Double quantization on coadjoint representations of simple Lie groups and its orbits

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1 Introduction

Quantum groups can be considered as symmetry objects of certain "quantum spaces" described by noncommutative algebra of functions. This point of view was developed, for example, in [RTF] and [Ma]. Here we study the inverse problem: given the quantum group corresponding to a Lie group G, we want to define a "quantum space" corresponding to a given classical G-manifold.

Let M be a manifold with an action of a Lie group G, $\mathfrak{g}$ the Lie algebra of G, and $U_h(\mathfrak{g})$ the quantized universal enveloping algebra. Let $\mathcal{A}$ be the sheaf of function algebras on M. It may be a sheaf of smooth, analytic, or algebraic functions. For shortness, we simply call $\mathcal{A}$ a function algebra. The algebra $\mathcal{A}$ is of course invariant under the induced action of the bialgebra $U(\mathfrak{g})$.

We consider the following two general problems.

The first problem. Does there exist a deformation quantization, $\mathcal{A}_h$, of $\mathcal{A}$, which is invariant under the action of the quantum group $U_h(\mathfrak{g})$?

The second problem. Suppose $\mathcal{A}_t$ is a $U(\mathfrak{g})$ invariant quantization of $\mathcal{A}$. Does there exist a two parameter quantization, $\mathcal{A}_{t,h}$, of $\mathcal{A}$ such that $\mathcal{A}_{t,0} = \mathcal{A}_t$, which is invariant under $U_h(\mathfrak{g})$?

In this paper, we study the first and the second problems for two cases. The first case, when M is the coadjoint representation of a simple Lie group. The second case, when M is a semisimple orbit in this representation. This paper is motivated by papers [Do2] and [DGS] where we started to study these problems. In this paper we develop results of [Do2] and [DGS] and present some additional results.

The paper is organized as follows.

In Section 2 we recall some facts about quantum groups and related categories, which are essential for a strict formulation of our problems and for our approach to $U_h(\mathfrak{g})$

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invariant quantization of algebras. In particular, we use the Drinfeld category with non-trivial associativity constraint determined by an invariant element $\Phi_h \in U(\mathfrak{g})^{\otimes 3}[[h]]$ and show that the problem of $U_h(\mathfrak{g})$ invariant quantization is equivalent to the problem of deforming the function algebra in such a way that the deformed algebra to be G invariant and Φ_h associative (see Subsection 2.3).

Subsection 2.4 is very important for the paper. In this subsection we give, for all commutative algebras, a description of Poisson brackets related to one and two parameter $U_h(\mathfrak{g})$ invariant quantizations. We show the following. If $\mathcal{A}_h$ is a $U_h(\mathfrak{g})$ invariant quantization, the corresponding Poisson bracket, p, on M has to be a difference of two brackets, $p = f - r_M$. Here r_M is the so called r-matrix bracket obtained from a classical r-matrix $r \in \wedge^2\mathfrak{g}$ with the help of the action morphism $\mathfrak{g} \to \mathrm{Vect}(M)$. So, the Schouten bracket $[\![r_M, r_M]\!]$ is equal to the image φ_M of the invariant element $\varphi \in \wedge^3\mathfrak{g}$. The bracket f is $U(\mathfrak{g})$ invariant and such that $[\![f, f]\!] = -\varphi_M$. Of course, any invariant bracket, f, is compatible with r_M, so that $[\![p, p]\!] = 0$.

We see that for existence of the family $\mathcal{A}_h$ one needs existence of an invariant bracket f on M such that

$$[\![f, f]\!] = -\varphi_M. \tag{1.1}$$

Note that the manifold M endowed with the bracket $p = f - r_M$ is a Poisson manifold with a Poisson action of G, where G is considered to be the Poisson-Lie group with Poisson structure defined by r. We shall not use this fact in the paper.

Similarly, given a two parameter quantization, $\mathcal{A}_{t,h}$, a pair of compatible Poisson brackets is determined. These brackets are: the bracket $p = f - r_M$ considered above and a $U(\mathfrak{g})$ invariant Poisson bracket, s, the initial term of the $U(\mathfrak{g})$ invariant quantization $\mathcal{A}_t$. We may perceive the family $\mathcal{A}_{t,h}$ as a $U_h(\mathfrak{g})$ invariant quantization of the Poisson bracket s.

We assume that s is given in advance and determined, for example, by a G invariant simplectic structure on M. From the compatibility of p and s (this means $[\![p, s]\!] = 0$) follows that

$$[\![f, s]\!] = 0. \tag{1.2}$$

So, for existence of the family $\mathcal{A}_{t,h}$ one needs existence of an invariant bracket f on M such the both equations (1.1) and (1.2) hold.

Thus, our problems divide into two steps. The first step is looking for invariant brackets f on M satisfying either (1.1) (in case of the first problem) or both (1.1) and (1.2) (in case of the second problem). The second step is quantizing these brackets.

In Section 3 we consider the one and two parameter quantization on $M = \mathfrak{g}^*$, the coadjoint representation of a simple Lie algebra $\mathfrak{g}$. As a function algebra on $\mathfrak{g}^*$, we take the symmetric algebra $S\mathfrak{g}$. It turns out that the cases $\mathfrak{g} = sl(n)$ and $\mathfrak{g} \neq sl(n)$ are quite different.

We prove that for $\mathfrak{g} \neq sl(n)$ the two parameter family which is a $U_h(\mathfrak{g})$ invariant quantization of the Lie bracket on $S\mathfrak{g}$ does not exist. Moreover, as a conjecture we state that in this case even a one parameter $U_h(\mathfrak{g})$ invariant quantization of $S\mathfrak{g}$ does not exist.

In the case $\mathfrak{g} = sl(n)$, the two parameter quantization of $S\mathfrak{g}$ exists. Moreover, the picture looks like in the classical case. Recall that in the classical case, the natural one parameter $U(\mathfrak{g})$ invariant quantization of $S\mathfrak{g}$ is given by the family $(S\mathfrak{g})_t = T(\mathfrak{g})[t]/J_t$, where J_t is the ideal generated by the elements of the form $x \otimes y - \sigma(x \otimes y) - t[x,y]$, $x, y \in \mathfrak{g}$, σ is the permutation. By the PBW theorem, $(S\mathfrak{g})_t$ is a free module over $\mathbb{C}[t]$. We have $(S\mathfrak{g})_0 = S\mathfrak{g}$, so this family of quadratic-linear algebras gives a $U(\mathfrak{g})$ invariant quantization of $S\mathfrak{g}$. It is obvious that the Poisson bracket, s, related to this quantization is the Lie bracket on $\mathfrak{g}^*$.

We show that for $\mathfrak{g} = sl(n)$ this picture can be extended to the quantum case. Namely, there exist deformations, σ_h and $[\cdot,\cdot]_h$, of both the mappings σ and $[\cdot,\cdot]$ such that the two parameter family of algebras $(S\mathfrak{g})_{t,h} = T(\mathfrak{g})[[h]][t]/J_{t,h}$, where $J_{t,h}$ is the ideal generated by the elements of the form $x \otimes y - \sigma_h(x \otimes y) - t[x,y]_h$, $x, y \in \mathfrak{g}$, gives a $U_h(\mathfrak{g})$ invariant quantization of the Lie bracket s on $\mathfrak{g}^*$. In this case, the corresponding bracket f from (1.2) is a quadratic bracket which is, up to a factor, a unique nontrivial invariant map $\wedge^2\mathfrak{g} \to S^2\mathfrak{g}$.

Taking $t = 0$ we obtain the family $(S\mathfrak{g})_h$ which is a quadratic algebra over $\mathbb{C}[[h]]$. This algebra can be called the quantum symmetric algebra (or quantum polynomial algebra on $\mathfrak{g}^*$). We show (Subsection 3.4) that $(S\mathfrak{g})_h$ can be included in the deformed graded differential algebra (deformed de Rham complex). In Subsection 3.5 we prove that the family $(S\mathfrak{g})_{t,h}$ can be restricted on the maximal semisimple orbits in $\mathfrak{g}^*$ to give a two parameter quantization on these orbits.

In Section 4 we study the problems of one and two parameter quantization on semisimple orbits in $\mathfrak{g}^*$ for all simple Lie algebras $\mathfrak{g}$. First of all, we classify all the brackets f satisfying (1.1) and both (1.1) and (1.2) for s being the Kirillov-$\mathcal{A}_{t,h}$ (KKS) bracket on the orbit. After that, we construct quantizations of these brackets.

Let M be a semisimple orbit. In Subsection 4.1 we prove that the brackets f satisfying (1.1) form a $\dim H^2(M)$-dimensional variety. We give a description of this variety and prove (in Subsection 4.3) that almost all these brackets can be quantized. So, we obtain for M a $\dim H^2(M)$ parameter family of non-equivalent one parameter quantizations. Note that in [DG2] we have built one of these quantizations, the quantization of the so called Sklyanin-Drinfeld Poisson bracket.

It turns out that brackets f satisfying (1.1) and (1.2) exist not for all orbits. We call an orbit M *good* if there exists a bracket f satisfying (1.1) and (1.2) for the Kirillov-Kostant-Souriau (KKS) bracket s.

In Subsection 4.1 we give the following classification of the semisimple good orbits for all simple $\mathfrak{g}$, [DGS].

In the case $\mathfrak{g} = sl(n)$ all semisimple orbits are good. (Actually we prove that in this case all orbits are good.)

For $\mathfrak{g} \neq sl(n)$ all symmetric orbits (which are symmetric spaces) are good. In this case $\varphi_M = 0$, so r_M itself is a Poisson bracket compatible with s.

Only in the case $\mathfrak{g}$ of type D_n and E_6 (except of A_n) there are good orbits different from the symmetric ones. For such orbits $\varphi_M \neq 0$.

We show that brackets f on a good orbit satisfying (1.1) and (1.2), form a one parameter family.

In Subsection 4.2 we consider cohomologies of an invariant complex with the differential given by the Schouten bracket with the bivector f. These cohomologies are needed for our construction of quantization.

In Subsection 4.3 we construct one and two parameter quantizations for semisimple orbits. According to our approach, as a first step we construct a G invariant Φ_h associative quantization, i.e., a quantization in the Drinfeld category with non-trivial associativity constraint given by Φ_h. Note that the bracket f from (1.1) can be considered as a "Poisson bracket" in that category. As a second step, we make a passage to the category with trivial associativity to obtain the associative $U_h(\mathfrak{g})$ invariant quantization. We applied this method earlier for quantizing the function algebra on the highest weight orbits in irreducible representations of G, the algebra of sections of linear vector bundles over flag manifolds, and the function algebra on symmetric spaces, [DGM], [DG1], [DS1].

I put in the text some questions which naturally appeared by exposition. They are open (for me) and seem to be important.

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2 Preliminaries

2.1 Quantum groups

We shall consider quantum groups in sense of Drinfeld, [Dr2], as deformed universal enveloping algebras. If $U(\mathfrak{g})$ is the universal enveloping algebra of a complex Lie algebra $\mathfrak{g}$, then the quantum group (or quantized universal enveloping algebra) corresponding to $U(\mathfrak{g})$ is a topological Hopf algebra, $U_h(\mathfrak{g})$, over $\mathbb{C}[[h]]$, isomorphic to $U(\mathfrak{g})[[h]]$ as a topological $\mathbb{C}[[h]]$ module and such that $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(\mathfrak{g})$ as a Hopf algebra over $\mathbb{C}$. In particular, the deformed comultiplication in $U_h(\mathfrak{g})$ has the form

$$\Delta_h = \Delta + h\Delta_1 + o(h), \tag{2.1}$$

where Δ is the comultiplication in the universal enveloping algebra $U(\mathfrak{g})$. One can prove, [Dr2], that $\Delta_1 : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is such a map that $\Delta_1 - \sigma\Delta_1 = \delta$ (σ is the usual permutation) being restricted on $\mathfrak{g}$ gives a map $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ which is a 1-cocycle and defines the structure of a Lie coalgebra on $\mathfrak{g}$ (the structure of a Lie algebra on the dual space $\mathfrak{g}^*$). The pair $(\mathfrak{g}, \delta)$ is considered as a quasiclassical limit of $U_h(\mathfrak{g})$.

In general, a pair $(\mathfrak{g}, \delta)$, where $\mathfrak{g}$ is a Lie algebra and δ is such a 1-cocycle, is called a Lie bialgebra. It is proven, [EK], that any Lie bialgebra $(\mathfrak{g}, \delta)$ can be quantized, i.e., there exists a quantum group $U_h(\mathfrak{g})$ such that the pair $(\mathfrak{g}, \delta)$ is its quasiclassical limit.

A Lie bialgebra $(\mathfrak{g}, \delta)$ is said to be a coboundary one if there exists an element $r \in \wedge^2$, called the classical r-matrix, such that $\delta(x) = [r, \Delta(x)]$ for $x \in \mathfrak{g}$. Since δ defines a Lie coalgebra structure, r has to satisfy the so-called classical Yang-Baxter equation which can by written in the form

$$[\![r, r]\!] = \varphi, \tag{2.2}$$

where $[\![\cdot, \cdot]\!]$ stands for the Schouten bracket and $\varphi \in \wedge^3\mathfrak{g}$ is an invariant element. We denote the coboundary Lie bialgebra by $(\mathfrak{g}, r)$.

In case $\mathfrak{g}$ is a simple Lie algebra, the most known is the Sklyanin-Drinfeld r-matrix:

$$r = \sum_\alpha X_\alpha \wedge X_{-\alpha},$$

where the sum runs over all positive roots; the root vectors X_α are chosen is such a way that $(X_\alpha, X_{-\alpha}) = 1$ for the Killing form $(\cdot, \cdot)$. This is the only r-matrix of weight zero, [SS], and its quantization is the Drinfeld-Jimbo quantum group. A classification of all r-matrices for simple Lie algebras was given in [BD].

We are interested in the case when $\mathfrak{g}$ is a semisimple finite dimensional Lie algebra. In this case, from results of Drinfeld and Etingof and Kazhdan one can derive the following

Proposition 2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then*

a) any Lie bialgebra $(\mathfrak{g}, \delta)$ is a coboundary one;

b) the quantization, $U_h(\mathfrak{g})$, of any coboundary Lie bialgebra $(\mathfrak{g}, r)$ exists and is isomorphic to $U(\mathfrak{g})[[h]]$ as a topological $\mathbb{C}[[h]]$ algebra;

c) the comultiplication in $U_h(\mathfrak{g})$ has the form

$$\Delta_h(x) = F_h\Delta(x)F_h^{-1}, \qquad x \in U(\mathfrak{g}), \tag{2.3}$$

where $F_h \in U(\mathfrak{g})^{\otimes 2}[[h]]$ and can be chosen in the form

$$F_h = 1 \otimes 1 + \frac{h}{2}r + o(h). \tag{2.4}$$

Proof. a) follows from the fact that $H^1(\mathfrak{g}, \wedge^2\mathfrak{g}) = 0$. From the fact that $H^2(\mathfrak{g}, U(\mathfrak{g})) = 0$ follows that $U(\mathfrak{g})$ does not admit any nontrivial deformations as an algebra, (see [Dr1]), which proves b). From the fact that $H^1(\mathfrak{g}, U(\mathfrak{g})^{\otimes 2}) = 0$ follows that any deformation of the algebra morphism $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ appears as a conjugation of Δ. In particular, the comultiplication in $U_h(\mathfrak{g})$ looks like (2.3) with some F_h such that $F_0 = 1 \otimes 1$.

From the coassociativity of Δ_h follows that F_h satisfies the equation

$$(F_h \otimes 1) \cdot (\Delta \otimes id)(F_h) = (1 \otimes F_h) \cdot (id \otimes \Delta)(F_h) \cdot \Phi_h \tag{2.5}$$

for some invariant element $\Phi_h \in U(\mathfrak{g})^{\otimes 3}[[h]]$.

The element F_h satisfying (2.3) and (2.4) can be obtained by correction of some F_h only obeying (2.3), [Dr2]. This procedure also makes use simple cohomological arguments and essentially (2.5). This proves c). $\square$

From (2.5) follows that if F_h has the form (2.4), then the coefficient by h for Φ_h vanishes. Moreover, as a coefficient by h^2 one can take the element φ from (2.2), i.e.,

$$\Phi_h = 1 \otimes 1 \otimes 1 + h^2 \varphi + o(h^2). \tag{2.6}$$

In addition, from (2.5) follows that Φ_h satisfies the pentagon identity

$$(id^{\otimes 2} \otimes \Delta)(\Phi_h) \cdot (\Delta \otimes id^{\otimes 2})(\Phi_h) = (1 \otimes \Phi_h) \cdot (id \otimes \Delta \otimes id)(\Phi_h) \cdot (\Phi_h \otimes 1). \tag{2.7}$$

Question 2.1. Let $(\mathfrak{g}, r)$ be a coboundary Lie bialgebra. Does there exist a quantization of it, $U_h(\mathfrak{g})$, such that $U_h(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$ as a topological $\mathbb{C}[[h]]$ algebra and the comultiplication has the form (2.3)?

From [Dr4] follows that if $[\![r, r]\!] = 0$, the answer to this question is positive.

2.2 Categorical interpretation

It is known that the elements constructed above have a nice categorical interpretation. First, recall some facts about the Drinfeld algebras and the monoidal categories determined by them.

Let A be a commutative algebra with unit, B a unitary A-algebra. The category of representations of B in A-modules, i.e. the category of B-modules, will be a monoidal category if the algebra B is equipped with an algebra morphism, $\Delta : B \to B \otimes_A B$, called comultiplication, and an invertible element $\Phi \in B^{\otimes 3}$ such that Δ and Φ satisfy the conditions (see [Dr2])

$$(id \otimes \Delta)(\Delta(b)) \cdot \Phi = \Phi \cdot (\Delta \otimes id)(\Delta(b)), \quad b \in B, \tag{2.8}$$

$$(id^{\otimes 2} \otimes \Delta)(\Phi) \cdot (\Delta \otimes id^{\otimes 2})(\Phi) = (1 \otimes \Phi) \cdot (id \otimes \Delta \otimes id)(\Phi) \cdot (\Phi \otimes 1). \tag{2.9}$$

Define a tensor product functor for the category of B modules $\mathcal{C}$, denoted $\otimes_{\mathcal{C}}$ or simply $\otimes$ when there can be no confusion, in the following way: given B-modules M, N, $M \otimes_{\mathcal{C}} N = M \otimes_A N$ as an A-module. The action of B is defined by

$$b(m \otimes n) = (\Delta b)(m \otimes n) = b_1 m \otimes b_2 n,$$

where $\Delta b = b_1 \otimes b_2$ (we use the Sweedler convention of an implicit summation over an index). The element $\Phi = \Phi_1 \otimes \Phi_2 \otimes \Phi_3$ defines the associativity constraint,

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P),\ a_{M,N,P}((m \otimes n) \otimes p) = \Phi_1 m \otimes (\Phi_2 n \otimes \Phi_3 p).$$

Again the summation in the expression for Φ is understood. By virtue of (2.8) Φ induces an isomorphism of B-modules, and by virtue of (2.9) the pentagon identity for monoidal categories holds. We call the triple (B, Δ, Φ) a Drinfeld algebra. The definition is somewhat non-standard in that we do not require the existence of an antipode. The category $\mathcal{C}$ of B-modules for B a Drinfeld algebra becomes a monoidal category. When it becomes necessary to be more explicit we shall denote $\mathcal{C}(B, \Delta, \Phi)$.

Let (B, Δ, Φ) be a Drinfeld algebra and $F \in B^{\otimes 2}$ an invertible element. Put

$$\widetilde{\Delta}(b) = F\Delta(b)F^{-1}, \quad b \in B, \tag{2.10}$$

$$\widetilde{\Phi} = (1 \otimes F) \cdot (id \otimes \Delta)(F) \cdot \Phi \cdot (\Delta \otimes id)(F^{-1}) \cdot (F \otimes 1)^{-1}. \tag{2.11}$$

Then $\widetilde{\Delta}$ and $\widetilde{\Phi}$ satisfy (2.8) and (2.9), therefore the triple $(B, \widetilde{\Delta}, \widetilde{\Phi})$ also becomes a Drinfeld algebra. We say that it is obtained by twisting from (B, Δ, Φ). It has an equivalent monoidal category of modules, $\widetilde{\mathcal{C}}(B, \widetilde{\Delta}, \widetilde{\Phi})$. Note that the equivalent categories $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ consist of the same objects as B-modules, and the tensor products of two objects are isomorphic as A-modules. The equivalence $\mathcal{C} \to \widetilde{\mathcal{C}}$ is given by the pair (Id, F), where $\mathrm{Id} : \mathcal{C} \to \widetilde{\mathcal{C}}$ is the identity functor of the categories (considered without the monoidal structures, but only as categories of B-modules), and $F : M \otimes_{\mathcal{C}} N \to M \otimes_{\widetilde{\mathcal{C}}} N$ is defined by $m \otimes n \mapsto F_1 m \otimes F_2 n$ where $F_1 \otimes F_2 = F$.

We are interested in the case when $A = \mathbb{C}[[h]]$, $B = U(\mathfrak{g})[[h]]$ where $\mathfrak{g}$ is a complex semisimple Lie algebra. In this case, all tensor products over $\mathbb{C}[[h]]$ are completed in h-adic topology.

We have two nontrivial Drinfeld algebras. The first is $(U(\mathfrak{g})[[h]], \Delta, \Phi_h)$, with the usual comultiplication and Φ_h from (2.5). The condition (2.8) means the invariantness of Φ_h, while (2.9) coincides with (2.7). The second Drinfeld algebra is $(U(\mathfrak{g})[[h]], \Delta_h, \mathbf{1})$. It obtaines by twisting of the first one by the element F_h from (2.3). The equation (2.11) follows from (2.5). The pair (Id, F_h) defines an equivalence between the corresponding monoidal categories $\mathcal{C}(U(\mathfrak{g})[[h]], \Delta, \Phi_h)$ and $\mathcal{C}(U(\mathfrak{g})[[h]], \Delta_h, \mathbf{1})$. The last is the category of representations of the quantum group $U_h(\mathfrak{g})$.

It is clear that reduction modulo h defines a functor from either of these categories to the category of representations of $U(\mathfrak{g})$ and the equivalence just described reduces to the identity modulo h. In fact, both categories are $\mathbb{C}[[h]]$-linear extensions (or deformations) of the $\mathbb{C}$-linear category of representations of $\mathfrak{g}$. Ignoring the monoidal structure the extension is a trivial one, but the associator Φ_h in the first case and the comultiplication Δ_h in the second case make the extension non-trivial from the point of view of monoidal categories.

2.3 $U_h(\mathfrak{g})$ ivariant quantizations of algebras

Let (B, Δ, Φ) be a Drinfeld algebra. Assume $\mathcal{A}$ is a B-module with a multiplication $\mu : \mathcal{A} \otimes_A \mathcal{A} \to \mathcal{A}$ which is a homomorphism of A-modules. We say that μ is Δ invariant if

$$b\mu(x \otimes y) = \mu\Delta(b)(x \otimes y) \quad \text{for } b \in B,\ x, y \in \mathcal{A}, \tag{2.12}$$

and μ is Φ associative, if

$$\mu(\Phi_1 x \otimes \mu(\Phi_2 y \otimes \Phi_3 z))) = \mu(\mu(x \otimes y) \otimes z) \quad \text{for } x, y, z \in \mathcal{A}. \tag{2.13}$$

Note, that a B-module $\mathcal{A}$ equipped with Δ invariant and Φ associative multiplication is an associative algebra in the monoidal category $\mathcal{C}(B, \Delta, \Phi)$. If $(B, \widetilde{\Delta}, \widetilde{\Phi})$ is a Drinfeld algebra twisted by (2.10) and (2.11), then the algebra $\mathcal{A}$ may be transfered into the equivalent category $\widetilde{\mathcal{C}}(B, \widetilde{\Delta}, \widetilde{\Phi})$: the multiplication $\widetilde{\mu} = \mu F^{-1} : M \otimes_A M \to M$ is $\widetilde{\Phi}$-associative and invariant in the category $\widetilde{\mathcal{C}}$.

Let $\mathcal{A}$ be a $U(\mathfrak{g})$ invariant associative algebra, i.e., an algebra with $U(\mathfrak{g})$ invariant multiplication μ in sense of (2.12). A deformation (or quantization) of $\mathcal{A}$ is an associative algebra, $\mathcal{A}_h$, which is isomorphic to $\mathcal{A}[[h]] = \mathcal{A} \otimes \mathbb{C}[[h]]$ (completed tensor product) as a $\mathbb{C}[[h]]$-module, with multiplication in $\mathcal{A}_h$ having the form $\mu_h = \mu + h\mu_1 + o(h)$. The algebra $U(\mathfrak{g})[[h]]$ is clearly acts on the $\mathbb{C}[[h]]$ module $\mathcal{A}_h$.

We will study quantizations of $\mathcal{A}$ which will be invariant under the comultiplication Δ_h. In other words, $\mathcal{A}_h$ will be an algebra in the category of representations of the quantum group $U_h(\mathfrak{g})$. It is clear from the previous Subsection that if $\mathcal{A}_h$ is such a quantization, then the multiplication $\mu_h F_h$ makes the module $\mathcal{A}[[h]]$ into an algebra in the category $\mathcal{C}(U(\mathfrak{g})[[h]], \Delta, \Phi_h)$, i.e., this multiplication is $U(\mathfrak{g})$ invariant and Φ_h associative.

We shall see that often it is easier to constract $U(\mathfrak{g})$ invariant and Φ_h associative quantization of $\mathcal{A}$. After that, the ivariant quantization with respect to any quantum group from Proposition 2.1 can be obtained by twisting by the appropriate F_h.

As an algebra $\mathcal{A}$ we may take an algebra $\mathcal{A}_t$ that is itself a $U(\mathfrak{g})$ invariant quantization of a commutative algebra $\mathcal{A}$. In this case, a $U_h(\mathfrak{g})$ invariant quantization of $\mathcal{A}_t$ is an algebra $\mathcal{A}_{t,h}$ over $\mathbb{C}[[t, h]]$.

2.4 Poisson brackets associated with the $U_h(\mathfrak{g})$ invariant quantization

Let $\mathcal{A}$ be a $U(\mathfrak{g})$ invariant commutative algebra with multiplication μ and $\mathcal{A}_h$ its quantization with multiplication $\mu_h = \mu + h\mu_1 + o(h)$. The Poisson bracket corresponding to the quantization is given by $\{a, b\} = \mu_1(a, b) - \mu_1(b, a)$, $a, b \in \mathcal{A}$.

In general, we call a skew-symmetric bilinear form $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ a bracket, if it satisfies the Leibniz rule in either argument when the other is fixed. The term Poisson bracket indicates that the Jacobi identity is also true.

A bracket of the form

$$\{a,b\}_r = (r_1 a)(r_2 b) = \mu r(a\otimes b) \qquad a,b\in\mathcal{A}, \tag{2.14}$$

where $r = r_1\otimes r_2$ (summation implicit) is the representation of r-matrix r, will be called an r-matrix bracket.

Assume $\mathcal{A}_h$ is a $U_h(\mathfrak{g})$ invariant quantization, i.e., the multiplicatin μ_h is Δ_h invariant. We shall show that in this case the Poisson bracket $\{\cdot,\cdot\}$ has a special form. Suppose f and g are two brackets on $\mathcal{A}$. Define their Schouten bracket $[\![f,g]\!]$ as

$$[\![f,g]\!](a,b,c) = f(g(a,b),c) + g(f(a,b),c) + \text{cyclic permutations of } a,b,c. \tag{2.15}$$

Then $[\![f,g]\!]$ is a skew-symmetric map $\mathcal{A}^{\otimes 3}\to\mathcal{A}$. We call f and g compatible if $[\![f,g]\!]=0$.

Proposition 2.2. *Let $\mathcal{A}$ be a $U(\mathfrak{g})$ invariant commutative algebra and $\mathcal{A}_h$ a $U_h(\mathfrak{g})$ invariant quantization. Then the corresponding Poisson bracket has the form*

$$\{a,b\} = f(a,b) - \{a,b\}_r \tag{2.16}$$

where $f(a,b)$ is a $U(\mathfrak{g})$ invariant bracket.

The brackets f and $\{\cdot,\cdot\}_r$ are compatible and $[\![f,f]\!] = -\varphi_{\mathcal{A}}$, where $\varphi_{\mathcal{A}}(a,b,c) = (\varphi_1 a)(\varphi_2 b)(\varphi_3 c)$ and $\varphi_1\otimes\varphi_2\otimes\varphi_3 = \varphi\in\wedge^3\mathfrak{g}$ is the invariant element from (2.2).

Proof. Let the comultiplication for $U_h(\mathfrak{g})$ have the form (2.1). Let $\mathcal{A}$ be a commutative algebra with the $U(\mathfrak{g})$ invariant multiplication μ. Suppose $\mathcal{A}_h$ is a $U_h(\mathfrak{g})$ invariant quantization of $\mathcal{A}$. This means that the deformed multiplication has the form

$$\mu_h = \mu + h\mu_1 + o(h) \tag{2.17}$$

and satisfies the relation

$$x\mu_h(a\otimes b) = \mu_h\Delta_h(x)(a\otimes b) \qquad \text{for} \qquad x\in U(\mathfrak{g}),\ a,b\in\mathcal{A}. \tag{2.18}$$

Substituting (2.1) and (2.17) in (2.18) and collecting the terms by h we obtain

$$\mu_1(a\otimes b) = \mu\Delta(x)(a\otimes b) + m\Delta_1(x)(a\otimes b).$$

Subtracting from this equation the similar one with permuting a and b and making use that Δ is commutative and $\delta = \Delta_1 - \sigma\Delta_1$ is skew-commutative, we derive that the Poisson bracket $p = \{\cdot,\cdot\}$ has to satisfy the property

$$xp(a\otimes b) = p\Delta(x)(a\otimes b) + \mu\delta(x)(a\otimes b), \qquad x\in U(\mathfrak{g}). \tag{2.19}$$

Let us prove that the bracket $f(a,b) = \{a,b\} + \{a,b\}_r$ is $U(\mathfrak{g})$ invariant. Indeed, from (2.14) we have for $x\in U(\mathfrak{g})$, $a,b\in\mathcal{A}$

$$x\mu r(a\otimes b) = \mu\Delta(x)r(a\otimes b) = \mu r\Delta(x)(a\otimes b) - \mu[r,\Delta(x)](a\otimes b).$$

Using this expression, (2.19), and the fact that $\delta(x) = [r, \Delta(x)]$, we obtain

$$xf = xp + x\mu r = (p\Delta(x) + \mu[r, \Delta(x)]) + (\mu r\Delta(x) - \mu[r, \Delta(x)]) =$$
$$= p\Delta(x) + \mu r\Delta(x) = f\Delta(x),$$

which proves the invariantness of f.

So, we have $\{a, b\} = f(a, b) - \{a, b\}_r$, as required.

It is easy to check that any bracket of the form $\{a, b\} = (X_1 a)(X_2 b) = \mu(X_1 a, X_2 b)$, for $X_1 \otimes X_2 \in \mathfrak{g} \wedge \mathfrak{g}$, is compatible with any invariant bracket. In particular, an r-matrix bracket is compatible with f. In addition, $\{\cdot, \cdot\}$ is a Poisson bracket, so its Schouten bracket with itself is equal to zero. Using this and the fact that the Schouten bracket of r-matrix bracket with itself is equal to $\varphi_{\mathcal{A}}$, we obtain from (2.16) that $[\![f, f]\!] = -\varphi_{\mathcal{A}}$. □

Remark 2.1. Let $\mathcal{A}$ be the function algebra on a G-manifold M, where the Lie group G corresponds to the Lie algebra $\mathfrak{g}$. It is easy to see that condition (2.19) with $\delta(x) = [r, \Delta(x)]$ is equivalent to the condition that the pair (M, p) becomes a $(G, \tilde{r})$-Poisson manifold, where $\tilde{r}$ is the Poisson structure on G defined by the r-matrix r: $\tilde{r} = r' - r''$, where r' and r'' are the left- and right-invariant bivector fields on G corresponding to r. It is known that $\tilde{r}$ makes G into a Poisson-Lie group. So Proposition 2.2 gives a description of Poisson structures p on M making (M, p) into a $(G, \tilde{r})$-Poisson manifold.

We shall also consider two parameter quantizations of algebras. A two parameter quantization of an algebra $\mathcal{A}$ is an algebra $\mathcal{A}_{t,h}$ isomorphic to $\mathcal{A}[[t, h]]$ as a $\mathbb{C}[[t, h]]$ module and having a multiplication in the form

$$\mu_{t,h} = \mu + t\mu_1' + h\mu_1'' + o(t, h).$$

With such a quantization, one associates two Poisson brackets: the bracket $s(a, b) = \mu_1'(a, b) - \mu_1'(b, a)$ along t, and the bracket $p(a, b) = \mu_1''(a, b) - \mu_1''(b, a)$ along h. It is easy to check that p and s are compatible Poisson brackets, i.e., the Schouten bracket $[\![p, s]\!] = 0$.

A pair of compatible Poisson brackets we call a Poisson pencil.

Corollary 2.1. *Let $\mathcal{A}_{t,h}$ be a two parameter $U_h(\mathfrak{g})$ invariant quantization of a commutative algebra $\mathcal{A}$ such that $\mathcal{A}_{t,0}$ is a one parameter $U(\mathfrak{g})$ invariant quantization of $\mathcal{A}$ with Poisson bracket s. Then the $U_h(\mathfrak{g})$ invariant quantization $\mathcal{A}_{0,h}$ has a Poisson bracket p of the form (2.16): $p = f - \{\cdot, \cdot\}_r$, where f is an invariant bracket such that $[\![f, f]\!] = -\varphi_{\mathcal{A}}$ and compatible with s, i.e.,*

$$[\![f, s]\!] = 0. \tag{2.20}$$

Proof. For the two parameter quantization, the Poisson brackets p and s form a Poisson pencil, hence must be compatible. Also, s is a $U(\mathfrak{g})$ invariant bracket, so that s is compatible with the r-matrix bracket $\{\cdot, \cdot\}_r$. It follows from (2.16) that s has to be compatible with f. □

In what follows, we shall often call $\mathcal{A}_{t,h}$ a $U_h(\mathfrak{g})$ invariant quantization (or double quantization) of the invariant Poisson bracket s, or of the Poisson pencil s and p.

Remark 2.2. As we have seen in Subsection 2.3, to construct a $U_h(\mathfrak{g})$ invariant quantization of $\mathcal{A}$ is the same that to construct a $U(\mathfrak{g})$ invariant Φ_h associative quantization of $\mathcal{A}$. We shall see that the last problem often turns out to be simpler (see Subsection 4.3). We observe that if $p = f - \{\cdot,\cdot\}_r$ is an admissible Poisson bracket for $U_h(\mathfrak{g})$ invariant quantization, then the invariant bracket f with the property $[\![f,f]\!] = -\varphi_{\mathcal{A}}$ may be considered as a "Poisson bracket" of quantization in the category with Φ_h defining the associativity constraint. Also, the pair f, s is a Poisson pencil in that category.

3 Double quantization on coadjoint representations

In this section we study a two parameter (or double) quantization on coadjoint representations of simple Lie algebras.

Let $\mathfrak{g}$ be a complex Lie algebra. Then, the symmetric algebra $S\mathfrak{g}$ can be considered as a function algebra on $\mathfrak{g}^*$. The algebra $U(\mathfrak{g})$ is included in the family of algebras $(S\mathfrak{g})_t = T(\mathfrak{g})[t]/J_t$, where J_t is the ideal generated by the elements of the form $x\otimes y - \sigma(x\otimes y) - t[x,y]$, $x,y\in\mathfrak{g}$, σ is the permutation. By the PBW theorem, $(S\mathfrak{g})_t$ is a free module over $\mathbb{C}[t]$. We have $(S\mathfrak{g})_0 = S\mathfrak{g}$, so this family of quadratic-linear algebras gives a $U(\mathfrak{g})$ invariant quantization of $S\mathfrak{g}$ by the Lie bracket s.

It turns out that for $\mathfrak{g} = sl(n)$ this picture can be extended to the quantum case, [Do2]. Namely, there exist deformations, σ_h and $[\cdot,\cdot]_h$, of both the mappings σ and $[\cdot,\cdot]$ such that the two parameter family of algebras $(S\mathfrak{g})_{t,h} = T(\mathfrak{g})[[h]][t]/J_{t,h}$, where $J_{t,h}$ is the ideal generated by the elements of the form $x\otimes y - \sigma_h(x\otimes y) - t[x,y]_h$, $x,y\in\mathfrak{g}$, gives a $U_h(\mathfrak{g})$ invariant quantization of the Lie bracket s on $\mathfrak{g}^*$. In this case, the corresponding bracket f from (2.20) is a quadratic bracket which is, up to a factor, a unique nontrivial invariant map $\wedge^2\mathfrak{g}\to S^2\mathfrak{g}$.

We shall show that for other simple Lie algebras, double quantizations of the Lie brackets do not exist.

We give two constructions of the algebra $(S\mathfrak{g})_{t,h}$. The first construction uses an idea from the paper [LS] on a quantum analog of Lie algebra for $sl(n)$. The second construction using the so called reflection equations (RE), [KS], [Maj], is presented in Remark 3.4..

3.1 Quantum Lie algebra for $U_h(sl(n))$

Let $U_h(\mathfrak{g})$ be a quantized universal enveloping algebra for a Lie algebra $\mathfrak{g}$. We consider $U_h(\mathfrak{g})$ as a $U_h(\mathfrak{g})$ module with respect to the left adjoint action: $\mathrm{ad}(x)y = x_1 y\gamma(x_2)$, where $x,y\in U_h(\mathfrak{g})$, $\Delta_h(x) = x_1\otimes x_2$ (summation implicit).

There were attemptions to define quantum Lie algebras as deformed standard classical embeddings of $\mathfrak{g}$ into $U_h(\mathfrak{g})$ obeying some additional properties, [DG], [LS].

In the classical case, there is probably the following way (not using comultiplication) to distinguish the standard embedding $\mathfrak{g} \to U(\mathfrak{g})$ from other invariant embeddings: with respect to this embedding $U(\mathfrak{g})$ is a quadratic-linear algebra. So, we give the following (working) definition of quantum Lie algebras.

Definition 3.1. Let $\mathfrak{g}_h$ be a subrepresentation of $U_h(\mathfrak{g})$, which is a deformation of the standard embedding of $\mathfrak{g}$ in $U(\mathfrak{g})$. We call $\mathfrak{g}_h$ a quantum Lie algebra, if the kernel of the induced homomorphism $T(\mathfrak{g}_h) \to U_h(\mathfrak{g})$ is defined by (deformed) quadratic-linear relations.

We are going to show that the quantum Lie algebra exists in case $\mathfrak{g} = sl(n)$. On the other hand, if such an algebra exists for some Lie algebra $\mathfrak{g}$, then a double quantization of the Lie bracket on $\mathfrak{g}^*$ also exists. But, as we shall see, no double quantization exists for simple $\mathfrak{g} \neq sl(n)$. So, among simple finite dimensional Lie algebras, only $sl(n)$ has a quantum Lie algebra in our sense.

Our construction is the following. Let $R = R'_i \otimes R''_i \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ (completed tensor product) be the R-matrix (summation by i is assumed). It satisfies the properties [Dr2]

$$\Delta'_h(x) = R\Delta_h(x)R^{-1}, \quad x \in U_h(\mathfrak{g}), \tag{3.1}$$

where Δ_h is the comultiplication in $U_h(\mathfrak{g})$ and Δ'_h is the opposite one,

$$\begin{aligned} (\Delta_h \otimes 1)R &= R^{13}R^{23} = R'_i \otimes R'_j \otimes R''_i R''_j \\ (1 \otimes \Delta_h)R &= R^{13}R^{12} = R'_i R'_j \otimes R''_j \otimes R''_i, \end{aligned} \tag{3.2}$$

and

$$(1 \otimes \varepsilon)R = (\varepsilon \otimes 1)R = 1 \otimes 1, \tag{3.3}$$

where ε is the counit in $U_h(\mathfrak{g})$.

Consider the element $Q = Q'_i \otimes Q''_i = R^{21}R$. It follows from (3.1) that Q commutes with elements from $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ of the form $\Delta_h(x)$. This is equivalent for Q to be invariant under the adjoint action of $U_h(\mathfrak{g})$ on $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$.

Let V be an irreducible finite dimensional representation of $U_h(\mathfrak{g})$ and $\rho : U_h(\mathfrak{g}) \to \mathrm{End}(V)$ the corresponding map of algebras. Consider the dual space $\mathrm{End}(V)^*$ as a left $U_h(\mathfrak{g})$ module setting

$$(x\varphi)(a) = \varphi(\gamma(x_{(1)})ax_{(2)}),$$

where $\varphi \in \mathrm{End}(V)^*$, $a \in \mathrm{End}(V)$, $\Delta_h(x) = x_{(1)} \otimes x_{(2)}$ in Sweedler notions, and γ denotes the antipode in $U_h(\mathfrak{g})$.

Consider the map

$$f : \mathrm{End}(V)^* \to U_h(\mathfrak{g}) \tag{3.4}$$

defined as $\varphi \mapsto \varphi(\rho(Q_i')Q_i''$. From the invariance of Q it follows that f is a $U_h(\mathfrak{g})$ equivariant map, so $\overline{L} = \mathrm{Im}(f)$ is a $U_h(\mathfrak{g})$ submodule.

It follows from (3.2) that $\overline{L}$ is a left coideal in $U_h(\mathfrak{g})$, i.e., $\Delta(x) \in U_h(\mathfrak{g}) \otimes \overline{L}$ for any $x \in \overline{L}$. Indeed, $Q = R_i''R_j' \otimes R_i'R_j''$. Applying (3.2) we obtain

$$(1 \otimes \Delta_h)R^{21}R = R_i''R_j''R_k'R_l' \otimes R_i'R_l'' \otimes R_j'R_k'' \tag{3.5}$$

Let $\varphi \in \mathrm{End}(V)^*$. Define $\psi_{il} \in \mathrm{End}(V)^*$ setting $\psi_{il}(a) = \varphi(R_i''aR_l')$ for $a \in \mathrm{End}(V)$. Then $\Delta\varphi(R_i''R_j')R_i'R_j'' = R_i'R_l'' \otimes \psi_{il}(R_j''R_k')R_j'R_k''$, which obviously belongs to $U_h(\mathfrak{g}) \otimes \overline{L}$.

Recall, [Dr2], that $R = F_h^{21}e^{\frac{h}{2}\mathbf{t}}F_h^{-1}$. Here $\mathbf{t} = \sum_i t_i \otimes t_i$ is the split Casimir, where t_i form an orthonormal basis in $\mathfrak{g}$ with respect to the Killing form, $F = 1 \otimes 1 + \frac{h}{2}r + o(h)$ (see (2.4)), and r is a classical r-matrix. Therefore,

$$Q = R^{21}R = Fe^{h\mathbf{t}}F^{-1} = 1 \otimes 1 + h\mathbf{t} + \frac{h^2}{2}(\mathbf{t}^2 + [r, \mathbf{t}]) + o(h^2). \tag{3.6}$$

Denote by Tr the unique (up to a factor) invariant element in $\mathrm{End}(V)^*$. Let $Z_0 = \rho_0(\mathfrak{g})$, and denote by Z_h some $U_h(\mathfrak{g})$ invariant deformation of Z_0 in $\mathrm{End}(V)$. Then we have a decomposition $\mathrm{End}(V) = I \oplus Z_h \oplus W$, where I is the one dimensional invariant subspace generated by the identity map, W is a complement to $I \oplus Z_h$ invariant subspace. This gives a decomposition $\mathrm{End}(V)^* = I^* \oplus Z_h^* \oplus W^*$ where W^* consists of all the elements which are equal to zero on $I \oplus Z_h$. The space I^* is generated by Tr, and after normalizing in such a way that $\mathrm{Tr}(\mathrm{id}) = 1$, we obtain that $C_V = f(\mathrm{Tr})$ is of the form

$$C_V = \mathrm{Tr}(\rho(Q_i'))Q_i'' = 1 + h^2c + o(h^2), \tag{3.7}$$

where c is an invariant element of $U(\mathfrak{g})$. It follows from (3.3) that $\varepsilon(C) = 1$.

From (3.6) follows that the elements of $f(Z_h^*)$ have the form

$$z = hx + o(h), \quad x \in \mathfrak{g}, \tag{3.8}$$

hence the subspace $L_1 = h^{-1}f(Z_h^*)$ forms a subrepresentation of $U_h(\mathfrak{g})$ with respect to the left adjoint action of $U_h(\mathfrak{g})$ on itself, which is a deformation of the standard embedding of $\mathfrak{g}$ into $U(\mathfrak{g})$. It follows from (3.3) that $\varepsilon(L_1) = 0$.

The elements from $f(W^*)$ have the form $w = h^2b + o(h^2)$ and $\varepsilon(W^*) = 0$. Denote $L_2 = h^{-2}f(W^*)$.

So, $\overline{L} = \mathbb{C}C_V \oplus hL_1 \oplus h^2L_2 = \mathbb{C}C_V + hL$, where $L = L_1 \oplus hL_2$. Since $\overline{L}$ is a left coideal in $U_h(\mathfrak{g})$, for any $x \in \overline{L}$ we have

$$\Delta_h(x) = x_{(1)} \otimes x_{(2)} = z \otimes C_V + v \otimes x',$$

where $z, v \in U_h(\mathfrak{g})$, $x' \in L$. Applying to the both hand sides $(1 \otimes \varepsilon)$ and multiplying we obtain $x = x_{(1)}\varepsilon(x_{(2)}) = z\varepsilon(C_V) + v\varepsilon(x') = z$. So, z has to be equal to x. and we obtain

$$\Delta_h(x) = x_{(1)} \otimes x_{(2)} = x \otimes C_V + v \otimes x', \quad x, x' \in L. \tag{3.9}$$

From (3.9) we have for any $y \in L$

$$xy = x_{(1)} y \gamma(x_{(2)}) x_{(3)} = x_{(1)} y \gamma(x_{(2)}) C_V + v_{(1)} y \gamma(v_{(2)}) x'. \tag{3.10}$$

Introduce the following maps:

$$\begin{aligned} \sigma'_h : \quad & L \otimes L \to L \otimes L, \qquad x \otimes y \mapsto v_{(1)} y \gamma(v_{(2)}) \otimes x', \\ [\cdot, \cdot]'_h : \quad & L \otimes L \to L, \qquad x \otimes y \mapsto x_{(1)} y \gamma(x_{(2)}). \end{aligned} \tag{3.11}$$

We may rewrite (3.10) in the form

$$m(x \otimes y - \sigma'_h(x \otimes y)) - [x, y]'_h C_V = 0. \tag{3.12}$$

Observe now that, as follows from (3.7), C_V is an invertible element in $U_h(\mathfrak{g})$. Put $P = C_V^{-1}$. Transfer the maps (3.11) to the space $P \cdot L$, i.e., define

$$\begin{aligned} \sigma_h(Px, Py) &= (P \otimes P)\sigma'_h(x, y), \\ [Px, Py]_h &= P[x, y]'_h. \end{aligned}$$

From (3.9) we obtain

$$P_{(1)} x_{(1)} \otimes P_{(2)} x_{(2)}) = P_{(1)} x \otimes P_{(2)} C_V + P_{(1)} v \otimes P_{(2)} x'. \tag{3.13}$$

Using this relation and taking into account that P commutes with all elements from $U_h(\mathfrak{g})$, we obtain as in (3.10)

$$\begin{aligned} PxPy \;&=\; P_{(1)} x_{(1)} P y \gamma(x_{(2)}) \gamma(P_{(2)}) P_{(3)} x_{(3)} = && (3.14) \\ & P_{(1)} x_{(1)} P y \gamma(x_{(2)}) \gamma(P_{(2)}) P_{(3)} C_V + P_{(1)} v_{(1)} P y \gamma(v_{(2)}) \gamma(P_{(2)}) P_{(3)} x' = && (3.15) \\ & P[x, y]'_h + P^2 m \sigma'_h(x \otimes y) = [Px, Py]_h + m\sigma_h(Px \otimes Py)). && (3.16) \end{aligned}$$

This equality may be written as

$$m(x \otimes y - \sigma_h(x \otimes y)) - [x, y]_h = 0, \quad x, y \in C_V^{-1} L. \tag{3.17}$$

Define $L_V = C_V^{-1} L$. Let $T(L_V) = \oplus_{k=0}^{\infty} L_V^{\otimes k}$ be the tensor algebra over L_V. Notice, that $T(L_V)$ is not supposed to be completed in h-adic topology. Let J be the ideal in $T(L_V)$ generated by the relations

$$(x \otimes y - \sigma_h(x \otimes y)) - [x, y]_h, \quad x, y \in L_V. \tag{3.18}$$

Due to (3.17) we have a homomorphism of algebras over $\mathbb{C}[[h]]$

$$\psi_h : T(L_V)/J \to U_h(\mathfrak{g}), \tag{3.19}$$

extending the natural embedding $L_V \to U_h(\mathfrak{g})$ of $U_h(\mathfrak{g})$ modules..

Now we can prove

Proposition 3.1. *For $\mathfrak{g} = sl(n)$ the quantum Lie algebra exists.*

Proof. Apply the above construction to $V = \mathbb{C}^n[[h]]$, the deformed basic representation of $\mathfrak{g}$. In this case $\mathrm{End}(V) = I \oplus Z_h$, where Z_h is a deformed adjoint representation. So, $\mathfrak{g}_h = L_V = h^{-1}C_V^{-1}f(Z_h^*)$ is a deformation of the standard embedding of $\mathfrak{g}$ in $U(\mathfrak{g})$. It is easy to see that in this case σ_h is a deformation of the usual permutation: $\sigma_0(x\otimes y) = y\otimes x$, and $[\cdot,\cdot]_h$ is a deformation of the Lie bracket on $\mathfrak{g}$: $[x,y]_0 = [x,y]$, $x,y \in \mathfrak{g} \subset U(\mathfrak{g})$. Hence, at $h = 0$, the quadratic-linear relations (3.18) are exactly the defining relations for $U(\mathfrak{g})$, therefore the map (3.19) is an isomorphism at $h = 0$. It follows that (3.19) is an embedding. (Actually, (3.19) is essentially an isomorphism, i.e., it is an isomorphism after completion of $T(L_V)$ in h-adic topology.) So, the kernel of the map $T(L_h) \to U_h(\mathfrak{g})$ is defined by the quadratic-linear relations (3.18). $\square$

Remark 3.1. Quadratic-linear relations (3.18) can be obtained in another way. Note that equation (3.5) may be rewritten as

$$(1 \otimes \Delta_h)Q = R_{21}Q_{13}R_{12}. \tag{3.20}$$

Since Q commutes with all elements of the form $\Delta_h(x)$, $x \in U_h(\mathfrak{g})$, one derives from (3.20):

$$Q_{23}R_{21}Q_{13}R_{12} = R_{21}Q_{13}R_{12}Q_{23}. \tag{3.21}$$

Consider the element $Q_\rho = \rho(Q_1) \otimes Q_2$ as a $\dim(V) \times \dim(V)$ matrix with the entries from $U_h(\mathfrak{g})$. Applying to (3.21) operator $\rho \otimes \rho \otimes 1$, we obtain the following relation for Q_ρ:

$$(Q_\rho)_2\overline{R}_{21}(Q_\rho)_1\overline{R} = \overline{R}_{21}(Q_\rho)_1\overline{R}(Q_\rho)_2, \tag{3.22}$$

where $\overline{R} = (\rho\otimes\rho)R$ is a number matrix, the Yang-Baxter operator in $V\otimes V$. Replacing in this equation $\overline{R}$ by $S = \sigma\overline{R}$, we obtain that the matrix Q_ρ satisfies the following reflection equation (RE):

$$(Q_\rho)_2S(Q_\rho)_2S = S(Q_\rho)_2S(Q_\rho)_2. \tag{3.23}$$

It is clear that the entries of the matrix Q_ρ generate the image of the map (3.4). From (3.7) follows that Q_ρ has the form

$$Q_\rho = \mathrm{Id}_V\, \mathcal{C}_V + hB', \tag{3.24}$$

where B' has the form $B' = \sum D_i \otimes b_i$, D_i belong to the complement to $\mathbb{C}\,\mathrm{Id}_V$ submodule in $\mathrm{End}(V)$ and $b_i \in U_h(\mathfrak{g})$. Note that the entries of the matrix B' form the subspace L, whereas the entries of $B = \mathcal{C}_V^{-1} B'$ form the subspace L_V from (3.17). From (3.24) we obtain

$$\mathcal{C}_V^{-1} Q_\rho = \mathrm{Id} + hB. \tag{3.25}$$

Since the element $\mathcal{C}_V^{-1}$ belongs to the center of $U_h(\mathfrak{g})$, the matrix $\mathcal{C}_V^{-1} Q_\rho$ obeys the RE (3.23) as well. So, B satisfies the relation

$$(\mathrm{Id} + hB)_2 S (\mathrm{Id} + hB)_2 S = S(\mathrm{Id} + hB)_2 S (\mathrm{Id} + hB)_2. \tag{3.26}$$

One checks that (3.26), considered as a qudratic-linear relations for indetermined entries of B, is equivalent to (3.18) in the case $\mathfrak{g} = sl(n)$.

3.2 Double quantization on $sl(n)^*$

Introduce a new variable, t, and consider a homomorphism of algebras, $T(L_V)[t] \to U_h(\mathfrak{g})[t]$, which extends the embedding $t \cdot \imath : L_V[t] \to U_h(\mathfrak{g})[t]$, where $\imath$ stands for the standard embedding $L_V \to U_h(\mathfrak{g})$. From (3.17) follows that $t \cdot \imath$ factors through the homomorphism of algebras over $\mathbb{C}[[h]][t]$

$$\phi_{t,h} : T(L_V)[t]/J_t \to U_h(\mathfrak{g})[t], \tag{3.27}$$

where J_t is the ideal generated by the relations

$$(x \otimes y - \sigma_h(x \otimes y)) - t[x, y]_h, \quad x, y \in L_V. \tag{3.28}$$

Proposition 3.2. *For $\mathfrak{g} = sl(n)$ the algebra $(S\mathfrak{g})_{t,h} = T(L_V)[t]/J_t$ is a double quantization of the Lie bracket on $S\mathfrak{g}$.*

Proof. Since in this case $L_V = \mathfrak{g}_h$, from Proposition 3.1 follows that (3.27) is a monomorphism at $t = 1$. Due to the PBW theorem the algebra $\mathrm{Im}(\phi_{t,h})$ at the point $h = 0$ is a free $\mathbb{C}[t]$-module and is equal to

$$(S\mathfrak{g})_t = T(\mathfrak{g})/\{x \otimes y - y \otimes x - t[x, y]\}. \tag{3.29}$$

For $t = 0$ this algebra is the symmetric algebra $S\mathfrak{g}$, the algebra of algebraic functions on $\mathfrak{g}^*$. For $t \neq 0$ this algebra is isomorphic to $U(\mathfrak{g})$. Since $U_h(\mathfrak{g})$ is a free $\mathbb{C}[[h]]$-module, it follows that $\phi_{t,h}$ in (3.27) is a monomorphism of algebras over $\mathbb{C}[[h]][t]$ and $\mathrm{Im}(\phi_{t,h})$ is a free $\mathbb{C}[[h]][t]$-module isomorphic to

$$(S\mathfrak{g})_{t,h} = T(\mathfrak{g}_h)[t]/\{x \otimes y - \sigma_h(x \otimes y) - t[x, y]_h\}. \tag{3.30}$$

It is clear that $(S\mathfrak{g})_t = (S\mathfrak{g})_{t,0}$ is the standard quantization of the Lie bracket on $\mathfrak{g}^*$. $\square$

Call the algebra

$$(S\mathfrak{g})_h = (S\mathfrak{g})_{0,h} = T(\mathfrak{g}_h)/\{x \otimes y - \sigma_h(x \otimes y)\} \tag{3.31}$$

a quantum symmetric algebra (or quantum polynomial algebra on $\mathfrak{g}^*$). It is a free $\mathbb{C}[[h]]$ module and a quadratic algebra equal to $S\mathfrak{g}$ at $h = 0$.

Remark 3.2. Up to now, all our constructions were considered for the quantum group in sense of Drinfeld, $U_h(\mathfrak{g})$, defined over $\mathbb{C}[[h]]$. But one can deduce the results above for the quantum group in sense of Lusztig, $U_q(\mathfrak{g})$, defined over the algebra $\mathbb{C}[q, q^{-1}]$. We show, for example, how to obtain the quantum symmetric algebra over $\mathfrak{g}$. Let E be a Grassmannian consisting of subspaces in $\mathfrak{g}\otimes\mathfrak{g}$ of dimension equal to $\dim(\wedge^2\mathfrak{g})$, and $\mathcal{Z}$ the closed algebraic subset of E consisting of subspaces J such that $\dim(E \otimes J \cap J \otimes E) \geq \dim(\wedge^3\mathfrak{g})$. Let $\mathcal{X}$ be the algebraic subset in $\mathcal{Z} \times (\mathbb{C} \setminus 0)$ consisting of points (J, q) such that J is invariant under the action of $U_q(\mathfrak{g})$. The projection $\pi : \mathcal{X} \to \mathbb{C} \setminus 0$ is a proper map. It is clear that the fiber of this projection over $q = 1$ contains the point corresponding to the symmetric algebra $S\mathfrak{g}$ as an isolated point, because there are no quadratic $U(\mathfrak{g})$ invariant Poisson brackets on $S\mathfrak{g}$.

As follows from the existence of $(S\mathfrak{g})_h$ (completed situation at $q = 1$), the dimension of $\mathcal{X}$ is equal to 1. Hence, the projection $\pi : \mathcal{X} \to \mathbb{C}\setminus 0$ is a covering. For $x \in \mathcal{X}$ let J_x be the corresponding subspace in $\mathfrak{g} \otimes \mathfrak{g}$ and $(S\mathfrak{g})_x = T(\mathfrak{g})/\{J_x\}$ the corresponding quadratic algebra. Due to the projection π, the family $(S\mathfrak{g})_x$, $x \in \mathcal{X}$, is a module over $\mathbb{C}[q, q^{-1}]$. Since J_x is $U_{p(x)}(\mathfrak{g})$ invariant, $(S\mathfrak{g})_x$ is a $U_{p(x)}(\mathfrak{g})$ invariant algebra. Hence, after possible deleting from $\mathcal{X}$ some countable set of points, we obtain a family of quadratic algebras with the same dimensions of graded components as $S\mathfrak{g}$. So, the family $(S\mathfrak{g})_x$, $x \in \mathcal{X}$ can be considered as a quantum symmetric algebra over $U_q(\mathfrak{g})$.

Note also that the family $(S\mathfrak{g})_h$ can be thought of as a formal section of the map $\pi : \mathcal{X} \to (\mathbb{C} \setminus 0)$ over the formal neighborhood of point $q = 1$. It follows that there is also an analytic section of π over some neighborhood, U, of the point $q = 1$. If $(S\mathfrak{g})_h$ is a quantization with Poisson bracket $f - \{\cdot,\cdot\}_r$ (see Proposition 2.2), then a quantization with Poisson bracket $-f - \{\cdot,\cdot\}_r$ gives another section of π over U. Hence, in a neighborhood of the "classical" point $x_0 \in \mathcal{X}$, $\pi(x_0) = 1$, the space $\mathcal{X}$ has a singularity of type "cross".

3.3 Poisson pencil corresponding to $(S\mathfrak{g})_{t,h}$

Let $\mathfrak{g} = sl(n)$ and $(S\mathfrak{g})_{t,h}$ be the double quantization from Proposition 3.2.

Proposition 3.3. *The Poisson pencil corresponding to the quantization $(S\mathfrak{g})_{t,h}$ consists of two compatible Poisson brackets:*

s (along t) is the Lie bracket;

p (along h) is a quadratic Poisson bracket of the form $p = f - \{\cdot,\cdot\}_r$, where f is an invariant quadratic bracket which is a unique up to a factor invariant map $f : \wedge^2\mathfrak{g} \to S^2\mathfrak{g}$,

and $\{\cdot,\cdot\}_r$ is the r-matrix bracket. Moreover, $[\![s,f]\!]=0$ and $[\![f,f]\!]=-\overline{\varphi}$, where $\overline{\varphi}$ has the form $\overline{\varphi}(a,b,c)=[\varphi_1,a][\varphi_2,b][\varphi_3,c]$, and $\varphi=\varphi_1\wedge\varphi_2\wedge\varphi_3=[\![r,r]\!]$. Recall that φ is a unique up to a factor invariant element of $\wedge^3\mathfrak{g}$.

Proof. That s coincides with the Lie bracket is obvious from (3.29). From Corollary 2.1 we have $p=f-\{\cdot,\cdot\}_r$. Since $(S\mathfrak{g})_h$ is a quadratic algebra over $\mathbb{C}[[h]]$, p must be a quadratic bracket. But the r-matrix bracket $\{\cdot,\cdot\}_r$ is quadratic, too. Hence, f must be a quadratic invariant bracket. There is only one possibility for such a bracket: it must be a unique (up to a factor) nontrivial invariant map $f:\wedge^2\mathfrak{g}\to S^2\mathfrak{g}$. Now apply Proposition 2.2 and Corollary 2.1. □

Consider now the quadratic bracket f in more detail.

We say that a k-vector field, g, on a manifold M is strongly restricted on a submanifold $N\subset M$ if at any point of N the polyvector g can be presented as an exterior power of tangent vectors to N.

Consider the coadjoint action of the Lie group $G=SL(n)$ on $\mathfrak{g}^*=sl(n)^*$. We want to prove that the bracket f is strongly restricted on any orbit of G in $\mathfrak{g}$. It turns out that there is the following general fact.

Proposition 3.4. *Let G be a semisimple Lie group with its Lie algebra $\mathfrak{g}$, $s=[\cdot,\cdot]$ the Lie bracket on $\mathfrak{g}^*$. Let $f=\{\cdot,\cdot\}$ be an invariant bracket on $\mathfrak{g}^*$ such that the Schouten bracket $[\![s,f]\!]$ is a three-vector field, ψ, strongly restricted on an orbit $\mathcal{O}$ of G. Then f is strongly restricted on $\mathcal{O}$.*

Proof. Let $x,y,z\in\mathfrak{g}$. The invariance condition for $\{\cdot,\cdot\}$ means:

$$[x,\{y,z\}]=\{[x,y],z\}+\{y,[x,z]\}. \tag{3.32}$$

The Schouten bracket $[\![s,f]\!]$ is:

$$[x,\{y,z\}]+[y,\{z,x\}]+[z,\{x,y\}]+$$
$$\{x,[y,z]\}+\{y,[z,x]\}+\{z,[x,y]\}=\psi(x,y,z).$$

In the left hand side of this expression, the 1-st, 5-th, and 6-th terms are canceled due to (3.32), and we have

$$[y,\{z,x\}]+[z,\{x,y\}]+\{x,[y,z]\}=\psi(x,y,z).$$

Putting in this equation instead of $[y,\{z,x\}]$ its expression from (3.32), i.e., $\{[y,z],x\}+\{z,[y,x]\}$, we obtain, since the term $\{x,[y,z]\}$ is canceled:

$$\{z,[x,y]\}=[z,\{x,y\}]+\psi(x,y,z). \tag{3.33}$$

Now observe that, due to the Leibniz rule, equation (3.33) is valid for any $z\in S\mathfrak{g}$. To prove the proposition, it is sufficient to show that if z belongs to the ideal $I_{\mathcal{O}}$ defining the orbit $\mathcal{O}$, then $\{z,u\}$ also belongs to this ideal. Again, due to the Leibniz rule, it is sufficient to show this for $u \in \mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, there are elements $x, y \in \mathfrak{g}$ such that $[x,y] = u$. We have from (3.33)

$$\{z,u\} = \{z,[x,y]\} = [z,\{x,y\}] + \psi(x,y,z).$$

But $[z,\{x,y\}] \in I_{\mathcal{O}}$, since the Lie bracket is restricted on any orbit, $\psi(x,y,z) \in I_{\mathcal{O}}$ by hypothesis of the proposition. So, $\{z,u\} \in I_{\mathcal{O}}$. □

As a consequence we obtain

Proposition 3.5. *Let $\mathfrak{g} = sl(n)$. Then the bracket f from Proposition 3.3 is strongly restricted on any orbit of $SL(n)$.*

Proof. Follows from Propositions 3.3 and 3.4. □

Remark 3.3. According to Remark 2.1, this Proposition shows that in case $G = SL(n)$ any orbit in coadjoint representation has a Poisson bracket $p = f - r_M$ such that the pair (M,p) becomes a $(G,\tilde{r})$-Poisson manifold.

Remark 3.4. Recall that in case $\mathfrak{g} = sl(n)$ the tensor square $\mathfrak{g}\otimes\mathfrak{g}$, considered as a representation of $\mathfrak{g}$, has a decomposition into irreducible components which are contained in $\mathfrak{g}\otimes\mathfrak{g}$ with multiplicity one, except of the component isomorphic to $\mathfrak{g}$ having multiplicity two. Moreover, both the symmetric and skew-symmetric parts of $\mathfrak{g}\otimes\mathfrak{g}$ contain components, $\mathfrak{g}^1$ and $\mathfrak{g}^2$, isomorphic to $\mathfrak{g}$. Hence, the bracket f takes $\mathfrak{g}^2$ onto $\mathfrak{g}^1$ and all the other components to zero.

For $\mathfrak{g}$ simple not equal to $sl(n)$, the decomposition of $\mathfrak{g}\otimes\mathfrak{g}$ is multiplicity free, hence non-trivial invariant maps $\wedge^2\mathfrak{g} \to S^2\mathfrak{g}$ do not exist at all. It follows that for $\mathfrak{g} \neq sl(n)$, there do not exist quadratic algebras $(S\mathfrak{g})_h$ which are $U_h(\mathfrak{g})$ invariant quantizations of $S\mathfrak{g}$.

Question 3.1. Prove that there exist no one parameter $U_h(\mathfrak{g})$ invariant quantizations of $S\mathfrak{g}$ (not necessarily in the class of quadratic algebras) for all simple Lie algebras $\mathfrak{g} \neq sl(n)$.

Now we prove that for simple $\mathfrak{g} \neq sl(n)$, the double quantization does not exist (not necessarily in the class of quadratic-linear algebras).

Proposition 3.6. *Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra not equal to $sl(n)$. Then a $U_h(\mathfrak{g})$ invariant quantization of the Lie bracket on $\mathfrak{g}^*$ does not exist.*

Proof. If such a quantization exists, then from Corollary 2.1 follows that there exists an invariant bracket f on $\mathfrak{g}^*$ such that $[\![s,f]\!] = 0$ and $[\![f,f]\!] = -\overline{\varphi}$. Here s is the Lie bracket and $\overline{\varphi}$ is the three-vector field induced by φ (see Proposition 3.3). We show that such f does not exist. Observe that $\overline{\varphi}$ has type $(3,3)$, i.e., is a sum of terms of the view $b\,\partial_x\wedge\partial_y\wedge\partial_z$, where b is a homogeneous polynomial of degree 3. Observe also that the Schouten bracket of two polyvector fields of degrees (i,j) and (k,l) is a polyvector field of degree $(i+k-1, j+l-1)$. We shall write i for degree (i,j) when the second number, j, is clear from context.

It is obvious that on $\mathfrak{g}$ there are no invariant bivector fields of degree 0 and, up to a factor, there is a unique invariant bivector field of degree 1, the Lie bracket s itself. Since $\mathfrak{g} \neq sl(n)$, there are no bivector fields of degree 2 (see Remark 3.4). Therefore, f must be of the form: $f = s + f_1$, where f_1 is a bracket of degree ≥ 3. Since f is compatible with s and $[\![f,f]\!] = -\overline{\varphi}$, it must be $[\![f_1,f_1]\!] = -\overline{\varphi}$. But it is impossible, because $[\![f_1,f_1]\!]$ has at least degree 5. $\square$

3.4 Quantum de Rham complex on $(sl(n))^*$

Consider the algebra $\Omega^\bullet$ of differencial forms on $\mathfrak{g}^*$ with polynomial coefficients. This is a graded differential algebra with differential d of degree 1 which forms the de Rham complex

(3.34)

where Ω^k is the space of k-forms with polynomial coefficients.

We call a complex over $\mathbb{C}[[h]]$

$$\Omega_h^\bullet : (S\mathfrak{g})_h \xrightarrow{d_h} \Omega_h^1 \xrightarrow{d_h} \Omega_h^2 \xrightarrow{d_h} \cdots \tag{3.35}$$

a quantum (deformed) de Rham complex if it consists of $U_h(\mathfrak{g})$ invariant topologically free modules over $\mathbb{C}[[h]]$ and coincides with (3.34) at $h=0$.

Proposition 3.7. *Let $\mathfrak{g} = sl(n)$. Then the quantized polynomial algebra $(S\mathfrak{g})_h$ from (3.31) can be included in a $U_h(\mathfrak{g})$ invariant graded differential algebra, $\Omega_h^\bullet$, which form a quantum de Rham complex (3.35).*

Proof. First of all, define a quantum exterior algebra, $(\Lambda\mathfrak{g})_h$, an algebra of differential forms with constant coefficients. Let us modify the operator σ_h from (3.31). Since the representation $\mathfrak{g}_h^*$ is isomorphic to $\mathfrak{g}_h$, there exists a $U_h(\mathfrak{g})$ invariant bilinear form on $\mathfrak{g}_h$, deformed Killing form. This form can be naturally extended to all tensor degrees $\mathfrak{g}_h^{\otimes k}$. Let W_h^2 be the $\mathbb{C}[[h]]$ submodule in $\mathfrak{g}_h \otimes \mathfrak{g}_h$ orthogonal to $V_h^2 = \mathrm{Im}(\mathrm{id} \otimes \mathrm{id} - \sigma_h)$. Define an operator $\bar{\sigma}_h$ on $\mathfrak{g}_h \otimes \mathfrak{g}_h$ in such a way that it has the eigenvalues -1 on V_h^2 and 1 on W_h^2. It is clear that V_h^2 and W_h^2 are deformed skew symmetric and symmetric subspaces of $\mathfrak{g} \otimes \mathfrak{g}$.

Now observe that the third graded component in the quadratic algebra $(S\mathfrak{g})_h$ is the quotient of $\mathfrak{g}_h^{\otimes 3}$ by the submodule $V_h^2 \otimes \mathfrak{g}_h + \mathfrak{g}_h \otimes V_h^2$, hence this submodule and, therefore, the submodule $V_h^2 \otimes \mathfrak{g}_h \cap \mathfrak{g}_h \otimes V_h^2$ are direct submodules in $\mathfrak{g}_h^{\otimes 3}$, i.e., they have complement submodules. As the complement submodules one can choose the submodules $W_h^2 \otimes \mathfrak{g}_h \cap \mathfrak{g}_h \otimes W_h^2$ and $W_h^2 \otimes \mathfrak{g}_h + \mathfrak{g}_h \otimes W_h^2$, respectively, since they are complement at the point

$h = 0$ and W_h^2 is orthogonal to V_h^2 with respect to the Killing form extended to $\mathfrak{g}_h \otimes \mathfrak{g}_h$. Hence, $W_h^2 \otimes \mathfrak{g}_h + \mathfrak{g}_h \otimes W_h^2$ is a direct submodule. Also, the symmetric algebra $S\mathfrak{g}$ is Koszul. From a result of Drinfeld, [Dr3] (see also [DM]), follows that the quadratic algebra $(\Lambda\mathfrak{g})_h = T(\mathfrak{g}_h)/\{W_h^2\}$ is a free $\mathbb{C}[[h]]$ module, i.e., is a $U_h(\mathfrak{g})$-invariant deformation of the exterior algebra $\Lambda\mathfrak{g}$.

Call $(\Lambda\mathfrak{g})_h$ a quantum exterior algebra over $\mathfrak{g}$.

Define a quantum algebra of differential forms over $\mathfrak{g}^*$ as the tensor product $\Omega_h^\bullet = (S\mathfrak{g})_h \otimes (\Lambda\mathfrak{g})_h$ in the tensor category of representations of the quantum group $U_h(\mathfrak{g})$. The multiplication of two elements $a\otimes\alpha$ and $b\otimes\beta$ looks like $ab_1\otimes\alpha_1\beta$, where $b_1\otimes\alpha_1 = S(\alpha\otimes b)$ for $S = \sigma R$ being the permutation in that category. So, $\Omega_h^k = (S\mathfrak{g})_h \otimes (\Lambda^k\mathfrak{g})_h$.

As in the classical case, the algebras $(S\mathfrak{g})_h$ and $(\Lambda\mathfrak{g})_h$ can be embedded in $T(\mathfrak{g}_h)$ as a graded submodules in the following way. Call the submodule $W_h^k = (W_h^2 \otimes \mathfrak{g}_h \otimes \cdots \otimes \mathfrak{g}_h) \cap (\mathfrak{g}_h \otimes W_h^2 \otimes \mathfrak{g}_h \otimes \cdots \otimes \mathfrak{g}_h) \cap \cdots \cap (\mathfrak{g}_h \otimes \mathfrak{g}_h \otimes \cdots \otimes W_h^2)$ of $T^k(\mathfrak{g}_h)$ a k-th symmetric part of $T(\mathfrak{g}_h)$. It is clear that the natural map $\pi_W : T(\mathfrak{g}_h) \to (S\mathfrak{g})_h$ restricted to W_h^k is a bijection onto the k-degree component $(S^k\mathfrak{g})_h$ of $(S\mathfrak{g})_h$. Denote by $\pi'_W : (S^k\mathfrak{g})_h \to W_h^k$ the inverse bijection. Similarly we define V_h^k, the k-th skew symmetric part of $T(\mathfrak{g}_h)$, and the bijection $\pi'_V : (\Lambda^k\mathfrak{g})_h \to V_h^k$.

Now, define a differential d_h in $\Omega_h^\bullet$ as a homogeneous operator of degree $(-1, 1)$. It acts on an element, $a \otimes \omega$, of degree (k, m) in the following way. Let $a \otimes \omega = (a_1 \otimes \cdots \otimes a_k) \otimes (\omega_1 \otimes \cdots \otimes \omega_m)$ be its realization as an element from $W_h^k \otimes V_h^m$. Then the formula

$$d_h(a \otimes \omega) = (a_1 \otimes \cdots \otimes a_{k-1} \otimes \pi'_V\pi_V(a_k \otimes \omega_1 \otimes \cdots \otimes \omega_m) \tag{3.36}$$

presents the element $d_h(a\otimes\omega)$ through its realization in $W_h^{k-1} \otimes V_h^{m+1}$. It is obvious that $d_h^2 = 0$.

So, the graded differential algebra $\Omega_h^\bullet$ is constructed. It is easy to see that at the point $h = 0$ this algebra coincides with $\Omega^\bullet$. □

Note that the quantum de Rham complex is exact, because it is exact at $h = 0$.

3.5 Restriction of $(S\mathfrak{g})_{t,h}$ on orbits

In this section $G = SL(n)$, $\mathfrak{g} = sl(n)$.

Let M be an invariant closed algebraic subset in $\mathfrak{g}^*$ and A the algebra of algebraic functions on M. The algebra A can be presented as a quotient of $S\mathfrak{g}$ by some ideal, $S\mathfrak{g} \to A \to 0$.

We say that the quantization $(S\mathfrak{g})_{t,h}$ can be restricted on M if there exists a $U_h(\mathfrak{g})$ invariant quantization, $A_{t,h}$, of A, which can be presented as a quotient of $(S\mathfrak{g})_{t,h}$ by some ideal, $(S\mathfrak{g})_{t,h} \to A_{t,h} \to 0$.

Note that, on the infinitesimal level, there are no obstructions for $(S\mathfrak{g})_{t,h}$ to be restricted on M. Indeed, the Lie bracket on $\mathfrak{g}^*$ is strongly restricted on any orbit of G

and induces the Kirillov-Kostant-Souriau bracket on M. Also , by Proposition 3.5, the bracket f involved in the quantization along h is also strongly restricted on any orbit.

From [DS1], one can derive that the problem of restriction of $(S\mathfrak{g})_{t,h}$ is solved positively in case M is a minimal semisimple orbit, i.e., M is a hermitian symmetric space.

We are going to show here that the problem also has a positive solution for M being a maximal semisimple orbit, i.e., if M can be defined as a set of zeros of invariant functions from $S\mathfrak{g}$. Such orbits are the orbits of diagonal matrices with distinct elements on diagonal.

Proposition 3.8. *Let $\mathfrak{g} = sl(n)$. Then the family $(S\mathfrak{g})_{t,h}$ can be restricted on any maximal semisimple orbit in $\mathfrak{g}^*$.*

Proof. There exists an isomorphism of $U_h(\mathfrak{g})$ modules $(S\mathfrak{g})_h \to W_h$, where $W_h = \oplus_k W_h^k$, the direct sum of the k-th symmetric parts of $T(\mathfrak{g}_h)$ (see previous Subsection). Consider the composition $W_h[t] \to T(\mathfrak{g}_h)[t] \to (S\mathfrak{g})_{t,h}$, where the last map appears from (3.30). It is an isomorphism, since it is an isomorphism at the point $h = 0$. It follows that $(S\mathfrak{g})_{t,h}$ is isomorphic to $W_h[t]$ as a $U_h(\mathfrak{g})$-module,

Denote by $\mathcal{I}_{t,h}$ the submodule of $U_h(\mathfrak{g})$ invariant elements in $(S\mathfrak{g})_{t,h}$. It is obvious that $\mathcal{I}_{t,h}$ is isomorphic to $\oplus_k \mathcal{I}_h^k[t]$, where $\mathcal{I}_h^k$ is the invariant submodule in W_h^k. Hence, $\mathcal{I}_{t,h}$ is a direct free $\mathbb{C}[[h]][t]$ submodule in $(S\mathfrak{g})_{t,h}$. Moreover, $\mathcal{I}_{t,h}$ is a central subalgebra in $(S\mathfrak{g})_{t,h}$. Indeed, for a generic t the algebra $(S\mathfrak{g})_{t,h}$ can be invariantly embedded in $U_h(\mathfrak{g})$. But $\mathrm{ad}(U_h(\mathfrak{g}))$ invariant elements in $U_h(\mathfrak{g})$ form the center of $U_h(\mathfrak{g})$. Also, $\mathcal{I}_{t,h}$ as an algebra is isomorphic to $\mathcal{I}[[h]][t]$ with the trivial action of $U_h(\mathfrak{g})$, where $\mathcal{I} = \mathcal{I}_{0,0}$, the algebra of invariant elements in $S\mathfrak{g}$. This follows from the fact that $\mathcal{I}$ is a polynomial algebra, [Dix], and, therefore, admits no nontrivial commutative deformations.

By the Kostant theorem, [Dix], $U(\mathfrak{g})$ is a free module over its center. It follows that at the point $h = 0$ the module $(S\mathfrak{g})_{t,0}$ is a free module over the algebra $\mathcal{I}_{t,0}$. One can easily derive from this that $(S\mathfrak{g})_{t,h}$ is a free module over $\mathcal{I}_{t,h}$.

Now, let the maximal semisimple orbit M be defined by invariant elements from $\mathcal{I}$. Consider a character defined by M, the algebra homomorphism $\lambda : \mathcal{I} \to \mathbb{C}$ which takes each element from $\mathcal{I}$ to its value on M. Then, $\mathbb{C}$ may be considered as an $\mathcal{I}$-module, and the function algebra A on M is equal to $S\mathfrak{g}/\operatorname{Ker}(\lambda)S\mathfrak{g} = S\mathfrak{g} \otimes_{\mathcal{I}} \mathbb{C}$. Extend the character λ up to a character $\lambda_{t,h} : \mathcal{I}_{h,t} \to \mathbb{C}[[h]][t]$ in the trivial way and consider $\mathbb{C}[[h]][t]$ as a $\mathcal{I}_{h,t}$-module. The tensor product over $\mathcal{I}_{t,h}$

$$A_{t,h} = (S\mathfrak{g})_{t,h} \otimes \mathbb{C}[[h]][t]$$

is a $\mathbb{C}[[h]][t]$-algebra. It is a free $\mathbb{C}[[h]][t]$-module, since $(S\mathfrak{g})_{t,h}$ is a free one over $\mathcal{I}_{t,h}$.

It is obvious, that $A_{0,0} = A$, $A_{t,0}$ gives a quantization of the KKS bracket on M, and $A_{t,h}$ is a quotient algebra of $(S\mathfrak{g})_{t,h}$. ☐

In a next paper we shall prove that the quantization $(S\mathfrak{g})_{t,h}$ can be restricted on all semisimple orbits.

Question 3.2. Can be the quantization $(S\mathfrak{g})_{t,h}$ restricted on all orbits (not necessarily semisimple)?

As we have seen, the corresponding Poisson brackets are strongly restricted on all the orbits.

In next Section we consider the $U_h(\mathfrak{g})$ invariant quantizations on semisimple orbits in $\mathfrak{g}^*$ for all simple Lie algebras $\mathfrak{g}$. It turns out that in general, on a given orbit there are many nonequivalent quantizations which are not restrictions from a quantization on $\mathfrak{g}^*$. From this point of view, the quantization on maximal orbits described by Proposition (3.8) is a distinguished one.

4 The one and two parameter quantization on semisimple orbits in $\mathfrak{g}^*$

4.1 Pairs of brackets on semisimple orbits

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h}$ a fixed Cartan subalgebra. Let $\Omega \subset \mathfrak{h}^*$ be the system of roots corresponding to $\mathfrak{h}$. Select a system of positive roots, Ω^+, and denote by $\Pi \subset \Omega$ the subset of simple roots. Fix an element $E_\alpha \in \mathfrak{g}$ of weight α for each $\alpha \in \Omega^+$ and choose $E_{-\alpha}$ such that

$$(E_\alpha, E_{-\alpha}) = 1 \tag{4.1}$$

for the Killing form $(\cdot, \cdot)$ on $\mathfrak{g}$.

Let Γ be a subset of Π. Denote by $\mathfrak{h}^*_\Gamma$ the subspace in $\mathfrak{h}^*$ generated by Γ. Note, that $\mathfrak{h}^* = \mathfrak{h}^*_\Gamma \oplus \mathfrak{h}^*_{\Pi\setminus\Gamma}$, and one can identify $\mathfrak{h}^*_{\Pi\setminus\Gamma}$ and $\mathfrak{h}^*/\mathfrak{h}^*_\Gamma$ via the projection $\mathfrak{h}^* \to \mathfrak{h}^*/\mathfrak{h}^*_\Gamma$.

Let $\Omega_\Gamma \subset \mathfrak{h}^*_\Gamma$ be the subsystem of roots in Ω generated by Γ, i.e., $\Omega_\Gamma = \Omega \cap \mathfrak{h}^*_\Gamma$. Denote by $\mathfrak{g}_\Gamma$ the subalgebra of $\mathfrak{g}$ generated by the elements $\{E_\alpha, E_{-\alpha}\}$, $\alpha \in \Gamma$, and $\mathfrak{h}$. Such a subalgebra is called the Levi subalgebra.

Let G be a complex connected Lie group with Lie algebra $\mathfrak{g}$ and G_Γ a subgroup with Lie algebra $\mathfrak{g}_\Gamma$. Such a subgroup is called the Levi subgroup. It is known that G_Γ is a connected subgroup. Let M be a homogeneous space of G and G_Γ be the stabilizer of a point $o \in M$. We can identify M and the coset space G/G_Γ. It is known, that such M is isomorphic to a semisimple orbit in $\mathfrak{g}^*$. This orbit goes through an element $\lambda \in \mathfrak{g}^*$ which is just the trivial extension to all of $\mathfrak{g}^*$ (identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form) of a map $\lambda : \mathfrak{h}_{\Pi\setminus\Gamma} \to \mathbb{C}$ such that $\lambda(\alpha) \neq 0$ for all $\alpha \in \Pi \setminus \Gamma$. Conversely, it is easy to show that any semisimple orbit in $\mathfrak{g}^*$ is isomorphic to the quotient of G by a Levi subgroup.

The projection $\pi : G \to M$ induces the map $\pi_* : \mathfrak{g} \to T_o$, where T_o is the tangent space to M at the point o. Since the ad-action of $\mathfrak{g}_\Gamma$ on $\mathfrak{g}$ is semisimple, there exists an $\mathrm{ad}(\mathfrak{g}_\Gamma)$-invariant subspace, $\mathfrak{m} = \mathfrak{m}_\Gamma$, of $\mathfrak{g}$ complementary to $\mathfrak{g}_\Gamma$, and one can identify T_o

and $\mathfrak{m}$ by means of π_*. It is easy to see that subspace $\mathfrak{m}$ is uniquely defined and has a basis formed by the elements $E_\gamma, E_{-\gamma}$, $\gamma \in \Omega^+ \setminus \Omega_\Gamma$.

Let $v \in \mathfrak{g}^{\otimes m}$ be a tensor over $\mathfrak{g}$. Using the right and the left actions of G on itself, one can associate with v right and left invariant tensor fields on G denoted by v^r and v^l.

We say that a tensor field, t, on G is right G_Γ invariant, if t is invariant under the right action of G_Γ. The G equivariant diffeomorphism between M and G/G_Γ implies that any right G_Γ invariant tensor field t on G induces tensor field $\pi_*(t)$ on M. The field $\pi_*(t)$ will be invariant on M if, in addition, t is left invariant on G, and any invariant tensor field on M can be obtained in such a way. Let $v \in \mathfrak{g}^{\otimes m}$. For v^l to be right G_Γ invariant it is necessary and sufficient that v to be $\mathrm{ad}(\mathfrak{g}_\Gamma)$ invariant. Denote $\pi^r(v) = \pi_*(v^r)$ for any tensor v on $\mathfrak{g}$ and $\pi^l(v) = \pi_*(v^l)$ for any $\mathrm{ad}(\mathfrak{g}_\Gamma)$ invariant tensor v on $\mathfrak{g}$. Note, that tensor $\pi^r(v)$ coincides with the image of v by the map $\mathfrak{g}^{\otimes m} \to \mathrm{Vect}(M)^{\otimes m}$ induced by the action map $\mathfrak{g} \to \mathrm{Vect}(M)$. Any G invariant tensor on M has the form $\pi^l(v)$. Moreover, v clearly can be uniquely chosen from $\mathfrak{m}^{\otimes m}$.

Denote by $[\![v, w]\!] \in \wedge^{k+l-1}\mathfrak{g}$ the Schouten bracket of the polyvectors $v \in \wedge^k\mathfrak{g}$, $w \in \wedge^l\mathfrak{g}$, defined by the formula

$$[\![X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l]\!] = \sum (-1)^{i+j}[X_i, Y_j] \wedge X_1 \wedge \cdots \hat{X}_i \cdots \hat{Y}_j \cdots \wedge Y_l,$$

where $[\cdot,\cdot]$ is the bracket in $\mathfrak{g}$. The Schouten bracket is defined in the same way for polyvector fields on a manifold, but instead of $[\cdot,\cdot]$ one uses the Lie bracket of vector fields. We will use the same notation for the Schouten bracket on manifolds. It is easy to see that $\pi^r([\![v, w]\!]) = [\![\pi^r(v), \pi^r(w)]\!]$, and the same relation is valid for π^l.

Denote by $\overline{\Omega}_\Gamma$ the image of Ω in $\mathfrak{h}^*_{\Pi\setminus\Gamma}$ without zero. It is clear that $\Omega_{\Pi\setminus\Gamma}$ can be identified with a subset of $\overline{\Omega}_\Gamma$ and each element from $\overline{\Omega}_\Gamma$ is a linear combination of elements from $\Pi \setminus \Gamma$ with integer coefficients which all are either positive or negative. Thus, the subset $\overline{\Omega}_\Gamma^+ \subset \overline{\Omega}_\Gamma$ of the elements with positive coefficients is exactly the image of Ω^+. We call elements of $\overline{\Omega}_\Gamma$ quasiroots and the images of $\Pi \setminus \Gamma$ simple quasiroots.

Proposition 4.1. *The space $\mathfrak{m}$ considered as a $\mathfrak{g}_\Gamma$ representation space decomposes into the direct sum of subrepresentations $\mathfrak{m}_{\bar\beta}$, $\bar\beta \in \overline{\Omega}_\Gamma$, where $\mathfrak{m}_{\bar\beta}$ is generated by all the elements E_β, $\beta \in \Omega$, such that the projection of β is equal to $\bar\beta$. This decomposition have the following properties:*

a) all $\mathfrak{m}_{\bar\beta}$ are irreducible;

b) for $\bar\beta_1, \bar\beta_2 \in \overline{\Omega}_\Gamma$ such that $\bar\beta_1 + \bar\beta_2 \in \overline{\Omega}_\Gamma$ one has $[\mathfrak{m}_{\bar\beta_1}, \mathfrak{m}_{\bar\beta_2}] = \mathfrak{m}_{\bar\beta_1+\bar\beta_2}$;

c) for any pair $\bar\beta_1, \bar\beta_2 \in \overline{\Omega}_\Gamma$ the representation $\mathfrak{m}_{\bar\beta_1} \otimes \mathfrak{m}_{\bar\beta_2}$ is multiplicity free.

Proof. Statements a) and b) are proven in [DGS]. Statement c) follows from the fact that all the weight subspaces for all $\mathfrak{m}_{\bar\beta}$ have the dimension one (see N.Bourbaki, Groupes et algèbres de Lie, Chap. 8.9, Ex. 14). $\square$

Since $\mathfrak{g}_\Gamma$ contains the Cartan subalgebra $\mathfrak{h}$, each $\mathfrak{g}_\Gamma$ invariant tensor over $\mathfrak{m}$ has to be of weight zero. It follows that there are no invariant vectors in $\mathfrak{m}$. Hence, there are no invariant vector fields on M.

Consider the invariant bivector fields on M. From the above, such fields correspond to the $\mathfrak{g}_\Gamma$ invariant bivectors from $\wedge^2\mathfrak{m}$. Note, that any $\mathfrak{h}$ invariant bivector from $\wedge^2\mathfrak{m}$ has to be of the form $\sum c(\alpha)E_\alpha \wedge E_{-\alpha}$.

Proposition 4.2. *A bivector $v \in \wedge^2\mathfrak{m}$ is $\mathfrak{g}_\Gamma$ invariant if and only if it has the form $v = \sum c(\alpha)E_\alpha \wedge E_{-\alpha}$ where the sum runs over $\alpha \in \Omega^+ \setminus \Omega_\Gamma$, and for two roots α, β which give the same element in $\mathfrak{h}^*/\mathfrak{h}_\Gamma^*$ one has $c(\alpha) = c(\beta)$.*

Proof. Follows from Proposition 4.1 and condition (4.1) ☐

This proposition shows, that coefficients of an invariant element $v = \sum c(\alpha)E_\alpha \wedge E_{-\alpha}$ depend only of the image of α in $\overline{\Omega}_\Gamma^+$, denoted $\bar{\alpha}$, so v can be written in the form $v = \sum c(\bar{\alpha})E_\alpha \wedge E_{-\alpha}$. Let $v \in \wedge^2\mathfrak{m}$ be of the form $v = \sum c(\bar{\alpha})E_\alpha \wedge E_{-\alpha}$, where the sum runs over $\alpha \in \Omega^+ \setminus \Omega_\Gamma$. Denote by θ the Cartan automorphism of $\mathfrak{g}$. Then, v is θ anti-invariant, i.e., $\theta v = -v$. Hence, any $\mathfrak{g}_\Gamma$ invariant bivector is θ anti-invariant. If $v, w \in \wedge^2\mathfrak{m}$ are $\mathfrak{g}_\Gamma$ invariant, then $[\![v, w]\!]$ is θ invariant and is of the form $[\![v, w]\!] = \sum e(\bar{\alpha}, \bar{\beta})E_{\alpha+\beta} \wedge E_{-\alpha} \wedge E_{-\beta}$ where roots α, β are both negative or both positive and $e(\bar{\alpha}, \bar{\beta}) = -e(-\bar{\alpha}, -\bar{\beta})$. Hence, to calculate $[\![v, w]\!]$ for such v and w it is sufficient to calculate coefficients $e(\bar{\alpha}, \bar{\beta})$ for positive $\bar{\alpha}$ and $\bar{\beta}$.

Define by φ_M the invariant three-vector field on M determined by the invariant element $\varphi \in \wedge^3\mathfrak{g}$. A direct computation shows (see [DGS]) that the Schouten bracket of bivector $v = \sum c(\bar{\alpha})E_\alpha \wedge E_{-\alpha}$ with itself is equal to $K^2\varphi_M$ for a complex number K, if and only if the following equations hold

$$c(\bar{\alpha} + \bar{\beta})(c(\bar{\alpha}) + c(\bar{\beta})) = c(\bar{\alpha})c(\bar{\beta}) + K^2 \tag{4.2}$$

for all the pairs of positive quasiroots $\bar{\alpha}, \bar{\beta}$ such that $\bar{\alpha} + \bar{\beta}$ is a quasiroot. So, if $c(\bar{\alpha})$ and $c(\bar{\beta})$ are given and $c(\bar{\alpha}) + c(\bar{\beta}) \neq 0$,

$$c(\bar{\alpha} + \bar{\beta}) = \frac{c(\bar{\alpha})c(\bar{\beta}) + K^2}{c(\bar{\alpha}) + c(\bar{\beta})}. \tag{4.3}$$

Proposition 4.3. *Let $(\bar{\alpha}_1, \cdots, \bar{\alpha}_k)$ be the k-tuple of all simple quasiroots. Given a k-tuple of complex numbers $(c_1, ..., c_k)$, assign to each $\bar{\alpha}_i$ the number c_i. Then*

a) for almost all k-tuples of complex numbers (except an algebraic subset in $\mathbb{C}^k$ of lesser dimension) equations (4.3) uniquely define numbers $c(\bar{\alpha})$ for all positive quasiroots $\bar{\alpha} = \sum \bar{\alpha}_i$ such that the bivector $f = \sum c(\bar{\alpha})E_\alpha \wedge E_{-\alpha}$ satisfies the condition

$$[\![f, f]\!] = K^2\varphi_M;$$

b) when $K = 0$, the solution described in part a) defines a Poisson bracket on M. Numbers $c(\bar{\alpha})$ give a solution of (4.2) if and only if there exists a linear form $\lambda \in \mathfrak{h}^*_{\Pi\setminus\Gamma}$ such that

$$c(\bar{\alpha}) = \frac{1}{\lambda(\bar{\alpha})} \tag{4.4}$$

for all quasiroots $\bar{\alpha}$.

Proof. See [DGS]. □

Remark 4.1. This proposition shows that invariant brackets f on M defined by part a) of the proposition form a k-dimensional variety, $\mathcal{X}_K$, where k is the number of simple quasiroots. On the other hand, $k = \dim H^2(M)$, [Bo]. If K is regarded as indeterminate, then f forms a $k+1$ dimensional variety, $\mathcal{X} \subset \mathbb{C}^k \times \mathbb{C}$, (component $\mathbb{C}$ corresponds to K). Subvariety $\mathcal{X}_0$ corresponds to $K = 0$, i.e., consists of Poisson brackets. It is easy to see that all the Poisson brackets with $c(\bar{\alpha}) = 1/\lambda(\bar{\alpha}) \neq 0$ are nondegenerate. Since $\mathcal{X}$ is connected, it follows that almost all brackets f (except an algebraic subset in $\mathcal{X}$ of lesser dimension) are nondegenerate as well.

Remark 4.2. Equations (4.3) show that when $c(\bar{\alpha}) + c(\bar{\beta}) = 0$, there appears a harm for determining $c(\bar{\alpha} + \bar{\beta})$ from given $c(\bar{\alpha})$ and $c(\bar{\beta})$. Nevertheless, it is easy to derive from equations (4.2) that

(*) If $c(\bar{\alpha}) + c(\bar{\beta}) = 0$ then necessarily $c(\bar{\alpha}) = \pm K$, $c(\bar{\beta}) = \mp K$.

So it is naturally to consider the quasiroots $\bar{\alpha}$ where $c(\bar{\alpha})$ are equal to $\pm K$ or not separately.

Let $c(\bar{\alpha})$, $\bar{\alpha} \in \overline{\Omega}_\Gamma$, be a solution of equations (4.2) (we assume $c(-\bar{\alpha}) = -c(\bar{\alpha})$). It is easy to derive from equations (4.2) the following properties.

(**) If $c(\bar{\alpha}) = \pm K$ and $c(\bar{\beta}) \neq \pm K$, then $c(\bar{\alpha} + \bar{\beta}) = \pm K$ and $c(\bar{\alpha} - \bar{\beta}) = \pm K$;

(***) If $c(\bar{\alpha}) = \pm K$ and $c(\bar{\beta}) = \pm K$, then $c(\bar{\alpha} + \bar{\beta}) = \pm K$.

Let $\overline{\Omega}'_\Gamma \subset \overline{\Omega}_\Gamma$ be the subset of quasiroots $\bar{\alpha}$ such that $c(\bar{\alpha}) \neq \pm K$. From (**) follows that $\overline{\Omega}'_\Gamma$ is a linear subset, i.e., $\overline{\Omega}'_\Gamma = \overline{\Omega}_\Gamma \cap \operatorname{span}(\overline{\Omega}'_\Gamma)$, where $\operatorname{span}(\overline{\Omega}'_\Gamma)$ is the vector subspace of $\mathfrak{h}^*/\mathfrak{h}^*_\Gamma$ generated by $\overline{\Omega}'_\Gamma$. Let $(\bar{\alpha}_1, \cdots, \bar{\alpha}_k)$ be a k-tuple of elements from $\overline{\Omega}'_\Gamma$ that form a basis of $\operatorname{span}(\overline{\Omega}'_\Gamma)$. Since by (*) $c(\bar{\alpha}) + c(\bar{\beta}) \neq 0$ for any $\bar{\alpha}, \bar{\beta} \in \overline{\Omega}'_\Gamma$, all $c(\bar{\alpha})$, $\bar{\alpha} \in \overline{\Omega}'_\Gamma$, can be found from (4.3) using the initial values $c_i = c(\bar{\alpha}_i)$, as in Proposition 4.3.

Note that since $c_i \neq \pm K$, there are uniquely defined complex numbers $\lambda_i \neq 0, 1$ such that $c(\bar{\alpha}_i) = c_i = K\psi(\lambda_j)$, where

$$\psi(x) = \frac{x+1}{x-1}.$$

Using the formula

$$\psi(xy) = \frac{\psi(x)\psi(y) + 1}{\psi(x) + \psi(y)},$$

it is easy to derive that if $\lambda : \overline{\Omega}'_\Gamma \to \mathbb{C}^*$ is the multiplicative map (such that if $\bar{\alpha}, \bar{\beta}, \bar{\alpha}+\bar{\beta} \in \overline{\Omega}'_\Gamma$ then $\lambda(\bar{\alpha}+\bar{\beta}) = \lambda(\bar{\alpha})\lambda(\bar{\beta})$) defined by $c(\bar{\alpha}_i) = \lambda_i$, then the solution of (4.3) is given by the formula

$$c(\bar{\alpha}) = K\psi(\lambda(\bar{\alpha})), \qquad \bar{\alpha} \in \overline{\Omega}'_\Gamma. \tag{4.5}$$

For correctness of this formula, one needs that the map λ to be regular, i.e., that λ to satisfy the condition: if $\bar{\alpha}, \bar{\beta}, \bar{\alpha}+\bar{\beta} \in \overline{\Omega}'_\Gamma$ then $\lambda(\bar{\alpha})\lambda(\bar{\beta}) = 1$ only when $\bar{\alpha} = -\bar{\beta}$.

From property (**) follows that the numbers $c(\bar{\alpha})$ define a function on the set $\pi(\overline{\Omega}_\Gamma)$, where π is the natural map $\mathfrak{h}^*/\mathfrak{h}^*_\Gamma \to (\mathfrak{h}^*/\mathfrak{h}^*_\Gamma)/\mathrm{span}(\overline{\Omega}'_\Gamma)$. This function has values $\pm K$. Let $X \subset \pi(\overline{\Omega}_\Gamma)$ be the subset where this function has value K. From property (***) follows that X is a semilinear subset. It means that if $x_1, x_2 \in X$ and $x_1 + x_2 \in \pi(\overline{\Omega}_\Gamma)$ then $x_1 + x_2 \in X$, and $X \cap (-X) = \emptyset$, $X \cup (-X) = \pi(\overline{\Omega}_\Gamma)$.

The arguments above lead to the following description of the variety $\mathcal{Z}_K$ of all solutions of (4.2) (or, what is the same, the variety of invariant brackets f on M such that $[\![f, f]\!] = K^2\varphi_M$).

Proposition 4.4. *Variety $\mathcal{Z}_K$ splits into stratas. Each strata is defined by choosing a linear subset $\overline{\Omega}'_\Gamma$ of $\overline{\Omega}_\Gamma$ and a semilinear subset X of $\pi(\overline{\Omega}_\Gamma)$. Points of this strata are parameterized by the multiplicative regular maps $\lambda : \overline{\Omega}'_\Gamma \to \mathbb{C}^*$.*

Let the data $(\overline{\Omega}'_\Gamma, X, \lambda)$ corresponds to a point of $\mathcal{Z}_K$. Then the coefficients $c(\bar{\alpha})$ of f are determined in the following way. If $\bar{\alpha} \in \overline{\Omega}'_\Gamma$ then $c(\bar{\alpha})$ is found by (4.5). If $\pi(\bar{\alpha}) \in X$ then $c(\bar{\alpha}) = K$. If $\pi(\bar{\alpha}) \in -X$ then $c(\bar{\alpha}) = -K$.

Of course, in case $K = 0$ the choose of X does not matter: a strata of $\mathcal{Z}_0$ is determined only by choosing $\overline{\Omega}'_\Gamma$.

Note also that the description of $\mathcal{Z}_K$ given in the proposition does not depend on choosing a basis in $\overline{\Omega}_\Gamma$. The variety $\mathcal{X}_K$ from the previous remark forms an open everywhere dense subset of $\mathcal{Z}_K$ and does depend on choosing a basis. According to Remark 2.1 this proposition describes all the $(G, \tilde{r})$-Poisson structures on semisimple orbits.

Now we fix a Poisson bracket $s = \sum(1/\lambda(\bar{\alpha}))E_\alpha \wedge E_{-\alpha}$, where λ is a fixed linear form, and describe the invariant brackets $f = \sum c(\bar{\alpha})E_\alpha \wedge E_{-\alpha}$ which satisfy the conditions

$$[\![f, f]\!] = K^2\varphi_M \qquad \text{for} \quad K \neq 0, \tag{4.6}$$
$$[\![f, s]\!] = 0.$$

Direct computation shows that the condition $[\![f, s]\!] = 0$ is equivalent to the system of equations for the coefficients $c(\bar{\alpha})$ of f

$$c(\bar{\alpha})\lambda(\bar{\alpha})^2 + c(\bar{\beta})\lambda(\bar{\beta})^2 = c(\bar{\alpha}+\bar{\beta})\lambda(\bar{\alpha}+\bar{\beta})^2 \tag{4.7}$$

for all the pairs of positive quasiroots $\bar{\alpha}, \bar{\beta}$ such that $\bar{\alpha}+\bar{\beta}$ is a quasiroot.

Definition 4.1. Let M be an orbit in $\mathfrak{g}^*$ (not necessarily semisimple). We call M a *good* orbit, if there exists an invariant bracket, f, on M satisfying the conditions (4.6) for s the Kirillov-Kostant-Souriau (KKS) Poisson bracket on M.

So, a semisimple orbit M is a good orbit if and only if equations (4.2) and (4.7) are compatible, i.e., have a common solution.

Proposition 4.5. *The good semisimple orbits are the following:*

a) For $\mathfrak{g}$ of type A_n all semisimple orbits are good.

b) For all other $\mathfrak{g}$, the orbit M is good if and only if the set $\Pi \setminus \Gamma$ consists of one or two roots which appear in representation of the maximal root with coefficient 1.

c) The brackets f on good orbits form a one-dimensional variety: all such brackets have the form

$$\pm f_0 + ts,$$

where $t \in \mathbb{C}$ and f_0 is a fixed bracket satisfying (4.6).

Proof. See [DGS]. □

Remark 4.3. From Proposition (3.5) follows that for $\mathfrak{g} = sl(n)$ all orbits (not only semisimple) are good ones. In addition, if an orbit, M, is such that $\varphi_M = 0$, then M is good: one can take $f = 0$. In [GP] there is a classification of orbits for all simple $\mathfrak{g}$, for which $\varphi_M = 0$.

Question 4.1. Let $\mathfrak{g}$ be a simple Lie algebra. Are all orbits in $\mathfrak{g}^*$ good? If not, what is a classification of good orbits?

4.2 Cohomologies defined by invariant brackets

In the next subsection we prove the existence of a $U_h(\mathfrak{g})$ invariant quantiztion of the Poisson brackets described above using the methods of [DS1]. This requires us to consider the 3-cohomology of the complex $(\Lambda^\bullet(\mathfrak{g}/\mathfrak{g}_\Gamma))^{\mathfrak{g}_\Gamma} = (\Lambda^\bullet\mathfrak{m})^{\mathfrak{g}_\Gamma}$ of $\mathfrak{g}_\Gamma$ invariants with differential given by the Schouten bracket with the bivector $f \in (\Lambda^2\mathfrak{m})^{\mathfrak{g}_\Gamma}$ from Proposition 4.3 a),

$$\delta_f : u \mapsto [\![f, u]\!] \qquad \text{for} \quad u \in (\Lambda^\bullet\mathfrak{m})^{\mathfrak{g}_\Gamma}.$$

The condition $\delta_f^2 = 0$ follows from the Jacobi identity for the Schouten bracket together with the fact that $[\![f, f]\!] = K^2\varphi_M$. Denote these cohomologies by $H^k(M, \delta_f)$, whereas the usual de Rham cohomologies are denoted by $H^k(M)$.

Recall (see Remark 4.1) that the brackets f satisfying $[\![f, f]\!] = K^2\varphi$ form a connected variety $\mathcal{X}$ which contains a submanifold $\mathcal{X}_0$ of Poisson brackets.

Proposition 4.6. *For almost all $f \in \mathcal{X}$ (except an algebraic subset of lesser dimension) one has*

$$H^k(M, \delta_f) = H^k(M)$$

for all k. In particular, $H^k(M, \delta_f) = 0$ for odd k.

Proof. First, let v be a Poisson bracket, i.e., $v \in \mathcal{X}_0$. Then the complex of polyvector fields on M, $\Theta^\bullet$, with the differential δ_v is well defined. Denote by $\Omega^\bullet$ the de Rham complex on M. Since none of the coefficients $c(\bar{\alpha})$ of v are zero, v is a nondegenerate bivector field, and therefore it defines an $\mathcal{A}$-linear isomorphism $\tilde{v} : \Omega^1 \to \Theta^1$, $\omega \mapsto v(\omega, \cdot)$, which can be extended up to the isomorphism $\tilde{v} : \Omega^k \to \Theta^k$ of k-forms onto k-vector fields for all k. Using Jacobi identity for v and invariance of v, one can show that $\tilde{v}$ gives a G invariant isomorphism of these complexes, so their cohomologies are the same.

Since $\mathfrak{g}$ is simple, the subcomplex of $\mathfrak{g}$ invariants, $(\Omega^\bullet)^\mathfrak{g}$, splits off as a subcomplex of $\Omega^\bullet$. In addition, $\mathfrak{g}$ acts trivially on cohomologies, since for any $g \in G$ the map $M \to M$, $x \mapsto gx$, is homotopic to the identity map, (G is a connected Lie group corresponding to $\mathfrak{g}$). It follows that cohomologies of complexes $(\Omega^\bullet)^\mathfrak{g}$ and $\Omega^\bullet$ coincide.

But $\tilde{v}$ gives an isomorphism of complexes $(\Omega^\bullet)^\mathfrak{g}$ and $(\Theta^\bullet)^\mathfrak{g} = ((\Lambda^\bullet \mathfrak{m})^{\mathfrak{g}_\Gamma}, \delta_v)$. So, cohomologies of the latter complex coincide with de Rham cohomologies, which proves the proposition for v being Poisson brackets.

Now, consider the family of complexes $((\Lambda^\bullet \mathfrak{m})^{\mathfrak{g}_\Gamma}, \delta_v)$, $v \in \mathcal{X}$. It is clear that δ_v depends algebraicly on v. It follows from the uppersemicontinuity of $\dim H^k(M, \delta_v)$ and the fact that $H^k(M) = 0$ for odd k, [Bo], that $H^k(M, \delta_v) = 0$ for odd k and almost all $v \in \mathcal{X}$. Using the uppersemicontinuity again and the fact that the number $\sum_k (-1)^k \dim H^k(M, \delta_v)$ is the same for all $v \in \mathcal{X}$, we conclude that $\dim H^k(M, \delta_v) = \dim H^k(M)$ for even k and almost all v. □

Remark 4.4. Call $f \in \mathcal{X}$ admissible, if it satisfies Proposition 4.6. From the proof of the proposition follows that the subset $\mathcal{D}$ such that $\mathcal{X} \setminus \mathcal{D}$ consists of admissible brackets does not intersect with the subset $\mathcal{X}_0$ consisting of Poisson brackets. It follows from this fact that for each good orbit there are admissible f compatible with the KKS bracket. Indeed, let M be a good orbit and $f_0 + ts$ the family from Proposition 4.5 c) satisfying (4.6) for a fixed K. Then for almost all numbers t this bracket is admissible. In fact, this family is contained in the two parameter family $uf_0 + ts$. By $u = 0$, $t \neq 0$ we obtain admissible brackets. So, there exist $u_0 \neq 0$ and t_0 such that the bracket $u_0 f_0 + t_0 s$ is admissible. It follows that the bracket $f_0 + (t_0/u_0)w$ is admissible, too. So, in the family $f_0 + ts$ there is an admissible bracket, and we conclude that almost all brackets in this family (except a finitely many) are admissible.

For the proof of existence of two parameter quantization for the cases D_n and E_6 in the next subsection, we will use the following result on invariant three-vector fields.

Denote by θ the Cartan automorphism of $\mathfrak{g}$.

Lemma 4.1. *For either D_n or E_6 and one of the subsets, Γ, of simple roots such that G_Γ defines a good orbit, any $\mathfrak{g}_\Gamma$ and θ invariant element v in $\Lambda^3\mathfrak{m}$ is a multiple of φ_M, that is,*

$$\left(\Lambda^3(\mathfrak{m})\right)^{\mathfrak{g}_\Gamma} \cong \langle\varphi_M\rangle.$$

Proof. In this case the system of positive quasiroots consists of $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\alpha}+\bar{\beta}$, where $\bar{\alpha}$, $\bar{\beta}$ are the simple quasiroots. From Proposition 4.1 follows that invariant elements in $\mathfrak{m}_{\bar{\alpha}} \otimes \mathfrak{m}_{\bar{\beta}} \otimes \mathfrak{m}_{-\bar{\alpha}-\bar{\beta}}$ and $\mathfrak{m}_{-\bar{\alpha}} \otimes \mathfrak{m}_{-\bar{\beta}} \otimes \mathfrak{m}_{\bar{\alpha}+\bar{\beta}}$ form subspaces of dimension one, I_1 and I_2. Moreover, all the ivariant elements of $\Lambda^3\mathfrak{m}$ are lying in I_1+I_2. Since θ takes I_1 onto I_2, there is only one-dimansional θ invariant subspace in I_1+I_2, which is necessarily generated by φ_M. □

4.3 $U_h(\mathfrak{g})$ invariant quantizations in one and two parameters

In this subsection we prove the existence of one and two parameter $U_h(\mathfrak{g})$ invariant quantization of the function algebras $\mathcal{A}$ on semisimple orbits, M, in $\mathfrak{g}^*$. By Proposition 2.2, the one parameter quantization has the Poisson bracket of the form

$$f(a,b) - \{a,b\}_r, \qquad [\![f,f]\!] = -\varphi_M. \tag{4.8}$$

We show that the one parameter quantization exists for all semisimple orbits and all f constructed in Proposition 4.3 a) and satisfying Proposition 4.6.

For two parameter quantization, there are two compatible Poisson brackets: the KKS bracket s and the bracket of the form (4.8) with the additional condition

$$[\![f,s]\!] = 0. \tag{4.9}$$

We show that the two parameter quantization exists for good orbits in cases D_n and E_6 and for almost all f satisfying (4.8) and (4.9).

Note that in subsection 3.5 we have proven that in case A_n the two parameter quantization exists for maximal semisimple orbits. In a next paper we shall prove the same for all semisimple orbits.

We remind the reader of the method in [DS1]. The first step is to construct a $U(\mathfrak{g})$ invariant quantization in the category $\mathcal{C}(U(\mathfrak{g})[[h]],\Delta,\Phi_h)$. Then we use the equivalence given by the pair (Id,F_h) between the monoidal categories $\mathcal{C}(U(\mathfrak{g})[[h]],\Delta,\Phi_h)$ and $\mathcal{C}(U(\mathfrak{g})[[h]],\Delta_h,\mathbf{1})$ to define a $U_h(\mathfrak{g})$ invariant quantization, either $\mu_h F_h^{-1}$ in the one parameter case or $\mu_{t,h}F_h^{-1}$ in the two parameter case (see Subsections 2.2 and 2.3). In the following we often write Φ for Φ_h.

Proposition 4.7. *Let $\mathfrak{g}$ be a simple Lie algebra, M a semisimple orbit in $\mathfrak{g}^*$. Then, for almost all (in sense of Proposition 4.6) $\mathfrak{g}$ invariant brackets f satisfying $[\![f,f]\!] = -\varphi_M$, there exists a multiplication μ_h on $\mathcal{A}$*

$$\mu_h(a,b) = ab + (h/2)f(a,b) + \sum_{n\geq 2} h^n \mu_n(a,b)$$

which is $U(\mathfrak{g})$ invariant (equation 2.12)) and Φ associative (equation (2.13)).

Proof. To begin, consider the multiplication $\mu^{(1)}(a,b) = ab + (h/2)f(a,b)$. The corresponding obstruction cocycle is given by

$$obs_2 = \frac{1}{h^2}(\mu^{(1)}(\mu^{(1)} \otimes id) - \mu^{(1)}(id \otimes \mu^{(1)})\Phi)$$

considered modulo terms of order h. No $\frac{1}{h}$ terms appear because f is a biderivation and, therefore, a Hochschild cocycle. The fact that the presence of Φ does not interfere with the cocyle condition and that this equation defines a Hochschild 3-cocycle was proven in [DS1]. It is well known that if we restrict to the subcomplex of cochains given by differential operators, the differential Hochschild cohomology of $\mathcal{A}$ in dimension p is the space of p-polyvector fields on M. Since $\mathfrak{g}$ is reductive, the subspace of $\mathfrak{g}$ invariants splits off as a subcomplex and has cohomology given by $(\Lambda^p\mathfrak{m})^{\mathfrak{gr}}$. The complete antisymmetrization of a p-tensor projects the space of invariant differential p-cocycles onto the subspace $(\Lambda^p\mathfrak{m})^{\mathfrak{gr}}$ representing the cohomology. The equation $[\![f,f]\!] + \varphi_M = 0$ implies that obstruction cocycle is a coboundary, and we can find a 2-cochain μ_2, so that $\mu^{(2)} = \mu^{(1)} + h^2\mu_2$ satisfies

$$\mu^{(2)}(\mu^{(2)} \otimes id) - \mu^{(2)}(id \otimes \mu^{(2)})\Phi = 0 \bmod h^2.$$

Assume we have defined the deformation $\mu^{(n)}$ to order h^n such that Φ associativity holds modulo h^n, then we define the $(n+1)^{\text{st}}$ obstruction cocycle by

$$obs_{n+1} = \frac{1}{h^{n+1}}(\mu^{(n)}(\mu^{(n)} \otimes id) - \mu^{(n)}(id \otimes \mu^{(n)})\Phi) \bmod h.$$

In [DS1] (Proposition 4) we showed that the usual proof that the obstruction cochain satisfies the cocycle condition carries through to the Φ associative case. The coboundary of obs_{n+1} appears as the h^{n+1} coefficient of the signed sum of the compositions of $\mu^{(n+1)}$ with obs_{n+1}. The fact that $\Phi = 1 \bmod h^2$ together with the pentagon identity implies that the sum vanishes identically, and thus all coefficients vanish, including the coboundary in question. Let $obs'_{n+1} \in (\Lambda^3\mathfrak{m})^{\mathfrak{gr}}$ be the projection of obs_{n+1} on the totally skew symmetric part, which represents the cohomology class of the obstruction cocycle. The coefficient of h^{n+2} in the same signed sum, when projected on the skew symmetric part, is $[\![f, obs'_{n+1}]\!]$ which is the coboundary of obs'_{n+1} in the complex $((\Lambda^\bullet\mathfrak{m})^{\mathfrak{gr}}, \delta_f = [\![f,.]\!])$. Thus obs'_{n+1} is a δ_f cocycle. By Proposition 4.6, this complex has zero cohomology. Now we modify $\mu^{(n+1)}$ by adding a term $h^n\mu_n$ with $\mu_n \in (\Lambda^2\mathfrak{m})^{\mathfrak{gr}}$ and consider the $(n+1)^{\text{st}}$ obstruction cocycle for $\mu'^{(n+1)} = \mu^{(n+1)} + h^n\mu_n$. Since the term we added at degree h^n is a Hochschild cocyle, we do not introduce a h^n term in the calculation of $\mu^{(n)}(\mu^{(n)} \otimes id) - \mu^{(n)}(id \otimes \mu^{(n)})\Phi$ and the totally skew symmetric projection h^{n+1} term has been modified by $[\![f, \mu_n]\!]$. By choosing μ_n appropriately, we can make the $(n+1)^{\text{st}}$ obstruction cocycle represent the zero cohomology class, and we are able to continue the recursive construction of the desired deformation. $\square$

Now we prove the existence of a two parameter deformation for good orbits in the cases D_n and E_6.

Proposition 4.8. *Given a pair of $\mathfrak{g}$ invariant brackets, f, v, on a good orbit in D_n or E_6 satisfying $[\![f,f]\!] = -\varphi_M$, $[\![f,v]\!] = [\![v,v]\!] = 0$, there exists a multiplication $\mu_{h,t}$ on $\mathcal{A}$*

$$\mu_{t,h}(a,b) = ab + (h/2)f(a,b) + (t/2)v(a,b) + \sum_{k,l\geq 1} h^k t^l \mu_{k,l}(a,b)$$

which is $U(\mathfrak{g})$ invariant and Φ associative.

Proof. The existence of a multiplication which is Φ associative up to and including h^2 terms is nearly identical to the previous proof. Both f and v are anti-invariant under the Cartan involution θ. We shall look for a multiplication $\mu_{t,h}$ such that $\mu_{k,l}$ is θ anti-invariant and skew-symmetric for odd $k+l$ and θ invariant and symmetric for even $k+l$.

So, suppose we have a multiplication defined to order n,

$$\mu_{t,h}(a,b) = ab + h\mu_1(a,b) + t\mu_1'(a,b) + \sum_{k+l\leq n} h^k t^l \mu_{k,l}(a,b),$$

with mentioned above invariance properties and Φ associative to order h^n.

Further we shall suppose that Φ has the properties: It is invariant under the Cartan involution θ and $\Phi^{-1} = \Phi_{321}$. Such Φ always can be choosen, [DS2]. Using these properties for Φ, direct computation shows that the obstruction cochain,

$$obs_{n+1} = \sum_{k=0,\dots,n+1} h^k t^{n+1-k} \beta_k,$$

has the following invariance properties: For odd n, obs_{n+1} is θ invariant and $obs_{n+1}(a,b,c) = -obs_{n+1}(c,b,a)$, and for even n, and obs_{n+1} is θ anti-invariant and $obs_{n+1}(a,b,c) = obs_{n+1}(c,b,a)$.

Hence, the projection of obs_{n+1} on $(\Lambda^3\mathfrak{m})^{\mathfrak{g}_\Gamma}$ is equal to zero for even n. It follows that all the β_k are Hochschild coboundaries, and the standard argument implies that the multiplication can be extended up to order $n+1$ with the required properties.

For odd n, Lemma 4.1 shows that the projection on $(\Lambda^3\mathfrak{m})^{\mathfrak{g}_\Gamma}$ has the form

$$obs_{n+1} = \left(\sum_{k=0,\dots,n+1} a_k h^k t^{n+1-k}\right)\varphi_M.$$

The KKS bracket is given by the two-vector

$$v = \sum_{\alpha\in\Omega^+\setminus\Omega_\Gamma} \frac{1}{\lambda(\bar{\alpha})} E_\alpha \wedge E_{-\alpha}.$$

Setting

$$w = \sum_{\alpha\in\Omega^+\setminus\Omega_\Gamma} \lambda(\bar{\alpha}) E_\alpha \wedge E_{-\alpha},$$

gives

$$[\![v, w]\!] = -3\varphi_M.$$

Defining

$$\mu'^{(n)} = \mu^{(n)} + \frac{a_0}{3} t^n w,$$

the new obstruction cohomology class is

$$obs'_{n+1} = (\sum_{k=1,\dots,n+1} a_k h^k t^{n+1-k})\varphi_M.$$

Finally we define

$$\mu''^{(n)} = \mu'^{(n)} + \sum_{k=1,\dots,n+1} a_k h^{k-1} t^{n+1-k})f$$

and get an obstruction cocycle which is zero in cohomology. Now the standard argument implies that the deformation can be extended to give a Φ associative invariant multiplication with the required properties of order $n+1$.

So, we are able to continue the recursive construction of the desired multiplication. $\square$

Using the Φ_h associative multiplications μ_h and $\mu_{t,h}$ from Propositions 4.7 and 4.8 and the equivalence between the monoidal categories $\mathcal{C}(U(\mathfrak{g})[[h]], \Delta, \Phi_h)$ and $\mathcal{C}(U(\mathfrak{g})[[h]], \widetilde{\Delta}, \mathbf{1})$ given by the pair (Id, F_h) (see Section 2), one can define $U_h(\mathfrak{g})$ invariant multiplications, either $\mu_h F_h^{-1}$ in the one parameter case or $\mu_{t,h} F_h^{-1}$ in the two parameter case.

Remark 4.5. After [Ko], the philosophy is that there are no obstructions for quantizations of Poisson brackets on manifolds. In this connection, the following question arises:

Question 4.2. Let M be a G-manifold on which there exists an invariant connection. Given a G invariant Poisson bracket, v, on M, does there exist a G invariant quantization of v?

In case M is a homogeneous manifold the bracket v has a constant rank, and such a quantization can be obtained by Fedosov's method, [Fed], [Do1].

Another question which relates to the topic of this paper is the following.

Question 4.3. Let M be a G-manifold on which there exists an invariant connection, $U(\mathfrak{g})$ the corresponding to G universal enveloping algebra, and $\Phi_h \in (U(\mathfrak{g}))^{\otimes 3}[[h]]$ an invariant element of the form (2.6) obeying the pentagon identity (2.7). Let f be an invariant bracket on M satisfying $[\![f, f]\!] = -\varphi_M$. Does there exist a $U(\mathfrak{g})$ (or G) invariant and Φ_h associative quantization of f (as in Proposition 4.7)?

Note that if the answer to this Question is positive, then the answer to Question 2.1 is also positive: we take for M the group G itself and consider it as a G-manifold by left multiplication.

References

[Dr1] V.G.Drinfeld, *On almost-cocommutative Hopf algebras*, Leningrad Math. J., 1 (1990), 321–342.

[Dr2] V.G.Drinfeld, *Quasi-Hopf algebras*, Leningrad Math. J., 1 (1990), 1419–1457.

[Dr3] V.G. Drinfeld, On quadratic commutator relations in the quasiclassical case, Math. physics and functional analysis, Kiev, Naukova Dumka, 1986.

[Dr4] V.G. Drinfeld, *Constant quasiclassical solutions of the Yang-Baxter quantum equation*, Dokl. Akad. Nauk SSSR 273 (1983), no. 3, 531–535.

[BD] A.A.Belavin and V.G.Drinfeld, *Triangle Equations and simple Lie algebras*, Soviet Scientific Review, section C 4 (1984), Harwood Academic Publishers, 93–165.

[RTF] N.Yu.Reshetikhin, L.A.Takhtadzhyan, and L.D. Faddeev, *Quantization of Lie groups and Lie algebras*, (Russian) Algebra i Analiz 1 (1989), no. 1, 178–206.

[Ma] Yu.i.Manin, "Topics in Noncommutative Geometry", Princeton, 1991.

[EK] P.Etingof and D.Kazhdan, *Quantization of Lie bialgebras, I*, Selecta Math., 2(1996), no. 1, 1–41.

[KS] P.P.Kulish and E.K.Sklyanin, *Algebraic structures related to reflection equations*, J. Phys. A 25 (1992), no. 22, 5963–5975.

[Fed] B.Fedosov, "Deformation Quantization and Index Theory", Academie Verlag, Berlin, 1996.

[SS] S.Shnider and S.Stenberg, "Quantum Groups", Int. Press, 1993.

[Maj] S.Majid, "Foundations of Quantum Group Theory", Cambridge Univ. Press, 1995.

[LS] V.Lyubashenko and A.Sudbery, *Generalized Lie algebras of type A_n*, J. Math. Phys. 39 (1998), no. 6, 3487–3504.

[DG] G.W.Delius and M.D.Gould, *Quantum Lie algebras, their existence, uniqueness and q-antisymmetry*, Comm. Math. Phys. 185 (1997), no. 3, 709–722.

[Do1] J.Donin, *On the quantization of Poisson brackets*, Adv. Math. 127 (1997), no. 1, 73–93.

[Do2] J.Donin, *Double quantization on the coadjoint representation of $sl(n)^*$*, Czechoslovak J. of Physics, 47 (1997), no 11 , 1115–1122.

[DGM] J.Donin, D.Gurevich, and S.Majid, *R-matrix brackets and their quantization*, Ann.de l'Institut d'Henri Poincaré, Phys. Theor. 58 (1993), no. 2, 235–246.

[DG1] J.Donin and D.Gurevich, *Quasi-Hopf Algebras and R-Matrix Structure in Line Bundles over Flag Manifols*, Selecta Mathematica Sovietica 12 (1993, no. 1, 37–48.

[DG2] J.Donin and D.Gurevich, *Some Poisson structures associated to Drinfeld-Jimbo R-matrices and their quantization*, Israel Math. J., 92, (1995), no. 1, 23–32.

[DS1] J.Donin and S.Shnider, *Quantum symmetric spaces*, J. of Pure and Applied Algebra, 100 (1995), 103–115.

[DS2] J.Donin and S.Shnider, *Cohomological construction of quantized universal enveloping algebras*, Trans. Amer. Math. Soc., 349 (1997), no. 4, 1611–1632.

[DGS] J.Donin, D.Gurevich, and S.Shnider, *Double Quantization on Some Orbits in the Coadjoint Representations of Simple Lie Groups*, Commun. Math. Phys., 204 (1999), no. 1, 39–60.

[DM] J.Donin and L.Makar-Limanov, *Quantization of quadratic Poisson brackets on a polynomial algebra of three variables*, J. Pure Appl. Algebra 129 (1998), no. 3, 247–261.

[GP] D.Gurevich and D.Panyushev, *On Poisson pairs associated to modified R-matrices*, Duke Math. J., v. 73 (1994), no 1.

[Ko] M.Kontsevich, Deformation quantization on Poisson manifolds, I, q-alg/9709040.

[Bo] A.Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. Math., 57 (1953), 115–207.

[Dix] J.Dixmier, “Algèbra enveloppanes”, Cahiers scientifiques, vol. 37, Cauthier-Villars, Paris 1974.