# *Homotopy Actions and Cohomology of Finite Groups*

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# Homotopy Actions and Cohomology of Finite Groups

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## Introduction

Let $X$ be a connected topological space, and let $H(X)$ be the monoid of homotopy equivalences of $X$. The group of self-equivalences of $X$, $E(X)$, is defined to be $\pi_0 H(X)$. A homomorphism $\alpha : G \to E(X)$ is called a homotopy action of $G$ on $X$. Equivalently, the assignment of a self-homotopy equivalence $\alpha(g) : X \to X$ to each $g \in G$ such that $\alpha(g_1 g_2) \sim \alpha(g_1)\alpha(g_2)$ and $\alpha(1) \sim 1_X$ is also called a homotopy action. Since it is easier to construct self-homotopy equivalences rather than homeomorphisms of $X$, it is natural to consider the questions of existence of actions first on the homotopy level, (i.e. homotopy actions) and then try to find an equivalent topological action. A topological G-action $\varphi$ on $Y$ is said to be equivalent to a homotopy action $\alpha$ on $X$, if there exists a homotopy equivalence $f : Y \to X$ which commutes with $\varphi$ and $\alpha$ up to homotopy, i.e. $f$ is homotopy equivariant (for short, $f$ is an h-G-map). This is the point of view taken in [16] and the motivation for $G$. Cooke's study of the question:

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Question 1. Given a homotopy action $\alpha$ on $X$, when is $(X,\alpha)$ equivalent to a topological action?

The problem is quickly and efficiently turned into a lifting problem: A homomorphism $\alpha : G \to E(X)$ yields a map $B\alpha : BG \to BE(X)$. On the other hand the exact sequence of monoids $H_1(X) \to H(X) \to E(X)$ yields a fibration $BH_1(X) \to BH(X) \to BE(X)$.

Theorem (G. Cooke) [16]. $(X,\alpha)$ is equivalent to a topological action if and only if $B\alpha : BG \to BE(X)$ lifts to $BH(X)$ in the fibration $BH(X) \to BE(X)$.

Note that if $X$ does not have a "homotopically simple structure", e.g. if $X$ is not a $K(\pi,n)$ and $\dim X<\infty$, then $\pi_1(BH_1(X))$ is exceedingly difficult to calculate, and the above lifting problem will have infinitely many a priori non-zero obstructions. However, if $G$ is a finite group (and we will assume this throughout) and $X$ is localized away from the prime divisors of $|G|$, e.g. if $\pi_1(X) = 1$ and $X$ is rational, then all the obstructions vanish, and any such $(X,\alpha)$ is equivalent to a topological action. Algebraically, this can be interpreted by the fact that all the relevant RG-modules (where $R$ is a ring of characteristic prime to $|G|$) are semi-simple and consequently cohomologically trivial. Thus the interest lies in the "modular case", (i.e. when a prime divisor of $|G|$ divides the characteristic of $R$) and the inetgral case $R = \mathbb{Z}$.

In comparison with topological actions, homotopy actions have very little structure in general. For instance, there are no analogues of "fixed point sets", "orbit spaces" or "isotropy groups". This makes a general study of homotopy actions a difficult task. Notwithstanding, there has been some applications to problems in homotopy theory and geometric (differential) topology (e.g. [5] [6] [16] [22] [34] [35] for a sample).

Given a homotopy functor $h$ and a homotopy action of $G$, say $(X,\alpha)$, we obtain a "representation of $G$". E.g. if $X \simeq K(\pi,n)$ and $h = \pi_n$, then $\pi_n(X) \cong \pi$ becomes a $\mathbb{Z}G$-module. In this case, any $\mathbb{Z}G$-module $\pi$ also gives rise to a homotopy G-action on $X \simeq K(\pi,n)$, and in fact a topological G-action.

For spaces which are not homotopically easy to understand (such as most manifolds and finite dimensional spaces) homology and cohomology provide a more useful representation module. From this point of view, spaces with a single non-vanishing homology, known as Moore spaces, are the simplest to study. For simplicity, suppose we are given a $\mathbb{Z}G$-module $M$ which is $\mathbb{Z}$-free. Then it is easy to see that there exists a homotopy action $\alpha$ of $G$ on a bouquet of spheres $X$ such that $\bar{H}_*(X) \cong M$ as $\mathbb{Z}G$-modules. We say that "$(X,\alpha)$ realizes $M$", or that $M$ is realizable by $(X,\alpha)$. An obstruction theory argument shows that the question of realizability of $\mathbb{Z}G$-modules by homotopy G-actions on Moore spaces has a 2-torsion obstruction ([7] [22]) which can be identified with appropriate cohomological invariants of the $\mathbb{Z}G$-module $M$ ([7] P. Vogel, unpublished). In relation with the question of how close these homotopy actions are to topological actions, one should mention the following well-known problem attributed to Steenrod [26]:

<u>Question 2</u>. Is an integral representation of $G$ realizable by a G-action on a Moore space?

There has been some partial progress in answering the above question and we refer the reader to [3] [9] [13] [22] [30] [32] [33] and their references. In an attempt to understand homotopy actions, we will specialize and apply the methods of this paper to the above problem. Thus constructions and the study of the counterexamples for Question 2 in this paper should be regarded as a method of producing and investigating "invariants of homotopy actions" for more general spaces.

As mentioned above, the usual notion of transformation groups such as fixed points, isotropy groups, and orbit spaces do not carry over to homotopy actions as such. Therefore, we will try to attach other invariants, mostly of cohomological nature, to both G-spaces and homotopy G-actions, and compare them. For topological actions these invariants are naturally (and expectedly) related to fixed point sets and isotropy groups (whenever they are well-defined). Thus we have placed special emphasis on topological actions with some finiteness condition on the underlying space (e.g. finite cohomological dimension) as well as G-actions with collapsing spectral sequence in their Borel construction. On the algebraic side, our feeling is that the category of integral (modular) representations of $G$ which arise as homology (cohomology) of G-spaces is an important part of the category of all representations, and its algebraic study is worthwhile in its own right. The projectivity criterion (Thm. 2.1) as well as the complexity criterions (Sec. 3) and their consequences are some steps in this direction.

In comparing homotopy and topological actions, we will study:

Question 3. When is a representation of $G$ realizable by the homology of a G-space?

As we will see below, there are integral (and modular) representations of $G$ which are not realizable via the homology of any G-space (we do not restrict ourselves to Moore spaces). On the other hand, there are representations which are not realizable by G-actions on Moore spaces but they can still be realized by G-actions on other spaces (Section 5). All these representations arise from homotopy actions. These examples show that, even for homologically simple spaces, such as bouquet of spheres, the collection of integral representation of $G$ on $H_*(X)$ induced by a homotopy action $\alpha : G \to E(X)$ does not by itself decide whether $(X,\alpha)$ is equivalent to a topological action. It is the interrelationship of all $H_i(X)$ as $\mathbb{Z}G$-modules which determines the realizability in this case (Section 5 ). In the applications of homotopy actions to differential topological problems, one often needs to find finite dimensional G-spaces which realize a given homotopy action. The solution to the lifting problem mentioned earlier in the introduction, provides an infinite dimensional free G-space. In this context, the following problem is often necessary to answer:

Question 4. Suppose $X$ is homotopy equivalent to a finite dimensional space and $\varphi : G \times X \to X$ is an action. When does there exist a finite dimensional G-space $K$ and a G-map $f : X \to K$ inducing homotopy equivalence?

We study this problem and the related question Question 3 by "reduction to p-groups". This is the subject of a future paper. In particular, one has satisfactory characterizations for groups with periodic cohomology and some other classes of groups which includes nilpotent groups or some of the alternating groups.

Notation and conventions. All rings are commutative with unit. $\mathbb{F}_p$ is the field with p-element, where p always denotes a prime number, and k is a field of characteristic $p > 0$ (often an algebraic closure of $\mathbb{F}_p$ ). For a finite group $G$, $H_G$ denotes the ring $\oplus_i H^{2i}(G;k)$ if p is odd and $H_G = \oplus_i H^i(G;k)$ if $p = 2$. $\hat{H}^*$ denotes Tate cohomology [14] and the terminologies in this context are in [14] and [28]. $\mathbb{Z}_p = \mathbb{Z}/p\,\mathbb{Z}$ = integers (mod p) . The localization of a ring R with respect to the multiplicative subset generated by an element $\gamma \in R$ is denoted by $R[\gamma^{-1}]$ . For an ideal J in a ring R , rad(J) is the radical of J and if M is an R-module, Ann(x) is the annihilating ideal of $x \in M$ . The dual of a k-algebra A is denoted by $A^*$ . For an RG-module M and a subgroup H , $M|RH$ denotes the restriction to H . The terminology and conventions in topological group actions are taken from [10] and [19] and those related to homotopy actions are to be found in [16]. For example $E_G$ is the contractible free G-space and $E_G \times_G X$ is the Borel construction of a G-space X . If a G-space X needs to have a base point in the context, we replace X by its suspension $\Sigma X$ and take $x \in X^G \neq \emptyset$ , unless X is already endowed with a base point. Many of the statemets which are phrased in terms of cohomology have their counterparts in homology and we have avoided repeating this fact. The spaces X are not necessarily CW complexes unless otherwise specified. We may use sheaf cohomology for more general situations and the proofs are still valid (with some mild modification if necessary). The basic reference is [27] part I in particular its appendix, and we have used Quillen's terminology and notation when appropriate. E.g. $cd_p(X)$ means cohomological dimension of X (mod p) .

The bibliography contains the references which have been available to us, at least in some written form. Otherwise they have been mentioned in the context.

## Section 1. Localization and Projectivity

In this section we present a variation on P.A. Smith's theorem as a consequence of Quillen's version of the localization theorem of Borel (cf. [19] or [27]). The statements are not as general as they could be because we will present different proofs when the cohomological finiteness of the G-spaces are not assumed. These finiteness assumptions are necessary when applying the localization theorem. There is an analogy between the finiteness assumptions of this section on the level of orbit spaces and the weaker finiteness assumptions for cohomology in the following sections. There is also a localization-type argument implicit in the arguments of sections 2 and 3 which are explicit in the context of this section. The special cases treated differently in this section will hopefully serve to give motivation and some insight into the more algebraic arguments of the following sections. The basic reference for some details of the assertions of this sections (as well as the terminology and the notation) is [10]. More general forms of the localization theorem are discussed in [19].

<u>1.1 Proposition</u>. Let $G$ be a finite group and let $X$ be a connected G-space which is either compact, or $cd_p(X/G) < \infty$ for a fixed prime $p$. Assume that for each subgroup $C \subset G$ in order $p$, $H^i(X;\mathbb{F}_p)$ is a cohomologically trivial $\mathbb{F}_pC$-module for all $i > 0$. Then the p-singular set of $X$, $S_p(X) = \bigcup_P X^P$, where $P$ ranges over non-trivial p-subgroups of $G$, satisfies $\bar{H}^*(S_p(X);\mathbb{F}_p) = 0$.

<u>Proof</u>: Let $C \subset G$ and $|C| = p$, and let $\gamma \in H^2(C;\mathbb{F}_p)$ be the polynominal generator. Without loss of generality, we may assume that $X^G \neq \emptyset$, hence $X^C \neq \emptyset$. Choose $x \in X^G \subset X^C$. The Serre spectral sequence of the Borel construction $(X,x) \to E_C \times_C (X,x) \to BC$ collapse since $H^i(BC;H^j(X,x;\mathbb{F}_p)) = 0$ for $i > 0$ and all $j$ by cohomological triviality. Thus $H_C^*(X,x;\mathbb{F}_p) \cong H^0(BC;H^*(X,x;\mathbb{F}_p))$. Localization with respect to $\gamma$ shows ([27]):

$$H_C^*(X,x;\mathbb{F}_p)[\gamma^{-1}] \cong H^*(BC;H^*(X,x))[\gamma^{-1}]$$

$$\cong \hat{H}^*(C;H^*(X,x)) = 0 ,$$

(by the hypothesis of cohomological triviality) where $\hat{H}^*$ denotes Tate cohomology. By the localization theorem

$$H_C^*(X^C,x;\mathbb{F}_p)[\gamma^{-1}] \cong H_C^*(X,x;\mathbb{F}_p)[\gamma^{-1}] = 0 .$$

Since $H_C^*(X^C,x;\mathbb{F}_p)[\gamma^{-1}] \cong H^*(X^C,x;\mathbb{F}_p)\otimes_{\mathbb{F}_p} \hat{H}^*(C;\mathbb{F}_p)$, it follows that $H^*(X^C,x;\mathbb{F}_p) = 0$.

For any subgroup $K \subseteq G$, such that $|K| = p^r$ and $K \supseteq C$, it follows that $X^K \neq \emptyset$ and $\bar{H}^*(X^K;\mathbb{F}_p) = 0$ by an induction. Since this holds for every cyclic p-subgroup $C \subseteq G$, one has $\bar{H}^*(X^K;\mathbb{F}_p) = 0$ for all subgroups $K \subseteq G$, $K \neq 1$. An inductive argument using Mayer-Vietoris sequences yields the desired conclusion. ∎

We will be particularly interested in the class of G-spaces for which the Serre spectral sequence of their Borel construction collapses. This is formulated as condition (DSBC) (degenerate spectral sequence of Borel construction) below.

CONDITION (DSBC): Let $X$ be a G-space and let $A \subset G$ be a subgroup. We say that $X$ satisfies the condition (DSBC) for $A$ if the Serre spectral sequence of the fibration $X \to E_A \times_A X \to BA$ (in the Borel construction of the A-space $X$) collapses.

1.2 Proposition. Let $p$ be a prime divisor of order of $G$, and suppose that $X$ is a connected G-space such that either $X$ is compact or that $cd_p(X/G) < \infty$. Assume that:

(1) $X$ satisfies condition (DSBC) for each maximal elementary abelian subgroup $A \subseteq G$.

(2) The p-singular set $S_p(X)$ satisfies: $S_p(X) \neq \emptyset$ and $\bar{H}^*(S(X);\mathbb{F}_p) = 0$.

Then $\bar{H}^*(X;\mathbb{F}_p)$ is cohomologically trivial as an $\mathbb{F}_pG$-module.

Proof: Let $A$ be any p-elementary abelian rank $t$ subgroup, and let $e_A \in H^{2t}(A;\mathbb{F}_p)$ be the product of the $t$ 2-dimensional polynomial generators in $H^2(A;\mathbb{F}_p)$, (cf. [27] Part I). Since $S_p(X)^A = X^A$ and (2) implies that $\bar{H}^*(X^A,x;\mathbb{F}_p) = 0$ (where $x \in X^G \neq \emptyset$ is the base point), it follows that $H^*_A(X,x;\mathbb{F}_p)[e_A^{-1}] = 0$, by the localization theorem ([27] Part I). Since the Serre spectral sequence of $(X,x) \to E_A \times_A (X,x) \to BA$ collapses by (1), we may localize the $E_2$-term with respect to $e_A$ and conclude that $H^*(BA;H^*(X,x;\mathbb{F}_p))[e_A^{-1}] = 0$. But $H^*(BA;H^*(X,x;\mathbb{F}_p))[e_A^{-1}] \cong \hat{H}^*(A;H^*(X,x;\mathbb{F}_p))$. Since this is true for all p-elementary abelian groups $A \subseteq G$, $|A| = p^r$, it follows that $H^*(X,x;\mathbb{F}_p)$ is cohomologically trivial over all p-elementary abelian subgroups of $G$. By Chouinard's theorem (cf. [15] and [20]) $H^*(X,x;\mathbb{F}_p)$ is cohomologically trivial over $G$ (see the introduction to section 2). ∎

We obtain a special case of Theorem 2.1 as a corollary:

1.3 Corollary. Suppose that $X$ is a connected G-space with the following properties:

(1) Either $X$ is compact or $cd_p(X/G) < \infty$ for each $p$ dividing order $G$.

(2) $X$ satisfies condition (DSBC) for each p-elementary abelian subgroup $A \subseteq G$. Then $\bar{H}^*(X)$ is $\mathbb{Z}G$-projective if and only if $\bar{H}^*(X)|\mathbb{Z}C$ is $\mathbb{Z}C$-projective for each subgroup $C \subseteq G$ of prime order. In particular, this conclusion holds if $X$ is a Moore space which satisfies (1).

Proof: By 1.1 and 1.2, the cohomological triviality of $\bar{H}^*(X)$ over $G$ is equivalent to the cohomological triviality of $\bar{H}^*(X;\mathbb{F}_p)$ for all cyclic subgroups of order $p$. But a $\mathbb{Z}G$-module is $\mathbb{Z}G$-projective if and only if it is $\mathbb{Z}$-free and cohomologically trivial (cf. [28]). ∎

## Section 2. The Projectivity Criteria

Let $G$ be a finite group. Sylow(G) denotes the set of Sylow subgroups, and $G_p \in$ Sylow(G) denotes a p-Sylow subgroup. Let $R$ be a ring and $RG$ be the group algebra over $R$. In studying the cohomological properties of RG-modules, it is necessary to have a good understanding of projective modules. The following two theorems have played important roles in the "local-to-global" arguments.

(1) Rim [28]: A $\mathbb{Z}G$-module is $\mathbb{Z}G$-projective if and only if $M|\mathbb{Z}G_p$ is $\mathbb{Z}G_p$-projective for all $G_p \in$ Sylow(G).

(2) Chouinard [15] (See also Jackowski [20]): A $\mathbb{Z}G$-module $M$ is $\mathbb{Z}G$-projective if and only if $M|\mathbb{Z}E$ is $\mathbb{Z}E$-projective for all p-elementary abelian groups.

Chouinard's theorem is particularly useful in the problems related to cohomological properties of $M$, since the cohomology of elementary abelian groups are well-understood, whereas the cohomology ring of a general p-group is far more complicated and has remained mysterious as yet.

Thus, the projectivity of a $\mathbb{Z}G$-module $M$ is detected by its restrictions to the elementary abelian subgroups. Now suppose that $M$ is a kE-module, where $E$ is p-elementary of rank $n$ (i.e. of order $p^n$), and where $k$ is a field of characteristic $p$. (For simplicity, assume that $k$ is algebraically closed, although for the most part this assumption is not used.)

It is tempting to look for a projectivity criterion for $M$ in terms of a family of proper subgroups of $E$. In general there is no such criterion if we consider only subgroups of $E$. However, there is such a characterization if we include a certain family of well-behaved subgroups of $kE$. This is basically the content of a result due to Dade [17]. To describe this, let $I$ be the augmentation ideal: $0 \to I \to kE \overset{\xi}{\to} k \to 0$ and choose an $\mathbb{F}_p$-basis for $E$, say $\{e_1,\ldots,e_n\} \subset E$. Let $A = (a_{ij})$ be a non-singular $n \times n$ matrix over $k$ and define the homomorphism $\psi_A: kE \to kE$ by:

$$\psi_A(e_i) = 1 + \sum_{j=1}^{n} a_{ji}(e_j - 1) .$$

Then $\psi_A$ is an automorphism since $A$ is non-singular. In [11] J. Carlson called subgroups of order $p^m$ in $kE$, $m \leq n$, generated by $\{\psi_A(e_1),\ldots,\psi_A(e_m)\}$, "shifted subgroups" of $kE$. Such subgroups are p-elementary abelian and for $m = n$, $\{\psi_A(e_1),\ldots,\psi_A(e_n)\}$ generate $kE$ as a k-algebra. A cyclic subgroup $S$ of the shifted subgroup $\langle\psi_A(e_1),\ldots,\psi_A(e_n)\rangle$ is called a "shifted cyclic subgroup" and any generator of $S$ is called a "shifted unit". From now on we assume that all kE-modules are finite dimensional over $k$.

(3) Dade [17]: A kE-module $M$ is kE-projective if and only if $M|kS$ is kS-projective for every shifted cyclic subgroup of $kE$.

(Since $kE$ is a local ring, projective, injective, cohomologically trivial, and free modules coincide [28]). In fact, one can show that $M|kS$ is kS-projective if and only if $M|kS'$ is kS'-projective provided that the shifted units generating $S$ and $S'$ are congruent modulo $I^2$. This leads to the following more intrinsic definition of shifted subgroups and units [11] [8]. Let $L$ be an n-dimensional k-subspace of $I$ such that $I = L \oplus I^2$. Then every element $\ell \in L$ satisfies $\ell^p = 0$, and a k-basis of $L$ generates $kE$ as a k-algebra. Consequently, for any $\ell \in L$, $1 + \ell$ is a shifted unit and for any k-basis of $L$, say $\{\ell_1,\ldots,\ell_n\}$, the p-elementary subgroup generated by $\{1+\ell_1,\ldots,1+\ell_n\}$ is a shifted subgroup. J. Carlson attached a global invariant to a kE-module $M$, by taking the set $V_L^r(M)$ consisting of all nonzero $\ell \in L$ for which $M|k\langle 1+\ell\rangle$ is not $k\langle 1+\ell\rangle$-free (where $\langle 1+\ell\rangle$ is the group generated by $1+\ell$) together with zero. He showed that this is an affine algebraic variety and exhibited many beautiful properties of $V_L^r(M)$, called "the rank variety of $M$" (cf. [11]). Carlson conjectured that $V_L^r(M)$ is isomorphic to the cohomology variety of $M$, $V_E(M)$ (called the Quillen variety and inspired by Quillen's ideas in [27]), and he showed that $V_L^r(M)$ injects into $V_E(M)$. The Quillen variety $V_E(M)$ is the affine variety in $k^n$ defined by the ideal of elements in the commutative graded ring $H_E \equiv \oplus_i H^{2i}(E;k)$ which annihilate the $H_E$-module $H^*(E;M)$ ($H_E = \oplus_i H^i(E;k)$ when $E$ is a 2-group. The conjecture of Carlson is proved by Avrunin-Scott [8], and as a corollary $V_L^r(M)$ is independent of $L$ up to isomorphism. Thus the projectivity criterion of Dade which can be detected "locally" by shifted units, has the following "global formulation". From now on we drop the subscript $L$ in $V_L^r(M)$.

(4) Carlson [11]: $M$ is kE-free if and only if $V^r(M) = 0$.

This motivates the search for a projectivity criterion for $\mathbb{Z}G$-modules which appear as (reduced) homology of G-spaces. It turns out that the family of cyclic subgroups of order $p$ of $G$ detects the projectivity (and cohomological triviality). Thus "the geometry of $M$" is determined by a restricted class of subgroups of $G$ in this case, and gives an idea of how restricted the category of realizable $\mathbb{Z}G$-modules is. This is not true for homology of all G-spaces, rather a special class which includes Moore spaces. The projectivity criterion for the homology of more general G-spaces should be described in terms of "global invariants" attached to a G-space. The specific nature of a G-action on a space $X$ determines a certain interrealationship between $H_i(X)$ and $H_j(X)$ as $\mathbb{Z}G$-modules, and this fact is not detectable by simply considering the graded module $\oplus_i \tilde{H}_i(X)$. The examples of the following sections will elaborate more on this point.

2.1 Theorem. Suppose $X$ is a connected G-space which satisfies the condition (DSBC) for each p-elementary abelian subgroup $A \subseteq G$. Let $M$ be the $\mathbb{Z}G$-module determined by the G-action on the total homology of $X$ in positive dimensions. Then $M$ is $\mathbb{Z}G$-projective if and only if $M|\mathbb{Z}C$ is $\mathbb{Z}C$-projective for each subgroup $C \subset G$ of prime order. (Similarly for cchomological triviality).

2.2 Corollary. Suppose the $\mathbb{Z}G$-module $M$ appears as the homology of a Moore G-space. Then $M$ is $\mathbb{Z}G$-projective if and only if $M$ is $\mathbb{Z}C$-projective for each cyclic subgroup of $G$.

We will give two proofs of the above theorem. The first is in the spirit of transformation group theory and while it is quite elementary it reveals the topological nature of this criterion. The second proof is in a more general setting and hopefully will provide some motivation for introducing and emphasis on the global invariants of a G-space.

2.3 Corollary. Suppose $X_1$ and $X_2$ are connected G-spaces, both of which satisfy (DSBC) as in (2.1) and suppose $f : X_1 \to X_2$ is a G-map. Let $M_1$ and $M_2$ denote the total reduced homology of $X_1$ and $X_2$ as $\mathbb{Z}G$-modules and let $\varphi : M_1 \to M_2$ be the $\mathbb{Z}G$-homomorphism induced by $f$. Then there are $\mathbb{Z}G$-projective modules $P_1$ and $P_2$ such that $M_1 \oplus P_1 \cong M_2 \oplus P_2$ if and only if $\varphi_* : \hat{H}^i(C;M_1) \to \hat{H}^i(C;M_2)$ are isomorphisms for $i = 0,1$, and all cyclic subgroups $C \subseteq G$ of prime order.

## Section 3. Varieties associated to a G-space

Let $k$ be an algebraically closed field of characteristic $p>0$, and let $G$ be a p-elementary abelian group of rank $n$. For a connected G-space $X$, we will assume $X^G \neq \emptyset$ (when needed) and $x \in X^G$ is the base point. As far as homological invariants of $X$ are concerned at this point, this will be no restriction, since we acn always suspend the action. For a kG-module $M$, the rank variety $V^r(M)$ reveals much about its cohomological invariants. Thus, we are tempted to consider the rank variety $V^r(\oplus_i H^i(X,x;k)$ and investigate its influence on the topology of the G-space $X$. However, the more directly related variety, (when we have sufficient knowledge about the G-action) is the "support variety" $V_G(X)$.

In [27], Quillen studied cohomological varieties arising from equivariant cohomology rings $H_G^*(X;k)$ for a G-space $X$ (cohomology with constant coefficients), and he proved his celebrated stratification theorem among other results. According to Quillen's stratification theorem, the cohomological variety of a G-space $X$ for a general finite group $G$ has a piecewise description in terms of varieties arising from elementary abelian subgroups of $G$. Inspired by this work of Quillen, Avrunin-Scott in [8] defined the cohomological variety $V_G(M)$ for a finitely generated kG-module $M$ and proved an anloguous stratification theorem for $V_G(M)$ in terms of elementary abelian subgroups of $G$. Here, $V_G(M)$ is the largest support (in Max $H_G$) of the $H_G$-module $H^*(G,N\otimes M)$ where $N$ ranges over all finitely generated kG-modules. Avrunin-Scott's stratification theorem may be regarded as generalizing the special case of Quillen's result for the G-space X=point to the equivariant cohomology with local coefficients $H^*_G$ (point;M) (the kG-module M replacing the constant coefficients k of Quillen). The stratification of support varieties in the case of equivariant cohomology with local coeeficients $H^*_G(X;M)$ for a G-space X (whose orbit space X/G has finite cohomological dimension over k ) is carried out by Stefan Jackowski in [21] under the extra hypothesis that M is a kG-algebra. Jackowski's theorem yields a topological proof of Avrunin-Scott theorem in the spirit of Quillen's original approach.

Such stratification theorems describe the above mentioned cohomological varieties of a general finite group G in terms of elementary abelian subgroups of G . When G is an elementary abelian group, $V_G(X)$ is the affine algebraic variety defined by the annihilator ideal in $H_G$ of $H^*_G(X,x;k)$ . For the rest of this section, we will assume that G is an elementary abelian group. The corresponding results and notions for the case of a general finite group is obtained from this basic case and the appropriate stratification theorem. Elaboration of these ideas will appear elsewhere.

While one hopes that $V_G(X) \cong V^r_G(\oplus_i H^i(X,x))$ , this turns out to be true only for a restricted, but nevertheless important class of G-spaces. For a G-space with $H^i(X) \neq 0$ for only finitely many i (and some mildly more general class), it turns out that one can define a different, (but related) rank variety in a natural way. This is done by associating to X a $\mathbb{Z}$G-module defined up to a suitable stable equivalence. The $V^r_G(X)$ is defined to be the rank variety of this module (tensored with k ). The isomorphism $V_G(X) = V^r_G(X)$ will show that the "cohomological support variety" is also a "rank variety" and as such, it will enjoy the properties of rank varieties.

Following [5], call two G-spaces $X_1$ and $X_2$ "freely equivalent", if there exists a G-space Y such that $X_i \subset Y$ , and $Y-X_i$ are free G-spaces with $cd_p(Y-X_i) < \infty$ for $i = 1,2$ . This defines an equivalence relation between G-spaces. We may also consider the case when $Y/X_i$ is compact if $cd(y-X_i)=\infty$ with appropriate modifications.

<u>3.1 Lemma</u>. Suppose $X_1$ and $X_2$ are freely equivalent. Then $V_G(X_1) \cong V_G(X_2)$ .

Proof: Compare the Leray spectral sequences for $E_G\times_G X_i \to X_i/G$ with $E_G\times_G Y \to Y/G$ where $X_i$ and $Y$ are as above, $Y-X_i$ = free G-space [27]. It follows that $V_G(X_i) \cong V_G(Y)$ . ∎

3.2 Proposition. Suppose $H^i(X;k) \neq 0$ for only finitely many $i$ . Then $V_G(X) \subset V_G^r(\oplus_i H^i(X,x;k))$ . If $X$ satisfies the condition (DSBC) for $G$, then $V_G(X) \cong V_G^r(\oplus_i H^i(X,x;k))$ .

Proof: Proceed by induction on $\nu(X) \stackrel{\text{def}}{\equiv}$ number $\{i|H^i(X,x;k) \neq 0\}$ . For $\nu(X) = 1$ , $X$ is a Moore space and the spectral sequence of $(X,x) \to E_G\times_G(X,x) \to BG$ degnerates to one line, which shows that $V_G(X) \cong V_G(\oplus_j H^j(X,x;k)$ ($\equiv$ its support variety). By Avrunin-Scott's proof of J. Carlson's conjecture [8], the latter is isomorphic to $V_G^r(\oplus_j H^j(X,x;k))$. Suppose the assertion is true whenever $\nu(X) < m$ , $m > 1$ . Given $X_1$ with $\nu(X_1) = m$ , we add free G-cells to $X_1$ to obatin the G-space $Y$ so that $Y-X$ is free, $\dim(Y-X) < \infty$ , and $\nu(Y) < m$ . For example, kill the first non-vanishing homology, say $H_\ell(X,x;k)$, using Serre's version of the Hurewicz theorem, (after suspending $X$ , if needed). Then $V_G(X) \cong V_G(Y)$ since $X$ and $Y$ are freely equivalent (Lemma 3.1) and $V_G(Y) \cong V_G^r(\oplus_j H^j(Y,x;k))$ by induction. On the other hand, $V_G^r(Y) \subseteq V_G^r(X)$. This follows again because $(Y/X)^G$ = point and $\dim(Y/X) < \infty$ . Alternatively, if we kill $H_\ell(X,x;k)$ (the first non-vanishing) to obtain $Y$ , we have the exact sequence:

$$0 \to H_{\ell+1}(X;k) \to H_{\ell+1}(Y;k) \to F \to H_\ell(X;k) \to 0$$

where $F$ is a free kG-module, and

$$H_i(X;k) \cong H_i(Y;k) \quad \text{for} \quad i > \ell+1 .$$

For every shifted cyclic subgroup $S$ of $kG$ for which $H^i(X,x;k)|kS$ is kS-free, $H^i(Y,x;k)|kS$ will also be kS-free by Schanuel's lemma. Hence $V_G^r(\oplus_j H^j(Y,x;k)) \subset V_G^r(\oplus_i H^i(X,x;k))$ as desired.

If $X$ satisfies the condition (DSBC) for $G$ , then in the Serre spectral sequence of $X \to E_G\times_G X \to BG$, $E_2^{p,q} = E_\infty^{p,q}$ . Thus $\mathrm{rad}(\mathrm{Ann}\, H_G^*(X,x;k)) \cong \mathrm{rad}(\mathrm{Ann}\, H^*(G,H^*(X,x;k)))$ by a simple calculation and a filtration argument. Since $\mathrm{rad}(\mathrm{Ann}\, H^*(G,H^*(X,x;k))) \cong \bigcap_i \mathrm{rad}(\mathrm{Ann}\, H^*(G;H^i(X,x;k)))$ , it follows that

$$V_G(X) \cong V_G(\oplus_i H^i(X,x;k)) \cong \bigcup_i V_G(H^i(X,x;k)) \cong \bigcup_i V_G^r(H^i(X,x;k)) \cong$$

$V_G^r(\oplus_i H^i(X,x;k))$

(where the isomorphism between $V_G$ and $V_G^r$ of $H^i(X,x;k)$ is due Avrunin-Scott's theorem again). ∎

The second assertion of 3.2 is not true in general. The examples in the following sections illustrate this point.

The above observations lead us to define a kG-module $M(X)$ for each G-space $X$ with $H^i(X;k) \neq 0$ for only finitely many $i$, such that $V_G(X) \cong V_G^r(M(X))$. Since for Moore spaces $X$, $V_G(X) \cong V_G^r(H^*(X,x;k))$, we embed $X$ in a "mod k" Moore G-space $Y$ freely equivalent to it. This is possible since $H^i(X;k) = 0$ for large $i$ and we can add free G-cells inductively using Serre's Hurewicz theorem. Let $M(X) \equiv H_*(Y,x;k)$. Although $M(X)$ is not well-defined, $H^*(G;M(X)^*)$ and $H_G^*(X,x;k)$ are isomorphic modulo $H_G$-torsion. Hence $V_G(X) \cong V_G(M(X)^*) \cong V_G^r(M(X)^*) \cong V_G^r(M(X))$ and $V_G(X)$ has a description as a rank variety.

The module $M(X)$ is well-defined only in a "stable sense". For a kG-module $L$, define $\omega^0(L) \cong L$, and $\omega^1(L) \equiv \omega(L)$ by the exact sequence $0 \to \omega(L) \to F \to L \to 0$, where $F$ is kG-free, and $\omega^{i+1}(L) \equiv \omega(\omega^i(L))$. These modules are stably well-defined by Schanuel's lemma (cf. e.g. Swan's Springer-Verlag LNM 76).

<u>3.3 Proposition</u>. Suppose $X$ is a G-space such that $H^i(X;k) \neq 0$ for finitely many $i$. Let $Y_1$ and $Y_2$ be two mod k Moore G-spaces freely equivalent to $X$. Then there are integers $s$ and $t \geq 0$, such that $\omega^s(H^*(Y_1,x;k))$ is stably isomorphic to $\omega^t(H^*(Y_2,x;k))$. (Call this $\omega$-stability for short.)

<u>Proof</u>: Choose a G-space $Z$ freely equivalent to $Y_1$ and $Y_2$ and containing $Y_1$ and $Y_2$, and such that $H_i(Z,x;k) = 0$ for $i \neq \ell$, $\ell \gg$ nonzero dimensions in $H^*(Y_j;k)$ for $j = 1,2$. Then $C_*(Z/Y_i;k)$ are free kG-modules except for $* = 0$, where the base point naturally defines a split augmentation $C_0(Z/Y_i;k) \underset{\longleftarrow}{\xrightarrow{\varepsilon_i}} k \to 0$. $C_*(Z/Y_i;k)$ has homology (mod k) nonzero only in two dimensions above $0$, corresponding to $H_\ell(Z;k)$ and $H_*(Y_i,x;k)$. An appropriate application of the Schanuel's lemma shows that $\omega^t(H_*(Y_1,x;k)) \cong H_\ell(Z;k) \cong \omega^s(H_*(Y_2,x;k))$ for some integers $t,s \geq 0$. ∎

<u>3.4 Corollary</u>. Given a G-space $X$ with $H^i(X;k) = 0$ for sufficiently

large $i$, there exists a kG-module $M(X)$ which is well-defined up to $\omega$-stability and $V_G(X) \cong V_G^r(M(X))$. □

The $\omega$-stable class of $M(X)$ is in fact a "composite extension" of various $\omega^{s_i}(H_i(X;k))$ for all $i > 0$ and appropriate integers $s_i \geq 0$. This means that if $0 < i(1) < i(2) < \ldots < i(m)$ are the dimensions where $H_i(X;k) \neq 0$, then there are integers $s(1), \ldots, s(m)$ and extensions:
$0 \to H_{i(j+1)}(X;k) \to L_{i(j+1)} \to \omega^{s(j)}(L_{i(j)}) \to 0$ for $j = 1, \ldots, m$, and where $L_{i(1)} \equiv H_{i(1)}(X;k)$ and $M(X) \cong \omega^t L_{i(m)}$ for some $t \geq 0$. Let us refer to this construction as "an $\omega$-composite extension".

We have the following formal corollary:

3.5 Corollary. Suppose that $H^i(X;k) = 0$ for all sufficiently large $i$, and suppose $X$ has a homotopy G-action $\alpha : G \to E(X)$. Then $(X,\alpha)$ is equivalent to a topological G-action only if some $\omega$-composite extension $L$ of the kG-modules $H_i(X;k)$ (as given by $\alpha$) is realizable by a mod $k$ Moore G-space. ■

While this corollary seems to be a formal consequence of definitions, it does lead to the following theorem which will be proved in section 5.

3.6 Theorem. There exist decomposable kG-modules $M$ which are realizable by homotopy G-actions, but they are not realizable by the homology of any G-space $X$.

Next, we apply the above results to give a proof of Theorem 2.1.

Proof of Theorem 2.1: Let $M = \bigoplus_{i>0} H_i(X)$. Then, if $M$ is $\mathbb{Z}G$-projective, clearly $M$ is $\mathbb{Z}C$-projective for any subgroup, in particular cyclic subgroups of $G$. Conversely, suppose $M$ is $\mathbb{Z}C$-projective for all such $C \subseteq G$ as in the theorem. Let $M' = \bigoplus_{i>0} H_i(X;k)$. By Chouinard's theorem, it suffices to consider the case where $G$ is p-elementary abelian, and we will assume this for the sequel. Since $X$ satisfies the condition (DSBC) for $G$, one has $V_G(X) \cong V_G^r(M')$, by Proposition 3.2. At this point one has several (basically equivalent) ways of finishing the proof. The first is somewhat longer, but more illuminating, and we will refer to it in the applications.

First argument: $V_G(X)$ is defined via the radical of the annihilator of $H^*_G(X,x;k)$, say $J$, in $H_G$, which is the intersection of associated prime ideals $\mathrm{Ann}_{H_G}(\alpha)$, for $\alpha \in H^*_G(X,x;k)$. Since associated primes are closed under the Steenrod algebra, a theorem of Landweber [24] and [25] (generalizing a theorem of Serre [29]; see also [1]) shows that they are generated by two dimensional classes in $\bigoplus_{i>0} H^{2i}(G;\mathbb{F}_p) \subset H_G$. Landweber's proof is for $\mathbb{F}_p$-coefficients throughout, but one can easily check that his arguments goes through with k-coefficients and the same conclusion. (The invariance of associated primes under the Steenrod algebra has been observed by several authors [25] [31] [18]). Thus $J$ is defined by linear equations with $\mathbb{F}_p$-coefficients. Consequently $V_G(X)$ as well as $V^r_G(M')$ are $\mathbb{F}_p$-rational, (i.e., a union of subvarieties defined by linear equations with $\mathbb{F}_p$-coefficients). For a shifted cyclic subgroup $S \subset kG$, $V^r_S(M'|kS) \cong V^r_G(M') \cap \mathrm{tr}_{S,G}(V^r_S(k))$ (cf. [8]) where $\mathrm{tr}_{S,G}$ is the transfer. It follows that for each shifted cyclic subgroup which is not a subgroup of $G$, $S \cap G = \{1\}$ and $\mathrm{tr}_{G,S}(V^r_G(k)) \cap V^r_G(M') = 0$. (Here we assume to have chosen a k-vector space $L$ such that $I = L \oplus I^2$, $I$ = augmentation ideal, as described in Section 2.) Hence $V^r_G(M')$ is detected by the shifted cyclic subgroups $S$ such that $S \cap \{G\} \neq \{1\}$, i.e. cyclic subgroups of $G$. By the hypothesis, $M'|kS$ is kS-free for all such $S \subset G$. Thus, $V^r_G(M') = 0$ and $M'$ is kG-free. Since $H^i(X,x)$ is $\mathbb{Z}C$-projective, it is $\mathbb{Z}$-free. The long exact sequence of cohomology associated to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_p \to 0$ breaks into short exact sequences:

$$0 \to H^i(X;\mathbb{Z}) \xrightarrow{xp} H^i(X;\mathbb{Z}) \to H^i(X;\mathbb{F}_p) \to 0 .$$

But for all $A \subseteq G$, $\hat{H}^*(A;H^*(X,x;\mathbb{F}_p)) = 0$ ($\hat{H}^*$ = Tate cohomology and kG-projectivity implies $\mathbb{F}_pG$-cohomological triviality [14]). Hence $\hat{H}^*(A,H^*(X,x))$ is p-divisible, which means that it vanishes for all $A \subseteq G$. Therefore $H^*(X,x)$ is $\mathbb{Z}G$-projective, being $\mathbb{Z}$-free and $\mathbb{Z}$-cohomologically trivial [28].

Second argument: An inductive argument using Cartan's formula shows that the annihilating ideal of $H^*_G(X,x;k)$ is invariant under the Steenrod algebra, as in G. Carlsson [13]. A theorem of Serre [29] then shows that the variety $V_G(X)$ is $\mathbb{F}_p$-rational. Hence $V^r_G(M')$ is rational using Proposition 3.2. The rest of the proof is as in the first argument and the details are left to the reader. ∎

3.7 Addendum. The examination of the proof shows that in fact the statement of Theorem 2.1 remains valid, if we replace $\mathbb{Z}$-coefficients by k-coefficients as well as $\mathbb{Z}G$- and $\mathbb{Z}C$-projective by kG- and kC-free respectively. Thus one needs that $H^i(X;k) = 0$ for all sufficiently large $i$, instead of the stronger statement with $\mathbb{Z}$-coefficients. ∎

The above proof also suggests that as in J. Carlson [12], one can determine the complexity of $H^*(X,x;k)$ by the dimension of the variety $V_G^r(\oplus_i H^i(X,x;k)) = V_G(X)$ for this particular case. This is the counterpart of Theorem 2.1 for non-projective modules.

Let $p$ be a fixed prime and let $k$ be a field of characteristic $p$, say algebraically closed for convenience sake. We denote by $cx_G(M)$ the complexity of the kG-module $M$ (cf. [2] [23] [12]).

3.8 Theorem. Let $X$ be a connected G-space which satisfies the condition (DSBC) for each maximal elementary abelian p-subgroup $A \subseteq G$ and $H^*(-;k)$. Let $M \cong \bigoplus_{i>0} H_i(X;k)$ with the induced kG-module structure. Suppose $cx_G(M) = r$. Then there exists a p-elementary abelian subgroup $E \subseteq G$ of rank $r$ such that $cx_E(M|kE) = r$.

Proof: By Alperin-Evens [2], $cx_G(M) = \max_A\{cx_A(M|kA) \,|\, A \subseteq G$ maximal p-elementary abelian$\}$. Thus we may assume that $G$ is elementary abelian. Since $V_G^r(M) \cong V_G(X)$ is rational as in the proof of Theorem 2.1 above, dim $V_G^r(M)$ is the maximum dimension of the rational linear subvarieties whose union is $V_G^r(M)$. Let $V_0$ be one such linear maximum dimensional subspace of $k^n \cong V_G^r(k)$, (where we assumed $n = \text{rank } G$) and let $E = G \cap V_0$ be the set of rational points of $V_0$. Then rank $E = \dim V_0$ since $V_0$ is rational. On the other hand, $tr_{E,G}(V_E^r(M|kE)) \cong V_0$ (cf. [8] and [11] for details) and $cx_E(M|kE) = \dim V_0 = \text{rank } E$. ∎

Let $G$ be a p-elementary abelian group of rank $n$. In [23], Ove Kroll proves that if $cx_G(M) = t$ for a kG-module $M$, then there exists a shifted subgroup $\Gamma \subset kG$ of rank $n-t$ such that $M|k\Gamma$ is $k\Gamma$-free. J. Carlson's proof of Kroll's theorem [12] is in essence a "transversality argument" in the following sense. Since $cx_G(M) = t$, $\dim V_G^r(M) = t$, and it is always possible to find an $(n-t)$-dimensional linear subspace $L$ of $k^n \cong V_G^r(k)$ which is in "transverse position" to $V_G^r(M)$, (i.e. it has intersection $\{0\}$.) Now restriction to the shifted subgroup $\Gamma$ which is obtained from any k-basis of $L$ yields $\dim V^r(M|k\Gamma) = \dim(L \cap V_G^r(M)) = 0$, which means that $M|k\Gamma$ is $k\Gamma$-free.

When $V_G^r(M)$ is rational, one would like to find a subgroup $\Gamma \subseteq G$ with the above property. But this is not possible in general as it can be seen from the following simple example:

3.9 Example. Let $M = \oplus_E (kG \otimes_{kE} k)$ where $E$ runs over all cyclic subgroups of $G$. Then $cx_G(M) = 1$ and $M|kA$ is not kA-free for any non-trivial subgroup $A \subseteq G$.

However, the first argument of the proof of Theorem 2.1 above reveals that we can give a counterpart to Kroll's theorem in a particular case.

Call a G-space $X$ "k-primary", if the radical of the annihilator ideal of $H_E^*(X,x;k)$ in $H_E$ is prime for all maximal p-elementary abelian subgroups of $G$. (Here $k$ is a field of characteristic $p$ again.) Recall p-rank $(G) \stackrel{\text{def}}{=} \max$ {rank of elementary abelian p-subgroup $E \subseteq G$}.

3.10 Theorem. Suppose p-rank $(G) = n$ and $X$ is a connected k-primary G-space which satisfies the condition (DSBC) for all maximal p-elementary abelian subgroups and $H^*(-;k)$-coefficients. Also, assume that $H^i(X;k) = 0$ for all sufficiently large $i$. Then there exists a p-elementary abelian subgroup $E \subseteq G$ such that rank $E = n - \max_i \{cx_G(H_i(X,x;k))\}$ and $H_i(X,x;k)$ is kE-free for all $i$.

Proof: As in the preceding theorems, it suffices to assume that $G$ is p-elementary abelian (Alperin-Evens [2]). By Proposition 3.2 $V_G^r(\oplus_i H^i(X,x;k)) \cong V_G(X)$. Since $X$ is k-primary, the first argument in the proof of Theorem 2.1 shows that $V_G^r(\oplus_i H_i(X,x;k) \cong V_G^r(\oplus_i H^i(X,x;k))$ consists of one rational linear subvariety of $k^n \cong V_G^r(k)$, and its dimension equals to $cx_G(\oplus H_i(X,x;k)) = \max_{i>0} cx_G(H_i(X;k))$. Hence there is a rational linear subspace $L$ transverse to $V_G^r(H_i(X;k))$, and we may choose $\dim L = n - \max_{i>0} cx_G(H_i(X;k))$. Let $E$ be the subgroup of $G$ whose $\mathbb{F}_p$-generators gives an $\mathbb{F}_p$-basis for $L$. This is the desired subgroup. ∎

3.11 Remark. One can modify the above argument to weaken the hypothesis that "$X$ is k-primary" or that "$X$ satisfies (DSBC)", etc. But these hypotheses cannot be removed altogether by the above example 3.9 and the example in Sections 4 and 5.

Section 4. Applications to Steenrod's problem

In this section we consider the special case of G-actions of Moore spaces. Suppose $M$ is a finitely generated $\mathbb{Z}$-free $\mathbb{Z}G$-module. Then $M$ is determined by a homomorphism $\rho : G \to GL(n,\mathbb{Z})$ , where $n = \operatorname{rank}_{\mathbb{Z}}(M)$. Suppose that $X$ is homotopy equivalent to a bouquet of spheres of dimension $k \geq 2$ , and $H_k(X) \cong \mathbb{Z}^n$ . Then $E(X) \equiv \pi_0 H(X) \cong GL(n,\mathbb{Z})$ by obstruction theory. Thus $\rho$ induces a homomorphism $\alpha : G \to E(X)$ such that the homotopy action $(X,\alpha)$ realizes the $\mathbb{Z}G$-module $M$ . More generally, if $\operatorname{Tor}_1^{\mathbb{Z}}(M,\mathbb{Z}_2) = 0$ , or if $G$ is of odd order, then an obstruction theory argument (cf. [22]) shows that any homomorphism $\rho : G \to GL(n,\mathbb{Z})$ (which induces the $\mathbb{Z}G$-module structure of $M$ ) can be lifted to a homomorphism $\alpha : G \to E(X)$ . Thus the homotopy action $(X,\alpha)$ realizes $M$ .

On the other hand, given $M$ , we have the $\mathbb{Z}$-free $\mathbb{Z}G$-module $M'$ from the exact sequence $0 \to M' \to F \to M \to 0$ , where $F$ is a free $\mathbb{Z}G$-module. It is not difficult to see that $M$ is realizable by a Moore G-space, if and only if $M'$ is realizable by a Moore G-space. Thus, as far as the question of realizability of $\mathbb{Z}G$-modules is concerned, one can consider $\mathbb{Z}$-free $\mathbb{Z}G$-modules with no loss of generality. Therefore, the realizability of modules by homotopy actions does not pose a difficult problem in the contexts where one is primarily interested in realizability by topological G-actions.

In passing, let us mention that the obstructions for realizability of a $\mathbb{Z}G$-module by a homotopy action on a Moore space has been studied by P. Vogel [7] (unpublished). Vogel has shown that for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ , there is an $\mathbb{F}_2[G]$-module which is not realizable by a homotopy action on a Moore space:

4.1 Example (P. Vogel) [7]. Regard $\mathbb{Z}_2 \times \mathbb{Z}_2$ as the 2-Sylow subgroup of $GL(2,\mathbb{F}_4)$ , i.e. as $2 \times 2$ upper triangular matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $x$ belongs to the field with 4 elements. The natural action of $GL(2,\mathbb{F}_4)$ by left multiplication on the column vectors of $M = (\mathbb{F}_4)^2$ makes $M$ into a $\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_2]$-module. Vogel's obstruction theory shows that this modules is not realizable by a homotopy action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on a Moore space.

4.2 Construction and Examples. Let $k$ be an algebraic closure of $\mathbb{F}_p$ , and let $G = \mathbb{Z}_p \times \mathbb{Z}_p$ be generated by $e_1$ and $e_2$ . Let $I$ be the augmentation ideal and choose the k-vector space $L$ such that $I = L \oplus I^2$, with $\{\ell_1,\ell_2\}$ a k-basis for $L$, (as in Section 2). Then for almost all choices of $\alpha = (\alpha_1,\alpha_2) \in k^2$, the shifted unit $u_\alpha = 1+\alpha_1\ell_1+\alpha_2\ell_2$ generates a shifted subgroup $S \equiv \langle u_\alpha\rangle$ of order $p$ such that $S \cap G = \{1\}$. (Cf. Carlson [11] for details on shifted subgroups). More explicitly, for a (finite) Galois extension $K$ of $\mathbb{F}_p$, choose $\alpha_1,\alpha_2 \in K$ such that $u_\alpha = 1+\alpha_1(e_1-1) + \alpha_2(e_2-1)$ satisfies $u_\alpha-1 \notin I^2$ and a $u_\alpha \not\equiv g \pmod{I^2}$ for any $g \in G$. The condition $1-u_\alpha \notin I^2$ ensures that $kG$ is $kS$-free, and $S \equiv \langle u_\alpha\rangle \subset kG$ can be treated like an ordinary subgroup as far as induction and restriction is concerned [11]. In particular, Mackey's formula and Shapiro's Lemma are valid.

Recall that for the local ring $kG$, projective, injective, cohomologically trivial, and free modules coincide. First we need the following:

<u>4.3 Lemma</u>. (i) There exists an indecomposable $kG$-module $M_0$ such that $M_0$ is $kC$-projective for all cyclic subgroups $C \subset G$, but $M_0$ is not $kG$-projective.

(ii) There exists a finitely generated $\mathbb{Z}$-free $\mathbb{Z}G$-module $M_1$ which is $\mathbb{Z}C$-projective for all cyclic subgroups $C \subset G$, but $M_1$ is not $\mathbb{Z}G$-projective.

(iii) There exists an indecomposable $\mathbb{Z}G$-module $M$ with the same properties as in (ii) above.

(iv) In above part (iii), one may choose $M$ such that $k \otimes M \cong M' \oplus Q$, where $M'$ is an indecomposable $kG$-module, and $Q$ is $kG$-free.

<u>Proof</u>: (i) The above discussion, for (almost all) $u_\alpha$ chosen with $S = \langle u_\alpha\rangle$, one has $S \cap G = \{1\}$ and $kG$ is a free $kS$-module. Let $M_0 = kG \otimes_{kS} k$ be the induced module. Then for each $C \subset G$, $|C| = p$, $C \cap S = \{1\}$. Hence $\hat{H}^*(C,M_0) = 0$ by Mackey's formula. But $\hat{H}(G,M_0) \cong \hat{H}^*(S;k) \neq 0$ by Shapiro's Lemma. Since $kG$ is local, a cohomologically trivial $kG$-module is $kG$-free (= $kG$-projective). Thus (i) is proved.

(ii) One can choose $u_\alpha$ such that $u_\alpha = 1+\alpha_1(e_1-1) + \alpha_2(e_2-1)$, where $\alpha_1$ and $\alpha_2$ lie in a finite Galois extension of $\mathbb{F}_p$, say $k_1$, and $\langle u_\alpha\rangle = S$ still satisfies the same properties as in (i). Let $M_0 = k_1G \otimes_{k_1S} k_1$ be the $k_1G$-module which is $k_1C$-free for each $C \neq G$ but not $k_1G$-free as in (i). Consider the exact sequence $0 \to M_1 \to (\mathbb{Z}G)^t \to M_0 \to 0$. The long exact sequence of cohomology

$$\ldots\ldots \to \hat{H}^i(C,M_1) \to \hat{H}^i(C,(\mathbb{Z}G)^t) \to \hat{H}^i(C,M_0) \to \ldots\ldots$$

shows that $M_1$ is $\mathbb{Z}C$-projective for all $C \subset G$ $C \neq G$, and $M_1$ is not $\mathbb{Z}G$-projective.

(iii) Let $M_1 = M_1^1 \oplus \ldots\ldots \oplus M_1^r$ be a decomposition in terms of indecomposable $\mathbb{Z}G$-modules. Then all $M_1^j$ are $\mathbb{Z}C$-projective, but at least one of them is not $\mathbb{Z}G$-projective, say $M_1^1$. Then $M_1^1$ satisfies (ii) and it is also indecomposable.

(iv) Tensor the exact sequence of (ii) by $k$:

$$0 \to k \otimes M_1 \to (kG)^t \to k \otimes M_0 \to 0 .$$

Note that we can choose $M_0$ so that $k \otimes M_0$ is indecomposable. (Briefly: $\dim_k k \otimes M_0 = \dim_k KG/kS = [G:S] = p$, and since $k \otimes M_0 | kC$ is projective, the dimension over $k$ of each $kC$-indecomposable summand, and hence each $kG$-indecomposable summand must be divisible by $p$.) In the short sequence:

$$0 \to M' \to P \to k \otimes M_0 \to 0$$

where $P$ is the projective cover of $k \otimes M_0$, $M'$ is also indecomposable, since $k \otimes M_0$ is indecomposable. Hence Schanuel's Lemma shows that $k \otimes M_1 \cong M' \oplus$ (projective). ∎

<u>4.4 Theorem</u>. Suppose $G$ is a finite group such that $G \supset \mathbb{Z}_p \times \mathbb{Z}_p$. Then:

(I) there exists a $kG$-module $M_0'$ which satisfies (i) of Lemma 4.3.

(II) There exists a $\mathbb{Z}G$-module $M'$ which satisfies (iv) of Lemma 4.3. Further, it is not possible to find a Moore G-space $X$ such that $\bar{H}_*(X;k) \cong M_0'$ as $kG$-modules.

Similarly, there does not exist a Moore G-space $X$ such that $\bar{H}_*(X;\mathbb{Z}) \cong M'$ as $\mathbb{Z}G$-modules.

<u>Proof</u>: Let $M_0$ be the $k[\mathbb{Z}_p \times \mathbb{Z}_p]$-module of Lemma 4.3(i). Let $M_0' \equiv kG \otimes_{k[\mathbb{Z}p \times \mathbb{Z}p]} M_0$. Since $S \cap C = \{1\}$, Mackey's formula shows that for each $C \subset G$, $|C| =$ prime, $M_0'/kC$ is $kC$-cohomologically trivial, hence $kC$-free. But $M_0'$ is not $kG$-free since it is not $k[\mathbb{Z}_p \times \mathbb{Z}_p]$-free, as $M_0' | \mathbb{Z}_p \times \mathbb{Z}_p$ has $M_0$ as a direct summand, (or apply Shapiro's lemma).

(II) Let $M$ be as in Lemma 4.3 (iv), and let $M' = \mathbb{Z}G \otimes_{\mathbb{Z}[\mathbb{Z}p \times \mathbb{Z}p]} M$. The assertion follows as in part (I). Now the non-existence of the Moore G-spaces realizing these G-modules is a consequence of the projectivity criterion Theorem 2.1. ∎

(Compare 4.4 with G. Carlsson's theorem [13].)

The case $G = Q_{2^n}$ = generalized quaternionic group of order $2^n$ is somewhat different, because the maximal elementary abelian subgroup of $Q_{2^n}$ is the subgroup of order two generated by the central element $\tau \in Q_{2^n}$. Therefore kG-projectivity (or $\mathbb{Z}G$-projectivity) of a module is completely decided by the restriction to $k\langle\tau\rangle$ or $\mathbb{Z}\langle\tau\rangle$. Therefore Theorem 2.1 does not help directly in this situation. In the sequel, we present first a proof of non-realizability of a kG-module by Moore G-space (similarly for a $\mathbb{Z}G$-module) in the finite dimensional case, and we will use the geometric intuition of this case to remove the finite dimensionality restriction with a different proof.

<u>4.5 Proposition</u>. Let $G$ be the quaternionic group of order $2^n$, $n \geq 3$. Then there exists a $\mathbb{Z}G$-module $M$ such that $M$ is not $\mathbb{Z}G$-isomorphic to the (reduced) homology of a finite dimensional Moore G-space $X$. Similarly, $k \otimes M$ is not $\mathbb{Z}G$-isomorphic to $\bar{H}_*(X;k)$.

<u>Proof</u>: Let $\tau \in G$ be the central element of order 2 and let $\tau$ generate $T \cong \mathbb{Z}_2 \subset G$. Then $G/T \cong D_{2^{n-1}}$, the dihedral group of order $2^{n-1}$. Let $M$ be the module over $\mathbb{Z}_2 \times \mathbb{Z}_2$ constructed as in Lemma 4.3 (iv) above and let $N = \mathbb{Z}[D_{2^{n-1}}] \underset{\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_2]}{\otimes} M$. Then $M$ is not $\mathbb{Z}[D_{2^{n-1}}]$-isomorphic to $\bar{H}_*(X_0)$ for any Moore G-space $X_0$. In fact, $k_1 \otimes M$ is not $k_1[D_{2^{n-1}}]$-isomorphic to $\bar{H}_*(X_0;k_1)$ for any field $k_1$ of characteristic 2.

Consider $N$ as a $\mathbb{Z}Q_{2^n}$-module, where $T$ acts trivially on $N$. (To get a G-module on which all elements of $G$ act non-trivially take $\mathbb{Z}G \oplus N$, or the group of n-cocycles in a minimal projective resolution of $N$ over $\mathbb{Z}G$.). Suppose there exists a finite dimensional Moore G-space $Y$ such that, $Y^G \neq \emptyset$ and $H_*(Y) \cong N$ as $\mathbb{Z}G$-module. Then $Y^T$ is a $D_{2^{n-1}}$-space of finite dimension, and since the Serre spectral sequence of $Y \to E_T \times_T Y \to BT$ collapses, $\bar{H}^*(Y^T;k_1) \cong \bar{H}^*(T;N) \otimes_{\hat{H}^*(T;k_1)} k_1$ as in the proof of Proposition 1.1. Using this periodicity of $H^*(T;N)$, it follows that $\bar{H}^*(Y^T;k_1) \cong (N \otimes k_1)^\tau / (1+\tau)(N \otimes k_1) \cong N \otimes k_1$. But this means that $N \otimes k_1$ is realized by the $D_{2^{n-1}}$-space $Y^T$, i.e. $\bar{H}^*(Y^T;k_1) \cong N \otimes k_1$. By Proposition 1.2 or Theorem 4.4 this cannot happen. ∎

Alternatively, the $\omega$-stable module $M(Y^T)$ up to $\omega$-stability is $k_1G$-isomorphic to $N \oplus Q$ where $N$ is the indecomposable factor and $Q$ is kG-free. This is the case because $\bar{H}_*(Y^T;k_1)$ has only one decomposable $k_1G$-module $N$ as a summand which is not $k_1G$-free. Thus the construction $M(X)$ and the definition of $\omega$-composite extensions shows that any $\omega$-composite extension of various $\bar{H}_i(Y^T;k_1)$ is of the form $N \oplus Q$ up to $\omega$-stability. Now the Projectivity criterion Theorem 2.1 of Theorem 4.4 shows that $N \oplus Q$ of $\omega^j(N) \oplus Q$ cannot occur as $H_*(L;k_1)$ for any Moore $D_2n$-1-space $L$. This contradiction shows that such a Moore G-space cannot exist.

The proof of the above implies the finite dimensional case of the following corollary. (The details are left to the reader).

<u>4.6 Corollary</u>. If $G \supseteq Q_{2^n}$, then there are $\mathbb{Z}G$-modules which are not $\mathbb{Z}G$-isomorphic to the reduced homology of a Moore G-space. □

Now we proceed to give a different proof which shows that such Moore G-spaces cannot exist regardless of their dimensions.

Since every quaternionic 2-group contains the quaternionic group $Q_8$ of order 8, we will prove the theorem for $Q_8$ and deduce the result for $Q_{2^n}$, $n \geq 3$ from it. Suppose that $X$ is any Moore G-space, where $G = Q_8$, such that $H^*(X,x) \cong M$, $(x \in X^G \neq \emptyset)$. Let $M$ be a $\mathbb{Z}Q_8$-module which is $\mathbb{Z}$-free, and $T \equiv \langle\tau\rangle \subset Q_8$ acts trivially on $M$, and let $A = Q_8/T \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ induce a $\mathbb{Z}A$-module structure on $M$. Consider the Borel construction $(W,W_0) = E_G \times_T (X,x)$ which carries a free A-action. The Serre spectral sequence $(X,x) \to (W,W_0) \to BT$ collapses and $H^*(W,W_0;k_1) \cong H^*(T,M \otimes k_1)$. Denote $M \otimes k_1$ by $M_1$. Since $T$ acts trivially on $M_1$, it follows that $H^*(T,M_1) \cong H^*(T,k_1) \otimes M \cong H^*(W,W_0,k_1)$.

Now consider the Borel construction $E_A \times_A (W,W_0) \to BA$. In the spectral sequence of this fibration, $E_2^{p,0} = 0$ for all $p$ and $E_2^{p,1} \cong H^p(A;H^1(W,W_0)) \cong H^p(A;M_1)$. On the other hand, $E_A \times_A (W,W_0) \simeq (W/A, W_0/A)$ since $A$ acts freely, and $(W/A,W_0/A) = E_G \times_G (X,x)$. Hence $E_2^{1,1} \cong E_\infty^{1,1} \cong H_G^1(X,x;k_1) \cong H^1(G;M_1)$. The $H_A$-module structure of $E_G \times_G (X,x)$ is also related to the $H_G$-structure by the following commutative diagram:

$$\begin{array}{ccc} E_G \times_G (X,x) & \longleftrightarrow & E \times_A (W,W_0) \\ \downarrow & & \downarrow \\ BG & \longrightarrow & BA . \end{array}$$

At this point, let $M_1 \equiv k_1A \otimes_{k_1S} k_1$ , and note that $H^*(A;M_1) \cong H^*(S;k_1) \cong k_1[g_\alpha]$ for $g_\alpha \in H^1(S;k_1)$. Let the corresponding generated be denoted by $\gamma \in H^1(A;M_1)$ . Then $\mathrm{rad}(\mathrm{Ann}(\gamma))$ in $H_A$ is the ideal $J = (\alpha_1 y + \alpha_2 x)$ .

On the other hand, let $C$ be the cyclic group of order 4 in $k_1[Q_8]$ given by the extension $T \to C \to S$ . If we regard $k_1$ as a trivial module over $kS$ on which $T$ acts trivially also, it follows that $k_1A \otimes_{k_1S} k_1 | k_1C \cong k_1Q_8 \otimes_{k_1C} k_1 | k_1C$ .

Thus, $H^*(Q_8;M_1) \cong H^*(C;k_1)$ , and in the Lyndon-Hochschild-Serre spectral sequence of $T \to C \to S$ , $H^1(S;k_1) \xrightarrow{\cong} H^1(C;k_1)$ while all other $H^1(S;k_1)$ map to zero in $H^i(C;k_1)$ .

Since the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d] & S \arrow[d] \\ T \arrow[r] & Q_8 \arrow[r] & A \end{tikzcd}$$

commutes, we may identify $g_\alpha \in H^1(S;k_1)$ with a generator $g \in H^1(C;k_1) \cong H^1(Q_8;M_1)$ . Under this identifiaction, $g \in H^1_{Q8}(X,x;k_1) = H^1(E \times_A (W,W_0);k_1)$ , is identified with $\gamma \in H^1(A;M_1) \cong H^1(S;k_1)$ .

<u>4.7 Assertion</u>: $\mathrm{rad}(\mathrm{Ann}(g)) = J$ in $H_A$ .

<u>Proof</u>: It suffices to show that $f = \alpha_1 y + \alpha_2 x$ belongs to $\mathrm{Ann}(g)$ since $f$ generates $J$ . But $f^t.\gamma = 0$ since $f \in \mathrm{Ann}(\gamma) = J$ for some $t \geq 0$ . The naturality of all the identifications made above shows that $f^t.\gamma = 0 \Longleftrightarrow f^t.g = 0 \Longleftrightarrow f^t.g = 0 \Rightarrow (f) = \mathrm{rad}(\mathrm{Ann}(g))$ .

On the other hand, $\mathrm{rad}(\mathrm{Ann}(g))$ must be invariant under Steenrod algebra, being an associated prime for the module $H^*_{Q8}(X,x;k_1)$ over $H_A$ . Hence its variety must be $\mathbb{F}_p$-rational by Serre's theorem [29], and $J$ is not rational over $\mathbb{F}_p$ by the choice of $\alpha$ . This contradiction establishes the theorem.

<u>4.8 Remark</u>. An alternative proof using a complexity argument is briefly as follows. In the spectral sequence with $E_2^{p,q} = H^p(A;H^q(W,W_0))$ which converges to $H^*(E_G \times_G (X,x) \cong H^*(C;k_1)$ , for $p+q$ = constant, $E_\infty^{p,q} \neq 0$ only for one pair $(p,q)$ . Thus multiplication by $f^t$ shifts the filtration in $E_\infty$ . But since there is only one non-zero term, it

follows that an appropriate power of $f^t$ kills the $E_\infty$-term in this case. This shows that the radical of the annihilator of the module contains $f$. Hence the $H_A$-variety of $X$ is the intersection of the line $\ell$ given by $f$ with possible other lines. If this intersection does not include $\ell$, then it must be zero dimensional, and one argues that $M$ must be $\mathbb{Z}_2 \times \mathbb{Z}_2$-projective accordingly, which is a contradiction again.

The above results show the following theorem, due to Carlsson for $G = \mathbb{Z}_p \times \mathbb{Z}_p$ [13] and to Vogel for $G \supset Q_8$ (to appear) using calculations with the Steenrod algebra. An exposition of Vogel's theorem can be found in [9].

4.9 Theorem. If all $\mathbb{Z}G$-modules are realizable by Moore G-spaces, then $G$ is "metacyclic", i.e. all Sylow subgroups of $G$ are cyclic.

4.10 Remark. Jackowski, Vogel and several others have observed that Carlsson's counterexample for $\mathbb{Z}_p \times \mathbb{Z}_p$ implies that for $G \supset \mathbb{Z}_p \times \mathbb{Z}_p$ the induced module is also a counterexample.

## Section 5. Some Examples

We have seen how to construct examples of $\mathbb{Z}G$-modules which are not realizable by Moore G-spaces. These also give examples of homotopy actions on Moore spaces which are not equivalent to a topological action. The question arises whether these lead to criteria for homotopy actions on more general spaces to be equivalent to topological actions. It is helpful to consider the case of spaces which are bouquets of Moore spaces of different dimensions. We will briefly investigate the possibility of realizing a given $\mathbb{Z}G$-module $M$ by a topological action on such a space. This module $M$ arises from a homotopy action $(X,\alpha)$ and as a consequence our examples reveal some properties of homotopy actions on such spaces. Note that if a $\mathbb{Z}G$-module $M$ is indecomposable, then $M$ can be realized only by a Moore G-space. Thus to get new examples, we will consider decomposable modules.

By means of a simple construction using the modules of Section 4 and the theory of Sections 2 and 3, we will show that for $G \supset \mathbb{Z}_p \times \mathbb{Z}_p$ the following hold.

(5.1) There is a $\mathbb{Z}G$-module $M = M_1 \oplus M_2$, where $M_i \neq 0$ are indecomposable, such that neither $M$ nor $M_i$ are realizable by Moore G-spaces.

(5.2) There is an $(n-1)$-connected finite G-CW complex $X$ of dimension $n+1$ such that $\oplus_i \tilde{H}_i(X) = M$ as $\mathbb{Z}G$-module. Call this action $\varphi : G \times X \to X$ .

(5.3) $X$ is homotopy equivalent to a bouquet of spheres of dimension $n$ and $n+1$ , but $(X,\varphi)$ is not G-homotopy equivalent to a bouquet of spheres, with a G-action.

(5.4) Let $P$ be the projective cover of $M_1$ and $0 \to \Omega(M_1) \to P \to M_1 \to 0$ be an exact sequence of $\mathbb{Z}G$-modules. Then an extension of $M_1$ and $\Omega(M_1)$ is realizable by a finite dimensional Moore G-space. Similarly for $M$ . This extension is non-trivial necessarily.

(5.5) We may choose $M_1 = M_2$ in the above.

(5.6) Since $\Omega(M_1)$ is not realizable by a Moore space either, we have also examples of modules $M_1$ and $M_1' = \Omega(M_1)$ such that $M_1 \oplus M_1'$ is not realizable by a topological action on a Moore space, but some non-trivial extension of $M_1$ and $M_1'$ is realizable by a Moore G-space.

(5.7) We may construct examples where $M_1 = \Omega(M_1)$ in the above.

(5.8) There is a homotopy action of $G$ , say $\alpha$ , on a finite bouquet of n-spheres $L$ , such that $(L,\alpha)$ and any suspension of this h-action $(\Sigma^i L,\Sigma^i\alpha)$ are not equivalent to topological actions. But $(L\vee\Sigma L,\alpha\vee\Sigma\alpha)$ is equivalent to a topological action.

(5.9) $V_G(X) \neq V_G^r(X)$ , thus the inclusion $V_G(X) \subset V_G^r(X)$ of Proposition 3.2 cannot be improved (even for finite dimensional spaces). Here the varieties are taken over $kG$ . Here $V_G(X) = 0$ while $\oplus_i H_i(X,x;k)$ is not kG-free.

(5.10) Radicals of the annihilators in $H_G$ of $H_G^*(X,x;k)$ and $H^*(G;H^*(X,x;k))$ are not equal.

(5.11) We may choose $M_i$ such that the projectivity criterion 2.1 does not apply to $X$ . This will follow because we will choose $M_i$ such that $\oplus H_i(X,x)|\mathbb{Z}C$ is $\mathbb{Z}C$-projective for all $C \subset G$ , $|C|$ = prime, but $\oplus_i H_i(X,x)$ is not $\mathbb{Z}G$-projective. Thus Theorem 2.1 cannot be extended to all G-spaces without additional hypotheses (even for finite dimensional G-spaces).

(5.12) For appropriate choices of $M_1$ and $M_2$ , $M = M_1 \oplus M_2$ will not be realizable by any G-space, $M_i \neq 0$ , $i = 1,2$ .

5.13 Example. It suffices to consider $G = \mathbb{Z}_p \times \mathbb{Z}_p$ , and the above assertions (whenever applicable) hold for $G \supset \mathbb{Z}_p \times \mathbb{Z}_p$ or $G \supset Q_8$ . Consider the $\mathbb{Z}G$-module $M_1$ constructed in Theorem 4.4. For some of the assertions such as (5.5), (5.6), and (5.7), let $p = 2$ , otherwise $p$ is any prime. We may choose $M_1$ to be $\mathbb{Z}$-free and $\mathbb{Z}G$-indecomposable. From the exact sequence:

$$(5.14) \qquad 0 \to M_2 \to (\mathbb{Z}G)^r \xrightarrow{\varphi} (\mathbb{Z}G)^s \to M_1 \to 0$$

it follows that $M_2|\mathbb{Z}C$ is $\mathbb{Z}C$-projective for all $C \subset G$ , $|C| =$ prime while $M_2$ is not $\mathbb{Z}G$-projective, since $M_1$ is not $\mathbb{Z}G$-projective. Therefore $M_2$ is not realizable by a Moore G-space either. Let $M = M_1 \oplus M_2$ . The same holds for $M$ .

We may take bouquets of $s$ and $r$ free G-orbits of n-spheres,

i.e. $X_1 = \bigvee_{i=1}^{s} (G_+ \wedge S^n)_i$ and $X_2 = \bigvee_{j=1}^{r} (G_+ \wedge S^n)_j$ .

There exists a G-map $f : X_2 \to X_1$ such that $f_*: H_n(X_2) \to H_n(X_1)$ can be identified with the $\mathbb{Z}G$-homomorphism $\varphi : (\mathbb{Z}G)^r \to (\mathbb{Z}G)^s$ after appropriate identifications $H_n(X_1) \cong (\mathbb{Z}G)^s$ and $H_n(X_2) \cong (\mathbb{Z}G)^r$ . Then the mapping cone of $f$ is a finite G-space $X$ which satisfies (5.1) and (5.2) above, in view of the exact sequence (5.14), (5.1) and (5.2) imply (5.3).

$\eta$ The projective cover of $M_1$ , namely $P$ , satisfies $0 \to P \to F_1 \to F_2 \to 0$ where $F_1$ and $F_2$ are $\mathbb{Z}G$-free (not necessarily finitely generated). Thus $P$ can be realized via the mapping cone $X_0$ of the G-map $g : \bigvee_i (G_+ \wedge S^{n-1})_i \to \bigvee_j (G_+ \wedge S^{n-1})_j$ corresponding to $\eta$ (i.e. $g_* = \eta$ in $H_{n-1}(-;\mathbb{Z})$ ). $X_0$ is also free off the base point. In the exact sequence:

$$(5.15) \qquad 0 \to M_1' \to P \xrightarrow{\psi} M_1 \to 0$$

the homomorphism $\psi$ can be realized by a G-map $f': X_0 \to X$ which induces $f_*' : H_n(X_0) \to H_n(X)$ , $f_*' = \psi$ , by equivariant obstruction theory (or see [3]). The mapping cone of $f'$ , say $Y$ , is a Moore G-space and $H_{n+1}(Y)$ is the extension in the sequence:

$$(5.16) \qquad 0 \to M_2 \to H_{n+1}(Y) \to M_1' \to 0$$

Thus an extension of $M_1'$ and $M_2$ is realizable by the Moore G-space $Y$. This proves (5.4). Since $M_1$ is a periodic module by construction, by taking $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ we can fulfill (5.5) - (5.7). If we wish to choose $M_1 \cong M_2$ for odd $p$, just take the exact sequence

$$(5.17) \qquad 0 \to M_1 \to P_1 \xrightarrow{\zeta} P_2 \to M_1 \to 0$$

where $P_i$ are projective covers, and $\mathrm{Ker}\zeta = M_1$ since $M_1$ is chosen to be indecomposable. (5.17) exists due to periodicity of $M_1$. This is the analogue of (5.14) and we can use $P_i$ instead of $F_i$, $i=1,2$.

Since all these modules are realizable by homotopy actions (obstruction theory), the assertion (5.8) follows easily from the previous ones.

To see (5.9), note that $V_G^r(k \otimes M_1)$ is not $\mathbb{F}_p$-rational by the construction (cf. Section 4). Thus $V_G^r(\oplus_i \bar{H}_i(X;k)) \cong V_G^r(k \otimes (M_1 \oplus M_2)) = V_G^r(M_1)$ is not rational over $\mathbb{F}_p$. But $V_G(X)$ is rational over $\mathbb{F}_p$ (see the proof of 2.1). Thus $V_G(X) \neq V_G^r(\oplus_i H_i(X;k))$. Since $V_G^r(M_1)$ is only one line, in this case it follows that $V_G(X) = 0$ in fact. Except for (5.12) which will be proved below separately, the other assertions follow from the above discussion and elementary considerations.

Again, in the following $G \supset \mathbb{Z}_p \times \mathbb{Z}_p$ or $Q_8$.

<u>5.18 Theorem</u>. There exists a decomposable $\mathbb{Z}G$-module $M$ which cannot be realized by the total reduced homology of any G-space. There are homotopy actions $(X,\alpha)$ realizing $M$, and all such $(X,\alpha)$ are not equivalent to topological actions.

<u>Proof</u>: As before, we may assume $G = \mathbb{Z}_p \times \mathbb{Z}_p$ and the general case follows from this case. Choose $u_\alpha$ and $u_\beta$ as in Theorem 4.4, such that $u_\alpha \not\equiv u_\beta \pmod{I^2}$ and the lines in $k^2$ given by $u_\alpha$ and $u_\beta$ are distinct. Corresponding to these choices we get indecomposable $\mathbb{Z}$-free $\mathbb{Z}G$-modules $M_\alpha$ and $M_\beta$ whose rank varieties are the lines determined by $u_\alpha$ and $u_\beta$. Neither $M_\alpha$ nor $M_\beta$ is realizable by a Moore G-space using the projectivity criterion 2.1. For the same reason, $\omega^t(M_\alpha)$, $\omega^s(M_\beta)$ or any direct sum of them are not realizable by Moore G-spaces (see Section 3). Any $\omega$-composite of $M_\alpha$ and $M_\beta$ is of the form:

$$(5.19) \qquad 0 \to \omega^t(M_\alpha) \to U \to \omega^s(M_\beta) \to 0$$

and this extension is determined by a class $\eta \in \mathrm{Ext}^1_{\mathbb{Z}}(\omega^t(M_\alpha), \omega^s(M_\beta))$. By tensoring with $k$, we get

$$(5.20) \qquad 0 \to \omega^t(M_\alpha \otimes k) \to U \otimes k \to \omega^s(M_\beta \otimes k) \to 0$$

and a corresponding class $\eta' \in \mathrm{Ext}^1_{kG}(\omega^t(M_\alpha \otimes k), \omega^s(M_\beta \otimes k))$. We claim that this class vanishes, so that (5.20) is split and $U \otimes k \cong \omega(M_\beta \otimes k) \oplus \omega^t(M_\alpha \otimes k)$. But this follows from the fact that $\mathrm{Ext}^1_{kG}(\omega^t(M_\alpha \otimes k), \omega^s(M_\beta \otimes k)) \cong H^1(G, \omega^t(M_\alpha \otimes k)^* \otimes \omega^s(M_\beta \otimes k)) = 0$, where $*$ means dual with respect to $k$. The last assertion is a consequence of J. Carlson tensor product formula ([11] Theorem 5.6) as follows. The rank variety of $\omega^t(M_\alpha \otimes k)^*$ is seen to be the same as $V^r_G(\omega^t(M_\alpha \otimes k)) = V^r_G(M_\alpha \otimes k)$ by the definition of $V^r$, and $V^r_G(\omega^t(M_\alpha \otimes k)^* \otimes \omega^s(M_\beta \otimes k)) = V^r_G(M_\alpha \otimes k) \cap V^r_G(M_\alpha \otimes k) = 0$ by the choice of $\alpha$ and $\beta$. Hence $\omega^t(M_\alpha \otimes k)^* \otimes \omega^s(M_\beta \otimes k)$ is kG-free by (4) of Section 2, and $\eta' = 0$ as a consequence. Now suppose $M = M_\alpha \oplus M_\beta$ is realizable by a G-space. Then an $\omega$-composite of $M_\alpha \otimes k$ and $M_\beta \otimes k$ is realizable by a Moore G-space by Corollary 3.5. By the above discussion, any such $\omega$-composite is split and it cannot be realized by a Moore G-space since it does not satisfy the projectivity criterion (Theorem 2.1).

Since $M$ is realizable by a homotopy action, the second assertion follows. ∎

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