On sum-normalised cohomology of categories, twisted homotopy pairs, and universal Toda brackets

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MPI / 95-22

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Introduction

This paper describes new links between cohomology of categories, groupoid enriched categories, and homotopy theory of fibres and cofibres.

For a category $\mathbf{C}$ with sums we introduce in (2.10), (2.15) a natural transformation of cohomology groups

$$H^{n+1}(\mathbf{C}, D) \xrightarrow{\lambda_{\mathrm{sum}}} H^n(\mathrm{Twist}(\mathbf{C}), \widehat{D})$$

where $\mathrm{Twist}(\mathbf{C})$ is the twisted version of the category of pairs in $\mathbf{C}$. The construction of λ_{sum} relies on the *normalisation theorem* A.9 in the appendix which shows that cochains can be assumed to respect sums. This is a new kind of normalisation result extending classical normalisation with respect to identities and generalising that with respect to zero maps [5]. There are dual results for categories with products, yielding the dual transformation λ_{prod}.

Given a groupoid-enriched category with sums we introduce the category of *twisted homotopy pairs* which generalises the category of homotopy pairs studied by Hardie [8, 9]. We show that

$$H^3(\mathbf{C}, D) \xrightarrow{\lambda_{\mathrm{sum}}} H^2(\mathrm{Twist}(\mathbf{C}), \widehat{D})$$

takes the class represented by a groupoid enriched category to the class represented by its category of twisted homotopy pairs; see (3.15). Actually this correspondence is the motivation for studying the transformation λ_{sum}.

If $\mathbf{C}$ is the homotopy category of suspensions, resp. loop spaces, then the associated classical groupoid-enriched category given by maps and homotopies represents an element

$$T_\Sigma \in H^3(\mathbf{C}, D_\Sigma), \quad \text{resp.} \quad T_\Omega \in H^3(\mathbf{C}, D_\Omega)$$

termed the *universal Toda bracket*. This determines all classical triple Toda brackets in $\mathbf{C}$ [5].

In homotopy theory a space is often obtained as a homotopy cofibre $C(f)$ of an attaching map f or dually as a homotopy fibre $P(f')$ of a classifying map f'. Therefore it is a classical problem to describe homotopy classes of maps

$$C(f) \xrightarrow{F} C(g) \qquad \text{resp.} \qquad P(f') \xrightarrow{F'} P(g')$$

and their composites only in terms of the homotopy classes of the attaching maps or classifying maps repsectively. Studying this problem leads inevitably to the theory of this paper; solutions are described in (4.7), (5.7) where we show that maps F or F' are equivalent to twisted homotopy pairs. This improves considerably the classical method of constructing such maps by homotopy pairs.

It is well known that examples of maps F and F' are given by 'extensions' and 'coextensions' and that these are related to classical Toda brackets. In fact we show how the universal Toda bracket determines such homotopy categories of maps between cofibres, resp. fibres; see (4.8), (5.8). For this we use the transformation λ_{sum}, resp. λ_{prod}, and the result that maps between cofibres or fibres can be constructed by twisted homotopy pairs.

1 Cohomology of categories

Recall from [1, 4] that the *category of factorisations* $F\mathbf{C}$ on a category $\mathbf{C}$ is the category with objects the morphisms $f : A \to X$ of $\mathbf{C}$ and morphisms $(\alpha, \beta) : f \to g$ the 'factorisations' given by commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle\beta}\uparrow & & \downarrow{\scriptstyle\alpha} \\ B & \xrightarrow{g} & Y \end{array}$$

with the composition $(\alpha, \beta)(\alpha', \beta') = (\alpha\alpha', \beta'\beta)$. A *natural system* D on $\mathbf{C}$ is then a functor from $F\mathbf{C}$ to the category of abelian groups. We write D_f for the abelian group $D(f)$ and α_*, β^* for the induced homomorphisms

$$D_f \xrightarrow{\alpha_*} D_{\alpha f} \qquad\qquad D_f \xrightarrow{\beta^*} D_{f\beta}$$

given by $D(\alpha, 1)$ and $D(1, \beta)$ respectively.

If D is a natural system on a category $\mathbf{C}$ recall that the cohomology of $\mathbf{C}$ with coefficients in D is defined as follows. First let $\mathrm{Ner}(\mathbf{C})$ be the simplicial nerve of $\mathbf{C}$, given in dimension $n \geq 1$ by the set of all sequences $\sigma = (\sigma_1, \ldots, \sigma_n)$

$$X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{\sigma_n} X_n$$

of n composable morphisms in $\mathbf{C}$, and in dimension 0 by $\mathrm{Ob}(\mathbf{C})$. The face maps are defined by

$$\begin{aligned} d_0(\sigma) &= (\sigma_2, \ldots, \sigma_n) \\ d_k(\sigma) &= (\sigma_1, \ldots, \sigma_k\sigma_{k+1}, \ldots, \sigma_n) \\ d_n(\sigma) &= (\sigma_1, \ldots, \sigma_{n-1}) \end{aligned}$$

and the degeneracies by insertion of identities. We will write $|\sigma|$ for the composite $\sigma_1 \ldots \sigma_n$. Now let $F^* = F^*(\mathbf{C}, D)$ be the cochain complex with F^n the abelian group of all functions

$$\mathrm{Ner}(\mathbf{C})_n \xrightarrow{c} \bigcup \{D_g \ : \ g \in \mathrm{Mor}(\mathbf{C})\}$$

with $c\sigma \in D_{|\sigma|}$. Addition in F^n is defined pointwise, and the coboundary $\delta : F^{n-1} \to F^n$ is defined by

$$(\delta c)\sigma = (\sigma_1)_* c(d_0\sigma) + \sum_{i=1}^{n-1}(-1)^i c(d_i\sigma) + (-1)^n(\sigma_n)^* c(d_n\sigma)$$

Then the cohomology groups of $\mathbf{C}$ with coefficients in D are defined by

$$H^n(\mathbf{C}, D) = H^n(F^*(\mathbf{C}, D), \delta)$$

for $n \geq 0$. An equivalence of categories $\phi : \mathbf{K} \to \mathbf{C}$ induces by [4] an isomorphism of cohomology groups:

$$H^n(\mathbf{K}, \phi^* D) \cong H^n(\mathbf{C}, D)$$

A *sum* (or *coproduct*) of objects X_k, $1 \leq k \leq r$, in a category $\mathbf{C}$ is an object $X = X_1 \vee \ldots \vee X_r$ of $\mathbf{C}$ together with morphisms $i_k : X_k \to X$ such that pre-composition by the i_k induces natural bijections of hom-sets

$$i^* = (i_1^*, \ldots, i_r^*) : \mathbf{C}(X, Z) \cong \mathbf{C}(X_1, Z) \times \ldots \times \mathbf{C}(X_r, Z)$$

Some applications of the cohomology of categories may be found in [7, 10, 11, 12].

Definition 1.1 Suppose D is a natural system on a category $\mathbf{C}$. Let (X, i) be a sum in $\mathbf{C}$ and $f : X \to Y$ a morphism of $\mathbf{C}$. There are homomorphisms

$$D_f \xrightarrow{\ i_k^*\ } D_{fi_k}$$

which define a homomorphism

$$D_f \xrightarrow{\ i^*\ } \bigoplus_{k=1}^{r} D_{fi_k}$$

by $(i^* a)_k = i_k^*(a)$. We say D is *compatible with sums* if i^* is an isomorphism of groups for each such morphism f of $\mathbf{C}$ and sum diagram (X, i).

In the appendix we will show that the cohomology $H^n(\mathbf{C}, D)$ admits a "normalisation theorem" in the case that D is compatible with sums.

2 Pairs, twisted pairs and the natural transformation λ

Let $\mathbf{C}$ be a category with finite sums, that is, with binary sums $A \vee B$ and an initial object $*$. Suppose that $*$ is also a terminal object. For objects A, B of $\mathbf{C}$ the *zero morphism* $0 = 0_{AB} : A \to B$ is given by $A \to * \to B$. For $f : A \to X$, $g : B \to X$ we write $(f, g) : A \vee B \to X$ for the unique morphism with $(f, g)i_A = f$ and $(f, g)i_B = g$.

Definition 2.1 A morphism $\xi : A \to X \vee Y$ in $\mathbf{C}$ is *trivial on* Y if the composite $(0, 1)\xi : A \to X \vee Y \to Y$ is the zero morphism.

$$\begin{array}{ccc} A & \xrightarrow{\ \xi\ } & X \vee Y \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle (0,1)} \\ & & Y \end{array}$$

In particular the composite $i_X \zeta : A \to X \to X \vee Y$ is trivial on Y for every morphism $\zeta : A \to X$ of $\mathbf{C}$.

Definition 2.2 The *twisted pair category* $\mathrm{Twist}(\mathbf{C})$ on $\mathbf{C}$ is the category with objects the morphisms f of $\mathbf{C}$ and morphisms $(\xi, \eta) : f \to g$ given by commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\ \xi\ } & X \vee Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (g,1)} \\ B & \xrightarrow{\ \eta\ } & Y \end{array}$$

where ξ is trivial on Y. Composition is defined by

$$(\xi, \eta)(\xi', \eta') \quad = \quad (\bar{\xi}\xi', \eta\eta')$$

where $\overline{\xi} : A \vee B \to X \vee Y$ is given by $(\xi, i_Y \eta)$. One readily checks that this is a well-defined category. In fact an alternative description of the morphisms $f \to g$ of Twist(C) is given by pairs of commutative diagrams

$$\begin{array}{ccc} B & \xrightarrow{\eta} & Y \\ {\scriptstyle i_B}\downarrow & & \downarrow{\scriptstyle i_Y} \\ A \vee B & \xrightarrow{\overline{\xi}} & X \vee Y \\ {\scriptstyle (0,1)}\downarrow & & \downarrow{\scriptstyle (0,1)} \\ B & \xrightarrow{\eta} & Y \end{array} \qquad \begin{array}{ccc} A \vee B & \xrightarrow{\overline{\xi}} & X \vee Y \\ {\scriptstyle (f,1)}\downarrow & & \downarrow{\scriptstyle (g,1)} \\ B & \xrightarrow{\eta} & Y \end{array}$$

with the composition given by horizontal composition of these diagrams.

Definition 2.3 A natural system D on **C** is said to be *strongly* compatible with sums if D is compatible with sums and has the following additional properties:

1. for each $\xi : A \to X \vee Y$ in **C** which is trivial on Y, the homomorphism $(0,1)_* : D(\xi) \to D(0_{AY})$ is surjective; we write $D(\xi)_2$ for the kernel.

$$D(\xi)_2 \hookrightarrow D(\xi) \xrightarrow{(0,1)_*} D(0_{AY})$$

2. for each sum $X \vee Y$ and each morphism $\zeta : A \to X$ in **C**, the homomorphism $(i_X)_* : D(\zeta) \to D(i_X\zeta)$ has image $D(i_X\zeta)_2$.

Example 2.4 The natural system D_Σ used in (4.4) is strongly compatible with sums since the homomorphisms

$$D_\Sigma(f) = [\Sigma X, Y] \xrightarrow{g_*} D_\Sigma(gf) = [\Sigma X, Z]$$

are defined simply by $a \mapsto ga$.

A natural system D on **C** defines a natural system $\widehat{D}$ on the category Twist(C) of twisted pairs as follows. For morphisms $(\xi, \eta) : f \to g$ of Twist(C) consider the subgroup

$$f^* D(\eta) \; + \; (g,1)_* D(\xi)_2 \;\; \subseteq \;\; D(\eta f)$$

and define $\widehat{D}(\xi, \eta)$ to be the quotient

$$\widehat{D}(\xi, \eta) \;\; = \;\; D(\eta f) \;/\; f^* D(\eta) + (g,1)_* D(\xi)_2 \tag{2.5}$$

For $a \in D(\eta f)$ we write $[a] \in \widehat{D}(\xi, \eta)$ for the corresponding coset.

Consider morphisms (ξ, η), (ξ', η') of Twist(C), with composite $(\overline{\xi}\xi', \eta\eta')$.

$$\begin{array}{ccccc} X_0 \vee Y_0 & \xleftarrow{\overline{\xi}} & X_1 \vee Y_1 & \xleftarrow{\xi'} & X_2 \\ {\scriptstyle (f_0,1)}\downarrow & & \downarrow{\scriptstyle (f_1,1)} & & \downarrow{\scriptstyle f_2} \\ Y_0 & \xleftarrow[\eta]{} & Y_1 & \xleftarrow[\eta']{} & Y_2 \end{array} \tag{2.6}$$

Lemma 2.7 Suppose D is compatible with sums and $a \in D(\eta(f_1, 1))$ satisfies $i_{X_1}^*(a) = 0$. Then $\xi'^*(a) \in (f_0, 1)_* D(\bar{\xi}\xi')_2$.

Proof: Let $b = i_{Y_1}^*(a) \in D(\eta)$ and let $b' \in D(\bar{\xi})$ correspond to $(0, (i_{Y_0})_* b) \in D(\xi) \oplus D(i_{Y_0}\eta)$. Then

$$(f_0, 1)_* b' = a \in D((f_0, 1)\bar{\xi}) = D(\eta(f_1, 1))$$

since they agree under $i_{X_1}^*$ and $i_{Y_1}^*$. Furthermore

$$(0, 1)_* b' = (0, 1)^* b \in D((0, 1)\bar{\xi}) = D(\eta(0, 1))$$

Thus $\xi'^*(a) = (f_0, 1)_*(\xi'^* b')$ with $(0, 1)_*(\xi'^* b') = \xi'^*(0, 1)^* b = 0$ since ξ' is trivial on Y_1. □

For morphisms $(\kappa, \rho) : A \vee B \to Y$ in **C** we write α_1 for the monomorphism

$$\alpha_1 : D(\kappa) \hookrightarrow D(\kappa) \oplus D(\rho) \xrightarrow{\cong} D(\kappa, \rho)$$

Proposition 2.8 Suppose D is compatible with sums. Then the groups $\widehat{D}(\xi, \eta)$ and the induced homomorphisms

$$\begin{aligned}(\xi, \eta)_* : \quad & \widehat{D}(\xi', \eta') \to \widehat{D}(\bar{\xi}\xi', \eta\eta'), \quad [a] \mapsto [\eta_* a] \\ (\xi', \eta')^* : \quad & \widehat{D}(\xi, \eta) \to \widehat{D}(\bar{\xi}\xi', \eta\eta'), \quad [a] \mapsto [\xi'^* \alpha_1(a)]\end{aligned}$$

form a well-defined natural system on the category Twist(**C**).

Proof: The only part which is not straight-forward is showing that $(\xi', \eta')^*$ takes the subgroup $f_1^* D(\eta)$ to zero. Let $b \in D(\eta)$ and let

$$a = \alpha_1(f_1^* b) - (f_1, 1)^* b \in D(\eta(f_1, 1))$$

Then $\xi'^*(f_1, 1)^* b = f_2^* \eta'^* b \in f_2^* D(\eta\eta')$. Also $i_{X_1}^* a = 0$ and so we have $\xi'^*(a) \in (f_0, 1)_* D(\bar{\xi}\xi')_2$ by lemma 2.7. Thus

$$(\xi', \eta')^* f_1^* b = \xi'^* \alpha_1(f_1^* b) = \xi'^*(f_1, 1)^* b + \xi'^* a$$

is zero in the quotient as required. □

If D is strongly compatible with sums then the groups $\widehat{D}(\xi, \eta)$ have the following alternative description. Consider the subgroup

$$(0^* \oplus f^*)\, D(\eta) + ((0, 1)_* \oplus (g, 1)_*)\, D(\xi) \subseteq D(0_{AY}) \oplus D(\eta f)$$

Then $\widehat{D}(\xi, \eta)$ is given by the quotient

$$\widehat{D}(\xi, \eta) = \frac{D(0_{AY}) \oplus D(\eta f)}{(0^* \oplus f^*)\, D(\eta) + ((0, 1)_* \oplus (g, 1)_*)\, D(\xi)} \tag{2.9}$$

Since $0^* = 0$ and $(0, 1)_* : D(\xi) \to D(0_{AY})$ is onto this agrees with the definition in (2.5). For an element $(b', b) \in D(0_{AY}) \oplus D(\eta f)$ we write $[b', b] \in \widehat{D}(\xi, \eta)$ for the corresponding element of the quotient.

We can now state the main theorem of this section.

Theorem 2.10 Suppose D is a natural system on **C** which is strongly compatible with sums. Then there is a well-defined natural transformation

$$H^{n+1}(\mathbf{C}, D) \xrightarrow{\lambda_{\text{sum}}} H^n(\text{Twist}(\mathbf{C}), \widehat{D})$$

given by (2.11) below.

There are homomorphisms

$$F^{n+1}(\mathbf{C}, D) \xrightarrow{\lambda_{\text{sum}}} F^n(\text{Twist}(\mathbf{C}), \widehat{D})$$

for $n \geq 0$ defined as follows. For $\sigma \in \text{Ner}(\text{Twist}(\mathbf{C}))_n$ given by $(\xi_i, \eta_i): f_i \to f_{i-1}$, $f_i: X_i \to Y_i$, one has $(n+1)$-simplices $\lambda_i\sigma \in \text{Ner}(\mathbf{C})$ with $|\lambda_i\sigma| = \eta_1 \ldots \eta_n f_n : X_n \to Y_0$ by

$$\lambda_i\sigma = \begin{cases} ((f_0, 1), \bar{\xi}_1, \ldots, \bar{\xi}_{n-1}, \xi_n) & i = 0 \\ (\eta_1, \ldots, \eta_i, (f_i, 1), \bar{\xi}_{i+1}, \ldots, \bar{\xi}_{n-1}, \xi_n) & 1 \leq i \leq n-1 \\ (\eta_1, \ldots, \eta_n, f_n) & i = n \end{cases}$$

Simplices $\lambda_i'\sigma$ with $|\lambda_i'\sigma| = 0 : X_n \to Y_0$ are defined similarly by replacing the f_i by $0 : X_i \to Y_i$. Then for c_{n+1} an $(n+1)$-cochain on $\mathbf{C}$ we define an n-cochain $\lambda_{\text{sum}}(c_{n+1})$ on $\text{Twist}(\mathbf{C})$ by

$$(2.11) \qquad (\lambda_{\text{sum}} c_{n+1})(\sigma) = \left[\sum_{i=0}^{n} (-1)^i c_{n+1}(\lambda_i'\sigma), \ \sum_{i=0}^{n} (-1)^i c_{n+1}(\lambda_i\sigma) \right]$$

where we use the definition of $\widehat{D}$ in (2.9).

The *pair category* $\text{Pair}(\mathbf{C})$ on $\mathbf{C}$ is the category with objects the morphisms f of $\mathbf{C}$ and morphisms $f \to g$ given by pairs of morphisms (ζ, η) such that $g\zeta = \eta f$. There is an inclusion

$$(2.12) \qquad \text{Pair}(\mathbf{C}) \xhookrightarrow{\iota} \text{Twist}(\mathbf{C})$$

which is the identity on objects and takes (ζ, η) to $(i_X\zeta, \eta)$. Recall from [3] that a natural system D on $\mathbf{C}$ induces a natural system $D^\#$ on $\text{Pair}(\mathbf{C})$ with

$$D^\#(\zeta, \eta) = D(\eta f) \ / \ f^* D(\eta) + g_* D(\zeta)$$

and that there is a natural transformation

$$H^{n+1}(\mathbf{C}, D) \xrightarrow{\lambda} H^n(\text{Pair}(\mathbf{C}), D^\#)$$

Proposition 2.13 Suppose D is strongly compatible with sums. Then there is a well defined natural isomorphism of natural systems

$$\tau : \iota^* \widehat{D} \xrightarrow{\cong} D^\#$$

induced by the identity on D.

Proof: For (ζ, η) in $\text{Pair}(\mathbf{C})$ we have

$$(\iota^* \widehat{D})(\zeta, \eta) = \widehat{D}(i_X\zeta, \eta) = D(\eta f) \ / \ f^* D(\eta) + (g, 1)_* D(i_X\zeta)_2$$

But as $D(i_X\zeta)_2 = (i_X)_* D(\zeta)$ we have $(g, 1)_* D(i_X\zeta)_2 = g_* D(\zeta)$ and the result follows. □

We thus have a natural homomorphism between cohomology groups

$$(2.14) \qquad H^n(\text{Twist}(\mathbf{C}), \widehat{D}) \xrightarrow{\iota_* \tau^*} H^n(\text{Pair}(\mathbf{C}), D^\#)$$

As an addendum to theorem 2.10 we have

Addendum 2.15 If D is strongly compatible with sums then the natural transformation λ factors through λ_{sum}, as shown in the following diagram:

$$\begin{array}{ccc} H^{n+1}(\mathbf{C},D) & \xrightarrow{\lambda_{\text{sum}}} & H^n(\text{Twist}(\mathbf{C}),\widehat{D}) \\ & \searrow^{\lambda} & \downarrow{\scriptstyle \iota_* \tau^*} \\ & & H^n(\text{Pair}(\mathbf{C}),D^{\#}) \end{array}$$

The intricate proof in the appendix of theorem 2.10 and its addendum requires the normalisation theorem A.9. In the following section we describe various topological interpretations of the natural transformation λ_{sum}.

3 Homotopy pairs and twisted homotopy pairs

A *track category*

$$T \rightrightarrows \mathbf{K} \xrightarrow{p} \mathbf{C}$$

is a groupoid-enriched category $T\mathbf{K}$ together with a functor $p : \mathbf{K} \to \mathbf{C}$ which is the identity on objects, is full, and satisfies $p(f) = p(g)$ on morphisms if and only if $T(f,g)$ is non-empty. Here $T(f,g)$ is the set of 2-morphisms $f \to g$ for $f,g : A \to B$ in $\mathbf{K}$. The category $\mathbf{C}$ is termed the quotient category of $T\mathbf{K}$, and is also denoted by $\mathbf{K}/\simeq$.

Example 3.1 Let I be the unit interval in the category $\mathbf{Top}^*$ of pointed topological spaces. For X a pointed space, let $IX = I \times X / I \times \{*\}$ be the reduced cylinder on X. For maps $f,g : X \to Y$ in $\mathbf{Top}^*$ let

$$T(f,g) \quad = \quad [IX,Y]^{(f,g)}$$

be the set of homotopy classes rel. $X \vee X$ of maps $H : IX \to Y$ with

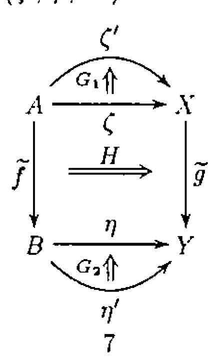
$$(f,g) \ = \ H(i_0,i_1) \ : X \vee X \to IX \to Y$$

This defines a track category

$$T \rightrightarrows \mathbf{Top}^* \xrightarrow{p} \mathbf{Top}^*/\simeq$$

with quotient category the homotopy category of pointed topological spaces.

Let $T\mathbf{K}$ be a track category with quotient category $\mathbf{C}$. For each morphism f of $\mathbf{C}$ we choose a fixed morphism $\tilde{f}$ in $\mathbf{K}$ with $p(\tilde{f}) = f$. Recall from [8] that the category Hopair($T\mathbf{K}$) of homotopy pairs in $T\mathbf{K}$ is the category with objects the morphisms f of $\mathbf{C}$ and morphisms $\{\zeta,\eta,H\} : f \to g$ given by equivalence classes of 3-tuples (ζ,η,H) with $H \in T(\eta\tilde{f},\tilde{g}\zeta)$. The equivalence relation on the morphisms is defined by $(\zeta,\eta,H) \sim (\zeta',\eta',H')$ if there exist tracks $G_1 \in T(\zeta,\zeta')$, $G_2 \in T(\eta',\eta)$ such that H' is the composite track



Moreover applying the functor $p : \mathbf{K} \to \mathbf{C}$ to a morphism in Hopair($T\mathbf{K}$) gives a commutative diagram in $\mathbf{C}$, and we have a functor

$$\text{Hopair}(T\mathbf{K}) \xrightarrow{\ \widehat{p}\ } \text{Pair}(\mathbf{C}) \tag{3.2}$$

which is the identity on objects and is full.

We now generalise this to twisted pairs, under certain conditions on $T\mathbf{K}$. We assume that finite sums exist in $\mathbf{K}$ and $\mathbf{C}$ and are preserved by p, and that $*$ is both initial and terminal in $\mathbf{K}$ and $\mathbf{C}$.

Definition 3.3 The track structure of $T\mathbf{K}$ is *compatible with sums* if for each sum $A \vee B$ in $\mathbf{K}$ the induced groupoid homomorphism

$$T\mathbf{K}(A \vee B, X) \xrightarrow{(i_A^*, i_B^*)} T\mathbf{K}(A, X) \times T\mathbf{K}(B, X)$$

is an isomorphism, and if the groupoids $T\mathbf{K}(*, X)$ and $T\mathbf{K}(X, *)$ are just the trivial group.

For morphisms $f, f' : A \to X$, $g, g' : B \to X$ in $\mathbf{K}$ and tracks $G \in T(f, f')$, $H \in T(g, g')$ we write $(G, H) \in T\mathbf{K}(A \vee B, X)$ for the corresponding track from (f, g) to (f', g').

Definition 3.4 Let $T\mathbf{K}$ be a track category compatible with sums, with $\mathbf{C}$ the corresponding quotient category. The category Hotwist($T\mathbf{K}$) of *twisted homotopy pairs* in $T\mathbf{K}$ is the category with objects the morphisms f of $\mathbf{C}$, and morphisms $\{\xi, \eta, H_0, H\} : f \to g$ the equivalence classes of 4-tuples (ξ, η, H_0, H) given by morphisms ξ, η of $\mathbf{K}$ and tracks H_0, H as shown in the following diagrams

$$\begin{array}{ccc} A & \xrightarrow{\ \xi\ } & X \vee Y \\ {\scriptstyle 0}\downarrow & \overset{H_0}{\Longrightarrow} & \downarrow{\scriptstyle (0,1)} \\ * & \xrightarrow[\ 0\ ]{} & Y \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\ \xi\ } & X \vee Y \\ {\scriptstyle \tilde{f}}\downarrow & \overset{H}{\Longrightarrow} & \downarrow{\scriptstyle (\tilde{g},1)} \\ B & \xrightarrow[\ \eta\ ]{} & Y \end{array} \tag{3.5}$$

The equivalence relation on such 4-tuples is defined by $(\xi, \eta, H_0, H) \sim (\xi', \eta', H_0', H')$ if there exist tracks $G_1 \in T(\xi, \xi')$, $G_2 \in T(\eta', \eta)$ such that H_0', H' are the composite tracks

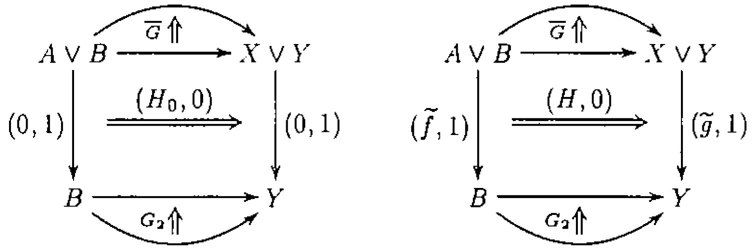
$$\begin{array}{ccc} & \overset{\xi'}{\frown} \ {\scriptstyle G_1 \Uparrow} & \\ A & \xrightarrow{\ \xi\ } & X \vee Y \\ {\scriptstyle 0}\downarrow & \overset{H_0}{\Longrightarrow} & \downarrow{\scriptstyle (0,1)} \\ * & \xrightarrow[\ 0\ ]{} & Y \end{array} \qquad \begin{array}{ccc} & \overset{\xi'}{\frown} \ {\scriptstyle G_1 \Uparrow} & \\ A & \xrightarrow{\ \xi\ } & X \vee Y \\ {\scriptstyle \tilde{f}}\downarrow & \overset{H}{\Longrightarrow} & \downarrow{\scriptstyle (\tilde{g},1)} \\ B & \xrightarrow{\ \eta\ } & Y \\ & \underset{\eta'}{\smile} \ {\scriptstyle G_2 \Uparrow} & \end{array} \tag{3.6}$$

respectively.

Since the track structure is compatible with sums we can equivalently define the morphisms via the diagrams

$$\begin{array}{ccc} A \vee B & \xrightarrow{(\xi, i_Y \eta)} & X \vee Y \\ {\scriptstyle (0,1)}\downarrow & \overset{(H_0, 0)}{\Longrightarrow} & \downarrow{\scriptstyle (0,1)} \\ B & \xrightarrow[\ \eta\ ]{} & Y \end{array} \qquad \begin{array}{ccc} A \vee B & \xrightarrow{(\xi, i_Y \eta)} & X \vee Y \\ {\scriptstyle (\tilde{f},1)}\downarrow & \overset{(H, 0)}{\Longrightarrow} & \downarrow{\scriptstyle (\tilde{g},1)} \\ B & \xrightarrow[\ \eta\ ]{} & Y \end{array}$$

subject to the equivalence relation indicated by



where $\overline{G}$ is a track of the form $(G_1, -(i_Y)_* G_2)$. Then we can define composition in Hotwist($T\mathbf{K}$) by the horizontal composition of such diagrams.

Proposition 3.7 Let $T\mathbf{K}$ be a track category with quotient category $\mathbf{C}$ and compatible with sums. Then there is a well defined functor

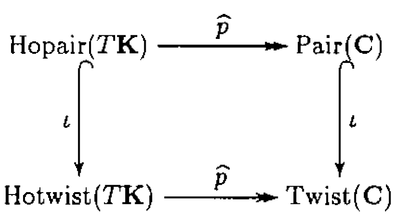
$$\text{Hotwist}(T\mathbf{K}) \xrightarrow{\widehat{p}} \text{Twist}(\mathbf{C})$$

which is the identity on objects and is full.

Proof: Given a morphism $(\xi, \eta, H_0, H) : f \to g$ of Hotwist($T\mathbf{K}$) we get via p a well defined morphism $(p\xi, p\eta) : f \to g$ of Twist($\mathbf{C}$); this defines $\widehat{p}$. Also $\widehat{p}$ is full: given (ξ, η) in Twist($\mathbf{C}$) we can choose tracks $H_0 \in T(0_{AY}, (0,1)\tilde{\xi})$, $H \in T(\tilde{\eta}\tilde{f}, (\tilde{g}, 1)\tilde{\xi})$ and we have $\widehat{p}(\tilde{\xi}, \tilde{\eta}, H_0, H) = (\xi, \eta)$. □

This is a generalisation of (3.2); there is an inclusion ι of Hopair($T\mathbf{K}$) into Hotwist($T\mathbf{K}$) with $\iota\{\zeta, \eta, H\} = \{i_Y\zeta, \eta, 0, i_Y H\}$, and the following diagram commutes:

(3.8)



We show now that this map ι is a map of linear extensions of categories.

Definition 3.9 Compare [1, 4, 5]. Suppose D is a natural system on a category $\mathbf{C}$.

A *linear extension* of $\mathbf{C}$ by D

$$D \xrightarrow{+} \mathbf{K} \xrightarrow{p} \mathbf{C}$$

consists of a category $\mathbf{K}$ with the same objects as $\mathbf{C}$, a functor $p : \mathbf{K} \to \mathbf{C}$ which is the identity on objects and is full, and for each morphism f of $\mathbf{C}$ a transitive effective action $+$ of $D(f)$ on $p^{-1}(f)$ satisfying the *linear distributivity law*

$$(g_0 + b)(f_0 + a) \quad = \quad g_0 f_0 + g_* a + f^* b$$

A *linear track extension* of $\mathbf{C}$ by D

$$D \xrightarrow{+} T \rightrightarrows \mathbf{K} \xrightarrow{p} \mathbf{C}$$

consists of a track category $T\mathbf{K}$ whose quotient category is $\mathbf{C}$, and for each morphism f of $\mathbf{K}$ an isomorphism of groups

$$\sigma_f : D(pf) \cong T(f, f)$$

such the natural system respects the compositions in $T\mathbf{K}$:

$$\begin{aligned} g \cdot \sigma_f(a) &= \sigma_{gf}((pg)_* a) \\ \sigma_g(b) \cdot f &= \sigma_{gf}((pf)^* b) \\ H + \sigma_f(a) &= \sigma_h(a) + H \quad \text{for } H \in T(f,h) \end{aligned}$$

Any fibration category or cofibration category [1] gives rise to such linear track extensions; compare (4.4).

In [3] it is shown that the category of homotopy pairs associated to a linear track extension $T\mathbf{K}$ yields a linear extension of categories

$$D\# \overset{+}{\rightarrowtail} \mathrm{Hopair}(T\mathbf{K}) \overset{\widehat{p}}{\twoheadrightarrow} \mathrm{Pair}(\mathbf{C}) \tag{3.10}$$

This extends to the category Hotwist($T\mathbf{K}$) as follows.

Proposition 3.11 Let D be a natural system on $\mathbf{C}$ which is strongly compatible with sums, and let $\mathbf{C}$ be the quotient category of a track category $T\mathbf{K}$ where the track structure is compatible with sums as in definition 3.3. If $T\mathbf{K}$ is part of a linear track extension

$$D \overset{+}{\rightarrowtail} T \rightrightarrows \mathbf{K} \overset{p}{\twoheadrightarrow} \mathbf{C}$$

then (3.8) and (3.10) are part of a map of linear extensions of categories

$$\begin{array}{ccccc} D\# & \overset{+}{\rightarrowtail} & \mathrm{Hopair}(T\mathbf{K}) & \overset{\widehat{p}}{\twoheadrightarrow} & \mathrm{Pair}(\mathbf{C}) \\ \downarrow{\scriptstyle \tau^{-1}\iota^*} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ \widehat{D} & \overset{+}{\rightarrowtail} & \mathrm{Hotwist}(T\mathbf{K}) & \overset{\widehat{p}}{\twoheadrightarrow} & \mathrm{Twist}(\mathbf{C}) \end{array}$$

where $\widehat{D}$ acts by

$$\{\xi, \eta, H_0, H\} + [a] \quad = \quad \{\xi, \eta, H_0, H + \sigma_{\eta\bar{f}}(a)\}$$

for $a \in D(p(\eta)f)$ as in (2.5), or equivalently by

$$\{\xi, \eta, H_0, H\} + [b', b] \quad = \quad \{\xi, \eta, H_0 + \sigma_{0_{AY}}(b'), H + \sigma_{\eta\bar{f}}(b)\}$$

for $(b', b) \in D(0_{AY}) \oplus D(p(\eta)f)$ as in (2.9).

Proof: We first show the action is well-defined. If $[b', b] = 0$ we have

$$(b', b) \quad = \quad ((0,1)_* y,\ f^* x + (g,1)_* y)$$

for $x \in D(p\eta)$, $y \in D(p\xi)$. Then

$$\begin{aligned} H_0 + \sigma_{0_{AY}}(b') &= \sigma_{(0,1)\xi}((0,1)_* y) + H_0 &&= (0,1) \cdot \sigma_\xi(y) + H_0 \\ H + \sigma_{\eta\bar{f}}(b) &= \sigma_{(\bar{g},1)\xi}((g,1)_* y) + H + \sigma_{\eta\bar{f}}(f^* x) &&= (\bar{g},1) \cdot \sigma_\xi(y) + H + \sigma_\eta(x) \cdot \tilde{f} \end{aligned}$$

and so by the definition of the equivalence relation in (3.6) we have

$$\{\xi, \eta, H_0, H\} + [b', b] \quad = \quad \{\xi, \eta, H_0, H\} \tag{3.12}$$

as required. Conversely the same argument in reverse shows that if (3.12) holds then $[b', b] = 0$; thus the action is effective.

For transitivity consider morphisms

$$\{\xi, \eta, H_0, H\},\ \{\xi', \eta', H_0', H'\} : f \to g$$

in Hotwist($T\mathbf{K}$) with $(p\xi, p\eta) = (p\xi', p\eta')$. Choose $G_1 \in T(\xi, \xi')$, $G_2 \in T(\eta', \eta)$ and then we have

$$\{\xi, \eta, H_0, H\} \quad = \quad \{\xi', \eta', H_0'', H''\}$$

where now H_0'', H'' are the composite tracks shown in (3.6). Now define $(b', b) \in D(0_{AY}) \oplus D(p(\eta)f)$ by

$$\sigma_{0_{AY}}(b') = -H_0'' + H_0' \quad \text{and} \quad \sigma_{\eta \tilde{f}}(b) = -H'' + H'$$

so that

$$\{\xi, \eta, H_0, H\} + [b', b] \;=\; \{\xi', \eta', H_0'', H''\} + [b', b] \;=\; \{\xi', \eta', H_0', H'\}$$

as required. □

Two linear track extensions $T\mathbf{K}$ are termed equivalent if they are objects in the same connected component of the category Track($\mathbf{C}, D$), where morphisms $T\mathbf{K} \to T'\mathbf{K}'$ in this category are given by groupoid-enriched functors which commute with the isomorphisms $\sigma_{f,f}$ and the functors p. Two linear extensions $\mathbf{K}$, $\mathbf{K}'$ are termed equivalent if there is an isomorphism $\mathbf{K} \cong \mathbf{K}'$ which respects the actions of D and induces the identity on $\mathbf{C}$. Writing $M^2(\mathbf{C}, D)$ and $M^3(\mathbf{C}, D)$ for set of equivalence classes of linear extensions and of linear track extensions respectively, we have

Theorem 3.13 There are natural bijections

$$M^n(\mathbf{C}, D) \quad \cong \quad H^n(\mathbf{C}, D)$$

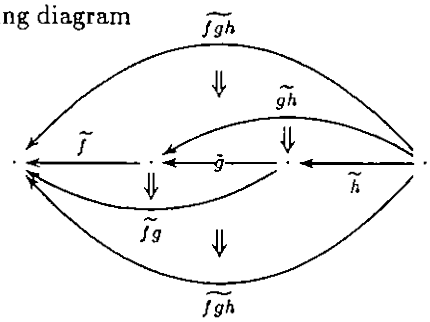
for $n = 2, 3$.

Proof: See [4, 5]. The 2-cocycle Δ corresponding to a linear extension $\mathbf{K}$ measures the non-functoriality of the function $\mathbf{C} \to \mathbf{K}$, $f \mapsto \tilde{f}$, and takes $(f, g) \in \text{Ner}(\mathbf{C})_2$ to $\Delta_{f,g} \in D(fg)$ given by

$$\widetilde{fg} \quad = \quad \tilde{f}\tilde{g} + \Delta_{f,g}$$

in $\mathbf{K}$. The 3-cocycle corresponding to a linear track extension $T\mathbf{K}$ measures the non-associativity of lifting composites from $\mathbf{C}$ to $T\mathbf{K}$; it takes $(f, g, h) \in \text{Ner}(\mathbf{C})_3$ to $a \in D(fgh)$ where $\sigma_{fgh}(a)$ is given by tracks as in the following diagram

(3.14)



in $T\mathbf{K}$. □

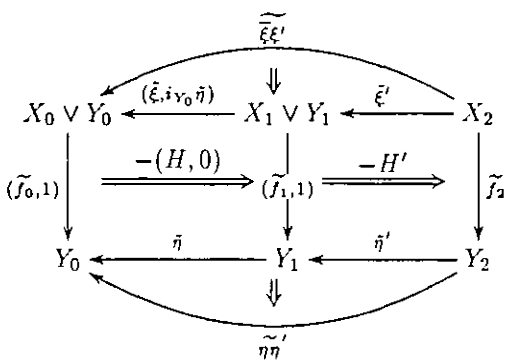
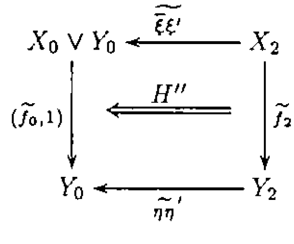
We now have the following crucial application of the natural transformation λ_{sum} of theorem 2.10. This was our original motivation for the study of this transformation.

Theorem 3.15 The map

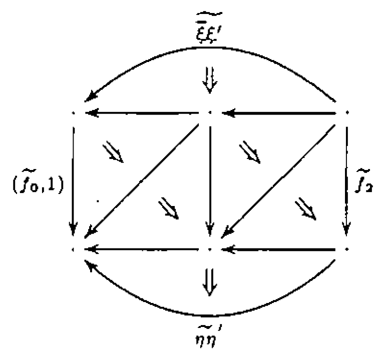
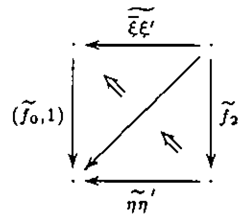
$$H^3(\mathbf{C}, D) \xrightarrow{\lambda_{\text{sum}}} H^2(\text{Twist}(\mathbf{C}), \widehat{D})$$

in (2.11) takes the class of the linear track extension $T\mathbf{K}$ to the class of the corresponding linear extension Hotwist($T\mathbf{K}$) of proposition 3.11.

Proof: The cocycle $\Delta \in H^2(\mathrm{Twist}(\mathbf{C}), \widehat{D})$ corresponding to $\mathrm{Hotwist}(T\mathbf{K})$ is defined as follows. For morphisms (ξ, η), (ξ', η') and $(\xi\xi', \overline{\eta}\eta')$ of $\mathrm{Twist}(\mathbf{C})$ as in (2.6) we choose corresponding lifts $(\widetilde{\xi}, \widetilde{\eta}, H_0, H)$, $(\widetilde{\xi}', \widetilde{\eta}', H_0', H')$ and $(\widetilde{\xi\xi'}, \widetilde{\eta\eta}', H_0'', H'')$ in $\mathrm{Hotwist}(T\mathbf{K})$. Then $\Delta((\xi, \eta), (\xi', \eta')) = [b', b]$ where $\sigma_{\eta\eta' f_2}(b)$ is a composite of tracks given by the sum of the following diagrams.

and $\sigma_{0_{X_0 Y_2}}(b')$ similarly. However adding the diagonals $(\widetilde{\eta f_1}, \widetilde{\eta})$, $\widetilde{\eta' f_2}$ and $\widetilde{\eta\eta' f_2}$ before choosing the tracks we see that $\sigma(b)$ can also be described by diagrams

Thus b is an element which represents

$$c((f_0, 1), \overline{\xi}, \xi') - c(\eta, (f_1, 1), \xi') + c(\eta, \eta', f_2)$$

for $c \in H^3(\mathbf{C}, D)$ the cohomology class of the linear track extension $T\mathbf{K}$, as described in (3.14). Together with the corresponding statement for b' this shows $\lambda_{\mathrm{sum}}(c) = \Delta$ as required. □

4 Universal Toda brackets and twisted maps between cofibres

Let $\mathbf{Top}^*$ be the track category in example 3.1, with quotient category $\mathbf{Top}^*/\simeq$. We consider the *cofibre functor*

(4.1) $$\mathrm{Hotwist}(\mathbf{Top}^*) \xrightarrow{\;C\;} \mathbf{Top}^*/\simeq$$

which carries an object f of $\mathrm{Hotwist}(\mathbf{Top}^*)$ to the mapping cone (or homotopy cofibre) $C(f)$ of $\widetilde{f}$. Here $\widetilde{f} : A \to B \in \mathbf{Top}^*$ represents the homotopy class $f : A \to B \in \mathbf{Top}^*/\simeq$, and the mapping cone of $\widetilde{f}$ is the pushout

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & CA \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle \pi_{\widetilde{f}}} \\ B & \xrightarrow[i_{\widetilde{f}}]{} & C(f) \end{array}$$

where $CA = IA/i_1A$ is the cone on A.

Suppose $\{\xi, \eta, H_0, H\} : f \to g$ is a morphism of Hotwist(**Top***) and let H_1 be the homotopy

$$\begin{array}{ccc} X & \xrightarrow{\ 0\ } & Y \\ {\scriptstyle \tilde{g}}\downarrow & \xRightarrow{H_1} & \downarrow{\scriptstyle i_{\tilde{g}}} \\ Y & \xrightarrow{\ i_{\tilde{g}}\ } & C(g) \end{array}$$

given by $\pi_{\tilde{g}} : CX \to C(g)$. Then we define $F = C(\xi, \eta, H_0, H)$

$$C(f) \xrightarrow{\ F\ } C(g)$$

to be the map with $Fi_{\widetilde{f}} = i_{\tilde{g}}\eta$ and with $F\pi_{\widetilde{f}}$ given by the following sum of the homotopies H, H_0, H_1:

$$\text{(4.2)} \qquad \begin{array}{ccccc} & & \overset{0}{\frown} & & \\ & & -H_0 \Uparrow & & \\ A & \xrightarrow{\ \xi\ } & X \vee Y & \xrightarrow{\ (0,1)\ } & Y \\ {\scriptstyle \widetilde{f}}\downarrow & \xRightarrow{H} & (\widetilde{g}, 1)\downarrow & \xRightarrow{(H_1, 0)} & \downarrow{\scriptstyle i_{\tilde{g}}} \\ B & \xrightarrow{\ \eta\ } & Y & \xrightarrow{\ i_{\tilde{g}}\ } & C(g) \end{array}$$

Such a map $F : C(f) \to C(g)$ is termed a *twisted map* in [1].

Proposition 4.3 The cofibre functor in (4.1) is well-defined.

Proof: Certainly $F\pi_{\widetilde{f}}$ defines a map $CA \to C(g)$ with $(F\pi_{\widetilde{f}})i = i_{\tilde{g}}\eta\widetilde{f} = (Fi_{\widetilde{f}})\widetilde{f}$, and so F is a map from the pushout $C(f)$ to $C(g)$. Now suppose $F' = C(\xi', \eta', H_0', H')$ and we have G_1, G_2 as in (3.6) so that

$$\{\xi, \eta, H_0, H\} \quad = \quad \{\xi', \eta', H_0', H'\}$$

Then the contributions of G_1 to H' and $-H_0'$ in $F'\pi_{\widetilde{f}}$ cancel, giving

$$F'\pi_{\widetilde{f}} \quad = \quad F\pi_{\widetilde{f}} + i_{\tilde{g}}G_2\widetilde{f}$$

We therefore have a homotopy $F' \simeq F$ given by

$$IC(f) \xrightarrow{\ q\ } CA \vee_A IB \xrightarrow{F\pi_{\widetilde{f}} \vee_A G_2} C(g)$$

where $CA \vee_A IB$ is the pushout of $i_1\widetilde{f} : A \to IB$ along $i : A \to CA$ and q is induced by the quotient $ICA \to CA$. □

Our main example of a linear track extension is the following. Let $\mathbf{Top}^*_\Sigma$ be a full subcategory of the category $\mathbf{Top}^*$ of pointed topological spaces such that all objects of $\mathbf{Top}^*_\Sigma$ are suspensions. Let $\mathbf{Top}^*_\Sigma/\simeq$ be the homotopy category of $\mathbf{Top}^*_\Sigma$ and let T be the track structure on $\mathbf{Top}^*_\Sigma$ given as in example 3.1 by

$$T(f,g) \;=\; [IX,Y]^{(f,g)}$$

for $f,g : X \to Y$. Then there is a natural system D_Σ on $\mathbf{Top}^*_\Sigma/\simeq$ such that $\mathbf{Top}^*_\Sigma$ is part of a linear track extension

$$(4.4) \qquad D_\Sigma \overset{+}{\rightarrowtail} T \rightrightarrows \mathbf{Top}^*_\Sigma \overset{p}{\twoheadrightarrow} \mathbf{Top}^*_\Sigma/\simeq$$

The natural system D_Σ on $\mathbf{Top}^*_\Sigma/\simeq$ is defined by the homotopy groups

$$D_\Sigma(f) \;=\; [\Sigma X, Y]$$

for $f : X \to Y$, with the homomorphisms $g_* : D(f) \to D(gf)$ given by $a \mapsto ga$ as in example 2.4. The homomorphisms $f^* : D(g) \to D(gf)$ are more complicated to define. See section 3 of [3] for details.

The cohomology class represented by (4.4) via theorem 3.13 is the *universal Toda bracket*

$$T_\Sigma \;\in\; H^3(\mathbf{Top}^*_\Sigma/\simeq, D_\Sigma)$$

which depends only on the homotopy category $\mathbf{Top}^*_\Sigma/\simeq$.

Now let $\mathcal{X}_\Sigma$ be a class of maps in $\mathbf{Top}^*_\Sigma/\simeq$, and let $\mathrm{Twist}(\mathcal{X}_\Sigma)$ and $\mathrm{Hotwist}(\mathcal{X}_\Sigma)$ be the full subcategories of $\mathrm{Twist}(\mathbf{Top}^*_\Sigma/\simeq)$ and $\mathrm{Hotwist}(\mathbf{Top}^*_\Sigma)$ respectively whose objects are in $\mathcal{X}_\Sigma$. Then by proposition 4.4, definition 3.4 and proposition 3.11 we have a linear extension of categories

$$(4.5) \qquad \widehat{D}_\Sigma \overset{+}{\rightarrowtail} \mathrm{Hotwist}(\mathcal{X}_\Sigma) \overset{\widehat{p}}{\twoheadrightarrow} \mathrm{Twist}(\mathcal{X}_\Sigma)$$

From theorem 3.15 we have the following result which shows that this extension is determined by the universal Toda bracket.

Theorem 4.6 The universal Toda bracket T_Σ determines the cohomology class

$$\langle \mathrm{Hotwist}(\mathcal{X}_\Sigma)\rangle \;\in\; H^2(\mathrm{Twist}(\mathcal{X}_\Sigma), \widehat{D}_\Sigma)$$

corresponding to the linear extension of categories (4.5). In fact this class is a restriction of $\lambda_{\mathsf{sum}}(T_\Sigma)$.

The cofibre functor C in (4.1) is compatible with the linear extension (4.5) above in the following sense. Let $\mathbf{C}(\mathcal{X}_\Sigma)$ be the full subcategory of $\mathbf{Top}^*/\simeq$ whose objects arise as mapping cones $C(f)$ of maps $f \in \mathcal{X}_\Sigma$.

Theorem 4.7 Let $a \geq 3$ and suppose $\mathcal{X}_\Sigma$ is a class of maps $h : A \to B$ between suspensions $A = \Sigma A'$, $B = \Sigma B'$ of CW complexes A', B' such that A is $(a-1)$-connected, B is simply-connected, $\dim(A) \leq 2a-1$ and $\dim(B) \leq a-1$. Then there is a map of linear extensions of categories

$$\begin{array}{ccccc}
\widehat{D}_\Sigma & \overset{+}{\rightarrowtail} & \mathrm{Hotwist}(\mathcal{X}_\Sigma) & \twoheadrightarrow & \mathrm{Twist}(\mathcal{X}_\Sigma) \\
{\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle C} & & \Vert \\
\Gamma/I & \overset{+}{\rightarrowtail} & \mathbf{C}(\mathcal{X}_\Sigma) & \twoheadrightarrow & \mathrm{Twist}(\mathcal{X}_\Sigma)
\end{array}$$

where C as in (4.1) is full, and τ is a surjective natural transformation. If $\dim(A) \leq 2a-2$ for all $h : A \to B \in \mathcal{X}_\Sigma$ then C and τ are isomorphisms.

Note that this is a significant improvement of the corresponding theorem for homotopy pairs in [3], which assumes higher connectivity for B.

As in theorem 4.6 we also have:

Corollary 4.8 Suppose $\mathcal{X}_\Sigma$ is given as in theorem 4.7. Then the homotopy category $\mathbf{C}(\mathcal{X}_\Sigma)$ is determined by the universal Toda bracket T_Σ. That is, the cohomology class

$$\langle \mathbf{C}(\mathcal{X}_\Sigma)\rangle \quad \in \quad H^2(\mathrm{Twist}(\mathcal{X}_\Sigma), \Gamma/I)$$

is the image under τ_* of the restriction of $\lambda_{\mathrm{sum}}(T_\Sigma)$ to $\mathrm{Twist}(\mathcal{X}_\Sigma)$.

Proof of Theorem 4.7: The assumptions on $\mathcal{X}_\Sigma$ show that $\mathbf{C}(\mathcal{X}_\Sigma) = \mathrm{TWIST}(\mathcal{X}_\Sigma)$ where the right-hand side is defined in [1, V.3.14]. Hence the result follows from (V.7.17,18) in [1], where also an explicit description of Γ/I is given when $\dim(A) = 2a - 1$. □

As an application of theorem 4.7 we consider the following category of CW-spaces with only two non-trivial homology groups. Let $2 \leq b \leq a$ and let $\mathbf{H}(b, a+1)$ be the full homotopy category of all simply-connected CW-spaces with homology groups in degree b and $a+1$, that is

$$H_b(X) = B, \quad H_{a+1}(X) = A, \quad \text{and} \quad \widetilde{H}_i(X) = 0 \quad \text{for } i \neq a+1, b$$

It is well known that X is the mapping cone of a map $h : M(A,a) \to M(B,b)$ between Moore spaces and it is an old problem of algebraic topology to determine the category $\mathbf{H}(b, a+1)$ by use of such attaching maps h; compare [6]. The next result yields such a classification of $\mathbf{H}(b, a+1)$.

Theorem 4.9 Let $2 \leq b < a - 1$ and let $\mathcal{X}_\Sigma^{b,a}$ be the class of all homotopy classes

$$M(A,a) \xrightarrow{\;h\;} M(B,b)$$

where A and B are abelian groups. Then there is a linear extension of categories

$$\begin{array}{ccccc} \widehat{D}_\Sigma & \overset{+}{\rightarrowtail} & \mathrm{Hotwist}(\mathcal{X}_\Sigma^{b,a}) & \twoheadrightarrow & \mathrm{Twist}(\mathcal{X}_\Sigma^{b,a}) \\ & & \downarrow\cong & & \\ & & \mathbf{C}(\mathcal{X}_\Sigma^{b,a}) & & \\ & & \downarrow\cong & & \\ & & \mathbf{H}(b, a+1) & & \end{array}$$

where the vertical arrows are equivalences of categories. As in 4.8 the cohomology class of the extension is determined by the universal Toda bracket T_Σ.

We point out that for $b \geq 3$ the category $\mathrm{Twist}(\mathcal{X}_\Sigma^{b,a})$ in theorem 4.9 coincides with $\mathrm{Pair}(\mathcal{X}_\Sigma^{b,a})$ and that in this case the theorem is also treated in [3]. For $b = 2$ however one has to use twisted maps to describe $\mathbf{H}(b, a+1)$. Theorem 4.9 implies the following result on the classification of homotopy types with two homology groups.

Corollary 4.10 Let $2 \leq b < a - 1$. For $b \geq 3$ the isomorphism types in $\mathrm{Pair}(\mathcal{X}_\Sigma^{b,a})$ are in 1-1 correspondence with the homotopy types in $\mathbf{H}(b, a+1)$. For $b \geq 2$ the isomorphism types in $\mathrm{Twist}(\mathcal{X}_\Sigma^{b,a})$ are in 1-1 correspondence with the homotopy types in $\mathbf{H}(b, a+1)$.

The case $b \geq 3$ of the corollary is an old result of Brown–Copeland [6].

5 Universal Toda brackets and twisted maps between fibres

This section is dual to section 4. Let $\mathbf{C}^{\mathrm{op}}$ be the *opposite category* of a category $\mathbf{C}$. This construction is the basis of duality in category theory; for example sums in $\mathbf{C}^{\mathrm{op}}$ are just products in $\mathbf{C}$. This leads to the following dual notion of the category Twist$(-)$ in 2.2.

Let $\mathbf{C}$ be a category with finite products, that is, with binary products $A \times B$ and a terminal object $*$. Suppose that $*$ is also an initial object. We define the category Twist$'(\mathbf{C})$ by the dual of definition 2.2.

$$\text{Twist}'(\mathbf{C}) \quad = \quad (\text{Twist}(\mathbf{C}^{\mathrm{op}}))^{\mathrm{op}} \tag{5.1}$$

Thus a morphism $(\xi, \eta) : f \to g$ of Twist$'(\mathbf{C})$ is given by commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\ \xi\ } & X \\ {\scriptstyle (1,f)}\downarrow & & \downarrow{\scriptstyle g} \\ A \times B & \xrightarrow[\ \eta\ ]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} A & & \\ {\scriptstyle (1,0)}\downarrow & \searrow{\scriptstyle 0} & \\ A \times B & \xrightarrow[\ \eta\ ]{} & Y \end{array}$$

in $\mathbf{C}$.

Dualising definitions 1.1 and 2.3 we have the notion of a natural system which is strongly compatible with products. Such a natural system D on $\mathbf{C}$ induces a natural system $\widehat{D}'$ on Twist$'(\mathbf{C})$ by the dual of (2.5). Then theorem 2.10 becomes

Theorem 5.2 Suppose D is a natural system on $\mathbf{C}$ which is strongly compatible with products. Then there is a well-defined natural transformation

$$H^{n+1}(\mathbf{C}, D) \xrightarrow{\lambda_{\mathrm{prod}}} H^n(\text{Twist}'(\mathbf{C}), \widehat{D}')$$

The dual of addendum 2.15 says also that λ factors through λ_{prod}; the proofs require normalisation with respect to products as in theorem A.11 in the appendix.

We have the notion of the opposite of a track category, and we say a track category $T\mathbf{K}$ is compatible with products if $T\mathbf{K}^{\mathrm{op}}$ is compatible with sums as in definition 3.3. Then similarly to 5.1 we define

$$\text{Hotwist}'(T\mathbf{K}) \quad = \quad (\text{Hotwist}(T\mathbf{K}^{\mathrm{op}}))^{\mathrm{op}}$$

for $T\mathbf{K}$ compatible with products. For $T\mathbf{K}$ a linear track extension of $\mathbf{C}$, Hotwist$'(T\mathbf{K})$ is a linear extension of Twist$'(\mathbf{C})$ dually to proposition 3.11, and λ_{prod} takes the cohomology class representing $T\mathbf{K}$ to that representing Hotwist$'(T\mathbf{K})$ by the dual of theorem 3.15.

In this section we consider the track category $\mathbf{Top}^*$ and the *fibre functor*

$$\text{Hotwist}'(\mathbf{Top}^*) \xrightarrow{\ P\ } \mathbf{Top}^*/\simeq \tag{5.3}$$

which carries an object f of Hotwist$'(\mathbf{Top}^*)$ to the fibre (or homotopy fibre) $P(f)$ of $\tilde{f}$ where again $\tilde{f}$ represents the homotopy class f. For this recall that $P(f)$ is constructed dually to $C(f)$ in section 4 using the duality of I-categories and P-categories in [1].

We obtain the dual of the linear track extension (4.4) as follows. Let $\mathbf{Top}^*_\Omega$ be a full subcategory of the category $\mathbf{Top}^*$ of pointed topological spaces such that all objects of $\mathbf{Top}^*_\Omega$ are loop spaces. Let $\mathbf{Top}^*_\Omega/\simeq$ be the homotopy category of $\mathbf{Top}^*_\Omega$ and let T be the track structure on $\mathbf{Top}^*_\Omega$ given as in example 3.1. Then there is a natural system D_Ω on $\mathbf{Top}^*_\Omega/\simeq$ such that $\mathbf{Top}^*_\Omega$ is part of a linear track extension

$$(5.4)\qquad D_\Omega \overset{+}{\rightarrowtail} T \rightrightarrows \mathbf{Top}^*_\Omega \overset{p}{\twoheadrightarrow} \mathbf{Top}^*_\Omega/\simeq$$

The natural system D_Ω on $\mathbf{Top}^*_\Omega/\simeq$ is defined by the homotopy groups

$$D_\Omega(f) \;=\; [X, \Omega Y]$$

for $f : X \to Y$, with the homomorphisms $f^* : D(g) \to D(gf)$ given by $a \mapsto af$. The homomorphisms $g_* : D(f) \to D(gf)$ are more complicated to define; see section 3 of [3] for details. We point out that D_Ω does not coincide with D_Σ in (4.4).

The cohomology class represented by (5.4) via theorem 3.13 is the *universal Toda bracket*

$$\mathcal{T}_\Omega \;\in\; H^3(\mathbf{Top}^*_\Omega/\simeq, D_\Omega)$$

which depends only on the homotopy category $\mathbf{Top}^*_\Omega/\simeq$.

Now let $\mathcal{X}_\Omega$ be a class of maps in $\mathbf{Top}^*_\Omega/\simeq$, and let $\mathrm{Twist}'(\mathcal{X}_\Omega)$ and $\mathrm{Hotwist}'(\mathcal{X}_\Omega)$ be the full subcategories of $\mathrm{Twist}'(\mathbf{Top}^*_\Omega/\simeq)$ and $\mathrm{Hotwist}'(\mathbf{Top}^*_\Omega)$ respectively whose objects are in $\mathcal{X}_\Omega$. Then we have the linear extension

$$(5.5)\qquad \widehat{D}'_\Omega \overset{+}{\rightarrowtail} \mathrm{Hotwist}'(\mathcal{X}_\Omega) \overset{\widehat{p}}{\twoheadrightarrow} \mathrm{Twist}'(\mathcal{X}_\Omega)$$

which is dual to (4.5). As a dual of theorem 4.6 we get

Theorem 5.6 The universal Toda bracket $\mathcal{T}_\Omega$ determines the cohomology class

$$\langle \mathrm{Hotwist}'(\mathcal{X}_\Omega)\rangle \;\in\; H^2(\mathrm{Twist}'(\mathcal{X}_\Omega), \widehat{D}'_\Omega)$$

corresponding to the linear extension of categories (5.5). In fact this class is a restriction of $\lambda_{\mathrm{prod}}(\mathcal{T}_\Omega)$.

The fibre functor P in (5.3) is compatible with the linear extension (5.5) above in the following sense. Let $\mathbf{P}(\mathcal{X}_\Omega)$ be the full subcategory of $\mathbf{Top}^*/\simeq$ whose objects arise as homotopy fibres $P(f)$ of maps $f \in \mathcal{X}_\Omega$.

Theorem 5.7 Let $a \geq 3$ and suppose $\mathcal{X}_\Omega$ is a class of maps $h : B \to A$ between loop spaces $A = \Omega A'$, $B = \Omega B'$ of CW complexes A', B' such that A is $(a-1)$-connected, B is simply-connected, $\pi_i(A) = 0$ for $i \geq 2a-1$ and $\pi_i(B) = 0$ for $i \geq a-1$. Then there is an isomorphism of linear extensions of categories

$$\begin{array}{ccccc}
\widehat{D}'_\Omega & \overset{+}{\rightarrowtail} & \mathrm{Hotwist}'(\mathcal{X}_\Omega) & \twoheadrightarrow & \mathrm{Twist}'(\mathcal{X}_\Omega) \\
\tau \downarrow \cong & & \cong \downarrow P & & \| \\
\Gamma/I & \overset{+}{\rightarrowtail} & \mathbf{P}(\mathcal{X}_\Omega) & \twoheadrightarrow & \mathrm{Twist}'(\mathcal{X}_\Omega)
\end{array}$$

where P is defined by (5.3).

Proof: This is a consequence of [1, V.10.19] and the exact sequence 3.3 in [2]. Details of the proof are somewhat sophisticated but are based on the material in section V.10 of [1]. □

Corollary 5.8 Suppose $\mathcal{X}_\Omega$ is given as in theorem 5.7. Then the homotopy category $\mathbf{P}(\mathcal{X}_\Omega)$ is determined by the universal Toda bracket $\mathcal{T}_\Omega$. That is, the cohomology class

$$\langle \mathbf{P}(\mathcal{X}_\Omega)\rangle \;\in\; H^2(\mathrm{Twist}'(\mathcal{X}_\Omega), \Gamma/I)$$

is the image under τ_* of the restriction of $\lambda_{\mathrm{prod}}(\mathcal{T}_\Omega)$ to $\mathrm{Twist}'(\mathcal{X}_\Omega)$.

A Appendix

Sum normalised cohomology of categories

Let C be a (small) category. Since we will always want to have explicit structure maps for sums in C, we make the following definition.

Definition A.1 The category Sum(C) of finite sum diagrams in C is the category with objects all pairs (X, i) with $i = (i_1, \ldots, i_r)$ an r-tuple of morphisms $i_k : X_k \to X$ which gives X the structure of a sum in C. Morphisms $f : (X, i) \to (Y, j)$ in Sum(C) are just morphisms $f : X \to Y$ in C.

The forgetful functor

$$\mathrm{Sum}(\mathbf{C}) \xrightarrow{\ \phi\ } \mathbf{C}$$

given by $(X, i) \mapsto X$, $f \mapsto f$, is an equivalence of categories; an inverse to ϕ is given by the functor ψ which carries X to the *trivial sum diagram* $(X, 1_X)$. We therefore have an isomorphism of cohomology groups

$$H^n(\mathrm{Sum}(\mathbf{C}), \phi^* D) \;\cong\; H^n(\mathbf{C}, D) \tag{A.2}$$

Dually one can define the category Product(C) of all finite product diagrams (X, p) in C; the defining property for products is that post-composing with the morphisms $p_k : X \to X_k$ induces natural bijections of hom-sets

$$p_* : \mathbf{C}(Z, X) \cong \mathbf{C}(Z, X_1) \times \ldots \times \mathbf{C}(Z, X_r)$$

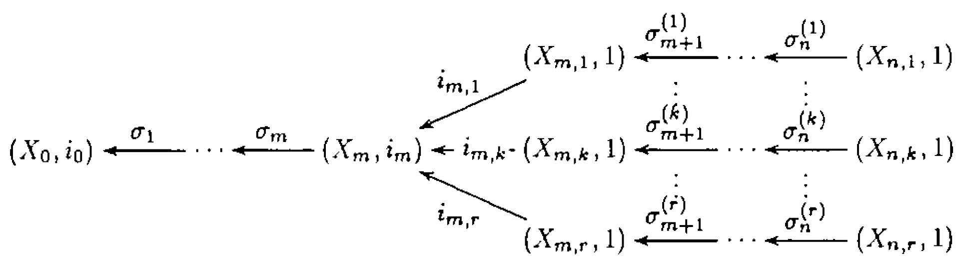
All the definitions and results for sums in this section will have dual formulations for products, with dual proofs.

Definition A.3 A cochain $c \in F^n(\mathrm{Sum}(\mathbf{C}), \phi^* D)$ is said to *respect the sum diagrams* of C if

$$i^*_{n,k} c(\sigma) \;=\; c(\sigma_1, \ldots, \sigma_{m-1}, \sigma_m i_{m,k}, \sigma^{(k)}_{m+1}, \ldots, \sigma^{(k)}_n) \quad \text{for } 1 \le k \le r \tag{A.4}$$

for each $0 \le m \le n$ and whenever $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathrm{Ner}(\mathrm{Sum}(\mathbf{C}))_n$ is such that each σ_j is a sum $\sigma_j = \bigvee_{k=1}^r \sigma_j^{(k)}$ for $j > m$, or in the case $m = n$ that the source of σ_n is a sum $X_n = \bigvee_{k=1}^r X_{n,k}$. If $m = 0$ then (A.4) is supposed to mean $i^*_{n,k} c(\sigma) = i_{0,k,*} c(\sigma_1^{(k)}, \ldots, \sigma_n^{(k)})$.

(A.5)



The collection of cochains on Sum(C) which respect the sum diagrams is written $F^*_{\mathrm{sum}}(\mathbf{C}, D)$.

For a simplex σ as above we will write $\sigma i_{m,k}$ and $(\sigma, i_{m,k})$ for the simplices

$$\begin{aligned} \sigma i_{m,k} &= (\sigma_1, \ldots, \sigma_{m-1}, \sigma_m i_{m,k}, \sigma^{(k)}_{m+1}, \ldots, \sigma^{(k)}_n) \\ (\sigma, i_{m,k}) &= (\sigma_1, \ldots, \sigma_{m-1}, \sigma_m, i_{m,k}, \sigma^{(k)}_{m+1}, \ldots, \sigma^{(k)}_n) \end{aligned}$$

of dimension n and $n+1$ respectively. We write equations (A.4) as $i^*_{n,k} c(\sigma) = c(\sigma i_{m,k})$, or $i^*_{n,k} c(\sigma) = i_{0,k,*} c(\sigma^{(k)})$ when $m = 0$.

Lemma A.6 $F^*_{\text{sum}}(\mathbf{C}, D)$ is a sub-cochain complex of $F^*(\text{Sum}(\mathbf{C}), \phi^* D)$.

Proof: Given $c \in F^{n-1}$ respecting sum diagrams we must show δc respects sum diagrams also. Suppose $\sigma \in \text{Ner}(\text{Sum}(\mathbf{C}))_n$ as in (A.5), with $m \neq 0, n$. Then $i^*_{n,k}(\delta c)(\sigma)$ is given by

$$\text{(A.7)} \qquad i^*_{n,k}\sigma_{1*}c(d_0\sigma) + \textstyle\sum_{i=1}^{n-1}(-1)^i i^*_{n,k}c(d_i\sigma) \; + (-1)^n i^*_{n,k}\sigma^*_n c(d_n\sigma)$$

Now $i^*_{n,k}c(d_i\sigma) = c(d_i(\sigma i_{m,k}))$ for $i < n$ and

$$i^*_{n,k}\sigma^*_n c(d_n\sigma) \;=\; \sigma_n^{(k)^*} i^*_{n-1,k}c(d_n\sigma) \;=\; \sigma_n^{(k)^*} c(d_n(\sigma i_{m,k}))$$

Thus (A.7) becomes

$$\sigma_{1*}c(d_0(\sigma i_{m,k})) + \sum_{i=1}^{n-1}(-1)^i c(d_i(\sigma i_{m,k})) \; + (-1)^n \sigma_n^{(k)^*} c(d_n(\sigma i_{m,k}))$$

which is $(\delta c)(\sigma i_{m,k})$ as required. The proofs for the special cases $m = 0$ and $m = n$ are similar. □

Thus we can define the cohomology groups of $\mathbf{C}$ with respect to the sum diagrams by

$$\text{(A.8)} \qquad H^n_{\text{sum}}(\mathbf{C}, D) \;=\; H^n(F^*_{\text{sum}}(\mathbf{C}, D), \delta_{|F^*_{\text{sum}}})$$

The main result of this section is the following normalisation theorem.

Theorem A.9 If D is compatible with sum diagrams then there is a natural isomorphism

$$H^n_{\text{sum}}(\mathbf{C}, D) \;\cong\; H^n(\mathbf{C}, D)$$

Proof: We show that the inclusion of cochain complexes

$$F^*_{\text{sum}}(\mathbf{C}, D) \;\subseteq\; F^*(\text{Sum}(\mathbf{C}), \phi^* D)$$

induces an isomorphism of cohomology groups

$$H^n_{\text{sum}}(\mathbf{C}, D) \;\cong\; H^n(\text{Sum}(\mathbf{C}), \phi^* D)$$

Then applying the isomorphism (A.2) we get the theorem.

We define for each $(n+1)$-cochain c on $\text{Sum}(\mathbf{C})$ an n-cochain γ_c such that $c + \delta\gamma_c$ respects sums if δc does. Let $\sigma \in \text{Ner}(\text{Sum}(\mathbf{C}))_n$. If the source (X_n, i) is a trivial sum diagram then we put $\gamma_c(\sigma) = 0$; otherwise there is a least m such that σ has the form of (A.5) and we define $\gamma_c(\sigma)$ by

$$i^*_{n,k}\gamma_c(\sigma) \;=\; \sum_{t=m}^{n}(-1)^t c(\sigma_1, \ldots, \sigma_t,\, i_{t,k},\, \sigma^{(k)}_{t+1}, \ldots, \sigma^{(k)}_n) \;=\; \sum_{t=m}^{n}(-1)^t c(\,\sigma,\; i_{t,k}\,)$$

for $1 \leq k \leq r$. This is well-defined since i^*_n is an isomorphism. The source of $(\sigma, i_{t,k})$ is the trivial sum diagram on $X_{n,k}$, so $\gamma_c(d_j(\sigma, i_{t,k})) = 0$ unless $j = t = n$, and hence

$$(\partial\gamma_c)(\sigma, i_{t,k}) \;=\; 0 \quad \text{for } t < n$$
$$(\partial\gamma_c)(\sigma, i_{n,k}) \;=\; (-1)^{n+1} i^*_{n,k}\gamma_c(\sigma) \;=\; \sum_{t=m}^{n}(-1)^{n+t+1}c(\,\sigma,\; i_{t,k}\,)$$

Putting $c' = c + \delta\gamma_c$ we thus have

$$\text{(A.10)} \qquad \sum_{t=m}^{n}(-1)^t c'(\sigma, i_{t,k}) \;=\; 0$$

For $\sigma \in \mathrm{Ner}(\mathbf{C})_{n+1}$ and m minimal such that $\sigma_t = \bigvee_{k=1}^{r} \sigma_t^{(k)}$ for $m < t \leq n+1$ we assume inductively

$$i_{n+1,k}^{*} c'(\sigma) \quad = \quad c'(\sigma i_{t,k}) \quad \text{for } m \leq t < s$$

which holds trivially for $s = m$. Assuming also that c' is normalised with respect to identities we have $c'(\sigma, i_{t,k}) = 0$ for $m \leq t < s$ in (A.10). Together with $d_{t+1}(\sigma, i_{t,k}) = d_{t+1}(\sigma, i_{t+1,k})$ this implies

$$\sum_{t=s}^{n+1} (-1)^t (\delta c')(\sigma, i_{t,k}) \quad = \quad c'(\sigma i_{s,k}) - i_{n+1,k}^{*} c'(\sigma)$$

But this is zero if $\delta c'$ is normalised with respect to sums and identities. □

Dually we can consider the cohomology of the category Product(**C**) of finite product diagrams $(X, (p_1, \ldots, p_r))$ in **C**:

$$H^n(\mathrm{Product}(\mathbf{C}), \phi^* D) \quad = \quad H^n(F^*(\mathrm{Product}(\mathbf{C}), \phi^* D), \delta)$$

The natural system D is compatible with products if the homomorphisms

$$D_f \xrightarrow{\;p_*\;} \bigoplus_{k=1}^{r} D_{p_k f}$$

are always isomorphisms and a cochain $c \in F^*(\mathrm{Product}(\mathbf{C}), \phi^* D)$ is compatible with products if the equations

$$p_{n,k_*} c(\sigma) \quad = \quad c(p_{m,k}\sigma) \quad (1 \leq k \leq r)$$

hold whenever appropriate. Considering only those cochains compatible with products we have a sub-cochain complex

$$F^*_{\mathrm{prod}}(\mathbf{C}, D) \quad \subseteq \quad F^*(\mathrm{Product}(\mathbf{C}), \phi^* D)$$

by the dual of lemma A.6. The dual of theorem A.9 is the following.

Theorem A.11 If D is compatible with product diagrams then there is a natural isomorphism

$$H^n_{\mathrm{prod}}(\mathbf{C}, D) \quad \cong \quad H^n(\mathbf{C}, D)$$

The natural transformation λ_{sum}

Given the normalisation theorem A.9 above we are now able to prove theorem 2.10 and its addendum 2.15.

Proof of Theorem 2.10: Recall that the homomorphism

$$F^{n+1}(\mathbf{C}, D) \xrightarrow{\;\lambda_{\mathrm{sum}}\;} F^n(\mathrm{Twist}(\mathbf{C}), \widehat{D})$$

is defined by

$$(\lambda_{\mathrm{sum}} c_{n+1})(\sigma) \quad = \quad \left[\sum_{i=0}^{n} (-1)^i c_{n+1}(\lambda_i' \sigma), \;\; \sum_{i=0}^{n} (-1)^i c_{n+1}(\lambda_i \sigma) \right]$$

for $\sigma \in \mathrm{Ner}(\mathrm{Twist}(\mathbf{C}))_n$ given by $(\xi_i, \eta_i)\colon f_i \to f_{i-1}$, $f_i\colon X_i \to Y_i$, and where

$$\lambda_i\sigma = \begin{cases} ((f_0,1),\overline{\xi}_1,\ldots,\overline{\xi}_{n-1},\xi_n) & i=0 \\ (\eta_1,\ldots,\eta_i,(f_i,1),\overline{\xi}_{i+1},\ldots,\overline{\xi}_{n-1},\xi_n) & 1 \le i \le n-1 \\ (\eta_1,\ldots,\eta_n,f_n) & i=n \end{cases}$$

and $\lambda'_i\sigma$ similarly, replacing the f_i by $0 : X_i \to Y_i$.

We show that λ_{sum} is a degree -1 cochain map and hence induces a well-defined map of cohomology groups; that is, we prove

$$\text{(A.12)} \qquad \delta\lambda_{\mathrm{sum}} + \lambda_{\mathrm{sum}}\delta = 0$$

The following relations between the functions λ_i and the simplicial face maps d_i are clear:

$$\begin{aligned} d_i\lambda_i\sigma &= d_i\lambda_{i-1}\sigma, \\ d_j\lambda_i\sigma &= \lambda_{i-1}d_j\sigma, \;\; j<i \\ d_j\lambda_i\sigma &= \lambda_i d_{j-1}\sigma, \;\; i+1<j\le n \end{aligned}$$

Thus on expanding the left-hand side of (A.12) by the definitions of λ_{sum} and δ most terms will cancel, leaving $(\delta\lambda_{\mathrm{sum}} + \lambda_{\mathrm{sum}}\delta)(c_n) = [b', b]$ where

$$\begin{aligned} b' &= (0,1)_*c_n d_0\lambda'_0\sigma - 0^*_n c_n d_{n+1}\lambda'_n\sigma + \sum_{i=0}^{n-1}(-1)^{n+i}\xi^*_n(\alpha_1 c_n\lambda'_i d_n\sigma - c_n d_{n+1}\lambda'_i\sigma) \\ b &= (f_0,1)_*c_n d_0\lambda_0\sigma - f^*_n c_n d_{n+1}\lambda_n\sigma + \sum_{i=0}^{n-1}(-1)^{n+i}\xi^*_n(\alpha_1 c_n\lambda_i d_n\sigma - c_n d_{n+1}\lambda_i\sigma) \end{aligned}$$

Since $d_0\lambda_0\sigma = d_0\lambda'_0\sigma$ and $d_{n+1}\lambda_n\sigma = d_{n+1}\lambda'_n\sigma$ the first two terms of b' and of b together are zero in the quotient $\widehat{D}(|\sigma|)$. Consider the elements

$$\begin{aligned} a'_i &= \alpha_1 c_n\lambda'_i d_n\sigma - c_n d_{n+1}\lambda'_i\sigma \\ &= \alpha_1 c_n(\eta_1,\ldots,\eta_i,(0,1),\overline{\xi}_{i+1},\ldots,\overline{\xi}_{n-2},\xi_{n-1}) - c_n(\eta_1,\ldots,\eta_i,(0,1),\overline{\xi}_{i+1},\ldots,\overline{\xi}_{n-2},\overline{\xi}_{n-1}) \\ a_i &= \alpha_1 c_n\lambda_i d_n\sigma - c_n d_{n+1}\lambda_i\sigma \\ &= \alpha_1 c_n(\eta_1,\ldots,\eta_i,(f_i,1),\overline{\xi}_{i+1},\ldots,\overline{\xi}_{n-2},\xi_{n-1}) - c_n(\eta_1,\ldots,\eta_i,(f_i,1),\overline{\xi}_{i+1},\ldots,\overline{\xi}_{n-2},\overline{\xi}_{n-1}) \end{aligned}$$

Assuming c_n respects sums we have $i^*_{X_{n-1}}(a'_i) = 0$ and so

$$\xi^*_n(a'_i) = \xi^*_n(0,1)^* i^*_{Y_{n-1}}(a'_i) = 0$$

Also $i^*_{X_{n-1}}(a_i) = 0$ and so $\xi^*_n(a_i) \in (f_0,1)_*D(\overline{\xi}_1\ldots\overline{\xi}_{n-1}\xi_n)_2$ by lemma 2.7. Thus $[b',b] = 0$ and (A.12) holds. □

Proof of Addendum 2.15: Let $\sigma = (\sigma_1,\ldots,\sigma_n)$ be an n-simplex of $\mathrm{Ner}(\mathrm{Pair}(\mathbf{C}))$ with $\sigma_k = (\zeta_k,\eta_k) : f_k \to f_{k-1}$ for $1 \le k \le n$. Then $\iota_*\sigma \in \mathrm{Ner}(\mathrm{Twist}(\mathbf{C}))$ is given by

$$\iota\sigma_k = (\xi_k,\eta_k) = (i_{X_{k-1}}\zeta_k,\eta_k)$$

and we have $\overline{\xi}_k = \zeta_k \vee \eta_k : X_k \vee Y_k \to X_{k-1} \vee Y_{k-1}$. Therefore

$$\begin{aligned} c_{n+1}(\lambda_i\iota_*\sigma) &= c_{n+1}(\eta_1,\ldots,\eta_i,(f_i,1),\overline{\xi}_{i+1},\ldots,\overline{\xi}_{n-1},\xi_n) \\ &= i^*_{X_n}c_{n+1}(\eta_1,\ldots,\eta_i,(f_i,1),\zeta_{i+1}\vee\eta_{i+1},\ldots,\zeta_n\vee\eta_n) \\ &= c_{n+1}(\eta_1,\ldots,\eta_i,f_i,\zeta_{i+1},\ldots,\zeta_n) \end{aligned}$$

since we can assume c_{n+1} respects sums. Assuming also it is normalised with respect to zero maps we have $c_{n+1}(\lambda'_i\iota_*\sigma) = 0$. Thus

$$(\lambda_{\mathrm{sum}}c_{n+1})(\iota_*\sigma) = \left[0\,,\; \sum_{i=0}^{n}(-1)^i c_{n+1}(\eta_1,\ldots,\eta_i,f_i,\zeta_{i+1},\ldots,\zeta_n)\right]$$

and $\tau^*\iota_*\lambda_{\mathrm{sum}}$ is just λ as defined in [3] as required. □

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March 1995.