AMOEBAS OF ALGEBRAIC VARIETIES

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The notion of amoebas for algebraic varieties was introduced in 1994 by Gelfand, Kapranov and Zelevinski [7]. Some traces of amoebas were appearing from time to time, even before the formal introduction, as auxiliary tools in several problems (see e.g. [3]). After 1994 amoebas have been seen and studied in several areas of mathematics, from algebraic geometry and topology to complex analysis and combinatorics.

In particular, amoebas provided a very powerful tool for studying topology of algebraic varieties. The purpose of this survey is to summarize our current state of knowledge about amoebas and to outline their applications to real algebraic geometry and adjacent areas. Most proofs are omitted here. An expanded version of this survey is currently under preparation jointly with Oleg Viro [19].

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1. Definition and basic properties of amoebas

1.1. **Definitions.** Let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety. Recall that $\mathbb{C}^* = \mathbb{C} \setminus 0$ is the group of complex numbers under multiplication. Let Log : $(\mathbb{C}^*)^n \to \mathbb{R}^n$ be defined by $\operatorname{Log}(z_1, \dots, z_n) \to (\log|z_1|, \dots, \log|z_n|)$.

Definition 1 (Gelfand-Kapranov-Zelevinski [7])**.** The *amoeba* of V is $\mathcal{A} = \operatorname{Log}(V) \subset \mathbb{R}^n$.

Proposition 1 (Gelfand-Kapranov-Zelevinski [7])**.** *The amoeba $\mathcal{A} \subset \mathbb{R}^n$ is a closed set with a non-empty complement.*

If $\mathbb{C}T \supset (\mathbb{C}^*)^n$ is a closed n-dimensional toric variety and $\bar{V} \subset \mathbb{C}T$ is a compactification of V then we say that $\mathcal{A}$ is the amoeba of $\bar{V}$ (recall that $\mathcal{A}$ is also the amoeba of $V = \bar{V} \cap (\mathbb{C}^*)^n$). Thus we can speak about amoebas of projective varieties once the coordinates in $\mathbb{CP}^n$, or at least an action of $(\mathbb{C}^*)^n$, is chosen.

If $\mathbb{C}T$ is equipped with a $(\mathbb{C}^*)^n$-invariant symplectic form then we can also consider the corresponding moment map $\bar{\mu} : \mathbb{C}T \to \Delta$ (see [2],[7]), where Δ is the convex polyhedron associated to the toric variety $\mathbb{C}T$ with the given symplectic form. The polyhedron Δ is a subset of $\mathbb{R}^n$ but it is well defined

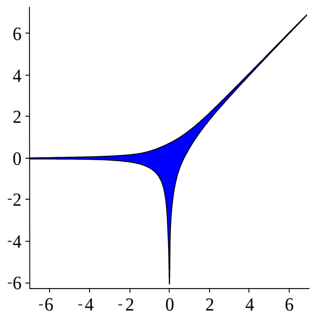

FIGURE 1. [19] The amoeba of the line $\{x+y+1=0\} \subset (\mathbb{C}^*)^2$.

only up to a translation. In this case we can also define the *compactified amoeba* of $\bar{V}$.

Definition 2 (Gelfand-Kapranov-Zelevinski [7]). The *compactified amoeba* of V is $\bar{\mathcal{A}} = \bar{\mu}(V) \subset \Delta$.

FIGURE 2. [17] Amoeba $\mathcal{A}$ and compactified amoeba $\bar{\mathcal{A}}$.

Remark 1. Maps $\bar{\mu}|_{(\mathbb{C}^*)^n}$ and Log are submersions and have the same real n-tori as fibers. Thus $\mathcal{A}$ is mapped diffeomorphically onto $\bar{\mathcal{A}} \cap \operatorname{Int}\Delta$ under a reparametrization of $\mathbb{R}^n$ onto $\operatorname{Int}\Delta$.

Using the compactified amoeba we can describe the behavior of $\mathcal{A}$ near infinity. Note that each face Δ' of Δ determines a toric variety $\mathbb{C}T' \subset \mathbb{C}T$. Consider $\bar{V}' = \bar{V} \cap \mathbb{C}T'$. Let $\bar{\mathcal{A}}'$ be the compactified amoeba of $\bar{V}'$.

Proposition 2 (Gelfand-Kapranov-Zelevinski [7]). *We have* $\bar{\mathcal{A}}' = \bar{\mathcal{A}} \cap \Delta'$.

This proposition can be used to describe the behavior of $\mathcal{A} \subset \mathbb{R}^n$ near infinity.

1.2. **Amoebas at infinity.** Consider a linear subspace $L \subset \mathbb{R}^n$ parallel to Δ' and with $\dim L = \dim \Delta'$. Let $H \subset \mathbb{R}^n$ be a supporting hyperplane for the convex polyhedron Δ at the face Δ', i.e. a hyperplane such that $\Delta \cap H = \Delta'$. Let $\vec{v}$ be an outwards normal vector to H. Let $\mathcal{A}_t^{\Delta'}$, $t > 0$, be the intersection of L with the result of translation of $\mathcal{A}$ by $-t\,\vec{v}$.

Proposition 3. *The subsets* $\mathcal{A}_t^{\Delta'}$ *converge in the Hausdorff metric to* $\mathcal{A}'$ *when* $t \to \infty$.

This proposition can be informally restated in the case $n = 2$. In this case Δ is a polygon and the amoeba $\mathcal{A}$ develops "tentacles" perpendicular to the sides of Δ (see Figure 1.3). The number of tentacles perpendicular to a side of Δ is equal to the integer length of this side, i.e. one plus the number of the lattice points in the interior of the side.

Note that we may assume (by passing to a different toric variety $\mathbb{C}T$ if needed) that V does not pass through the vertices of $\mathbb{C}T$, i.e. the fixed points of the $(\mathbb{C}^*)^n$-action. Thus we get the following corollary.

Corollary 4. *For a generic choice of the slope of a line ℓ in $\mathbb{R}^n$ the intersection $\mathcal{A} \cap \ell$ is compact.*

1.3. **Amoebas of hypersurfaces: concavity and topology of the complement.** Hypersurfaces case was treated by Forsberg, Passare and Tsikh in [6]. In this case V is a zero set of a single polynomial $f(z) = \sum_j a_j z^j, a_j \in \mathbb{C}$. Here we use the multiindex notations $z = (z_1, \dots, z_n)$, $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$ and $z^j = z_1^{j_1} \dots z_n^{j_n}$. Let

$$\Delta = \text{Convex hull}\{j \mid a_j \neq 0\} \subset \mathbb{R}^n \tag{1}$$

be the Newton polyhedron of f.

Theorem 5 (Forsberg-Passare-Tsikh [6]). *Each component of $\mathbb{R}^n \setminus \mathcal{A}$ is a convex domain in $\mathbb{R}^n$. There exists a locally constant function*

$$\text{ind} : \mathbb{R}^n \setminus \mathcal{A} \to \Delta \cap \mathbb{Z}^n$$

which maps different components of the complement of $\mathcal{A}$ to different lattice points of Δ.

Corollary 6 (Forsberg-Passare-Tsikh [6]). *The number of components of $\mathbb{R}^n \setminus \mathcal{A}$ is never greater then the number of lattice points of Δ.*

Theorem 5 and Proposition 3 indicate the dependence of the amoeba on the Newton polyhedron.

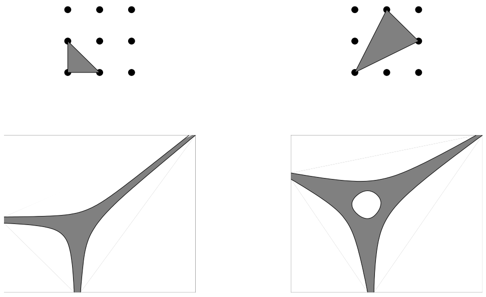

FIGURE 3. Amoebas together with their Newton polyhedra.

The inequality of Corollary 6 is sharp. This sharpness is a special case of Theorem 17 from section 2. Also examples of amoebas with the maximal number of the components of the complement are supplied by Theorem 49 from section 4.

The concavity of $\mathcal{A}$ is equivalent to concavity of its boundary. The boundary $\partial\mathcal{A}$ is contained in the critical value locus of $\mathrm{Log}|_V$. The following proposition also takes care of some interior branches of this locus.

Proposition 7 (Mikhalkin [17]). *Let $D \subset \mathbb{R}^n$ be an open convex domain and V' be a connected component of $\mathrm{Log}^{-1}(D) \cap V$. Then $D \setminus \mathrm{Log}(V')$ is convex.*

1.4. **Amoebas in higher codimension: concavity.** The amoeba of a hypersurface is of full dimension in $\mathbb{R}^n$, $n > 1$, unless its Newton polyhedron Δ is contained in a line. The boundary $\partial\mathcal{A}$ at its generic point is a smooth $(n-1)$-dimensional submanifold. Its normal curvature form has no negative squares with respect to the outwards normal (because of convexity of components of $\mathbb{R}^n \setminus \mathcal{A}$). This property can be generalized to the non-smooth points in the following way.

Definition 3. An open interval $D^1 \subset L$, where L is a straight line in $\mathbb{R}^n$, is called a *supporting 1-cap* for $\mathcal{A}$ if

- $D^1 \cap \mathcal{A}$ is non-empty and compact;
- there exists a vector $\vec{v} \in \mathbb{R}^n$ such that the translation of D^1 by $\epsilon\vec{v}$ is disjoint from $\mathcal{A}$ for all sufficiently small $\epsilon > 0$.

The convexity of the components of $\mathbb{R}^n \setminus \mathcal{A}$ can be reformulated as stating that *there are no 1-caps for $\mathcal{A}$*.

Similarly we may define higher-dimensional caps.

Definition 4. An open round disk $D^k \subset L$ of radius $\delta > 0$ in a k-plane $L \subset \mathbb{R}^n$ is called a *supporting k-cap* for $\mathcal{A}$ if

- $D^k \cap \mathcal{A}$ is non-empty and compact;
- there exists a vector $\vec{v} \in \mathbb{R}^n$ such that the translation of D^k by $\epsilon\vec{v}$ is disjoint from $\mathcal{A}$ for all sufficiently small $\epsilon > 0$.

Consider now the general case, where $V \subset (\mathbb{C}^*)^n$ is l-dimensional. Let $k = n - l$ be the codimension of V. The amoeba $\mathcal{A}$ is of full dimension in $\mathbb{R}^n$ if $2l \geq n$. The boundary $\partial\mathcal{A}$ at its generic point is a smooth $(n-1)$-dimensional submanifold. Its normal curvature form may not have more than $k-1$ negative squares with respect to the outwards normal. To see that note that a composition of $\mathrm{Log}|_V : V \to \mathbb{R}^n$ and any linear projection $\mathbb{R}^n \to \mathbb{R}$ is a pluriharmonic function.

Note that this implies that there are no k-caps for $\mathcal{A}$ at its smooth points. It turns out that there are no k-caps for $\mathcal{A}$ at the non-smooth points as well and also in the case of $2l < n$ when $\mathcal{A}$ is $2l$-dimensional.

Proposition 8 (Local higher-dimensional concavity of $\mathcal{A}$). *If $V \subset (\mathbb{C}^*)^n$ is of codimension k then $\mathcal{A}$ does not have supporting k-caps.*

A global statement generalizing convexity of components was recently found by André Henriques [9].

Definition 5 (Henriques [9]). A subset $\mathcal{A} \subset \mathbb{R}^n$ is called *k-convex* if for any k-plane $L \subset \mathbb{R}^n$ the induced homomorphism $H_{k-1}(L \setminus \mathcal{A}) \to H_{k-1}(\mathbb{R}^n \setminus \mathcal{A})$ is injective.

Theorem 9 (Global higher-dimensional concavity of $\mathcal{A}$, cf. [9]). *If $V \subset (\mathbb{C}^*)^n$ is of codimension k then $\mathcal{A}$ is k-convex.*

A proof of a weaker version of this statement is contained in [9]. Theorem 9 can be deduced from its local version, Proposition 8.

1.5. Amoebas in higher codimension: topology of the complement.

Recall that in the hypersurface case each component of $\mathbb{R}^n \setminus \mathcal{A}$ is connected and that there are not more than $\#(\Delta \cap \mathbb{Z}^n)$ such components. The correspondence between the components of the complement and the lattice points of Δ can be viewed as a cohomology class $\alpha \in H^0(\mathbb{R}^n \setminus \mathcal{A}; \mathbb{Z}^n)$ whose evaluation on a point in each component of $\mathbb{R}^n \setminus \mathcal{A}$ is the corresponding lattice point.

Similarly, when V is of codimension k there exists a natural class (cf. [27])

$$\alpha \in H^{k-1}(\mathbb{R}^n \setminus \mathcal{A}; H^k(T^n)),$$

where T^n is the real n-torus, the fiber of Log, $H^k(T^n) = H^k((\mathbb{C}^*)^n)$. The value of α on each $(k-1)$-cycle C in $\mathbb{R}^n \setminus \mathcal{A}$ and k-cycle C' in T^n is the linking number in $\mathbb{C}^n \supset (\mathbb{C}^*)^n$ of $C \times C'$ and the closure of V.

The cohomology class α corresponds to the linking with the fundamental class of V. Consider now the linking with smaller-dimensional homology of V.

Note that for an l-dimensional variety $V \subset (\mathbb{C}^*)^n$ we have $H_j(V) = 0$, $j > l$. Similarly, $H^c_j(V) = 0$, $j < l$, where H^c stands for homology with closed support. The linking number in $\mathbb{R}^n$ composed with $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ defines the following pairing

$$H^c_l(V) \times H_{k-1}(\mathbb{R}^n \setminus \mathcal{A}) \to \mathbb{Z}.$$

Together with the Poincaré duality between $H^c_l(V)$ and $H_l(V)$ this pairing defines the homomorphism

$$\iota : H_{k-1}(\mathbb{R}^n \setminus \mathcal{A}) \to H_l(V).$$

Question 1. Is ι injective?

Recall that a subspace $L \subset H_l(V)$ is called *isotropic* if the restriction of the intersection form to L is trivial.

Proposition 10. *The image $\iota(H_{k-1}(\mathbb{R}^n \setminus \mathcal{A}))$ is isotropic in $H_l(V)$.*

Remark . A positive answer to Question 1 together with Proposition 10 would produce an upper bound for the dimension of $H_{k-1}(\mathbb{R}^n \setminus \mathcal{A})$.

One may also define similar linking forms for $H_j(\mathbb{R}^n \smallsetminus \mathcal{A})$, $j \neq k-1$ (if $j > k-1$ then we can use ordinary homology $H_{n-j-1}(V)$ instead of homology with closed support) .

The answer to Question 1 is currently unknown even in the case when $V \subset (\mathbb{C}^*)^2$ is a curve. In this case V is a Riemann surface and it is defined by a single polynomial. Let Δ be the Newton polygon of V. The genus of V is equal to the number of lattice points strictly inside Δ (see [15]) while the number of punctures is equal to the number of lattice points on the boundary of Δ). Thus the dimension of a maximal isotropic subspace of $H_1(V)$ is equal to $\#(\Delta \cap \mathbb{Z}^2)$ and Question 1 agrees with Corollary 6 for this case.

2. Some analysis on amoebas

This section outlines the results obtained by Passare and Rullgård in [22], [26] and [27].

We assume that $V \subset (\mathbb{C}^*)^n$ is a hypersurface in this section. Thus $V = \{f = 0\}$ for a polynomial $f : (\mathbb{C}^*)^n \to \mathbb{C}$ and we can consider $\Delta \subset \mathbb{R}^n$, the Newton polyhedron of V (see 1.3).

2.1. The Ronkin function N_f.

Since f is a holomorphic function, $\log|f| : (\mathbb{C}^*)^n \smallsetminus V \to \mathbb{R}$ is a pluriharmonic function. Furthermore, if we set $\log(0) = -\infty$ then we have a plurisubharmonic function

$$\log|f| : (\mathbb{C}^*)^n \to \mathbb{R} \cup \{-\infty\},$$

which is, obviously, strictly plurisubharmonic over V. Recall that a function F in a domain $\Omega \subset \mathbb{C}^n$ is called plurisubharmonic if its restriction to any complex line L is subharmonic, i.e. the value of F at each point $z \in L$ is smaller or equal than the average of the value of F along a small circle in L around z.

Let $N_f : \mathbb{R}^n \to \mathbb{R}$ be the push-forward of $\log|f|$ under the map $\operatorname{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$, i.e.

$$N_f(x_1, \dots, x_n) = \frac{1}{(2\pi i)^n} \int\limits_{\operatorname{Log}^{-1}(x_1,\dots,x_n)} \log|f(z_1, \dots, z_n)| \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n},$$

cf. [25]. This function was called *the Ronkin function* in [22]. It is easy to see that it takes real (finite) values even over $\mathcal{A} = \operatorname{Log}(V)$ where the integral is singular.

Proposition 11 (Ronkin-Passare-Rullgård [22], [25])**.** *The function $N_f : \mathbb{R}^n \to \mathbb{R}$ is convex. It is strictly convex over $\mathcal{A}$ and linear over each component of $\mathbb{R}^n \smallsetminus \mathcal{A}$.*

This follows from plurisubharmonicity of $\log|f| : (\mathbb{C}^*)^n \to \mathbb{R}$, its strict plurisubharmonicity over V and its pluriharmonicity in $(\mathbb{C}^*)^n \smallsetminus V$. Indeed the convexity of a function in a connected real domain is just a real counterpart of plurisubharmonicity. A harmonic function of one real variable has to be linear and thus a function of several real variables is real-plurisubharmonic if and only if it is convex. Over each connected component of $\mathbb{R}^n \setminus \mathcal{A}$ the function is linear as the push-forward of a pluriharmonic function.

Remark 2. Note that just the existence of a convex function N_f, which is strictly convex over $\mathcal{A}$ and linear over components of $\mathbb{R}^n \setminus \mathcal{A}$, implies that each component of $\mathbb{R}^n \setminus \mathcal{A}$ is convex.

Thus the gradient $\nabla N_f : \mathbb{R}^n \to \mathbb{R}^n$ is constant over each component E of $\mathbb{R}^n \setminus \mathcal{A}$. Recall the classical Jensen's formula in complex analysis

$$\frac{1}{2\pi i} \int\limits_{|z|=e^x} \log|f(z)| \frac{dz}{z} = Nx + \log|f(0)| - \sum_{k=1}^{N} \log|a_k|,$$

where $a_1, \dots, a_N$ are the zeroes of f in $|z| < e^x$, if $f(0) \neq 0$ and $f(z) \neq 0$ if $|z| = e^x$. This formula implies that $\nabla N_f(E) \in \mathbb{Z}^n \cap \Delta$.

Proposition 12 (Passare-Rullgård [22]). *We have*

$$\operatorname{Int} \Delta \subset \nabla N_f(\mathbb{R}^n) \subset \Delta,$$

where $\operatorname{Int}\Delta$ *is the interior of the Newton polyhedron.*

Recall that Theorem 5 associates a lattice point to each component of $\mathbb{R}^n \setminus \mathcal{A}$.

Proposition 13 (Passare-Rullgård [22]). *We have*

$$\nabla N_f(E) = \operatorname{ind}(E)$$

for each component E *of* $\mathbb{R}^n \setminus \mathcal{A}$.

2.2. **The spine of amoeba.** Passare and Rullgård [22] used N_f to define *the spine* of amoeba. Recall that N_f is piecewise-linear on $\mathbb{R}^n \setminus \mathcal{A}$ and convex in $\mathbb{R}^n$. Thus we may define a superscribed convex linear function N_f^∞ by letting

$$N_f^\infty = \max_E N_E,$$

where E runs over all components of $\mathbb{R}^n \setminus E$ and $N_E : \mathbb{R}^n \to \mathbb{R}$ is the linear function obtained by extending $N_f|_E$ to $\mathbb{R}^n$ by linearity.

Definition 6 (Passare-Rullgård [22]). The spine $\mathcal{S}$ of amoeba is the corner locus of N_f^∞, i.e. the set of points in $\mathbb{R}^n$ where N_f^∞ is not locally linear.

Note that $\mathcal{S} \subset \mathcal{A}$ and that s is a piecewise-linear polyhedral complex. The following theorem shows that $\mathcal{S}$ is indeed a spine of $\mathcal{A}$ in the topological sense.

Theorem 14 (Passare-Rullgård [22], [27]). *The spine* $\mathcal{S}$ *is a strong deformational retract of the amoeba* $\mathcal{A}$.

Thus each component of $\mathbb{R}^n \setminus \mathcal{S}$ (i.e. each maximal open domain where N_f^∞ is linear) contains a unique component of $\mathbb{R}^n \setminus \mathcal{A}$.

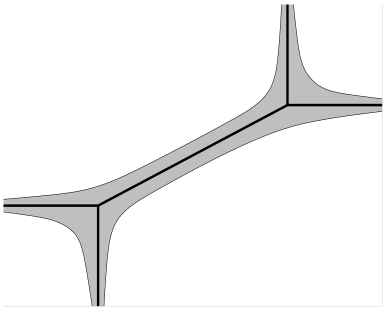

FIGURE 4. An amoeba and its spine.

2.3. **The spine $\mathcal{S}$ as a non-Archimedian amoeba.** Let K be an arbitrary field with a norm. Let $K^* = K \setminus \{0\}$ and $V \subset (K^*)^n$ be an algebraic variety, i.e. the zero set of a system of polynomial equations in K. The definition of amoeba still makes sense in this setup: $\mathcal{A}_K = \mathrm{Log}(V)$, where $\mathrm{Log} : (K^*)^n \to \mathbb{R}^n$ is defined by $\mathrm{Log}(z_1, \dots, z_n) = (\log|z_1|, \dots, \log|z_n|)$.

Recall that a real-valued *valuation* of a field K is a function $v : K^* \to \mathbb{R}$ such that $v(zw) = v(z) + v(w)$ and $-v(z + w) \geq \max\{-v(z), -v(w)\}$. A field with such a valuation is called *non-Archimedian*. Note that e^{-v} is a (multiplicative) norm and $\mathrm{Log} : (K^*)^n \to \mathbb{R}^n$ is given by $\mathrm{Log}(z_1, \dots, z_n) = (-v(z_1), \dots, -v(z_n))$.

Suppose that K is algebraically closed and that $v : K^* \to \mathbb{R}$ is surjective.

Example 1 (cf. [13]). Let K be the real-power *Puiseux series* in t, i.e. the field whose elements are formal power series $b(t) = \sum\limits_r b_r t^r$, where $b_r \in \mathbb{C}$ and the set of powers r is bounded from below and is contained in a finite union of arithmetic progression. This is an algebraically closed non-Archimedian field. The valuation of $b(t)$ is given by the smallest power of t which appears in the series.

Unlike the complex case the amoeba of a hypersurface $V = \{f = 0\} \subset (K^*)^n$ is completely determined by the norms of the coefficients of the defining polynomial f. Let $f(z) = \sum\limits_j a_j z^j$, where $z \in (K^*)^n$ and $j \subset \mathbb{Z}^n$ is a multiindex. A function $j \mapsto v(a_j))$ can be considered as a partially defined function on $\mathbb{R}^n$ (defined only on the finite set of lattice points j). Its Legendre transform

$$N_f^K(x) = \max_j \{jx - v(a_j)\},$$

where jx is the scalar product of j and x in $\mathbb{R}^n$, is a convex piecewise-linear function $\mathbb{R}^n \to \mathbb{R}$.

Theorem 15 (Kapranov [13]). *The amoeba $\mathcal{A}_K$ coincides with the corner locus of the piecewise-linear function N_f^K (cf. Definition 6). In particular, $\mathcal{A}_K$ is completely determined by the norms of the coefficients of f.*

2.4. **Non-Archimedian amoebas as a counterpart of algebraic hypersurfaces.** Subsets $\mathcal{A}_K \subset \mathbb{R}^n$ may be treated in a similar way we treat algebraic hypersurfaces in $(\mathbb{C}^*)^n$. Theorem 15 ensures that the choice of non-Archimedian field K is irrelevant here as long as K is algebraically closed and its valuation is onto $\mathbb{R}$.

Let us fix a Newton polyhedron Δ. The space of all complex polynomials which have Δ as its Newton polyhedron is $\mathbb{C}^N$, where $N = \#(\Delta \cap \mathbb{Z}^n)$. Polynomials which are different by multiplication by a constant give the same hypersurface. Thus the hypersurfaces space is $\mathbb{CP}^{N-1}$.

By Theorem 15 the amoeba $\mathcal{A}_K$ is determined solely by the valuations of the coefficients of the polynomial defining $V \subset (K^*)^n$. A valuation on the monomials from Δ are functions $\Delta \cap \mathbb{Z}^n \to \mathbb{R}$. They form the space $\mathbb{R}^N$. Valuation functions which are different by adding a constant give the same non-Archimedian amoebas. Thus non-Archimedian amoebas are parametrized by $\mathbb{R}^{N-1}$.

Remark 3. In general, the space of non-Archimedian amoebas is not $\mathbb{R}^{N-1}$ but a quotient of $\mathbb{R}^{N-1}$. If the valuation function $v : \Delta \cap \mathbb{Z}^n \to \mathbb{R}$ is not convex then we may vary a little v at some points keeping the Legendre transform the same.

Remark 4 (Non-Archimedian amoebas and Enumerative Geometry)**.** In a seminar talk in Paris, November 2000, Kontsevich noted a possibility of using non-Archimedian amoebas in enumerative geometry. As an example consider the problem of counting the number n_d of rational curves of degree d in $\mathbb{CP}^2$ which pass through $3d-1$ fixed generic points. A generic complex polynomial defines a curve of genus $(d-1)(d-2)/2$. The polynomials defining rational curves form a subset of codimension $(d-1)(d-2)/2$ and thus the rational curves form a $(3d-1)$-dimensional space (the space of curves has dimension one less than the dimension of the space of corresponding polynomials).

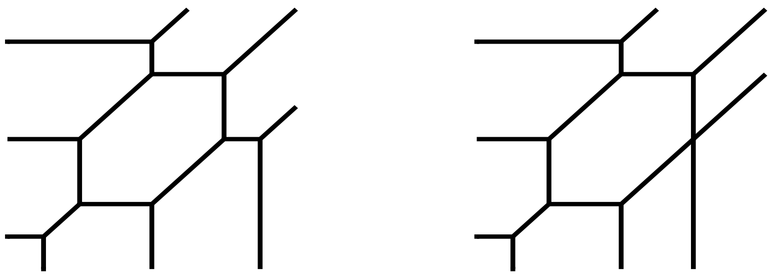

FIGURE 5. A smooth "non-Archimedian cubic amoeba" and a rational "non-Archimedian cubic amoeba".

Consider the space of non-Archimedian amoebas corresponding to curves of degree d in $\mathbb{P}^2$. This means that the Newton polygon Δ is the triangle with vertices $(0,0)$, $(d,0)$ and $(0,d)$. It may be deduced from Theorem 15 that a generic amoeba is a 3-valent graph which is homotopy equivalent to a wedge of up to $(d-1)(d-2)/2$ circles. Amoebas with $(d-1)(d-2)/2$ 4-valent vertices form a subset of codimension $(d-1)(d-2)/2$. These amoebas play the role of rational curves. Through generic $(3d-1)$ points in $\mathbb{R}^2$ there are n_d of different amoebas of this kind.

As an exercise the reader may check that there is a unique non-Archimedian amoeba of degree 1 through any 2 generic points in $\mathbb{R}^2$. The 2 points are special for this problem if they belong to the same horizontal, vertical or slope 1 line in $\mathbb{R}^2$. There is an infinite number of degree 1 non-Archimedian amoebas through the 2 points if they are special.

2.5. **Spine of amoebas and some functions on the space of complex polynomials.** Now we return to the study of the spine $\mathcal{S} \subset \mathcal{A}$ of a complex amoeba. The spine $\mathcal{S}$ itself a certain amoeba over a non-Archimedian field K. It does not matter what is the field K as long as the corresponding hypersurface over K has the coefficients $a_j \in K$ with the correct valuations. We can find these valuations from N_f^∞ by taking its Legendre transform. Since N_f^∞ is obtained as a maximum of a finite number of linear function with integer slopes its Legendre transform has a support on a convex lattice polyhedron $\Delta \subset \mathbb{R}^n$. Let $c_\alpha \in \mathbb{R}$, $\alpha \in \Delta \cap \mathbb{Z}^n$ be the value of the Legendre transform of N_f^∞ at α. To present $\mathcal{S}$ as a non-Archimedian amoeba we choose $a_j \in K$ such that $v(a_j) = c_\alpha$.

For each $\alpha \in \Delta \cap \mathbb{Z}^n$ let U_α be the space of all polynomials whose Newton polyhedron is contained in Δ and whose amoeba contains a component of the complement of index α. The space of all polynomials whose Newton polyhedron is contained in Δ is isomorphic to $\mathbb{C}^N$, where $N = \#(\Delta \cap \mathbb{Z}^n)$. The subset $U_\alpha \subset \mathbb{C}^N$ is an open domain. Note that c_α defines a real-valued function on U_α. This function was used by Rullgård [26], [27] for the study of geometry of U_α.

2.6. **Geometry of U_α.** Fix $\alpha \in \Delta \cap \mathbb{Z}^n$. Consider the following function in the space $\mathbb{C}^N$ of all polynomials f whose Newton polyhedron is contained in Δ

$$u_\alpha(f) = \inf_{x\in\mathbb{R}^n} \frac{1}{(2\pi i)^n} \int\limits_{\mathrm{Log}^{-1}(x)} \log\left|\frac{f(z)}{z^\alpha}\right| \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}, z \in (\mathbb{C}^*)^n.$$

Rullgård [26] observed that this function is plurisubharmonic in $\mathbb{C}^N$ while pluriharmonic over U_α. Indeed, over U_α there is a component $E_\alpha \subset \mathbb{R}^n \setminus \mathcal{A}$ corresponding to α and $u_\alpha = \operatorname{Re}\Phi_\alpha$, where

$$\Phi_\alpha = \frac{1}{(2\pi i)^n} \int\limits_{\mathrm{Log}^{-1}(x)} \log\left(\frac{f(z)}{z^\alpha}\right) \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}, x \in E_\alpha$$

is a $(\mathbb{C}/2\pi i\mathbb{Z})$-valued holomorphic function. Note that over $\mathrm{Log}^{-1}(E_\alpha)$ we can choose a holomorphic branch of $\log(\frac{f(z)}{z^\alpha})$ and that Φ_α does not depend on the choice of $x \in E_\alpha$. Therefore, U_α is pseudo-convex.

Note that U_α is invariant under the natural $\mathbb{C}^*$-action in $\mathbb{C}^N$. Let $\mathcal{C} \subset \mathbb{CP}^{N-1}$ be the complement of the image of U_α under the projection $\mathbb{C}^N \to \mathbb{CP}^{N-1}$.

Theorem 16 (Rullgård [26]). *For any line $L \subset \mathbb{CP}^{n-1}$ the set $L \cap \mathcal{C}$ is non-empty and connected.*

The next theorem describes how the sets U_α with different $\alpha \in \Delta \cap \mathbb{Z}^n$ intersect. It turns out that for any choice of subdivision $\Delta \cap \mathbb{Z}^n = A \cup B$ with $A \cap B = \emptyset$ the sets $\bigcup\limits_{\alpha \in A} U_\alpha$ and $\mathbb{C}^N \smallsetminus \bigcup\limits_{\beta \in B} U_\beta$ intersect. A stronger statement was found by Rullgård. Let $A, B \subset \Delta \cap \mathbb{Z}^n$ be disjoint sets. The set $A \cup B \subset \Delta \cap \mathbb{Z}^n$ defines a subspace $\mathbb{C}^{\#(A \cup B)} \subset \mathbb{C}^N$.

Theorem 17 (Rullgård [26]). *For any $\#(A \cup B)$-dimensional space L parallel to $\mathbb{C}^{\#(A\cup B)}$ the intersection $L \cap \bigcup\limits_{\alpha \in A} U_\alpha \cap \mathbb{C}^N \smallsetminus \bigcup\limits_{\beta \in B} U_\beta$ is non-empty.*

2.7. **The Monge-Ampère measure and the symplectic volume.**

Definition 7 (Passare-Rullgård [22]). The Monge-Ampère measure on $\mathcal{A}$ is the pull-back of the Lebesgue measure on $\Delta \subset \mathbb{R}^n$ under ∇N_f.

Indeed by Proposition 11 the Monge-Ampère measure is well-defined. Furthermore, we have the following proposition.

Proposition 18 (Passare-Rullgård [22]). *The Monge-Ampère measure has its support on $\mathcal{A}$. The total Monge-Ampère measure of $\mathcal{A}$ is* $\operatorname{Vol} \Delta$.

By Definition 7 the Monge-Ampère measure is given by the determinant of the Hessian of N_f. By convexity of N_f its Hessian $\operatorname{Hess} N_f$ is a non-negatively defined matrix-valued function. The trace of $\operatorname{Hess} N_f$ is the Laplacian of N_f, it gives another natural measure supported on $\mathcal{A}$. Note that $\omega = \sum\limits_{k=1}^{n} \frac{dz_k}{z_k} \wedge \frac{d\bar{z}_k}{\bar{z}_k}$ is a symplectic form on $(\mathbb{C}^*)^n$ invariant with respect to the group structure. The restriction $\omega|_V$ is a symplectic form on V. Its $(n-1)$-th power divided by $(n-1)!$ is a volume form called *the symplectic volume* on the $(n-1)$-manifold V.

Theorem 19 (Passare-Rullgård [22]). *The measure on $\mathcal{A}$ defined by the Laplacian of N_f coincides with the push-forward of the symplectic volume on V, i.e. for any Borel set A*

$$\int\limits_A \Delta N_f = \int\limits_{\operatorname{Log}^{-1}(A) \cap V} \omega^{n-1}.$$

This theorem appears in [22] as a particular case of a computation for *the mixed Monge-Ampère operator*, the symmetric multilinear operator associating a measure to n functions $f_1, \dots, f_n$ (recall that by our convention n is the number of variables) and such that its value on $f, \dots, f$ is the Monge-Ampère measure from Definition 7. The total mixed Monge-Ampère measure for $f_1, \dots, f_n$ is equal to the mixed volume of the Newton polyhedra of $f_1, \dots, f_n$ divided by $n!$.

Recall that this mixed volume divided by $n!$ appears in the Bernstein formula [4] which counts the number of common solutions of the system of equations $f_k = 0$ (assuming that the corresponding hypersurfaces intersect transversely). Passare and Rullgård found the following local analogue of the Bernstein formula which also serves as a geometric interpretation of the mixed Monge-Ampère measure. Note that the complex torus $(\mathbb{C}^*)^n$ acts on polynomials of n variables. The value of $t \in (\mathbb{C}^*)^n$ on $f : (\mathbb{C}^*)^n \to \mathbb{C}$ is the composition $f \circ t$ of the multiplication by t followed by application of f. In particular, the real torus $T^n = \operatorname{Log}^{-1}(0) \subset (\mathbb{C}^*)^n$ acts on polynomials of n variables.

Theorem 20 (Passare-Rullgård [22]). *The mixed Monge-Ampère measure for $f_1, \dots, f_n$ of a Borel set $A \subset \mathbb{R}^n$ is equal to the average number of solutions of the system of equations $f_k \circ t_k = 0$ in $\operatorname{Log}^{-1}(E) \subset (\mathbb{C}^*)^n$, $t_k \in T^n$, $k = 1, \dots, n$.*

The number of solution of this system of equations does not depend on t_k as long as the choice of t_k is generic. Thus Theorem 20 produces the Bernstein formula when $E = \mathbb{R}^n$.

2.8. **The area of a planar amoeba.** The computations of the previous subsection can be used to obtain an upper bound on amoeba's area in the case when $V \subset (\mathbb{C}^*)^2$ is a curve. With the help of Theorem 20 Passare and Rullgård [22] showed that in this case the Lebesgue measure on $\mathcal{A}$ is not greater than π^2 times the Monge-Ampère measure. In particular we have the following theorem.

Theorem 21 (Passare-Rullgård [22]). *If $V \subset (\mathbb{C}^*)^2$ is an algebraic curve then*

$$\operatorname{Area} \mathcal{A} \leq \pi^2 \operatorname{Area} \Delta.$$

This theorem is specific for the case $\mathcal{A} \subset \mathbb{R}^2$. Non-degenerate higher-dimensional amoebas of hypersurfaces have infinite volume. This follows from Proposition 3 since the area of the cross-section at infinity must be separated from zero.

3. Applications to real algebraic geometry

3.1. **The first part of Hilbert's 16th problem.** Most applications considered here are in the framework of Hilbert's 16th problem. Consider the classical setup of its first part, see [10]. Let $\mathbb{R}\bar{V} \subset \mathbb{RP}^2$ be a smooth algebraic curve of degree d. *What are the possible topological types of pairs $(\mathbb{RP}^2, \mathbb{R}\bar{V})$ for a given d?*

Since $\mathbb{R}\bar{V}$ is smooth it is homeomorphic to a disjoint union of circles. All of these circles must be contractible in $\mathbb{R}P^2$ (such circles are called *the ovals*) if d is even. If d is odd then exactly one of these circles is non-contractible. Therefore, the topological type of $(\mathbb{RP}^2, \mathbb{R}\bar{V})$ (also called *the* *topological arrangement* of $\mathbb{R}\bar{V}$ in $\mathbb{RP}^2$) is determined by the number of components of $\mathbb{R}\bar{V}$ together with the information on the mutual position of the ovals.

The possible number of components of $\mathbb{R}\bar{V}$ was determined by Harnack [8]. He proved that it cannot be greater than $\frac{(d-1)(d-2)}{2}+1$. Furthermore he proved that for any number

$$l \leq \frac{(d-1)(d-2)}{2}+1$$

there exists a curve of degree d with exactly l components as long as $l>0$ in the case of odd d (recall that for odd d we always have to have a non-contractible component).

Note that each oval separates $\mathbb{RP}^2$ into its *interior*, which is homeomorphic to a disk, and its *exterior*, which is homeomorphic to a Möbius band. If the interiors of the ovals intersect then the ovals are called *nested*. Otherwise the ovals are called *disjoint*. Hilbert's problem started from a question whether a curve of degree 6 which has 11 ovals (the maximal number according to Harnack) can have all of the ovals disjoint. This question was answered negatively by Petrovsky [23] who showed that at least two ovals of a sextic must be nested if the total number of ovals is 11.

In general the number of topological arrangements of curves of degree d grows exponentially with d. Even for small d the number of the possible types is enormous. Many powerful theorems restricting possible topological arrangements were found for over 100 years of history of this problem, see, in particular, [23], [1], [24]. A powerful *patchworking* construction technique [28] counters these theorems. The complete classifications is currently known for $d \leq 7$, see [28].

The most restricted turn out to be curves with the maximal numbers of components, i.e. with $l=\frac{(d-1)(d-2)}{2}+1$. Such curves were called *M-curves* by Petrovsky. However, even for M-curves, the number of topological arrangements grows exponentially with d.

The situation becomes different if we consider $\mathbb{RP}^2$ as a toric surface, i.e. as a compactification of $(\mathbb{R}^*)^2$. Recall that $\mathbb{RP}^2 \smallsetminus (\mathbb{R}^*)^2$ consists of three lines l_0, l_1 and l_2 which can be viewed as coordinate axes for homogeneous coordinates in $\mathbb{RP}^2$. Thus we have three affine charts for $\mathbb{RP}^2$. The intersection of all three charts is $(\mathbb{R}^*)^2 \subset \mathbb{RP}^2$. We denote $\mathbb{R}V=\mathbb{R}\bar{V}\cap(\mathbb{R}^*)^2$. The complexification $V\subset(\mathbb{C}^*)^2$ is the complex hypersurface defined by the same equation as $\mathbb{R}V$. Thus we are in position to apply the content of the previous sections of the paper to the amoeba of V.

In [17] it was shown (with the help of amoebas) that for each d the topological type of the pair $(\mathbb{RP}^2,\mathbb{R}\bar{V})$ is unique as long as the curve $\mathbb{R}\bar{V}$ is maximal in each of the three affine charts of $\mathbb{RP}^2$. Furthermore, the diffeomorphism type of the triad $(\mathbb{RP}^2;\mathbb{R}\bar{V},l_0\cup l_1\cup l_2)$ is unique. In subsection 3.5 we formulate this maximality condition and sketch the proof of uniqueness. A similar statement holds for curves in other toric surfaces. The Newton polygon Δ plays then the rôle of the degree d. In subsections 3.6 and 3.7 we describe an analogous but weaker statement towards uniqueness.

3.2. **Relation to amoebas: the real part $\mathbb{R}V$ as a subset of the critical locus of $\operatorname{Log}|_V$ and the logarithmic Gauss map.** Suppose that the hypersurface $V \subset (\mathbb{C}^*)^n$ is defined over real numbers (i.e. by a polynomial with real coefficients). Denote its real part via $\mathbb{R}V = V \cap (\mathbb{R}^*)^n$. We also assume that V is non-singular. Let $F \subset V$ be the critical locus of the map $\operatorname{Log}|_V : V \to \mathbb{R}^n$. It turns out that the real part $\mathbb{R}V$ is always contained in F.

Proposition 22 (Mikhalkin [17]). *$\mathbb{R}V \subset F$.*

This proposition indicates that the amoeba must carry some information about $\mathbb{R}V$. The proof of this proposition makes use of the *logarithmic Gauss map*.

Note that since $(\mathbb{C}^*)^n$ is a Lie group there is a canonical trivialization of its tangent bundle. If $z \in (\mathbb{C}^*)^n$ then the multiplication by z^{-1} induces an isomorphism $T_z(\mathbb{C}^*)^n \approx T_1(\mathbb{C}^*)^n$ of the tangent bundles at z and $1 = (1, \dots, 1) \in (\mathbb{C}^*)^n$.

Definition 8 (Kapranov [12]). The *logarithmic Gauss map* is a map

$$\gamma : V \to \mathbb{CP}^{n-1}.$$

It sends each point $z \in V$ to the image of the hyperplane $T_zV \subset T_z(\mathbb{C}^*)^n$ under the canonical isomorphism $T_z(\mathbb{C}^*)^n \approx T_1(\mathbb{C}^*)^n = \mathbb{C}^n$.

The map γ is a composition of a branch of a holomorphic logarithm $(\mathbb{C}^*)^n \to \mathbb{C}^n$ defined locally up to translation by $2\pi i$ with the usual Gauss map of the image of V. We may define γ explicitly in terms of the defining polynomial f for V by logarithmic differentiation formula. If $z = (z_1, \dots, z_n) \in V$ then

$$\gamma(z) = [< \nabla f, z >] = [\frac{\partial f}{\partial z_1} z_1 : \dots : \frac{\partial f}{\partial z_n} z_n] \in \mathbb{CP}^{n-1}.$$

Lemma 23 (Mikhalkin [17]). *$F = \gamma^{-1}(\mathbb{RP}^{n-1})$*

To justify this lemma we recall that $\operatorname{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ is a smooth fibration and V is non-singular. Thus $z \in V$ is critical for $\operatorname{Log}|_V$ if and only if the tangent vector space to V and the tangent vector space to the fiber torus $\gamma^{-1}(\gamma(z))$ intersect along an $(n-1)$-dimensional subspace. Such points are mapped to real points of $\mathbb{CP}^{n-1}$ by γ.

Note that this lemma implies Proposition 22. If V is defined over $\mathbb{R}$ then γ is equivariant with respect to the complex conjugation and maps $\mathbb{R}V$ to $\mathbb{RP}^{n-1}$.

3.3. **Compactification: a toric variety associated to a hypersurface in $(\mathbb{C}^*)^n$.** A hypersurface $V \subset (\mathbb{C}^*)^n$ is defined by a polynomial $f : \mathbb{C}^n \to C$. If the coefficients of f are real then we define the real part of V by $\mathbb{R}V = V \cap (\mathbb{R}^*)^n$. Recall that the Newton polyhedron $\Delta \subset \mathbb{R}^n$ of V is an integer convex polyhedron obtained as the convex hull of the indices of monomials participating in f, see (1) in subsection 1.3.

Let $\mathbb{C}T_\Delta \supset (\mathbb{C}^*)^n$ be the toric variety corresponding to Δ, see e.g. [7] and let $\mathbb{R}T_\Delta \supset (\mathbb{R}^*)^n$ be its real part. We define $\bar{V} \subset \mathbb{C}T_\Delta$ as the closure of V in $\mathbb{C}T_\Delta$ and we denote via $\mathbb{R}\bar{V}$ its real part.

Note that $\bar{V}$ may be singular even if V is not. Nevertheless $\mathbb{C}T_\Delta$ is, in some sense, the best toric compactification of $(\mathbb{C}^*)^n$ for V. Namely, $\bar{V}$ does not pass via the points of $\mathbb{C}T_\Delta$ corresponding to the vertices of Δ and therefore it does not have singularities there. Furthermore, $\mathbb{C}T_\Delta$ is minimal among such toric varieties, since $\bar{V}$ intersect any line in $\mathbb{C}T_\Delta$ corresponding to an edge of Δ.

Thus we may naturally compactify the pair $((\mathbb{C}^*)^n, V)$ to the pair $(\mathbb{C}T_\Delta, \bar{V})$. In such a setup the polyhedron Δ plays the rôle of the degree in $\mathbb{C}T_\Delta$. Indeed, two integer polyhedra Δ define the same toric variety $\mathbb{C}T_\Delta$ if their corresponding faces are parallel. But the choice of Δ also fixes the homology class of $\bar{V}$ in $H_{2n-2}(\mathbb{C}T_\Delta)$.

The simplest example is the projective space $\mathbb{CP}^n$. The corresponding Δ is, up to translation and the action of $SL_n(\mathbb{Z})$ the simplex defined by equations $z_j > 0$, $z_1 + \dots + z_n < d$. Thus in this case Δ is parametrized by a single natural number d which is the degree of $\bar{V} \subset \mathbb{CP}^n$.

3.4. **Maximality condition for $\mathbb{R}V$.** The inequality $l \le \frac{(d-1)(d-2)}{2}$ discovered by Harnack for the number l of components of a curve $\mathbb{R}\bar{V}$ is a part of a more general *Harnack-Smith inequality.* Let X be a topological space and let Y be the fixed point set of a of a continuous involution on X. Denote by $b_*(X;\mathbb{Z}_2) = \dim H_*(X;\mathbb{Z}_2)$ the total $\mathbb{Z}_2$-Betti number of X.

Theorem 24 (P. A. Smith, see e.g. the appendix in [30])**.**

$$b_*(Y;\mathbb{Z}_2) \le b_*(X;\mathbb{Z}_2).$$

Corollary 25. $b_*(\mathbb{R}\bar{V};\mathbb{Z}_2) \le b_*(\bar{V};\mathbb{Z}_2)$, $b_*(\mathbb{R}V;\mathbb{Z}_2) \le b_*(V;\mathbb{Z}_2)$.

Note that Theorem 24 can also be applied to pairs which consist of a real variety and real subvariety and other similar objects.

Definition 9 (Rokhlin [24])**.** A variety $\mathbb{R}\bar{V}$ is called an *M-variety* if

$$b_*(\mathbb{R}\bar{V};\mathbb{Z}_2) = b_*(\bar{V};\mathbb{Z}_2).$$

E.g. if $\bar{V} \subset \mathbb{CP}^2$ is a smooth curve of degree d then $\bar{V}$ is a Riemann surface of genus $g = \frac{(d-1)(d-2)}{2}$. Thus $b_*(\bar{V};\mathbb{Z}_2) = 2 + 2g$. On the other hand, $b_*(\mathbb{R}\bar{V};\mathbb{Z}_2) = 2l$, where l is the number of (circle) components of $\mathbb{R}\bar{V}$.

Let $\mathbb{R}V \subset (\mathbb{R}^*)^n$ be an algebraic hypersurface, Δ be its Newton polyhedron, $\mathbb{R}T_\Delta$ be the toric variety corresponding to Δ and $\mathbb{R}\bar{V} \subset \mathbb{R}T_\Delta$ the closure of $\mathbb{R}V$ in $\mathbb{R}T_\Delta$. We denote with $V \subset (\mathbb{C}^*)^n$ and $\bar{V} \subset \mathbb{C}T_\Delta$ the complexifications of these objects. Recall (see e.g. [7]) that each (closed) k-dimensional face Δ' of Δ corresponds to a closed k-dimensional toric variety $\mathbb{R}T_{\Delta'} \subset \mathbb{R}T_\Delta$ (and, similarly, $\mathbb{C}T_{\Delta'} \subset \mathbb{C}T_\Delta$). The intersection $V_{\Delta'} = \bar{V} \cap \mathbb{C}T_{\Delta'}$ is itself a hypersurface in the k-dimensional toric variety $\mathbb{C}T_{\Delta'}$ with the Newton polyhedron Δ'. Its real part is $\mathbb{R}V_{\Delta'} = V_{\Delta'} \cap \mathbb{R}\bar{V}$.

Denote with $\operatorname{St}\Delta' \subset \partial\Delta$ the union of all the closed faces of Δ containing Δ'. Denote $V_{\operatorname{St}\Delta'} = \bigcup\limits_{\Delta''\subset \operatorname{St}\Delta'} V_{\Delta'}$ and $\mathbb{R}V_{\operatorname{St}\Delta'} = V_{\operatorname{St}\Delta'} \cap \mathbb{R}T_{\Delta}$.

Definition 10 (Mikhalkin [21])**.** A hypersurface $\mathbb{R}\bar{V} \subset \mathbb{C}T_{\Delta}$ is called *torically maximal* if the following conditions hold

- $\mathbb{R}\bar{V}$ is an M-variety, i.e. $b_*(\mathbb{R}\bar{V};\mathbb{Z}_2) = b_*(\bar{V};\mathbb{Z}_2)$;
- the hypersurface $\bar{V} \cap \mathbb{C}T_{\Delta'} \subset \mathbb{C}T_{\Delta}$ is torically maximal for each face $\Delta' \subset \Delta$ (inductively we assume that this notion is already defined in smaller dimensions);
- for each face $\Delta' \subset \Delta$ we have $b_*(\mathbb{R}V \cup \mathbb{R}V_{\operatorname{St}\Delta'}, \mathbb{R}V_{\operatorname{St}\Delta'};\mathbb{Z}_2) = b_*(V \cup V_{\operatorname{St}\Delta'}, V_{\operatorname{St}\Delta'};\mathbb{Z}_2)$.

Consider a linear function $h : \mathbb{R}^n \to \mathbb{R}$. A facet $\Delta' \subset \Delta$ is called *negative* with respect to h if the image of its outward normal vector under h is negative. We define $\mathbb{C}T^- = \bigcup\limits_{\text{negative } \Delta'} \mathbb{C}T_{\Delta'}$. In these formula we take the union over all the closed facets Δ' negative with respect to h. Let $V^- = \bar{V} \cap \mathbb{C}T^-$ and $\mathbb{R}V^- = V^- \cap \mathbb{R}\bar{V}$.

We call a linear function $h : \mathbb{R}^n \to \mathbb{R}$ generic if its kernel does not contain vectors orthogonal to facets of Δ.

Proposition 26. *If a hypersurface $\mathbb{R}\bar{V} \subset \mathbb{R}T_{\Delta}$ is torically maximal then for any generic linear function h we have*

$$b_*(\mathbb{R}V \cup \mathbb{R}V^-, \mathbb{R}V^-;\mathbb{Z}_2) = b_*(V \cup V^-, V^-;\mathbb{Z}_2).$$

3.5. **Curves in the plane.**

3.5.1. *Curves in $\mathbb{RP}^2$ and their bases.* Note that if $\mathbb{R}V \subset (\mathbb{R}^*)^2$ is a torically maximal curve then the number of components of $\mathbb{R}\bar{V}$ coincides with the genus of $\mathbb{C}\bar{V}$. In other words (cf. 3.1) $\mathbb{R}\bar{V}$ is an M-curve.

We start by reformulating the maximality condition of Definition 10 for the case of curves in the projective plane. Let $\mathbb{R}C \subset \mathbb{RP}^2$ be a non-singular curve of degree d.

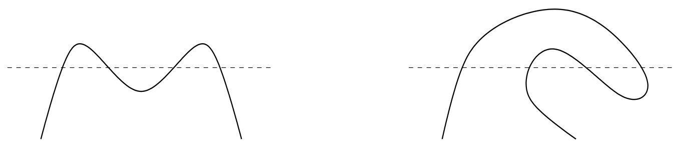

FIGURE 6. Possible bases for a real quartic curve.

Definition 11 (Brusotti [5])**.** Let α be an arc (i.e. an embedded closed interval) in $\mathbb{R}C$. The arc α is called a *base* (or a *base of rank 1*, see [5]) if there exists a line $L \subset \mathbb{RP}^2$ such that the intersection $L \cap \alpha$ consists of d distinct points.

Note if three lines L_1, L_2, L_3 in $\mathbb{RP}^2$ are generic, i.e. they do not pass through the same point, then $=\mathbb{RP}^2 \setminus (L_1 \cup L_2 \cup L_3) = (\mathbb{R}^*)^2$. We call such $(\mathbb{R}^*)^2$ a *toric chart* of $\mathbb{RP}^2$. Thus $\mathbb{R}V = \mathbb{R}C \setminus (L_1 \cup L_2 \cup L_3)$ is a curve in $(\mathbb{R}^*)^2$. If $\mathbb{R}C$ does not pass via $L_j \cap L_k$ then the Newton polygon of $\mathbb{R}V$ (for any choice of coordinates (x, y) in $(\mathbb{R}^*)^2$ extendable to affine coordinates in $\mathbb{R}^2 = \mathbb{RP}^2 \setminus L_j$ for some j) is the triangle $\Delta_d = \{x \geq 0\} \cap \{y \geq 0\} \cap \{x + y \leq d\}$.

Proposition 27 (Mikhalkin [21])**.** *The curve $\mathbb{R}C \subset \mathbb{RP}^2$ is maximal in some toric chart of $\mathbb{RP}^2$ if and only if $\mathbb{R}C$ is an M-curve with three disjoint bases.*

Many M-curves with one or two disjoint bases are known (see e.g. [5]). However there is (topologically) only one known example of curve with three disjoint bases, namely the first M-curve constructed by Harnack [8]. Theorem 28 asserts that this example is the only possible.

Definition 12 (simple Harnack curve in $\mathbb{RP}^2$, cf. [8], [18])**.** A non-singular curve $\mathbb{R}C \subset \mathbb{RP}^2$ of degree d is called a (smooth) simple Harnack curve if it is an M-curve and

- all ovals of $\mathbb{R}C$ are disjoint (i.e. have disjoint interiors, see 3.1) if $d = 2k - 1$ is odd;
- one oval of $\mathbb{R}C$ contains $\frac{(k-1)(k-2)}{2}$ ovals in its interior while all other ovals are disjoint if $d = 2k$ is even.

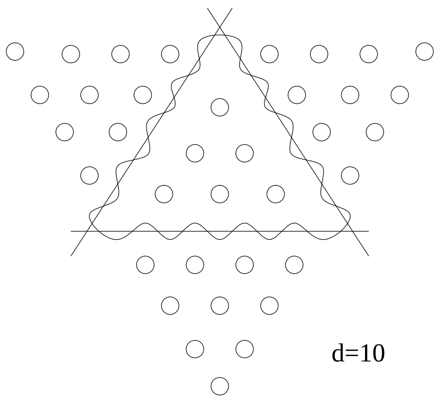

FIGURE 7. [17] A simple Harnack curve.

Theorem 28 (Mikhalkin [17])**.** *Any smooth M-curve $\mathbb{R}C \subset \mathbb{RP}^2$ with at least three base is a simple Harnack curve.*

There are several topological arrangements of M-curves with fewer than 3 bases for each d (in fact, their number grows exponentially with d). There is a unique (Harnack) topological arrangement of an M-curve with 3 bases by Theorem 28. In the same time 3 is the highest number of bases an M-curve of sufficiently high degree can have as the next theorem shows.

Theorem 29 (Mikhalkin [17]). *No M-curve in $\mathbb{RP}^2$ can have more than 3 bases if $d \geq 3$.*

3.5.2. *Curves in real toric surfaces.* Theorem 28 has a generalization applicable to other toric surfaces. Let $\mathbb{R}V \subset (\mathbb{R}^*)^2$ be a curve with the Newton polygon Δ. The sides of Δ correspond to lines $L_1, \dots, L_n$ in $\mathbb{R}T_\Delta$. We have $\mathbb{R}V = \mathbb{R}\bar{V} \smallsetminus (L_1 \cup \dots \cup L_n$.

Theorem 30 (Mikhalkin [17]). *The topological arrangement of a torically maximal curve is unique for each Δ. More precisely, the topological type of the triad $(\mathbb{R}T_\Delta; \mathbb{R}\bar{V}, L_1 \cup \dots \cup L_n)$ and, in particular, the topological type of the pair $((\mathbb{R}^*)^2, \mathbb{R}V)$ depends only on Δ as long as $\mathbb{R}V$ is a torically maximal curve.*

A torically maximal curve $\mathbb{R}\bar{V}$ is a counterpart of a simple Harnack curve for $\mathbb{R}T_\Delta$. All of its components except for one are ovals with disjoint interiors. The remaining component is not homologous to zero unless Δ is even (i.e. obtained from another lattice polygon by a homothety with coefficient 2). If Δ is even the remaining component is also an oval whose interior contains $g(V)$ ovals of $\mathbb{R}V$. Recall that, by Khovanskii's formula [15], $g(V)$ coincides with the number of lattice points in the interior of Δ.

Theorem 31 (Harnack, Itenberg-Viro [8], [11]). *For any Δ there exists a curve $\mathbb{R}V \subset (\mathbb{R}^*)^2$ which is torically maximal and has Δ as its Newton polygon.*

As in Definition 12 we call such curves *simple Harnack curves*, cf. [18].

3.5.3. *Geometric properties of algebraic curves in $(\mathbb{R}^*)^2$.* It turns out that the simple Harnack curves have peculiar geometric properties, but they are better seen after a logarithmic reparametrization $\operatorname{Log}|_{(\mathbb{R}^*)^2} : (\mathbb{R}^*)^2 \to \mathbb{R}^2$. A point of $\mathbb{R}V$ is called a logarithmic inflection point if it corresponds to an inflection point of $\operatorname{Log}(\mathbb{R}V) \subset \mathbb{R}^2$ under Log.

Theorem 32 (Mikhalkin [17]). *The following conditions are equivalent.*

- $\mathbb{R}V \subset (\mathbb{R}^*)^2$ *is a simple Harnack curve.*
- $\mathbb{R}V \subset (\mathbb{R}^*)^2$ *has no real logarithmic inflection points.*

Remark 5 (cf. [17]). Recall that by Proposition 22 $\operatorname{Log}(\mathbb{R}V)$ is contained in the critical value locus of $\operatorname{Log}|_V$. The map $\operatorname{Log}|_V : V \to \mathbb{R}^2$ is a surface-to-surface map in our case and its most generic singularities are folds. By Proposition 7 the folds are convex. Thus a logarithmic inflection point of $\mathbb{R}V$ must correspond to a higher singularity of $\operatorname{Log}|_V$.

In [17] it was noted that there are two types of stable (surviving small deformations of $\mathbb{R}V$) logarithmic inflection points of $\mathbb{R}V$. The first one, *junction*, and correspond to intersection of $\mathbb{R}V$ with a branch of imaginary folding curve. A junction logarithmic inflection point can be found at the curve $y = (x-1)^2 + 1$. Note that the image of the imaginary folding curve under the complex conjugation is also a folding curve. Thus over its image we have a double fold.

The second type, *pinching*, corresponds to intersection of $\mathbb{R}V$ with a circle $E \subset V$ that gets contracted by Log. Such circles E survive if we deform V in the class of hypersurfaces with real coefficients but disappear under a generic small perturbation if we allow the coefficients to become imaginary.

The circle E intersect $\mathbb{R}V$ at two points. These points belong to different quadrants of $(\mathbb{R}^*)^2$, but have the same absolute values of their coordinates. Both of these points are logarithmic inflection points.

FIGURE 8. [17] A pinching point and a junction point.

Proposition 33. *The logarithmic image* $\mathrm{Log}(\mathbb{R}V)$ *is trivial in the closed support homology group* $H_1^c(\mathbb{R}^2)$.

Thus the curve $\mathrm{Log}(\mathbb{R}V)$ spans a surface in $(\mathbb{R}^*)^2$. Theorem 21 has the following corollary.

Corollary 34. *The area of any region spanned by branches of* $\mathrm{Log}(\mathbb{R}V)$ *is smaller than* $\mathrm{Area}\,\Delta$.

The situation is especially simple for the logarithmic image of a simple Harnack curve.

Proposition 35 ([17])**.** *If* $\mathbb{R}V$ *is a simple Harnack curve then* $\mathrm{Log}\,|_{\mathbb{R}V}$ *is an embedding and* $\mathrm{Log}\,\mathbb{R}V = \partial\mathcal{A}$.

Thus in this case $\mathcal{A}$ coincides with the region spanned by the whole curve $\mathrm{Log}(\mathbb{R}V)$. Furthermore, in [18] it was shown that simple Harnack curves maximize the area of this region.

Theorem 36 (Mikhalkin-Rullgård, [18])**.** *If* $\mathbb{R}V$ *is a simple Harnack curve then* $\mathrm{Area}\,\mathcal{A} = \mathrm{Area}\,\Delta$.

In the opposite direction we have the following theorem. We say that a curve $V \subset (\mathbb{C}^*)^2$ is real up to translation if there exists $a \in (\mathbb{C}^*)^2$ such that aV is defined by a polynomial with real coefficients. We denote the corresponding real part with $\mathbb{R}V$. (Note that in general this real part might depend on the choice of translation.)

Theorem 37 (Mikhalkin-Rullgård [18]). *If* $\operatorname{Area}\mathcal{A} = \operatorname{Area}\Delta > 0$ *and* V *is non-singular and transverse to the lines (coordinate axes) in* $\mathbb{C}T_\Delta$ *corresponding to the sides of* Δ *then* V *is real up to translation in a unique way and* $\mathbb{R}V$ *is a simple Harnack curve.*

Furthermore, in [18] it was shown that the only singularities that V can have in the case $\operatorname{Area}\mathcal{A} = \operatorname{Area}\Delta > 0$ are ordinary real isolated double points.

3.6. Surfaces in the 3-space.

3.6.1. *Topological uniqueness for torically maximal surfaces.* Let $\mathbb{R}V \subset (\mathbb{R}^*)^3$ be an algebraic surface with the Newton polyhedron $\Delta \subset \mathbb{R}^3$. Let $\mathbb{R}\bar{V} \subset \mathbb{R}T_\Delta$ be its compactification.

Recall (see Definition 10) that if $\mathbb{R}V$ is a torically maximal surface then $b_*(\mathbb{R}\bar{V};\mathbb{Z}_2) = b_*(\bar{V};\mathbb{Z}_2)$, i.e. $\mathbb{R}\bar{V}$ is an M-surface.

Theorem 38 (Mikhalkin [21]). *Given a Newton polyhedron* Δ *the topological type of a torically maximal surface* $\mathbb{R}\bar{V} \subset \mathbb{R}T_\Delta$ *is unique.*

To describe the topological type of $\mathbb{R}\bar{V}$ it is useful to compute the total Betti number $b_*(\bar{V};\mathbb{Z}_2)$ in terms of Δ. Note that by the Lefschetz hyperplane theorem $b_*(\bar{V};\mathbb{Z}_2) = \chi(\bar{V})$.

We denote by $\operatorname{Area}\partial\Delta$ the total area of the faces of Δ. Each of these faces sits in a plane $P \subset \mathbb{R}^3$. The intersection $P \cap \mathbb{Z}^3$ determines the area form on P. This area form is translation invariant and such that the area of the smallest lattice parallelogram is 1.

Similarly we denote by $\operatorname{Length}\operatorname{Sk}^1\Delta$ the total length of all the edges of Δ. Again, each edge sits in a line $L \subset \mathbb{R}^3$. The intersection $L \cap \mathbb{Z}^3$ determines the length on L by setting the length of the smallest lattice interval 1.

Proposition 39. $b_*(V;\mathbb{Z}_2) = 6\operatorname{Vol}\Delta - 2\operatorname{Area}\partial\Delta + \operatorname{Length}\operatorname{Sk}^1\Delta.$

This proposition follows from Khovanskii's formula [15].

Theorem 40 (Mikhalkin [21]). *A torically maximal surface* $\mathbb{R}\bar{V}$ *consists of* $p_g + 1$ *components, where* p_g *is the number of points in the interior of* Δ. *There are* p_g *components homeomorphic to 2-spheres and contained in* $(\mathbb{R}^*)^3$. *These spheres bound disjoint spheres in* $(\mathbb{R}^*)^3$. *The remaining component is homeomorphic to*

- *a sphere with* $b_*(V;\mathbb{Z}_2) - 2p_g(V) - 2$ *Möbius bands in the case when* Δ *is odd (i.e. cannot be presented as* $2\Delta'$ *for some lattice polyhedron* Δ'*);*
- *a sphere with* $\frac{1}{2}b_*(V;\mathbb{Z}_2) - p_g(V) - 1$ *handles in the case* Δ *is even.*

Remark 6. Not for every Newton polyhedron Δ a torically maximal surface $\mathbb{R}V \subset (\mathbb{R}^*)^3$ exists. The following example is due to B. Bertrand. Let $\Delta \subset \mathbb{R}^3$ be the convex hull of $(1,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,0,2k+1)$. If $k > 0$ then there is no M-surface $\mathbb{R}\bar{V}$ with the Newton polyhedron Δ. In particular, there is no torically maximal surface $\mathbb{R}V$ for Δ.

Example 2. There are 3 different topological types of smooth M-quartics in $\mathbb{RP}^3$ (see [14]). They realize all topological possibilities for maximal real structures on abstract K3-surfaces. Namely, such real surface may be homeomorphic to

- the disjoint union of 9 spheres and a surface of genus 2;
- the disjoint union of 5 spheres and a surface of genus 6;
- the disjoint union of a sphere and a surface of genus 10.

Theorem 40 asserts that only the last type can be a torically maximal quartic in $\mathbb{RP}^3$. More generally, only the last type can be a torically maximal surface is a toric 3-fold $\mathbb{R}T_\Delta$.

3.6.2. *Geometric properties of maximal algebraic surfaces in* $(\mathbb{R}^*)^3$. Recall the classical geometric terminology. Let $S \subset \mathbb{R}^3$ be a smooth surface. We call a point $x \in S$ *elliptic*, *hyperbolic* or *parabolic* if the Gauss curvature of S at x is positive, negative or zero.

Remark 7. Of course we do not actually need to use the Riemannian metric on S do define these points. Here is an equivalent definition without referring to the curvature. Locally near x we can present S as the graph of a function $\mathbb{R}^2 \to \mathbb{R}$. If the Hessian form of this function at x is degenerate then we call x parabolic. If not, the intersection of S with the tangent plane at x is a real curve with an ordinary double point in x. If this point is isolated we call x elliptic. If it is an intersection of two real branches of the curve we call it hyperbolic.

We say that a point $x \in \mathbb{R}V \subset (\mathbb{R}^*)^3$ is *logarithmically* elliptic, hyperbolic or parabolic if it maps to such point under $\operatorname{Log}|_{(\mathbb{R}^*)^3} : (\mathbb{R}^*)^3 \to \mathbb{R}^3$.

Generically for a smooth surface in $\mathbb{R}^3$ the parabolic locus, i.e. the set of parabolic points, is a 1-dimensional curve. So is the logarithmic parabolic locus for a surface in $(\mathbb{R}^*)^3$. In a contrast to this we have the following theorem for torically maximal surfaces. Note that torically maximal surfaces form an open subset in the space of all surfaces with a given Newton polyhedron.

Theorem 41 (Mikhalkin [21])**.** *The logarithmic parabolic locus of a torically maximal surface consists of a finite number of points.*

Note that such a zero-dimensional locus cannot separate the surface $\mathbb{R}V$. Thus each component of $\mathbb{R}V$ is either logarithmically elliptic (all its points except finitely many are logarithmically elliptic) or logarithmically hyperbolic (all its points except finitely many are logarithmically hyperbolic).

Corollary 42 (Mikhalkin [21])**.** *Every compact component of* $\mathbb{R}V$ *is diffeomorphic to a sphere.*

This corollary is a part of Theorem 40.

Remark 8 (logarithmic monkey saddles of $\mathbb{R}V$)**.** The Hessian at the isolated parabolic points $\operatorname{Log}(\mathbb{R}V)$ vanishes. Generic parabolic points sitting on hyperbolic components of $\mathrm{Log}(\mathbb{R}V)$ look like so-called Monkey saddles (given in some local coordinates (x, y, z) by $z = x(y^2 - x^2)$).

Logarithmic monkey saddles do not appear on generic *smooth* surfaces in $(\mathbb{R}^*)^3$. But they do appear on generic *real algebraic* surfaces in $(\mathbb{R}^*)^3$. In particular, they appear on every torically maximal surface of sufficiently high degree.

The counterpart on the elliptic components of $\mathrm{Log}(\mathbb{R}V)$, the *imaginary monkey saddles*, are locally given by $z = x(y^2 + x^2)$.

3.7. **Hypersurfaces of higher dimension.** Let $\mathbb{R}V \subset (\mathbb{R}^*)^n$ be a hypersurface and $n \geq 4$. Theorems 30 and 38 have a weaker version for these dimensions.

Theorem 43 (Mikhalkin [21]). *If $\mathbb{R}V$ is torically maximal then every compact component of $\mathbb{R}V$ is a sphere. All these $(n-1)$-spheres bound disjoint n-balls in $(\mathbb{R}^*)^n$.*

The following theorem is a counterpart of Theorem 41 and a weaker version of Theorem 32.

Theorem 44 (Mikhalkin [21]). *The parabolic locus of $\mathrm{Log}(\mathbb{R}V) \subset \mathbb{R}^n$ is of codimension 2 if $\mathbb{R}V$ is torically maximal.*

3.8. **Maximality conditions for non-Archimedian amoebas.** Let $\mathcal{A}_K \subset \mathbb{R}^n$ be a non-Archimedian amoeba (see 2.4) whose Newton polyhedron is Δ.

Proposition 45. *The number of vertices of $\mathcal{A}_K$ is not greater than $n! \operatorname{Vol} \Delta$.*

This proposition can be deduced from Theorem 15 and the fact that the smallest possible volume of a convex lattice polyhedron is $\frac{1}{n!}$.

Definition 13. A non-Archimedian amoeba $\mathcal{A}_K$ is called *maximal* if the number of its vertices equals to $n! \operatorname{Vol} \Delta$.

Remark 9. For some choices of Δ maximal amoebas do not exist. We can take, for instance, $\Delta \subset \mathbb{R}^3$ to be the tetrahedron with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(0, 0, k)$. Any valuation on the corresponding coefficients would have to be linear. Its Legendre transform would have just one vertex while $\operatorname{Vol} \Delta = \frac{k}{n!}$. Note that these polyhedra were used by B. Bertrand to show absence of real maximal surfaces, see Remark 6.

Nevertheless, if the toric variety corresponding to Δ is a projective space or a product of projective spaces then maximal non-Archimedian amoebas exist. This statement is implicitly contained in [11].

4. Patchworking of amoebas, Maslov's dequantization and topology of complex algebraic varieties

4.1. **Patchworking polynomial.** In 1979 Viro discovered a *patchworking* technique for construction of real algebraic hypersurfaces, see [28].

Fix a convex lattice polyhedron $\Delta \in \mathbb{R}^n$. Choose a function $v : \Delta \cap \mathbb{Z}^n \to \mathbb{R}$. The graph of v is a discrete set of points in $\mathbb{R}^n \times \mathbb{R}$. The overgraph is a family of parallel rays. Thus the convex hull of the overgraph is a semi-infinite polyhedron $\tilde{\Delta}$. The facets of $\tilde{\Delta}$ which project isomorphically to $\mathbb{R}^n$ define a subdivision of Δ into smaller convex lattice polyhedra Δ_k.

Let $F(z) = \sum\limits_{j\in\Delta} a_j z^j$ be a generic polynomial in the class of polynomial whose Newton polyhedron is Δ. The *truncation* of F to Δ_k is $F_{\Delta_k} = \sum\limits_{j\in\Delta_k} a_j z^j$. The *patchworking polynomial* f is defined by formula

$$f_t^v(z) = \sum_j a_j t^{v(j)} z^j, \tag{2}$$

$z \in \mathbb{R}^n$, $t > 1$ and $j \in \mathbb{Z}^n$.

Consider the hypersurfaces V_{Δ_k} and V_t in $(\mathbb{C}^*)^n$ defined by F_{Δ_k} and f_t^v. If F has real coefficients then we denote $\mathbb{R}V_{\Delta_k} = V_{\Delta_k} \cap (\mathbb{R}^*)^n$ and $\mathbb{R}V_t = V_t \cap (\mathbb{R}^*)^n$. Viro's patchworking theorem [28] asserts that for large values of t the hypersurface $\mathbb{R}V_t$ can be obtained from $\mathbb{R}V_{\Delta_k}$ by a certain patchworking procedure. The same holds for amoebas of the hypersurfaces V_t and $\mathbb{R}V_{\Delta_k}$. In fact patchworking of real hypersurfaces can be interpreted as the real version of patchworking of amoebas (cf. Appendix in [17]). Below we describe a special case of amoeba's patchworkings in terms of the so-called dequantization.

4.2. **Maslov's dequantization.** It was noted by Viro in [29] that patchworking is related to so-called *Maslov's dequantization* of positive real numbers.

Recall that a *quantization* of a semiring R is a family of semirings R_h, $h \geq 0$ such that $R_0 = R$ and $R_t \approx R_s$ as long as $s, t > 0$, but R_0 is not isomorphic to R_t. The semiring R_h with $h > 0$ is called a *quantized* version of R_0.

Maslov (see [16]) observed that the "classical" semiring $\mathbb{R}_+$ of real positive number is a quantized version of some other ring in this sense Let R_h be the set of positive numbers with the usual multiplication and with the addition operation $z \oplus_h w = (z^{\frac{1}{h}} + w^{\frac{1}{h}})^h$ for $h > 0$ and $z \oplus_h w = \max\{z, w\}$ for $h = 0$. Note that

$$\lim_{h\to 0}(z^{\frac{1}{h}} + w^{\frac{1}{h}})^h = \max\{z, w\}$$

and thus this is a continuous family of arithmetic operations.

The semiring R_1 coincides with the standard semiring $\mathbb{R}_+$. The isomorphism between $\mathbb{R}_+$ and R_h with $h > 0$ is given by $z \mapsto z^h$. On the other hand the semiring R_0 is not isomorphic to $\mathbb{R}_+$ since it is idempotent, indeed $z + z = \max\{z, z\} = z$.

4.3. **Logarithmic dequantization.** Alternatively we may define the dequantization deformation with the help of the logarithm. The logarithm $\log_t$, $t > 1$, induces a semiring structure on $\mathbb{R}$ from $\mathbb{R}_+$,

$$x \oplus_t y = \log_t(t^x + t^y),\ x \otimes_t y = x + y,\ x, y \in \mathbb{R}.$$

Similarly we have $x \oplus_\infty y = \max\{x, y\}$. Let $R_t^{\log}$ be the resulting semiring.

Proposition 46. *The map* $\log : R_h \to R_t^{\log}$, *where* $t = e^{\frac{1}{h}}$, *is an isomorphism.*

4.4. **Patchworking as a dequantization.** The patchworking polynomial (2) can be viewed as a deformation of the polynomial f_1^v. We define a similar deformation with the help of Maslov's dequantization. Instead of deforming the coefficients we keep coefficients the same and deform arithmetic operations as in 4.2 and 4.3.

Choose any coefficients α_j, $j \in \Delta$. Let $\phi_t : (R_t^{\log})^n \to R_t^{\log}$, $t \geq e$, be a polynomial whose coefficients are α, i.e.

$$\phi_t(x) = \bigoplus_t (\alpha_j + jx), \ x \in \mathbb{R}^n.$$

Let $\mathrm{Log}_t : (\mathbb{C}^*)^n \to \mathbb{R}^n$ be defined by $(x_1, \dots, x_n) = (\log|z_1|, \dots, \log|z_n|)$.

Proposition 47 (Maslov [16],Viro [29])**.** *The function* $f_t = (\log_t)^{-1} \circ \phi_t \circ \mathrm{Log}_t : (\mathbb{R}_+)^n \to \mathbb{R}_+$ *is a polynomial with respect to the standard arithmetic operations in* $\mathbb{R}_+$,

$$f_t(z) = \sum_j t^{\alpha_j} z^j.$$

This is a special case of the patchworking polynomial (2). The coefficients α_j define the function $v : \Delta \cap \mathbb{Z}^n \to \mathbb{R}$.

4.5. **Limit set of amoebas.** Let $V_t \subset (\mathbb{C}^*)^n$ be the zero set of f_t and let $\mathcal{A}_t = \mathrm{Log}_t(V_t) \subset \mathbb{R}^n$. Note that $\mathcal{A}_t$ is the amoeba of V_t scaled $\log t$ times. Note also that the family $f_t = \sum\limits_j t^{\alpha_j} z^j$ can be considered as a single polynomial whose coefficients are powers of t. In particular we may treat it as a polynomial over the field of Puiseux series, i.e. a non-Archimedian field (see Example 1). Let $\mathcal{A}_K$ be the corresponding non-Archimedian amoeba.

We have a uniform convergence of the addition operation in $R_t^{\log}$ to the addition operation in $R_\infty^{\log}$. As it was observed by Viro it follows from the following inequality

$$\max\{x, y\} \leq x \oplus_t y = \log_t(t^x + t^y) \leq \max\{x, y\} + \log_t 2.$$

More generally, we have the following lemma.

Lemma 48.

$$\max_{j \in \Delta}(\alpha_j + jx) \leq \phi_t(x) \leq \max_{j \in \Delta}(\alpha_j + jx) + \log N,$$

where N *is the number of lattice points in* Δ.

The following theorem is a corollary of this lemma.

Theorem 49 (Mikhalkin [20], Rullgård [27])**.** *The subsets* $\mathcal{A}_t \subset \mathbb{R}^n$ *tend in the Hausdorff metric to* $\mathcal{A}_K$.

Note that by Theorem 15 $\mathcal{A}_K$ is obtained by patchworking of the amoebas of the truncations of f_e to smaller polyhedra Δ_k (see 4.1).

4.6. **Torus fibrations for algebraic hypersurfaces in** $(\mathbb{C}^*)^n$**.** Recall that as long as a hypersurface $V \cap (\mathbb{C}^*)^n$ is non-singular its diffeomorphism type depends only on its Newton polyhedron Δ. Theorem 49 implies that for large values of t the amoeba $\mathcal{A}_t$ is contained in a regular neighborhood W of $\mathcal{A}_K$.

The space $\mathcal{A}_K$ has a natural cellular decomposition which turns $\mathcal{A}_K$ to an $(n-1)$-dimensional CW-complex. The decomposition comes from piecewise-linear embedding of $\mathcal{A}_K$ into $\mathbb{R}^n$ (cf. Theorem 15). Each k-cell of $\mathcal{A}_K$ is contained in an affine k-subspace of $\mathbb{R}^n$.

Proposition 50 (Mikhalkin [20])**.** *Let $z \in \mathcal{A}_K$ be a point of an open $(n-1)$-cell. Let $\rho : W \to \mathcal{A}_K$ be a regular neighborhood retraction such that its restriction to a neighborhood of z is a smooth submersion. For sufficiently large $t > 0$ the composition*

$$\lambda : V_t \stackrel{\mathrm{Log}_t}{\to} W \stackrel{\rho}{\to} \mathcal{A}_K$$

is submersive near z and $\lambda^{-1}(z)$ is diffeomorphic to a smooth $(n-1)$-torus.

If $\mathcal{A}_K$ is maximal (see Definition 13) then the map λ can be further improved. Let $z \in \mathcal{A}_K$.

Definition 14. Let M be a manifold, $N \subset \mathbb{R}^n$ be a piecewise-smooth CW-complex and $\lambda : M \to N \subset \mathbb{R}^n$ be a smooth map. Let $x \in L$ be a point. By the *degeneration type* of λ near x we mean the equivalence class of the restriction $\lambda^{-1}(U) \to U$ of λ to a small open ball $U = N \cap D_x(\epsilon)$ near x. Two smooth maps $W \to U \subset D_x(\epsilon)$ and $W' \to U' \subset D_{x'}(\epsilon')$ are equivalent if there exist diffeomorphisms $W \stackrel{\approx}{\to} W'$ and $D_x(\epsilon) \stackrel{\approx}{\to} D_{x'}(\epsilon')$ which take the first map to the second map.

Theorem 51 (Mikhalkin [20])**.** *Suppose that $\mathcal{A}_K$ is maximal. There exists a regular neighborhood retraction $\rho : W \to \mathcal{A}_K$ such that for sufficiently large $t > 0$ the composition*

$$\lambda = \rho \circ (\mathrm{Log}_t |_{V_t}) : V_t \to \mathcal{A}_K$$

is a singular torus fibration in the following sense

- *the restriction of λ to any open cell of $\mathcal{A}_K$ is a trivial fibration;*
- *the fiber of λ over an $(n-1)$-cell is T^{n-1};*
- *the degeneration type of λ at $x \in \mathcal{A}_K$ depends only on the dimension of the open cell containing x.*

The fiber of λ over an open k-cell of $\mathcal{A}_K$ is a $(n-1)$-dimensional CW-complex that can be embedded to the n-torus T^n. The fiber over an open $(n-2)$-cell is the product of a θ-graph (i.e. the graph with 2 vertices and 3 edges joining them) and a torus T^{n-2}. In addition we have the following properties.

- *The base $\mathcal{A}_K$ is homotopy equivalent to a wedge of p_g spheres S^{n-1}, where p_g is the number of lattice points in the interior of Δ.*
- *The induced homomorphism*
$$\lambda^* : H^{n-1}(\mathcal{A}_K; \mathbb{Z}) \to H^{n-1}(V; \mathbb{Z})$$
is a monomorphism.

4.7. **Torus fibrations for complex projective hypersurfaces.** This theorem admits a compactified version. Let $\bar{V} \subset \mathbb{C}T_\Delta$ be the compactification of V. A non-Archimedean amoeba corresponding to Δ can be compactified as well. Recall (see Remark 1) that the moment maps for the symplectic spaces $(\mathbb{C}^*)^n$ and $\mathbb{C}T_\Delta$ define a reparametrization $\mathbb{R}^n \overset{\approx}{\to} \operatorname{Int}\Delta$. The *compactified non-Archimedean amoeba* Π is the closure in Δ of the image of a non-Archimedean amoeba under this reparametrization.

Note that Π admits a natural cellular structure. To each cell we can associate two indices. One index is its dimension k. The other is the dimension l of the (open) face of Δ containing the cell.

Definition 15. An $(n-1)$-dimensional cellular space Π is called a *special spine* if a small neighborhood of a point $x \in \Pi$ from an open k-cell is homeomorphic to the direct product of $\mathbb{R}^k$ and the cone over the $(n-k-2)$-skeleton of the $(n-k)$-dimensional simplex.

The space Π is called a *special spine with corners* if for each open k-dimensional cell there exists an integer number l, $k < l \leq n$ with the following property. A small neighborhood of a point $x \in \Pi$ from this cell is homeomorphic to the direct product of $\mathbb{R}^k \times [0, +\infty)^{n-l}$ and the cone over the $(l-k-2)$-skeleton of the $(l-k)$-dimensional simplex. Note that a (-1)-skeleton is always empty. Such a k-dimensional cell is called a *(k,l)-cell.*

Example 3. A 1-dimensional special spine is a 3-valent graph. A 1-dimensional special spine with corners is a 3- and 1-valent graph.

Proposition 52. *If $\mathcal{A}_K$ is a maximal non-Archimedean amoeba then Π is a special spine with corners.*

Remark 10. The term "special spine" comes from Topology. Let X be an n-manifold (possibly with boundary or even with corners). An $(n-1)$-dimensional CW-complex $S \subset X$ is called a spine of X if the complement $X \smallsetminus (S \cup \partial X)$ is a disjoint union of open n-balls.

Originally the term "special spine" referred to a spine which satisfies to additional properties specified in Definition 15. Now the this term is also used (in particular, in this paper) also for CW-complexes without any ambient space. Note that in our case Π is a spine of Δ in the topological sense.

We introduce the following definition for the next theorem.

Definition 16 (Mikhalkin [20]). A map $\lambda : M \to \Pi$ is called a *manifold fibration over a special spine* $\Pi \subset \Delta$ *with corners*, where $\Delta \subset \mathbb{R}^n$ is a convex polyhedron, it

- M is a manifold;
- $\lambda : M \to \Pi \subset \Delta$ is a smooth map;
- the restriction of λ to each open cell of Π is a smooth trivial fibration (a submersion over an open cell);
- the degeneration type (see Definition 14) of λ at a point x from an open (k,l)-cell depends only on k and l.

Note that an $(n-1)$-dimensional cell is always a $(n-1,n)$-cell.

Remark 9 states that maximal non-Archimedean amoebas exist in the case when $\mathbb{C}T_\Delta$ is a projective space or a product of projective spaces. If $\mathcal{A}_K$ is maximal then the corresponding compactified non-Archimedean amoeba Π is

Theorem 53 (Mikhalkin [20]). *Let $\bar{V} \subset \mathbb{CP}^n$ be a non-singular hypersurface. There exists a special spine Π with corners and a manifold fibration $\bar{\lambda} : \bar{V} \to \Pi$ over Π such that*

- *the general fiber of $\bar{\lambda}$ (i.e. the fiber over an open $(n-1)$-dimensional cell) is a smooth $(n-1)$-dimensional torus;*
- *the homotopy type of Π is the wedge of p_g copies of S^{n-1}, where $p_g = h^{n-1,0}$ is the geometric genus of $\bar{V}$;*
- *the induced homomorphism $\lambda^* : H^{n-1}(\Pi;\mathbb{Z}) \to H^{n-1}(\bar{V};\mathbb{Z})$ is a monomorphism.*

Addendum 54 (Mikhalkin [20]). *Here is a partial description of special fibers of $\bar{\lambda}$ from Theorem 53.*

- *The fiber of $\bar{\lambda}$ over an $(k,k+1)$-cell, $k < n$, is a smooth k-dimensional torus;*
- *the fiber of $\bar{\lambda}$ over an $(k,k+2)$ cell, $k < n-1$, is a product of the θ-graph (i.e. the graph with 2 vertices and 3 edges joining them) and a $(k-1)$-torus;*
- *more generally, the fiber of $\bar{\lambda}$ over a (k,l)-cell is an $(l-1)$-dimensional CW-complex whose topology depends only on k and l and such that it can be embedded to an l-dimensional torus.*

Addendum 55 (Mikhalkin [20]). *Let $x \in \Pi$ be a point from a (k,l)-cell and $U \ni x$ be a regular neighborhood of x in Π. The inverse image $\bar{\lambda}^{-1}(U)$ is diffeomorphic to the product of $\mathbb{R}^k \times [0,+\infty)^{n-l}$ and $\mathbb{C}P^{l-k-1}$ minus $l-k+1$ hyperplanes in general position.*

4.8. **Decomposition of projective hypersurfaces into pairs of pants.** Let S be a closed Riemann surface. An *open pair of pants* is an open manifold diffeomorphic to the two sphere S^2 minus 3 points. A (closed) *pair of pants* is a compact surface of genus 0 with 3 boundary components. It is easy to see that an open pair of pants is a pair of pants without its boundary.

A *pair of pants decomposition for S* is given by a collection of disjoint embedded circles such that each component of their complement is an open pair of pants.

Let $p_1, \dots, p_m \in S$ are distinct points. A *pair of pants decomposition for* $(S; p_1, \dots, p_m)$ is given by a collection of disjoint embedded circles such that each component of their complement in $S \setminus \bigcup_j \{p_j\}$ is an open pair of pants.

Proposition 56. *To a pair of pants decomposition of S we may canonically associate a manifold fibration $\lambda : S \to \Pi$ over a 3-valent graph Π.*

To a pair of pants decomposition of $(S; p_1, \dots, p_m)$ we may canonically associate a manifold fibration $\lambda : S \to \Pi$ over a 3- and 1-valent graph Π.

Note that there is a natural fibration of a pair of pants over a Y-shaped graph such that the boundary components are fibers over 1-valent vertices and the fiber over the 3-valent vertex is a θ-shaped graph.

In the opposite direction we have the following proposition.

Proposition 57. *Let $S \to \Pi$ be a manifold fibration over a 3-valent graph Π such that the fibers over 3-valent vertices are θ-shaped graphs. Then the inverse images of the midpoints of the edges give a pair of pants decomposition for S.*

Let $S \to \Pi$ be a manifold fibration over a 3- and 1-valent graph Π such that the fibers over 3-valent vertices are θ-shaped graphs and the fibers over 1-valent vertices are points $p_1, \dots, p_m$. Then the inverse images of the midpoints of the edges connecting 3-valent vertices give a pair of pants decomposition for $(S; p_1, \dots, p_m)$.

The graph Π can be interpreted as combinatorial data needed for gluing pairs of pants to obtain S.

Proposition 58. *The surface S may be recovered from Π by the following procedure.*

1. *Take a disjoint union of pairs of pants, one pair of pants for each 3-valent vertex of Π.*
2. *For each edge connecting 3-valent vertices identify some boundary components of the corresponding pairs of pants.*
3. *Collapse the remaining boundary components (those corresponding to 1-valent vertices to points.*

Definition 17 (Mikhalkin [20])**.** An open l-dimensional pair of pants is an open manifold diffeomorphic to $\mathbb{CP}^l$ minus $l+2$ hyperplanes in general position.

Note that the arrangement of $l+2$ hyperplanes in general position is unique up to the natural action of $PSL(l+1, \mathbb{C})$.

Definition 18 (Mikhalkin [20])**.** An l-dimensional pair of pants P_l is an compact manifold (with corners) diffeomorphic to $\mathbb{CP}^l$ minus the union of small tubular neighborhoods $l+2$ hyperplanes in general position. A *closed facet* of P_l is the intersection of ∂P_l and the boundary of the tubular neighborhood of one of the $l+2$ hyperplanes. A *closed m-face* of P_l is the intersection of $l-m$ facets in ∂P_l. An *open m-face* is a closed m-face minus all smaller-dimensional faces.

Note that an open m-face of P_l is an open manifold diffeomorphic to the open m-dimensional pair of pants P_m times the real $(l-m)$-torus T^{l-m}. Note also that an open pair of pants is a pair of pants minus its boundary.

Remark 11. We can collapse a part of the boundary of P_l corresponding to a m facets of P_l. The result of collapse is $\mathbb{CP}^l$ minus the union of small tubular neighborhoods of the remaining $l+2-m$ hyperplanes. Thus we add back the tubular neighborhoods of the hyperplane corresponding to collapsing facets.

Theorem 53 can be interpreted as a higher-dimensional pair of pants decomposition for smooth projective hypersurfaces thanks to the following corollary from Addendum 55.

Corollary 59. *Let $x \in \Pi$ be a $(0,n)$-cell of Π and $U \ni x$ be a regular neighborhood of x. The inverse image $\bar{\lambda}^{-1}(U)$ is diffeomorphic to an open $(n-1)$-dimensional pair of pants.*

The polyhedral complex Π may be interpreted as combinatorial data needed for gluing pairs of pants to construct $\bar{V}$ in a fashion similar to Proposition 58. We start from a disjoint union of $(n-1)$-dimensional pairs of pants, one for each $(0,n)$-cell of Π. Each $(1,n)$-cell is an edge connecting $(0,n)$-vertices.

Each $(0,n)$-vertex is adjacent to $n+1$ edges of Π corresponding to $n+1$ facets of P_{n-1}. Similarly, it is adjacent to $\binom{n+1}{k}$ k-faces of Π corresponding to $\binom{n+1}{k}$ $(n-k-1)$-faces of P_{n-1}. For each $(1,n)$-edge we identify corresponding closed facets of the pairs of pants corresponding to the endpoints.

Our identification is subject to the following additional condition. For each k-cell e of Π we consider all $(0,n)$-vertices adjacent to e. Each of the corresponding $(n-1)$-dimensional pair of pants contain an $(n-k-1)$-faces corresponding to e. All these $(n-k-1)$-faces have to be identified.

To get $\bar{V}$ from the result of this identification we have to collapse the boundary as in Remark 11.

References

[1] V. I. Arnold, *The situation of ovals of real plane algebraic curves, the involutions of four-dimensional smooth manifolds, and the arithmetic of integral quadratic forms*, Funkcional. Anal. i Priložen. **5** (1971) no. 3, 1-9.

[2] M. F. Atiyah, *Angular momentum, convex polyhedra and algebraic geometry*, Proc. Edinburgh Math. Soc., **26** (1983) 121-133.

[3] G. M. Bergman, *The logarithmic limit set of an algebraic variety*, Trans. AMS, **157** (1971), 459-469.

[4] D. Bernstein, *The number of roots of a system of equations*, Functional Anal. Appl. **9** (1975), 183-185.

[5] L. Brusotti, *Curve generatrici e curve aggregate nella costruzione di curve piane d'ordine assegnato dotate del massimo numero di circuiti,* Rend. Circ. Mat. Palermo **42** (1917), 138-144.

[6] M. Forsberg, M. Passare, A. Tsikh, *Laurent determinants and arangements of hyperplane amoebas,* Advances in Math. **151** (2000), 45–70.

[7] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[8] A. Harnack, *Über Vieltheiligkeit der ebenen algebraischen Curven,* Math. Ann. **10** (1876), 189-199.

[9] A. Henriques, *An analogue of convexity for complements of amoebas of varieties of higher codimensions*, Preprint, Berkeley, May 2001.

[10] D. Hilbert, *Mathematische Probleme*, Arch. Math. Phys. (3) **1** (1901), 213-237.

[11] I. Itenberg, O. Viro, *Maximal real algebraic hypersurfaces of projective space*, in preparation.

[12] M. M. Kapranov, *A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map*, Math. Ann. **290** (1991), 277-285

[13] M. M. Kapranov, *Amoebas over non-Archimedian fields*, Preprint, 2000.

[14] V. M. Kharlamov, *On the classification of nonsingular surfaces of degree 4 in* $\mathbb{RP}^3$ *with respect to rigid isotopies*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 1, 49-56.

[15] A. G. Khovanskii, *Newton polyhedra and toric varieties,* Funkcional. Anal. i Priložen. **11** (1977), no. 4, 56 - 64.

[16] V. P. Maslov, *New superposition principle for optimization problems*, in Seminaire sur les Equations auc Dérivées Partielles 1985/6, Centre Mathématiques de l'École Polytechnique, Palaiseau, 1986, exposé 24.

[17] G. Mikhalkin, *Real algebraic curves, moment map and amoebas*, Ann. of Math. **151** (2000), 309 - 326.

[18] G. Mikhalkin, H. Rullgård, *Amoebas of maximal area*, Intern. Math. Res. Notices **9** (2001), 441-451.

[19] G. Mikhalkin, O. Viro, *Geometry of Amoebas* (in preparation).

[20] G. Mikhalkin, *Complex algebraic hypersurfaces as torus fibrations* (in preparation).

[21] G. Mikhalkin, *Maximal real algebraic hypersurfaces* (in preparation)

[22] M. Passare, H. Rullgård, *Amoebas, Monge-Ampère measures, and triangulations of the Newton polytope.* Preprint, Stockholm University, 2000.

[23] I. G. Petrovsky, *On the topology of real plane algebraic curves*, Ann. Math. **39** (1938), 187-209.

[24] V. A. Rohlin, *Congruences modulo* 16 *in Hilbert's sixteenth problem*, Funkcional. Anal. i Priložen. **6** (1972), no. 4, 58-64.

[25] L. Ronkin, *On zeroes of almost periodic functions generated by holomorphic functions in a multicircular domain*, to appear in "Complex Analysis in Modern Mathematics", Fazis, Moscow, 2000, 243-256.

[26] H. Rullgård, *Stratification des espaces de polynômes de Laurent et la structure de leurs amibes*, C. R. Acad. Sci. Paris, Série I, **331** (2000), 355-358.

[27] H. Rullgård, *Polynomial amoebas and convexity*, Preprint, Stockholm University, 2001.

[28] O. Ya. Viro, *Real plane algebraic curves: constructions with controlled topology,* Leningrad Math. J. **1** (1990), no. 5, 1059-1134.

[29] O. Ya. Viro, *Dequantization of Real Algebraic Geometry on a Logarithmic Paper,* Proceedings of the European Congress of Mathematicians (2000).

[30] G. Wilson, *Hilbert's sixteenth problem*, Topology **17** (1978), 53-73.

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