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ABELIAN VARIETIES OVER FINITE FIELDS AS BASIC ABELIAN VARIETIES

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Abstract. In this note we show that any basic abelian variety with additional structures over an arbitrary algebraically closed field of characteristic $p > 0$ is isogenous to another one which is defined over a finite field. We also show that the category of abelian varieties over finite fields up to isogeny can be embedded into the category of basic abelian varieties with suitable endomorphism structures. Using this connection, we derive a new mass formula for an orbit of polarized abelian surfaces over a finite field.

1. Introduction

In this note we work on abelian varieties over fields of positive characteristic, particularly on basic abelian varieties with additional structures. Let p be a prime number. Basic abelian varieties with fixed additional structures (endomorphisms, a polarization and a level structure) over a field of characteristic $p > 0$ are conceptually defined as the points in a moduli space of PEL-type over $\overline{\mathbb{F}}_p$ which land in the minimal Newton stratum (Rapoport-Zink [6] and Rapoport [5]). The group-theoretic definition was introduced by Kottwitz [1]. This is a geometric notion, that is, an abelian variety with additional structures is basic if and only if its base change to any algebraically closed field extension is also basic. As landing in the same Newton stratum is an isogeny property, an abelian variety with additional structures which is isogenous (compatible with the additional structures) to a basic abelian variety with the same additional structures is also basic.

Let B be a finite-dimensional semi-simple $\mathbb{Q}$-algebra with a positive involution $*$ and O_B an order in B stable under $*$. A polarized abelian O_B-variety is a triple (A, λ, ι) where (A, λ) is a polarized abelian variety and $\iota : O_B \to \mathrm{End}(A)$ is a ring monomorphism which is compatible with λ. We recall the definition of basic polarized abelian O_B-varieties (A, λ, ι) in Section 2.

Basic abelian varieties with additional structures share many similar properties as supersingular abelian varieties without additional structures have. For example, like supersingular abelian varieties, one can formulate a geometric mass for a finite orbit of basic abelian varieties and relate this geometric mass to an arithmetic mass. A typical example is the Deuring-Eichler mass formula. We refer to [15] for more discussions in this aspect. In this paper we prove the following result, which may be regarded as another analogue of supersingular abelian varieties.

Theorem 1.1. *Let $\underline{A} = (A, \lambda, \iota)$ be a basic polarized abelian O_B-variety over an algebraically closed field k of characteristic $p > 0$. Then there exists a polarized*

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abelian O_B-variety $\underline{A}' = (A', \lambda', \iota')$ over a finite field κ and an O_B-linear isogeny $\varphi : A' \otimes_\kappa k \to A$ over k that preserves the polarizations.

The second part of this note studies the converse. We show that any abelian variety over a finite field can be regarded as a basic abelian variety with suitable endomorphism structures. More precisely, if A is an abelian variety over $\mathbb{F}_q$ and let $F = \mathbb{Q}(\pi_A)$ be the $\mathbb{Q}$-algebra generated by its Frobenius endomorphism π_A, then the abelian variety A together with the F-action is a basic abelian F-variety (Proposition 4.1). See Remark 3.2 for the definition of basic abelian B-varieties. The original definition of basic abelian varieties with additional structures requires both structures of endomorphisms and polarizations. However, like supersingular abelian varieties, polarizations play no role in the supersingularity.

Let $\mathcal{A}_{\mathbb{F}_q}$ denote the category of abelian varieties over $\mathbb{F}_q$ up to isogeny, and let $\mathcal{B}^{\rm rig}$ be the category of basic abelian varieties with rigidified endomorphisms over $\overline{\mathbb{F}}_p$ up to isogeny, defined in Section 4. We prove the following result.

Theorem 1.2. *There is a functor Φ which embeds the category $\mathcal{A}_{\mathbb{F}_q}$ as a full subcategory of $\mathcal{B}^{\rm rig}$.*

Theorem 1.2 connects (polarized) abelian varieties over a finite field $\mathbb{F}_q$ with basic (polarized) abelian F-varieties over $\overline{\mathbb{F}}_p$ for a suitable commutative semi-simple $\mathbb{Q}$-algebra F. This connection is useful when the $\mathbb{Q}$-algebra F is fixed. In this case one considers abelian varieties over $\mathbb{F}_q$ whose endomorphism rings contain the maximal order O_F. Then we can embed the set of such kind of abelian varieties over $\mathbb{F}_q$ into the basic locus of a moduli space of polarized abelian O_F-varieties; see Lemma 5.1 and (5.2) for details. Below is a example where we use this embedding to derive a mass formula for a class of polarized abelian surfaces over $\mathbb{F}_p$.

Choose a simple abelian varieties A_0 over the prime finite field $\mathbb{F}_p$ whose Frobenius endomorphism π_0 satisfying $\pi_0^2 = p$. Then A_0 is a superspecial abelian surface, i.e. the base change $A_0 \otimes \overline{\mathbb{F}}_p$ is isomorphic to the product of two supersingular elliptic curves. Let us consider the set Λ of isomorphism classes of principally polarized simple abelian surfaces (A, λ) over $\mathbb{F}_p$ which is isogenous to A_0. Put $F = \mathbb{Q}(\pi_0) = \mathbb{Q}(\sqrt{p})$ and O_F its ring of integers. Let $\Lambda^{\max} \subset \Lambda$ the subset of (A, λ) such that $O_F \subset \mathrm{End}(A)$. We can show that $\Lambda^{\max}$ is nonempty set and that the mass $\mathrm{Mass}(\Lambda^{\max})$ of $\Lambda^{\max}$

$$\mathrm{Mass}(\Lambda^{\max}) := \sum_{(A,\lambda)\in\Lambda^{\max}} |\,\mathrm{Aut}(A, \lambda)|^{-1} \tag{1.1}$$

is equal to the mass of a finite orbit S of the superspecial locus of a Hilbert modular surface modulo p. Furthermore, using the geometric mass formula for the superspecial orbits [12], we obtain the mass formula

$$\mathrm{Mass}(\Lambda^{\max}) = \frac{\zeta_F(-1)}{4} \tag{1.2}$$

for $p > 2$, where $\zeta_F(s)$ the Dedekind zeta function of F. See Section 5.2 for details.

The paper is organized as follows. In Section 2 we recall the definition of basic abelian varieties with additional structures. The proof and some consequences of Theorem 1.1 are given in Section 3. In Section 4 we show that any abelian variety over a finite field, together with the action of the center of its endomorphism algebra, is a basic abelian variety. This connection allows us to make Theorem 1.2.

In the last section we restrict ourselves to a special example of an isogeny class of simple supersingular abelian surfaces. We compute the associated mass (1.1) by a geometric mass formula in [12].

Notations. If M is a $\mathbb{Z}$-module or a $\mathbb{Q}$-module and ℓ is a prime, we write $M_\ell := M \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ or $M_\ell = M \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, respectively. For any perfect field k of characteristic $p > 0$, denote by $W(k)$ the ring of Witt vectors over k, $B(k)$ the fraction field of $W(k)$, σ the Frobenius map on $W(k)$ and $B(k)$ induced by $\sigma : k \to k,\ x \mapsto x^p$. If F is a finite product of number fields, denote by O_F the maximal order in F. For an abelian variety A over a field k, write $\mathrm{End}(A) = \mathrm{End}_k(A)$ for the endomorphism ring of A over k and $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ for the endomorphism algebra of A over k.

2. Basic abelian varieties with additional structures

In this section we recall basic abelian varieties with additional structures introduced by Kottwitz [1]. Our reference is Rapoport-Zink [6, p.11, p. 281 and 6.25, p. 291].

2.1. **Settings.** Let B be a finite-dimensional semi-simple algebra over $\mathbb{Q}$ with a positive involution $*$, and let O_B be any order of B stable under $*$.

Recall that a non-degenerate $\mathbb{Q}$-valued skew-Hermitian B-space is a pair (V, ψ) where V is a left faithful finite B-module, and $\psi : V \times V \to \mathbb{Q}$ is a non-degenerate alternating pairing such that $\psi(bx, y) = \psi(x, b^* y)$ for all $b \in B$ and all $x, y \in V$.

A *polarized abelian O_B-variety* (resp. *polarized abelian B-variety*) is a triple $\underline{A} = (A, \lambda, \iota)$, where (A, λ) is a polarized abelian variety and $\iota : O_B \to \mathrm{End}(A)$ (resp. $\iota : B \to \mathrm{End}^0(A)$) is a ring monomorphism such that $\lambda\iota(b^*) = \iota(b)^t \lambda$ for all $b \in O_B$.

Let $\underline{A}$ be a polarized abelian O_B-variety over a field k. For any prime ℓ (not necessarily invertible in k), we write $\underline{A}(\ell)$ for the associated ℓ-divisible group with additional structures $(A[\ell^\infty], \lambda_\ell, \iota_\ell)$, where λ_ℓ is the induced quasi-polarization from $A[\ell^\infty]$ to $A^t[\ell^\infty] = A[\ell^\infty]^t$ (the Serre dual), and $\iota_\ell : (O_B)_\ell \to \mathrm{End}(A[\ell^\infty])$ the induced ring monomorphism. If $\ell \neq \mathrm{char}(k)$, let $T_\ell(A)$ denote the ℓ-adic Tate module of A, $V_\ell := T_\ell(A) \otimes \mathbb{Q}_\ell$, and let

$$\rho_\ell : \mathcal{G}_k \to \mathrm{GU}_{B_\ell}(V_\ell, e_\lambda) \tag{2.1}$$

be the associated Galois representation. Here $\mathcal{G}_k := \mathrm{Gal}(k_s/k)$ is the Galois group of k, k_s a separably closure of k, and

$$\mathrm{GU}_{B_\ell}(V_\ell, e_\lambda) := \{g \in \mathrm{Aut}_{B_\ell}(V_\ell) \,|\, e_\lambda(gx, gy) = c\, e_\lambda(x, y) \text{ for some } c \in \mathbb{Q}_\ell^\times \}$$

is the group of B_ℓ-linear similitudes with respect to the Weil pairing

$$e_\lambda = e_{\lambda,\ell} : T_\ell(A) \times T_\ell(A) \to \mathbb{Z}_\ell(1),$$

where

$$\mathbb{Z}_\ell(1) := \varprojlim \mu_{p^m}(k_s)$$

is the Tate twist.

If k is a perfect field of characteristic p, let $M(\underline{A})$ denote the covariant Dieudonné module of $\underline{A}$ with the additional structures and put $N(\underline{A}) := M(\underline{A}) \otimes_{W(k)} B(k)$, the rational Dieudonné module (or the isocrystal) with the additional structures.

In this note we only consider the objects $\underline{A} = (A, \lambda, \iota)$ so that there is a non-degenerate skew-Hermitian B-space (V, ψ) such that $2 \dim A = \dim_{\mathbb{Q}} V$. That is, we require the dimension g of A have the property that there exists a complex g-dimensional polarized abelian O_B-variety. For example we exclude the case where A is a supersingular elliptic curve and B is the quaternion $\mathbb{Q}$-algebra ramified precisely at $\{p, \infty\}$.

2.2. **Basic abelian varieties.** Let k be any field of characteristic p and let $\bar{k}$ be an algebraic closure of k. Put $W := W(\bar{k})$ and $L := B(\bar{k})$. Let (V_p, ψ_p) be a $\mathbb{Q}_p$-valued non-degenerate skew-Hermitian B_p-module. A polarized abelian O_B-variety $\underline{A}$ over $\bar{k}$ is said to be *related to* (V_p, ψ_p) if there is a $B_p \otimes_{\mathbb{Q}_p} L$-linear isomorphism $\alpha : N(\underline{A}) \simeq (V_p, \psi_p) \otimes_{\mathbb{Q}_p} L$ which preserves the pairings for a suitable identification $L(1) \simeq L$.

Let $G_p := \mathrm{GU}_{B_p}(V_p, \psi_p)$ be the algebraic group over $\mathbb{Q}_p$ of B_p-linear similitudes with respect to the pairing ψ_p. A choice α gives rise to an element $b \in G_p(L)$ so that one has an isomorphism of isocrystals with additional structures $N(\underline{A}) \simeq (V_p \otimes L, \psi_p, b(\mathrm{id} \otimes \sigma))$. Let $[b]$ be the σ-conjugacy class of b in $G_p(L)$. The decomposition of $V_p \otimes L$ into isotypic components (the components of single slope) induces a $\mathbb{Q}$-graded structure, and thus defines a (slope) homomorphism $\nu_{[b]} : \mathbf{D} \to G_p$ over some unramified finite extension $\mathbb{Q}_{p^s}$ of $\mathbb{Q}_p$, where $\mathbf{D}$ is the pro-torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$.

Definition 2.1. (1) A polarized abelian O_B-variety $\underline{A}$ over $\bar{k}$ is said to be *basic with respect to* (V_p, ψ_p) if

(a) $\underline{A}$ is related to (V_p, ψ_p), and
(b) the slope homomorphism $\nu_{[b]} : \mathbf{D} \to G_p$ is central.

(2) The object $\underline{A}$ over $\bar{k}$ is said to be *basic* if it is basic with respect to (V_p, ψ_p) for some non-degenerate skew-Hermitian B_p-space (V_p, ψ_p).

(3) A polarized abelian O_B-variety $\underline{A}$ over any field k is said to be *basic* if its base change $\underline{A} \otimes_k \bar{k}$ is basic.

Clearly a polarized abelian O_B-variety $\underline{A}$ is basic if (and only if) its polarized abelian B-variety is so. Two polarized abelian B-varieties $\underline{A}_1$ and $\underline{A}_2$ are said to be *isogenous*, denote $\underline{A}_1 \sim \underline{A}_2$, if there is a B-linear isogeny $\varphi : A_1 \to A_2$ such that the pull-back $\varphi^* \lambda_2$ is a $\mathbb{Q}$-multiple of λ_1. Clearly the property for an object $\underline{A}$ being basic is an isogeny property. From the definition it is also easy to see that this is a geometric notion: an object $\underline{A} = (A, \lambda, \iota)$ over k is basic if and only if it base change $\underline{A} \otimes_k k_1$ is basic for any algebraically closed field $k_1 \supset k$.

3. Proof of Theorems 1.1 and its corollaries

3.1. We need some properties of basic abelian varieties with additional structures. Let (V, ψ) be a non-degenerate skew-Hermitian B-space and let $G := \mathrm{GU}_B(V, \psi)$ be the algebraic group over $\mathbb{Q}$ of B-linear similitudes with respect to the pairing ψ.

Let F be the center of B and F_0 be the subfield fixed by the involution on F, which we denote by $a \mapsto \bar{a}$. Let Σ_p be the set of primes $\mathbf{p}$ of F over p, and for a prime $\mathbf{p}|p$, write $\mathrm{ord}_{\mathbf{p}}$ the corresponding p-adic valuation normalized so that $\mathrm{ord}_{\mathbf{p}}(p) = 1$. Let $F_p := F \otimes \mathbb{Q}_p = \prod_{\mathbf{p}|p} F_{\mathbf{p}}$ be the decomposition as a product of local fields. For each isocrystal N with an F_p-linear action, let

$$N = \oplus_{\mathbf{p}|p} N_{\mathbf{p}}$$

be the decomposition with respect to the F_p-action.

Lemma 3.1 (Rapoport-Zink)**.** *Notations as above.*

(1) *The center Z of G is the algebraic group over $\mathbb{Q}$ whose group of $\mathbb{Q}$-rational points is*

$$Z(\mathbb{Q}) = \{g \in F^\times;\ g\bar{g} \in \mathbb{Q}^\times\}.$$

(2) *Let N be an isocrystal with additional structures and suppose that it is related to $(V \otimes \mathbb{Q}_p, \psi)$. Then N is basic with respect to $(V \otimes \mathbb{Q}_p, \psi)$ if and only if each component $N_{\mathbf{p}}$ is isotypic. In particular, if N is basic, then $N_{\mathbf{p}}$ is supersingular for primes $\mathbf{p}$ with $\mathbf{p} = \bar{\mathbf{p}}$.*

Proof. The statement (1) and the only if part of the statement (2) are proved in 6.25 of [6]. The if part is easier: as each $N_{\mathbf{p}}$ is isotypic, say of slope r/s, the action of $s\nu(p)$ on $N_{\mathbf{p}}$ is a scalar and thus the slope homomorphism ν must be central.

Remark 3.2. Lemma 3.1 provides a simple criterion to check whether a polarized abelian B-variety $\underline{A} = (A, \lambda, \iota)$ is basic. Note that the assertion of the statement (2) only depends on the underlying structure of B-action, but not on the equipped polarization structure. Therefore, it makes sense to call an abelian B-variety (A, ι) *basic* if for any B-linear polarization λ the polarized abelian B-variety (A, λ, ι) is basic in Definition 2.1. A B-linear polarization λ on an abelian B-variety (A, ι) always exists [2, Lemma 9.2].

It follows from Lemma 3.1 that an abelian B-variety (A, ι) is basic if and only if the abelian F-variety $(A, \iota|_F)$ is basic, where $\iota|_F$ is the restriction of ι to F.

The following two lemmas are reorganized from [6, 6.26-6.29]; proofs are provided only for the reader's convenience.

Lemma 3.3. *Given any set $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ of rational numbers with $0 \le \lambda_{\mathbf{p}} \le 1$ and $\lambda_{\mathbf{p}} + \lambda_{\bar{\mathbf{p}}} = 1$, then there is a positive integer s and $u \in O_F[1/p]^\times$ such that*

$$u\bar{u} = q, \quad \text{and} \quad \operatorname{ord}_{\mathbf{p}} u = s\lambda_{\mathbf{p}},\ \forall\, \mathbf{p} \in \Sigma_p,$$

where $q = p^s$.

Proof. Consider the map

$$\operatorname{ord} : O_F \left[\frac{1}{p}\right]^\times \to \mathbb{Z}^{\Sigma_p}, \quad u \mapsto (\operatorname{ord}_{\mathbf{p}}(u))_{\mathbf{p} \in \Sigma_p}.$$

By Dirichlet's unit theorem, the image is of finite index. Therefore, there is a positive integer s such that there is an element $u \in O_F[1/p]^\times$ so that $\operatorname{ord}_{\mathbf{p}}(u) = s\lambda_{\mathbf{p}} =: r_{\mathbf{p}}$ for all $\mathbf{p} \in \Sigma_p$. Let $q = p^s$ and $u' := qu/\bar{u}$, then one computes

$$\operatorname{ord}_{\mathbf{p}} u' = 2r_{\mathbf{p}}, u'\overline{u}' = q^2.$$

Replacing u by u' and q by q^2, one gets the desire results. ∎

Lemma 3.4. *Fix $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ and $q = p^s$ as in Lemma 3.3. Then there is a positive integer n such that for any basic polarized abelian O_B-variety $\underline{A}$ over a finite extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ with slopes $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$, the n-th power of relative Frobenius morphism π_A^n lies in $\iota(F)$.*

Proof. We first prove that the statement holds for one such object $\underline{A}$. Let M be the Dieudonné module of $\underline{A}$. Within the isogeny class, we can choose $\underline{A}$ so that $F^s M_{\mathbf{p}} = p^{r_{\mathbf{p}}} M_{\mathbf{p}}$ for all $\mathbf{p} \in \Sigma_p$, where $r_{\mathbf{p}} = s\lambda_{\mathbf{p}}$. Let u be as in Lemma 3.3, then $\iota(u)^{-m}\pi_A$ as an automorphism of A that preserves the polarization. Therefore, a power of it is the identity.

Let $C := \mathrm{End}^0_B(A)$. By the result we just proved, the algebra C is independent of the choice of $\underline{A}$ and it has center $\iota(F)$. Therefore, there is a positive integer n so that any roots of unity ζ in C satisfies $\zeta^n = 1$.

Repeat the same proof above and we get $\pi_A^n \in \iota(F)$ for all such objects $\underline{A}$. ∎

3.2. Proof of Theorem 1.1. It suffices to show that A has smCM, that is, any maximal commutative semi-simple $\mathbb{Q}$-subalgebra of $\mathrm{End}^0(A)$ has degree $2\dim A$. Then by a theorem of Grothendieck (see a proof in [4] or in [11]) there exists an abelian variety A' over a finite field κ and an isogeny $\varphi : A' \otimes_\kappa k \to A$ over k. Replacing A' by one in its isogeny class if necessary, we may assume that A' admits an action ι' of O_B so that the isogeny φ is O_B-linear. Take the pull-back polarization λ' on A', which is clearly defined over a finite field extension of κ.

Let $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ be the set of slopes for $\underline{A}$. Let $q = p^s$ and a positive integer n be as in Lemmas 3.3 and 3.4. Let k_0 be a field of finite type of $\mathbb{F}_q$ for which $\underline{A}$ is defined. The abelian variety $\underline{A}$ extends to a polarized abelian O_B-scheme $\underline{\mathbf{A}}$ over a subring R of k_0 with $\mathrm{Frac}(R) = k_0$, which is smooth and of finite type over $\mathbb{F}_q$. Put $S = \mathrm{Spec}\, R$. Let s be a closed point of S and η the generic point. By Grothendieck's specialization theorem, the special fiber $\underline{\mathbf{A}}_s$ over s also has the same slopes $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$, and hence is basic.

We identify the endomorphism rings $\mathrm{End}_{k_0}(A) = \mathrm{End}_R(\mathbf{A}) \subset \mathrm{End}(\mathbf{A}_{\bar{s}})$, and write ι for the O_B-actions on these abelian varieties. Let

$$\rho_\ell : \pi_1(S, \bar{\eta}) \to \mathrm{Aut}(T_\ell(A_{\bar{\eta}}))$$

be the associated ℓ-adic representation. The action of $\mathrm{Gal}(\bar{\eta}/\eta)$ on $T_\ell(A_{\bar{\eta}})$ factors through ρ_ℓ. Again we identify the Tate modules $T_\ell(\mathbf{A}_{\bar{s}}) = T_\ell(\mathbf{A}_{\widetilde{S}_{\bar{s}}}) = T_\ell(A_{\bar{\eta}})$, where $\widetilde{S}_{\bar{s}}$ is the (strict) Henselization of S at $\bar{s}$.

Let π_{A_s} be the relative Frobenius morphism on $\mathbf{A}_s$ and Frob_s the geometric Frobenius element in $\pi_1(S, \bar{\eta})$ corresponding to the closed point s. We have

(i) $\pi^n_{A_s} \in \iota(F) \subset \mathrm{End}(T_\ell(\mathbf{A}_{\bar{s}}))$, by Lemma 3.4;
(ii) $\rho_\ell(\mathrm{Frob}^n_s) = \pi^n_{A_s}$ lies in the center $Z(\mathbb{Q}_\ell)$ of $\mathrm{GU}_{B_\ell}(T_\ell(A_{\bar{\eta}}), \langle\,,\rangle)$, by identifying the Tate modules and (i);
(iii) the Frobenius elements Frob_s for all closed points s generate a dense subgroup of $\pi_1(S, \bar{\eta})$.

Let $G_\ell := \rho_\ell(\pi_1(S, \bar{\eta}))$ be the ℓ-adic monodromy group. Let $m_n : G_\ell \to G_\ell$ be the multiplication by n. It is an open mapping and the image of m_n contains an open subgroup U of G_ℓ. Clearly U lies in the center $Z(\mathbb{Q}_\ell)$ by (ii) and (iii). Replacing k_0 by a finite extension, we have $G_\ell \subset Z(\mathbb{Q}_\ell)$. Let $\mathbb{Q}_\ell[\pi]$ be the (commutative) subalgebra of $\mathrm{End}(T_\ell(A_{\bar{\eta}}))$ generated by G_ℓ. By Zarhin's theorem [16], $\mathbb{Q}_\ell[\pi]$ is semi-simple and commutative, and $\mathrm{End}_{\mathbb{Q}_\ell[\pi]}(T_\ell(A_{\bar{\eta}})) = \mathrm{End}(A) \otimes \mathbb{Q}_\ell$. This shows that any maximal commutative semi-simple subalgebra of $\mathrm{End}^0(A)$ has degree $2g$. This completes the proof. ∎

3.3. Some consequences. In [15] we defined a class of polarized abelian B-varieties, called of arithmetic type there, which are those the main result [15, Theorem 2.2] can extend over algebraically closed fields. We related these abelian B-varieties with basic abelian B-varieties in the case where the ground field k is $\overline{\mathbb{F}}_p$; see [15, Theorem 4.5]. Theorem 1.1 extends this result to that over an arbitrary algebraically closed field k of characteristic $p > 0$.

Recall that we call a polarized abelian B-variety (A, λ, ι) over an algebraically closed field k of characteristic $p > 0$ *of arithmetic type* if there is a model $(A_0, \lambda_0, \iota_0)$ of (A, λ, ι) over a field k_0 finitely generated over $\mathbb{F}_p$ such that the associated Galois representation $\rho_\ell : \mathcal{G}_{k_0} \to \mathrm{GU}_B(V_\ell(A_0), e_{\lambda,\ell})$ (Section 2.1) is central for some prime $\ell \neq p$ (or equivalently for all primes $\ell \neq p$, see [15, Proposition 3.10]). It is shown in [15, Section 3] that this is again a geometric notion and this notion only depends on the underlying abelian B-variety (A, ι) but not on the carried polarization structure λ.

Theorem 3.5. *An abelian B-variety (A, ι) over an algebraically closed field k of characteristic $p > 0$ is of arithmetic type if and only if it is basic.*

PROOF. By Theorem 1.1, there is an abelian B-variety (A_0, ι_0) over $\overline{\mathbb{F}}_p$ and an B-linear isogeny $\varphi : (A_0, \iota_0) \otimes_{\overline{\mathbb{F}}_p} k \to (A, \iota)$. As a result we can reduce the statement to the case when $k = \overline{\mathbb{F}}_p$ and this is Theorem 4.5 of [15]. ∎

Proposition 3.6 (cf. [6, Corollary 6.29])**.** *Let K be a finite-dimensional semi-simple $\mathbb{Q}$-algebra that admits a positive involution. Let (A, ι) and (A', ι') be two abelian K-varieties over an algebraically closed field k of characteristic $p > 0$. Then we have*

$$\mathrm{Hom}_K(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \simeq \mathrm{Hom}_K(V_\ell(A), V_\ell(A')) \quad \forall\, \ell \neq p,$$

and

$$\mathrm{Hom}_K(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \simeq \mathrm{Hom}_K((N, F), (N', F)),$$

where N and N' are the isocrystals with additional structures associated to (A, ι) and (A', ι'), respectively.

PROOF. By Theorem 1.1, there are abelian K-varieties (A_0, ι_0) and (A_0', ι_0') over $\overline{\mathbb{F}}_p$ such that $(A_0, \iota_0) \otimes_{\overline{\mathbb{F}}_p} k \sim (A, \iota)$ and $(A_0', \iota_0') \otimes_{\overline{\mathbb{F}}_p} k \sim (A', \iota')$. We have a natural isomorphism $\mathrm{Hom}_K(A_0, A_0') \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Hom}_K(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}$, $V_\ell(A_0) = V_\ell(A)$ and $V_\ell(A_0') = V_\ell(A')$, and the identification $\mathrm{Hom}((N_0, F), (N_0', F)) = \mathrm{Hom}((N, F), (N', F))$, where N_0 and N_0' are the isocrystals with additional structures associated to (A_0, ι_0) and (A_0', ι_0'), respectively. Therefore, we are reduced to prove the statement when $k = \overline{\mathbb{F}}_p$, which is proved by Rapoport-Zink (see [6, Corollary 6.29, p. 293]). ∎

4. A correspondence

4.1. Let $\mathbb{F}_q$ be the finite field of $q = p^s$ elements. Let $\mathcal{A}_{\mathbb{F}_q}$ denote the category of abelian varieties over $\mathbb{F}_q$ up to isogeny. Let $\mathcal{B}$ be the category defined as follows, which we call the category of basic abelian varieties with endomorphisms over $\overline{\mathbb{F}}_p$ up to isogeny . The objects of $\mathcal{B}$ consist of all triples (F, A, ι), where

- F is a finite-dimensional commutative semi-simple $\mathbb{Q}$-algebra that admits a positive involution, and

- (A, ι) is a basic abelian F-variety over $\overline{\mathbb{F}}_p$.

For any two objects $\underline{A}_1 = (F_1, A_1, \iota_1)$ and $\underline{A}_2 = (F_2, A_2, \iota_2)$ in $\mathcal{B}$, a morphism $\varphi : \underline{A}_1 \to \underline{A}_2$ is a pair $(\varphi, \widetilde{\varphi})$, where

- $\widetilde{\varphi} : F_1 \to F_2$ is a usual $\mathbb{Q}$-algebra homomorphism (that is, it is not required for $\widetilde{\varphi}$ mapping the identity 1_{F_1} of F_1 to 1_{F_2}), and
- φ is an element in $\mathrm{Hom}(A_1, A_2) \otimes \mathbb{Q}$ which is (F_1, F_2)-equivariant in the sense that $\varphi \circ \iota_1(a) = \iota_2(\widetilde{\varphi}(a)) \circ \varphi$ holds for all $a \in F_1$.

Note that if a usual $\mathbb{Q}$-algebra homomorphism $\widetilde{\varphi} : F_1 \to F_2$ is surjective, then $\widetilde{\varphi}(1_{F_1}) = 1_{F_2}$, i.e. it becomes a ring homomorphism. Clearly two objects $\underline{A}_1$ and $\underline{A}_2$ are isomorphic in $\mathcal{B}$ if and only if there is a $\mathbb{Q}$-algebra isomorphism $\widetilde{\varphi} : F_1 \simeq F_2$, and a (F_1, F_2)-equivariant quasi-isogeny $\varphi : A_1 \to A_2$ over $\overline{\mathbb{F}}_p$.

The category $\mathcal{B}$ is not yet good enough from comparing the study of abelian varieties with additional structures; there are too many additional morphisms $\widetilde{\varphi}$ among the fields F. For example, when $F_1 = F_2 = F$, the usual notion of morphisms between two abelian F-varieties always requires $\widetilde{\varphi}$ be identity in morphisms of $\mathcal{B}$ but not an arbitrary automorphism. We introduce another category $\mathcal{B}^{\mathrm{rig}}$, which we call the category of basic abelian varieties with rigidified endomorphisms over $\overline{\mathbb{F}}_p$ up to isogeny. The objects of $\mathcal{B}^{\mathrm{rig}}$ are tuples (F, x, A, ι) over $\overline{\mathbb{F}}_p$, where (F, A, ι) is an object in $\mathcal{B}$ and $x \in F$ is an element that generates F over $\mathbb{Q}$. Suppose (F, x, A, ι) is an object, let $\mathbb{Q}[t] \to F$ be the natural surjective map from the polynomial ring $\mathbb{Q}[t]$ onto F sending t to x, and let $f : \mathbb{Q}[t] \to \mathrm{End}^0(A)$ be the morphism composing with the map ι. For any two objects $\underline{A}_i = (F_i, x_i, A_i, \iota_i)$ for $i = 1, 2$ in $\mathcal{B}^{\mathrm{rig}}$, a morphism $\varphi : \underline{A}_1 \to \underline{A}_2$ in $\mathcal{B}^{\mathrm{rig}}$ is an element $\varphi \in \mathrm{Hom}(A_1, A_2) \otimes \mathbb{Q}$ such that $\varphi \circ f_1(a) = f_2(a) \circ \varphi$ for all $a \in \mathbb{Q}[t]$, where $f_i : \mathbb{Q}[t] \to \mathrm{End}^0(A)$ are the maps associated as above. In the case when $F_1 = F_2 = F$, we would have

$$\mathrm{Hom}_F((A_1, \iota_1), (A_2, \iota_2)) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}_{\mathcal{B}^{\mathrm{rig}}}((F, x, A_1, \iota_1), (F, x, A_2, \iota_2))$$

for any element x generates F over $\mathbb{Q}$, which recovers the usual notion of morphisms of abelian F-varieties (though we may not really want the additional structure x).

We shall embed $\mathcal{A}_{\mathbb{F}_q}$ as a full subcategory of $\mathcal{B}^{\mathrm{rig}}$. We first prove the following connection.

Proposition 4.1. *Let A be an abelian variety over $\mathbb{F}_q$ and π_A its relative Frobenius endomorphism. Put $F := \mathbb{Q}(\pi_A)$ and $\iota : F \to \mathrm{End}^0(A)$ is the inclusion. Then the abelian F-variety (A, ι) is basic.*

PROOF. Suppose that the finite field k has $q = p^s$ elements. Let $A \sim \Pi_{i=1}^t A_i^{n_i}$ be the decomposition into components up to isogeny, where each abelian variety A_i is simple and $A_i \not\sim A_j$ for any $i \neq j$. Let π_i be the relative Frobenius endomorphism of A_i and put $F_i := \mathbb{Q}(\pi_i)$. Then we have $F = \prod_i^t F_i$. Let $\Sigma_{p,i}$ be the set of the primes $\mathbf{p}$ of F_i over p. Thus, Σ_p is the disjoint union of its subsets $\Sigma_{p,i}$ for $i = 1, \dots, t$. Let N (resp. N_i) be the isocrystal with additional structures associated to the abelian F-variety $\underline{A} = (A, \iota)$ (resp. $\underline{A}_i = (A_i, \iota_i)$). Clearly if $\mathbf{p} \in \Sigma_{p,i}$ then $N_{\mathbf{p}} = N_{i,\mathbf{p}}^{n_i}$. In particular, $N_{\mathbf{p}}$ is isotypic for all $\mathbf{p} \in \Sigma_p$ if and only if $N_{i,\mathbf{p}}$ is isotypic for all i and all $\mathbf{p} \in \Sigma_{p,i}$. It follows from Lemma 3.1 that $\underline{A}$ is basic if and only if $\underline{A}_i$ is basic for all $i = 1, \dots, t$. Therefore, it suffices to show the statement when A is simple. In this case as $F^s = \pi$ and $\pi \in F_{\mathbf{p}}$, the component $N_{\mathbf{p}}$ has slope $\mathrm{ord}_{\mathbf{p}}(\pi)/s$. ∎

Using Lemma 3.1, if K is any commutative semi-simple $\mathbb{Q}$-subalgebra of the endomorphism algebra $\mathrm{End}^0(A)$ which is stable under a Rosati involution and contains F, then (A,i), where $i : K \subset \mathrm{End}^0(A)$, is also a basic abelian K-variety. The way to make A as a basic abelian variety with endomorphism structures in Proposition 4.1 is, after a base change, the most "economical" one, that is, it uses the least endomorphism structure.

Proposition 4.2. *Let A be an abelian variety over $\mathbb{F}_q$ such that $\mathrm{End}(A) = \mathrm{End}(\overline{A})$, where $\overline{A} = A \otimes \overline{\mathbb{F}}_p$. Suppose that (A,ι) is a basic abelian K-variety for a commutative semi-simple $\mathbb{Q}$-algebra K which admits a positive involution. Then $\iota(K)$ contains the center F of the endomorphism algebra $\mathrm{End}^0(A)$.*

PROOF. Let π be the relative Frobenius endomorphism of A. Then for any positive integer n one has $F = \mathbb{Q}(\pi^n)$ as F is the center of the endomorphism algebra $\mathrm{End}^0(A \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n})$. Now using Lemma 3.4, there is a positive integer n so that π^n is contained in $\iota(K)$. As a result, the center F is contained in $\iota(K)$. ∎

Now we define a functor $\Phi : \mathcal{A}_{\mathbb{F}_q} \to \mathcal{B}^{\mathrm{rig}}$ as follows. To each abelian variety A over $\mathbb{F}_q$ we associates the tuples $(F, \pi_A, \overline{A}, \iota)$, where π_A is the relative Frobenius endomorphism of A, $F := \mathbb{Q}(\pi_A)$, $\overline{A} := A \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_p$ and $\iota : F \to \mathrm{End}^0(\overline{A})$ is the inclusion. Clearly we have the associated map

$$\Phi_* : \mathrm{Hom}(A_1, A_2) \otimes \mathbb{Q} \to \mathrm{Hom}_{\mathcal{B}^{\mathrm{rig}}}(\Phi(A_1), \Phi(A_2)) \tag{4.1}$$

as $\varphi \circ \iota_1(\pi_{A_1}) = \iota_2(\pi_{A_2}) \circ \varphi$ for any map φ in the source.

Theorem 4.3. *The functor $\Phi : \mathcal{A}_{\mathbb{F}_q} \to \mathcal{B}^{\mathrm{rig}}$ is fully faithful.*

PROOF. Let A_1 and A_2 be two abelian varieties over $\mathbb{F}_q$, and let $\underline{A}_i := (F_i, \pi_i, \overline{A}_i, \iota_i)$ be the associated object in $\mathcal{B}^{\mathrm{rig}}$ for $i = 1, 2$. We must show that the associated map Φ_* in (4.1) is bijective. It is clear that Φ_* is injective. Let $\overline{f} : \overline{A}_1 \to \overline{A}_2$ be an element in $\mathrm{Hom}_{\mathcal{B}^{\mathrm{rig}}}(\Phi(A_1), \Phi(A_2))$, particularly $\pi_2 \overline{f} = \overline{f} \pi_1$. As $\sigma_q(\overline{f}) = \pi_2 \overline{f} \pi_1^{-1} = \overline{f}$, where $\sigma_q \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$ is the Frobenius map, the morphism $\overline{f}$ is defined over $\mathbb{F}_q$. ∎

4.2. We restrict the functor Φ to the objects which have a common center of their endomorphism algebras. We choose any abelian variety A_0 over $\mathbb{F}_q$. Let π_0 be the relative Frobenius endomorphism of A_0 over $\mathbb{F}_q$, $p(t) \in \mathbb{Z}[t]$ its minimal polynomial over $\mathbb{Q}$ and $F := \mathbb{Q}[t]/(p(t))$. A commutative semi-simple $\mathbb{Q}$-algebra F arising in this way is called a *q-Weil $\mathbb{Q}$-algebra.*

Let $\mathcal{A}_{\pi_0, \mathbb{F}_q}$ denote the category of abelian varieties A over $\mathbb{F}_q$ up to isogeny such that the minimal polynomial of the relative Frobenius endomorphism of A is equal to $p(t)$. In other words, an abelian variety A over $\mathbb{F}_q$ in $\mathcal{A}_{\pi_0, \mathbb{F}_q}$ shares the common simple components of A_0.

Let $\mathcal{B}_F$ denote the category of basic abelian F-varieties over $\overline{\mathbb{F}}_p$ up to isogeny. Similarly we define a functor

$$\Phi_F : \mathcal{A}_{\pi_0, \mathbb{F}_q} \to \mathcal{B}_F, \quad A \mapsto (\overline{A}, \iota), \tag{4.2}$$

where $\overline{A} := A \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_p$ and $\iota : F \to \mathrm{End}^0(\overline{A})$ is the ring monomorphism sending t to π_A.

By Theorem 4.3, we have the following result.

Proposition 4.4. *For any q-Weil $\mathbb{Q}$-algebra $F=\mathbb{Q}(\pi_0)$, the functor $\Phi_F:\mathcal{A}_{\pi_0,\mathbb{F}_q}\to\mathcal{B}_F$ is fully faithful.*

Remark 4.5. The functor Φ_F is not essentially surjective usually. For example take $q=p^2$ and $\pi_0=p\zeta_6$ and let $p\equiv 1 \pmod 3$. The corresponding abelian variety A_0 is a simple supersingular abelian surface, and any object in $\mathcal{A}_{\pi_0,\mathbb{F}_q}$ is isogenous to a finite product of copies of A_0. However, as $F=\mathbb{Q}(\sqrt{-3})$ and p splits in F, there is an ordinary elliptic curve E over $\overline{\mathbb{F}}_p$ so that there is an isomorphism $i:F\simeq \mathrm{End}^0(E)$. Clearly (E,i) is in $\mathcal{B}_F$ but it does not land in the essential image of the functor Φ_F.

5. A mass formula

5.1. Within a simple isogeny class. Let π be a q-Weil number, $F=\mathbb{Q}(\pi)$ the number field generated by π over $\mathbb{Q}$, and O_F the ring of integers in F. Let $\mathrm{Isog}(\pi)$ denote the simple isogeny class corresponding to π by the Honda-Tate theory [8]. Let A_0 be an abelian variety over $\mathbb{F}_q$ in $\mathrm{Isog}(\pi)$ and put $d:=\dim(A_0)$.

Let $\Lambda(\pi)$ denote the set of isomorphism classes of abelian varieties over $\mathbb{F}_q$ in $\mathrm{Isog}(\pi)$, and let $\Lambda(\pi)^{\max}\subset\Lambda(\pi)$ be the subset consisting of all abelian varieties A so that the ring O_F is contained in $\mathrm{End}(A)$. Let $\mathbf{B}_{d,O_F}$ denote the set of isomorphism classes of d-dimensional basic abelian O_F-varieties over $\overline{\mathbb{F}}_p$.

The following lemma clearly follows from Proposition 4.4.

Lemma 5.1. *The association $A\mapsto(\overline{A},\iota)$ induces an injection $\Phi_\pi:\Lambda(\pi)^{\max}\to\mathbf{B}_{d,O_F}$.*

If $A\in\Lambda(\pi)^{\max}$ is an abelian variety over $\mathbb{F}_q$ and $(\overline{A},\iota)$ the corresponding basic abelian O_F-variety over $\overline{\mathbb{F}}_p$, then clearly any O_F-linear polarization $\overline{\lambda}$ on $(\overline{A},\iota)$ descends uniquely to a polarization λ on A over $\mathbb{F}_q$. Particularly, the map $\lambda\mapsto\overline{\lambda}$ gives rise to a one-to-one correspondence between polarizations on A and O_F-linear polarizations on $(\overline{A},\iota)$ over $\overline{\mathbb{F}}_p$. It follows that A admits a principal polarization if and only if $(\overline{A},\iota)$ admits a principal O_F-linear polarization. Moreover, we also have a natural isomorphism of finite groups

$$\mathrm{Aut}(A,\lambda)\simeq\mathrm{Aut}(\overline{A},\overline{\lambda},\iota). \tag{5.1}$$

Let $\Lambda(\pi)^{\max}_1$ be the set of isomorphism classes of principally polarized abelian varieties (A,λ) over $\mathbb{F}_q$ such that the underlying abelian variety A belongs to $\Lambda(\pi)^{\max}$. The set $\Lambda(\pi)^{\max}_1$ could be empty; nevertheless, it is finite. Indeed $\Lambda(\pi)^{\max}_1$ is contained in the finite set $\mathcal{A}_{d,1}(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of the Siegel modular varieties $\mathcal{A}_{d,1}$,

Let $\mathcal{A}_{d,O_F,1}$ be the moduli space over $\overline{\mathbb{F}}_p$ of d-dimensional principally polarized abelian O_F-varieties, and let $\mathbf{B}_{d,O_F,1}\subset\mathcal{A}_{d,O_F,1}(\overline{\mathbb{F}}_p)$ be its basic locus. Then the map Φ_π induces an injective map

$$\Phi_\pi:\Lambda(\pi)^{\max}_1\to\mathbf{B}_{d,O_F,1}. \tag{5.2}$$

We have the following commutative diagram

$$\begin{array}{ccc}\Lambda(\pi)^{\max}_1 & \xrightarrow{\Phi_\pi} & \mathbf{B}_{d,O_F,1}\\ \downarrow & & \downarrow\\ \Lambda(\pi)^{\max} & \xrightarrow{\Phi_\pi} & \mathbf{B}_{d,O_F},\end{array}$$

where the vertical maps forget the polarization.

The mass of $\Lambda(\pi)_1^{\max}$ is defined by

$$\mathrm{Mass}(\Lambda(\pi)_1^{\max}) := \sum_{(A,\lambda)\in\Lambda(\pi)_1^{\max}} |\operatorname{Aut}(A,\lambda)|^{-1} \tag{5.3}$$

if it is nonempty, and to be zero otherwise. For any finite subset $S \subset \mathcal{A}_{d,O_F,1}(\overline{\mathbb{F}}_p)$, the mass of S is defined by

$$\mathrm{Mass}(S) := \sum_{(\overline{A},\overline{\lambda},\iota)\in S} |\operatorname{Aut}(\overline{A},\overline{\lambda},\iota)|^{-1} \tag{5.4}$$

if S is nonempty and $\mathrm{Mass}(S) = 0$ otherwise. From (5.1) we have the equality

$$\mathrm{Mass}(\Lambda(\pi)_1^{\max}) = \mathrm{Mass}(\mathrm{Im}\Phi_\pi). \tag{5.5}$$

5.2. An example: $\pi = \sqrt{p}$. We consider a example when $\pi = \sqrt{p}$ and prove the following result.

Theorem 5.2. *When $\pi = \sqrt{p}$, the finite set $\Lambda(\pi)_1^{\max}$ is nonempty and*

$$\mathrm{Mass}(\Lambda(\pi)_1^{\max} = \frac{o}{4}\,\zeta_{\mathbb{Q}(\sqrt{p})}(-1), \tag{5.6}$$

where $o = 8$ or 1 according as $p = 2$ or not.

We need a general result. We refer to [10, Section 1] for Dieudonné modules and Dieudonné modules with additional structures.

Proposition 5.3. *Let F be a totally real field, $\mathcal{O} := O_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and k an algebraically closed field of characteristic $p > 0$.*

(1) *Let $\underline{M} = (M, \iota_M, \langle\,,\rangle)$ be a supersingular separably quasi-polarized Dieudonné $\mathcal{O}$-module over k satisfying the following condition*

$$(*) \quad \operatorname{tr}(\iota_M(a)) \cdot [F : \mathbb{Q}] = (\operatorname{rank}_W M) \cdot \operatorname{tr}_{F/\mathbb{Q}}(a), \quad \forall\, a \in O_F.$$

Then there is a supersingular principally polarized abelian O_F-variety $\underline{A} = (A, \lambda, \iota)$ over k such that the associated Dieudonné module $M(\underline{A})$ with additional structures is isomorphic to $\underline{M}$.

(2) *Assume that p is totally ramified in F and that $\dim A = [F : \mathbb{Q}]$. For any supersingular Dieudonné $\mathcal{O}$-module $\underline{M} = (M, \iota_M)$ over k, there is a principally polarized abelian O_F-variety $\underline{A} = (A, \lambda, \iota)$ over k such that the associated Dieudonné $\mathcal{O}$- module $\underline{M}(A, \iota)$ is isomorphic to $\underline{M}$.*

Proof. (1) By [13, Theorem 1.1], there is a polarized abelian O_F-variety $\underline{A} = (A, \lambda, \iota)$ such that $M(\underline{A}) \simeq \underline{M}$. We need an elementary fact that there is a self-dual skew-Hermitian $O_F \otimes \mathbb{Z}_\ell$-lattice (L_ℓ, ψ_ℓ) so that there is an isomorphism $\varphi_\ell : (L_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \psi_\ell) \simeq (V_\ell(A), e_{\lambda,\ell})$ for all primes $\ell \neq p$. Note that any two skew-Hermitian F_ℓ-spaces are isomorphic provided they are free over F_ℓ of the same rank. We leave the existence of an $O_F \otimes \mathbb{Z}_\ell$-free self-dual skew-Hermitian $O_F \otimes \mathbb{Z}_\ell$-module to the reader. Then there is an abelian O_F-variety (A', ι') and a prime-to-p degree O_F-linear quasi-isogeny $\varphi' : (A', \iota') \to (A, \iota)$ such that $\varphi'_*(T_\ell(A')) = \varphi_\ell(L_\ell)$ for all $\ell \neq p$. Take $\lambda' = \varphi^*\lambda$ and then λ' is a principal polarization. The object (A', λ', ι') is a desired one.

(2) Since there is only one prime of O_F over p, the condition $(*)$ is satisfied. By [10, Proposition 2.8] the Dieudonné $\mathcal{O}$-module $\underline{M}$ admits a separable $\mathcal{O}$-linear

quasi-polarization, noting that an equivalent condition (5) of loc. cit. is satisfied when p is totally ramified. Then the statement follows from (1). ∎

Now let $F = \mathbb{Q}(\sqrt{p})$. The prime p is ramified in F with ramification index $e = 2$. Clearly any member A in $\Lambda(\pi)^{\max}$ is a superspecial abelian surface over $\mathbb{F}_p$. The Dieudonné module $M = M(A)$ of A is a rank 4 free $\mathbb{Z}_p$-module together with a $\mathbb{Z}_p$-linear action by O_F. Therefore, $M \simeq \mathcal{O}^2$ with $\mathcal{O} = O_F \otimes \mathbb{Z}_p = \mathbb{Z}_p[\sqrt{p}]$ on which the Frobenius $\mathcal{F}$ and the Verschiebung $\mathcal{V}$ both operate by $\sqrt{p}$. From this the Lie algebra $\mathrm{Lie}(A) = M/\mathcal{V}M$ of A is isomorphic to $\mathbb{F}_p \oplus \mathbb{F}_p$ as an O_F/p-module. In other words, A has Lie type $(1, 1)$ in the terminology of [10, Section 1]. Therefore, the injective map $\Phi_\pi : \Lambda(\pi)^{\max} \to \mathbf{B}_{2,O_F}$ factors through the subset $\mathbf{S} \subset \mathbf{B}_{2,O_F}$ of superspecial abelian O_F-surfaces of Lie type $(1, 1)$.

We first claim that the induced map

$$\Phi_\pi : \Lambda(\pi)^{\max} \to \mathbf{S} \tag{5.7}$$

is bijective. Fix a member $A_0 \in \Lambda(\pi)^{\max}$. By Waterhouse [9, Theorem 6.2], there is a natural bijection between the set $\Lambda(\pi)^{\max}$ and the set $\mathrm{Cl}(\mathrm{End}(A_0))$ of right ideal classes. Since the map Φ_π is injective, it suffices to show that $\mathbf{S}$ has the same cardinality of $\mathrm{Cl}(\mathrm{End}(A_0))$. Note that the isomorphism classes of (unpolarized) *superspecial* Dieudonné $\mathcal{O}$-modules are uniquely determined by their Lie types [12, Lemma 3.2]. It follows that the Dieudonné modules and Tate modules of any two members in $\mathbf{S}$ are mutually isomorphic (compatible with the actions of O_F). By [12, Theorem 2.5] (the unpolarized version), there is a natural bijection $\mathbf{S} \simeq \mathrm{Cl}(\mathrm{End}_{O_F}(\overline{A}_0))$. Since we have $\mathrm{End}(A_0) = \mathrm{End}_{O_F}(\overline{A}_0)$, our claim is proved.

Let $\mathbf{S}_1 \subset \mathbf{B}_{2,O_F,1}$ be the subset consisting of objects (A, λ, ι) so that the underlying abelian O_F-surface (A, ι) belongs to $\mathbf{S}$. Proposition 5.3 implies that $\mathbf{S}_1$ is nonempty. Consider the commutative diagram

$$\begin{array}{ccc} \Lambda(\pi)_1^{\max} & \xrightarrow{\Phi_\pi} & \mathbf{S}_1 \\ {\scriptstyle f_\Lambda}\downarrow & & {\scriptstyle f_{\mathbf{S}}}\downarrow \\ \Lambda(\pi)^{\max} & \xrightarrow[\simeq]{\Phi_\pi} & \mathbf{S} \end{array} \tag{5.8}$$

Note that a member A in $\Lambda(\pi)^{\max}$ admits a principal polarization if and only if $\Phi_\pi(A) = (\overline{A}, \iota)$ admits a principal O_F-linear polarization. Moreover, the equivalence classes of principal polarizations on A are in bijection with the equivalence classes of principal O_F-linear polarizations on $(\overline{A}, \iota)$. It follows that the diagram (5.8) is cartesian, which particularly implies that the map $\Phi_\pi : \Lambda(\pi)_1^{\max} \simeq \mathbf{S}_1$ is an isomorphism. Thus, we have proved $\mathrm{Mass}(\Lambda(\pi)_1^{\max}) = \mathrm{Mass}(\mathbf{S}_1)$.

Now we use the mass formula for $\mathrm{Mass}(\mathbf{S}_1)$ [12, Theorem 3.9] and get

$$\mathrm{Mass}(\Lambda(\pi)_1^{\max}) = \mathrm{Mass}(\mathbf{S}_1) = \frac{o}{4}\zeta_F(-1), \tag{5.9}$$

where $o = 8$ or 1 according as $p = 2$ or not. This proves Theorem 5.2.

5.3. Fiber of the map $f_{\mathbf{S}}$. We describe the fiber of the map $f_{\mathbf{S}}$ in (5.8). Suppose (A, λ_0, ι) is a member in $\mathbf{S}_1$. Put $D := \mathrm{End}^0_{O_F}(A)$ and $O_D := \mathrm{End}_{O_F}(A)$; D is the quaternion F-algebra ramified only at the two real places of F and O_D is a maximal order. Note that the canonical involution $*$ is the unique positive involution on D. Therefore the Rosati involution induced by any O_F-linear polarization must be

$*$. Suppose λ is another O_F-linear principal polarization, then $\lambda = \lambda_0 a$ for some element $a \in O_D^\times$ with $a = a^*$ that is totally positive, or for some $a \in O_{F,+}^\times$, the set of totally positive units in O_F. Suppose $b \in \mathrm{Aut}_{O_F}(A)$ is an O_F-linear automorphism. Then

$$b^*(\lambda_0 a) = b^t \lambda_0 a b = \lambda_0 \lambda_0^{-1} b^t \lambda_0 b a = \lambda_0 (b^* b) a.$$

Therefore, the set of equivalence classes of principal O_F-linear polarizations on (A, ι) is in bijection with the set $O_{F,+}^\times / \mathrm{Nr}(O_D^\times)$, where $\mathrm{Nr} : O_D \to O_F$ is the reduced norm. In other words, we yield an isomorphism

$$f_{\mathbf{S}}^{-1}(A, \iota) \simeq O_{F,+}^\times / \mathrm{Nr}(O_D^\times). \tag{5.10}$$

Now the group $O_{F,+}^\times / \mathrm{Nr}(O_D^\times)$ is a homomorphism image of $O_{F,+}^\times / O_F^{\times,2}$. The latter group has 1 or 2 elements according as the fundamental unit ϵ of F has norm -1 or not. Therefore, if $N(\epsilon) = -1$, then $f_{\mathbf{S}}^{-1}(A, \iota)$ has one element. Otherwise, the fiber $f_{\mathbf{S}}^{-1}(A, \iota)$ has at most two elements.

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