Geometry of differential operators on Weyl manifolds

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We define natural operators of Laplace type for a Weyl manifold which transform conformally. We use the asymptotics of the heat equation for these operators to construct global invariants in Weyl geometry.

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0. Introduction

Relations between conformal and projective structures are of particular interest in both mathematics and in mathematical physics. Weyl (1922) attempted a unification of gravitation and electromagnetism in a model of space-time geometry combining both structures. His particular approach failed for physical reasons but his model is still studied in mathematics (see for example Folland (1970), Higa (1993), Pedersen & Swann (1991)) and in mathematical physics (see for example Hitchin (1982)).

In this paper, we shall investigate the close relations between a Weyl geometry and the so called Codazzi structure which is constructed from a conformal and a projective structure using the Codazzi equations. We shall also study a class of operators of Laplace type on Weyl manifolds. We shall investigate their properties under gauge transformations and the asymptotic expansion of the associated heat equation trace.

Let $\mathcal{C}$ be a conformal class of semi Riemannian metrics on a smooth manifold M of dimension $m \geq 2$. A second order partial differential operator D is said to be of Laplace type if the leading symbol of D is given by $h \in \mathcal{C}$. Let $C_+^\infty(M)$ be the space of smooth positive functions on M; this is a group under pointwise multiplication and will be our gauge group. This group acts on $\mathcal{C}$; if $\beta \in C_+^\infty(M)$ and if $h \in \mathcal{C}$, then $h \mapsto {}_\beta h = h\beta$. We suppose given some auxiliary geometric structure $\mathcal{S}$ on which $C_+^\infty(M)$ also acts. For $h \in \mathcal{C}$ and $s \in \mathcal{S}$, we assume given a natural operator $D = D\{h, s\}$ on M which is of Laplace type. Let $D \mapsto {}_\beta D := D\{{}_\beta h, {}_\beta s\}$ and let $\mathcal{M}(\beta)$ be function multiplication. An operator D is said to transform conformally if

$$ {}_\beta D = \mathcal{M}(\beta^a) \circ D \circ \mathcal{M}(\beta^b) \text{ for } a + b = -1. \qquad (0.1)$$

The conformal Laplacian is an example of such an operator; there is no auxiliary geometric structure which is required. Let d be the exterior derivative and let δ_h be the co-derivative; this is the adjoint of d relative to the metric h. Let ${}^h\Delta := \delta_h d$ on $C^\infty(M)$ be the ordinary Laplacian. Let $\tau(h)$ be the scalar curvature of h and let ${}^h\Box := {}^h\Delta + (m-2)\tau(h)/4(m-1)$ be the conformal Laplacian. Then ${}^h\Box$ transforms conformally, see for example Branson & Orsted (1986):

$$ {}^h_\beta\Box = \mathcal{M}(\beta^{-1-a}) \circ {}^h\Box \circ \mathcal{M}(\beta^a) \text{ for } a = (m-2)/4. \tag{0.2} $$

Let D be an operator of Laplace type. Suppose that h is positive definite. If $t > 0$, the operator e^{-tD} is of trace class in L^2 and as $t \downarrow \infty$, there is an asymptotic expansion of the form

$$ \mathrm{Tr}_{L^2} e^{-tD} \approx \Sigma_n a_n(D) t^{(n-m)/2}, \tag{0.3} $$

see for example Gilkey (1994). The invariants $a_n(D)$ are locally computable. If D transforms conformally, the conformal index theorem of Branson & Orsted (1986) and of Parker & Rosenberg (1987) shows that $a_m(D) = a_m({}_\beta D)$ so this is a gauge invariant of the geometric structures involved if h is positive definite. It is possible to do an analytic continuation and extend these invariants to the semi Riemannian category as well.

This paper deals with Codazzi and Weyl structures; these are defined in §1. In §2, we will define several natural operators of Laplace type for these structures which transform conformally. In §3, we will compute the invariants of the heat equation for $m = 2$ and for $m = 4$ to illustrate the global invariants which arise.

It is a pleasant task to thank M. Kriele and V. Perlik for helpful discussions on Weyl geometries.

1. Weyl and Codazzi Structures

We begin our discussion by introducing some notational conventions.

1.a Weyl manifolds: Fix a torsion free connection ${}^{\mathcal{W}}\nabla$, called the Weyl connection, on the tangent bundle of M. Let h be a semi-Riemannian metric on M. We assume there exists a 1-form $\hat\theta = \hat\theta_h$ so that

$$ {}^{\mathcal{W}}\nabla h = 2\hat\theta_h \otimes h. \tag{1.1} $$

Let $\mathcal{C} = \mathcal{C}\{\mathcal{W}\}$ be the conformal class defined by h and let $\mathcal{T} = \mathcal{T}\{\mathcal{W}\}$ be the corresponding collection of 1-forms $\hat\theta_h$. We shall identify metrics which differ by a constant positive factor so there is a bijective correspondence between elements of $\mathcal{C}$ and of $\mathcal{T}$. We will call the triple $\mathcal{W} := ({}^{\mathcal{W}}\nabla, \mathcal{C}, \mathcal{T})$ a Weyl structure on M and we will call $(M, \mathcal{W})$ a Weyl manifold. Let

$$ h \mapsto {}_\beta h := \beta h \text{ and } \hat\theta \mapsto {}_\beta\hat\theta := \hat\theta + d\ln\beta/2 \text{ for } \beta \in C^\infty_+(M) \tag{1.2} $$

define an action of the gauge group. We note that equation (1.1) is preserved by gauge equivalence and that $C^\infty_+(M)$ acts transitively on $\mathcal{C}$ and on $\mathcal{T}$.

Let u, v, ... be vector fields on M and let ${}^h\nabla$ be the Levi-Civita connection of h. Let θ be the vector field dual to the 1-form $\hat\theta$, i.e. $h(w,\theta) = \hat\theta(w)$. Let

$$ \alpha(u,v,w) := h(({}^{\mathcal{W}}\nabla_u - {}^h\nabla_u)v, w). $$

Since ${}^{\mathcal{W}}\nabla$ and ${}^{h}\nabla$ are torsion free, $\alpha(u,v,w) = \alpha(v,u,w)$. Since ${}^{h}\nabla h = 0$ and since ${}^{\mathcal{W}}\nabla$ satisfies equation (1.1), we have

$$\begin{aligned} &\alpha(u,v,w) + \alpha(u,w,v) + 2\hat{\theta}(u)h(v,w) = 0 \\ &\alpha(u,v,w) = -\hat{\theta}(u)h(v,w) - \hat{\theta}(v)h(u,w) + \hat{\theta}(w)h(u,v) \\ &{}^{\mathcal{W}}\nabla_u v = {}^{h}\nabla_u v - \hat{\theta}(u)v - \hat{\theta}(v)u + h(u,v)\theta. \end{aligned} \tag{1.3}$$

Conversely, if equation (1.3) is satisfied, then ${}^{\mathcal{W}}\nabla = {}^{\mathcal{W}}\nabla\{h,\hat{\theta}\}$ is a torsion free connection and equation (1.1) is satisfied. We generate a Weyl structure from an arbitrary semi-Riemannian metric h and from an arbitrary 1-form $\hat{\theta}$ by using equation (1.3) to define ${}^{\mathcal{W}}\nabla$ and using the action of $C^\infty_+(M)$ defined in equation (1.2) to generate the classes $\mathcal{C}$ and $\mathcal{T}$; see Weyl (1922) or Folland (1970) for further details.

1.b Curvature: We use the sign convention of Kobayashi & Nomizu (1963). Let

$$\begin{aligned} &{}^{\mathcal{W}}F := d\hat{\theta}_h, \\ &{}^{\mathcal{W}}R(u,v) := {}^{\mathcal{W}}\nabla_u {}^{\mathcal{W}}\nabla_v - {}^{\mathcal{W}}\nabla_v {}^{\mathcal{W}}\nabla_u - {}^{\mathcal{W}}\nabla_{[u,v]}, \\ &{}^{\mathcal{W}}K(u,v)w := {}^{\mathcal{W}}R(u,v)w - {}^{\mathcal{W}}F(u,v)w. \end{aligned} \tag{1.4}$$

By equation (1.2), ${}^{\mathcal{W}}F$ is a gauge invariant. The curvature ${}^{\mathcal{W}}R$ of ${}^{\mathcal{W}}\nabla$ and the Weyl directional curvature ${}^{\mathcal{W}}K$ are also gauge invariants. Let $h \in \mathcal{C}(\mathcal{W})$. Then

$$\begin{aligned} &h(z, {}^{\mathcal{W}}R(u,v)z) = {}^{\mathcal{W}}F(u,v)h(z,z), \\ &h({}^{\mathcal{W}}K(u,v)w, w) = 0, \\ &h({}^{\mathcal{W}}F(u,v)w, z) = h({}^{\mathcal{W}}F(u,v)z, w), \\ &h({}^{\mathcal{W}}K(u,v)w, z) = -h({}^{\mathcal{W}}K(u,v)z, w). \end{aligned}$$

Let $H^p(M)$ denote the de Rham cohomology groups of M. If ${}^{\mathcal{W}}F = 0$, then $[\hat{\theta}_h] \in H^1(M)$ is a gauge invariant.

Theorem 1.1. (Higa (1993)). *The following assertions are equivalent:*

(i) We have ${}^{\mathcal{W}}F(\mathcal{W}) = 0$ *and* $[\hat{\theta}_h] = 0$ *in* $H^1(M)$.

(ii) There exists $h \in \mathcal{C}\{\mathcal{W}\}$ *such that* ${}^{\mathcal{W}}\nabla h = 0$; *i.e.* ${}^{\mathcal{W}}\nabla$ *is the Levi-Civita connection of* h.

1.c Projective equivalence: Two torsion free connections ∇ and $\check{\nabla}$ are said to be projectively equivalent if their unparametrized geodesics coincide or equivalently (see Eisenhart (1964)) if there exists a 1-form $\hat{\Theta}$ so that

$$\check{\nabla}_v u - \nabla_v u = \hat{\Theta}(u)v + \hat{\Theta}(v)u. \tag{1.5}$$

Let $Ric(u,v) = \mathrm{Tr}(w \mapsto R(w,u)v)$ be the Ricci curvature of ∇. Let

$$\rho(u,v) := (Ric(u,v) + Ric(v,u))/2$$

be the symmetrized Ricci curvature. A connection ∇ is said to be Ricci symmetric if, and only if, $Ric = \rho$. Note that ∇ is Ricci symmetric if, and only if, ∇ locally admits a parallel volume form; see Pinkall et al (1994).

1.d Codazzi manifolds: A torsion free connection ${}^*\nabla$ and a semi-Riemannian metric h are said to satisfy the Codazzi equation or to be Codazzi compatible if

$$({}^*\nabla_u h)(v,w) = ({}^*\nabla_v h)(u,w). \tag{1.6}$$

A projective class $\mathcal{P}$ of torsion free connections and a conformal class $\mathcal{C}$ of semi-Riemannian metrics are said to be Codazzi compatible if there exists ${}^*\nabla \in \mathcal{P}$ and $h \in \mathcal{C}$ which are Codazzi compatible. We extend the action of the gauge group to define ${}^*\nabla \mapsto {}^*_\beta\nabla$ where ${}^*_\beta\nabla$ is defined by taking $\hat{\Theta} = d\ln\beta$ in equation (1.5);

$${}^*_\beta\nabla_u v = {}^*\nabla_u v + d\ln\beta(u)v + d\ln\beta(v)u. \tag{1.7}$$

The Codazzi equations are preserved by gauge equivalence. A Codazzi structure $\mathcal{K}$ on M is a pair $(\mathcal{C},\mathcal{P})$ where the conformal class of semi-Riemannian metrics $\mathcal{C}$ and the projective class $\mathcal{P}$ are Codazzi compatible; we shall call $(M,\mathcal{K})$ a Codazzi manifold.

Suppose that $(h, {}^*\nabla)$ are Codazzi compatible. Let $C := {}^*\nabla - {}^h\nabla$ be a (1.2) tensor and let $\hat{C}$ be the associated cubic form. Since ${}^*\nabla$ and ${}^h\nabla$ are torsion free, C is a symmetric (1.2) tensor and $\hat{C}(u,v,w) = \hat{C}(v,u,w)$. The Codazzi equation (1.6) and this symmetry then shows $\hat{C}(u,v,w) = \hat{C}(w,v,u)$. Thus $\hat{C}$ is totally symmetric.

Conversely, let h be a semi-Riemannian metric and let $\hat{C}$ be a totally symmetric cubic form. Let ${}^*\nabla := {}^h\nabla + C$ where C is the associated symmetric (1.2) tensor field. Since $\hat{C}$ is symmetric, ${}^*\nabla$ is torsion free and the Codazzi equations (1.6) are satisfied. Let $\nabla := {}^h\nabla - C$. Note that $({}^*\nabla, h, \nabla)$ form a conjugate triple, i.e.

$$uh(v,w) = h(\nabla_u v, w) + h(v, {}^*\nabla_u w).$$

1.e Relating Codazzi and Weyl structures: Let $\mathcal{W}$ be a Weyl structure. We use the gauge group $C^\infty_+(M)$ to generate a Codazzi structure as follows. Let $C = C\{h,\hat{\theta}\}$ and $\hat{C} = \hat{C}\{h,\hat{\theta}\} := h(C(u,v),w)$ be the symmetric (1.2) tensor field and associated totally symmetric cubic form:

$$\begin{aligned} C(u,v) &:= \hat{\theta}(u)v + \hat{\theta}(v)u + h(u,v)\theta, \\ \hat{C}(u,v,w) &:= \hat{\theta}(u)h(v,w) + \hat{\theta}(v)h(w,u) + \hat{\theta}(w)h(u,v). \end{aligned} \tag{1.8}$$

We note that we can recover $\hat{\theta}$ from equation (1.8);

$$\hat{\theta}(u) = (m+2)^{-1}\mathrm{Tr}(v \mapsto C(u,v)). \tag{1.9}$$

If $\beta \in C^\infty_+(M)$, let ${}_\beta C = C\{{}_\beta h, {}_\beta\hat{\theta}\}$ and ${}_\beta\hat{C} = \hat{C}\{{}_\beta h, {}_\beta\hat{\theta}\}$.

Lemma 1.2. *Let h be a semi-Riemannian metric on M and let $\hat{\theta}$ be a 1 form. Let ${}^{\mathcal{W}}\nabla = {}^{\mathcal{W}}\nabla\{h,\hat{\theta}\}$ and $C = C\{h,\hat{\theta}\}$ be the associated Weyl connection and symmetric (1.2) tensor field. We use h and $\hat{\theta}$ to generate connections $\nabla = \nabla\{h,\hat{\theta}\}$ and ${}^*\nabla = {}^*\nabla\{h,\hat{\theta}\}$ with the following properties:*

(i) The connections ${}^*\nabla := {}^h\nabla + C$ and $\nabla := \hat{\nabla} - C$ are torsion free, and $(\nabla, h, {}^*\nabla)$ forms a conjugate triple.

(ii) We have ${}^{\mathcal{W}}\nabla$ and ${}^*\nabla$ are projectively equivalent.

(iii) We have h and ${}^*\nabla$ are Codazzi compatible.

(iv) We have ${}^*\nabla\{{}_\beta h, {}_\beta\hat{\theta}\} = {}_\beta({}^*\nabla\{h,\hat{\theta}\})$.

Proof. The first assertion follows from the total symmetry of the tensor $\hat{C}$. Equations (1.3) and (1.8) imply

$$^{\mathcal{W}}\nabla_u v - {}^*\nabla_u v = -2\hat{\theta}(u)v - 2\hat{\theta}(v)u \tag{1.10}$$

so $^{\mathcal{W}}\nabla_u$ and $^*\nabla$ are projectively equivalent. Since $(^*\nabla_u h)(v,w) = -2\hat{C}(u,v,w)$, the Codazzi equation (1.6) now follows from the total symmetry of C. We use equations (1.2) and (1.10) to complete the proof by checking equation (1.7) holds:

$$\begin{aligned} {}^*\nabla\{_\beta h, {}_\beta\theta\}_u v - {}^*\nabla\{h,\hat{\theta}\}_u v &= 2({}_\beta\hat{\theta} - \hat{\theta})(u)v + 2({}_\beta\hat{\theta} - \hat{\theta})(v)u \\ &= d\ln\beta(u)v + d\ln\beta(v)u. \end{aligned}$$

■

If $\mathcal{W}$ is a Weyl structure, we can use Lemma 1.2 to define an associated Codazzi structure $\mathcal{K}\{\mathcal{W}\}$. Conversely let $\mathcal{K}$ be a Codazzi structure. Let

$$C := {}^*\nabla - {}^h\nabla \text{ and } \hat{\theta}(u) := (m+2)^{-1}\mathrm{Tr}(v \mapsto C(u,v)).$$

Let $\mathcal{W} = \mathcal{W}\{h,\hat{\theta}\}$ be the associated Weyl structure defined by equation (1.3). If $\beta \in C_+^\infty(M)$, let ${}_\beta h = \beta h$ and let ${}_\beta(^*\nabla)$ be defined by equation (1.7). Let v_i be a local orthogonal frame; $h(v_i,v_j) = 0$ for $i \neq j$. Let $\hat{\Theta} = d\ln\beta$. Then

$$\begin{aligned} \hat{\theta}(u) &= \Sigma_i h((^*\nabla_u - {}^h\nabla_u)v_i, v_i)/h(v_i,v_i) \\ &= \Sigma_i h(^*\nabla v_i, v_i)/h(v_i,v_i) - \Sigma_i u(h(v_i,v_i))/2h(v_i,v_i). \\ {}_\beta\hat{\theta}(u) - \theta(u) &= \Sigma_i h(^*_\beta\nabla_u - {}^*\nabla_u)v_i, v_i)/2h(v_i,v_i) \\ &\quad -\Sigma_i(u(\beta h(v_i,v_i))/\beta h(v_i,v_i) - u(h(v_i,v_i))/2h(v_i,v_i)) \\ &= \Sigma_i\hat{\Theta}(u)h(v_i,v_i)/2h(v_i,v_i) \\ &\quad +\hat{\Theta}(v_i)h(u,v_i)/2h(v_i,v_i) - m\hat{\Theta}(u)/2 = \hat{\Theta}(u)/2. \end{aligned}$$

This is the transformation law given in equation (1.2) and thus the Weyl structure is invariantly defined. Since $\hat{\theta}\{h, C\{h,\hat{\theta}\}\} = \hat{\theta}$, $\mathcal{W}\{\mathcal{K}\{\mathcal{W}\}\} = \mathcal{W}$ so we may recover the Weyl structure from the associated Codazzi structure. However, if we start with a Codazzi structure $\mathcal{K}$, then $C\{h,\hat{\theta}\{h,C\}\} \neq C$ in general so $\mathcal{K}\{\mathcal{W}\{\mathcal{K}\}\} \neq \mathcal{K}$. For a given Codazzi structure $\mathcal{K}$, let

$$\hat{\theta}(h)(u) := (m+2)^{-1}\mathrm{Tr}(v \mapsto C(u,v)) \text{ where } C := {}^*\nabla - {}^h\nabla.$$

Use equation (1.3) to generate a Weyl structure $\mathcal{W}$ from $\{h,\hat{\theta}\}$, let $C_1 := C_1(h,\hat{\theta})$ be the symmetric (1.2) tensor defined by equation (1.8), and let $\hat{\gamma} := \hat{C}_1 - \hat{C}$. Then $\mathrm{Tr}(v \mapsto \gamma(u,v)) \equiv 0$ i.e. γ is apolar. The set of all Codazzi structures $(\mathcal{C},\mathcal{P})$ giving rise to a given Weyl structure is parametrized by the apolar (1.2) tensors.

2. Second order differential operators on Weyl manifolds

2.a Relating curvatures: Let ∇ be a torsion free connection and let h be a semi-Riemannian metric on M. Let $\tau(h,\nabla) := \mathrm{Tr}_h{}^\nabla\rho$ be the contraction of the Ricci tensor of ∇. We let $\tau(h) = \tau(h, {}^h\nabla)$. Let δ_h be the co-derivative defined by h; $\delta_h\hat{\theta} = -(^h\nabla_i\hat{\theta})^i$. Let $||\hat{\theta}||_h^2$ be the norm^2 of $\hat{\theta}$. We omit the proof of the following Lemma as it is a straightforward application of formulas from Eisenhart (1964).

Lemma 2.1.

(i) Let h be a semi-Riemannian metric on M and let $\hat\theta$ be a 1 form. Let ${}^{\mathcal{W}}\nabla = {}^{\mathcal{W}}\nabla\{h,\hat\theta\}$ be the associated Weyl connection, let $C = C\{h,\hat\theta\}$ be the associated symmetric (1.2) tensor field, and let ${}^*\nabla = {}^h\nabla + C$ be the associated connection which is projectively equivalent to ${}^{\mathcal{W}}\nabla$. Then

(a) $\tau(h, {}^{\mathcal{W}}\nabla) = \tau(h) - 2(m-1)\delta_h\hat\theta - (m-1)(m-2)||\hat\theta||_h^2$

(b) $\tau(h, {}^*\nabla) = \tau(h) + (m-1)(m+2)||\hat\theta||_h^2$.

(ii) Let $\check\nabla$ and ∇ be projectively equivalent. Let $\hat\Theta_{;vu} = u(\hat\Theta(v)) - \hat\Theta(\nabla_u v)$ denote the components of the covariant derivative $\nabla\hat\Theta$. Then

(a) $\check R(u,v)w = R(u,v)w + \hat\Theta_{;wu}v - \hat\Theta_{;wv}u + d\hat\Theta(u,v)w + \hat\Theta(v)\hat\Theta(w)u$
$-\hat\Theta(u)\hat\Theta(w)v$.

(b) $\check\rho(u,w) = \rho(u,w) - (m-1)(\hat\Theta_{;uw} + \hat\Theta_{;wu})/2 + (m-1)\hat\Theta(u)\hat\Theta(w)$.

2.b The normalized Hessian: Let ∇ be a torsion free connection on the tangent bundle of M. Let $S^2(T^*M)$ denote the space of symmetric (0.2) tensors. We define the Hessian $H = H\{\nabla\}$, the normalized Hessian $\mathcal{H} = \mathcal{H}(\nabla)$, and the trace of the normalized Hessian $\mathcal{D} = \mathcal{D}\{h,\nabla\}$ by

$$\begin{aligned} &H(f)(u,v) := u(v(f)) - df(\nabla_u v) : C^\infty(M) \to S^2(T^*M),\\ &\mathcal{H}(f) := H(f) + (m-1)^{-1}f\rho : C^\infty(M) \to S^2(T^*M),\\ &\mathcal{D}f := -\mathrm{Tr}_h(\mathcal{H}) : C^\infty(M) \to C^\infty(M). \end{aligned} \tag{2.1}$$

There is another way to think of the operator $\mathcal{D}$ which is useful. If ψ is a 1 form, then $\delta_h\psi = -h^{ij\,h}\nabla_i\psi_j = -\mathrm{Tr}_h{}^h\nabla\psi$. We generalize this operator to define $\delta_{h,\nabla} = -\mathrm{Tr}_h{}^h\nabla$ and the associated Laplacian $\Delta^0_{h,\nabla} = \delta_{h,\nabla}d$. Then

$$\begin{aligned} &\nabla df(u,v) = u(df(v)) - df(\nabla_u v) = H(f)(u,v),\\ &\mathcal{D} = \Delta^0_{h,\nabla} - \tau(h,\nabla)/(m-1). \end{aligned}$$

Thus $\mathcal{D}$ generalizes the conformal Laplacian to Weyl and Codazzi geometry.

Lemma 2.2. *Let $(R,\rho,H,\mathcal{H},\mathcal{D})$ and $({}_\beta R, {}_\beta\rho, {}_\beta H, {}_\beta\mathcal{H}, {}_\beta\mathcal{D})$ be defined by (h,∇) and $({}_\beta h, {}_\beta\nabla)$, where ∇ is torsion free.*

(i) ${}_\beta R(u,v)w = R(u,v)w + \beta H(\beta^{-1})(v,w)u - \beta H(\beta^{-1})(u,w)v$.

(ii) ${}_\beta\rho = \rho + (m-1)\beta H(\beta^{-1})$.

(iii) $\beta^{-1}{}_\beta H(\beta f) = H(f) - f\beta H(\beta^{-1})$.

(iv) $\beta^{-1}{}_\beta\mathcal{H}(\beta f) = \mathcal{H}(f)$.

(v) ${}_\beta\mathcal{D}(f) = \mathcal{D}(\beta^{-1}f)$.

Proof. Let $\phi = \ln\beta$, let $\phi_u = u(\phi)$, let $\phi_{uv} = (uv)(\phi)$, and let $\hat\Theta = d\ln\beta$. Then

$$\begin{aligned} (\hat\Theta_{;wu} - \hat\Theta(u)\hat\Theta(w))v &= (\phi_{uw} - \phi_{\nabla_u w} - \hat\Theta(u)\hat\Theta(w))v\\ &= -\beta H(\beta^{-1})(u,w)v \end{aligned}$$

Since $d\hat\Theta = 0$ and $\hat\Theta_{;uw} = \hat\Theta_{;wu}$, (i) and (ii) follow from Lemma 2.1. We compute:

$$\begin{aligned} \beta^{-1}{}_\beta H(\beta f)(u,v) &= \beta^{-1}(\beta f)_{(uv)} - \beta^{-1}({}_\beta\nabla_u v)(\beta f)\\ &= f_{uv} + \phi_u f_v + \phi_v f_u + \phi_{uv}f + \phi_u\phi_v f - \phi_{(\nabla_u v)}f - f_{(\nabla_u v)}\\ &\quad -2\phi_u\phi_v f - \phi_v f_u - \phi_u f_v \end{aligned}$$

$$= H(f)(u,v) - f\beta(\beta^{-1})_{(uv)} + f\beta(\nabla_u v)\beta^{-1}$$
$$= H(f)(u,v) - f\beta H(\beta^{-1})(u,v).$$

This proves (iii); (iv) follows from (iii). We complete the proof by computing ${}_\beta\mathcal{D}(\beta f) = -\mathrm{Tr}_{\beta h}({}_\beta\mathcal{H}(\beta f)) = -\beta^{-1}\mathrm{Tr}_h({}_\beta\mathcal{H}(\beta f)) = -\mathrm{Tr}_h(\mathcal{H}(f)) = \mathcal{D}(f)$. ∎

2.c Natural operators: Let $\mathcal{K}$ be a Codazzi structure, let $(h, {}^*\nabla) \in \mathcal{K}$, and let ${}^\mathcal{W}\nabla$ be the Weyl connection defined by $\mathcal{K}$. We recall the definition of the normalized Hessian from §2.b and define:

(i) Let ${}^*\mathcal{D} := \mathcal{D}\{h, {}^*\nabla\}$ be the trace of the normalized Hessian of ${}^*\nabla$.

(ii) Let ${}^\mathcal{W}\mathcal{D} := \mathcal{D}\{h, {}^\mathcal{W}\nabla\}$ be the trace of the normalized Hessian of ${}^\mathcal{W}\nabla$.

(iii) Let ${}^\mathcal{W}\Delta := -\mathrm{Tr}_h {}^\mathcal{W}\nabla d$ be the scalar Laplacian of ${}^\mathcal{W}\nabla$.

(iv) Let ${}^h\Box := -\mathrm{Tr}_h\delta_h d + (m-2)\tau(h)/4(m-1)$ be the conformal Laplacian.

The following Lemma is now immediate:

Lemma 2.3. *The operators ${}^*\mathcal{D}$, ${}^\mathcal{W}\mathcal{D}$, ${}^\mathcal{W}\Delta$, and ${}^h\Box$ transform conformally:*

(i) ${}^\mathcal{W}_\beta\mathcal{D} = \mathcal{M}(\beta^{-1}){}^\mathcal{W}\mathcal{D}$.

(ii) ${}^*_\beta\mathcal{D} = {}^*\mathcal{D}\mathcal{M}(\beta^{-1})$.

(iii) ${}^\mathcal{W}_\beta\Delta = \mathcal{M}(\beta^{-1}){}^\mathcal{W}\Delta$.

(iv) ${}^h_\beta\Box = \mathcal{M}(\beta^{-1-a}){}^h\Box\mathcal{M}(\beta^a)$ *for* $a = (m-2)/4$.

2.d Heat equation asymptotics: Suppose that h is positive definite. We let $x = (x^1, ..., x^m)$ be local coordinates on a closed manifold M of dimension m. Let $\partial_i = \partial/\partial x^i$. Let

$$D := -(h^{ij}\partial_i\partial_j + A^k\partial_k + B) \tag{2.2}$$

be an operator of Laplace type on $C^\infty(M)$ where A^k and B are smooth functions on M. The invariants $a_n(D)$ defined in equation (0.3) are locally computable; they vanish for n odd, see Gilkey (1994) for details. Let $d\nu_h$ be the measure determined by h. For n even, there exist smooth local invariants $a_n(x, D)$ so that

$$a_n(D) = \textstyle\int_M a_n(x,D)d\nu_h(x). \tag{2.3}$$

We refer to Gilkey (1975, 1994) for the proof of the following assertion giving combinatorial formulas for these invariants. Let $\tau(h)$, $||{}^h\rho||_h^2$, and $||{}^hR||_h^2$ be the scalar curvature, the norm^2 of the Ricci curvature, and norm^2 of the full curvature tensor for the Levi-Civita connection defined by h.

Lemma 2.4. *Let h be a Riemannian metric. Let D be an operator of Laplace type as given in equation (2.2). Let ${}^h\Gamma_{ij}{}^k$ be the Christofel symbols of the Levi-Civita connection ${}^h\nabla$. There exists a unique connection $\nabla = \nabla\{D\}$ on $C^\infty(M)$ and a unique function $E = E\{D\} \in C^\infty(M)$ so that $D = -(\mathrm{Tr}(\nabla^2) + E)$. Let $\omega = \omega\{D\}$ and $\Omega = \Omega\{D\}$ be the connection 1-form and curvature of ∇. Then*

(i) $\omega_i = h_{ij}(A^j + {}^h\Gamma_{kl}{}^j h^{kl})/2$.

(ii) $E = B - h^{ij}(\partial_i\omega_j + \omega_i\omega_j + {}^h\Gamma_{ij}{}^k\omega_k)$.

(iii) $a_0(x,D) = (4\pi)^{-m/2}$.

(iv) $a_2(x,D) = 6^{-1}(4\pi)^{-m/2}(\tau(h) + 6E)$.

(v) $a_4(x,D) = 360^{-1}(4\pi)^{-m/2}\{60\tau(h)_{;kk} + 60\tau(h)E + 180E^2$
$+30g\Omega_{ij}\Omega_{ij} + 12\tau(h)_{;kk} + 5\tau(h)^2 - 2||{}^h\rho||_h^2 + 2||{}^hR||_h^2\}$.

We compute the endomorphism E and the curvature Ω for the four natural operators discussed in Lemma 2.3.

Lemma 2.5. *We have*
(i) $E\{{}^*\mathcal{D}\} = \{(m+2)\tau(h, {}^{\mathcal{W}}\nabla) - (m-2)\tau(h)\}/4(m-1)$.
(ii) $\Omega\{{}^*\mathcal{D}\} = -(m+2){}^{\mathcal{W}}F/2$.
(iii) $E\{{}^{\mathcal{W}}\mathcal{D}\} = -(m-2)\delta_h\hat{\theta}/2 - (m-2)^2||\hat{\theta}||_h^2/4 + (m-1)^{-1}\tau(h, {}^{\mathcal{W}}\nabla)$.
(iv) $\Omega\{{}^{\mathcal{W}}\mathcal{D}\} = -(m-2){}^{\mathcal{W}}F/2$.
(v) $E\{{}^{\mathcal{W}}\Delta\} = -(m-2)\delta_h\hat{\theta}/2 - (m-2)^2||\hat{\theta}||_h^2/4$, *and* $\Omega\{{}^{\mathcal{W}}\Delta\} = -(m-2){}^{\mathcal{W}}F/2$.
(vi) $E\{{}^h\square\} = -(m-2)\tau(h)/4(m-1)$, *and* $\Omega\{{}^h\square\} = 0$.

Proof. Let ∂_i be a local coordinate frame for the tangent bundle. We compute

$$ {}^*H(f)(\partial_i, \partial_j) = \partial_i\partial_j f - {}^*\Gamma_{ij}{}^k\partial_k f $$
$$ \mathrm{Tr}_h({}^*H + (m-1)^{-1}{}^*\rho) = -h^{ij}(\partial_i\partial_j - {}^*\Gamma_{ij}{}^k\partial_k + (m-1)^{-1}{}^*\rho_{ij}). $$

Consequently $A^k\{{}^*\mathcal{D}\} = -h^{ij}{}^*\Gamma_{ij}{}^k$ and $B\{{}^*\mathcal{D}\} = (m-1)^{-1}\tau(h, {}^*\nabla)$. Therefore

$$ \omega\{{}^*\mathcal{D}\}_i = h_{ij}h^{kl}({}^h\Gamma_{kl}{}^j - {}^*\Gamma_{kl}{}^j)/2 = -h_{ij}h^{kl}C_{kl}{}^j/2 = -(m+2)\hat{\theta}_i/2. $$

We compute $\Omega\{{}^*\mathcal{D}\}$ using equation (1.4). We compute $E\{{}^*\mathcal{D}\}$:

$$ \begin{aligned} E\{{}^*\mathcal{D}\} &= B - h^{ij}(\omega_{j;i} + \omega_i\omega_j) \\ &= \tau(h, {}^*\nabla)/(m-1) - (m+2)\delta_h\hat{\theta}/2 - (m+2)^2||\hat{\theta}||_h^2/4 \\ &= \tau(h)/(m-1) + (m+2)||\hat{\theta}||_h^2 - (m+2)\delta_h\hat{\theta}/2 - (m+2)^2||\hat{\theta}||_h^2/4 \\ &= \{4\tau(h) + (m+2)(m-1)(-2\delta_h\hat{\theta} - (m-2)||\hat{\theta}||_h^2)\}/4(m-1) \\ &= \{4\tau(h) + (m+2)(\tau(h, {}^{\mathcal{W}}\nabla) - \tau(h))\}/4(m-1). \end{aligned} $$

We use equation (1.3) and argue as above to see

$$ \begin{aligned} \omega\{{}^{\mathcal{W}}\mathcal{D}\}_i = \omega\{{}^{\mathcal{W}}\Delta\}_i &= h_{ij}h^{kl}({}^h\Gamma_{kl}{}^j - {}^{\mathcal{W}}\Gamma_{kl}{}^j)/2 \\ &= -h_{ij}h^{kl}\alpha_{kl}{}^j/2 = -(m-2)\hat{\theta}_i/2. \end{aligned} $$

The computation of E and Ω for ${}^{\mathcal{W}}\mathcal{D}$ and ${}^{\mathcal{W}}\Delta$ is now immediate. The connection defined by $\delta_h d$ is flat and the endomorphism defined by $\delta_h d$ is zero. Thus we have that $\Omega\{{}^h\square\} = 0$ and $E\{{}^h\square\} = -(m-2)\tau(h)/4(m-1)$. ■

3. Invariants of Codazzi and Weyl structures

3.a The conformal index theorem: We refer to Branson & Orsted (1986) and to Parker & Rosenberg (1987) for the proof of the first assertion in the following Lemma. The second assertion follows from the first assertion and from Lemma 2.3.

Lemma 3.1.
(i) Let D be an operator of Laplace type. Let ${}_\beta D = \mathcal{M}(\beta^a) \circ D \circ \mathcal{M}(\beta^{-1-a})$. Then $a_m({}_\beta D) = a_m(D)$ and $a_{m-2}(x, {}_\beta D) = \beta^{(2-m)/2}a_{m-2}(x, D)$.

(ii) We have that $a_m({}^*\mathcal{D})$, $a_m({}^{\mathcal{W}}\mathcal{D})$, $a_m({}^{\mathcal{W}}\Delta)$, and $a_m({}^h\square)$ are gauge invariants of a Codazzi structure $\mathcal{K}$.

3.b Heat invariants: We apply Lemma 2.4 in dimensions $m = 2$ and $m = 4$; we refer the reader to Gilkey (1975) for the formulas which would permit a similar calculation in dimension $m = 6$. Let $\chi(M)$ be the Euler-Poincare characteristic of M. The Chern Gauss Bonnet theorem yields

$$\chi(M^2) = (4\pi)^{-1} \int_M \tau(h)(x) d\nu_h(x)$$
$$\chi(M^4) = (32\pi^2)^{-1} \int_M \{||{}^hR||_h^2 - 4||{}^h\rho||_h^2 + \tau(h)^2)\}(x) d\nu_h(x)$$

Theorem 3.2. *Let* $\dim(M) = 2$. *Then*
(i) $a_2({}^*\mathcal{D}) = \chi(M)/6 + (4\pi)^{-m/2} \int_M \tau(h, {}^{\mathcal{W}}\nabla)(x) d\nu_h(x)$.
(ii) $a_2({}^{\mathcal{W}}\mathcal{D}) = \chi(M)/6 + (4\pi)^{-m/2} \int_M \tau(h, {}^{\mathcal{W}}\nabla)(x) d\nu_h(x)$.
(iii) $a_2({}^{\mathcal{W}}\Delta) = \chi(M)/6$.
(iv) $a_2({}^h\Box) = \chi(M)/6$.

Proof. If $m = 2$, then $E\{{}^*\mathcal{D}\} = \tau(h, {}^{\mathcal{W}}\nabla)$. Thus

$$a_2({}^*\mathcal{D}) = (4\pi)^{-1}/6 \int_M \{\tau(h) + 6\tau(h, {}^{\mathcal{W}}\nabla)\}(x) d\nu(x).$$

This proves the first assertion, the others follow similarly. ■

Theorem 3.3. *Let* $\dim(M) = 4$. *Let* ${}^hW$ *be the Weyl conformal curvature. Then*
(i) $a_4({}^*\mathcal{D}) = -\chi(M)/180 + (4\pi)^{-2}(360)^{-1} \int_M \{3||{}^hW||_h^2 + 270||{}^{\mathcal{W}}F||_h^2 + 45\tau(h, {}^{\mathcal{W}}\nabla)^2\} d\nu_h(x)$.
(ii) $a_4({}^{\mathcal{W}}\mathcal{D}) = -\chi(M)/180 + (4\pi)^{-2}(360)^{-1} \int_M \{3||{}^hW||_h^2 + 30||{}^{\mathcal{W}}F||_h^2 + 45\tau(h, {}^{\mathcal{W}}\nabla)^2\} d\nu_h(x)$.
(iii) $a_4({}^{\mathcal{W}}\Delta) = -\chi(M)/180 + (4\pi)^{-2}(360)^{-1} \int_M \{3||{}^hW||_h^2 + 30||{}^{\mathcal{W}}F||_h^2 + 5\tau(h, {}^{\mathcal{W}}\nabla)^2\} d\nu_h(x)$.
(iv) $a_4({}^h\Box) = -\chi(M)/180 + (4\pi)^{-2}(360)^{-1} \int_M \{3||{}^hW||_h^2\} d\nu_h(x)$.

Proof. We use Lemmas 2.4 and 2.5. Let $\mathcal{E} := ||{}^hR||_h^2 - 4||{}^h\rho||_h^2 + \tau(h)^2$ be the normalized integrand of the Chern Gauss Bonnet theorem in dimension 4. We complete the proof by computing:

$$||{}^hW||_h^2 := ||{}^hR||_h^2 - 2||{}^h\rho||_h^2 + \tau(h)^2/3,\ 2||{}^hR||_h^2 - 2||{}^h\rho||_h^2 = 3{}^hW - \mathcal{E},$$
$$E\{{}^*\mathcal{D}\} = (3\tau(h, {}^{\mathcal{W}}\nabla) - \tau(h))/6,\ E\{{}^{\mathcal{W}}\mathcal{D}\} = (3\tau(h, {}^{\mathcal{W}}\nabla) - \tau(h))/6,$$
$$E\{{}^{\mathcal{W}}\Delta\} = (\tau(h, {}^{\mathcal{W}}\nabla) - \tau(h))/6,\ E\{{}^h\Box\} = -\tau(h)/6,$$
$$5\tau(h)^2 + 60\tau(h)E\{{}^*\mathcal{D}\} + 180E\{{}^*\mathcal{D}\}^2 = 45\tau(h, {}^{\mathcal{W}}\nabla)^2,$$
$$5\tau(h)^2 + 60\tau(h)E\{{}^{\mathcal{W}}\mathcal{D}\} + 180E\{{}^{\mathcal{W}}\mathcal{D}\}^2 = 45\tau(h, {}^{\mathcal{W}}\nabla)^2,$$
$$5\tau(h)^2 + 60\tau(h)E\{{}^{\mathcal{W}}\Delta\} + 180E\{{}^{\mathcal{W}}\Delta\}^2 = 5\tau(h, {}^{\mathcal{W}}\nabla)^2,$$
$$5\tau(h)^2 + 60\tau(h)E\{{}^h\Box\} + 180E\{{}^h\Box\}^2 = 0\tau(h, {}^{\mathcal{W}}\nabla)^2,$$
$$30||\Omega\{{}^*\mathcal{D}\}||_h^2 = 270||{}^{\mathcal{W}}F||_h^2,\ 30||\Omega\{{}^{\mathcal{W}}\mathcal{D}\}||_h^2 = 30||{}^{\mathcal{W}}F||_h^2,$$
$$30||\Omega\{{}^{\mathcal{W}}\Delta\}||_h^2 = 30||{}^{\mathcal{W}}F||_h^2,\ 30||\Omega\{{}^h\Box\}||_h^2 = 0.$$
■

Remark 3.4. *We can illustrate Lemma 3.1 by computing*

$$a_2(x, {}^*\mathcal{D}\{_\beta h, _\beta\theta\}) = \beta^{-1} a_2(x, {}^*\mathcal{D}),\ a_2(x, {}^{\mathcal{W}}\mathcal{D}\{_\beta h, _\beta\theta\}) = \beta^{-1} a_2(x, {}^{\mathcal{W}}\mathcal{D}),$$
$a_2(x, {}^{\mathcal{W}}\Delta) = 0$, *and* $a_2(x, {}^h\Box) = 0$.

3.c Global invariants: If f is a scalar invariant, let $f[M] := \int_M f(x)d\nu_h(x)$. The Euler characteristic is a topological invariant of M which does not depend on the Codazzi structure. The following Corollary is now immediate:

Corollary 3.5.
(i) The invariants $\tau(h, {}^{\mathcal{W}}\nabla)^2[M]$, $||^{\mathcal{W}}F||_h^2[M]$, *and* $||^hW||_h^2[M]$ *of a Weyl structure on* M *are determined by* $\chi(M)$ *and by the spectrum of the operators* ${}^*\mathcal{D}$, ${}^{\mathcal{W}}\mathcal{D}$, *and* ${}^{\mathcal{W}}\Delta$.
(ii) We have $32\pi^2\chi(M^4) \geq 45\tau(h, {}^{\mathcal{W}}\nabla)^2[M] + 270||^{\mathcal{W}}F||_h^2[M] - (4\pi)^2 360 a_4({}^*\mathcal{D})$ *with equality if, and only if, the class* $\mathcal{C}$ *is conformally flat.*
(iii) We have $32\pi^2\chi(M^4) \geq 45\tau(h, {}^{\mathcal{W}}\nabla)^2[M] + 3||^hW||_h^2[M] - (4\pi)^2 360 a_4({}^*\mathcal{D})$ *with equality if, and only if, the length curvature* ${}^{\mathcal{W}}F \equiv 0$.

Remark 3.6. *We can use the formulas of Theorems 3.2 and 3.3 to extend the invariants* a_m *from the Riemannian to the semi-Riemannian category; this can be done for any* m.

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