them to chains y_i in Y; this gives the definition of bv_p given in 5.2.1. □

A.2. Whitehead (semi-exact) triples. According to Proposition A.2.1 the relations $\operatorname{tr} \circ \Delta^{-r+1} \circ \operatorname{pr}$ form a sequence of differential relations in the sense of Puppe, see [P], which generates Kalinin's spectral sequence. Thus the latter can be derived from the Smith exact sequence. Below, in A.3, we describe this derived spectral sequence in a direct way. For this purpose, we need a slight modification of the machinery of exact couples (see [Ma] or [McL]). Some elements of this modification are contained in [Wh].

A.2.1. Definition. A *Whitehead (semi-exact) triple* $C = (H, D, \tilde D; \alpha, \beta, \gamma)$ is a triangle

$$C\colon \qquad \begin{array}{ccc} & H & \\ {}^{\beta}\nearrow & & \searrow^{\gamma} \\ \tilde D & \xleftarrow{\ \alpha\ } & D \end{array}$$

of abelian groups and homomorphisms which is exact at D and $\bar{D}$ and such that (1) D is a subgroup of $\bar{D}$, and (2) $\gamma \circ \beta = 0$.

The *homology group* of $\mathcal{C}$ is $\hat{H}(\mathcal{C}) = \operatorname{Ker} \gamma / \operatorname{Im} \beta$.

Given such a $\mathcal{C}$, one can define the *derived triple* $\mathcal{C}' = (H', D', \bar{D}'; \alpha', \beta', \gamma')$ as follows. Consider $d = \beta \circ \gamma \colon H \to H$ and let $H' = \operatorname{Ker} d / \operatorname{Im} d$, $\bar{D}' = \operatorname{Im} \alpha$, and $D' = \bar{D}' \cap D$. The new maps α' and γ' are induced by α and γ respectively, and β' is given by $\beta' = \beta \circ \alpha^{-1}$ (in the sense that $\beta' d' = \beta d$, where $d' = \alpha d$).

A.2.2. Proposition. *The derived triple of a Whitehead triple is well defined and is a Whitehead triple. There is an exact sequence*

$$0 \longrightarrow \bar{D}'/D' \longrightarrow \bar{D}/D \longrightarrow \hat{H}(\mathcal{C}') \longrightarrow \hat{H}(\mathcal{C}) \longrightarrow 0,$$

where the middle homomorphism $\bar{D}/D \to \hat{H}(\mathcal{C}')$ is induced by β, and the others are the natural projections.

Proof. Both the assertions can be proved by diagram chasing in the initial Whitehead triangle. (The exact sequence in question is, in fact, the Ker – Coker sequence for $D \hookrightarrow \bar{D} \xrightarrow{\beta} \operatorname{Ker} \gamma$.) □

Due to A.2.2, starting with a Whitehead triple $\mathcal{C} = \mathcal{C}^1$, one can define the sequence $\mathcal{C}^p = (\mathcal{C}^{p-1})'$ of derived triples and, as usual, its limit

$$\mathcal{C}^\infty: \qquad \begin{array}{ccc} & H^\infty & \\ {\scriptstyle \beta^\infty = 0} \nearrow & & \searrow {\scriptstyle \gamma^\infty} \\ \bar{D}^\infty = \bigcap_{p \geqslant 1} \operatorname{Im} \alpha^p & \xleftarrow{\quad \alpha^\infty \quad} & D^\infty = D \cap \bar{D}^\infty, \end{array}$$

which is still a Whitehead triple with $\beta^\infty = 0$ and α^∞ onto. The terms H^r of $\mathcal{C}^r$ and differentials $d^r \colon H^r \to H^r$ defined above form a spectral sequence, which converges to H^∞. From the second part of A.2.2 it follows that there are two filtrations

$$\begin{aligned} \hat{H}(\mathcal{C}^\infty) \supset \hat{H}^0 \supset \hat{H}^1 \supset \cdots \quad &, \text{ where } \hat{H}^p = \operatorname{Ker}[\hat{H}(\mathcal{C}^\infty) \to \hat{H}(\mathcal{C}^{p+1})], \text{ and} \\ \bar{D}/D = \hat{\mathcal{F}}^0 \supset \hat{\mathcal{F}}^1 \supset \cdots \quad &, \text{ where } \hat{\mathcal{F}}^p = \operatorname{Im}[\bar{D}^{p+1}/D^{p+1} \to \bar{D}/D], \end{aligned}$$

and the isomorphisms

$$\hat{\mathcal{F}}^p / \hat{\mathcal{F}}^{p+1} \xrightarrow{\cong} \hat{H}^p / \hat{H}^{p+1}$$

induced by β^{p+1} establish an isomorphism of the associated graded groups

$$\operatorname{Gr}_{\hat{\mathcal{F}}}[(\bar{D}/D)/(\bar{D}^\infty/D^\infty)] \xrightarrow{\cong} \operatorname{Gr}_{\hat{H}} \operatorname{Ker}[\hat{H}(\mathcal{C}^\infty) \to \hat{H}(\mathcal{C})].$$

Remark. Note that, in general, $\hat{H}^0 \neq \hat{H}(\mathcal{C}^\infty)$ and $\bigcap_{p\geqslant 0} \hat{H}^p \neq 0$.

A.2.3. Proposition. *Let $\mathcal{C}$ be an exact triple (i.e., $\hat{H}(\mathcal{C}) = 0$) with nilpotent α (i.e., there is a positive integer n such that $\alpha^n y = 0$ on all the elements $y \in D$ on which it is well defined). Then there are finite filtrations $0 = \hat{\mathcal{F}}^{n+1} \subset \cdots \subset \hat{\mathcal{F}}^1 \subset \hat{\mathcal{F}}^0 = \bar{D}/D$ and $0 = \hat{H}^{n+1} \subset \cdots \subset \hat{H}^1 \subset \hat{H}^0 = H^\infty$ and an isomorphism $\operatorname{Gr}_{\hat{\mathcal{F}}}(\bar{D}/D) \cong \operatorname{Gr}_{\hat{H}} H^\infty$ of the associated graded groups.*

Proof. By the hypothesis, α^n is identically zero on D^n. Thus $\bar{D}^{n+1} = D^{n+1} = 0$ and $\gamma^{n+1} = 0$; hence, $\mathcal{C}^\infty = \mathcal{C}^{n+1}$ and $\hat{H}(\mathcal{C}^\infty) = H^\infty = H^{n+1} = \hat{H}(\mathcal{C}^{n+1})$, and, in particular, $\hat{H}^{n+1} = 0$. Besides, $\hat{H}(\mathcal{C}) = 0$, and thus $\hat{H}^0 = H^\infty$. □

A.3. The spectral sequence derived from the Smith exact sequence. Given a $\mathbb{Z}/2$-space (Y, c), its Smith exact sequence obviously forms an exact triple

$$\begin{array}{ccc} & H_*(Y) & \\ {\scriptstyle \text{in}_* + \text{tr}} \nearrow & & \searrow {\scriptstyle \text{rel} \circ \text{pr}_*} \\ H_*(\operatorname{Fix} c) \oplus H_*(Y'; \operatorname{Fix} c) & \xleftarrow{\quad \Delta \quad} & H_*(Y'; \operatorname{Fix} c). \end{array}$$

If both Y' and $\operatorname{Fix} c$ have homotopy type of finite dimensional cell complexes, their homology are graded groups with degree bounded from both above and below, and Δ, being a map of degree -1, is nilpotent. Hence, according to Proposition A.2.3, one obtains a spectral sequence $H^1_* = H_*(Y) \Rightarrow H_*(\operatorname{Fix} c)$. Instead of the nonhomogeneous filtration $\mathcal{F}^p$ and Viro homomorphisms $\operatorname{bv}_p \colon \mathcal{F}^p \to H^\sim_p$ Proposition A.2.3 provides the homogeneous filtrations $\hat{\mathcal{F}}^p_q$ on $H_q(\operatorname{Fix} c)$ and $\hat{H}^p_q$ on $H^\sim_q$ and isomorphisms $\beta^p_q \colon \hat{\mathcal{F}}^p_q/\hat{\mathcal{F}}^{p+1}_q \to \hat{H}^p_{q+p}/\hat{H}^{p+1}_{q+p}$.

Let $\mathcal{F}^p_q$ be the image of $\mathcal{F}^{q+p} \cap H_{\leqslant q}(\operatorname{Fix} c)$ in $\mathcal{F}^{q+p}/\mathcal{F}^{q+p+1}$. For p fixed the groups $\mathcal{F}^i_{p-i}$ form a filtration on $\mathcal{F}^p/\mathcal{F}^{p+1}$, and simple comparing Proposition A.1.1 and definitions in A.2 gives the following result:

A.3.1. Proposition. *The above spectral sequence coincides with Kalinin's spectral sequence $H^p_*(Y)$, $p \geqslant 1$, and there are isomorphisms $\mathcal{F}^p_q/\mathcal{F}^{p+1}_{q-1} \cong \hat{\mathcal{F}}^p_q/\hat{\mathcal{F}}^{p+1}_q$ which make the following diagram commute:*

$$\begin{array}{ccc} \mathcal{F}^p_q/\mathcal{F}^{p+1}_{q-1} & \xrightarrow{\operatorname{bv}_{q+p}} & \hat{H}^p_{q+p}/\hat{H}^{p+1}_{q+p} \\ \cong \downarrow & & \parallel \\ \hat{\mathcal{F}}^p_q/\hat{\mathcal{F}}^{p+1}_q & \xrightarrow{\beta^p_q} & \hat{H}^p_{q+p}/\hat{H}^{p+1}_{q+p} \end{array}$$

Remark. In particular, Proposition A.3.1 gives an alternative proof of the fact that Viro homomorphisms induce isomorphisms of the associated graded groups.

Remark. Note that, like $H^p_*(Y)$ and d^p_*, the Viro homomorphisms and filtration $\mathcal{F}^p$ can be derived algebraically from the Smith exact sequence (cf. Proposition A.1.1). To do that one can use the following properties of the original exact triple: E, $\bar{D}$, and D are graded groups with bounded degree; β and γ are homomorphisms of degree 0; α is a homomorphism of degree -1; and $\bar{D}$ is represented as a direct sum $\bar{D} = D \oplus (\bar{D}/D)$.

Appendix B. Kalinin's intersection form

B.1. The local case. Kalinin's spectral sequence and, in particular, Viro homomorphisms admit an obvious relative version. We make use of such a version to do some calculations in a neighborhood of the fixed point set. Then, in the next subsection, we apply the result obtained to establish (in the global case) a relation between Kalinin's intersection form $(a, b) \mapsto (\mathrm{bv}_* a \circ \mathrm{bv}_* b)$, $a, b \in H_*(F)$ and the Poincarè intersection form $(x, y) \mapsto x \circ y$, $x, y \in H_*(F)$, see Theorem B.2.1..

B.1.1. Lemma. *Let ν be an m-dimensional vector bundle over a finite cell complex F, and let T and ∂T be the associated disk and sphere bundles, respectively, supplied with the antipodal involution. Then the homology filtration $\mathcal{F}^*$ associated with Kalinin's spectral sequence of $(T, \partial T)$ is given by $\mathcal{F}^{m+p} = w(\nu)^{-1} \cap H_{\geqslant p}(F)$, where $w(\nu) = 1 + w_1(\nu) + w_2(\nu) + \dots$ is the total Stiefel-Whitney class of ν.*

Proof. Given a topological space Y with involution $c\colon Y \to Y$ and an integer k, $0 \leqslant k \leqslant \infty$, denote by Y_k the twisted product

$$Y_k = Y \times S^k/\{(y, s) \sim (cy, gs)\}, \tag{B.1.2}$$

where $g\colon S^k \to S^k$ is the antipodal involution on the standard sphere S^k. It is clear (see, e.g., [D]) that T_k and $(\partial T)_k$ are, respectively, the disk and the sphere bundles associated with $\nu \otimes \eta$ over $F_k = F \times \mathbb{R}\mathrm{p}^k$, where η is the tautological linear bundle over $\mathbb{R}\mathrm{p}^k$. Let $h_i \in H_i(\mathbb{R}\mathrm{p}^k)$ be the generators. (We let $h_i = 0$ for $i < 0$ or $i > k$.) In [D] it is shown that a sufficient condition for a class $\sum x_i$, $x_i \in H_i(F)$, to belong to $\mathcal{F}_q$ is that the image of $\sum x_i \otimes h_{q-1-i}$ in $H_{q-1}(T_q, \partial T_q)$ under the inclusion map $H_*(F_q) \to H_*(T_q, \partial T_q)$ should vanish. (In [D] only the absolute case is considered, but the proof works in the relative case without any change.) The inclusion map $H_*(F_q) \to H_*(T_q, \partial T_q)$ is equal to the composition of the multiplication by $w_m(\nu \otimes \eta) = \sum w_i(\nu) \otimes h^{m-i}$ and Thom isomorphism, and spelling out the product $w_m(\nu \otimes \eta) \cap \sum x_i \otimes h_{q-1-i}$ and taking into account the coefficients of those of h_j which are not identically zero in $H_*(\mathbb{R}\mathrm{p}^q)$ shows that the above sufficient condition is equivalent to $w(\nu) \cap \sum x_i \in H_{\geqslant q-m}(F)$, i.e., $\sum x_i \in w(\nu)^{-1} \cap H_{\geqslant q-m}(F)$. *A priori*, the subgroup obtained is only a portion of $\mathcal{F}^q$, but comparing the dimensions shows that, in fact, these two subgroups coincide. ◻

B.1.3. Corollary. *Let F, ν, T, and ∂T be as in Lemma B.1.1, and let $\mathrm{th}\colon H_{q+m}(T,\partial T) \to H_q(F)$ be the Thom isomorphism. Then for any class $a \in H_q(F)$ one has $\mathrm{bv}_{q+m}(w^{-1}(\nu)\cap a) = \mathrm{th}^{-1} a$.*

Proof. The result has actually been proved for the case when F is a q-dimensional polyhedron with $H_q(F) = \mathbb{Z}/2$, and a is the generator of the latter group: in this case $w^{-1}(\nu)\cap a$ is the only nontrivial element in $\mathcal{F}^{q+m}$, $\mathrm{th}^{-1} a$ is the only nontrivial element in $H_{q+m}(T,\partial T)$, and $\mathrm{bv}_{q+m}\colon \mathcal{F}^{q+m} \to H_{q+m}(T,\partial T)$ is an isomorphism. In general, one can find a singular q-dimensional polyhedron $f\colon P \to F$ with $H_q(P)$ generated by a single element $[P]$ so that $a = f_*[P]$. The result follows then from the naturality of bv_* and th. ☐

B.2. The global case.

B.2.1. Theorem. *Let Y be a smooth closed N-dimensional manifold with a smooth involution $c\colon Y \to Y$, and let $F = \operatorname{Fix} c$ be the fixed point set of c. Then for any two classes $a \in \mathcal{F}^p$ and $b \in \mathcal{F}^q$ one has*

$$w(\nu)\cap(a\circ b) \in \mathcal{F}^{p+q-N}$$

and

$$\mathrm{bv}_p\, a \circ \mathrm{bv}_q\, b = \mathrm{bv}_{p+q-N}[w(\nu)\cap(a\circ b)],$$

where $w(\nu)$ is the total Stiefel-Whitney class of the normal bundle ν of F in Y.

First, let us prove the following lemma:

B.2.2. Lemma. *Let Y, c, and F be as above. Denote by $D_Y\colon H^*(Y) \to H_*(Y)$ and $D_F\colon H^*(F) \to H_*(F)$ the Poincaré duality maps in Y and F respectively, and by $D_c\colon H^*(F) \to H_*(F)$ the map $\alpha \mapsto \alpha\cap(w^{-1}(\nu)\cap[F])$. Then:*

(1) *D_c induces isomorphisms $\mathcal{F}_{N-p} \to \mathcal{F}^p$;*
(2) *given $x \in \mathcal{F}_p$, one has $\mathrm{bv}^{N-p}(D_Y^{-1}\,\mathrm{bv}_p\, x) \equiv D_c^{-1} x \mod \mathcal{F}_{N-p-1}$.*

Proof. From the naturality of Kalinin's spectral sequence and Corollary B.1.3 it follows that the only nontrivial element of $\mathcal{F}^N$ is $w^{-1}(\nu)\cap[F]$ and, hence, $[Y] = \mathrm{bv}_N(w^{-1}(\nu)\cap[F])$. Thus, D_c is the multiplication by the generator of $\mathcal{F}^N$; hence, it maps $\mathcal{F}_{N-p}$ to $\mathcal{F}^p$. Furthermore, D_c is an isomorphism (as composition of Poincaré duality and multiplication by an invertible element), and comparing the dimensions shows that so is its restriction to $\mathcal{F}_{N-p} \to \mathcal{F}^p$. (Recall that $\dim \mathcal{F}_{N-p} = \dim \mathcal{F}^p$ due to 5.3.1 and 5.3.2.)

From the above it follows that $D_c\,\mathrm{bv}^{N-q}(D_Y^{-1}\,\mathrm{bv}_q\, x) \in \mathcal{F}_p$, and one has (see 5.3.1 (4)):

$$\mathrm{bv}_p(D_c\,\mathrm{bv}^{N-p}(D_Y^{-1}\,\mathrm{bv}_p\, x)) = D_Y^{-1}\,\mathrm{bv}_p\, x \cap [Y] = \mathrm{bv}_p\, x;$$

since $\operatorname{Ker}\mathrm{bv}_p = \mathcal{F}^{p+1}$, this gives $D_c\,\mathrm{bv}^{N-p}(D_Y^{-1}\,\mathrm{bv}_p\,x) \equiv x \mod \mathcal{F}^{p+1}$. □

Proof of Theorem B.2.1. By the definition, $w(\nu) \cap (a \circ b) = D_c^{-1} a \cap b \in \mathcal{F}_{N-p} \cap \mathcal{F}^q \subset \mathcal{F}^{p+q-N}$, and a direct calculation using Lemma B.2.2 (2) shows that $\mathrm{bv}_{p+q-N}(D_c^{-1} a \cap b) = D_Y^{-1}\,\mathrm{bv}_p\, a \cap \mathrm{bv}_q\, b = \mathrm{bv}_p\, a \circ \mathrm{bv}_q\, b$ □

We would like to also mention the following immediate consequence of B.1.1 and B.1.3:

B.2.3. Proposition. *Let Y, c, F, and ν be as in Theorem* B.2. *Pick a component $F_i \subset F$ of dimension $(N-m)$, and denote by $\mathrm{in}_i\colon F_i \to Y$ the inclusion. Then $\mathcal{F}^q \cap H_*(F_i) \subset w^{-1}(\nu) \cap H_{\geqslant q-m}(F_i)$, and for any class $a \in \mathcal{F}^q$ one has $\mathrm{in}_i^!\,\mathrm{bv}_q\, a = [w(\nu) \cap a]_{q-m}\big|_{F_i}$, where $\mathrm{in}^!$ is the inverse Hopf homomorphism and $[\,\cdot\,]_{q-m}$ stands for the $(q-m)$-dimensional component of a nonhomogeneous homology class.*

Proof. The first statement follows from the naturality of the filtration and Lemma B.1.1 applied to $\nu|_{F_i}$. To prove the second one just note that $\mathrm{in}_i^!$ is the composition of the relativization homomorphism $H_q(Y) \to H_q(T_i, \partial T_i)$ and Thom isomorphism $H_q(T_i, \partial T_i) \to H_{q-m}(F_i)$, and apply Corollary B.2.2. □

Appendix C.
Viro homomorphisms and Steenrod operations

Let Y be a good topological space (see the first paragraph of Section 5) with an involution c.

Recall the original construction of Kalinin's spectral sequence (see [Ka] or [D] for details). Consider the fibration $Y_\sim \to \mathbb{R}p^\infty = S^\infty/a$ (see B.1.2 for the definition of $Y_\sim$) and its Borel-Serre spectral sequence $H^1_{*\,*} = H_*(Y) \otimes H_*(\mathbb{R}p^\infty) \Rightarrow H_*(Y_\sim)$. For q big enough, the cap-product by the generator $h \in H^1(\mathbb{R}p^\infty)$ defines isomorphisms $H^r_{p,q+1} \to H^r_{pq}$ and $H_{q+1}(Y_\infty) \to H_q(Y_\sim)$, see [Bo]. Furthermore, the composition of the Kunneth formula and the homomorphism induced by the inclusion $(\operatorname{Fix} c)_\sim = \operatorname{Fix} c \times \mathbb{R}p^\infty \hookrightarrow Y_\sim$ defines an isomorphism $H_q(Y_\infty) = H_*(\operatorname{Fix} c)$, *loc. cit.* By definition, Kalinin's spectral sequence is the stabilization $H^r_p(Y) = \varprojlim H^r_{pq} \Rightarrow H_*(\operatorname{Fix} c) = \varprojlim H_q(Y_\infty)$.

Since in this paper we are mainly dealing with homology groups, let us consider homology Steenrod operations $\mathrm{Sq}_t\colon H_p(Y) \to H_{p-t}(Y)$, which, at least when Y has finite homology in each dimension, are just dual to the cohomology Steenrod squares $\mathrm{Sq}^t\colon H^{p-t}(Y) \to H^p(Y)$ (see [SE] for details).

As it is known (see, for example, [McC, Theorem 6.10]), the Borel-Serre spectral sequence $H_*(Y) \otimes H_*(\mathbb{R}p^\infty) \Rightarrow H_*(Y_\sim)$ respects Steenrod operations:

C.1. Lemma. *In the Borel-Serre spectral sequence $H_*(Y) \otimes H_*(\mathbb{R}p^\infty) \Rightarrow H_*(Y_\sim)$, there are some natural homomorphisms (homology Steenrod operations) $\mathrm{Sq}_t\colon H^r_{*,*} \to H^r_{*-t,*}$, $t \geqslant 0$, so that*

(1) Sq_t *commute with the differentials, i.e.,* $\mathrm{Sq}_t \circ d^r = d^r \circ \mathrm{Sq}_t$;

(2) *for each $r \geqslant 1$ the operations on H^{r+1}_{**} coincide with those induced by Sq_t on H^r_{**} via (1);*
(3) *the operations on $H^1_{**} = H_*(Y) \otimes H_*(\mathbb{R}\mathrm{p}^\infty)$ are defined by the Steenrod operations on Y via $\mathrm{Sq}_t(x \otimes h_s) = \mathrm{Sq}_t\, x \otimes h_s$;*
(4) *Sq_t converge to the Steenrod operations in Y_∞.*

C.2. Definition. Given $x \in H_p(Y)$ and $t \geqslant 0$, define the *weighed Steenrod operation* $\widehat{\mathrm{Sq}}_t x = \sum_{0 \leqslant j \leqslant t} \binom{P-p}{t-j} \mathrm{Sq}_j\, x$, where $P > p + t$ is a power of 2. (The binomial coefficients do not depend on P, see, e.g., Lemma I.2.6 in [SE].)

C.3. Theorem. *The Steenrod operations naturally descend to $H^\infty_*(Y)$ so that for any $x \in \mathcal{F}^p$ and $t \geqslant 0$ one has $\mathrm{Sq}_t \,\mathrm{bv}_p\, x = \mathrm{bv}_{p-t}\, \widehat{\mathrm{Sq}}_t x$.*

Proof. Pick some $P > t + \dim Y$ which is a power of 2. Since H^r_{pq} does not depend on $q \gg 0$, one can replace the stabilization homomorphisms $\cap h\colon H^r_{p,q+1} \to H^r_{pq}$ with $\cap h^P\colon H^r_{p,q+P} \to H^r_{pq}$, which commute with Sq_t and, hence, induce some operation on $H^r_p(Y)$. (Indeed, under the assumption on P one has $\mathrm{Sq}^j h^P = 0$ for $1 \leqslant j \leqslant t$, and Cartan formula applies.) The induced operation depends on the initial row; if one starts with $\mathrm{Sq}_t\colon H^r_{pP} \to H^r_{p-t,P}$, it obviously coincides with that induced by $\mathrm{Sq}_t\colon H_p(Y) \to H_{p-t}(Y)$.

Since Sq_t is natural, it only remains to evaluate it on, say, $H_P(\mathrm{Fix}\, c)_\sim = H_P(\mathrm{Fix}\, c \times \mathbb{R}\mathrm{p}^\infty)$. For an element $x \otimes h_{P-p}$, $x \in H_p(\mathrm{Fix}\, c)$, the Cartan formula gives

$$\mathrm{Sq}_t\, x = \sum_{0 \leqslant j \leqslant t} \mathrm{Sq}_j\, x \otimes \mathrm{Sq}_{t-j}\, h_{P-p} = \sum_{0 \leqslant j \leqslant t} \binom{P-p}{t-j} \mathrm{Sq}_j\, x \otimes h_{P-p-t+j},$$

and, after dropping the h factors, one obtains $\widehat{\mathrm{Sq}}_t x$. □

Applying the last result to the Bockstein homomorphism Sq_1 gives:

C.4. Corollary. *For any class $x = \sum_{i \leqslant p} x_i \in \mathcal{F}^p$ one has*

$$\mathrm{Sq}_1 \,\mathrm{bv}_p \left(\textstyle\sum_{i \leqslant p} x_i\right) = \mathrm{bv}_{p-1} \left(\textstyle\sum_{i \leqslant p} \mathrm{Sq}_1\, x_i + \sum_{j > 0} x_{p-2j}\right).$$

C.5. Corollary. *If $p = 2$, then (see the notation in 5.4) $\mathrm{Sq}_1 \,\mathrm{bv}_2[C_i] = \mathrm{bv}_1\, w_1(C_i)$ and $\mathrm{Sq}_1 \,\mathrm{bv}_2(o + \langle C_i - C_j\rangle) = \mathrm{bv}_1\, o$.*

Remark. In general, $H^\sim_{p-1}(Y)$ is a subquotient group of $H_{p-1}(Y)$. If it is a subgroup, Corollary C.4 allows to find all the classes in $H^\sim_p(Y)$ which have an integral representative. For example, this applies when $p = 2$, Y is connected, and c_* acts trivially on $H_1(Y)$.

Appendix D.
'Generalized Enriques surfaces' with $w_2(E) = 0$

In this section we assume that E satisfies all the axioms of generalized Enriques surfaces (see 1.2) except the requirement $w_2(E) \neq 0$, i.e., E is the orbit space of a generalized $K3$-surface X by a fixed point free holomorphic involution $\tau: X \to X$, and $w_2(E) = 0$. As in the case $w_2 \neq 0$, the components of $E_{\mathbb{R}}$ may be of one of the types S_g, V_g, or T_g (see 1.3). Note that $E_{\mathbb{R}}$ has no nonorientable components of odd genus (i.e., V_{2g+1}), as the fundamental class of such a component would have square 1.

Obviously, all the results of Sections 4 and 6, with '$w_2(E)$' replaced with 'generator of $\operatorname{Tors}_2 H_2(E;\mathbb{Z})$', are still valid for this class of surfaces.

D.1. Surfaces with nonorientable real part. Below, for brevity, we say that $E_{\mathbb{R}}$ or $E_{\mathbb{R}}^{(i)}$ is of type I if its fundamental class belongs to the image of $\operatorname{Tors}_2 H_2(E;\mathbb{Z})$ in $H_2(E)$.

D.1.1. Theorem (cf. Theorem 2.1.2). *If $E_{\mathbb{R}}$ is nonorientable, then $E_{\mathbb{R}}$ consists of a single half and the restriction $X_{\mathbb{R}} \to E_{\mathbb{R}}$ of the projection $X \to E$ is the orientation double covering (i.e., there is no components of type T_g). Besides, E is an $(M-d)$-surface, $d \geqslant 2$, and*

(1) *if $d = 2$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pmod{16}$ and $E_{\mathbb{R}}$ is of type I;*
(2) *if $d = 3$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 2 \pmod{16}$;*
(3) *if $d = 4$ and $\chi(E_{\mathbb{R}}) \equiv \sigma(E) + 8 \pmod{16}$, then $E_{\mathbb{R}}$ is of type I.*

Proof. Pick a nonorientable component $C_1 \subset E_{\mathbb{R}}^{(1)}$ and a disorienting cycle $\alpha \in H_1(C_1)$. If $E_{\mathbb{R}}^{(2)} \neq \varnothing$, say, $E_{\mathbb{R}}^{(2)} \supset C_2$, then $[\operatorname{bv}_2(\alpha + (C_1 - C_2))]^2 = 1$. If there is an orientable component C_3 and a class $\beta \in H_1(C_3)$ which does not vanish in $H_1(E)$, then $[\operatorname{bv}_2(\alpha + \beta)]^2 = 1$. In both the cases we constructed a class $x \in H_2^{\sim}(E)$ with $x^2 = 1$; this contradicts to the assumption that $w_2(E) = 0$.

From Corollary C.5 it follows that, under the hypotheses of the theorem, $\operatorname{bv}_2 x$ can be represented by an integral cycle for any $x \in \mathcal{F}^2$. Indeed, since $E_{\mathbb{R}}$ has only one half, there are no classes of the form $\operatorname{bv}_2(\alpha + (C_i - C_j))$, and, since $w_2(E) = 0$, each nonorientable component C_i of $E_{\mathbb{R}}$ is of even genus (otherwise one would have $(\operatorname{bv}_2[C_i])^2 = 1$); hence, $\operatorname{bv}_1 w_1(C_i) = 0$, and C.5 applies. Thus, $H_2^{\sim}(E)$ is a subquotient of $\hat{H}(H_2(E;\mathbb{Z}))$, and, since $H_2(E)$ does have nonintegral classes, $\dim \mathcal{D}^- < d$ (see Lemma 5.1.2 for the definition of $\mathcal{D}^-$). Due to the (mod 2)-congruence (Lemma 5.1.2) it must be $\dim \mathcal{D}^- = d - 2$, and the rest of the proof repeats that of Theorem 2.2.1. ☐

D.2. Surfaces with orientable real part.

D.2.1. Theorem (cf. Theorems 2.1.2 and 2.2.1). *If E is an $(M-d)$-surface with orientable real part and either $E_{\mathbb{R}}$ is trivially covered by $X_{\mathbb{R}}$ (i.e., there* *is no components of type T_g) or $E_{\mathbb{R}}$ consists of a single half, then $d \geqslant 2$ and*

(1) *if $d = 2$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pmod{16}$ and $E_{\mathbb{R}}$ is of type I;*
(2) *if $d = 3$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 2 \pmod{16}$;*
(3) *if $d = 4$ and $\chi(E_{\mathbb{R}}) \equiv \sigma(E) + 8 \pmod{16}$, then $E_{\mathbb{R}}$ is of type I.*

D.2.2. Theorem (cf. Theorem 2.2.2). *Let E be an $(M-3)$-surface with $E_{\mathbb{R}} = kS$. Then $E_{\mathbb{R}} = \{4pS\} \sqcup \{(4q+1)S\}$, both the halves being nonempty unless $k \equiv 1 \pmod 8$.*

D.2.3. Theorem (cf. Theorem 2.2.3). *Let $E_{\mathbb{R}} = T_g \sqcup kS$. Suppose that E is an $(M-d)$-surface and $\chi(E_{\mathbb{R}}) \equiv \sigma(E) + 2\delta \pmod{16}$. Then for all the values of (d, δ) listed in Table 1 in 2.2 one has $E_{\mathbb{R}} = \{T_g \sqcup k^{(1)}S\} \sqcup \{k^{(2)}S\}$, where $k^{(2)} \pmod 4$ is given in the table and $k^{(2)} \neq 0$ with the possible exception of the case $d = 2$, $\delta = 0$, $E_{\mathbb{R}}$ is of type I. Besides, there are the following additional restrictions:*

(1) *if $d = 0$, then both the halves (as well as $E_{\mathbb{R}}$ itself) are of type I;*
(2) *if $d = 0$, then $k^{(1)} \neq 0$ unless $k \equiv 0 \pmod 8$;*
(3) *if $d = 1$ and $k^{(1)} = 0$, then either $k \equiv \delta \pmod 8$, or $k \equiv 0 \pmod 4$ and $E_{\mathbb{R}}^{(2)}$ is of type I.*

Proof of Theorems D.2.1, D.2.2, and D.2.3 repeats that of Theorems 2.2.1, 2.2.2, and 2.2.3 respectively and is based on the modification of Lemmas 6.2 and 6.1 given below (Lemmas D.2.4 and D.2.5 respectively). □

D.2.4. Lemma. *There is at least one and at most two relations between the images of the components of $[E_{\mathbb{R}}]$ in $H_2^{\infty}(E)/\operatorname{Tors}_2 H_2(E;\mathbb{Z})$. One relation is $[E_{\mathbb{R}}] = 0$; the only other possible relation is $[E_{\mathbb{R}}^{(1)}] = [E_{\mathbb{R}}^{(2)}] = 0$.*

Proof. The relation $[E_{\mathbb{R}}] = 0$ follows from Lemma 5.4.1. An element vanishes in $H_2^{\infty}(E)/\operatorname{Tors}_2 H_2(E;\mathbb{Z})$ if and only if it annihilates all the integral classes in $H_2^{\infty}(E)$, i.e., all the classes except $\operatorname{bv}_2(\alpha + \langle C_i - C_j \rangle)$ with $\operatorname{bv}_1\langle C_i - C_j\rangle = \operatorname{bv}_1 \alpha \neq 0$ (see Corollary C.5). If $\operatorname{bv}_2[C_1] + \cdots = 0$ is a relation, $C_1 \subset E_{\mathbb{R}}^{(1)}$, and there is another component $D_1 \subset E_{\mathbb{R}}^{(1)}$, then $\operatorname{bv}_2(C_1 - D_1) \circ (\operatorname{bv}_2[C_1] + \dots) = 1$, which is a contradiction. □

D.2.5. Lemma. *Let E be an $(M-d)$-surface with orientable real part, and let $\mathcal{D}^-$ be the discriminant form of the sublattice of conj-skew-invariant vectors in $H_2(E;\mathbb{Z})/\operatorname{Tors}$. Then:*

$d = \dim \mathcal{D}^-$ *if $E_{\mathbb{R}}$ has a component T_g and both the halves are nonempty;*

$d = 2 + \dim \mathcal{D}^-$ *if either*

(1) *$E_{\mathbb{R}}$ has a component T_g and one of the halves is empty, or*
(2) *$E_{\mathbb{R}}$ has no components T_g and both the halves are nonempty;*

$d = 2 + \dim \mathcal{D}^-$ *or* $4 + \dim \mathcal{D}^-$ *if $E_{\mathbb{R}}$ has no components T_g and one of the halves is empty.*

Proof is essentially the same as that of Lemma 6.1, with Lemmas 6.3.3 and 6.3.4 replaced with the following statements (where the first one follows from Corollary C.5, and the second one is obvious):

D.2.6. *$E_{\mathbb{R}}$ being orientable,* $\mathrm{Tors}_2\, H_2(E;\mathbb{Z})$ *does not vanish in $H_2^{\infty}(E)$ (i.e., there is a nonintegral class in $H_2^{\infty}(E)$) if and only if $E_{\mathbb{R}}$ has a component T_g and both the halves are nonempty.*

D.2.7. *$E_{\mathbb{R}}$ being orientable, $H_1^{\infty}(E) \neq 0$ if and only if either*

(1) *$E_{\mathbb{R}}$ has a components T_g, or*
(2) *both the halves are nonempty.*

Remark. Probably, one can enforce Theorem D.2.3 taking into account the class represented by the component T_g.

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