

THE SCHOTTKY PROBLEM

Ron Donagi

Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26
5300 Bonn 3
BRD

MPI/87-56

THE SCHOTTKY PROBLEM

Ron Donagi
Max-Planck-Institut
Gottfried-Claren-Str. 26
D-5300 Bonn 3

Introduction

0. Theta functions
1. Andreotti-Mayer
 - 1.1 $\text{Sing}(\theta)$ for Jacobians
 - 1.2 The Andreotti-Mayer loci
 - 1.3 Jacobians are a component of Andreotti-Mayer
 - 1.4 Further results
2. Schottky-Jung
 - 2.1 Prym varieties
 - 2.2 The Schottky loci
 - 2.3 Jacobians are a component of Schottky
 - 2.4 Conjectures
3. Novikov
 - 3.1 More conjectures
 - 3.2 Trisecants and the KP hierarchy
 - 3.3 Andreotti-Mayer vs. Novikov
4. Andreotti-Mayer in low genus
 - 4.1 $\text{Sing}(\theta)$ for a Prym
 - 4.2 Prym is proper
 - 4.3 $\text{Sing}(\theta)$ for generalized Pryms
 - 4.4 The tetragonal construction
5. Schottky-Jung in low genus
 - 5.1 Symmetry of the theta maps
 - 5.2 Schottky in genus ≤ 5

Introduction

The Schottky problem is the problem of characterizing the Jacobians of algebraic curves among all principally polarized abelian varieties. The three most successful approaches to this problem are associated with the names of Schottky-Jung, Andreotti-Mayer, and Novikov. The special properties used to characterize Jacobians are, respectively, the existence of Prym varieties, the large singular locus of a Jacobian theta divisor, and the differential equations satisfied by Jacobian theta functions.

The purpose of these notes is to describe how closely each approach is known to characterize Jacobians, and especially to relate these three a priori independent approaches.

Novikov's approach has come the closest. The Novikov Conjecture, proved by Shiota, says that an indecomposable abelian variety X is a Jacobian if there are 3 translation-invariant vector fields D_1, D_2, D_3 on X and a scalar d so that the theta function of X satisfies the "KP" differential equation corresponding to these constants. In some ways, though, this solution is still not satisfactory. First, it is not clear how to convert the KP equation into explicit equations in the natural coordinates on moduli space, the theta nulls. Then there is the question of eliminating the choices (D_1, D_2, D_3, d) involved, and the fact that the locus we get in \mathcal{A}_g consists of Jacobians \mathcal{J}_g together with all decomposable abelian varieties. Finally, from an algebro-geometric point of view, Novikov's condition may be considered to be "too strong". We discuss the differential-equations approach in Chapter 3, and in particular we propose a stronger version (3.1) of the Novikov Conjecture, i.e. a weaker condition which should suffice to characterize Jacobians among indecomposable abelian varieties.

The Andreotti-Mayer condition, on the other hand, is too weak. The locus defined by it contains Jacobians as an irreducible component, but does contain other components. Beauville and Debarre have shown that the Andreotti-Mayer locus contains Novikov's locus as well as several of its variants. We discuss this approach in general in Chapter 1, then return to it in Chapter 4 with an analysis of the components of the Andreotti-Mayer locus in genus ≤ 5 .

The strength of the Schottky-Jung approach is somewhere in between. The conjecture is much stronger (i.e. the condition satisfied by Jacobians is much weaker) than Novikov's, yet I am aware of no evidence against it. A precise version of the conjecture is stated in (2.11). It implies the strong version of Novikov (3.1), as well as the four conjectures (2.13-2.16) of van Geemen and van der Geer, and various analogues. Again, we first discuss the general theory, including a proof that Jacobians are a component of Schottky, in Chapter 2, and then in Chapter 5 we sketch proofs of Conjecture (2.11) in genus 4 (Igusa's theorem) and genus 5.

These notes evolved from lecture series which I gave at the CIME meeting in Montecatini and at UNAM in Mexico City, in the spring and summer of 1985. The lectures included background material on moduli spaces and their compactifications (Satake-Baily-Borel, toroidal, stable curves), the algebraic theory of theta functions, the theory of Prym varieties, and the three approaches to the Schottky problem. The original version of these notes has become much too long for the present format; I hope it will appear in the near future as a book on the moduli of curves and abelian varieties. The present version is more or less an extract from the last chapters of the book.

I heartily thank Eduardo Sernesi and Felix Recillas for the invitations to give the lectures which started this project, and for the warmth of their hospitality. I am also grateful to the many people with whom I discussed the Schottky problem over the years, including Arbarello, Beauville, Clemens, Debarre, Mumford, van Geemen, van der Geer and many others. special thanks go to the Max-Planck-Institut für Mathematik which provided the perfect conditions for completing this work, and to Karin Deutler for the excellent typing job.

CHAPTER 0

Theta functions

We recall some notation and results from the theory of theta functions. The reader may prefer to skim through this introductory chapter, or to skip it and refer to it later as needed.

The moduli space \mathcal{A}_g of g -dimensional principally polarized abelian varieties (ppav) is the quotient

$$\mathcal{A}_g := \mathbb{H}_g / \Gamma_g^{(1)}$$

where \mathbb{H}_g is Siegel's half space

$$\mathbb{H}_g := \{ \Omega \text{ symmetric } g \times g \text{ complex matrix, } \text{im}(\Omega) > 0 \},$$

and

$$\Gamma_g^{(1)} := \text{Sp}(2g, \mathbb{Z})$$

is the integral symplectic group, acting on \mathbb{H}_g properly discontinuously. For a subgroup

$$\Gamma_g^{\text{level}} \subset \Gamma_g^{(1)}$$

of finite index we have the corresponding level moduli space

$$\mathcal{A}_g^{\text{level}} := \mathbb{H}_g / \Gamma_g^{\text{level}},$$

a finite branched cover of \mathcal{A}_g .

The action of Γ_g^{level} on \mathbb{H}_g lifts to a properly discontinuous action of $\Gamma_g^{\text{level}} \times \mathbb{Z}^{2g}$ on $\mathbb{H}_g \times \mathbb{C}^g$, where

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g^{\text{level}}$$

acts on:

$$(0.1) \quad \begin{aligned} & \cdot \mathbb{H}_g \quad \text{by } \Omega \longmapsto (A\Omega + B)(C\Omega + D)^{-1} \\ & \cdot \mathbb{C}^g \quad \text{by } {}^t(C\Omega + D)^{-1} \\ & \cdot \mathbb{Z}^{2g} \quad \text{by } {}^t\gamma^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}. \end{aligned}$$

We let \mathfrak{X}_g denote the quotient. It maps naturally to $\mathfrak{A}_g^{\text{level}}$, and for high enough level (i.e. small enough Γ_g^{level} in Γ_g) it gives a universal abelian variety, i.e. the fiber over the isomorphism class $[X]$ is isomorphic to X . On the other hand, if $(-1) \in \Gamma_g^{\text{level}}$ then the generic fiber is the Kummer variety

$$(0.2) \quad K(X) := X/(\pm 1).$$

The level groups which we will encounter are:

(0.3) The principal congruence subgroup

$$\Gamma^{(n)} := \{\gamma \in \Gamma^{(1)} \mid \gamma \equiv 1 \pmod{n}\}.$$

The quotient $\mathfrak{A}_g^{(n)}$ parametrizes ppav's A with a chosen basis for the finite group A_n of points of order n in A . For $n \geq 3$, the action of $\Gamma^{(n)}$ is fixed point free (so $\mathfrak{A}_g^{(n)}$ is non-singular) and $\mathfrak{X}_g^{(n)}$ is a universal abelian variety. For $n = 1, 2$, $\mathfrak{A}_g^{(n)}$ has quotient singularities and the generic fiber of $\mathfrak{X}_g^{(n)} \rightarrow \mathfrak{A}_g^{(n)}$ is a Kummer

(0.4) The theta group

$$\Gamma_g^{(2,4)} := \{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g^{(2)} \mid \text{diag } {}^tAC \equiv \text{diag } {}^tBD \equiv 1 \pmod{4}\}.$$

This sits between $\Gamma_g^{(2)}$ and $\Gamma_g^{(4)}$. The universal object $\mathfrak{X}_g^{(2,4)}$ is a family of Kummer varieties.

(0.5) Fix a primitive vector $v \in \mathbb{Z}^{2g}$, and let

$$\mathfrak{H}_g := \{ \gamma \in \Gamma \mid \gamma v \equiv v \pmod{2} \},$$

$$\mathfrak{H}_g^{(2,4)} := \{ \gamma \in \Gamma^{(2,4)} \mid \gamma v \equiv v \pmod{4} \}.$$

The corresponding moduli spaces will be denoted $\mathfrak{A}_g, \mathfrak{A}_g^{(2,4)}$. Here \mathfrak{A}_g parametrizes isomorphism classes of ppav's $A \in \mathcal{A}_g$ with a marked, non-zero point of order 2, $\mu \in A_2$, and $\mathfrak{A}_g^{(2,4)}$ has a similar interpretation with a level (2,4)-structure and a point of order 4.

The natural coordinates on these spaces are given by the various types of theta function. Riemann's theta function is

$$\vartheta : \mathbb{H}_g \times \mathbb{C}^g \longrightarrow \mathbb{C}$$

(0.6)

$$\vartheta(\Omega, z) := \sum_{n \in \mathbb{Z}^g} \exp \pi i ({}^t n \Omega n + 2 {}^t n z).$$

For given Ω , it is a section of a line bundle $L = \mathcal{O}_X(\vartheta)$ on the ppav $X = X_\Omega$. L is in the principal polarization on X ; taking a different Ω with the same X may cause L to be changed via translation by a point of order 2 in X . The line bundle L^2 therefore depends on $X \in \mathcal{A}_g$ alone.

The theta divisor $\theta \subset X$ is given by the vanishing of ϑ . It is a "theta characteristic", i.e. a symmetric divisor representing the principal polarization. Its translates by points of order 2 in X are the other theta characteristics.

For $\epsilon, \delta \in \mathbb{Q}^g$, we have the "theta functions with characteristics":

$$\vartheta \left[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix} \right] (\Omega, z) := \sum_{n \in \mathbb{Z}^g} \exp \pi i ({}^t (n+\epsilon) \Omega (n+\epsilon) + 2 {}^t (n+\epsilon) (z+\delta))$$

(0.7)

$$= \exp \pi i ({}^t \epsilon \Omega \epsilon + 2 {}^t \epsilon (z+\delta)) \cdot \vartheta(\Omega, z+\Omega \epsilon + \delta).$$

The space of sections $H^0(X_\Omega, L^k)$ is k^g -dimensional, and an explicit basis is given by the "k-th order theta functions":

$$(0.8) \quad \vartheta_k[\epsilon](\Omega, z) := \vartheta \begin{bmatrix} \epsilon/k \\ 0 \end{bmatrix} (k\Omega, kz)$$

for $\epsilon \in (\mathbb{Z}/k\mathbb{Z})^g$ (they depend on ϵ only modulo k).

We will make much use of the map given by the second order theta functions

$$(0.9) \quad \vartheta_2 : \mathbb{H}_g \times \mathbb{C}^g \longrightarrow U_g$$

$$\vartheta_2(\Omega, z)_\epsilon := \vartheta_2[\epsilon](\Omega, z)$$

(where U_g is the standard representation of the Heisenberg group, or we can think of it as shorthand notation for \mathbb{C}^{2g} .) We also consider the projectivized version of ϑ_2 :

$$\chi : \mathbb{H}_g \times \mathbb{C}^g \longrightarrow \mathbb{P}(U_g),$$

called the Kummer map. From the transformation properties of ϑ it follows that χ factors through the universal Kummer variety:

$$(0.10) \quad \chi : \mathfrak{X}_g^{(2,4)} \longrightarrow \mathbb{P}(U_g).$$

Two other theta maps which we will need are

$$\alpha : \mathfrak{A}_g^{(2,4)} \longrightarrow \mathbb{P}(U_g)$$

and

$$\beta : \mathfrak{A}_g^{(2,4)} \longrightarrow \mathbb{P}(U_{g-1})$$

obtained by restricting χ to the 0-section of

$$\mathfrak{a}_g^{(2,4)} \rightarrow \mathfrak{a}_g^{(2,4)},$$

respectively to the natural section (which is torsion of order 4) of

$$\mathfrak{A}_g^{(2,4)} \rightarrow \mathfrak{A}_g^{(2,4)}.$$

At level- ∞ , these maps are given explicitly by:

$$(0.11) \quad \alpha(\Omega)_\epsilon := \vartheta_2[\epsilon](\Omega, 0) = \vartheta \begin{bmatrix} \epsilon/2 \\ 0 \end{bmatrix} (2\Omega, 0)$$

$$\beta(\Omega)_\epsilon := \vartheta \begin{bmatrix} \epsilon/2 & 0 \\ 0 & 1/2 \end{bmatrix} (2\Omega, 0).$$

The entries of α are called the theta nulls. Finally, we can get rid of the annoying level on the left, by dividing by the Galois group

$$G_g(2,4) := \Gamma_g^{(1)} / \Gamma_g^{(2,4)}$$

of $\mathfrak{a}_g^{(2,4)}$ over \mathfrak{a}_g , on the right. We get maps:

$$(0.12) \quad \alpha : \mathfrak{a}_g \rightarrow \mathbb{P}(U_g) / G_g(2,4)$$

$$\beta : \mathfrak{A}_g \rightarrow \mathbb{P}(U_{g-1}) / G_{g-1}(2,4).$$

($G_g(2,4)$ has a natural, "Heisenberg" action on $\mathbb{P}(U_g)$. We omit the details.)

There is one basic identity relating ϑ to the second-order theta functions:

(0.13) Riemann's Quadratic Identity:

$$\vartheta(\Omega, z+w) \cdot \vartheta(\Omega, z-w) = \sum_{\sigma \in (\mathbb{Z}/2\mathbb{Z})^g} \vartheta_2[\sigma](\Omega, z) \cdot \vartheta_2[\sigma](\Omega, w).$$

Geometrically, this gives an identification of the Kummer map χ with a "dual" map χ^* . For fixed X , χ is simply the map given by the linear system $|O_X(2\theta)| = |L^2|$ on X :

$$\chi : X \longrightarrow \mathbb{P}H^0(X, \mathcal{O}_X(2\theta))^*.$$

On the other hand, fix a theta characteristic θ , e.g. by choosing a period matrix Ω for X and taking θ as $\{z \mid \vartheta(\Omega, z) = 0\}$. We get a map

$$\chi^* : X \longrightarrow \mathbb{P}H^0(X, \mathcal{O}_X(2\theta))$$

$$x \longmapsto \theta_x + \theta_{-x}$$

where θ_x means the translate of θ by x . Riemann's Quadratic Identity can be reformulated as:

(0.14) Kummer Identification Theorem There is a (natural) isomorphism $\mathbb{P}H^0(X, \mathcal{O}_X(2\theta)) \xrightarrow{\sim} \mathbb{P}H^0(X, \mathcal{O}_X(2\theta))^*$ which takes χ to χ^* .

We mention one more property of the theta function. It satisfies the following analogue of the heat equation:

$$(0.15) \quad \frac{\partial \vartheta}{\partial \Omega_{ij}} = \frac{1}{2\pi i(1+\delta_{ij})} \frac{\partial^2 \vartheta}{\partial z_i \partial z_j}.$$

More algebraically, this gives a natural identification of the tangent space to moduli,

$$T_X \mathcal{M}_g,$$

with the dual of the "quadratic differentials"

$$S^2 T_0 X.$$

CHAPTER 1

Andreotti-Mayer

The approach based on the singularities of θ is the least successful of the three approaches which we consider. Historically it was the first, and conceptually it is the easiest, so we start with it. We define the Andreotti-Mayer locus and sketch the proof that Jacobians form a component. We also touch on some rather deep results of Green and Welters which are closely related to the Andreotti-Mayer approach but also reappear elsewhere in these notes.

§ 1.1 Sing(θ) for Jacobians

The basic property of Jacobians used in the approach of Andreotti-Mayer is:

(1.1.) Proposition. Let θ be the theta divisor of a Jacobian $J(C)$ of a curve $C \in \mathcal{M}_g$. Then:

$$\text{Dim}(\text{Sing}(\theta)) \geq g - 4.$$

This is based on Riemann's Theorem and Riemann's Singularity Theorem, which say that $\theta, \text{Sing}(\theta)$ on a Jacobian can be described in terms of linear systems on the underlying curve:

$$\theta \approx W_{g-1}^0$$

$$\text{Sing}(\theta) \approx W_{g-1}^1$$

where W_d^r is the subvariety of $\text{Pic}^d(C) \approx J(C)$ consisting of line bundles of degree d on C with $h^0 \geq r + 1$. The well-developed theory of linear systems on curves [ACGH] provides many ways to estimate the dimensions of these varieties:

(1) This is a special case of the Existence Theorem in Brill-Noether theory, since for

$$r = 1, \quad d = g - 1$$

we have

$$\rho := g - (r + 1)(g - d + r) = g - 2 \cdot 2 = g - 4.$$

(2) An elementary argument for this special case is based on checking that for any divisor

$$D_0 = p_1 + \dots + p_{g-3} \in S^{g-3}C$$

there is a divisor

$$D = D_0 + p_{g-2} + p_{g-1} \in S^{g-1}C$$

such that $h^0(D) \geq 2$. Generically the inequality will be an equality, so we lose only 1 dimension in mapping to $\text{Pic}(C)$.

(3) We give another argument which introduces the very important relationship of $\text{Sing}(\theta)$ with quadrics of rank 4. Assume we are given

$$\xi \in W_{g-1}^1 \setminus W_{g-1}^2,$$

i.e. a double point of θ . The projectivized tangent cone $\mathbb{P}T_\xi(\theta)$ is thus a quadric Q . By Riemann's Singularity Theorem,

$$Q = \bigcup_{D \in |\xi|} \text{span}(\phi(D))$$

where ϕ is the canonical map of C . Q therefore contains a 1-parameter family of linear subspaces of codimension 1 in Q , or of codimension 2 in \mathbb{P}^{g-1} . Therefore Q is a quadric of rank ≤ 4 . Its vertex is therefore a linear subspace of codimension ≤ 4 which is contained in $T_\xi(\text{Sing}(\theta))$.

§ 1.2. The Andreotti-Mayer loci

We want to define a series of loci

$$\mathcal{N}_g^k \subset \mathcal{A}_g$$

given by the property:

$$"A \in \mathcal{N}_g^k \text{ if } \dim(\text{Sing}(\theta_A)) \geq k".$$

To avoid problems of existence and smoothness of the universal abelian variety \mathcal{X}_g , we work initially at level ∞ , i.e. in \mathbb{H}_g . We define:

$$\mathcal{B}_g := \{ (\Omega, z) \in \mathcal{X}_g^{(\infty)} = \mathbb{H}_g \times \mathbb{C}^g / \mathbb{Z}^{2g} \mid z \in \text{Sing}(\theta_\Omega) \}$$

$$\pi : \mathcal{B}_g \longrightarrow \mathcal{A}_g^{(\infty)} = \mathbb{H}_g, \text{ the natural projection.}$$

$$\mathcal{B}_g^k := \{ (\Omega, z) \in \mathcal{B}_g \mid \dim_{(\Omega, z)} \pi^{-1}(\Omega) \geq k \} \quad k = 0, 1, \dots$$

$$\mathcal{N}_g^k := \pi(\mathcal{B}_g^k) \subset \mathcal{A}_g^{(\infty)} = \mathbb{H}_g$$

$$\pi^k : \mathcal{B}_g^k \longrightarrow \mathcal{N}_g^k, \text{ the restriction of } \pi.$$

By construction, \mathcal{N}_g^k is in \mathbb{H}_g , but it is clearly $\Gamma^{(1)}$ -invariant, so it determines a locus in \mathcal{A}_g which we also denote \mathcal{N}_g^k . The Andreotti-Mayer locus is then [AM]:

$$\mathcal{AM}_g := \mathcal{N}_g^{g-4} \subset \mathcal{A}_g.$$

§ 1.3 Jacobians are a component of Andreotti-Mayer

(1.2) Theorem [AM] \mathcal{J}_g is an irreducible component of \mathcal{AM}_g .

Proposition (1.1) tells us that $\mathcal{J}_g \subset \mathcal{AM}_g$. The idea is to show that at a generic $C \in \mathcal{M}_g$ we have an equality of tangent spaces

$$T_{J(C)} \mathcal{J}_g \approx T_{J(C)} \mathcal{AM}_g,$$

or equivalently that the conormal spaces agree. The heat equation gives an interpretation of quadrics in canonical space as cotangent directions, at $J(C)$, to \mathcal{A}_g . With this interpretation, the conormal to \mathcal{A}_g becomes the space

$$I_2 := \ker(S^2 H^0(\omega_C) \rightarrow H^0(\omega_C^{\otimes 2}))$$

of quadrics through the canonical curve. We claim that the conormal to \mathcal{A}_g is given by

$$I_2(\theta) := \text{span}\left(\left.\frac{\partial^2 \theta}{\partial z_i \partial z_j}\right|_{\xi} \mid \xi \in \text{Sing}(\theta)\right).$$

Note that $I_2(\theta)$ is a subspace of I_2 , by Riemann's Singularity Theorem.

(1.3) Lemma Let

$$\begin{aligned} \bar{X} &\text{ be a curve in } \mathcal{A}_g^0, \\ X &:= \pi(\bar{X}) \subset \mathcal{H}_g^0, \text{ its projection,} \\ (x, \xi) &\in \bar{X}, \text{ a point.} \end{aligned}$$

Then the tangent cone $T_x X \subset T_x \mathcal{H}_g$ to the curve X at x is contained in the hyperplane

$$\left.\left(\frac{\partial \theta}{\partial \Omega}\right)\right|_{(x, \xi)} = 0.$$

(This lemma follows immediately when we differentiate θ along X , using the vanishing

$$\theta = \frac{\partial \theta}{\partial z} = 0.)$$

We can conclude that $I_2(\theta)$ is contained in the conormal to \mathcal{A}_g at $J(C)$, by combining the lemma with the heat equation (0.15) and with some sort of irreducibility assumption, for instance it suffices to assume:

- (A1) $\text{Sing}(\theta)$ is precisely $(g - 4)$ -dimensional.
 (A2) $\text{Sing}(\theta)$ is irreducible.

(These assumptions imply that for any curve $X \subset \mathcal{M}_g$ passing through the point $J(C) \in \mathcal{J}_g$, and any $\xi \in \text{Sing}(\theta_C)$, there is a lift $X_\xi \subset \mathcal{M}_g^{g-4}$ of X passing through $(J(C), \xi)$, so we can apply (1.3) to X_ξ .)

To prove Theorem (1.2) it therefore suffices to exhibit a curve X satisfying (A1), (A2) and:

(A3) $I_2 = I_2(\theta)$.

The argument clearly breaks down without the irreducibility (A2), since quadrics coming from points ξ in different components of $\text{Sing}(\theta)$ could give directions normal to different curves X in \mathcal{M}_g . Still, we may weaken (A2) to:

(A2') For each component Ξ of $\text{Sing}(\theta)$, the quadrics

$$\left\langle \left(\frac{\partial^2 g}{\partial z_i \partial z_j} \right) \Big|_{\xi} \mid \xi \in \Xi \right\rangle$$

span $I_2(\theta)$.

In the original proof [AM], Andreotti and Mayer consider trigonal curves C . Here (A2) fails, but (A1) and (A2') are easy: $\text{Sing}(\theta)$ consists of two components, each $(g-4)$ -dimensional. One is

$$\langle L = L_0 \otimes T \mid L_0 \in W_{g-4}^0, T = \text{the trigonal bundle} \rangle,$$

the other is its image under the involution

$$L \longmapsto \omega_C \otimes L^{-1}.$$

Symmetry implies that the components span the same subspace of $I_2(\theta)$, which is therefore all of $I_2(\theta)$. This explicit description of $\text{Sing}(\theta)$ then allows direct verification of (A3), proving the theorem.

QED

§ 1.4 Further results

It turns out that all three assumptions made in the proof of Theorem (1.2) hold, at least generically. We discuss these next.

(A1) By (1.1), we know that $\dim(\text{Sing}(\theta))$ is always at least $g - 4$. An easy dimension count shows that equality must hold generically. A theorem of Martens [ACGH, p. 191] says that equality holds if and only if C is non-hyperelliptic.

(A2) The irreducibility of $\text{Sing}(\theta)$ for generic C also follows from Brill-Noether theory. In fact, the Fulton-Lazarsfeld Connectedness Theorem together with Gieseker's Smoothness Theorem [ACGH, pp. 212 and 214] imply that W_d^r is irreducible for generic C whenever the Brill-Noether number ρ is ≥ 1 .

A more precise result is known in our case, when $r = 1$: Teixidor [Tx] shows that W_d^1 is irreducible except when C is trigonal, bielliptic (branched double cover of an elliptic curve) or a certain type of curve of genus 5.

(A3) Andreotti and Mayer showed that $I_2(\theta) = I_2$ for trigonal C , hence for generic C . There are several other loci where the equality can be checked directly, e.g. for bielliptic curves. The best result was proved by Mark Green:

(1.4) Theorem [G] For any non-hyperelliptic curve C of genus $g \geq 4$, the space I_2 of quadrics through the canonical curve $\phi(C)$ is spanned by the tangent cones to θ_C at its double points, i.e.

$$I_2(\theta) = I_2.$$

In particular, this implies that I_2 is spanned by quadrics of rank ≤ 4 , since we saw in the third proof of (1.1) that tangent cones to θ at double points are quadrics of rank ≤ 4 . This also produces a simple proof of Torelli's Theorem, in fact a recipe for recovering a curve (not hyperelliptic, trigonal or a plane quintic) from its Jacobian: the canonical curve $\phi(C)$ is the intersection of the tangent cones to θ at its double points.

Next we describe a result of Welters' which is closely related to Green's Theorem. Given a curve C , we define three loci in $J(C)$:

$$F_C := \bigcap_{D \in |2\theta|, m_0(D) \geq 4} (D),$$

$$F'_C := \bigcap_{\xi \in \text{Sing}(\theta)} (\theta_\xi \cup \theta_{-\xi}),$$

$$F''_C := \bigcap_{\xi \in \text{Sing}(\theta)} (\theta_\xi) = \{a \in J(C) \mid a + \text{sing}(\theta) \subset \theta\}.$$

(1.5) Theorem (Welters [We]) For a curve C of genus g , the surface $(C - C) \subset J(C)$ is equal to:

- (1) F_C , if $g = 3$ or $g \geq 5$.
- (2) F'_C , if $g \geq 5$ and C is not trigonal
- (3) F''_C , if $g \geq 5$.

When C is of genus 4, it has two trigonal bundles T_0, T_1 (possibly equal); in this case

$$F_C = (C - C) \cup \pm(T_0 - T_1).$$

We observe that for non-trigonal C , (2) \Rightarrow (1), since

$$(\theta_\xi \cup \theta_{-\xi}) \in \{D \in |2\theta|, m_0(D) \geq 4\}.$$

By Teixidor's results [Tx] on the irreducibility of $\text{Sing}(\theta)$, we can also deduce (3) \Rightarrow (2), so the main difficulty is in proving (3).

One connection with Green's Theorem is given by the following weak version of (1.5), which follows from (1.4):

(1.6) Corollary (Weak version of Welters' Theorem) $C - C$ is a component of F_C ($g \geq 4$) and F'_C, F''_C ($g \geq 5$).

Since clearly $C - C$ is contained in the three loci, it suffices to show that they are 2-dimensional at 0. Green's Theorem says that

$$C = \cap \mathbb{P}T_{\xi}\theta,$$

hence set-theoretically

$$C = \mathbb{P}T_0(\cap_{\xi} (\theta_{\xi} \cup \theta_{-\xi}))$$

so the tangent cone is the cone over C , a surface (and of course, equal exactly to $T_0(C - C)$).

The main connection of the two theorems is in the proofs, both of which make heavy use of the geometry of the $(g - 1)$ -st symmetric product $S^{g-1}C$, which is a desingularization of θ .

CHAPTER 2
Schottky-Jung

After reviewing some basic properties of Prym varieties, we define the Schottky loci (there are several of them: \mathcal{H}_g , $\mathcal{H}_g^{(big)}$, \mathcal{H}_g) in §2.2, and show that Jacobians are in these loci. The main fact known about these loci is that Jacobians are actually a component; we sketch that in §2.3, and then conclude with a series of conjectures, all of which follow from what should be considered "The Schottky-Jung Conjecture", (2.11).

§ 2.1 Prym varieties

The property of Jacobians used in the Schottky-Jung approach is the existence of Prym varieties. In this section we briefly review the definition and some basic facts about Pryms.

Consider an unramified double cover

$$\pi : \tilde{C} \rightarrow C$$

of a curve $C \in \mathcal{H}_g$. By Hurwitz' formula, the genus of \tilde{C} is $2g - 1$. For given C , the set of double covers π is in 1 - 1 correspondence with the set

$$J_2(C) \setminus (0)$$

of nonzero points μ of order 2 in $J(C)$. There are induced maps on Jacobians,

$$\begin{aligned} \pi^* : J(C) &\rightarrow J(\tilde{C}) \\ Nm : J(\tilde{C}) &\rightarrow J(C). \end{aligned}$$

The kernel of π^* is $(0, \mu)$, where $\mu \in J_2(C)$ corresponds to π as above. The kernel of Nm also has two components which we denote

P, P^- , where $P \subset J(\tilde{C})$ is an abelian subvariety, and P^- a translate of P by a point of order 2. Since N_m is surjective, P is $(g-1)$ -dimensional. The principal polarization on $J(\tilde{C})$ induces twice a principal polarization on P ; more precisely:

(2.1) Wirtinger's Theorem [M1] Riemann's theta divisor $\tilde{\theta} \in J(\tilde{C})$ intersects P in twice a divisor E in the principal polarization:

$$\tilde{\theta} \cap P = 2 \cdot E.$$

In particular, we can think of P in a natural way as a ppav, $(P, E) \in \mathcal{A}_{g-1}$, called the Prym variety of (C, μ) . The assignment

$$(C, \mu) \longmapsto P = P(C, \mu)$$

gives a morphism of moduli spaces

$$\mathcal{P} : \mathcal{M}_g \rightarrow \mathcal{A}_{g-1}$$

called the Prym map.

Let J, \tilde{J} denote $J(C), J(\tilde{C})$ respectively, and let J', \tilde{J}' denote the respective torsors (= principal homogeneous spaces) of effective divisors in the principal polarizations of J, \tilde{J} . We have a pull-back map

$$\pi'^* : J' \rightarrow \tilde{J}'$$

and a pushforward

$$\pi'_* : \tilde{J}' \rightarrow \{\text{divisors in twice the principal polarization}\}.$$

(2.2) Splitting Theorem For any divisor $\theta \in J'$ in the principal polarization on J , the pushforward of $\pi'^*\theta$ splits:

$$\pi'_*(\pi'^*(\theta)) = \theta + \theta_\mu.$$

(2.3) Prym-Kummer Identification Theorem [M2] Let L_0 be a line bundle on C satisfying $L_0^{\otimes 2} \approx \mu \otimes \omega_C$, and let $L := \pi^* L_0$. (We think of L_0, L as elements of J', \tilde{J}' respectively.) Then:

- (1) L determines a subvariety $P_L \subset \tilde{J}'$, a translate of P .
- (2) L_0 determines a natural (i.e. equivariant under the action of the Heisenberg group) embedding

$$i_{L_0} : |2\Xi|^* \longrightarrow |2\theta|$$

(where Ξ, θ are the natural theta divisors on P, J).

(3) The Kummer map χ_P can be identified with π'_* , i.e. the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\chi_P} & |2\Xi|^* \\ \downarrow \int & & \downarrow \int i_{L_0} \\ P_L & \xrightarrow{\pi'_*} & |2\theta|. \end{array}$$

We also mention that the Abel-Jacobi map

$$AJ : \tilde{C} \longrightarrow \tilde{J}$$

induces an "Abel-Prym" map

$$AP : \tilde{C} \longrightarrow P.$$

While the "derivative" (Gauss map) of AJ is the canonical map, the derivative of AP is the Prym-canonical map $C \rightarrow P^{g-2}$ given by the linear system $\omega_C \otimes \mu$.

§ 2.2. The Schottky loci

At level- ∞ , we define the Schottky locus to be:

$$\mathcal{Y}_g^{(\infty)} := \beta^{-1}(\text{image } \alpha) \subset \mathbb{H}_g.$$

From the transformation properties of theta functions it follows that $\mathcal{Y}_g^{(\infty)}$ is the inverse of a locus $\mathcal{X}\mathcal{Y}_g \subset \mathcal{X}\mathcal{A}_g$, defined by

$$\begin{aligned} \mathcal{X}\mathcal{Y}_g &:= \beta^{-1}(\text{image } \alpha) \\ &= \left\{ (X, \mu) \in \mathcal{X}\mathcal{A}_g \mid \begin{array}{l} \chi_X(\frac{1}{2}\mu) = \chi_P(0) \text{ for some choice} \\ \text{of } \frac{1}{2}\mu \in X \text{ and some } P \in \mathcal{A}_{g-1} \end{array} \right\}. \end{aligned}$$

(The last condition can also be interpreted as an equality of sets in $\mathbb{P}(U_{g-1})/G_{g-1}$:

$$\chi_X(\frac{1}{2}\mu) = \chi_P(P_2)$$

with $\frac{1}{2}\mu := \{\lambda \in X_4 \mid 2\lambda = \mu\}$.) However, $\mathcal{Y}_g^{(\infty)}$ does not come from a locus in \mathcal{A}_g (equivalently, $\mathcal{Y}_g^{(\infty)}$ is $\mathcal{X}\Gamma$ -invariant but not necessarily $\Gamma^{(1)}$ -invariant). We therefore have two loci in \mathcal{A}_g :

$$\mathcal{Y}_g^{(\text{big})} := \{X \in \mathcal{A}_g \mid (X, \mu) \in \mathcal{X}\mathcal{A}_g \text{ for some } \mu \in X_2 \setminus 0\}$$

$$\mathcal{Y}_g := \{X \in \mathcal{A}_g \mid (X, \mu) \in \mathcal{X}\mathcal{A}_g \text{ for all } \mu \in X_2 \setminus 0\}.$$

(2.4) Schottky-Jung Theorem ([S], [SJ], [FR], [F], [M2]). $\mathcal{X}\mathcal{Y}_g \subset \mathcal{X}\mathcal{Y}_g^{(\infty)}$.

(2.5) Corollary. $\mathcal{Y}_g \subset \mathcal{Y}_g \subset \mathcal{Y}_g^{(\text{big})}$.

The point is, of course, the existence of Prym varieties. Both results follow from

(2.6) Schottky-Jung Identities. For $(C, \mu) \in \mathcal{X}\mathcal{A}_g$ with Prym variety $P(C, \mu)$, we have an equality (in $\mathbb{P}(U_{g-1})/G_{g-1}$):

$$\alpha(P(C, \mu)) = B(J(C), \mu).$$

(This equality can be lifted to $P(U_{g-1})$ by being careful to choose the right level-(2,4) structure on P corresponding to a given one on J . In this form (2.6) is known as the Schottky-Jung proportionality.)

This is just an analytic expression of the Splitting Theorem (2.2):

$$\pi'_*(\pi'^*(\theta)) = \theta + \theta_\mu,$$

where the LHS is interpreted via the Kummer-Prym Identification Theorem (2.3), and the RHS via the general Kummer Identification Theorem (0.14) (applied to the divisor $\theta_{\frac{1}{2}\mu}$ for some (any) choice of $\frac{1}{2}\mu$).

§ 2.3 Jacobians are a component of Schottky

The title result of this section was proved by van Geemen:

(2.7) Theorem [vG1] \mathcal{J}_g is an irreducible component of \mathcal{Y}_g .

In the sequel we will need a small improvement, with similar proof:

(2.8) Theorem [D2] $\mathcal{K}\mathcal{J}_g$ is an irreducible component of $\mathcal{K}\mathcal{Y}_g$, hence \mathcal{J}_g is an irreducible component of $\mathcal{Y}_g^{(\text{big})}$.

Both proofs are based on degeneration to the boundary of moduli space, so let us begin with recalling the Satake-Baily-Borel compactification

$$\bar{\mathcal{M}}_g^S \supset \bar{\mathcal{M}}_{g-1}^S \supset \dots \supset \bar{\mathcal{M}}_1^S \supset \mathcal{M}_0$$

where \mathcal{M}_0 is a point and

$$\bar{\mathcal{M}}_k^S \setminus \bar{\mathcal{M}}_{k-1}^S \approx \mathcal{M}_k.$$

Its boundary, ∂ , is therefore irreducible, and is just a compactification of \mathcal{A}_{g-1} . The corresponding compactifications of level moduli spaces have reducible boundaries. We need a more precise description of the boundary of $\mathcal{R}\mathcal{A}_g$.

Consider a corank -1 degeneration in $\mathcal{R}\mathcal{A}_g$ with general fiber (X, μ) , and let $\lambda \in X_2$ be the vanishing cycle (reduced mod. 2). In terms of the $\mathbb{Z}/2\mathbb{Z}$ -valued intersection pairing (= Weil pairing) on X_2 we have 3 possibilities:

- I. $\lambda = \mu$
- II. $\lambda \neq \mu, (\lambda, \mu) = 0$
- III. $(\lambda, \mu) \neq 0$.

These give (at least) 3 boundary components $\partial^I, \partial^{II}, \partial^{III}$ of $\overline{\mathcal{R}\mathcal{A}}_g^S$.

(2.9) Lemma ([D2], [VG2]) The boundary of $\overline{\mathcal{R}\mathcal{A}}_g^S$ has exactly 3 irreducible components, described as above. They are isomorphic to the Satake-Baily-Borel compactification of $\mathcal{A}_{g-1}, \mathcal{R}\mathcal{A}_{g-1}, \mathcal{A}_{g-1}$ respectively.

The idea for proving the theorems is then to analyze the boundary behavior of Schottky.

(2.10) Proposition. $\partial(\mathcal{R}\mathcal{Y}_g) = \partial^I \cup \partial^{III} \cup i_{II}(\mathcal{R}\mathcal{Y}_{g-1})$, where

$$i_{II} : \mathcal{R}\mathcal{A}_{g-1} \subset \overline{\mathcal{R}\mathcal{A}}_{g-1}^S \approx \partial^{II} \hookrightarrow \overline{\mathcal{R}\mathcal{A}}_g^S$$

is the natural inclusion.

The reason ∂^I and ∂^{III} are in $\overline{\mathcal{R}\mathcal{Y}}_g^S$ is that they are the boundary of the locus of products

$$\overline{\mathcal{R}\mathcal{A}}_1^S \times \mathcal{A}_{g-1},$$

which is in $\mathcal{R}\mathcal{Y}_g$ since

$$\beta(X \times Y, \pi_1^* \mu) = \beta(X, \mu) \times \alpha(Y),$$

and the latter becomes $\alpha(Y)$ if $(X, \mu) \in \overline{\mathcal{R}\mathcal{A}}_1^S$. ∂^{II} is not in this locus (since $\partial^{II} \overline{\mathcal{R}\mathcal{A}}_1^S$ is empty!), and van Geemen shows that

$$\overline{\mathcal{R}\mathcal{P}}_g \cap \partial^{II} = i_{II}(\overline{\mathcal{R}\mathcal{P}}_{g-1}).$$

The argument is now concluded by an induction. For Theorem (2.7), we need to show that the tangent cone to $\overline{\mathcal{F}}_g^S$ at a point $J(C)$ of $\mathcal{F}_{g-1} \subset \partial(\overline{\mathcal{F}}_g^S)$ is an irreducible component of the tangent cone to $\overline{\mathcal{F}}_g^S$ there. The latter is the intersection of the tangent cones to $\overline{\mathcal{R}\mathcal{P}}_g^S$ at the points $(J(C), \mu)$ for $\mu \in J_2 \setminus 0$, so it suffices to show the corresponding statement at a Jacobian point of any one of the 3 lifts $\partial^I, \partial^{II}, \partial^{III}$. van Geemen does this at ∂^I . The picture is as follows:

- For $X \in \mathcal{A}_{g-1} \subset \partial \overline{\mathcal{A}}_g^S$, the projectivized tangent cone $\text{PT}_X \overline{\mathcal{A}}_g^S$ is the Kummer variety $K(X) := X/(\pm 1)$.
- Let X denote also the corresponding point of $\partial^I \overline{\mathcal{R}\mathcal{A}}_g^S$. Then $\text{PT}_X \overline{\mathcal{R}\mathcal{A}}_g^S$ maps isomorphically (by the forgetful map $\overline{\mathcal{R}\mathcal{A}}_g^S \rightarrow \overline{\mathcal{A}}_g^S$) to $\text{PT}_X \overline{\mathcal{A}}_g^S \approx K(X)$.
- When $X = J(C)$, the subvariety

$$\text{PT}_{J(C)} \overline{\mathcal{F}}_g^S \subset \text{PT}_{J(C)} \overline{\mathcal{A}}_g^S$$

is a surface, the Abel-Jacobi image of $C - C$ in $K(X)$. (Ditto for $\text{PT}_{J(C)} \overline{\mathcal{R}\mathcal{P}}_g^S \subset \text{PT}_{J(C)} \overline{\mathcal{R}\mathcal{A}}_g^S$.)

- For any $X \in \mathcal{A}_{g-1} \subset \partial^I \overline{\mathcal{A}}_g^S$, $\text{PT}_X \overline{\mathcal{R}\mathcal{P}}_g^S$ can be computed by pulling back $T(\text{image } \alpha)$. It turns out to be the base locus, in $K(X)$, of the linear system

$$\begin{aligned} \Gamma_{00} &:= |O_X(2\theta) \otimes (\mathcal{F}_{\{0\}})^{\otimes 4}| \\ &= \{s \in |O_X(2\theta)| \mid \text{mult}_0(s) \geq 4\}. \end{aligned}$$

By Welters' Theorem (1.5), this base locus is known when $X = J(C)$ is a non-hyperelliptic Jacobian: it is again the surface $C - C$, except in genus 4 when it contains additionally the point $\pm(T_0 - T_1) \in K(X)$, where T_0, T_1 are the g_3^1 's on X . In any case, $C - C$ is a component of the base locus, proving (2.7).

In proving (2.8) we do not have the freedom to switch boundary components, so we must work at ∂^{II} . The map $K(X) \rightarrow \mathbb{P}(\Gamma_{00}^*)$ given by the linear system Γ_{00} is then replaced by a projected Kummer map

$$K(\tilde{X}) \xrightarrow{\pi \circ \chi} \mathbb{P}(U_{g-2})$$

where $\tilde{X} \rightarrow X$ is the double cover determined by μ , and

$$\pi : U_{g-1} \longrightarrow U_{g-2}$$

is the natural projection onto an eigenspace. The proof requires a second "blowup" (i.e. computation of tangent cone to the tangent cone), and is then reduced to an analogue of Welters' Theorem, a question on the linear system $|2\theta_p|$ on a Prym.

§ 2.4. Conjectures

Unfortunately, \mathcal{K}_g does have components other than Jacobians. We have already noted that

$$\mathcal{K}_g \supset \mathcal{K}_1 \times \mathcal{K}_{g-1} \quad ,$$

and more generally

$$\mathcal{K}_g \supset \mathcal{K}_{g-k} \times \mathcal{K}_k \quad , \quad k \geq 4. \quad |$$

(For $k \leq 3$ the RHS is in the closure of $\mathcal{R}\mathcal{P}_g$.)

For many purposes, the toroidal compactifications $\overline{\mathcal{M}}_g^t$ [AMRT], and especially Voronoi's, are more convenient than the highly-singular $\overline{\mathcal{M}}_g^S$. In corank 1 (i.e. at generic points of the boundary components), a toroidal compactification looks like the blowup of $\overline{\mathcal{M}}_g^S$ along its boundary. We thus have

$$\partial \overline{\mathcal{M}}_g^t \sim \mathcal{M}_{g-1}$$

and $\partial(\overline{\mathcal{M}}_g^t)$ has 3 components with analogous descriptions.

In the toroidal version, the symmetry of ∂^I and ∂^{III} breaks: if we define

$$\overline{\mathcal{R}\mathcal{P}}_g^t := \overline{\beta}^{-1} \quad (\text{image } \overline{\alpha})$$

for appropriate extensions $\overline{\alpha}, \overline{\beta}$ of α, β , then $\overline{\mathcal{R}\mathcal{P}}_g^t$ contains $\partial^I \overline{\mathcal{M}}_g^t$ but not $\partial^{III} \overline{\mathcal{M}}_g^t$. The point is that β extends to the Satake compactification near ∂^I , but only to the toroidal compactification near ∂^{III} . (This can be seen already on the Prym level. The Prym map $\mathcal{P} : \mathcal{M}_g \rightarrow \mathcal{M}_{g-1}$ extends to $\overline{\mathcal{P}} : \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{M}}_{g-1}$, where $\overline{\mathcal{M}}_g$ is a stable-curve compactification. Its boundary components $\partial^I, \partial^{III}$ map to \mathcal{M}_{g-1} with 2-dimensional fibers. The extension of \mathcal{P} to ∂^I depends only on the image point in \mathcal{M}_{g-1} (Wirtinger's reducible double covers), but the extension to ∂^{III} depends on the fiber (Fay's double covers with 2 branch points) (cf. §4.2). This implies the corresponding statements for β , since by the Schottky-Jung Identities (2.6), $\beta = \alpha \circ \mathcal{P}$.) The upshot is that $\overline{\mathcal{R}\mathcal{P}}_g^t$ contains $\partial^I \overline{\mathcal{M}}_g^t$, but is only guaranteed to contain the zero-section of $\partial^{III} \overline{\mathcal{M}}_g^t \sim \mathcal{M}_{g-1} \rightarrow \mathcal{M}_{g-1}$ over a general ppav (and a surface in $K(X)$ for $X = J(C)$ a Jacobian, by the previous analysis of \mathcal{P}).

Finally, we will see in Chapter 5 that \mathcal{H}_5 contains another component \mathcal{H}^0 , the moduli space of (intermediate Jacobians of) cubic threefolds with an even point of order 2. Assembling the pieces, we arrive at what we consider to be the natural formulation of the Schottky-Jung problem:

(2.11) Conjecture. The Schottky locus equals

$$\overline{\mathcal{H}}_g = \overline{\mathcal{H}}_g \cup \partial^I \overline{\mathcal{H}}_g \cup (\overline{\mathcal{H}}^0 \times \overline{\mathcal{A}}_{g-5}) \cup \bigcup_{k \geq 4} \overline{\mathcal{H}}_{g-k} \times \overline{\mathcal{A}}_k,$$

where $\overline{\quad}$ denotes (Voronoi's) toroidal compactification.

(2.12) Corollary (of the conjecture). $\mathcal{H}_g = \mathcal{J}_g$

(The conjectured components other than \mathcal{H}_g do not contain a complete fiber of \mathcal{H}_g over \mathcal{A}_g .)

If another component of \mathcal{H}_g is discovered, the conjecture will of course need to be modified. As van Geemen pointed out though, the normal direction at $x \in \mathcal{J}_4$ to $\mathcal{A}_4 \sim \partial^I \overline{\mathcal{H}}_5^S$ along the locus \mathcal{H}^0 of cubic threefolds is given precisely by the difference of trigonal bundles $\pm(T_0 - T_1) \in K(X)$ [Co]. Since this is the only exception to Welters' Theorem, one hopes that \mathcal{H}^0 is the only non-trivial component of \mathcal{H}_g other than Jacobians. (Of course, it is still possible that components exist which do not meet $\partial^I \mathcal{H}_g$, or meet it tangentially to one of the known components.)

In [vGvdG], van Geemen and van der Geer made (more or less) the following 4 conjectures (our versions of (2.13), (2.15) are slightly stronger):

Conjectures [vGvdG]

(2.13) The base locus of Γ_{00} in a Jacobian $J(C)$ is the surface $C - C$.

(2.14) The base locus of Γ_{00} in an indecomposable non-Jacobian X is (0) .

(2.15) The intersection $\chi_X(X) \cap (\overline{\text{image } \alpha})$ in a Jacobian $X = J(C)$ is the surface $\frac{1}{4}(C - C)$.

(2.16) The intersection $\chi_X(X) \cap (\overline{\text{image } \alpha})$ in an indecomposable non-Jacobian X is (0) .

Conjecture (2.13), with a slight modification, has since become Welters' Theorem. The base locus of Γ_{00} can also be described as the intersection

$$\chi_X(X) \cap T_X(\text{image } \alpha),$$

hence the analogy between the two pairs of conjectures.

(2.17) Proposition. The [vGvdG] conjectures follow from (2.11).

Indeed, these conjectures express the fact that, at Jacobian and non-Jacobian points of $\partial^{\text{I}} \overline{\mathcal{M}}_g^t$ and $\partial^{\text{III}} \overline{\mathcal{M}}_g^s$, the tangent cone to $\overline{\mathcal{M}}_g$ is the tangent cone to the known components in the RHS of (2.11). (X must be assumed indecomposable to avoid the stupid components in (2.11).)

By considering the behavior of \mathcal{M}_g at ∂^{II} , we can make one more conjecture (recall that $\partial^{\text{II}} \mathcal{M}_g$ is \mathcal{M}_{g-1} , not all of \mathcal{M}_{g-1}):

(2.18) Conjecture. Let $\tilde{C} \rightarrow C$ be an unramified double cover with Prym P . We have maps

$$\text{Kummer } \chi_P : P \rightarrow \mathbb{P}(U_{g-1})/G_{g-1}$$

$$\text{Projected Kummer } \pi \circ \chi_{\tilde{C}} : \tilde{C} \rightarrow \mathbb{P}(U_{g-1})/G_{g-1}.$$

Then the intersection of the images is the image of $S^{2\tilde{C}}/i$, which maps to $K(\tilde{C})$, $K(P)$ by the Abel-Jacobi, Abel-Prym maps respectively.

If we believe Conjecture (2.11) as a scheme-theoretic statement, we get stronger versions of the conjectures. For instance, we "blow up" Conjectures (2.13), (2.14) at 0 : for any ppav X , let

$$\Gamma_{000} := \{s \in \Gamma_{00} \mid \text{mult}_0(s) \geq 6\}.$$

Taking fourth-order terms gives an exact sequence

$$0 \rightarrow \Gamma_{000} \rightarrow \Gamma_{00} \rightarrow |O_{\mathbb{P}^{g-1}}(4)|,$$

so we can think of Γ_{00}/Γ_{000} as a linear system of quartics on $\text{PT}_0 X \approx \mathbb{P}^{g-1}$. From (2.11) we deduce:

(2.19) Conjecture. The base locus of the linear system Γ_{00}/Γ_{000} of quartics in $\mathbb{P}^{g-1} \approx \text{PT}_0 X$ is the canonical curve $\phi(C) \subset \mathbb{P}^{g-1}$, if $X = J(C)$ is a Jacobian, and is empty if X is an indecomposable ppav which is not a Jacobian.

The case of Jacobians follows from Welters' Theorem. For non-hyperelliptic curves it gives a very explicit prescription for recovering a curve from its Jacobian.

CHAPTER 3

Novikov

The theta function of a Jacobian satisfies a family of differential equations ("KP") which yield the best answer to the Schottky problem to date. The geometric explanation of these equations is based on the trisecants of a Jacobian Kummer variety; we discuss this in §3.2. Novikov's Conjecture (= Shiota's Theorem), saying that an abelian variety whose theta function satisfies KP is either a Jacobian or a product, is seen in §3.1 to follow from a more general conjecture which is in turn equivalent to Conjecture (2.19). We conclude with a brief description of the work of Beauville and Debarre which shows that the Novikov locus, of ppav's satisfying the KP equation (or various analogues), is contained in the Andreotti-Mayer locus, and in particular it contains the locus \mathcal{J}_g of Jacobians as a component.

§ 3.1 More conjectures

Our starting point in this section is Conjecture (2.19), itself a corollary of Conjecture (2.11). We interpret it first in terms of linear differential relations satisfied by the vector-valued second-order theta function

$$\vartheta_2 : \mathbb{H}_g \times \mathbb{C}^g \rightarrow U_g \approx \mathbb{C}^{2g}$$

whose projectivization gives the Kummer map χ , and then in terms of non-linear differential equations satisfied by ϑ itself.

(3.1) Conjecture (Differential Characterization of Jacobians)

An indecomposable ppav X is a Jacobian if and only if its second-order theta function satisfies a constant-coefficient linear differential relation (i.e. polynomial in constant vector fields on X) of the form

$$((D_1)^4 + (\text{lower order terms})) \vartheta_2(\Omega, z) |_{z=0} = 0,$$

where Ω is any period matrix for X (i.e. $\Omega \in \mathbb{H}_g$ maps to $X \in \mathcal{A}_g$) and D_1 is a constant vector field on X .

This conjecture is simply a reformulation of (2.19). An element of $\Gamma = |O_X(2\theta)|$ is a linear combination of the entries of θ_2 ; it is in Γ_{00} if and only if all derivatives of order < 4 of this combination vanish at 0. Hence all the quartics in Γ_{00}/Γ_{000} vanish at some $D_1 \in T_0X$ if and only if $D_1^4 \theta_2(\Omega, 0)$ is a linear combination of lower order operators applied to $\theta_2(\Omega, 0)$.

It is now natural to ask for the explicit form of the differential relations satisfied by Jacobian theta functions. Since the base locus of Γ_{00}/Γ_{000} in $J(C)$ is the canonical curve $\phi(C)$, we know that these equations are parametrized by C . We will find their explicit form in §3.2:

(3.2) Proposition (Differential Relations for Jacobian Theta Functions). Let Ω be a period matrix of a Jacobian $X = J(C)$. Then $\vartheta_2(\Omega, 0)$ satisfies precisely a one-dimensional family of inequivalent differential relations of the form (3.1). This family is parametrized by C ; the equation corresponding to $p \in C$ is of the form

$$(D_1^4 - D_1 D_3 + D_2^2 + d) \vartheta_2(\Omega, z) |_{z=0} = 0,$$

where d is a scalar constant, and the constant vector fields D_1, D_2, D_3 are determined by their values at $AJ(p)$ (image of p under Abel-Jacobi), where they span the osculating line, plane and solid to $AJ(C)$.

(3.3) Corollary (Novikov's Conjecture, Dubrovin's Form) An indecomposable ppav X is a Jacobian if and only if its second-order theta function satisfies a differential relation of the form (3.2).

This follows immediately from (3.1) and (3.2). Together the conjectures say that if ϑ_2 satisfies any equation of type (3.1) then we are on a Jacobian and the equation is of the form (3.2). Novikov's Conjecture has been proved by Shiota [Sh], but (3.1) is open.

The differential relations (3.1), (3.2) satisfied at 0 by the vector-valued ϑ_2 can be converted to a non-linear differential

equation satisfied by the (scalar valued) theta function. This follows immediately from Riemann's Quadratic Identity (0.13):

$$\vartheta(z+w)\vartheta(z-w) = \sum_{\sigma \in (\mathbb{Z}/2\mathbb{Z})^g} \vartheta_2[\sigma](z) \vartheta_2[\sigma](w).$$

We treat one of the variables, say w , as a constant, and apply a differential operator to both sides, then evaluate at $z = 0$; this gives a differential expression in $\vartheta(w)$, on the left, and on the right a linear combination of the entries of the vector obtained by applying the operator to ϑ_2 at $z = 0$. For instance, (3.2) becomes:

$$(3.4) \quad \begin{aligned} D_1^4 \vartheta \cdot \vartheta - 4D_1^3 \vartheta \cdot D_1 \vartheta + 3(D_1^2 \vartheta)^2 - D_1 D_3 \vartheta \cdot \vartheta + D_1 \vartheta \cdot D_3 \vartheta \\ + D_2^2 \vartheta \cdot \vartheta - (D_2 \vartheta)^2 + \frac{1}{2} d\vartheta^2 = 0. \end{aligned}$$

This is known as Hirota's bilinear form of the KP (= Kadomtsev-Petviashvili) equation. The standard form of this differential equation is:

$$(3.5) \quad (u_{xxx} + uu_x - u_t)_x + u_{yy} = 0.$$

Direct substitution shows that (3.4) for ϑ is equivalent to the KP equation (3.5) for $u := (\log \vartheta)_{xx}$.

§ 3.2. Trisecants and the KP hierarchy

The Kummer variety of a Jacobian, as embedded in $\mathbb{P}(U_g)$, has a four-dimensional family of trisecant lines. The KP equation (3.2), as well as a whole hierarchy of equations satisfied by Jacobian theta functions, express limiting cases of the existence of these trisecants. Our presentation here is based on ideas of Gunning, Welters and Arbarello-De Concini.

(3.6) Lemma. Let a, b, c, d be points of a curve C . The various translates of the divisor $\theta \in J(C)$ satisfy the following inclusions:

$$(1) \quad \theta \cap \theta_{a-b} \subset \theta_{a-c} \cup \theta_{d-b}$$

$$(2) \quad (\theta_{a+b-c-d} \cup \theta) \supset (\theta_{a-d} \cup \theta_{b-c}) \cap (\theta_{a-c} \cup \theta_{b-d})$$

(3)

$$\left[\theta_{\frac{a+b-c-d}{2}} \cup \theta_{\frac{-a-b+c+d}{2}} \right] \supset \left[\theta_{\frac{a-b+c-d}{2}} \cup \theta_{\frac{-a+b-c+d}{2}} \right] \cap \left[\theta_{\frac{a-b-c+d}{2}} \cup \theta_{\frac{-a+b+c-d}{2}} \right],$$

where the choices of halves are compatible, i.e. we fix one of the 2^{2g} values of $\frac{a+b-c-d}{2}$, and determine all other expressions accordingly:

$$\frac{a-b+c-d}{2} := \frac{a+b-c-d}{2} - b + c, \quad \frac{a-b-c+d}{2} := \frac{a+b-c-d}{2} - b + d, \text{ etc.}$$

Proof.

(1) follows from Riemann-Roch. (2) follows from (1) by expanding the RHS as union of four intersections: the inclusion of each in the LHS is equivalent to (1), with the letters permuted, after translation. (3) is equivalent to (2) via translation by the fixed value of $\frac{a+b-c-d}{2}$.

QED

This lemma is classical (Mumford [M3] attributes it to Weil), but its interpretation via trisecants was first noticed by Fay:

(3.7) Corollary [F] For $a, b, c, d \in C$, the three points

$$\chi\left(\frac{a+b-c-d}{2}\right), \chi\left(\frac{a-b+c-d}{2}\right), \chi\left(\frac{a-b-c+d}{2}\right)$$

of the Kummer are collinear. (The halves must be compatible as in (3.6)(3).)

The corollary is just a restatement of (3.6)(3), using the Kummer Identification Theorem (0.14).

We see that a Jacobian Kummer has a 4-dimensional family of trisecants. The group $J_2(C)$ of points of order 2 acts linearly on $P(U_g)$ inducing translation on $K(J_2(C))$, hence acts on the variety of trisecants. Let S be the quotient. It is clear from (3.7) that S is birationally equivalent to S^4C , and an easy additional computation shows that

$$S \approx S^4 C$$

biregularly. Let us see what happens to a trisecant as we bring the points a, b, c, d together:

Choose a point $a \in C$, and write down the Taylor expansion in \mathbb{C}^g of the Abel-Jacobi map near a , in terms of a coordinate t on C near a :

$$(3.8) \quad AJ(t) = AJ(a) + tD_1 + t^2D_2 + t^3D_3 + \dots$$

where D_1, D_2, \dots are constant vectors in \mathbb{C}^g . (We can also think of them as translation-invariant vector fields on $J(C)$.)

For general a, b, c, d , (3.7) says that the 3 vectors in U_g :

$$\vartheta_2\left(\frac{a+b-c-d}{2}\right), \vartheta_2\left(\frac{a-b+c-d}{2}\right), \vartheta_2\left(\frac{a-b-c+d}{2}\right)$$

are linearly dependent. Let us bring two of the points together, say $c \rightarrow a$. The 3 vectors become:

$$\vartheta_2\left(\frac{b-d}{2}\right), D_1\vartheta_2\left(\frac{b-d}{2}\right), \vartheta_2\left(\frac{b+d}{2} - a\right).$$

Next we may proceed in two different ways:

(A) Bring $d \rightarrow b$. The vectors become

$$\vartheta_2(0), D_1D'_1\vartheta_2(0), \vartheta_2(a - b)$$

where D'_1 is the first term in the Taylor expansion of $AJ(C)$ near b . (Recall that ϑ_2 is even, so its first derivatives at 0 vanish.)

Now let us take b near a , corresponding to the value t of the coordinate at a . Differentiating (3.8) gives.

$$D'_1 = D_1 + 2tD_2 + 3t^2D_3 + \dots$$

while $\vartheta_2(a - b)$ becomes

$$\vartheta_2(0) + t^2 D_1^2 \vartheta_2(0) + 2t^3 D_1 D_2 \vartheta_2(0) + t^4 (D_2^2 + 2D_1 D_3 + D_1^4) \vartheta_2(0) + \dots$$

or, subtracting $\vartheta_2(0) + t^2 D_1 D_1' \vartheta_2(0)$:

$$t^4 [(D_2^2 - D_1 D_3 + D_1^4) \vartheta_2 + \dots].$$

Setting this to equal a linear combination of $\vartheta_2(0)$ and $D_1 D_1' \vartheta_2(0)$ gives an infinite sequence of differential relations obtained by equating successive powers of t to 0. The leading term (coefficient of t^4) gives exactly Proposition (3.2). (D_3 may have to be replaced by a linear combination of D_3 and D_1 .)

(B) In the previous computation we brought d to b , i.e. considered fourtuples of the form (a, b, a, b) , resulting in the trisecant becoming a tangent line at 0 meeting $K(J(C))$ elsewhere (at $\vartheta_2(a - b)$). Instead, we may bring d to a , i.e. consider fourtuples (a, b, a, a) . The limiting trisecants now become flexes of the Kummer, at the point $\vartheta_2(\frac{a-b}{2})$, i.e. we obtain the linear dependence of

$$\vartheta_2(\frac{a-b}{2}), D_1 \vartheta_2(\frac{a-b}{2}), (D_2 + \frac{1}{2} D_1^2) \vartheta_2(\frac{a-b}{2}).$$

Again, we obtain an infinite set of differential relations satisfied by ϑ_2 . The first of these is again (3.2), but the relation of this sequence to the one described above is not clear. Arbarello and De Concini show in [AdC1] that this sequence of equations is a consequence of the "KP hierarchy", an infinite systems of PDE's, starting with (3.5), which can be interpreted as an infinite-dimensional completely integrable Hamiltonian system.

The fact that Jacobian theta functions satisfy the KP equation, indeed the KP hierarchy, was discovered by Krichever. That led Novikov to conjecture (3.3). Dubrovin observed in [Du] that the Hirota form (3.4) of the equations is equivalent to the differential relation (3.2), and proved that Jacobians form a component of the locus of

ppav's whose theta functions satisfy (3.2). Mulase [Mu] showed that an indecomposable ppav whose theta function satisfies the KP hierarchy is a Jacobian. Arbarello and De Concini showed [AdC1], based on earlier work of Gunning and Welters [We2], that a finite subset of this hierarchy suffices (the equations in (B) above corresponding to powers of t up to $6^g \cdot g! + 1$). The Novikov Conjecture itself was proved by Shiota [Sh]; a simplified proof is in [AdC2]. Various analogues have been proposed, but remain open. For instance, Welters asks in [We2] whether the existence of one trisecant of the Kummer variety forces it to come from a curve.

§ 3.3. Andreotti-Mayer vs. Novikov

(3.9) Theorem [BD] $A \in \mathcal{M}_g$ is in the Andreotti-Mayer locus \mathcal{M}_g if it satisfies any of the following conditions:

(1) There are distinct points $x, y, z \in A$ such that

$$\theta \cap \theta_z \subset \theta_x \cup \theta_y.$$

(2) The Kummer variety $K(A)$ has a trisecant.

(3) The theta function ϑ_A satisfies the KP equation (3.2).

Consider the map

$$R : A \setminus (0) \rightarrow \text{Div}(\theta)$$

sending $a \in A \setminus (0)$ to the divisor $\theta \cap \theta_a$ in θ . This extends to a morphism

$$R : \tilde{A} \rightarrow \text{Div}(\theta)$$

where \tilde{A} is the blowup of A at 0 : a point in the exceptional divisor, corresponding to a vector field D_1 on A , goes to the divisor

$$\{z \in A \mid \vartheta(z) = D_1 \vartheta(z) = 0\} \subset \theta.$$

The theorem of Beauville and Debarre follows from:

(3.9 bis) Theorem [BD]. Assume A satisfies:

(0) The divisor $R(a) \subset \theta$ is reducible for some $a \in \tilde{A}$.

Then either $A \in \mathcal{AM}_g$ or A contains an elliptic curve E such that $E \circ \theta = 2$.

Condition (1) implies (0) for $a = z \in A \setminus (0)$. By (0.14), condition (2) is equivalent either to (1) or to a limiting form, so it also implies (0). Finally, (3) implies (0) for a in the exceptional divisor, corresponding to the vector D_1 in (3.2). This is immediate from Hirota's version (3.4) of (3.2): setting $\vartheta = D_1 \vartheta = 0$ we get a product,

$$0 = 3(D_1^2 \vartheta)^2 - (D_2 \vartheta)^2 = (\sqrt{3}D_1^2 \vartheta - D_2 \vartheta)(\sqrt{3}D_1^2 \vartheta + D_2 \vartheta).$$

The idea for proving (3.9 bis) is that if $A \in \mathcal{AM}_g$ then θ is singular in codimension > 3 , hence is locally factorial; the reducible $R(a)$ is thus the sum $C + C'$ of two effective Cartier divisors in θ . These in turn come from divisors on A , and the resulting configuration forces the existence of E . Finally, the existence of E can be ruled out assuming conditions (1), (2) or (3).

CHAPTER 4

Andreotti-Mayer in low genus

In this chapter we present the results of [M2], [B] and [D1]: \mathcal{AM}_4 , the first non-trivial Andreotti-Mayer locus, consists of \mathcal{J}_4 and another divisor ($\mathcal{E}_{\text{null}}$) in \mathcal{A}_4 ; \mathcal{AM}_5 consists of \mathcal{J}_5 , products, and Pryms of bielliptic curves. The idea is to study $\text{Sing}(\theta)$ for Prym varieties and their degenerations, and to use the dominance of the Prym map to \mathcal{A}_g for $g \leq 5$.

§ 4.1 Sing(θ) for a Prym

(4.1) Theorem [M2] Let $P = P(C, \mu)$ be the Prym variety of $(C, \mu) \in \mathcal{AM}_g$. If P is in the Andreotti-Mayer locus \mathcal{AM}_{g-1} , then (C, μ) is one of the following:

- (a) hyperelliptic
- (b) trigonal
- (c) bielliptic (i.e. branched double cover of an elliptic curve)
- (d) $g = 5$, C has a vanishing even theta null L , and $L \otimes \mu$ is even (i.e. L satisfies $L^{\otimes 2} = \omega_C$, $h^0(L) = 2, h^0(L \otimes \mu) = 0$).
- (e) $g = 6$, C is a plane quintic curve, and μ (i.e. $h^0(\mu \otimes \mathcal{O}_C(1))$) is even.

The converse is also true. In case (a), P is a hyperelliptic Jacobian, in \mathcal{N}_{g-1}^{g-4} , or a product of two, in \mathcal{N}_{g-1}^{g-3} . In cases (b) and (e), P is a Jacobian (cf. Corollary (4.12) for (b)). In cases (c), (d), P is not a Jacobian, but it follows from the description of $\text{Sing}(\Xi)$ below that P is still in \mathcal{AM}_{g-1} .

The starting point for the proof is Wirtinger's Theorem (2.1):

$$\tilde{\theta} \cap P = 2\Xi.$$

Let $\tilde{C} \rightarrow C$ be the double cover given by μ . After translation, we can think of a point of P as given by a line bundle L on \tilde{C}

satisfying $Nm(L) = \omega_C$ (and a parity condition). There are two ways that L can represent a singular point of Ξ :

Type (1): $\text{mult}_L(\tilde{\theta}) \geq 4$.

Type (2): $\text{mult}_L(\tilde{\theta}) = 2$, and $T_L P \subset T_L \tilde{\theta}$.

(4.2) Lemma In any component of $\text{Sing}(\Xi)$ whose dimension is $\geq g - 5$, the generic point is of type (2).

This lemma allows Mumford to ignore type (1) singularities. Type (2) singularities can be described directly, and the question is transformed to finding all curves C on which

$$\dim (W_d^1) > d - 4$$

for some $d \leq g - 2$. By a theorem of Martens and Mumford ([M2, appendix] or [ACGH, Ch. IV, Theorems (5.1), (5.2)]), this occurs only for the exceptional curves listed in the theorem.

§ 4.2. Prym is proper

The Prym map

$$\mathcal{P} : \mathcal{RM}_g \rightarrow \mathcal{A}_{g-1}$$

is not proper, but can be made proper as follows. Let $\overline{\mathcal{RM}}_g$ denote the stable-curve compactification of \mathcal{RM}_g . By the universal extension property of the Satake-Baily-Borel compactification $\overline{\mathcal{A}}_{g-1}^S$ [Bo], there is an extension

$$\overline{\mathcal{P}} : \overline{\mathcal{RM}}_g \rightarrow \overline{\mathcal{A}}_{g-1}^S.$$

We then define $(\mathcal{RM}_g)_{\text{allowable}}$ to be the inverse image in $\overline{\mathcal{RM}}_g$ of the open subset \mathcal{A}_{g-1} . The resulting map

$$\mathcal{P}_{\text{allowable}} : (\mathcal{RM}_g)_{\text{allowable}} \rightarrow \mathcal{A}_{g-1}$$

is then a proper extension of \mathcal{P} .

It is more interesting to interpret this extension geometrically, i.e. to describe which degenerate double covers are allowable. There are 5 "types" of boundary components: first we have the 3 components $\partial^I, \partial^{II}, \partial^{III}$ of $\partial \mathcal{M}_g$ which are the restrictions of the corresponding components of $\partial \mathcal{M}_g$. Additionally, we have two families of boundary components consisting entirely of covers of reducible base-curves:

• ∂_k , for $1 \leq k \leq g-1$, parametrizes double covers $\tilde{C} \rightarrow C$ where C is reducible:

$$C = X \cup_p Y$$

with Y, X of genera $k, g-k$ respectively, meeting transversally at p , and \tilde{C} is the double cover corresponding to a point of order 2 $\mu \in J_2(Y) \setminus (0)$.

• $\partial_{k, g-k}$, for $1 \leq k \leq g-k$, parametrizes reducible covers with C as above but μ supported on both X and Y .

It is quite easy to see that $\partial^I, \partial^{III}$ and ∂_k are allowable, while $\partial^{II}, \partial_{k, g-k}$ are not. The degenerate double covers in ∂^I ("Wirtinger covers") are of the form

$$C := X/(p \sim q), \quad \tilde{C} := (X_0 \amalg X_1)/(p_0 \sim q_1, q_0 \sim p_1)$$

where $X \in \mathcal{M}_{g-1}$, $p, q \in X$, and X_0, X_1 are two copies of X . The limiting Prym in this case is just $J(X)$. The degenerate double covers in ∂^{III} ("Beauville covers") are of the form

$$C := X/(p \sim q), \quad \tilde{C} := \tilde{X}/(\tilde{p} \sim \tilde{q})$$

where $X \in \mathcal{M}_{g-1}$ and $\tilde{X} \rightarrow X$ is ramified at p, q . The limiting Prym $P(\tilde{C}/C)$ is just $P(\tilde{X}/X)$. (Fay showed in [F] that the Pryms of double covers with two branch points are ppav's.) A ∂_k -cover is of the form

$$C := Y \cup_p X, \quad \tilde{C} := X_0 \cup_{p_0} \tilde{Y} \cup_{p_1} X_1$$

and its limiting Prym is $J(X) \times P(\tilde{Y}/Y)$.

This takes care of "corank 1 degenerations" of double covers, but the same ideas extend to arbitrary degenerations: any stratum of $\overline{\mathcal{RM}}_g$ is locally the intersection of several boundary components (some of these components have self-intersection), and the result is allowable iff only allowable components are involved. More explicitly:

(4.3) Definition A branched double cover $\tilde{C} \rightarrow C$ of stable curves is:

(1) A stable Wirtinger degeneration if it is of the form

$$\begin{array}{c} \tilde{C} := (X_0 \coprod X_1) / (p_0 \sim q_1, q_0 \sim p_1) \\ \downarrow \\ C := X / (p \sim q), \end{array}$$

with X stable.

(2) An allowable reducible degeneration, if $C = Y \cup_p X$ (where Y is stable, $X = \coprod_{i \in I} X^i$ is the disjoint union of stable curves X^i , and the glueing set $p = \{p^i\}_{i \in I}$ contains one point in each X^i) and the corresponding cover is $\tilde{C} = X_0 \coprod_{p_0} \tilde{Y} \coprod_{p_1} X_1$ with $X_0 \approx X_1 \approx X$ and $\tilde{Y} \rightarrow Y$ any stable double cover.

(3) A stable Beauville degeneration if the branch points of the map $\tilde{X} \rightarrow X$ of normalizations are precisely the inverse images of the nodes.

The result is then:

(4.4) Theorem [B] A stable, branched double cover $\pi : \tilde{C} \rightarrow C$ is allowable if and only if it is either

(a) a stable Wirtinger,

$$C = X/(p \sim q)$$

with X treelike (i.e. the graph of components of X is a tree); or:

(b) allowable reducible,

$$C = Y \cup_p X, \quad \tilde{C} = X_0 \cup_{p_0} \tilde{Y} \cup_{p_1} X_1$$

where each connected component of X is treelike, and where

$$\tilde{Y} \rightarrow Y$$

is stable Beauville.

§ 4.3 Sing(θ) for generalized Pryms

Beauville has extended Mumford's analysis to allowable covers of singular curves:

(4.5) Theorem [B] Consider a stable Beauville degeneration $\tilde{C} \rightarrow C$ with Prym $P = P(\tilde{C}/C) \in \mathcal{A}_{g-1}$.

(1) If $P \in \mathcal{N}_{g-1}^{g-3}$, C is either hyperelliptic or a union $C = C_0 \cup C_1$ with $\#(C_0 \cap C_1) = 2$.

(2) If $P \in \mathcal{N}_{g-1}^{g-4}$, C is either hyperelliptic or hyperelliptic with 2 points identified.

- (3) If $P \in \mathcal{M}_{g-1} = \mathcal{N}_{g-1}^{g-5}$, C is one of the following:
- (a) trigonal
 - (b) hyperelliptic with two points identified
 - (c) bielliptic, $g \geq 6$
 - (d) $g = 5$, C has a vanishing theta null, $\tilde{C} \rightarrow C$ is even
 - (e) $g = 6$, a plane quintic with even double cover.
 - (f) hyperelliptic with two pairs of points identified.
 - (g) $g = 5$, a genus-4 curve with vanishing theta null and with a pair of points identified.
- (h), (i), (j), $C = C_0 \cup X_1$, $\#(X_0 \cap X_1) = 4$, and either:
- (h) neither X_0 nor X_1 is rational.
 - (i) X_0 is rational, X_1 hyperelliptic of genus ≥ 3 .
 - (j) X_0 is rational, X_1 is of genus 3, $\omega_{X_1} \approx \mathcal{O}_{X_1}(X_0 \cap X_1)$
(hence $g = 6$).

It was known already to Wirtinger [W] that $\mathcal{P} : \mathcal{M}_g \rightarrow \mathcal{A}_{g-1}$ is dominant for $g \leq 6$. (This is easiest to see by computing the differential of \mathcal{P} along the locus ∂^I of "Wirtinger covers".) Combining with §4.2, one gets

(4.6) Lemma \mathcal{P} allowable : (\mathcal{M}_g) allowable $\rightarrow \mathcal{A}_{g-1}$ is surjective for $g \leq 6$.

One can therefore completely analyze \mathcal{M}_g for $g \leq 5$: By Theorem (4.4), anything in \mathcal{M}_g is either a Wirtinger Prym (which is a Jacobian or product of Jacobians), or a product, or a stable Beauville Prym which is therefore in the list (4.5). Going through the list, Beauville deduces:

(4.7) Corollary. \mathcal{M}_4 has 2 irreducible components: \mathcal{P}_4 and the divisor $\mathcal{D}_{\text{null}}$ of ppav's with a vanishing theta null.

(4.8) Corollary. All components of \mathcal{M}_5 other than \mathcal{P}_5 are contained in the divisor $\mathcal{D}_{\text{null}}$.

(Unfortunately, θ_{null} is contained in \mathcal{N}_5^0 but not in $\mathcal{N}_5^1 = \mathcal{M}_5$.)

§ 4.4 The tetragonal construction

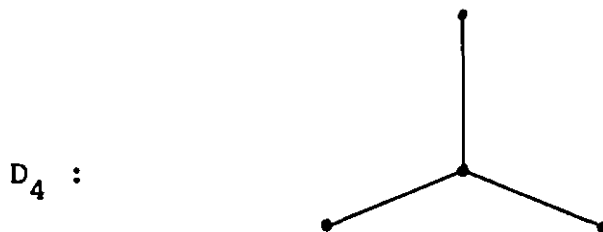
We describe a simple procedure, the tetragonal construction, which takes a tower

$$(4.9) \quad \tilde{C} \xrightarrow{\pi} C \xrightarrow{f} \mathbb{P}^1,$$

where f is a 4-sheeted branched cover (i.e. C is tetragonal) and π is an unramified double cover, and yields two new towers

$$(4.9)_i \quad \tilde{C}_i \xrightarrow{\pi_i} C_i \xrightarrow{f_i} \mathbb{P}^1 \quad i = 0, 1$$

of the same type. Such a tower is uniquely determined by a representation ρ of $\pi_1(\mathbb{P}^1 \setminus \{\text{branch points}\})$ in the Weyl group WD_4 of the Dynkin diagram:



(In general, the Weyl group WC_n is the group of signed permutations of n letters, and WD_n is its subgroup of index 2 consisting of even signed permutations.) Now D_4 has a special automorphism α , of order 3, (120° rotation), not present in any other D_n . This gives an outer automorphism α of WD_4 . Therefore representations of any group in WD_4 come in packages of three: $\rho, \alpha \circ \rho, \alpha^2 \circ \rho$. In particular, we get $(4.9)_i$ ($i = 0, 1$) starting with (4.9).

(More explicitly, starting with (4.9) we construct a

($16 = 2^4$)-sheeted branched cover $f_*\tilde{C} \rightarrow \mathbb{P}^1$ with a natural involution. This breaks into two components, each of degree 8 over \mathbb{P}^1 and invariant under the involution, yielding $(4.9)_1$, cf. [D1].)

(4.10) Theorem [D1] The tetragonal construction commutes with the Prym map:

$$P(\tilde{C}/C) \approx P(\tilde{C}_0/C_0) \approx P(\tilde{C}_1/C_1).$$

Consider the special case where π in (4.9) is the split double cover. The 16-sheeted branched cover $f_*\tilde{C}$ then splits into 5 components of degrees 1,4,6,4,1 respectively over \mathbb{P}^1 . The components of degree 4 make up $\tilde{C}_1 \rightarrow C_1$, which is isomorphic to $\tilde{C} \rightarrow C$. The remaining components give

$$\mathbb{P}^1 \coprod \tilde{T} \coprod \mathbb{P}^1 \longrightarrow T \coprod \mathbb{P}^1$$

where T is a trigonal curve and \tilde{T} its double cover. One sees easily that this special case sets up a bijection

$$(4.11) \quad \left\{ \begin{array}{l} C, \text{ a tetragonal curve} \\ \text{of genus } g \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} T, \text{ a trigonal curve of genus} \\ g+1 \text{ with an unramified} \\ \text{double cover } \tilde{T} \end{array} \right\}$$

This bijection, the trigonal construction, was described by Recillas [R]. Group theoretically it corresponds to the exceptional isomorphism

$$S_4 \xrightarrow{\sim} WD_3$$

which arises from the coincidence of Dynkin diagrams A_3, D_3 . (In general, the symmetric group S_n is the Weyl group WA_{n-1} .) Theorem (4.10) thus yields:

(4.12) Corollary [R] If (\tilde{T}, T) corresponds to C via the trigonal construction, then $P(\tilde{T}/T) \approx J(C)$.

(In particular, this shows that case (b) in Mumford's Theorem (4.1) leads to Jacobians.)

The tetragonal construction (4.9) (though not Theorem (4.10)) can be deduced from the trigonal construction (4.11): starting with a trigonal T of genus $g + 1$, choose a rank-2 isotropic subgroup of $(J(T))_2$, i.e. 3 points of order 2 μ, μ_0, μ_1 satisfying

$$(4.13) \quad \mu + \mu_0 + \mu_1 = 0, \quad (\mu, \mu_0) = 0 \in \mathbb{Z}/2\mathbb{Z}.$$

These points of order 2 determine double covers $\tilde{T}, \tilde{T}_0, \tilde{T}_1$ of T . Applying (4.11) we get 3 curves of genus g : C, C_0, C_1 , each with a g_4^1 . Finally, each of these curves comes with a point of order 2 in its Jacobian, hence a double cover: for C , this point is the common image in $J(C) \approx P(\tilde{T}/T)$ of μ_0 and μ_1 .

Using the tetragonal construction, we can obtain many identifications among Prym varieties of special curves. As an illustration, let us consider double covers of bielliptic curves. If C^0, C^1 are branched double covers of an elliptic curve E with disjoint branch loci, we form the fiber product

$$\tilde{C} := C^0 \times_E C^1.$$

\tilde{C} has 3 involutions $:\tau^i$, with quotient C^i , ($i = 0, 1$), and their composition $\tau := \tau^0 \circ \tau^1$ which is fixed-point free, yielding an unramified double cover $\tilde{C} \rightarrow C$ of a quotient curve C which is itself bielliptic. We say that $\tilde{C} \rightarrow C$ is a Cartesian cover.

Let $\mathfrak{B} = \mathfrak{B}_g$ be the moduli space of bielliptic curves of genus g . The space $\mathfrak{B}\mathfrak{B}_g$ of bielliptic curves with a double cover has $[\frac{g+1}{2}] + 1$ components:

- \mathcal{H}_i , $1 \leq i \leq \lfloor \frac{g+1}{2} \rfloor$, consists of Cartesian covers $\tilde{C} \rightarrow C$ arising from a pair of covers $C^0 \rightarrow E$, $C^1 \rightarrow E$ where C^0 is of genus i , C^1 of genus $g + 1 - i$.
- \mathcal{H}_0 consists of all non-Cartesian covers.

Each of these components is $(2g - 2)$ -dimensional. Since each bi-elliptic curve has a 1-dimensional family of g_4^1 's, we have ample room to play with the tetragonal construction. The result:

(4.14) Proposition

- (1) $\mathcal{P}(\mathcal{H}_1)$ is $(2g - 2)$ -dimensional and is also the locus of allowable Pryms of hyperelliptic curves with two pairs of points identified.
- (2) $\mathcal{P}(\mathcal{H}_i)$, for $i > 1$, is $(2g - 3)$ -dimensional, and is also the locus of Pryms of reducible allowable covers $\tilde{C} \rightarrow C$ where $C = X_0 \cup X_1$, $\tilde{C} = \tilde{X}_0 \cup \tilde{X}_1$, X is hyperelliptic of genus $i - 2$, X_1 is hyperelliptic of genus $g - 1 - i$.
- (3) $\mathcal{P}(\mathcal{H}_0) = \mathcal{P}(\mathcal{H}_1)$.

In particular, when $g = 6$, Beauville's list becomes quite short. (The announcement in [D1] is wrong. I thank O. Debarre for pointing this out.)

(4.15) Theorem. \mathcal{H}_5 consists of the 12-dimensional locus \mathcal{J}_5 of Jacobians, the 11-dimensional locus $\mathcal{A}_1 \times \mathcal{A}_4$ of products, and the 3 loci $\mathcal{P}(\mathcal{H}_i)$, $i = 1, 2, 3$, of Cartesian bielliptic Pryms (these have dimensions 10, 9, 9 respectively.)

CHAPTER 5

Schottky-Jung in low genus

In §5.1 we try to explain why the unexpected component \mathcal{X}^0 (of intermediate Jacobians of cubic threefolds with even point of order 2) pops into the Schottky locus \mathcal{H}_g in genus 5 (hence also for $g \geq 5$). This explains our formulation of the Schottky-Jung Conjecture (2.11).

The case $g = 4$ of the conjecture amounts to Igusa's Theorem. In the last section we sketch a new proof of this result, based on the various symmetries of Pryms and thetas, and in the same spirit we outline our recent proof (yet unpublished) of the conjecture in genus 5.

§ 5.1. Symmetry of the theta maps

In this section we discuss an extension of the tetragonal construction to arbitrary curves. Let $Q \in \mathcal{M}_{g+1}$ be a curve, and

$$\langle 0, \mu_0, \mu_1, \mu_2 = \mu_0 + \mu_1 \rangle$$

an isotropic rank-2 subgroup of $J_2(Q)$. For $i = 0, 1, 2$ we have a Prym variety

$$P_i = P(Q, \mu_i) \in \mathcal{A}_g$$

and on it a uniquely determined semiperiod ν_i , image of any μ_j ($j \neq i$) in P_i . The result is:

(5.1) Theorem [D3]. $\beta(P_i, \nu_i)$ is independent of $i = 0, 1, 2$.

This has many geometric applications. Taking $Q = T$ to be trigonal, we find ourselves in the situation of (4.14): each P_i is a Jacobian of a tetragonal curve, the three are related via the tetragonal construction, and this special case of (5.1) follows from (and is slightly weaker than) Theorem (4.10). Taking (Q, μ_0) to be a

Wirtinger (∂^I) degeneration, (5.1) becomes the Schottky-Jung identity (2.6).

Of interest to us is the case that Q is a plane quintic curve, μ_2 is an odd point of order 2, μ_0 (hence also μ_1) is even. By Theorem (4.1)(e), P_0 and P_1 are Jacobians of curves of genus 5, but P_2 is not.

Let $\mathcal{C} \subset \mathcal{A}_5$ be the 10-dimensional locus of Pryms of quintics with odd covers. (It is known ([CG], [M2], [T]) that this is precisely the locus of intermediate Jacobians of cubic threefolds.) Its lift to \mathcal{A}_5 splits into two components $\mathcal{C}^0, \mathcal{C}^1$, where \mathcal{C}^0 parametrizes precisely the pairs (P_2, ν_2) arising as above.

(5.2) Theorem [D3]. The locus \mathcal{C}^0 of intermediate Jacobians of cubic threefolds with an even point of order 2 is a component of the Schottky locus \mathcal{S}_5 .

The inclusion $\mathcal{C}^0 \subset \mathcal{S}_5$ follows immediately from the symmetry result (5.1) and the Schottky-Jung identities (2.6). The proof that \mathcal{C}^0 is actually a component was suggested by van Geemen. It is analogous to the proof that \mathcal{A} is a component of \mathcal{S} , Theorems (2.7) and (2.8): the closure of \mathcal{C}^0 meets $\partial^I \mathcal{A}_5 \approx \mathcal{A}_4$ in the locus of Jacobians \mathcal{J}_4 , and by a result of Collino [Co] the projectivized normal cone to ∂^I along \mathcal{C}^0 at $J(C) \in \mathcal{J}_4$ is given by the point

$$\pm(T_0 - T_1) \in K(J(C)),$$

where T_0, T_1 are the g_3^1 's on C . This is precisely the exceptional case in Welters' Theorem (1.5): for generic C , the point $\pm(T_0 - T_1)$ is not in $C - C$, hence forms a component of the base locus of Γ_{00} , which proves the theorem.

The theorem implies, of course, that $\mathcal{C} \subset \mathcal{S}_5^{(\text{big})}$. We will see below that \mathcal{C} is not in \mathcal{S}_5 .

§ 5.2. Schottky in genus ≤ 5

The theta map (0.12):

$$\beta : \mathcal{A}_g \longrightarrow \mathbb{P}(U_{g-1})/G_{g-1}$$

extends to a proper map on an appropriate toroidal compactification,

$$\beta : \overline{\mathcal{A}}_g^t \longrightarrow \mathbb{P}(U_{g-1})/G_{g-1}.$$

For $g \leq 5$ this map is surjective. Our strategy is to study the geometry of this map and to use it to completely describe $\overline{\mathcal{A}}_g$.

(5.3) Theorem For $C \in \mathcal{M}_3$, the inverse image

$$\beta^{-1}(\alpha(J(C)))$$

consists of two copies of the Kummer $K(J(C))$: one is

$$\phi^{-1}(J(C)) \subset \overline{\mathcal{M}}_4,$$

the other is the fiber over $J(C) \in \mathcal{M}_3$ of the natural map

$$\begin{array}{ccc} \partial^I \overline{\mathcal{A}}_4^t & \longrightarrow & \partial^I \overline{\mathcal{A}}_4^s \\ \Downarrow & & \Downarrow \\ \overline{\mathcal{A}}_3 & \longrightarrow & \overline{\mathcal{A}}_3. \end{array}$$

(5.4) Corollary (Igusa [I]). The Schottky locus \mathcal{S}_4 is irreducible, hence is precisely \mathcal{S}_4 . In particular,

$$\mathcal{S}_4^{(\text{big})} = \mathcal{S}_4 = \mathcal{S}_4.$$

Here is a sketch of the proof of (5.3). First, it is clear that the two copies of $K(J(C))$ are indeed in $\beta^{-1}(\alpha(J(C)))$. Consider the equivalence relation \sim on $\overline{\mathcal{A}}_4^t$ generated by the relation

" $(P_0, \nu_0) \approx (P_1, \nu_1)$ if they are related via the theta symmetry (5.1)".

One verifies that the equivalence class of $(C, \mu) \in \mathcal{K}_4$ consists of two copies of the Kummer $K(P(C, \mu))$, as in the statement of the theorem. We end up with a quotient map of β ,

$$\overline{\mathcal{K}}_4^t / \sim \longrightarrow \mathbb{P}^7 / G_3(2, 4) \quad ,$$

and by a degeneration argument we conclude that this map is an isomorphism over the image of α , so $\beta^{-1}(\alpha(J(C)))$ cannot contain anything new.

I would like to point out that the fiber of β over a point of $\mathbb{P}^7/G_3(2, 4)$ not in $\text{image}(\alpha)$ is not known. It is a deformation of the singular variety (consisting of two Kummers meeting along a surface) which is the fiber over a point of $\text{image}(\alpha)$, but it should be interesting to have an explicit description.

(5.5) Theorem The compactified map

$$\beta : \overline{\mathcal{K}}_5^t \longrightarrow \mathbb{P}^{15} / G_4(2, 4)$$

is generically finite of degree 119. Its Galois group is (contained in) $SO_8^-(2)$, the special orthogonal group preserving a quadric of Witt-defect 1 in $\mathbb{P}^7(\mathbb{F}_2)$.

(5.6) Theorem The closed Schottky locus $\overline{\mathcal{K}}_5^t$ has four components:

$$\overline{\mathcal{K}}_5^t = \overline{\mathcal{K}}_0^t \cup \overline{\mathcal{K}}_5^t \cup \partial^I \overline{\mathcal{K}}_5^t \cup (\overline{\mathcal{K}}_4^t \times \overline{\mathcal{K}}_1^t).$$

(I.e. Conjecture (2.11) is true in genus ≤ 5 .)

(5.7) Corollary $\mathcal{P}_5 = \mathcal{J}_5$.

The proofs, at present, are very complicated. They rely on detailed knowledge of the structure of the Prym maps

$$\mathcal{P}_g : \overline{\mathcal{R}\mathcal{M}}_g \longrightarrow \overline{\mathcal{M}}_{g-1}$$

for $g \leq 6$. This knowledge is obtained by applying the tetragonal construction to everything in sight. For instance, \mathcal{P}_6 is generically finite of degree 27 [DS] with Galois group

$$\text{WE}_6 \approx \text{SO}_6^-(2),$$

the symmetry group of lines on a cubic surface: two lines intersect or not according as the corresponding curves are obtained from each other by one tetragonal move or a sequence of two such moves. We can thus define an equivalence relation on $\overline{\mathcal{R}\mathcal{M}}_5$ by the theta symmetry (5.1), as in (5.3), but now we get a generically finite relation. Starting with any point of $\overline{\mathcal{R}\mathcal{M}}_5$ we get, in the first generation,

$$54 = 27 \cdot 2$$

equivalent objects. Theorem (5.5) involves showing that the second (and last) generation adds another 64 objects fitting together in a highly symmetric configuration, and that this equivalence spans the fibers of β . Theorem (5.6) then requires computation of the local degree of β on each of the 4 known components (these degrees are 1, 54, 64, 0 respectively), and checking the normal bundles to make sure that no extra components arise via blowup (i.e. contribute 0 to the degree).

REFERENCES

- [ACGH] E. ARBARELLO, M. CORNALBA, P. GRIFFITHS , J. HARRIS, Geometry of algebraic curves, vol. I, Springer-Verlag, New York (1985).
- [AdC1] E. ARBARELLO, C. DE CONCINI, On a set of equations characterizing Riemann matrices, Ann. of Math. 120 (1984), 119-140.
- [AdC2] E. ARBARELLO, C. DE CONCINI, Another proof of a conjecture of S.P. Novikov on periods and abelian integrals on Riemann surfaces, Duke Math. J. 54 (1987), 163-178.
- [AM] A. ANDREOTTI, A. MAYER, On period relations for abelian integrals on algebraic curves, Ann. Scuola Norm. Sup. Pisa 21 (1967), 189-238.
- [AMRT] A. ASH, D. MUMFORD, M. RAPOPORT , Y. TAI, Smooth compactification of locally symmetric varieties, Math. Sci. Press, Brookline (1985).
- [B] A. BEAUVILLE, Prym varieties and the Schottky problem. Inv. Math. 41 (1977), 149-196.
- [BD] A. BEAUVILLE, O. DEBARRE, Une relation entre deux approches du probleme de Schottky, preprint.
- [Bo] A. BOREL, Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem, J. Diff. Geo. 6(1972), 543-560.
- [CG] H. CLEMENS, P. GRIFFITHS, The intermediate Jacobian of the cubic threefold. Ann. Math. 95 (1972), 281-356.
- [Co] A. COLLINO, A cheap proof of the irrationality of most cubic threefolds. Bolletino U.M.I. (5) 16-B (1979), 451-465.
- [D1] R. DONAGI, The tetragonal construction. AMS Bull. 4 (1981), 181-185.
- [D2] R. DONAGI, Big Schottky, Inv. Math. 89 (1987), 569-599.
- [D3] R. DONAGI, Non-Jacobians in the Schottky loci, Ann. of Math. 126 (1987), 193-217.
- [DS] R. DONAGI, R. SMITH, The structure of the Prym map, Acta Math. 146 (1981), 25-102.
- [Du] B.A. DUBROVIN, Theta functions and non-linear equations, Russian Math. Surveys 36, n°2 (1981), 11-92.
- [F] J. FAY, Theta functions on Riemann Surfaces, Springer-Verlag, Berlin and New York, 1973, Lecture Notes, Vol. 352.

- [FR] H. FARKAS , H. RAUCH, Period relations of Schottky type on Riemann surfaces, Ann. of Math. 92 (1970), 434-461.
- [G] M. GREEN, Quadrics of rank four in the ideal of a canonical curve. Invent. Math. 75 (1984), 85-104.
- [I] J. IGUSA, On the irreducibility of Schottky's divisor, Journal of the Fac. of Science Tokyo, 28 (1981), 531-545.
- [M1] D. MUMFORD, Theta characteristics on an algebraic curve, Ann. Sci. E.N.S. 4 (1971), 181-192.
- [M2] D. MUMFORD, Prym varieties I. Contributions to Analysis, 325-350, New York, Acad. Press, 1974.
- [M3] D. MUMFORD, Curves and their Jacobians, Ann Arbor, University of Michigan Press, 1975.
- [Mu] M. MULASE, Cohomological structure in soliton equations and Jacobian varieties, J. Diff. Geo. 19 (1984), 403-430.
- [R] S. RECILLAS, Jacobians of curves with a g_4^1 are Prym varieties of trigonal curves, Bol. Soc. Math. Mexicana 19 (1974), 9-13.
- [S] F. SCHOTTKY, Zur Theorie der Abelschen Funktionen von vier Variablen, J. Reine und Angew. Math. 102 (1888), 304-352.
- [SJ] F. SCHOTTKY, H. JUNG, Neue Sätze über symmetrifunctionen und die Abelschen funktionen, S.-B. Berlin Akad. Wiss. (1909).
- [Sh] T. SHIOTA, Characterization of Jacobian varieties in terms of soliton equations. Inv. Math. 83 (1986), 333-382.
- [T] A. TJURIN, Five lectures on three dimensional varieties, Russ. Math. Surv. 27 (1972).
- [Tx] M. TEIXIDOR, For which Jacobi varieties is $\text{Sing } \theta$ reducible, preprint.
- [vG1] B. VAN GEEMEN, Siegel modular forms vanishing on the moduli space of curves, Inv. Math. 78 (1984), 329-349.
- [vG2] B. VAN GEEMEN, The Schottky problem and moduli spaces of Kummer varieties, U. of Utrecht thesis, 1985.
- [vGvdG] B. VAN GEEMEN, G. VAN DER GEER, Kummer varieties and the moduli space of curves, Am. J. of Math. 108 (1986), 615-642.
- [W] W. WIRTINGER, Untersuchungen über Thetafunktionen, Teubner, Berlin, 1895.
- [We] G. WELTERS, The surface $C - C$ on Jacobi varieties and 2nd order theta functions, preprint.
- [We2] G. WELTERS, A criterion for Jacobi varieties, Annals of Math. 120 (1984), 497-504.