Calculating p-adic orbital integrals on groups of symplectic similitudes in four variables

Michael Schröder

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-53225 Bonn

Germany

MPI / 94-120

Calculating p–adic orbital integrals on groups of symplectic similitudes in four variables

Michael Schröder

In this paper we address the problem of calculating orbital integrals on groups of symplectic similitudes over a local field. Orbital integrals usually appear when one wants to count the number of elements of arithmetic objects by analytic means. For instance, as a consequence of recent work of Kottwitz [**KoS**], the number of points mod p of certain moduli spaces of Abelian varieties over the field of rational numbers can be counted using adelic orbital integrals on the groups of symplectic similitudes. This fits into a bigger picture. To certain linear groups over number fields, in particular to the groups of symplectic similitudes, there are associated Shimura varieties which are algebraic varieties of both analytical and arithmetic nature. Since these varieties are analytical objects, there is an automorphic L–function in the sense of Langlands associated to them. On the other hand, being arithmetic objects they have a Hasse–Weil zeta function. Langlands conjectured essentially that these two functions should be equal, and he initiated the approach to prove this conjecture via the trace formula. Suitably modified, the conjecture was recently verified by this method for the groups of unitary similitudes in three variables [**M**]. The most intriguing difficulties encountered in the approach via the trace formula are caused by the local orbital integrals. It is in fact the orbital integrals over semisimple conjugacy classes which appear here and are to be dealt with.

A typical local orbital integral is defined as the integral of a Hecke operator over a conjugacy class in a reductive group. Recall that a Hecke operator is a smooth, compactly supported function on the group, bi–invariant under a particular maximal compact subgroup. In Langlands theory one furthermore seeks to replace systematically plain orbital integrals by certain stable orbital integrals. While orbital integrals are usually very hard to handle, stable orbital integrals are determined by invariant data and are easier to deal with.

In the first part of the paper we classify the stable conjugacy classes of maximal tori in the groups of symplectic similitudes $GSp(2n)$ and describe a set of representatives for them. We have to embed these tori into $GSp(2n)$. This we approach from a more general perspective. Taken up to conjugacy these embeddings are parametrized by the conjugacy classes within the corresponding stable conjugacy class. A problem is that the stable conjugacy class C of such a torus consists in general of several conjugacy classes. We construct for C a group which contains any of the conjugacy classes in C. On these groups we base a two–step method to calculate explicitly local orbital integrals in sections three and two. In section four we calculate an orbital integral using this method for one class of anisotropic tori in $GSp(4)$. We also find evidence for the qualitative description of the properties of orbital versus stable orbital integrals given above.

0. Notation: With the exeption of the first section, F will denote a non–archimedean local field with uniformizing element π, ring of integers $\mathcal{O}_F$ and residue field $k(F)$. We write $U(F)$ for the set of units in $\mathcal{O}_F$. The order on F is normalized such that $\operatorname{ord}\pi = 1$, and $|\pi| = 1/\#k(F) = q^{-1}$.

Let I be the involution on $M(2n, F)$, the $2n$ by $2n$ matrices with coefficients in F, defined by

$$I(g) = J^{-1} \cdot {}^t g \cdot J \qquad \text{with} \qquad J = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}.$$

The group $GSp(2n, F)$ of symplectic similitudes is the set of all g in $M(2n, F)$ such that ${}^t g \cdot J \cdot g = \mu(g) \cdot J$ or equivalently such that $I(g) \cdot g = \mu(g) \cdot E_{2n}$.

1. The tori τ_C and the groups H_C: In this section we show how to classify stable conjugacy classes C of maximal F–tori in $GSp(2n)$. We further construct a special representative τ_C of the tori in C. Then we identify a group H_C into which τ_C embeds naturally. The group H_C is unique up to conjugacy. One of the key properties of H_C is that any conjugacy class in C contains a conjugate in H_C of τ_C.

For this section only, we let F be any perfect field. We fix furthermore a stable conjugacy class C of maximal F–tori in $GSp(2n)$.

(1.1) We classify now the stable conjugacy classes C of maximal F–tori in $GSp(2n)$ in terms of algebras with involution. For this, let T be a torus in C and let s in $T(F)$ be a regular element in $GSp(2n)$. Note that s is regular in $GL(2n)$, too. The centralizer $C(s)$ of s in $M(2n, F)$ is isomorphic to the algebra $F[s]$, since $F[s]$ is a semisimple, commutative subalgebra of maximal dimension in $M(2n, F)$. Now $F[s]$ is isomorphic to the ring of polynomials $F[T]$ modulo the ideal generated by the minimal polynomial of s. Let $\mathcal{E} = \mathcal{E}_C$ denote the set consisting of the extension fields E of F defined by and indexed by all irreducible factors (taken with their multiplicities) of the minimal (=characteristic) polynomial of s. Then $F[s]$ is isomorphic to the direct product A_C of $\mathcal{E}$, so that we get

$$(1) \qquad A_C = \prod \Big\{ E : E \in \mathcal{E} \Big\} \cong F[T] \Big/ \begin{pmatrix} \text{minimal polynomial} \\ \text{of } s \text{ in } F[T] \end{pmatrix}.$$

We define the F–torus τ_C by taking the image of $T(F)$ in A_C as its F–rational points. So $\tau_C(F)$ consists of all elements in A_C which satisfy the symplecticity condition imposed by I:

$$(2) \qquad \tau_C(F) = \Big\{ x \in A_C : I(x) \cdot x = \mu \in F^* \Big\}.$$

One is thus led to study the action of I on the elements of $\mathcal{E}$ separately. We remark that this is done in [**S**, Kapitel 5] for a general semisimple element in $GSp(2n, F)$.

In the present situation, with s regular, there are clearly two possible cases for each E in $\mathcal{E}$

(GL) E belongs to a pair of fields (E, E') with E, E' in $\mathcal{E}$ which I interchanges. Note that $I(E) = E'$. Necessarily then, $E = E'$, and I restricts to an involution on $E \times E$. One checks that there is an automorphism σ_E of E such that $I(x, y) = (\sigma_E^{-1}(y), \sigma_E(x))$ on $E \times E$.

(U) I restricts to a non–trivial involution σ_E on E.

Therefore, let $\mathcal{G}$ denote the set consisting of all pairs (E, E') as in (GL) and let $\mathcal{U}$ denote the set consisting of all E as in (U). Then

$$\mathcal{E} = \mathcal{U} \bigcup \mathcal{G}$$

as disjoint union. We will identify below τ_C as a subtorus of the product of tori associated to the elements in $\mathcal{G}$ and $\mathcal{U}$. These are constructed by solving $I(x)x = \mu$ separately on each of the corresponding factors of A_C. We describe them now in a more precise way.

(1.2) The tori $\tau_{E\times E}$ associated to data in $\mathcal{G}$: We will show that the tori in each factor of A_C indexed by elements of $\mathcal{G}$ are in a natural way tori of general linear groups. So fix (E, E) in $\mathcal{G}$. Solving $(\mu, \mu) = I(x, y)\cdot(x, y) = (\sigma_E^{-1}(y)x, \sigma_E(x)y)$ on $E \times E$ shows that the F–torus $\tau_{E\times E}$ associated to this pair is isomorphic to $\mathrm{R}_{E/F}(\mathbf{G}_m)_E \times (\mathbf{G}_m)_F$, a typical anisotropic modulo center torus of $GL_F(E) \times F^*$. We will show that in fact $GL_F(E) \times F^*$ itself embeds naturally into $GSp(2n)$. More precisely, we claim that after fixing a suitable basis on $E \times E$ the following diagram

$$
(3)\qquad
\begin{array}{ccccc}
(x,\mu) & \longmapsto & (x, \mu\cdot\sigma_E(x^{-1})) & & \\
E^*\times F^* & \xrightarrow{\sim} & \tau_{E\times E}(F) & & \\
\downarrow & & \downarrow & & \\
GL_F(E)\times F^* & \xrightarrow{\sim} & L_{E\times E}(F) & \subseteq & GL_F(E)\times GL_F(E) \\
(x,\mu) & \longmapsto & (x, \mu\cdot {}^t x^{-1}) & &
\end{array}
$$

commutes, where the vertical maps are the canonical inclusions. This means in particular that σ_E has order two if $[E : F] \geq 2$. Note further that there is up to conjugacy at most one embedding of $\tau_{E\times E}$ into both $L_{E\times E}$ and $GSp(2n)$, since the first Galois cohomology of $\tau_{E\times E}$ vanishes by Hilbert 90. To construct such a map, and thereby prove the claim, recall that there is a standard way of embedding $GL_F(E)$ into $GSp(2n)$. We fix a symplectic form $\beta_{E\times E}$ on $E\times E$ such that there is a basis $(e_1, 0), \ldots, (e_\ell, 0), (0, f_1), \ldots, (0, f_\ell)$ of $E\times E$ symplectic with respect to this form. On the factor B of A_C complementary to $E \times E$ fix the symplectic form β_B induced by the symplectic involution I. Extend the symplectic basis on $E \times E$ to a basis of A_C symplectic with respect to $\beta_B \times \beta_{E\times E}$. Then $GL_F(E)$ embeds into a Levi factor of $GSp(2n)$ as described in (3). This completes the proof.

(1.3) The basic tori $\tau_b(E, \sigma_E)$ associated to data in $\mathcal{U}$: We will show that the tori in each factor of A_C indexed by elements of $\mathcal{U}$ come from tori in unitary groups. So let E in $\mathcal{U}$ and let σ_E be the restriction of I to E. Solving $\mu = I(x)\cdot x = \sigma_E(x)\cdot x$ on E we get the **basic torus** $\tau_b(E, \sigma_E)$ associated to (E, σ_E) which is defined by

$$
(4)\qquad \tau_b(E,\sigma_E)(F) = N_{E/E^+}^{-1}(F^*) = \left\{x \in E^* : N_{E/E^+}(x) \in F^*\right\}
$$

where E^+ is the fixed field of σ_E in E and $N_{E/E^+}(x) = \sigma_E(x)\cdot x$. In particular, each basic torus is contained in a typical torus of a group of unitary similitudes over E^+. This inclusion will be strict in general due to the conditions on the similarity character of $\tau_b(E, \sigma_E)$.

From a more functorial point of view, $\tau_b(E, \sigma_E)$ is defined by making the following pullback diagram

$$
(5)\qquad
\begin{array}{ccc}
\tau_b(E,\sigma_E) & \xrightarrow{incl} & \mathrm{R}_{E/F}(\mathbf{G}_m)_E \\
\downarrow{\scriptstyle \mu_E} & & \downarrow{\scriptstyle R_{E^+/F}(N_{E/E^+})} \\
(\mathbf{G}_m)_F & \xrightarrow{\Delta} & \mathrm{R}_{E^+/F}(\mathbf{G}_m)_{E^+}
\end{array}
$$

commute in which Δ is the diagonal embedding of $(\mathbf{G}_m)_F$ and μ_E denotes the similarity character.

In contrast to the general linear case considered in (1.2), it is a problem to construct embeddings of a basic torus into $GSp(2n)$, and a further problem to get all embeddings up to conjugacy. The second of these problems is is due to the easy fact that the first Galois cohomology group of such a torus is $(E^+)^*/F^* \cdot N_{E/E^+}(E^*)$, and so is not trivial in general. To address the first problem, we will describe a systematic way to construct embeddings via well–known modified trace forms.

For any non–zero element a in the (-1)–eigenspace E^- of σ_E on E, define the bilinear form $B_E(a)$ on E by

$$B_E(a)(x,y) = \mathrm{tr}_{E/F}\Big(x \cdot a \cdot \sigma_E(y)\Big). \tag{6}$$

This form is in fact symplectic: since a is non–zero it is non–degenerate, since a is in E^- it is alternating. It has the crucial property that $\tau_b(E,\sigma_E)(F) = N^{-1}_{E/E^+}(F^*)$ is the set of elements in E symplectic with respect to $B_E(a)$: for t in E^* and μ in F^* we have $B_E(a)(tx,ty) = \mu \cdot B_E(a)(x,y)$ on $E \times E$ if and only if $N_{E/E^+}(t)$ is in F^*. In this case $\mu = N_{E/E^+}(t)$.

The argument of (1.2) now shows mutatis mutandis that fixing a basis on E symplectic with respect to $B_E(a)$ defines a map to $GSp(2n)$, which embeds specifically the basic torus $\tau_b(E,\sigma_E)$ as a torus in $GSp(2n)$. In (1.7) we will address this embedding problem in full generality.

(1.4) The torus τ_C associated to C: We put together the tori associated to the single factors of A_C to obtain a better description of τ_C in terms of data from C only. From the description in (2) it is clear that τ_C is a subtorus of the product of all the tori $\tau_{E\times E}$ indexed by $\mathcal{G}$ and all the basic tori indexed by $\mathcal{U}$. More precisely, τ_C consists of all elements in this product having the same similarity factors. From a functorial point of view, τ_C is therefore defined by making the following pullback diagram

$$\begin{array}{ccc} \tau_C = \tau_C^U \times \tau_C^{GL} & \xrightarrow{incl} & \prod_{\mathcal{U}} \tau_b(E,\sigma_E) \times \prod_{\mathcal{G}} \tau_{E\times E} \\ \Big\downarrow \mu_C & & \Big\downarrow \Pi\mu_E \times \Pi\mu_{E\times E} \\ (\mathbf{G}_m)_F & \xrightarrow{\Delta} & {(\mathbf{G}_m)_F}^{\mathcal{U}} \times {(\mathbf{G}_m)_F}^{\mathcal{G}} \end{array} \tag{7}$$

commute, where Δ is the diagonal embedding of $(\mathbf{G}_m)_F$ and μ_* denote the various similarity characters. Thereby we get a natural decomposition of τ_C into a **unitary part** τ_C^U and a part τ_C^{GL} of **general linear type**. We would like to think that the similarity factor of an element in τ_C is determined by the similarity factors of its unitary part. If $\mathcal{U}$ is empty we set formally $\tau_C^U = (\mathbf{G}_m)_F$ to exclude pathologies.

(1.5) General aspects of the group H_C associated to C: Our motivation for defining the groups H_C comes from the unitary part of τ_C. This part of τ_C has a very special structure in that it is determined by tori of $GL(2)$. Namely, a basic torus is given by the subgroup $N^{-1}_{E/E^+}(F^*)$ of a typical torus E^* of $GL_2(E^+)$. But whereas E^* has up to conjugacy a unique embedding into $GL_2(E^+)$, an embedding of a basic torus into $GL_2(E^+)$ will not be unique up to conjugacy in general. So it seems plausible that the conjugacy classes within the stable conjugacy class of embeddings of τ_C^U into $GSp(2n)$ already can be seen in the product of the respective $GL_2(E^+)$. The group H_C is our attempt to make this notion rigorous. Again it naturally decomposes into a unitary part and a part of general linear type

$$H_C = H_C^U \times H_C^{GL}. \tag{8}$$

In analogy to the construction of τ_C^{GL}, there is a natural candidate for H_C^{GL}. Essentially this is the Levi factor of the symplectic group which contains τ_C^{GL}. More formally, H_C^{GL} is the group of all elements in the product $\prod_{\mathcal{G}} L_{E\times E}$ having the same similarity factors. The crucial step is now the construction of the unitary part, which will be given in the next section.

(1.6) Construction of the unitary part of H_C: Assume for simplicity $\mathcal{E} = \mathcal{U}$. First we have to put together the data used to define basic tori in (1.3). For any E in $\mathcal{E}$, let E^+ and E^- be the $(+1)$– and (-1)–eigenspaces of σ_E, respectively. This gives a decomposition $A_C = A_C^+ \oplus A_C^-$ in the $(+1)$– and (-1)–eigenspace of the symplectic involution $I \cong \prod \sigma_E$. For any invertible $a = (a_E)$ in A_C^- define the symplectic form $B_C(a)$ on the F–space A_C as the sum of the forms $B_E(a_E)$ in (6), so that $B_C(a)$ is given by

$$B_C(a)\Big((x_E),(y_E)\Big) = \sum\nolimits_E \mathrm{tr}_{E/K}\Big(x_E \cdot a_E \cdot \sigma_E(y_E)\Big) = \sum\nolimits_E B_E(a_E)\Big(x_E, y_E\Big). \tag{9}$$

The torus τ_C which we want to embed into $GSp(2n)$ (or better its F–rational points) sits in the product $\prod GL_{E^+}(E)$ which in turn embeds into the diagonal $\prod GL_F(E)$ of $GL_F(A_C)$. We take care of this product structure and first claim that for each E in $\mathcal{E}$ the following diagram

$$\begin{array}{ccc} H_E(F) = GL_2(E^+)\bigcap GSp(B_E(a_E)) & \longrightarrow & GSp(B_E(a_E)) \\ \big\downarrow \mathrm{det}_{E^+} & & \big\downarrow \mu \\ F^* & = & F^* \end{array} \tag{10}$$

commutes. In particular, H_E is then independent of the choice of the symplectic form $B_E(a_E)$ and

$$H_E(F) = \Big\{ g \in GL_{E^+}(E) : \det g \in F^* \Big\}. \tag{11}$$

This is most easily seen by fixing a basis e_+, e_- of E over E^+ with e_+ in E^+ and e_- in E^-. Since $\mathrm{tr}_{E/E^+}(e_-) = 0$ and e_-^2 is an element of E^+, we get $B_E(a_E)(x,y) = -\mathrm{tr}_{E/F}(a_E \cdot \det(x,y) \cdot e_+ e_-) = -\mathrm{tr}_{E/F}(a_E \cdot x \wedge y)$ for all x, y in E, and by abuse of notation in the last equality. Clearly then, g in $GL_2(E^+)$ satisfies $B_E(a_E)(gx, gy) = \mu \cdot B_E(a_E)(x,y)$ on $E \times E$ for an element μ in F if and only if $\det g$ is already in F. In this case $\mu = \det g$. This proves what we claimed in (10) and (11).

Remembering the hypothesis $\mathcal{E} = \mathcal{U}$, the group H_C^U is now defined by making the pullback diagram

$$\begin{array}{ccc} H_C^U & \longrightarrow & \prod_{E\in\mathcal{U}} H_E \\ \big\downarrow \mu_H & & \big\downarrow \prod R_{E^+/F}(\mathrm{det}_{E^+}) \\ (\mathbf{G}_m)_F & \xrightarrow{\Delta} & (\mathbf{G}_m)_F{}^{\mathcal{U}} \end{array} \tag{12}$$

commute where Δ is the diagonal embedding of $(\mathbf{G}_m)_F$. By construction, H_C^U is a subgroup of $GSp(B_C(a))$ for any a and contains τ_C^U in a natural way. This completes our construction of the unitary part of H_C.

(1.7) Proposition: *Each $GSp(2n,F)$-conjugacy class of the stable $GSp(2n)$-conjugacy class C contains a torus obtained by conjugating τ_C within H_C.*

(1.8) Corollary: *There is a bijection from the stable H_C-conjugacy class of τ_C modulo conjugation in $H_C(F)$ to the quotient C modulo conjugation in $GSp(2n,F)$.*

The corollary is an obvious consequence of the proposition. For the proof of the proposition, recall that $H^1(F,\tau_C) = \ker(H^1(F,\tau_C) \longrightarrow H^1(F, GSp(2n)))$ parametrizes the $GSp(2n,F)$-conjugacy classes of C and $\ker(H^1(F,\tau_C) \longrightarrow H^1(F,H_C))$ parametrizes the $H_C(F)$-conjugacy classes in the stable H_C-conjugacy class of τ_C. We are therefore reduced to show that $H^1(F,H_C)$ is trivial.
This is a problem on the unitary part of H_C only. So we are reduced to the unitary case $\mathcal{E} = \mathcal{U}$. But by (10) and (11), the kernel $H_C^{(1)}$ of the similarity character μ_H of H_C is in this case a product of special linear groups. The long exact sequence associated to $1 \longrightarrow H_C^{(1)} \longrightarrow H_C \xrightarrow{\mu_H} (\mathbf{G}_m)_F \longrightarrow 1$ now shows that $H^1(F,H_C)$ is in fact trivial. This completes the proof.

(1.9) There is actually in the unitary case $\mathcal{E} = \mathcal{U}$, too, a natural way to construct a set of representatives of tori as in the proposition. This is based on the embeddings of τ_C given by the symplectic forms $B_C(a)$ of (1.6)(9), and was first observed by Weissauer. We first describe the tori. Adopting the notation of (1.6), recall that $B_E(a_E)(x,y) = -\mathrm{tr}_{E/F}(a_E \cdot x \wedge y)$ for all x, y in E. Changing the basis e_+, e_- of E over E^+ by $\mathrm{diag}(1,b_E)$ in $GL_2(E^+)$ thus transforms $B_E(a_E)$ into $B_E(a_E b_E)$. Moreover, it clearly means conjugating $GSp(B_E(a_E))$ by $\mathrm{diag}(1,b_E)$. Switching from the single factors of the product A_C to A_C itself, we consider the tori

$$T_C(b) = \left(\mathrm{Int}\begin{pmatrix}1 & 0\\ 0 & b\end{pmatrix}\right) T$$

where $b = (\ldots, b_E, \ldots)$ is any unit of A_C^+, and T is the image of the torus τ_C under the embedding into $GSp(B_C(a))$ afforded by $B_C(a)$. Note that $T_C(b)$ is contained in $GSp(B_C(ab))$.
We claim that the tori $T_C(b)$, with b running through a set of representatives of $H^1(F,\tau_C)$, form a set of representatives of C modulo conjugacy as described in (1.8).
Recall for the proof that the conjugacy classes of tori within C correspond to the elements of $H^1(F,\tau_C)$. The long exact sequence associated to $1 \longrightarrow \tau_C^{(1)} \longrightarrow \tau_C \xrightarrow{\mu} (\mathbf{G}_m)_F \longrightarrow 1$ shows that $H^1(F,\tau_C)$ is a quotient of $\prod_{\mathcal{E}}(E^+)^*$. Any element β of $H^1(F,\tau_C)$ is thus represented by an invertible element $b = (\ldots, b_E, \ldots)$ in A_C^+. There exists $\xi = (\ldots, \xi_E, \ldots)$ in $\prod_{\mathcal{E}} (\mathbf{G}_m)_E(\overline{F})$ such that b is the norm of ξ. Then $h = \mathrm{diag}(1,b) \cdot \xi^{-1}$ is in $H_C^{(1)}$. For all σ in the absolute Galois group over E^+ we have $\beta_{E,\sigma} = h_E^{-1} \cdot \sigma(h_E) = \xi_E \cdot \sigma(\xi_E)^{-1}$. Thus (β_σ) gives a 1-cocycle in $\tau_C^{(1)}$ which represents $T_C(b)$ and maps to β in $H^1(F,\tau_C)$. This completes the proof.

2. The results for $GSp(4)$: In this section we first specialize the results of section 1 to the case of $GSp(4)$. More precisely, we want to describe the situation for stable conjugacy classes C in $GSp(4)$ which have a unitary part in the sense of section 1, and which are not contained in a proper Levi factor of $GSp(4)$. For these classes C we analyze in addition the space $H_C(F)\backslash GSp(4,F)/GSp(4,\mathcal{O}_F)$. It is remarkable that these quotients become very simple.

Checking now the possibilities in the classification explained in section 1, we find that there are exactly two types of stable conjugacy classes C as above: either the minimal polynomial of the class splits in two irreducible factors over F or the minimal polynomial of C is irreducible. In the first case we get tori of type $\mathrm{T_{3A}}$, in the second case basic tori. Their construction is reviewed below.

(2.1) Description of the group H and the tori $\tau_{E,L}$ of type $\mathrm{T_{3A}}$: We cover now the first of the two possibilities listed above. So let the minimal polynomial of C split in two irreducible factors. We obtain in this way a pair $\xi = (E, L)$ of quadratic extensions of F. By (1.4)(7) the maximal F–torus associated to C is the F–subtorus determined by

$$\tau_{E,L}(F) = \Big\{(x,y) \in E^* \times L^* : N_{E/F}(x) = N_{L/F}(y)\Big\} \tag{1}$$

of the F–torus $\mathrm{R}_{E/F}(\mathbf{G}_m)_E \times \mathrm{R}_{L/F}(\mathbf{G}_m)_L$ in $GL_F(E) \times GL_F(L)$. By (1.6)(11),(12) we get

$$H(F) = H_\xi(F) = \Big\{(h,h') \in GL_F(E) \times GL_F(L) : \det h = \det h'\Big\}. \tag{2}$$

We now construct a specific embedding of H in $GSp(4)$. Fix normalized primitive elements $\sqrt{A}$ of E and $\sqrt{B}$ of L over F, so that A and B are representatives in F of $F^*/(F^*)^2$ which both have orders 0 or 1. We take 1_E, 1_L, $\sqrt{A}$, $\sqrt{B}$ as symplectic orthonormal basis. Then $H(F)$ consists of all matrices of the form

$$[h,h'] = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix} \quad \text{with} \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ h' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tag{3}$$

in $GL_F(E) \times GL_F(L)$ satisfying the symplecticity condition $\det h = \det h'$. We will prove in section 5 the following

(2.2) Theorem: *Each double coset of $H(F)\backslash GSp(4,F)/GSp(4,\mathcal{O}_F)$ contains a unique element of the form*

$$g(\gamma) = \begin{pmatrix} E_2 & \gamma W \\ 0 & E_2 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with either $\gamma = 0$ or $\gamma = \pi^{-\ell}$ and $\ell > 0$ any natural number.

(2.3) Description of the groups H_ξ and the basic tori $\tau_b(\xi)$: In this section we cover the second type of stable conjugacy classes of maximal F–tori in $GSp(4)$ described above. Recall that any of these is associated to a pair $\xi = (E, \sigma_E)$ consisting of an extension E over F of degree four which has a non–trivial involution σ_E. By (1.6)(11) we now have

$$H_\xi(F) \cong \Big\{g \in GL(2,E^+) : \det g \in F^*\Big\}. \tag{4}$$

We construct again a specific embedding of this group into $GSp(4)$. Write $E = E^+(\sqrt{D}\,)$. Let $\sqrt{A}$ be a normalized primitive element over F of the fixed field E^+ of σ_E. Choosing a basis symplectic with respect to a form $B_\xi(a)$ for that $\mathrm{tr}_{E/F}(a\sqrt{D}) \neq 0$ as in [**S**, A.19.8], the group $H_\xi(F)$ embeds as the subgroup of $GSp(4,F)$ of all matrices of the form

$$\begin{pmatrix} a_1 & a_2A^{-1} & b_1 & b_2 \\ a_2 & a_1 & b_2 & b_1A \\ c_1 & c_2A^{-1} & d_1 & d_2 \\ c_2A^{-1} & c_1A^{-1} & d_2A^{-1} & d_1 \end{pmatrix}, \tag{5}$$

We remark that the torus associated to ξ is the basic torus $\tau_b(\xi) = \{x \in E^* : N_{E/E^+}(x) \in F^*\}$, as defined in (1.3)(4). We will prove in section 6 the following

(2.4) Theorem: *Let $\mathcal{H}_\xi = H_\xi(F)\backslash GSp(4,F)/GSp(4,\mathcal{O}_F)$ and for all $\ell \geq 1$ let*

$$g(\ell) = \begin{pmatrix} E_2 & S(\ell) \\ 0 & E_2 \end{pmatrix} \qquad \text{with} \qquad S(\ell) = \begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & 0 \end{pmatrix}.$$

If E^+ is unramified over F, a set of representatives for $\mathcal{H}_\xi$ is given by E_4 and $g(\ell)$ with $\ell \geq 1$. If E^+ is ramified over F, each double coset of $\mathcal{H}_\xi$ contains a unique element of the form $g(\ell)$ with $\ell \geq 1$.

3. A technique for calculating orbital integrals: In this section we indicate briefly which role the groups H_C constructed in (1.5) and (1.6) may play in calculating orbital integrals. So let τ_C be the image under any fixed embedding into $GSp(2n)$ of the torus constructed in (1.4), and let s be a regular, F–rational element of τ_C. Let $K = GSp(2n,\mathcal{O}_F)$. For any Hecke operator f on $GSp(2n,F)$ one has by [**Wa** I, p.477, A 1.2] and [**KoGL**, p.361f]

$$\begin{aligned} O_s^{GSp(2n)}(f) &\overset{\mathrm{def}}{=} \int_{\tau_C\backslash GSp(2n)} f(g^{-1}sg)\,d\dot{g} \\ &= \sum\nolimits_{x\in H_C\backslash GSp(2n)/K} \frac{\mathrm{vol}_{GSp(2n)}(K)}{\mathrm{vol}_{H_C}(H_C\cap xKx^{-1})} \int_{\tau_C\backslash H_C} (f\circ \mathrm{Ad}x^{-1})(h^{-1}sh)\,d\dot{h} \end{aligned} \tag{1}$$

where we identify the groups with their F–rational points, and measures are suitably normalized.

In some sense, this is a series expansion of an orbital integral on $GSp(2n,F)$ by a family of orbital integrals on the group $H_C(F)$. The group H_C is smaller and in addition embeds into a product of Weil restrictions of $GL(2)$. On the other hand, the functions in the orbital integrals on $H_C(F)$ may become considerably more complicated. The support of $f\circ\mathrm{Ad}\,x^{-1}$ is $\mathrm{supp}(f\circ\mathrm{Ad}\,x^{-1}) = H_C\cap(\mathrm{Ad}x)(\mathrm{supp}(f)) = H_C\cap x\cdot\mathrm{supp}(f)\cdot x^{-1}$. Furthermore, $f\circ\mathrm{Ad}\,x^{-1}$ is bi–invariant under $H_C\cap x\cdot K\cdot x^{-1}$, a group in general different from the maximal compact subgroup of $H_C(F)$.

4. Calculating the $GSp(4)$–orbital integral $O_s(T(\pi))$ for s in a torus of type $\mathrm{T}_{3\mathrm{A}}$: We want to show next that the abstract technique introduced in section 3 is a tool for calculating orbital integrals for the groups of symplectic similitudes. For this it is crucial to have a set of concrete representatives of the space $H_C(F)\backslash GSp(2n,F)/GSp(2n,\mathcal{O}_F)$. Given the results of (2.2) and (2.4), we therefore consider the group of symplectic similitudes in four variables $G = GSp(4)$. We choose the Hecke operator $T(\pi)$ defined as characteristic function of the double coset $K \cdot \mathrm{diag}(E_2, \pi E_2) \cdot K$ with $K = GSp(4, \mathcal{O}_F)$. We will furthermore assume that the local field F has odd residue characteristic.

(4.1) Embeddings of tori of type $\mathrm{T}_{3\mathrm{A}}$**:** We want to choose the semisimple element s in tori $\tau_{E,L}$ of type $\mathrm{T}_{3\mathrm{A}}$ which we introduced in (2.1). But there is a problem since $\tau_{E,L}$ may have two different non–conjugate embeddings into $GSp(4)$. For this, recall that the number of embeddings of $\tau_{E,L}$ into $GSp(4)$ up to conjugacy equals the number of conjugacy classes within the stable conjugacy class of $\tau_{E,L}$. Recall furthermore, that this last set is parametrized by the kernel of the natural map from $H^1(F, \tau_{E,L})$ into $H^1(F, G)$. Since $H^1(F, GSp(4))$ is trivial this kernel is $H^1(F, \tau_{E,L})$ itself. But in the local case, $H^1(F, \tau_{E,L})$ is by class field theory trivial for $E \neq L$ and cyclic of order two for $E = L$.
A set of representatives of the conjugacy classes of F–embeddings of $\tau_{E,L}$ into $GSp(4)$ was determined explicitly in [**S**, §11B]. They take their values in H, in accordance with (1.7). We fix the F–rational, semisimple element s in the image T of $\tau_{E,L}$ under any of these, so that

$$(1) \qquad s = [s_E, s_L] = \left[\begin{pmatrix} a & bD^{-1}A \\ bD & a \end{pmatrix}, \begin{pmatrix} a' & b'B \\ b' & a' \end{pmatrix}\right] \sim_{\overline{F}} \left[\begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}, \begin{pmatrix} \lambda' & \\ & \mu' \end{pmatrix}\right]$$

where $D = 1$ in the stable case $E \neq L$ and where $D \in \{1, \Theta\} \cong F^*/N_{E/F}(E^*)$ in the unstable case $E = L$. With this notation we have

(4.2) Theorem: *Let the residue characteristic of the local field F be odd and let s as in (1) be regular in $GSp(4)$.*

Necessary conditions for the $GSp(4)$–orbital integral $O_s(T(\pi))$ to be nonzero are: the similarity factor $\mu(s)$ has order 1 in F, the fields E and L are equal, and are both ramified over F.

If these conditions are satisfied, we have

$$\begin{aligned} O_s(T(\pi)) &\stackrel{\text{def}}{=} \int_{T\backslash GSp(4)} T(\pi)(g^{-1}sg)\, d\left(\frac{\mu_{GSp(4)}}{\mu_T}\right)(\dot{g}) \\ &= \frac{\mathrm{vol}_G(GSp(4, \mathcal{O}_F))}{\mathrm{vol}_T(T(\mathcal{O}_F))}\left(1 + \frac{2 \cdot \delta_D(s)}{|(\lambda + \mu) - (\lambda' + \mu')|} \cdot \frac{\xi_F(1)}{\xi_F(N)}\right) \end{aligned}$$

where we put $\delta_D(s) = 1$ if $-bD/b'$ is a quadratic residue modulo $\pi\mathcal{O}_F$ and $\delta_D(s) = 0$ otherwise, where $N = \mathrm{ord}_F((\lambda + \mu) - (\lambda' + \mu'))$, and $\xi_F(\ell) = 1/(1 - q^{-\ell})$ is the zeta function of F evaluated at ℓ.

Remarks: Taking for granted the validity of the "fundamental lemma", Kottwitz [**KoS**] was able to express the cardinality of the $\mathbf{F}_p$–rational points of the moduli space of principally polarized Abelian varieties of dimension g with level N structure as an adelic $GSp(2g)$–orbital integral. The Hecke operator in this integral is of the form $f^{(p)} \cdot T(p)$ with $f^{(p)}$ the characteristic function outside p of a certain congruence subgroup. Our calculation identifies now the following terms in the elliptic part of the trace formula: the (unstabilized) contribution from the local orbital integrals over $T(\pi)$ at the regular elements in tori of type $\mathrm{T}_{3\mathrm{A}}$.

The κ–orbital integral to s on $GSp(4)$ is up to a sign the difference of the orbital integrals to the two conjugates of s. In analogy to the conjectural "fundamental lemma" we thus get

$$\Delta(s)\cdot O_s^{\kappa}(T(\pi)) = SO_s^{H}(T(0,\pi))$$

with transfer factor $\Delta(s) = \pm|\lambda\mu|^{1/2}\cdot|(\lambda/\lambda'-1)\cdot(\lambda'/\lambda-1)\cdot(\mu/\lambda'-1)\cdot(\lambda'/\mu-1)|^{1/2}\cdot \xi_F(N)/\xi_F(1) = \pm|(\lambda+\mu)-(\lambda'+\mu')|\cdot\xi_F(N)/\xi_F(1)$, and with $T(0,\pi)$ as in (4.3)(4) below.

The proof of (4.2) will fill up the rest of this section. We will apply the techniques described in section 3 emphasizing the structure of our calculations.

(4.3) The operators $T(\ell,\pi)$ and the groups $H(\ell)$: We first give names to the concepts introduced in section 3 and then analyze their structure. Define for all integers $\ell\geq 0$

(2a) $$z(0) = E_4,$$

(2b) $$z(\ell) = \begin{pmatrix} 0 & \pi^{-\ell}E_2 \\ E_2 & 0\end{pmatrix} g(\pi^{-\ell}) = \begin{pmatrix} 0 & \pi^{-\ell}E_2 \\ E_2 & \pi^{-\ell}W\end{pmatrix} \quad\text{with}\quad W = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},\ \ell\geq 1,$$

(3) $$H(\ell) = H(F)\bigcap z(\ell)\cdot K\cdot z(\ell)^{-1},$$

(4) $$T(\ell,\pi) = T(\pi)\circ\left(\mathrm{Ad}\, z(\ell)^{-1}\right)\big|_{H(F)} \in \mathcal{H}\big(H(F),H(\ell)\big).$$

The family of all $z(\ell)$ is again a set of representatives for $H(F)\backslash GSp(4,F)/K$. We modified the elements $g(\gamma)$ since in this way we get a clearer picture of the groups $H(F)\cap x^{-1}\cdot K\cdot x$ and the supports of the pullbacks of $T(\pi)$. We discuss their structure in the next block of three results

(4.3.1) Proposition: *For any $\ell\geq 0$, the support of $T(\ell,\pi)$ is*

$$\mathrm{supp}\Big(T(\ell,\pi)\Big) = H(\ell)\cdot\left[\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix},\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}\right]\cdot H(\ell)$$

and $\mathrm{supp}(T(\ell,\pi))\subseteq\mathrm{supp}(T(\ell^*,\pi))\subseteq\mathrm{supp}(T(0,\pi))$ *for all* $\ell>\ell^*$.

(4.3.2) Lemma: *Let* pr *be the projection of $H(F)$ onto its first $GL(2)$-factor. Then for all $\ell\geq 0$, the sequence*

$$1\longrightarrow\{E_2\}\times\Gamma(\ell)\longrightarrow H(\ell)\xrightarrow{\mathrm{pr}} GL(2,\mathcal{O}_F)\longrightarrow 1$$

is exact, where $\Gamma(\ell)$ is the principal congruence subgroup of $SL(2,\mathcal{O}_F)$ of level π^ℓ. Furthermore

$$H(\ell) = \Big\{[X,Y]\in H(\mathcal{O}_F) : X\equiv {}^{W}Y(\mathrm{mod}\,\pi^\ell\mathcal{O}_F)\Big\},$$

where ${}^{W}Y = (\mathrm{Ad}\,W)(Y) = W\cdot Y\cdot W^{-1}$ *with W is as in* (2b).

(4.3.3) Symmetrization by the automorphism $1\times\Phi$ of H: Define

(5) $$(1\times\Phi)([h,h']) = \big(\mathrm{Ad}([E_2,W])\big)([h,h']) = \big[h,(\mathrm{Ad}\,W)(h')\big] = \big[h,\Phi(h')\big].$$

Then $H_\Phi(\ell) = (1\times\Phi)(H(\ell)) = \{[X,Y]\in H(\mathcal{O}_F) : X\equiv Y(\mathrm{mod}\,\pi^\ell\mathcal{O}_F)\}$ and the support of the pullback of $T(\ell,\pi)$ by $1\times\Phi$ is $H_\Phi(\ell)\cdot[\mathrm{diag}(1,\pi),\mathrm{diag}(1,\pi)]\cdot H_\Phi(\ell)$.

For the proofs we note that a straightforward calculation gives (4.3.2). To prove (4.3.1) we first indicate a general strategy to determine the $H(\ell)$–double cosets ξ in the support of the pull–back $f \circ (\mathrm{Ad}\, z(\ell)^{-1})$ to $H(F)$ of a Hecke operator f.
Choose a representative of ξ whose first $GL(2)$–component is a diagonal matrix $\mathrm{diag}(a_1, d_1)$ with pure π–powers a_1, d_1 and $\mathrm{ord}\, a_1 \leq \mathrm{ord}\, d_1$. We have to decide when $Y_{\ell,S} = z(\ell)^{-1} \cdot [\mathrm{diag}(a_1, d_1), \mathrm{diag}(d_1, a_1) \cdot S] \cdot z(\ell)$ is in the support of f, where S is chosen in $SL(2,F)/\Gamma(\ell)$. Using the filtration $\Gamma(\ell+1) \subseteq \Gamma(\ell) \subseteq \Gamma(0)$ one deals with this problem iteratively, starting with $\ell = 0$. For ℓ fixed, the first step is to decide for which parameters $Y_{\ell,S}$ has entries in $\mathcal{O}_F$. For these, as the second step, one then determines the elementary divisors.
Let $f = T(\pi)$. Then $a_1 = 1$, $d_1 = \pi$ imply that $Y_{\ell,S}$ has entries in $\mathcal{O}_F$ only if S is in $\Gamma(\ell)$.

(4.4) Necessary conditions on s: A necessary condition for each of the orbital integrals considered to be nonzero is that the similarity factor $\mu(s)$ has the same order 1 as $\mu(T(\pi))$. Recall that $\mu(s) = a^2 - b^2 A = (a')^2 - (b')^2 B$. By Hensel's lemma $\mu(s)$ therefore has order 1 only if the following conditions are satisfied: $\mathrm{ord}\, A = \mathrm{ord}\, B = 1$, $\mathrm{ord}\, a$, $\mathrm{ord}\, a' \geq 1$, and b, b' are both units. We assume from now on that these conditions are fulfilled. Note that in this case, E and L are ramified over F.
Finally we show that E and L are in fact equal, an observation due to Weselmann. To see this, recall that the quadratic extensions of F are parametrized by the three cosets in $F^*/(F^*)^2$ different from the identity. Under this correspondence, the ramified quadratic extensions of F are associated to the two cosets represented by π and $\xi \cdot \pi$ respectively where ξ is a non–square in $k(F)$. Assume that E and L are different, and that $A = \pi$ and $B = \xi \cdot \pi$. The two equations for $\mu(s)$ imply that $a^2 - (a')^2 = (b^2 - (b')^2 \cdot \xi) \cdot \pi$. Since b and b' are both units, reduction mod $\pi\mathcal{O}_F$ shows that $b^2 - (b')^2 \cdot \xi$ is also a unit. Since the order of $a^2 - (a')^2$ is at least two, we conclude that $A = B$.

In order to calculate the orbital integrals $O_s(T(\ell,\pi))$ we will use an iterative procedure. This is based on the support filtration (4.3.1). The key step is establishing (4.5).

(4.5) Proposition: *The support* $\{h \in H(F) : T(0,\pi)(h^{-1}sh) \neq 0\}$ *of* $O_s^H(T(0,\pi))$ *is*

$$T(\mathcal{O}_F)\backslash H(\mathcal{O}_F) = \Big\{T(\mathcal{O}_F) \cdot h : h \in H(\mathcal{O}_F)\Big\}.$$

Our proof of this proposition is based on the fact that $g_n = \mathrm{diag}(1, \pi^n)$ for $n \geq 0$ form a set of representatives of $\tau(F)\backslash GL(2,F)/GL(2,\mathcal{O}_F)$, when τ is the torus $\mathrm{R}_{E/F}(\mathbf{G}_m)_E$ in $GL(2)$.
Fix $[h, h']$ in the support set. By (4.3.1), (4.3.2) each factor of $[h^{-1}s_E h, (h')^{-1}s_L h']$ is in $\xi = GL(2,\mathcal{O}_F)\mathrm{diag}(1,\pi)GL(2,\mathcal{O}_F)$. Writing $h = tg_n k$ with t in E^* and k in $GL(2,\mathcal{O}_F)$ it follows that $g_n^{-1}s_E g_n$ is in ξ. Since b is a unit, an explicit calculation shows that this implies $n = 0$.
By construction, $y = t^{-1}h'k^{-1}$ is in $SL(2,F)$ and we have $T(0,\pi)([h,h']^{-1} \cdot s \cdot [h,h']) = T(0,\pi)([s_E, y^{-1}s_L y])$. As above, $y = t'k'$ with t' in $L^* = E^*$ and k' in $GL(2,\mathcal{O}_F)$. Then t' is a unit, so that its entries are in $\mathcal{O}_F$. Hence y is in $SL(2,\mathcal{O}_F)$. The reverse inclusion can be checked.

(4.6) The numbers $R(\ell, s)$: For $\ell \geq 1$, let

$$R(\ell,s) = \#\Big\{y \in SL(2,\mathcal{O}_F/\pi^\ell\mathcal{O}_F) : \big[s_E, y^{-1}s_L y\big] \in \mathrm{supp}\big(T(\ell,\pi)\big)\Big\} \tag{6}$$

so that $R(\ell, s)$ is the number of y in $SL(2, \mathcal{O}_F/\pi^\ell\mathcal{O}_F)$ such that $[s_E, y^{-1}s_L y]$ is in the support of $T(\ell, \pi)$.

Note that we identify $SL(2, \mathcal{O}_F/\pi^\ell\mathcal{O}_F)$ with $SL(2,\mathcal{O}_F)/\Gamma(\ell)$. We further choose the $k(F)$–space $\langle 1, \pi, \ldots, \pi^{\ell-1}\rangle$ generated by $1, \pi, \ldots, \pi^{\ell-1}$ as section for $\mathcal{O}_F/\pi^\ell\mathcal{O}_F$ in $\mathcal{O}_F$.

In this interpretation, each coset of $H(\mathcal{O}_F)/H(\ell)$ contains a unique element of the form $[1, y]$ with y in $SL(2, \mathcal{O}_F/\pi^\ell\mathcal{O}_F)$. So we get by (4.5) and §3(1) with the notations introduced above

(4.7) Proposition: *The orbital integrals $O_s(T(\ell, \pi))$ on H and $O_s(T(\pi))$ on $GSp(4)$ have the following expressions*

$$O_s(T(\ell,\pi)) = \frac{\mathrm{vol}_H(H(\ell))}{\mathrm{vol}_T(T(\mathcal{O}_F))} R(\ell, s),$$

$$O_s(T(\pi)) = \frac{\mathrm{vol}_G(GSp(4,\mathcal{O}_F))}{\mathrm{vol}_T(T(\mathcal{O}_F))}\left([EL:E] + \sum\nolimits_{\ell>0} R(\ell, s)\right).$$

(4.8) Characterizing $R(\ell, s)$ by congruence conditions: The formulas in (4.7) reduce the calculation of the orbital integrals $O_s(T(\pi, \ell))$, and so of $O_s(T(\pi))$, to calculating the numbers $R(\ell, s)$. Analyzing (6), it seems natural to look for congruence conditions which characterize the elements y counted by $R(\ell, s)$. For this, let $P = \mathrm{diag}(1, \pi)$, so that $s_E = h_E P$ for h_E in $GL(2, \mathcal{O}_F)$. Let y be in $SL(2, \mathcal{O}_F/\pi^\ell\mathcal{O}_F)$, taken as set of representatives for $SL(2, \mathcal{O}_F)/\Gamma(\ell)$ as described above. Then $[s_E, y^{-1}s_L y]$ is in the support of $T(\ell, \pi)$ if and only if

$$\Phi\left(y^{-1}s_L y\right) \in \Gamma(\ell)\cdot s_E \cdot \Gamma(\ell) \subseteq \begin{pmatrix} \pi\mathcal{O}_F & \pi U(F) \\ U(F) & \pi\mathcal{O}_F \end{pmatrix}. \tag{7}$$

The order conditions hold only for the elements

$$Y(\omega, \beta, \delta) = \begin{pmatrix} \omega & \gamma \\ \delta & \beta \end{pmatrix} \qquad \text{with} \qquad \gamma = \frac{\omega\beta - 1}{\delta} \tag{8a}$$

whose entries satisfy the order conditions

$$\omega \in \langle \pi, \ldots, \pi^{\ell-1}\rangle, \quad \beta \in \langle 1, \pi, \ldots, \pi^{\ell-1}\rangle, \quad \delta \in k(F)^* \oplus \langle \pi, \ldots, \pi^{\ell-1}\rangle. \tag{8b}$$

They are defined by the intersection of three quadrics in the affine space of dimension four over $\mathcal{O}_F/\pi^\ell\mathcal{O}_F$: multiply Y, with entries named as in (8), by $\omega - \delta\sqrt{A}$ on the left and let $\Delta(Y) = \omega^2 - \delta^2 A$, $z(Y) = \omega\gamma - \delta\beta A$. We obtain

$$\Phi\left(Y^{-1}s_L Y\right) = \begin{pmatrix} a' + zb' & b'\Delta \\ \dfrac{b'(A - z^2)}{\Delta} & a' - zb' \end{pmatrix}. \tag{9}$$

Their entries have orders as in (7) only if $\mathrm{ord}\,\Delta = 1$ and $\mathrm{ord}\, z \geq 1$. These conditions characterize the elements $Y(\omega, \beta, \delta)$.

We break our calculation of $R(\ell, s)$ into two steps. First we count by congruence relations the quadrics described above, then we count the points in the "fibres", i.e. count the points on any of these quadrics.

(4.8.1) Counting the number of quadrics: By construction $\Phi(Y^{-1}s_L Y)P^{-1}$ has entries in $\mathcal{O}_F$ for all $Y = Y(\omega, \beta, \delta)$. Thus the following criterion applies to determine for which (Δ, z) the $\Gamma(\ell)$–double cosets of $\Phi(Y^{-1}s_L Y)$ and $s_E = h_E P$ are equal.

Lemma: *Let g', g'' be in $GL(2, \mathcal{O}_F)$. Then $\Gamma(\ell) \cdot g'P \cdot \Gamma(\ell) = \Gamma(\ell) \cdot g''P \cdot \Gamma(\ell)$ if and only if there is β in $\pi^{\ell-1}k(F)$ such that*

$$g' \equiv g'' \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} (\operatorname{mod} \Gamma(\ell)).$$

We leave the proof to the reader. The congruences of the lemma in the present situation are equivalent to $a \equiv a' (\operatorname{mod} \pi^\ell \mathcal{O}_F)$, $0 \equiv z (\operatorname{mod} \pi^\ell \mathcal{O}_F)$ and

$$\Delta_0 = \frac{\Delta}{\pi} \equiv \frac{A}{\pi D} \frac{b}{b'} (\operatorname{mod} \pi^\ell \mathcal{O}_F), \qquad \left(\frac{b}{b'}\right)^2 \frac{A}{\pi} \equiv \frac{A - z^2}{\pi} (\operatorname{mod} \pi^\ell \mathcal{O}_F).$$

In this case, the values $\operatorname{mod} \pi^\ell \mathcal{O}_F$ of Δ_0, z are completely determined. Since $\operatorname{ord} z \geq \ell$ we obtain $(b/b')^2 \equiv 1 (\operatorname{mod} \pi^\ell)$.

Note that $a \equiv a' (\operatorname{mod} \pi^\ell \mathcal{O}_F)$ implies $b^2 \equiv (b')^2 (\operatorname{mod} \pi^\ell \mathcal{O}_F)$ and thus $b \equiv \varepsilon b' (\operatorname{mod} \pi^\ell \mathcal{O}_F)$ because of (4.4). Thus we get congruence conditions in $a = (\lambda + \mu)/2$ and $a' = (\lambda' + \mu')/2$ only.

(4.8.2) Counting the number of points in the fibres: We count, as a second step, the number of points on each of the quadrics above. It is well known, that for a smooth affine variety V over $\mathcal{O}_F$, the fibres of the natural maps $V(\mathcal{O}_F/\pi^{i+1}\mathcal{O}_F) \longrightarrow V(\mathcal{O}_F/\pi^i \mathcal{O}_F)$ have cardinality $\# k(F)^{\dim V}$, for all $i \geq 1$. The following result, proved by showing that the Jacobian has full rank, now gives the "fibre terms" of our counting argument and thus completes the proof of (4.2).

Lemma: *For z in $\pi\mathcal{O}_F$, Υ in $\pi U(F)$ and Δ_0 in $U(F)$, let Q_Υ be the zero set of $\Delta_0 = \pi^{-1}(\omega^2 - \delta^2 \Upsilon)$, $z = \omega\gamma - \delta\beta\Upsilon$, $1 = \omega\beta - \delta\gamma$ in the four-dimensional affine space. Then Q_Υ is a smooth variety over $\mathcal{O}_F$ and*

$$\# \left\{ (\omega, \beta, \delta, \gamma) \in Q_\Upsilon(\mathcal{O}_F/\pi^\ell \mathcal{O}_F) : \begin{matrix} \omega \equiv 0 (\operatorname{mod} \pi\mathcal{O}_F), \\ \delta \not\equiv 0 (\operatorname{mod} \pi\mathcal{O}_F) \end{matrix} \right\} = \begin{cases} 2 \cdot (\# k(F))^\ell & -\frac{\pi}{\Upsilon}\Delta_0 \in U(F)^2 \\ 0 & \text{otherwise.} \end{cases}$$

5. Proof of Theorem (2.2): For the proof fix the Borel group of $GSp(4)$ which consists of all matrices such that for each of them the two by two matrix in its lower left hand corner is zero, and the two by two matrix in its upper left hand corner is upper triangular. Using the Iwasawa decomposition of $GSp(4, F)$ one now starts with representatives in this Borel subgroup of $GSp(4, F)$. Multiplying on the left by suitable upper triangular matrices in $H(F)$ they can be modified to representatives of the form $h(a, b, 0)$ in the Heisenberg subgroup of $GSp(4, F)$ consisting of all matrices

$$h(a, b, c) = \begin{pmatrix} 1 & a & c & b \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 1 \end{pmatrix}$$

with a, b, c in F. Since $h(0,0,c)$ is in $H(F)$ for all c in F, the relations

$$h(a,b,0)\cdot h(a',b',0) = h(a+a', b+b', ab'-a'b),$$
$$h(0,0,-c)\cdot h(a,b,c) = h(a,b,0)$$

show that one may in fact choose representatives $g(a,b) = h(a,b,0)$ with a, b in the $k(F)$–space $\langle \dots, \pi^{-2}, \pi^{-1}\rangle$. Here we follow a suggestion of Weissauer for simplifying our original proof.

We reduce to pure π–powers a and b: Let $\alpha = ua$ and $\beta = wb$ for units u und w in $U(F)$. Then the following matrix is in $GSp(4, \mathcal{O}_F)$

$$g^{-1}(\alpha,\beta)\left[\begin{pmatrix} uw & 0 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} w & 0 \\ 0 & u\end{pmatrix}\right] g(a,b) = \mathrm{diag}(uw, w, 1, u).$$

So $g(\alpha,\beta)$ and $g(a,b)$ are in the same coset of $H(F)\backslash GSp(4,F)/GSp(4,\mathcal{O}_F)$.

By the same reasoning we reduce further to representatives $g(\gamma) = g(0,\gamma)$: In the case $\operatorname{ord} b \le \operatorname{ord} a$

$$g^{-1}(0,b)\left[\begin{pmatrix} 0 & b \\ b^{-1} & 0\end{pmatrix}, \begin{pmatrix} a & b \\ b^{-1} & 0\end{pmatrix}\right] g(a,b) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ b^{-1} & ab^{-1} & 0 & 1 \\ 0 & b^{-1} & 1 & 0\end{pmatrix}$$

is an element of $GSp(4,\mathcal{O}_F)$. For $\operatorname{ord} a < \operatorname{ord} b$

$$g^{-1}(0,a)\left[\begin{pmatrix} 0 & -a \\ a^{-1} & 0\end{pmatrix}, \begin{pmatrix} a & b \\ 0 & -a^{-1}\end{pmatrix}\right] g(a,b) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ a^{-1} & 1 & 0 & a^{-1}b \\ 0 & 0 & -1 & -a^{-1}\end{pmatrix}$$

is in $GSp(4,\mathcal{O}_F)$. To prove independence, assume that $g(\alpha)$ and $g(\beta)$ are in the same coset. This is equivalent to the existence of h in $H(F)$ such that $hg(\alpha)\cdot GSp(4,\mathcal{O}_F) = g(\beta)\cdot GSp(4,\mathcal{O}_F)$. Taking images of this $GSp(4,\mathcal{O}_F)$–coset under each element of a dual basis one obtains four equalities of ideals in $\mathcal{O}_F$. They translate into four conditions on the orders of the entries of $hg(\alpha)$ and $g(\beta)$. Distinguishing $\alpha\beta = 0$ and $\alpha\beta \neq 0$ one checks that α and β have the same orders and thus are in fact equal.

6. Proof of Theorem (2.4): One starts again with representatives in the Borel subgroup of $GSp(4,F)$ which was described in the preceding section. Their components in the Levi factor $\{\mathrm{diag}(A, \lambda {}^tA) : A \in GL(2,F),\ \lambda \in F^*\}$ can be reduced to matrices $\mathrm{diag}(g_\ell, g_\ell^{-1})$, where $g_\ell = \mathrm{diag}(1,\pi^\ell)$ with $\ell \ge 0$ form a set of representatives of $(E^+)^*\backslash GL(2,F)/GL(2,\mathcal{O}_F)$. Because of

$$\begin{pmatrix} 1 & 0 & b_1 & b_2 \\ 0 & 1 & b_2 & b_1 A \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & x & y \\ 0 & \pi^\ell & \pi^\ell y & \pi^\ell z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi^{-\ell}\end{pmatrix} = \begin{pmatrix} 1 & 0 & a+b_1 & \pi^{-\ell}(b_2+\pi^\ell y) \\ 0 & \pi^\ell & b_2 + \pi^\ell y & \pi^\ell z + \pi^{-\ell} b_1 A \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi^{-\ell}\end{pmatrix}$$

we can choose representatives with $y = z = 0$. After multiplying from the right by a suitable unipotent matrix in $GSp(4,\mathcal{O}_F)$ we can assume x in $\langle\dots,\pi^{-2},\pi^{-1}\rangle$ and obtain the matrices $g(\ell, x)$.

We now show that we can achieve $\ell = 0$: For $\ell \geq 1$

$$g(0,\pi^{-\ell})^{-1}\begin{pmatrix}\pi^{-\ell} & 0 & 0 & 0\\ 0 & \pi^{-\ell} & 0 & 0\\ 1 & 0 & \pi^{\ell} & 0\\ 0 & A^{-1} & 0 & \pi^{\ell}\end{pmatrix} g(\ell,0) = \begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & \pi^{\ell} & 0\\ 0 & A^{-1}\pi^{\ell} & 0 & 1\end{pmatrix}$$

is in $GSp(4,\mathcal{O}_F)$. For $x \neq 0$ let $z = -(1+\pi^{\ell})x^{-1}$. Then

$$g(0,-x\pi^{-\ell})^{-1}\begin{pmatrix}\pi^{-\ell} & 0 & 0 & 0\\ 0 & \pi^{-\ell} & 0 & 0\\ z & 0 & \pi^{\ell} & 0\\ 0 & zA^{-1} & 0 & \pi^{\ell}\end{pmatrix} g(\ell,x) = \begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ z & 0 & -1 & 0\\ 0 & zA^{-1} & 0 & 1\end{pmatrix}$$

is in $GSp(4,\mathcal{O}_F)$. We reduce to pure π–powers by the calculation

$$g(0,x)^{-1}\begin{pmatrix}E_2 & 0\\ 0 & uE_2\end{pmatrix} g(0,y) = \begin{pmatrix}E_2 & \begin{matrix} y-ux & 0\\ 0 & 0\end{matrix}\\ 0 & uE_2\end{pmatrix}.$$

The calculation

$$\begin{pmatrix}E_2 & \begin{matrix} -y & 0\\ 0 & -yA\end{matrix}\\ 0 & E_2\end{pmatrix} g(0,y) = \begin{pmatrix}E_2 & \begin{matrix} 0 & 0\\ 0 & -yA\end{matrix}\\ 0 & E_2\end{pmatrix}$$

eventually shows that representatives of $\mathcal{H}_{\xi}$ are of the form $E_4 = g(0,0)$ and $g(\ell) = g(0,\pi^{-\ell})$ with $\ell \geq 1$ for $\operatorname{ord} A = 0$, i.e., for E^+ unramified over F, and $g(\ell)$ with $\ell \geq 1$ for $\operatorname{ord} A = 1$.

To check their independence is tedious, but straightforward given the method indicated in §5.

7. Acknowledgements: *This paper is based in part on my PhD thesis. The proof of* (4.2) *owes much to the criticism of my advisor Prof Weissauer. I am most grateful to Prof Popp for an observation which got me started. I would like to thank Dr Seiler for invaluable moral and technical support over the past years. I am grateful to Professors Bifet, Harder, and Yui, and in particular to Dr Weselmann for very helpful comments on earlier versions of this paper. Finally, I wish to thank the Max-Planck-Institut für Mathematik in Bonn, and its director Professor Hirzebruch, for hospitality and support during the preparation of this paper.*

8. References

[KoGL] R. Kottwitz: Orbital Integrals on $GL(3)$, *Am. J. Math.* **102**(1980), 327-384

[KoS] R. Kottwitz: Shimura varieties and λ–adic representations, pp. 161–209 in *Automorphic forms, Shimura varieties, and L–functions I*, Clozel, Milne [eds.], Academic Press 1990

[M] R.P. Langlands, D. Ramakrishnan [eds.]: *The Zeta Functions of Picard Modular Surfaces*, Les Publications CRM, Montreal 1993

[S] M. Schröder: *Zählen der Punkte mod p einer Shimuravarietät zu* $GSp(4)$ *durch die* L^2*–Spurformel von Arthur*, Thesis, University of Mannheim, March 1993

[Wa] G. Warner: *Harmonic Analysis on Semi-Simple Lie Groups*, Springer 1972

CURRENT ADDRESS: MPI FÜR MATHEMATIK, GOTTFRIED–CLAREN–STRASSE 26, 53225 BONN