Quaternion structure on the moduli space of Yang-Mills connections

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1. This article is a continuation of [13]. We define in [13] a canonical Riemannian structure $\langle\ ,\ \rangle$ on the moduli space $\mathcal{M}^-$ of $SU(n)$- anti-self-dual connections over a compact oriented Riemannian 4-manifold (M, h). It was also showed that this Riemannian structure is Kähler when M is a complex Kähler surface. Moreover the formula of the Riemannian curvature tensor was derived by making use of the Green operators of the Laplacians associated to a connection. The method developed there is applied also to the moduli spaces of Einstein-Hermitian connections over a compact Riemann surface (i.e., a compact complex curve) to show that these moduli spaces admit the canonical Kähler metric of nonnegative holomorphic sectional curvature and hence of nonnegative scalar curvature.

Assume that (M, h) be a complex 2-torus with a flat metric or a K3 surface with a Ricci flat metric. As is well known, these spaces carry covariantly constant, globally defined almost complex structures $\{I_i\}_{i=1}^{3}$ satisfying $I_1I_2 = -I_2I_1 = I_3$, in other words, they admit a global orthonormal frame $\{\omega_i\}_{i=1}^{3}$ of the bundle Λ^2_+ of self-dual 2-forms which is covariantly constant. These spaces are characterized by that their holonomy group is contained in $S_p(1) = SU(2)$.

We say such a structure $\{I_i\}_{i=1}^{3}$ satisfying $I_1 I_2 = -I_2 I_1 = I_3$ to be a quaternion structure on M .

We will discuss in this article geometrical structure of the moduli spaces of anti-self-dual (or Einstein-Hermitian) connections over a 4-manifold (M, h) with a covariantly constant quaternion structure.

Note that only the complex 2-torus with flat metric and the K3 surface with Ricci flat metric are compact oriented Riemannian 4-manifolds admitting a covariantly constant quaternion structure.

We formulate the main results of this article in the following form.

Theorem 1. Let (M, h) be a complex 2-torus with a flat metric or a K3 surface with a Ricci flat metric. Let P be a G-principal bundle over M (G is a compact simple Lie group) and $\hat{M}^-$ the moduli space of irreducible anti-self-dual connections on P . Then the Riemannian manifold $(\hat{M}^-, < , >)$ admits a covariantly constant, quaternion structure. Namely its holonomy group is contained in $Sp(n)$, $4n = \dim_{\mathbb{R}} \hat{M}^-$.

Theorem 2. Let (M, h) be as in Theorem 1. and P a $U(n)$-principal bundle over M . Then the moduli space $\hat{M}_E$ of irreducible Einstein-Hermitian connections on P is endowed with a canonical Riemannian structure $<\ ,\ >$ so that its holonomy group is contained in a symplectic group.

While these spaces $(\hat{M}^-, <\ ,>)$ and $(\hat{M}_E, <\ ,\ >)$ are Ricci flat Kähler manifolds because of their holonomy groups by the aid of general theory on quaternionic Kähler manifolds ([4], [1] and [18]), we exhibit in § 5 a nontrivial Riemannian curvature identity from which the Ricci curvature actually vanishes.

<u>Theorem</u> 3. Let M be a complex 2-torus or a K3 surface admitting a Hodge metric (that is, a Kähler metric whose Kähler form Ω is cohomologuous to one pulled back the Kähler form of the Fubini-Study metric under an embedding $M \to P_N(\mathbb{C})$) . Let $M(r, c_1, c_2)$ be moduli of $[\Omega]$-stable holomorphic vector bundles over M with fixed rank r and Chern classes $c_i \in H^{2i}(M; \mathbb{Z})$, $i=1, 2$. Then $M(r, c_1, c_2)$ carries a canonical Kähler structure whose holonomy group is contained in $Sp(n)$.

By Yau's theorem [20] there is a Ricci flat Kähler metric h on M whose Kähler form is cohomologuous to Ω so that (M, h) carries a global, covariantly constant orthonormal frame of Λ^2_+ . Since the stability of vector bundles depends on the cohomology class $[\Omega]$ we have a one-to-one correspondence between $M(r, c_1, c_2)$ and the moduli space M_E of Einstein-Hermitian connections on a $U(n)$-principal bundle P over (M, h) with $c_i(P\times_\sigma \mathbb{C}^r)=c_i$, $i=1, 2$ ([14], [16] and [8]). Hence, Theorem 3 is a direct consequence of

Theorems 1 and 2.

REMARKS (i) It is conjectured that these spaces over a K3 surface are irreducible in the sense of deRham, that is, their holonomy group is exactly a symplectic group.

(ii) These moduli spaces are complex symplectic manifolds, i.e., admit non-degenerate holomorphic 2-forms, since their holonomy group is a subgroup of $Sp(n)$. Mukai proved in [17] that the moduli of simple (or stable) sheaves on an abelian surface or a K3 algebraic surface has a symplectic structure. Recently Kobayashi generalized Mukai's theorem with respect to moduli space of simple holomorphic vector bundles on a higher dimensional compact complex symplectic manifold ([15]). We can generalize Theorem 1 and 2 over a higher dimensional compact Riemannian manifold whose holonomy group is a subgroup of a symplectic group. However, the moduli spaces over such a manifold are generally no longer smooth manifolds because of cohomology groups of higher degree.

(iii) In general $M(r, c_1, c_2)$ is not a compact space. To compactify this space we must add (semi-)stable sheaves to it. The space $M(r, c_1, c_2)$ is a symplectic Kähler manifold due to [5] and [3] if it happens to be compact and nonsingular.

(iv) Let M^-_k be the moduli space of $SU(2)$-anti-self-dual connections with anti-instanton number $k=c_2(P\times_\delta \mathbb{C}^2)[M]$. Then its compactification $\overline{M_k}^-$ admits a structure of stratification ([9]);

$$\overline{M_k^-} \subseteq M^-_k \cup (M^-_{k-1} \times M) \cup (M^-_{k-2} \times M^{(2)}) \cup \ldots$$

$$\cup (M_1^- \times M^{(k-1)}) \cup M^{(k)}$$

here $M^{(i)}$ is the symmetric i-fold product of M. Since each stratum carries a canonical quaternion structure over a complex 2-torus with a flat metric or a K3 surface with a Ricci flat metric, this whole space must admit a quaternion structure consistent with this stratification.

(v) The Euclidean 4-space $\mathbb{R}^4$ is a typical 4-manifold admitting a covariantly constant quaternion structure. Due to [2] and [7] the moduli space $\tilde{M}_k^-$ of based (anti-)instantons on $\mathbb{R}^4$ carries a complex structure and its dimension $\dim_{\mathbb{R}} \tilde{M}_k^-$ is 4ℓ. The space $\tilde{M}_k^-$ should be endowed with a canonical quaternion structure.

(vi) At any reducible anti-self-dual connection the moduli space M^- is singular and can be actually described by the quotient of a slice $S^-_{A,\varepsilon}$ under the isotropy Γ_A which is mapped Γ_A -equivariantly to $\mathrm{Zero}(\phi)/_{\Gamma_A}$ of analytic functions $\phi; H^1_A \longrightarrow H^2_{+A} = H^0_A\, \omega_1 \oplus H^0_A\, \omega_2 \oplus H^0_A\, \omega_3$. However, the canonical Riemannian structure is well defined on $S^-_{A,\varepsilon}/_{\Gamma_A}$ and hence on $\mathrm{Zero}(\phi)/_{\Gamma_A}$.

2. Riemannian structure on the moduli spaces.

We give in this section a definition of the canonical Riemannian structure on the moduli space of anti-self-dual connections (for more detail, see [13]).

Let (M, h) be a complex 2-torus with a flat metric or a K3 surface with a Ricci flat metric. Let P be a C^∞ G-principal bundle (G ; a compact simple Lie group). We denote by $\mathcal{A}$ and $\mathcal{G}$ the set of all C^∞ connections on P and all C^∞ gauge transformations of P, respectively. With respect to the adjoint bundle $\mathfrak{g}_P = P\times_{Ad}\mathfrak{g}$ ($\mathfrak{g}$ is Lie algebra of G) we denote by $\Omega^p(\mathfrak{g}_P)$ the space of C^∞ $\mathfrak{g}_P$-valued p-forms; $\Omega^p(\mathfrak{g}_P) = \Gamma(M; \Lambda^p_M \otimes \mathfrak{g}_P)$, $p \geq 0$. $\mathcal{A}$ is an affine space associated to the vector space $\Omega^1(\mathfrak{g}_P)$. From the metric h together with the Killing form of $\mathfrak{g}$ we defi naturally a pointwise positive definite inner product on the bund $\Lambda^p_M \otimes \mathfrak{g}_P$ and then a global inner product on $\Omega^p(\mathfrak{g}_P)$. The inner product $(\ ,\ )$ is invariant under the action of $\mathcal{G}$ defined by $(g, \phi) \longrightarrow g(\phi) = Ad(g^{-1})\phi$.

For each connection A we have its curvature $F_A = dA + 1/_2[A\wedge A] \in \Omega^2(\mathfrak{g}_P)$; $F : \mathcal{A} \longrightarrow \Omega^2(\mathfrak{g}_P)$. From the Hodge operator $*$ induced from the canonical orientation of M the bundle of 2-for splits into $\Lambda^2_M = \Lambda^2_+ + \Lambda^2_-$, where $\Lambda^2_\pm$ denotes the eigenspace sub bundle of eigenvalue $\pm$ 1. A connection A is called anti-self-dual if $p_+ F_A = 0$, where $p_\pm$ is the orthogonal projection: $\Lambda^2_M \longrightarrow \Lambda^2_\pm$, $p_\pm(a) = 1/_2(a \pm * a)$. The set $\mathcal{A}^-$ of all anti-self-dua connections is a subset of $\mathcal{A}$ invariant under $\mathcal{G}$. Here the action of $\mathcal{G}$ on $\mathcal{A}$ is given by $(g, A) \rightarrow g(A) = L_g{-1}(dg) + Ad(g^{-1})A$. The quotient space $\mathcal{M}^- = \mathcal{A}^-/\mathcal{G}$, which we call moduli space of anti-self-dual connections, is considered as a space parametrizing $\mathcal{G}$ -orbits in $\mathcal{A}^-$. The moduli space is a subset of the quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ of connections on P .

We say connection A to be irreducible when the covariant derivative $d_A; \Omega^0(\mathfrak{g}_P) \longrightarrow \Omega^1(\mathfrak{g}_P)$ admits trivial kernel. Relative to the set $\hat{\mathcal{A}}$ of all irreducible connections it is known that the quotient space $\hat{\mathcal{B}} = \hat{\mathcal{A}}/\mathcal{G}$, a dense subset of $\mathcal{B}$, carries a Banach manifold structure, if we complete $\mathcal{A}$ and $\mathcal{G}$ with suitable Sobolev norms.

Since the tangent space $T_A\hat{\mathcal{A}}$, $A \in \hat{\mathcal{A}}$ is canonically identified with $\Omega^1(\mathfrak{g}_P)$, the $\mathcal{G}$ -invariant inner product (,) on it descends to a Riemannian structure < , > on $\hat{\mathcal{B}}$ so that the projection : $\hat{\mathcal{A}} \longrightarrow \hat{\mathcal{B}}$ is a Riemannian submersion. Its restriction to $\hat{\mathcal{M}}^- = \mathcal{M}^- \cap \hat{\mathcal{B}}$ defines a Riemannian structure. We call it a canonical Riemannian structure on $\hat{\mathcal{M}}^-$.

In order to study the structure < , > on $\hat{\mathcal{M}}^-$ more explicitly we define local coordinates on $\hat{\mathcal{M}}^-$ by the aid of slices and the Kuranishi map.

Denote by $\hat{\mathcal{A}}^- = \mathcal{A}^- \cap \hat{\mathcal{A}}$ the set of all irresucible anti-self-dual connections. We have the canonical projection $\pi : \hat{\mathcal{A}}^- \longrightarrow \hat{\mathcal{M}}^-$.

We use the symbol $[A]$ for an element $\pi(A)$ in $\hat{M}^-$.
For every $[A]\in\hat{M}^-$ there exists a subset $S^-_{A,\varepsilon}$ in $\hat{A}^-$, called a slice at A, such that $A\in S^-_{A,\varepsilon}$ and $\pi(S^-_{A,\varepsilon})$ gives a neighborhood of $[A]$ in $\hat{M}^-$ and moreover $\pi|_{S^-_{A,\varepsilon}} : S^-_{A,\varepsilon}\longrightarrow\pi(S^-_{A,\varepsilon})$ is a homeomorphism. Actually, $S^-_{A,\varepsilon}$ has the form of $\hat{A}^-\cap(A+(\mathbf{B}_\varepsilon\cap\operatorname{Ker} d_A^*))$, here $\mathbb{B}_\varepsilon$ is an ε-ball in $\Omega^1(\mathfrak{g}_P)$ with center at 0 and $d_A^* : \Omega^1(\mathfrak{g}_P)\longrightarrow\Omega^0(\mathfrak{g}_P)$ is the $(\ ,\ )$-formal adjoint of d_A. Note that the slice at A is identified with the subset in $\Omega^1(\mathfrak{g}_P)$ of the following form

$$\{\alpha\in\Omega^1(\mathfrak{g}_P)\ ;\ |\alpha|<\varepsilon\ ,\ d_A^*\alpha = 0 \text{ and } d_A^+\alpha + 1/_2[\alpha\wedge\alpha]^+ = 0\}\ ,$$

where $d_A^+ = p_+\circ d_A$ and $[.\wedge.]^+ = p_+\circ[.\wedge.]$.
We define next for a fixed connection A the Kuranishi map $\Xi_A : \Omega^1(\mathfrak{g}_P)\longrightarrow\Omega^1(\mathfrak{g}_P)$ by $\Xi_A(\alpha)=\alpha+d_A^{+*}G_A(1/_2[\alpha\wedge\alpha]^+)$. Here d_A^{+*} is the formal adjoint of d_A^+ and G_A is the Green operator of the Laplacian $\Delta^2_{+,A}=d_A^+d_A^{+*}:\Omega^2_+(\mathfrak{g}_P)=\Gamma(M;\Lambda^2_+\otimes\mathfrak{g}_P)\longrightarrow\Omega^2_+(\mathfrak{g}_P)$. Ξ_A is a C^∞ mapping in the sense of Frechêt derivatives.

Since M is a complex 2-torus or a K3 surface, we have $\operatorname{Ker}\Delta^2_{+,A} = \{0\}$ for any $A\in\hat{A}^-$ ([12, Proposition 2.3]). Therefore we have for every $A\in\hat{A}^-$ a small $\varepsilon>0$ so that Ξ_A mapps homeomorphically the slice $S^-_{A,\varepsilon}$ at A onto a neighborhood U_A of 0 in the first cohomology group $H^1_A = \operatorname{Ker}\Delta^1_A$, $\Delta^1_A = d_A d_A^* + d_A^{+*} d_A^+$.

PROPOSITION 2.1 ([13, Proposition 2.5]). The moduli space $\hat{M}^-$ is a C^∞ manifold with coordinate neighborhood system $\{(\Xi_A|_{S^-_{A,\varepsilon}}, U_A)\}_{A\in\hat{A}^-}$.

Now we will give the canonical Riemannian structure $<\ ,\ >$ on $\hat{M}^-$ in an explicit way.

The tangent space $T_A\hat{A}=\Omega^1(\mathfrak{g}_P)$, $A\in\hat{A}$ has a unique $(\ ,\ )$-orthogonal decomposition

$$T_A\hat{A} = V_A \oplus H_A\ ,\ V_A = \mathrm{Im}\ d_A\ ,\ H_A = \mathrm{Ker}\ d_A^* \tag{2.1}$$

to define a semi definite inner product $<\ ,\ >_A$ on $T_A\hat{A}$ by

$$<\beta\ ,\ \beta_1>_A = (\beta^h\ ,\ \beta_1{}^h)\ , \tag{2.2}$$

where β^h and $\beta_1{}^h$ are the horizontal components of β and β_1 , respectively. Then $<\ ,\ >$ defines indeed the Riemannian structure on $\hat{B}$ so that the projection: $\hat{A}\longrightarrow\hat{B}$ is a Riemannian submersion and the restriction of $<\ ,\ >$ to each of slices is the canonical Riemannian structure on $\hat{M}^-$ ([13, Proposition 3.1]).

In what follows we sometimes identify $A+\alpha$ and α for a fixed $A\in\hat{A}^-$ unless any confusion occurs.

The coordinate expression of $<\ ,\ >$ is given at each coordinate system $(\Xi_A|_{S^-_{A,\varepsilon}}, U_A)$ by

$$< X, Y >_x = <(\Xi_A)_*^{-1} X, (\Xi_A)_*^{-1} Y >_\alpha , x \in U_A, X, Y \in T_x U_A \tag{2.3}$$

here $x=\Xi_A(\alpha)$ with $A + \alpha \in S^-_{A,\varepsilon}$ and $(\Xi_A)_*^{-1}$ is an inverse of the differential $(\Xi_A)_* : T_\alpha S^-_{A,\varepsilon} \longrightarrow T_x U_A = H^1_A$.

We remark here that $T_\alpha S^-_{A,\varepsilon} = \text{Ker } d_A^* \cap \text{Ker } d^+_{A+\alpha}$ since $d^+_{A+\alpha}\beta = d_A{}^+\beta + [\alpha \wedge \beta]^+$ and hence

$$\beta^h = \mathbb{H}_{A+\alpha}\beta, \ \beta \in T_{A+\alpha} S^-_{A,\varepsilon} , \tag{2.4}$$

and further $(\Xi_A)_*$ coincides with $\mathbb{H}_A$ over $T_\alpha S^-_{A,\varepsilon}$ for each $A + \alpha \in S^-_{A,\varepsilon}$ where $\mathbb{H}_A$ is the orthogonal projection: $\Omega^1(\mathfrak{g}_P) \longrightarrow H^1_A$ ([13, Proposition 3.2]).

<u>PROPOSITION</u> 2.2. For each $A \in \hat{A}^-$ the coordinate system $(\Xi_A|S^-_{A,\varepsilon}, U_A)$ is Guassian normal at $\Xi_A(0) = 0$ with respect to the canonical Riemannian structure $< , >$.

The proof of this proposition is given at [13, Proposition 3.4].

3. Quaternion structure on the moduli spaces.

Let (M, h) be as in § 2, that is, a complex 2-torus with a flat metric or a K3 surface with a Ricci flat metric. The canonical line bundle K_M is trivial and the scalar curvature vanishes. Hence we have a covariantly constant, holomorphic 2-form ω on M. Thus, $\omega + \bar{\omega}$, $\sqrt{-1}(\omega - \bar{\omega})$ and the Kähler form Ω define a global, covariantly constant and pointwise orthonormal frame of the bundle Λ_+^2 (up to constant factors). Note that conversely a compact oriented Riemannian 4-manifold admitting such a frame is isomorphic to a complex 2-torus with a flat netric or a K3 surface with a Ricci flat metric ([10]).

As Λ_m^2 at $m \in M$ is isomorphic to $\mathfrak{so}(T_m) = \{u \in \mathrm{End}(T_m);\ h(uX, Y) + h(X, uY) = 0\}$, there exist then globally defined, skew-symmetric endomorphisms $\{I_i\}_{i=I}^3$ satisfying the following

(i) I_i is covariantly constant, $i = 1, 2, 3$,

(ii) I_i is an almost complex structure; $I_i^2 = -\mathrm{id}$, $i = 1, 2, 3$, (3.1)

(iii) $I_i I_j = \varepsilon_{ijk} I_k$, $i \neq j$ (3.2)

(ε_{ijk} is completely skew-symmetric and $\varepsilon_{123} = 1$) and the triple $\{\omega_i\}_{i=1}^3$ of 2-forms defined by $\omega_i(X, Y) = h(I_iX, Y)$ gives a global, covariantly constant, orthogonal frame of

Λ_+^2 ([6], [19, formulas (3.1), (3.2)]).

Each I_i induces an endomorphism of the contagent bundle $\Lambda_M^1 = T^*M$ by $(I_i\, a)X = a(I_i\, X)$ and hence a bundle endomorphism of the tensor product bundle $\Lambda_M^1 \otimes \mathfrak{g}_P$. We will use the same symbol for it. Note that I_i preserves (,) pointwise.

<u>PROPOSITION</u> 3.1 ([19]). The quaternion structure $\{I_i\}_{i=1}^3$ leaves invariant $\operatorname{Ker} \Delta_A^1$, $A \in \mathcal{A}$.

Since $\operatorname{Ker} \Delta_A^1$, $A \in \hat{\mathcal{A}}^-$ is identified with the tangent space $T_{[A]}\hat{\mathcal{M}}^-$ to the moduli space $\hat{\mathcal{M}}^-$, from Proposition 3.1 the restriction of $\{I_i\}_{i=1}^3$ to $\operatorname{Ker} \Delta_A^1$ defines in a natural way a quaternion structure on $\hat{\mathcal{M}}^-$.

The proposition is derived from the following lemmas.

<u>LEMMA</u> 3.2. $I_i(d_A\psi) = d_A^{+*}(\psi \otimes \omega_i)$, $i=1,2,3$, $\psi\in\Omega^0(\mathfrak{g}_P)$. (3.3)

Proof. Since $\psi \otimes \omega_I$ is a self-dual 2-form and ω_I is covariantly constant for $I = I_i$, $i=1,2,3$, we get

$$d_A^{+*}(\psi \otimes \omega_I) = -\sum_{i,s,t} h^{st}\nabla_s(\psi\, \omega_{ti})\, dx^i = -\sum \nabla_s\psi h^{st}\omega_{ti}dx^i ,$$

where $\omega_I = 1/2 \sum \omega_{ij} dx^i \wedge dx^j$ is given by $\omega_{ij} = -\omega_{ji} = \sum_k I_i{}^k h_{kj}$. Hence this reduces to $\sum I_i{}^s \nabla_s \psi dx^i$ which is equal to $I(d_A\psi)$.

Each $\Psi \in \Omega^2_+(g_P)$ is represented by $\sum_i \Psi^i \otimes \omega_i$, $\Psi^i \in \Omega^0(g_P)$. We say Ψ^i to be the i-th component of Ψ . Then we obtain

<u>LEMMA</u> 3.3 ([19, (3.3), (3.4)]).

(i) $\quad d_A{}^*(I_i\alpha) = -2$ {i-th component of $d_A{}^+\alpha$} , (3.4)

and hence the i-th component of $d_A{}^+(I_i\alpha) = 1/2 d_A{}^*\alpha$, i=1,2,3, and

(ii) $\quad$ the i-th component of $d_A{}^+(I_j\alpha) =$ (3.5)

$$\varepsilon_{ijk}\{\text{k-th component of } d_A{}^+\alpha\} \; i \neq j .$$

Proof. We give a proof by our terminology.

(i); We have from Lemma 3.2 that for $\psi \in \Omega^0(g_P)$ $(d_A{}^*I\alpha, \psi) = -(\alpha, I\, d_A\psi) = -(\alpha, d_A{}^{+*}(\psi \otimes \omega_I)) = -(d_A{}^+\alpha, \psi \otimes \omega_I) = -((d_A{}^+\alpha, \omega_I), \psi)$. Hence $d_A{}^* I\alpha = -(d_A{}^+\alpha, \omega_I)$, which implies (3.4).

(ii); $((d_A{}^+(I_j\alpha), \omega_i), \psi) = (d_A{}^+(I_j\alpha), \psi \otimes \omega_i) = (I_j\alpha, d_A{}^{+*}(\psi \otimes \omega_i)) = \varepsilon_{ijk}(d_A{}^+\alpha, \psi \otimes \omega_k)$. Then we have (3.5).

<u>PROPOSITION</u> 3.4. The quaternion structure $\{I_i\}_{i=1}^3$ on $\hat{M}^-$

is covariantly constant with respect to the Levi-Civita connection ∇ of $(\hat{M}^-, < , >)$.

From this proposition the holonomy group of the Riemannian manifold $(\hat{M}^-, < , >)$ is contained in a sypletic group. Hence we obtain Theorem 1.

The rest of this section is devoted to the proof of Proposition 3.4.

In order to get the covariant derivative of the almost complex structure I_i which is considered as an endomorphism of the tangent bundle $T\hat{M}^-$ we develop it along a slice and obtain an expression of the developed almost complex structure $J_{i,\alpha} : T_\alpha S^-_{A,\varepsilon} \longrightarrow T_\alpha S^-_{A,\varepsilon}$ for any $A+\alpha \in S^-_{A,\varepsilon}$. Of course we have $J_{i,\alpha} = I_i$ at $\alpha = 0$.

Fix a connection $A + \alpha_0$ in the slice $S^-_{A,\varepsilon}$ and let $S^-_{A',\varepsilon'}$ be another slice at $A' = A + \alpha_0$. Since $\pi(S^-_{A,\varepsilon}) \cap \pi(S^-_{A',\varepsilon'}) \neq \phi$, there exists a C^∞ mapping $g : S^-_{A,\varepsilon} \cap \pi^{-1}(\pi(S^-_{A',\varepsilon'})) \longrightarrow G$; $\alpha \longrightarrow g_\alpha$ such that $g_\alpha(A + \alpha)$ is in $S^-_{A',\varepsilon'}$ and $g_{\alpha_0}(A+\alpha_0) = A'$. This mapping g induces a transformation Ψ between these slices by $\Psi(A + \alpha) = g_\alpha(A + \alpha)$. Its differential $\Psi_{*\alpha} : T_{A+\alpha} S^-_{A,\varepsilon} \longrightarrow T_{\Psi(A+\alpha)} S^-_{A',\varepsilon'}$ is written as

$$\Psi_{*\alpha}(\beta) = g_\alpha(\beta) + d_{\Psi(A+\alpha)} \psi , \tag{3.6}$$

here $\psi \in \Omega^0(\mathfrak{g}_P)$ is uniquely determined from α and β ([13, § 3]). The almost complex structure $J = J_i$ at α_0 on the slice $S^-_{A,\varepsilon}$ must satisfy the following

$$J_{\alpha_0}(\beta) = ((\Psi_{*\alpha_0})^{-1}.I.\ (\Psi_{*\alpha_0}))(\beta),\ \beta \in T_{\alpha_0}S^-_{A,\varepsilon'} \tag{3.7}$$

since $T_{A'}S^-_{A',\varepsilon'} = \mathrm{Ker}\ \Delta^1_A$, and the almost complex structure developed along $S^-_{A',\varepsilon'}$ is just $I = I_i$ at $A' = A + \alpha_0$. Note that each $I_i : \Omega^1(\mathfrak{g}_P) \longrightarrow \Omega^1(\mathfrak{g}_P)$ commutes with any gauge transformation.

<u>LEMMA</u> 3.5. J_{α_0} has the following form;

$$J_{\alpha_0}(\beta) = I(\beta) + d_{A+\alpha_0}\phi + d^{+\,*}_{A+\alpha_0}(\psi \otimes \omega_I), \tag{3.8}$$

or

$$J_{\alpha_0}(\beta) = I(\beta^h) + d_{A+\alpha_0}\ \phi\ , \tag{3.9}$$

where I of the right hand side is the almost complex structure defined on $\Omega^1(\mathfrak{g}_P)$ and $\phi,\ \psi \in \Omega^0(\mathfrak{g}_P)$ are uniquely determined by α_0 and β .

Proof. From (3.3) an inverse $\Psi_{*\alpha_0}{}^{-1}$ of $\Psi_{*\alpha_0}$ is given by

$$\Psi_{*\alpha_0}{}^{-1}(\gamma) = g_{\alpha_0}{}^{-1}(\gamma) + d_{A+\alpha_0}\phi,\ \gamma \in T_{A'}S^-_{A',\varepsilon'}$$

here $\phi \in \Omega^0(\mathfrak{g}_P)$ is determined from α_0 and γ . Hence $J_{\alpha_0}(\beta) = g_{\alpha_0}{}^{-1}(I(\Psi_{*\alpha_0}\beta)) + d_{A+\alpha_0}\phi$ reduces to $(g_{\alpha_0}{}^{-1}Ig_{\alpha_0})(\beta) + g_{\alpha_0}{}^{-1}(I(d_{A+\alpha_0}\psi')) + d_{A+\alpha_0}\phi = I(\beta) + I(g_{\alpha_0}{}^{-1}(d_{A+\alpha_0}\psi')) +$

$d_{A+\alpha_0}\phi$. Since $g_{\alpha_0}^{-1}(d_{A+\alpha_0}\psi') = d_{A+\alpha_0}((g_{\alpha_0}^{-1})\psi')$, we have

$$J_{\alpha_0}(\beta) = I(\beta) + I(d_{A+\alpha_0}\psi) + d_{A+\alpha_0}\phi ,$$

for $\psi = g_{\alpha_0}^{-1}(\psi')$. Hence (3.8) is obtained from Lemma 3.2. The proof of (3.9) is given as follows. Decompose β as $\beta = \beta^v + \beta^h$, $\beta^v \in V_{A+\alpha_0}$, $\beta^h \in H_{A+\alpha_0}$. Because $d^+_{A+\alpha_0}(J_{\alpha_0}(\beta)) = 0$, $d^+_{A+\alpha_0}(I(\beta)) + \Delta^2_{+,A+\alpha_0}(\psi \otimes \omega_I) = 0$. Then $d^+_{A+\alpha_0}(I(\beta^v)) + \Delta^2_{+,A+\alpha_0}(\psi \otimes \omega_I) = 0$, since from (2.4) we have $I(\beta^h)) = I\,\mathbb{H}_{A+\alpha_0}(\beta) \in \operatorname{Ker}\Delta^1_{A+\alpha_0}$. But we have $\phi' \in \Omega^0(\mathfrak{g}_P)$ so that $\beta^v = d_{A+\alpha}\phi'$ and then from (3.3) $\Delta^2_{+,A+\alpha_0}((\phi'+\psi) \otimes \omega_I) = 0$ which concludes that $(\phi' + \psi) \otimes \omega_I = 0$ since $\operatorname{Ker}\Delta^2_+ = 0$. That is, $I(\beta^v) + d_{A+\alpha_0}^{+*}(\psi \otimes \omega_I) = 0$.

<u>REMARKS</u>. (i) From (3.9) and also (2.2) the quaternion structure preserves the canonical Riemannian structure.

(ii) (3.9) coincides with (4.8) in [13]. In [13] the formula (4.8) is showed by the aid of the canonical correspondence between $\hat{M}^-$ on an SU(n)-principal bundle and the moduli space M_h of holomorphic (0,1)-connections. We use here a method applicable to a case of general simple Lie group without any aid of M_h .

Proof of Proposition 3.4. Because for any $A \in \hat{A}^-$ $(\Xi_{A|S^-_{A,\varepsilon}}, U_A)$ is a coordinate system at [A] in $\hat{M}^-$ it suffices to show that

$\nabla J = 0$ at A, here $J = J_i$ over U_A is induced from J_i being developed along the slice under Ξ_A.

Let $\{X^1 \ldots\ldots X^\ell\}$ be an orthonormal basis of H_A^1, and $x = (x^1 \ldots\ldots x^\ell)$ the coordinate in $U_A \subset H_A^1$ corresponding to it and moreover $\partial/_{\partial x}i$, the canonical vector field on U_A, $i=1,\ldots\ldots\ell$. We show the following assertion from which the Proposition is obtained.

<u>ASSERTION</u>. At $x = 0$ $\nabla \partial/_{\partial x}i (J \partial/_{\partial x}j) = J(\nabla \partial/_{\partial x}i\, \partial/\partial x^j)$, $i, j = 1, \ldots \ell$.

The coordinate system is Gaussian normal at $x = 0$ from Proposition 2.2. Then $\nabla \partial/_{\partial x}i (J \partial/_{\partial x}j)$ at $x = 0$ coincides with $\sum_k (\partial/_{\partial x}i J_j^k)(0)(\partial/_{\partial x}k)_0$ and is equal to $d/_{dt|0} J_{c(t)} (\partial/_{\partial x}j)_{c(t)}$ where $c(t) = tX^i$ is a curve in U_A corresponding to $\partial/_{\partial x}i$. By its definition we have

$$J_{c(t)}(\partial/_{\partial x}i)_{c(t)} = (\Xi_A)_{*\alpha(t)} J_{\alpha(t)} (\Xi_A)_{*\alpha(t)}^{-1} (\partial/_{\partial x}j)_{c(t)},$$

where $\alpha(t)$ is a curve in $S^-_{A,\varepsilon}$ defined by $\Xi_A(\alpha(t)) = c(t)$. From the fact that $(\Xi_A)_{*\alpha} = \mathbf{H}_A$ over $T_\alpha S^-_{A,\varepsilon}$,

$$J_{c(t)}(\partial/_{\partial x}j)_{c(t)} = \mathbf{H}_A\, J_{\alpha(t)}(\gamma_{\alpha(t)}),$$

here we set $\gamma_{\alpha(t)} = (\Xi_{A*\alpha(t)})^{-1}(\partial/_{\partial x}j)_{c(t)}$, which is in $T_{\alpha(t)} S^-_{A,\varepsilon}$. Then

$$d/dt|_0 J_{c(t)}(\partial/\partial x^j)_{c(t)} = \mathbb{H}_A(d/dt|_0(J_{\alpha(t)}\gamma_{\alpha(t)})) .$$

From Lemma 3.5 we get ϕ_t and ψ_t in $\Omega^0(\mathfrak{g}_P)$ with parameter t such that

$$J_{\alpha(t)}(\gamma_{\alpha(t)}) = I(\gamma_{\alpha(t)}) + d_{A+\alpha(t)}\phi_t + d_{A+\alpha(t)}^{+*}(\psi_t \otimes \omega_I) .$$

Therefore the differential is

$$d/dt|_0 J_{\alpha(t)}(\gamma_{\alpha(t)}) = I(d/dt|_0 \gamma_{\alpha(t)})$$

$$+ d/dt|_0(d_{A+\alpha(t)}\phi_t + d_{A+\alpha(t)}^{+*}(\psi_t \otimes \omega_I))$$

$$= I(d/dt|_0 \gamma_{\alpha(t)}) + [X^i \wedge \phi_0] + d_A(d/dt|_0 \phi_t) +$$

$$([X^i \wedge (\psi_0 \otimes \omega_I)]) + d_A^{+*}((d/dt|_0 \psi_t) \otimes \omega_I) ,$$

here $([.\wedge.]) : \Omega^1(\mathfrak{g}_P) \times \Omega^2(\mathfrak{g}_P) \longrightarrow \Omega^1(\mathfrak{g}_P)$ implies the bracket operation together with the metric contraction, and also we used the fact that $\alpha(0) = 0$ and $d/dt|_0 \alpha(t) = X^i$ in the sense of Frechêt derivatives. Since $J = I$ at $\alpha = 0$ and I commutes with $\mathbb{H}_A$, we have $\phi_0 = \psi_0 = 0$ and hence

$$d/dt|_0 J_{c(t)}(\partial/\partial x^j)_{c(t)} = \mathbb{H}_A(I(d/dt|_0 \gamma_{\alpha(t)})) =$$

$$I(\mathbb{H}_A(d/dt|_0 \gamma_{\alpha(t)})) .$$

We have further $\mathbb{H}_A(d/_{dt|0}\gamma_{\alpha(t)}) = d/_{dt|0}\mathbb{H}_A\gamma_{\alpha(t)}) = d/_{dt|0}$ $(\partial/_{\partial x^j})_{c(t)}$ which is $\nabla \partial/_{\partial x^i}\, \partial/_{\partial x^j}$ at $x = 0$.

Therefore the assertion is verified.

4. Moduli space of Einstein-Hermitian connections.

We assume first that (M, h) is a compact complex surface with a Kähler metric. Let P be a C^∞ principal bundle over M with structure group $U(n)$. For each connection A on P we decompose its curvature F_A into the sum of the center part $1/_n$ $\mathrm{Tr}\, F_A \cdot \mathrm{id}_E$ and the trace free part $F_A^s = F_A - 1/_n\, \mathrm{Tr}\, F_A \cdot \mathrm{id}_E$ according to the decomposition of $\mathfrak{u}(n) = \mathfrak{c} + \mathfrak{su}(n)$. $\sqrt{-1}/_{2\pi}$ $\mathrm{Tr}\, F_A$ is a closed 2-form representing $c_1(E) \in H^2(M; \mathbb{Z})$; $E = P \times_\sigma \mathbb{C}^n$.

<u>DEFINITION</u> 4.1. A connection A on a $U(n)$-principal bundle P is called Einstein-Hermitian when $\mathrm{Tr}\, F_A$ is a harmonic 2-form of type $(1,1)$ and F_A^s is anti-self-dual $(p_+ F_A^s = 0)$.

Like as an anti-self-dual connection Einstein-Hermitian connection gives a connection minimizing absolutely the Yang-Mills functional if $\{c_2(E) - \frac{n-1}{n} c_1(E)^2\}[M] \geq 0$. For each Einstein-Hermitian connection A $\mathrm{Tr}\, A$ depends only on the bundle P from the uniqueness of harmonic form representing the cohomology class.

<u>PROPOSITION</u> 4.2. The following are equivalent for a connection A .

(i) A is Einstein-Hermitian,

(ii) F_A is a $\mathfrak{g}_P$-valued 2-form of type $(1,1)$ whose contraction ΛF_A with the Kähler form Ω is $\lambda.id_E$ for a constant λ .

Proof. (i)$\Longrightarrow$(ii) : This is obvious, since the harmonic form $Tr\ F_A$ splits into $p_+\ Tr\ F_A$ and $p_-\ Tr\ F_A$ and $p_+\ Tr\ F_A = a\ \Omega$ for a constant a , and moreover a 2-form Ψ is anti-self-dual if and only if it is type (1,1) and $\Lambda\Psi = 0$ ([11]).

(ii)$\Longrightarrow$(i) : We have $\Lambda F_A = \Lambda F_A{}^s + 1/_n(\Lambda Tr\ F_A).id_E$ which is equal to $\lambda.id_E$. Then the trace free part $\Lambda F_A{}^s$ vanishes and $\Lambda Tr\ F_A$ is constant. But this means over the Kähler surface M that the closed form $Tr\ F_A$ of type (1,1) is harmonic.

REMARK. Donaldson mentioned in [8] the equivalence in Proposition 4.2. Many authors define Einstein-Hermitian connection by Proposition 4.2, (ii).

We say a connection A on P to be irreducible if $\mathrm{Ker}\{d_A\ ;\ \Omega^0(\mathfrak{g}_P) \longrightarrow \Omega^1(\mathfrak{g}_P)\}$ is one-dimensional. We decompose the adjoint bundle $\mathfrak{g}_P$ into $\mathfrak{g}_P = \mathfrak{g}_P{}^c + \mathfrak{g}_P{}^s$ where $\mathfrak{g}_P{}^c = P\times_{Ad}\mathfrak{c}$ and $\mathfrak{g}_P{}^s = P\times_{Ad}\mathfrak{su}(n)$. Over $\mathfrak{g}_P{}^c d_A$ reduces to the ordinary differential d . The irreducibility of a connection A implies equivalently that the restriction of d_A t0 $\Omega^0(\mathfrak{g}_P{}^s)$ is injective.

Now we will describe a slice for moduli space $\hat{M}_E$ of irreducible Einstein-Hermitian connections on P .

PROPOSITION 4.3. Suppose that a connection A is Einstein-Hermitian. Then, a connection $A+\alpha$ is Einstein-Hermitian if and only if

$$d(\alpha^c) = 0$$

$$d_A^+(\alpha^s)+1/2[\alpha^s \wedge \alpha^s]^+ = 0 \qquad (4.1)$$

for the splitting $\alpha = \alpha^c + \alpha^s$.

Proof. Since $F_{A+\alpha} = F_A + d_A\alpha + \frac{1}{2}[\alpha \wedge \alpha]$, $A+\alpha$ is Einstein-Hermitian if and only if $\mathrm{Tr}(d_A\alpha + \frac{1}{2}[\alpha \wedge \alpha]) = 0$ and $P+((d_A\alpha + \frac{1}{2}[\alpha \wedge \alpha])^s) = 0$. These are equivalent to (4.1) because $d_A\alpha = d\alpha c + d_A\alpha^s$ and $[\alpha\wedge\alpha]^s = [\alpha^s\wedge\alpha^s]$, and also $\mathrm{Tr}(d_A\alpha) = \mathrm{Tr}(d\alpha^c)$.

From this proposition we can define a slice at A for $\hat{M}_E$. Denote by $S_{A,\varepsilon}$ a subset in $\Omega^1(\mathfrak{g}_P)$ of the following form

$\{\alpha = \alpha^c + \alpha^s \in \Omega^1(\mathfrak{g}_P)$; $|\alpha| < \varepsilon$, α satisfies $d_A{*}\alpha = 0$ and (4.1) $\}$

As α^c and α^s are chosen independently, $S_{A,\varepsilon}$ is written by a product $S^c_{A,\varepsilon_1} \times S^s_{A,\varepsilon_2}$, where

$$S^c_{A,\varepsilon_1} = \{\sqrt{-1}\, a.\mathrm{id}_E \text{ ; } a \text{ is a real harmonic 1-form on } M \text{ , } |a|<\varepsilon_1\} \qquad (4.2)$$

and

$$S^s_{A,\varepsilon_2} = \{\alpha^s \in \Omega^1(\mathfrak{g}_P{}^s) \; ; \; |\alpha^s| < \varepsilon_2 \; , \; d_A{}^* \alpha^s = 0 \; , \; d_A{}^+ \alpha^s + \frac{1}{2} [\alpha^s \wedge \alpha^s]^+ = 0 \} \; . \tag{4.3}$$

The quotient $S_{A,\varepsilon}/_{\Gamma_A}$ by the isotropy subgroup $\Gamma_A = \{g \in G; \; g(A) = A\}$ properly corresponds to a neighborhood of the space $\hat{M}_E$. However, we can neglect the action of Γ_A, since Γ_A is a subgroup consisting of constant abelian gauge transformations and its action on $\mathfrak{g}_P$ is trivial. Remark that $S^c_{A,\varepsilon_1} = \{0\}$ for M with $b_i(M)=0$.

With respect to the slice $S^s_{A,\varepsilon}$ at A we get the following elliptic complex

$$0 \longrightarrow \Omega^0(\mathfrak{g}^s_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}^s_P) \xrightarrow{d^+_A} \Omega^2_+(\mathfrak{g}^s_P) \longrightarrow 0 \tag{4.4}$$

to define a local coordinate on $\hat{M}_E$ at $[A]$ by making use of the Kuranishi map

REMARK. If we use here a notion of trace free anti-self-dual connection, then (4.3) and (4.4) lead us to define moduli of such connections $\hat{M}^s_E$. (In [8] such a connection was mentioned as a projectively anti-self-dual connection).

From (4.2) - (4.4) the moduli space $\hat{M}_E$ has a locally product structure; $\hat{M}_E \cong H^1{}_{deR}(M; \mathbb{R}) \times \hat{M}^s_E$. Of course each of them carries the canonical complex structure induced from M compatible with the product structure. The dimension of $\hat{M}^s_E$ is computed by the

Atiyah-Singer index theorem from (4.4) and in fact is given by

$$\dim_{\mathbb{C}} \hat{M}_E^s = \{ 2n\, c_2(E) - (n-1) c_1{}^2(E) \} [M] \\ -(n^2-1)\, Pa\,(M) \qquad (4.5)$$

($P_a\,(M) = 1-q(M) + P_g\,(M)$ is the arithmetic genus of M).

Assume finally that the base space is a complex 2-torus with a flat metric or a K3 surface with a Ricci flat metric.

Then $\hat{M}_E^s$ and hence $\hat{M}_E$ is a manifold with no singularities. By the aid of the same arguments in § 2 and 3 we obtain a canonical Riemannian structure $\langle\ ,\ \rangle$ and a covariantly constant quaternion structure. Therefore we have Theorem 2.

On the center part factor of $\hat{M}_E$ which is locally diffeomorphic to $H^1_{deR}\,(M;\, \mathbf{R})$, the canonical Riemannian structure reduces to a flat metric and the quaternion structure is directly inherited from the quaternion structure on T^*M .

5. Some curvature identity

We show that the canonical Riemannian structure on the moduli space $\hat{M}^-$ (or $\hat{M}^s_E$) is Ricci flat by making use some curvature identity.

The Riemannian curvature tensor R actually satisfies the following identity.

PROPOSITION 5.1.

$$< R (X,Y) Y,X > + \sum_{i=1}^{3} < R (X,I_iY) I_iY,X > = 0 , \qquad (5.1)$$

$X,Y \in H^1_A \cong T_{[A]}\hat{M}^-$. Here $\{I_i\}$ is of course the quaternion structure on $\hat{M}^-$ induced from (M, h) .

Like as holomorphic bisectional curvature on a Kähler manifold, the left hand side of (5.1) represents "quaternionic" bisectional curvature of two quaternion linear subspace V_X and V_Y in H^1_A , where V_X and V_Y are spanned by X and Y , respectively.

From (5.1) we can verify that the Ricci curvature vanishes.

LEMMA 5.2 $$[\alpha \wedge \alpha]^+ + \sum_{i=1}^{3} [I_i\alpha \wedge I_i\alpha]^+ = 0 , \qquad (5.2)$$

$\alpha \in \Omega^1(\mathfrak{g}_P)$.

Proof. Let $\{\omega_i\}_{i=1}^{3}$ be the triple of 2-forms corresponding to the quaternion structure $\{I_i\}_{i=1}^{3}$ on M. There exists at each $m \in M$ a coordinate $\{x^1, x^2, x^3, x^4\}$ so that it is Gaussian normal at m and $\omega_1 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$, $\omega_2 = dx^1 \wedge dx^3 + dx^4 \wedge dx^2$ and $\omega_3 = -(dx^1 \wedge dx^4 + dx^2 \wedge dx^3)$ at m. Then $\{I_i\}_{i=1}^{3}$ on T_m^*M has the following matrix representation:

$$I_1 = \left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right), \quad I_2 = \left(\begin{array}{cc|cc} & & -1 & \\ & & & 1 \\ \hline 1 & & & \\ & -1 & & \end{array}\right), \quad I_3 = \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right). \tag{5.3}$$

The formula (5.2) is derived by a straight computation and is nothing but the formula (3.6) in [19].

LEMMA 5.3 For any $X, Y \in \Omega^1(\mathfrak{g}_P)$ the i-th component of $[X \wedge Y]^+ = \frac{1}{2}\{X, I_i Y\}$, $i=1,2,3$. (5.4)

Here a skew-symmetric mapping $\{.,.\} : \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \longrightarrow \Omega^0(\mathfrak{g}_P)$ is defined by (5.5)

$$\{X, Y\} = \sum_{i,j} h^{ij}[X_i, Y_j]$$

Proof. The i-th component is $\frac{1}{2}([X \wedge Y], \omega_i)$ since $|\omega_i|^2 = 2$. Set $i=1$. Then

$$([X \wedge Y], \omega_1) = [X_1, Y_2] + [Y_1, X_2] + [X_3, Y_4] + [Y_3, X_4].$$

On the other hand, from (5.3)

$$I_1Y = -Y_1dx^2 + Y_2dx^1 - Y_3dx^4 + Y_4dx^3$$

and then

$$\{X,I_1Y\} = [X_1,Y_2] - [X_2,Y_1] + [X_3,Y_4] - [X_4,Y_3] .$$

(5.4) for $i=2$ and 3 are similarly obtained.

Proof of Proposition 5.1. We have in [13] the formula of the Riemannian curvature tensor;

$$< R(X,Y)\ Y,X > = 3(\{X,Y\},\ G_A(\{X,Y\})) - ([X\wedge Y]^+,\ G_A([X\wedge Y]^+))$$
$$+ ([X\wedge X]^+,\ G_A([Y\wedge Y]^+)) .$$

Then the left hand side of (5.1) is written by

$$3(\{X,Y\},\ G_A(\{X,Y\})) + 3\sum_i(\{X,I_iY\},\ G_A(\{X,I_iY\}))$$
$$-([X\wedge Y]^+,\ G_A([X\wedge Y]^+)) - \sum_i(X\wedge I_iY]^+,\ G_A([X\wedge I_iY]^+))$$
$$+([X\wedge X]^+,\ G_A([Y\wedge Y]^+)) + \sum_i([X\wedge X]^+,\ G_A([I_iY\wedge I_iY]^+)) .$$

The last two terms vanish because of (5.2) .

Since $[X\wedge Y]^+ = \frac{1}{2}\sum_i\{X,I_iY\}\omega_i$ and $G_A(\psi \otimes \omega_i) = 2(G_A\psi) \otimes \omega_i$,

$i=1,2,3$, $G_A([X\wedge Y]^+)$ reduces to $\sum_i G_A(\{X,I_iY\}\omega_i$ and then

$$([X\wedge Y]^+,\ G_A[X\wedge Y]^+) = \sum_i (\{X,I_iY\},\ G_A\{X,I_iY\}) \ . \tag{5.6}$$

So

$$([X\wedge I_iY]^+,\ G_A[X\wedge I_iY]^+) = \sum_j (\{X,I_jI_iY\},\ G_A\{X,I_jI_iY\})$$

$$= (\{X,Y\},\ G_A\{X,Y\}) + \sum_{j\neq i} (\{X,I_jY\},\ G_A\{X,I_jY\}) \ , \ i=1,2,3 \ .$$

Therefore we obtain (5.1).

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