DOUBLE HURWITZ NUMBERS AND THE INTERSECTION FORM ON THE MODULI SPACE IN GENUS 0

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Abstract. We give an explicit formula for the double Hurwitz numbers in genus zero counting the number of coverings $\mathbb{C}\mathbf{P}^1 \to \mathbb{C}\mathbf{P}^1$ with two branching points with a given branching behavior, as piecewise polynomials in the multiplicities of the preimages of the branching points.

§1. Introduction and results

Let $\mu = (\mu_1, \dots, \mu_m)$ be a partition of an integer d. The *Hurwitz number* h^g_μ is the number of genus g branched covers of $\mathbb{C}\mathbf{P}^1$ with branching corresponding to μ over a fixed point (usually identified with ∞) and an appropriate number of fixed simple branching points. Recall that each cover is counted with the weight c^{-1}, where c is the number of automorphisms of the cover.

Hurwitz numbers possess a rich structure explored by many authors in different fields, such as algebraic geometry, representation theory, integrable systems, combinatorics, and mathematical physics. Being very far of trying to describe all these achievements, we mention only the so-called ELSV-formula (see [ELSV1, ELSV2, GrVa])

$$h^g_\mu = \frac{(m+d+2g-2)!}{|\operatorname{Aut}\mu|}\prod_{i=i}^{m}\frac{\mu_i^{\mu_i}}{m_i!}\int_{\overline{\mathcal{M}}_{g,m}}\frac{1-\lambda_1+\lambda_2\cdots\pm\lambda_g}{(1-\mu_1\psi_1)\cdots(1-\mu_m\psi_m)},$$

where $\operatorname{Aut}\mu$ is the group of symmetries of the set μ, $\overline{\mathcal{M}}_{g,m}$ is the Deligne–Mumford compactification of the moduli space of genus g curves with m marked points, λ_k is a codimension k class, and ψ_i is a codimension 1 class (see the above mentioned papers for exact definitions).

Let $\nu = (\nu_1, \dots, \nu_n)$ be another partition of the same integer d. The *double Hurwitz number* $h^g_{\mu,\nu}$ is the number of genus g branched covers of $\mathbb{C}\mathbf{P}^1$ with branchings corresponding to μ and ν over two fixed points (in what follows we identify them with ∞ and 0, respectively) and an appropriate number of fixed simple branching points. We denote this number $r^g_{\mu,\nu}$; by the Riemann–Hurwitz theorem, $r^g_{\mu,\nu} = m+n+2g-2$. As before, each cover is counted with the weight c^{-1}, where c is the number of automorphisms of the cover. Evidently, $h^g_{\mu,\nu} = h^g_{\nu,\mu}$, so in what follows we assume, without loss of generality, that $m \leqslant n$.

Most of the known results concerning double Hurwitz numbers treat only the so-called one-part (or polynomial) case, when $m = 1$ and n is arbitrary. One-part double Hurwitz numbers in genus zero where studied in [ZL1] (see also [Z1] for an earlier version of the same result). It is proved there that

$$h^0_{(d),\nu} = \frac{(n-1)!}{|\operatorname{Aut}\nu|} d^{n-2}; \tag{1.1}$$

recall that $n-1 = r^0_{(d),\nu}$. Another way to get the same result is suggested in [Sh1]. The aim of this paper is to get an ELSV-type formula for $h^g_{(d),\nu}$. Two such formulas are derived; the first of them (see [Sh1, Sh3]) represents $h^g_{(d),\nu}$ as

$$h^g_{(d),\nu} = \frac{r!}{|\operatorname{Aut}\nu|} d^{r-1} \int_\Delta \psi_1^{r-1},$$

where $r = r^g_{(d),\nu} = n + 2g - 1$, Δ is a certain cycle in the moduli space $\overline{\mathcal{M}}_{g,n+1}$ and ψ_1 has the same meaning as in the ELSV-formula. In the case $g = 0$ the cycle Δ coincides with the whole space $\overline{\mathcal{M}}_{0,n+1}$, and since the integral is known to be equal to 1, we immediately get (1.1). The second formula (see [Sh1, Sh2]) represents $h^0_{(d),\nu}$ as

$$h^0_{(d),\nu} = \frac{d^{n-2}}{|\operatorname{Aut}\nu|} \int_{\overline{\mathcal{M}}_{0,d(n-1)+2}} \psi_1^{n-2}\Psi_2\cdots\Psi_{n+1}.$$

The definition of the classes Ψ_i is recursive and rather involved. The only example considered in [Sh1] is $n = 2$, in which case, after some effort, one gets the correct answer.

A different ELSV-type formula for the one-part double Hurwitz numbers is conjectured in [GJVa]. It is proved there for the cases $g = 0, 1$. The answers look as follows:

$$h^0_{(d),\nu} = \frac{(n-1)!}{|\operatorname{Aut}\nu|} d \int_{\overline{\mathcal{M}}_{0,n}} \frac{1}{(1-\nu_1\psi_1)\cdots(1-\nu_n\psi_n)},$$
$$h^1_{(d),\nu} = \frac{(n+1)!}{|\operatorname{Aut}\nu|} d \int_{\overline{\mathcal{M}}_{1,n+1}} \frac{1-\Lambda_2}{(1-\nu_1\psi_1)\cdots(1-\nu_n\psi_n)},$$

where ψ_i are the same as in the ELSV formula, and Λ_2 is a certain codimension 2 class pulled back from $\overline{\mathcal{M}}_{1,2}$. For higher genera, the conjectured formula looks very similar, however, it is yet not clear over which moduli space should the integral be taken. On the computational side, [GJVa] suggests the following result, which is a common generalization of (1.1) and [SSV, Th. 6]:

$$h^g_{(d),\nu} = \frac{r!}{|\operatorname{Aut}\nu|} d^{r-1} [t^{2g}] \prod_{k\geqslant 1} \left(\frac{\sinh kt/2}{kt/2}\right)^{c_k},$$

where $r = r^g_{(d),\nu}$, $\{c_k\}$ is a finite sequence defined solely by ν, and $[A]B$ is the coefficient of A in B. This result extend previous computations carried out in [Ku] for $n = 1, 2$.

Much less is known for general double Hurwitz numbers. In fact, there are two general results. First, it is proved in [Ok] that the exponent of the generating function for the double Hurwitz numbers is a τ-function for the Toda hierarchy of Ueno and Takasaki. Second, Theorem 2.1 in [GJVa] states that for fixed g, m, and n, double Hurwitz numbers are piecewise polynomial in variables $\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n$, and that the highest degree of this piecewise polynomial is constant and equal to $m + n + 4g - 3$. Moreover, it is proved in [GJVa] that for genus zero case this piecewise polynomial is homogeneous.

In a different direction, Theorem 4.1 of the same paper treats $h^g_{\mu,\nu}$ as a function of g for fixed partitions μ and ν. In principle, any such function can be obtained by recursive computation; the only case done explicitly is $m = n = 2$ and $\mu_1 > \nu_1 > \nu_2 > \mu_2$. Similar results are also obtained in [Ku] for all double Hurwitz numbers of degree at most 5.

Still another approach suggested in [GJVa] provides exact formulas for double Hurwitz numbers in genus zero when $m = 2$ or $m = 3$. In particular,

$$h^0_{(\mu_1,\mu_2),\nu} = \frac{n!}{d|\operatorname{Aut}\mu|} \sum \frac{l(\rho)! \prod_{j\geqslant 1} \rho_j}{|\operatorname{Aut}\rho||\operatorname{Aut}\sigma||\operatorname{Aut}\tau|} (\mu_1 - |\sigma|)(\mu_2 - |\tau|)\mu_1^{l(\sigma)-1}\mu_2^{l(\tau)-1},$$

where the summation is over partitions $\rho = (\rho_1, \dots, \rho_{l(\rho)})$, $\sigma = (\sigma_1, \dots, \sigma_{l(\sigma)})$, $\tau = (\tau_1, \dots, \tau_{l(\tau)})$ with $\rho \cup \sigma \cup \tau = \nu$ and $|\sigma| = \sum_{i=1}^{l(\sigma)} \sigma_i < \mu_1$, $|\tau| = \sum_{i=1}^{l(\tau)} \tau_i < \mu_2$.

Finally, the approach developed in [Z2] allows in principle to obtain formulas for $h^0_{\mu,\nu}$ as functions of the degree d for a very special choice of the partitions μ and ν: $\mu = (\alpha, 1, \dots, 1)$, $\nu = (\beta, 1, \dots, 1)$, α and β being fixed. For example, for $\alpha = \beta = 3$ one gets

$$h^0_{3^1 1^{d-3}, 3^1 1^{d-3}} = \frac{3}{4}(27d^2 - 137d + 180)\frac{d^{d-6}(2d-6)!}{(d-3)!}.$$

It is conjectured that for arbitrary α and β the number in question equals $P_{\alpha\beta}(d)d^{d-\alpha-\beta+1}(2d - \alpha - \beta)!/d!$, where $P_{\alpha\beta}$ is a polynomial of degree $\alpha + \beta - 2$ with rational coefficients; for example, $P_{33}(d) = \frac{3}{4}(d-1)(d-2)(27d^2 - 137d + 180)$.

In this note we give an explicit formula for the double Hurwitz numbers $h^0_{\mu,\nu}$ as a sum of homogeneous piecewise polynomials of degree $m + n - 3$ over a set of weighted trees. To formulate the main result we need to introduce some notation. Let $T = (V_T, E_T)$ be a tree with the vertex set V_T and the edge set E_T. A *weight function* is a map $w\colon V_T \cup E_T \to \mathbb{Z}_{\geqslant 0}$ satisfying the following properties:

$$\begin{gathered} \sum_{v \in V_T} w(v) = \sum_{e \in E_T} w(e) = W_T, \\ w(v) \geqslant \deg v - 3 \qquad \text{for any } v \in V_T, \end{gathered} \tag{1.2}$$

where $\deg v$ is the number of edges incident to v; observe that the second condition is nontrivial only for vertices of degree at least 4. The pair (T, w) is called a weighted tree, and $W_T = \sum_{e\in E_T} w(e)$ is the *total weight* of the weighted tree. Two weighted trees (T_1, w_1) and (T_2, w_2) are called isomorphic if there exists a bijection $V_{T_1} \to V_{T_2}$ preserving incidence relations and weights. We denote by $\mathcal{T}(s)$ the set of all weighted trees satisfying condition $|E_T| + W_T = s$ modulo this isomorphism. Slightly abusing notation, we will call the elements of $\mathcal{T}(s)$ weighted trees, and denote them (T, w).

Given a weighted tree $(T,w)\in\mathcal{T}(s)$ and a set S of size $s+3$, define a *labeling* ℓ as a function $\ell\colon S\to V_T$ satisfying condition

$$|\ell^{-1}(v)| = w(v) - \deg v + 3 \tag{1.3}$$

for any $v\in V_T$ (recall that the right hand side is always nonnegative due to (1.2)). Two labelings ℓ_1,ℓ_2 of a weighted tree (T,w) are called equivalent if there exists an automorphism φ of (T,w) such that $\ell_1=\varphi\circ\ell_2$. We denote by $\mathcal{L}(T,w)$ the set of all labelings of (T,w) modulo this equivalence. Abusing notation, we denote the elements of $\mathcal{L}(T,w)$ with the same letter ℓ and call them labelings. Let (T,w) be a weighted tree with a labeling ℓ. For any edge $e\in E_T$ define the equivalence relation $\sim_e$ on S by the following condition: $s\sim_e t$ if the unique simple path between $\ell(s)$ and $\ell(t)$ in T does not pass through e. Evidently, this equivalence relation has exactly two equivalence classes corresponding to the two subtrees obtained from T after deleting e. Combined with condition (1.3), this means that each edge of T defines a partition of S into two parts of size at least 2 each.

Given two partitions μ and ν as above, define $S=\{1,\dots,n,\bar1,\dots,\bar m\}$. The size of S is $m+n$, so it can be used for labeling trees in $\mathcal{T}(m+n-3)$. Given such a labeling, define the *defect* $\delta_\ell(e)$ of an edge $e\in E_T$ by the following expression:

$$\delta_\ell(e)=\sum_{i\sim_e\bar1}\nu_i-\sum_{\bar i\sim_e\bar1}\mu_i.$$

Finally, for any $e\in E_T$ put

$$\xi_\ell(e)=\begin{cases}\mu_1+(\delta_\ell(e))_+ & \text{if } 1\sim_e 2\not\sim_e \bar1,\\ (\delta_\ell(e))_+ & \text{otherwise,}\end{cases}$$

where $a_+=\max\{0,a\}$.

We are now ready to formulate our main result. Let $x_1,x_2,\dots$ and $y_1,y_2,\dots$ be two sets of formal variables corresponding to the edges $e_1,e_2,\dots$ and vertices $v_1,v_2,\dots$ numbered arbitrarily. For any edge $e\in E_T$ with the endpoints v_i and v_j put $y(e)=y_i+y_j$. Similarly, for any vertex $v\in V_T$ with the incident edges $e_{i_1},\dots,e_{i_{\deg v}}$ put $x(v)=x_{i_1}+\dots+x_{i_{\deg v}}$.

Theorem 1.1. *The double Hurwitz numbers $h^0_{\mu,\nu}$ with $m\geqslant1$, $n\geqslant2$ are given by*

$$h^0_{\mu,\nu}=\frac{r!(r-1)!}{|\operatorname{Aut}\mu||\operatorname{Aut}\nu|}\sum_{(T,w)\in\mathcal{T}(r-1)}(-1)^{W_T}C(T,w)\sum_{\ell\in\mathcal{L}(T,w)}\prod_{e\in E_T}\frac{\xi_\ell(e)^{w(e)+1}}{(w(e)+1)!}, \tag{1.4}$$

where

$$C(T,w)=\left[\prod_{i=1}^{k}x_i^{w(e_i)}\prod_{j=1}^{k+1}y_j^{w(v_j)}\right]\prod_{v\in V_T}x(v)^{w(v)}\prod_{e\in E_T}y(e)^{w(e)}, \tag{1.5}$$

$r=r^0_{\mu,\nu}=m+n-2$, and $[A]B$ denotes the coefficient of A in B.

Observe that the product $\prod_{e\in E_T}\xi_\ell(e)^{w(e)+1}$ is a homogeneous piecewise polynomial of degree $r-1$.

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2. Computations

2.1 One-part double Hurwitz numbers. Consider first the case of one-part double Hurwitz numbers, when $m = 1$, and hence $\mu_1 = d$. It is easy to see that under this condition $\delta_\ell(e) < 0$ for any tree T, any edge $e \in E_T$, and any labeling ℓ, and therefore, $\xi_\ell(e)$ is either d or 0. Let us say that a labeling ℓ is *nontrivial* if it gives a nontrivial contribution to the right hand side of (1.4). It is easy to see that the contribution of each nontrivial labeling is proportional to d^{n-2}. Moreover, each nontrivial labeling possesses the following property: $1 \sim_e 2$ and $1 \not\sim_e \bar{1}$ for any edge e. The only tree that admits a labeling possessing this property is a path. Denote by v_1 and v_{k+1} the endpoints of the path; all nontrivial labelings of the path are characterized by conditions

$$\ell(1) = \ell(2) = v_1, \qquad \ell(\bar{1}) = v_{k+1}. \tag{2.1}$$

Denote by $\mathcal{P}(s)$ the set of (equivalence classes of) all weighted paths (P, w) satisfying condition $|E_P| + W_P = s$, and by $\mathcal{L}^*(P, w)$ the set of (equivalence classes of) labelings of (P, w) satisfying condition (2.1). It follows from Theorem 1.1 and the above discussion that

$$h^0_{(d),\nu} = \frac{(n-1)!}{|\operatorname{Aut}\nu|} d^{n-2} I_n,$$

where

$$I_n = (n-2)! \sum_{(P,w)\in\mathcal{P}(n-2)} (-1)^{W_P} \frac{C(P,w)|\mathcal{L}^*(P,w)|}{\prod_{e\in E_T}(w(e)+1)!}.$$

Lemma 2.1. $I_n = 1$.

We were unable to find a combinatorial proof of Lemma 2.1. However, its algebro-geometric meaning is very transparent, and will be explained below in Section 3. As an immediate corollary we get expression (1.1) for the one-part double Hurwitz numbers.

2.2 Two-part double Hurwitz numbers: the extremal case. Let now $m = 2$, and assume in addition that $\mu_1 \geqslant \nu_1 \geqslant \cdots \geqslant \nu_n \geqslant \mu_2$. Similarly to the previous case, $\delta_\ell(e)$ is always negative. Therefore, the contribution of each nontrivial labeling is proportional to μ_1^{n-1}. The structure of graphs admitting nontrivial labelings, and the conditions characterizing nontrivial labelings are exactly the same as in the previous case. We therefore get

$$h^0_{(\mu_1,\mu_2),\nu} = \frac{n!}{|\operatorname{Aut}\mu||\operatorname{Aut}\nu|}\mu_1^{n-1} I_{n+1} \qquad \text{for } \mu_1 \geqslant \nu_1 \geqslant \cdots \geqslant \nu_n \geqslant \mu_2.$$

Using again Lemma 2.1 we arrive to the following result.

Corollary 2.2. *Assume that $\mu_1 \geqslant \nu_1 \geqslant \cdots \geqslant \nu_n \geqslant \mu_2$, then the two-part double Hurwitz numbers $h^0_{(\mu_1,\mu_2),\nu}$ are given by*

$$h^0_{(\mu_1,\mu_2),\nu} = \frac{n!}{|\operatorname{Aut}\mu||\operatorname{Aut}\nu|}\mu_1^{n-1}.$$

Observe that the only situation when the factor $|\operatorname{Aut}\mu|$ in the denominator is nontrivial arises when $n = 2$ and $\mu_1 = \mu_2 = \nu_1 = \nu_2$.

2.3 Five ramification points: complete description. The first case not covered by the above two formulas arises when $m+n=5$. Since $m \leqslant n$, and the case $m=1$ is covered by (1.1), we remain with the case $m=2$ and $\mu_1 \geqslant \mu_2$, $\nu_1 \geqslant \nu_2 \geqslant \nu_3$. Below we derive from Theorem 1.1 an explicit expression for double Hurwitz numbers in this case.

Since $r = m+n-2 = 3$, the summation in (1.4) goes over $\mathcal{T}(2)$. This set contains exactly two weighted trees shown on Fig. 1; the numbers above the trees are weights of edges, those below the trees are weights of vertices.

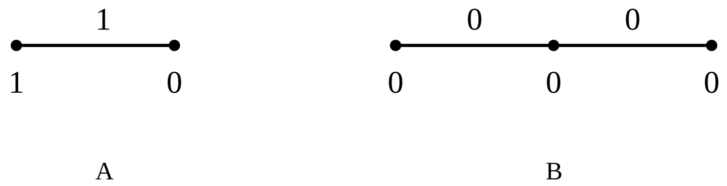

FIG. 1. THE SET $\mathcal{T}(2)$: A) (T_1, w_1); B) (T_2, w_2)

For the first tree one has $C(T_1, w_1) = [x_1 y_1](y_1 + y_2)x_1 = 1$, for the second one, $C(T_2, w_2) = [1]1 = 1$. To calculate the contributions of labelings, define $\delta(a,b)$, $a, b \in \{1, 2, 3, \bar{1}, \bar{2}\}$, by the following rules:

$$\begin{aligned} &\delta(i,j) = 0, && i,j = 1,2,3, \\ &\delta(\bar{i},\bar{j}) = 0, && i,j = 1,2, \\ &\delta(i,\bar{j}) = (-1)^j(\mu_j - \nu_i), && i = 1,2,3, \ j = 1,2. \end{aligned}$$

Consider an arbitrary edge e of T_1 or T_2. According to (1.3), it is incident to a unique vertex having exactly two preimages under each labeling. Let ℓ be a labeling, $a, b \in \{1, 2, 3, \bar{1}, \bar{2}\}$ be the two preimages. It is easy to check that $\delta_\ell(e) = \delta(a,b)$. Besides, $1 \sim_e 2 \not\sim_e \bar{1}$ if and only if $\{a,b\} = \{1,2\}, \{\bar{1},\bar{2}\}, \{3,\bar{1}\}$. Therefore, there are at most 8 nontrivial labelings of (T_1, w_1), and at most 8 nontrivial labelings of (T_2, w_2). Adding all the contributions, we get

$$h^0_{(\mu_1,\mu_2),(\nu_1,\nu_2,\nu_3)} = \frac{6}{|\operatorname{Aut}\mu||\operatorname{Aut}\nu|} P(\mu_1, \mu_2, \nu_1, \nu_2, \nu_3),$$

where

$$\begin{aligned} P(\alpha,\beta,\gamma,\delta,\varepsilon) &= \alpha^2 + 2\alpha((\beta-\gamma)_+ + (\beta-\delta)_+ + (\beta-\varepsilon)_+) + 2(\gamma-\alpha)_+((\beta-\delta)_+ + (\beta-\varepsilon)_+) \\ &+ 2(\delta-\alpha)_+(\beta-\varepsilon)_+ - (\beta-\gamma)_+^2 - (\beta-\delta)_+^2 - (\beta-\varepsilon)_+^2 - (\gamma-\alpha)_+^2 - (\delta-\alpha)_+^2; \end{aligned}$$

while deriving this formula it was taken into account that $\mu_1 + \mu_2 = \nu_1 + \nu_2 + \nu_3$ implies $(\nu_3 - \mu_1)_+ = 0$ and $(\nu_2 - \mu_1)_+(\mu_2 - \nu_1)_+ = 0$.

3. PROOFS

Denote by p_i, $i = 1, \dots, n$, the ramification points with multiplicities ν_i, and by $p_{\bar{j}}$, $j = 1, \dots, m$, the ramification points with multiplicities μ_j. Define the *Hurwitz* *space* $\mathcal{H}_{\mu,\nu}$ as the space of meromorphic functions on genus 0 curves with zeros at p_i of multiplicities ν_i and poles at $p_{\bar{i}}$ of multiplicities μ_i. More exactly, denote by $U = \overline{\mathcal{M}}_{0,m+n+1}$ the universal curve over $\overline{\mathcal{M}}_{0,m+n}$. Consider the relative sheaf $\mathcal{O}(\sum \mu_i p_{\bar{i}} - \sum \nu_i p_i)$, i.e., the sheaf of local sections of meromorphic functions on the fibers of U with poles at $p_{\bar{i}}$ of multiplicities μ_i and zeroes at p_i of multiplicities ν_i. By $\mathcal{H}_{\mu,\nu}$ we denote its push-forward to $\overline{\mathcal{M}}_{0,m+n}$.

It is easy to see that $\mathcal{H}_{\mu,\nu}$ is a vector bundle over $\overline{\mathcal{M}}_{0,m+n}$. The following result is an analog of Lemma 5.3 from [ELSV2].

Lemma 3.1.
$$h^0_{\mu,\nu} = \frac{(m+n-2)!}{|\operatorname{Aut}\mu||\operatorname{Aut}\nu|}\int_{\overline{\mathcal{M}}_{0,m+n}} s_{top}(\mathcal{H}_{\mu,\nu}),$$
where $s_{top}(\mathcal{H}_{\mu,\nu})$ is the top Segre class of $\mathcal{H}_{\mu,\nu}$.

The proof is similar to the proof of Lemma 5.3 from [ELSV2] (see also [LZ, Thm. 4] for a similar result).

Since $n \geqslant 2$, for a smooth curve we can choose a local coordinate x such that $x(p_1) = 0$, $x(p_2) = 1$, $x(p_{\bar{1}}) = \infty$. Therefore, any rational function with the prescribed ramifications can be written as

$$f(x) = C\frac{x^{\nu_1}(x-1)^{\nu_2}(x-a_3)^{\nu_3}\dots(x-a_n)^{\nu_n}}{(x-a_{\bar{2}})^{\mu_2}\dots(x-a_{\bar{m}})^{\mu_m}}, \tag{3.1}$$

where $a_j = x(p_j)$, $a_{\bar{j}} = x(p_{\bar{j}})$. We see that $\mathcal{H}_{\mu,\nu}$ is a line bundle, and the constant C is a coordinate in its fiber. It is important to note that, since the degree of functions in $\mathcal{H}_{\mu,\nu}$ is a constant, vanishing of an element of $\mathcal{H}_{\mu,\nu}$ does not mean that the corresponding map from a curve Γ to $\mathbb{C}\mathbf{P}^1$ becomes the constant map to point 0. Rather, it means that the coefficient C in (3.1) vanishes, and the corresponding function degenerates to a prestable map from a prestable curve $\tilde{\Gamma}$ to $\mathbb{C}\mathbf{P}^1$, where $\tilde{\Gamma} = \Gamma \cup \bigcup_{\bar{i}} \Gamma_{\bar{i}}$. A new irreducible component $\Gamma_{\bar{i}}$ emerges from each marked point $p_{\bar{i}}$. The restriction of the function on $\Gamma_{\bar{i}}$ has a form z^{μ_i}, where z is a local coordinate on $\Gamma_{\bar{i}}$ vanishing at the double point. The same holds true when functions vanish on irreducible components of singular curves.

Recall that the total Segre class of a line bundle is the inverse of its total Chern class. Since $\mathcal{H}_{\mu,\nu}$ is one-dimensional, we have $c(\mathcal{H}_{\mu,\nu}) = 1 + c_1(\mathcal{H}_{\mu,\nu})$, and hence the formula in Lemma 3.1 translates to
$$h^0_{\mu,\nu} = \frac{(m+n-2)!}{|\operatorname{Aut}\mu||\operatorname{Aut}\nu|}\int_{\overline{\mathcal{M}}_{0,m+n}} (-1)^{m+n-3} c_1^{m+n-3}(\mathcal{H}_{\mu,\nu}).$$

To calculate the first Chern class of $\mathcal{H}_{\mu,\nu}$, consider the section $\sigma = \{C = 1\}$. It is evident that zeroes and poles of σ are located on the boundary divisors of $\overline{\mathcal{M}}_{0,m+n}$. Given a subset $K \subset \{1,\dots,n,\bar{1},\dots,\bar{m}\}$ such that $2 \leqslant |K| \leqslant m+n-2$, denote by D_K the boundary divisor containing two-component curves with the points p_i, $i \in K$, and $p_{\bar{j}}$, $\bar{j} \in K$, lying on one of the components, and all the other marked point lying on the other component. Besides, denote $\nu_K = \sum_{i\in K}\nu_i$ and $\mu_K = \sum_{\bar{j}\in K}\mu_j$. Taking into account that $D_K = D_{K^c}$, where K^c is the complement of K, we may assume that $\bar{1} \in K$ and relate each divisor to one of the following two groups:

(i) $K \cap \{1,2\} \neq \varnothing$;

(ii) $K \cap \{1,2\} = \varnothing$.

The poles and zeros of the section σ are described by the following proposition.

Lemma 3.2. (a) *σ does not vanish on any of D_K.*

(b) *σ has a pole of order $(\nu_K-\mu_K)_+$ on any D_K related to group* (i).

(c) *σ has a pole of order $\mu_1+(\nu_K-\mu_K)_+$ on any D_K related to group* (ii).

Proof. To understand the behavior of σ in neighborhood of boundary components we consider the degeneration of σ along a 1-parameter family of curves. We denote the parameter by ε and assume that curves in the family are nonsingular for nonzero values of ε. For $\varepsilon=0$ a curve has at most two irreducible components.

Consider case (i) above. Our 1-parameter family is given by $x(p_i(\varepsilon)), x(p_{\bar{j}}(\varepsilon))\to a_0$, $|a_0|<\infty$, as $\varepsilon\to 0$ for $i,\bar{j}\in K^c$. We can assume $x(p_i)-a_0=\varepsilon A_i$, $x(p_{\bar{j}})-a_0=\varepsilon A_{\bar{j}}$ for some finite $A_i, A_{\bar{j}}$. The curve $\Gamma(\varepsilon)$ splits into two components $\Gamma_1\cup\Gamma_2$ as $\varepsilon\to 0$, such that $x(\Gamma_1\cap\Gamma_2)=a_0$, and $p_{\bar{1}}\in\Gamma_1$ whereas Γ_2 contains $p_i, p_{\bar{j}}$, $i,\bar{j}\in K^c$. Consider first the case $\nu_K-\mu_K>0$. Since the defect is positive for Γ_1, the nontrivial part is the restriction of σ on Γ_2. Choose a coordinate y on Γ_2 using $x-a_0=\varepsilon y$. Then

$$\sigma\to\frac{\prod_{i\in K^c}(\varepsilon(y-A_i))^{\nu_i}\prod_{i\in K}(a_0-a_i)^{\nu_i}}{\prod_{\bar{j}\in K^c}(\varepsilon(y-A_{\bar{j}}))^{\mu_j}\prod_{\bar{j}\in K\setminus\{\bar{1}\}}(a_0-a_{\bar{j}})^{\mu_j}}
=\frac{1}{\varepsilon^{\nu_K-\mu_K}}\frac{\prod_{i\in K^c}(y-A_i)^{\nu_i}\prod_{i\in K}(a_0-a_i)^{\nu_i}}{\prod_{\bar{j}\in K^c}(y-A_{\bar{j}})^{\mu_j}\prod_{\bar{j}\in K\setminus\{\bar{1}\}}(a_0-a_{\bar{j}})^{\mu_j}}.$$

Hence, σ has a pole of order $\nu_K-\mu_K$ on D_K.

Consider now the case $\nu_K-\mu_K<0$. Since the defect is positive for Γ_2, the nontrivial part is the restriction of σ on Γ_1. On Γ_1 we have

$$\sigma\to\frac{\prod_{i\in K^c}(x-a_0)^{\nu_i}\prod_{i\in K}(x-a_i)^{\nu_i}}{\prod_{\bar{j}\in K^c}(x-a_0)^{\mu_j}\prod_{\bar{j}\in K\setminus\{\bar{1}\}}(x-a_{\bar{j}})^{\mu_j}}=(x-a_0)^{\nu_K-\mu_K}\frac{\prod_{i\in K}(x-a_i)^{\nu_i}}{\prod_{\bar{j}\in K\setminus\{\bar{1}\}}(x-a_{\bar{j}})^{\mu_j}}.$$

Hence, σ has neither poles nor zeros on D_K. The same is true also in the case $\nu_K-\mu_K=0$.

Consider now case (ii) above. Our 1-parameter family is given by the condition $x(p_i(\varepsilon)), x(p_{\bar{j}}(\varepsilon))\to\infty$ as $\varepsilon\to 0$ for $i,\bar{j}\in K$. We can assume $x(p_i)=A_i/\varepsilon$, $x(p_{\bar{j}})=A_{\bar{j}}/\varepsilon$ for some finite $A_i, A_{\bar{j}}$. Consider first the case $\nu_K-\mu_K>0$. Then the nontrivial part of σ is the restriction on Γ_1, where we have

$$\sigma\to\frac{\prod_{i\in K}(x-A_i/\varepsilon)^{\nu_i}\prod_{i\in K^c}(x-a_i)^{\nu_i}}{\prod_{\bar{j}\in K\setminus\{\bar{1}\}}(x-A_{\bar{j}}/\varepsilon)^{\mu_j}\prod_{\bar{j}\in K^c}(x-a_{\bar{j}})^{\mu_j}}
=\frac{1}{\varepsilon^{\nu_K-\mu_K+\mu_1}}\frac{\prod_{i\in K}(-A_i)\prod_{i\in K^c}(x-a_i)^{\nu_i}}{\prod_{\bar{j}\in K\setminus\{\bar{1}\}}(-A_{\bar{j}})\prod_{\bar{j}\in K^c}(x-a_{\bar{j}})^{\mu_j}}.$$

Hence, σ has a pole of order $\nu_K-\mu_K+\mu_1$ at D_K. Finally, it is easy to see that for $\nu_K-\mu_K\leqslant 0$, σ has a pole of order μ_1 at D_K. $\square$

Slightly abusing notation, we use the same symbol D_K to denote the dual cohomology class of the boundary divisor D_K. Therefore, by Lemma 3.2,

$$c_1(\mathcal{H}_{\mu,\nu})
=-\left(\sum_{\substack{K\cap\{1,2\}\neq\varnothing\\ 2\leqslant|K|\leqslant m+n-2}}(\nu_K-\mu_K)_+D_K+\sum_{\substack{K\cap\{1,2\}=\varnothing\\ 2\leqslant|K|\leqslant m+n-2}}\left(\mu_1+(\nu_K-\mu_K)_+\right)D_K\right).$$

To complete the computation, it is convenient to introduce some additional notation. Let S be a finite set. A 2-*partition* of S is a pair (S_1, S_2) such that $S_1, S_2 \neq \varnothing$ and $S_1 = S_2^c$. Two 2-partitions (S_1, S_2) and (S_1', S_2') are called *non-crossing* if there exist S_i and S_j' such that $S_i \cap S_j' = \varnothing$. A family $\mathcal{C}$ of 2-partitions is called *crossing free* if any two 2-partitions in $\mathcal{C}$ are non-crossing.

Let $T = (V_T, E_T)$ be an arbitrary tree, and $\ell \colon S \to V_T$ be an arbitrary mapping. Each edge $e \in E_T$ defines two subtrees obtained from T by deleting e. The full preimages of the vertex sets of these subtrees under ℓ induce a 2-partition of S; in other words $s, t \in S$ belong to the same part if and only if $\ell(s)$ and $\ell(t)$ belong to the same subtree. It is easy to see that the family of such 2-partitions of S corresponding to all edges in E_T is crossing free; we denote it by $\mathcal{C}(T)$. By the Edmonds–Giles theorem [EdGi, Thm. 8.3], a family $\mathcal{C}$ is crossing free if and only if there exists a tree T and a mapping $\ell : S \to V_T$ such that $\mathcal{C} = \mathcal{C}(T)$. Moreover, this tree is unique when the number of its edges is minimal, that is, different edges define different 2-partitions of S. The mapping ℓ in this case is unique as well.

Let $M = D_{K_1}^{\gamma_1} \cdots D_{K_k}^{\gamma_k}$ be an arbitrary monomial of total degree $m + n - 3$ in variables $D_K \in H^2(\overline{\mathcal{M}}_{0,m+n})$. We assign to each D_K the 2-partition (K, K^c) of the set $S = \{1, \dots, n, \bar{1}, \dots, \bar{m}\}$, and to the monomial M, the family $\mathcal{C}_M$ of 2-partitions corresponding to $D_{K_1}, \dots, D_{K_k}$. According to [Ke, Thm. 1], $\int_{\overline{\mathcal{M}}_{0,m+n}} M$ can be non-zero only if $\mathcal{C}_M$ is crossing free.

Assume that $\mathcal{C}_M$ is crossing free, and let T_M, ℓ_M be the unique minimal tree and the mapping $S \to V_{T_M}$ corresponding to $\mathcal{C}_M$ by the Edmonds–Giles theorem. The geometric meaning of T_M and ℓ_M is transparent. Consider the $(k+1)$-component stable curve corresponding to the set $\{D_{K_1}, \dots, D_{K_k}\}$. The vertices of T_M are the components of this curve, the edges of T_M correspond to the double points, and the full preimage of a vertex under ℓ_M is the set of marked points lying on the corresponding component.

Denote by e_i the edge of T_M corresponding to D_{K_i}. Let us define a weight function $w_M \colon V_{T_M} \cup E_{T_M} \to \mathbb{Z}_{\geqslant 0}$. For an edge $e_i \in E_{T_M}$ put $w_M(e_i) = \gamma_i - 1$. For a vertex $v \in V_{T_M}$ put $w(v) = |\ell_M^{-1}(v)| + \deg v - 3$. It is easy to see that the weight function w_M thus defined satisfies both conditions in (1.2). The nonnegativity of $w_M(v)$ is evident for vertices with degree at least 3. For vertices of degree 2 it follows from the minimality of the tree, and for vertices of degree 1, from the stability condition $2 \leqslant |K| \leqslant m + n - 2$. Moreover, the total weight of the tree equals $m + n - 3 - |E_{T_M}|$, hence $(T_M, w_M) \in \mathcal{T}(m + n - 3)$. Finally, condition (1.3) is satisfied, and hence ℓ_M is a labeling of (T_M, w_M).

Lemma 3.3. *Let $M = D_{K_1}^{\gamma_1} \cdots D_{K_k}^{\gamma_k}$ be a monomial of total degree $m + n - 3$ such that $\mathcal{C}(M)$ is crossing free, and let $(T_M, w_M) \in \mathcal{T}(m + n - 3)$ be the corresponding weighted tree. Then*

$$\int_{\overline{\mathcal{M}}_{0,m+n}} M = (-1)^{W_{T_M}} C(T_M, w_M)$$

with $C(T, w)$ defined by (1.5).

Proof. Following [Ka], consider the linear system in variables $f_{v,e}$, $v \in V_{T_M}$, $e \in E_{T_M}$, $e \in \mathrm{st}(v)$:

$$f(v_1, e) + f(v_2, e) = w_M(e), \quad e = (v_1, v_2), \qquad \sum_{e \in \mathrm{st}(v)} f(v, e) = w_M(v),$$

where $\mathrm{st}(v)$ is the set of edges incident to v. It is easy to see that this system, though overdefined, has a unique solution. The main theorem of [Ka] claims that

$$\int_{\overline{\mathcal{M}}_{0,m+n}} M = \prod_{v\in V_{T_M}} (-1)^{w_M(v)} \frac{w_M(v)!}{\prod_{e\in \mathrm{st}(v)} f(v,e)!^2} \prod_{e\in E_{T_M}} w_M(e)!$$

provided all $f(v,e)$ are nonnegative, and $\int_{\overline{\mathcal{M}}_{0,m+n}} M = 0$ otherwise. This formula can be rewritten as

$$\int_{\overline{\mathcal{M}}_{0,m+n}} M \\ = (-1)^{W_{T_M}} \prod_{\substack{v\in V_{T_M}\\ e_i^v\in \mathrm{st}(v)}} \binom{w_M(v)}{f(v,e_1^v)\cdots f(v,e_{\deg v}^v)} \prod_{\substack{e\in E_{T_M}\\ e=(v_1^e,v_2^e)}} \binom{w_M(e)}{f(v_1^e,e)\; f(v_2^e,e)}.$$

Observe that according to the definition of a multinomial coefficient, $\binom{a}{b_1\cdots b_t} = 0$ if $b_i < 0$ for some i, $1 \leqslant i \leqslant t$. Therefore, the above formula is correct independently of the signs of $f(v,e)$'s. It remains to notice that the double product in the right hand side is exactly the coefficient $C(T_M, w_M)$ defined by (1.5). $\square$

Theorem 1.1 follows easily from the above discussion.

It remains to prove Lemma 2.1. Denote by $\psi_{\bar 1}$ the first Chern class of the line bundle over $\overline{\mathcal{M}}_{0,m+n}$ whose fiber is the cotangent space to the curve at $p_{\bar 1}$. It is well known that

$$\int_{\overline{\mathcal{M}}_{0,m+n}} \psi_{\bar 1}^{m+n-3} = 1. \tag{3.1}$$

We are going to prove that the above integral equals I_{m+n-3}. Indeed, it is easy to see that $\psi_{\bar 1} = \sum D_K$, where the sum is taken over all $K \subset \{1,\dots,n,\bar 1,\dots,\bar m\}$ such that $1,2 \notin K$, $\bar 1 \in K$, $2 \leqslant |K| \leqslant m+n-2$. Let M be a monomial in variables D_K of total degree $m+n-3$ with K satisfying the above conditions. It is clear from above that if M has a nontrivial contribution to the integral (3.1) then $\mathcal{C}(M)$ is crossing free. Moreover, the corresponding tree T_M is a path with the endpoints $\ell_M(1) = \ell_M(2)$ and $\ell_M(\bar 1)$. Indeed, if this is not the case then there exists an edge of T_M such that 1, 2 and $\bar 1$ are mapped by ℓ_M to the same of two subtrees defined by e, in a contradiction with the conditions defining possible subsets K. Observe that the above conditions on ℓ_M coincide with conditions (2.1) on the labeling of the path. Thus to get Lemma 2.1 it remains to apply Lemma 3.3.

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