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by

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Abstract In series of papers we have introduced and studied l-adic polylogarithms and l-adic iterated integrals which are analogues of the classical complex polylogarithms and iterated integrals in l-adic Galois realizations. In this note we shall show that in the generic case l-adic iterated integrals are linearly independent ver Q_l. In particular they are non trivial. This result can be view as analoguous of the statement that classical iterated integrals from 0 to z of sequences of one forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ are linearly independent over Q. We also study ramification properties of l-adic polylogarithms and the minimal quotient subgroup of G_K on which l-adic polylogarithms are defined. In the final sections of the paper we study l-adic sheaves and their relations with l-adic polylogarithms. We show that if an l-adic sheaf has the same monodromy representation as the classical complex polylogarithms then the action of G_K in stalks is given by l-adic polylogarithms.

Key words: Galois group, polylogarithms, fundamental group

1 Introduction

In this paper we study properties of l-adic iterated integrals and l-adic polylogarithms introduced in [W1] and [W2]. We describe briefly main results of the paper, though in the introduction we do not present them in full generality.

Let K be a number field, let $z \in K \setminus \{0,1\}$ or let z be a tangential point of $P^1_{\bar{K}} \setminus \{0,1,\infty\}$ defined over K and let γ be an l-adic path from $\overrightarrow{01}$ to z on $P^1_{\bar{K}} \setminus \{0,1,\infty\}$. For any $\sigma \in G_K$ we set

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$$f_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1^{\mathrm{et}}(P^1_{\bar K} \setminus \{0,1,\infty\}; \overrightarrow{01})_{\mathrm{pro}-l}.$$

Then we define l-adic iterated integrals from $\overrightarrow{01}$ to z. They are functions

$$l_b(z) : G_K \to Q_l$$

(they are coefficients of $f_\gamma(\ )$) and indices are taking values in a Hall base $\mathscr{B}$ of the free Lie algebra $\mathrm{Lie}(X,Y)$ on two generators X and Y. Let $\mathscr{B}_n$ be the set of elements of degree n in $\mathscr{B}$. Let $H_n \subset G_{K(\mu_{l^\infty})}$ be a subgroup of $G_{K(\mu_{l^\infty})}$ defined by the condition that all $l_b(z)$ and $l_b(\overrightarrow{10})$ vanish on H_n for all $b \in \bigcup_{i<n} \mathscr{B}_i$.

Our first result concerns linear independence of l-adic iterated integrals.

Theorem 1. *Let $z \in K \setminus \{0,1\}$. Assume that z is not a root of any equation of the form $z^p \cdot (1-z)^q = 1$, where p and q are integers such that $p^2+q^2 > 0$. Then the functions $l_b(z) : H_n \to Q_l$ for $b \in \mathscr{B}_n$ are linearly independent over Q_l.*

Our second result concerns the minimal quotient of G_K, on which l-adic polylogarithms $l_n(z)$ are defined and ramification properties of l-adic polylogarithms.

Let $z \in K \setminus \{0,1\}$. Consider the fields $K(\mu_{l^\infty})$ and $K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}})$. Let $M(K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}}))^{ab}_{l,1-z}$ be a maximal, abelian, pro-l, unramified outside l and $1-z$ extension of $K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}})$.

Theorem 2. *Let $z \in K \setminus \{0,1\}$. Assume that z is not a root of any equation of the form $z^p \cdot (1-z)^q = 1$, where p and q are integers such that $p^2+q^2 > 0$. Then we have:*

1. *The l-adic polylogarithm $l_n(z) : G_K \to Q_l$ factors through the group $Gal(M(K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}}))^{ab}_{l,1-z}/K)$.*
2. *The l-adic polylogarithm $l_n(z)$ ramifies only at prime divisors of the product $l \cdot z \cdot (1-z)$.*
3. *The l-adic polylogarithm $l_n(z)$ determines a non-trivial element in the group*

$$\mathrm{Hom}\big(Gal(M(K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}}))^{ab}_{l,1-z}/K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}})); Q_l\big).$$

Our third result concerns connections with a non-abelian Iwasawa theory though we are not sure if our terminology non-abelian Iwasawa theory is not exaggerated as a result is quite elementary.

Let us set $\mathscr{G} := Gal\big(M(K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}}))^{ab}_{l,1-z}/K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}})\big)$ and $\Phi := Gal(K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}})/K)$. The Galois group $\mathscr{G}$ is a Φ-module, hence it is also a $Z_l[[\Phi]]$-module. Therefore $\mathrm{Hom}(\mathscr{G}, Z_l)$ is also a $Z_l[[\Phi]]$-module.

Theorem 3. *Let $\mu \in Z_l[[\Phi]]$. Under the same assumptions as in Theorems 1 and 2 we have*

$$\mu(l_m(z)) = \Big(\int_\Phi \chi^m(x) d\mu\Big) l_m(z) + \sum_{k=1}^{m-1} \Big(\int_\Phi \frac{(-l(z)(x))^k}{k!} \chi^{m-k}(x) d\mu\Big) l_{m-k}(z). \quad (1)$$

In the final sections of the paper we study l-adic sheaves. We shall show that if an l-adic sheaf has the same monodromy representation as the classical complex polylogarithms then the Galois action in stalks is given by l-adic polylogarithms.

2 $P^1_{Q(\mu_n)} \setminus (\{0,\infty\} \cup \mu_n)$

In this section we recall some elementary results concerning Galois actions on fundamental groups in the special case of $P^1_{Q(\mu_n)} \setminus (\{0,\infty\} \cup \mu_n)$ (see [W3] and [DW]).

Let us fix a rational prime l. Let K be a number field containing the group μ_n of n-th roots of unity. Let $V := P^1_K \setminus (\{0,\infty\} \cup \mu_n)$. We denote by $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ the pro-l completion of the étale fundamental group of $V_{\bar{K}}$ based at $\overrightarrow{01}$. First we describe how to choose generators of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$. Let $\xi := \exp(\frac{2\pi i}{n})$. Let π_0 be the standard path from $\overrightarrow{01}$ to $\overrightarrow{10}$. Let x be a loop around 0 based at $\overrightarrow{01}$ in an infinitesimal neibourhood of 0. Let y_0' be a loop around 1 based at $\overrightarrow{10}$ and s_k a path from $\overrightarrow{01}$ to $\overrightarrow{0\xi^k}$ in infinitesimal neibourhoods of 1 and 0 respectively.

Let $r_k : V \to V$ be given by $r_k(z) = \xi^k \cdot z$. We set $y_0 := \pi_0^{-1} \cdot y_0' \cdot \pi_0$ and $y_k := s_k^{-1} \cdot ((r_k)_*(y_0)) \cdot s_k$ for $0 < k < n$. Then $x, y_0, y_1, \ldots, y_{n-1}$ are free generators of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$. Observe that $s_j^{-1} \cdot ((r_j)_*(y_k)) \cdot s_j = y_{k+j}$ if $k + j < n$ and $s_j^{-1} \cdot ((r_j)_*(y_k)) \cdot s_j = x^{-1} \cdot y_{k+j} \cdot x$ if $k + j \geq n$

Let $z \in V(K)$ or let z be a tangential point defined over K. Let γ be an l-adic path from $\overrightarrow{01}$ to z. We recall that for any $\sigma \in G_K$,

$$f_\gamma(\sigma)(x, y_0, \ldots, y_{n-1}) := \gamma^{-1} \cdot \sigma(\gamma). \tag{2}$$

Observe that $(r_k)_*(\gamma) \cdot s_k$ is a path from $\overrightarrow{01}$ to $\xi^k z$ and

$$f_{((r_k)_*(\gamma)) \cdot s_k}(\sigma) = f_\gamma(\sigma)(x, y_k, y_{k+1}, \ldots, y_{n-1}, x^{-1} \cdot y_0 \cdot x, \ldots, x^{-1} \cdot y_{k-1} \cdot x) \cdot x^{\frac{k(\chi(\sigma)-1)}{n}}. \tag{3}$$

Let

$$k : \pi_1(V_{\bar{K}}; \overrightarrow{01}) \to Q_l\{\{X, Y_0, \ldots, Y_{n-1}\}\}$$

be a continuous multiplicative embedding of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ into the Q_l-algebra of non-commutative formal power series $Q_l\{\{X, Y_0, \ldots, Y_{n-1}\}\}$ given by $k(x) = \exp(X)$ and $k(y_j) = \exp(Y_j)$ for $0 \leq j < n$.

Let $\pi(V_{\bar{K}}; z, \overrightarrow{01})$ be the $\pi_1(V_{\bar{K}}; \overrightarrow{01})$-torsor of l-adic paths from $\overrightarrow{01}$ to z. The map $\delta \to \gamma^{-1} \cdot \delta$ defines the bijection $t_\gamma : \pi(V_{\bar{K}}; z, \overrightarrow{01}) \to \pi_1(V_{\bar{K}}; \overrightarrow{01})$. Composing t_γ with the embedding k we get an embedding

$$k_\gamma : \pi(V_{\bar{K}}; z, \overrightarrow{01}) \to Q_l\{\{X, Y_0, \ldots, Y_{n-1}\}\}.$$

The Galois group G_K acts on $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ and on $\pi(V_{\bar{K}}; z, \overrightarrow{01})$. Hence we get two Galois representations

$$\varphi_{\overrightarrow{01}} : G_K \to \mathrm{Aut}(Q_l\{\{X, Y_0, \dots, Y_{n-1}\}\})$$

and

$$\psi_\gamma : G_K \to \mathrm{GL}(Q_l\{\{X, Y_0, \dots, Y_{n-1}\}\})$$

deduced from the action of G_K on $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ and on $\pi(V_{\bar{K}}; z, \overrightarrow{01})$ respectively.

Before going farther we fix the notation.
The set of Lie polynomials in $Q_l\{\{X, Y_0, \dots, Y_{n-1}\}\}$ we denote by $\mathrm{Lie}(X, Y_0, \dots, Y_{n-1})$. It is a free Lie algebra on $n+1$ generators $X, Y_0, \dots, Y_{n-1}$. The set of formal Lie power series in $Q_l\{\{X, Y_0, \dots, Y_{n-1}\}\}$ we denote by $\mathrm{L}(X, Y_0, \dots, Y_{n-1})$.

We denote by I_2 the closed Lie ideal of $\mathrm{L}(X, Y_0, \dots, Y_{n-1})$ generated by Lie brackets with two or more Y's. We shall use the following notation

$$[Y_k, X^{(1)}] := [Y_k, X] \ \text{ and } \ [Y_k, X^{(m)}] := [[Y_k, X^{(m-1)}], X] \ \text{ for } \ m > 1.$$

In an algebra the operator of the left (resp. right) multiplication by a we denote by L_a (resp. R_a).

We recall the definition of l-adic iterated integrals from [W1]. Let $\mathscr{B}$ be a Hall base of the free Lie algebra $\mathrm{Lie}(X, Y_0, \dots, Y_{n-1})$ on $n+1$ free generators $X, Y_0, \dots, Y_{n-1}$ and let $\mathscr{B}_m$ be the set of elements of degree m in $\mathscr{B}$. For $b \in \mathscr{B}$ we define l-adic iterated integrals

$$l_b(z)_\gamma : G_{K(\mu_{l^\infty})} \to Q_l$$

as follows. Let $\sigma \in G_{K(\mu_{l^\infty})}$. Then $(\log \psi_\gamma(\sigma))(1)$ is a Lie element, hence

$$(\log \psi_\gamma(\sigma))(1) = \sum_{b \in \mathscr{B}} l_b(z)_\gamma(\sigma) \cdot b.$$

More naively, for $\sigma \in G_K$ we define functions $li_b(z)_\gamma : G_K \to Q_l$ by the equality

$$\log \Lambda_\gamma(\sigma) = \sum_{b \in \mathscr{B}} li_b(z)_\gamma(\sigma) \cdot b, \tag{4}$$

where $\Lambda_\gamma(\sigma) := k(f_\gamma(\sigma))$.

With the representations $\varphi_{\overrightarrow{01}}$ and ψ_γ there are associated the filtrations $\{G_m = G_m(V, \overrightarrow{01})\}_{m \in N}$ and $\{H_m = H_m(V, z, \overrightarrow{01})\}_{m \in N}$ of G_K (see [W1], section 3, pp. 122-124).

We recall that

$$H_m = \{\sigma \in G_{K(\mu_{l^\infty})} \mid l_b(z)(\sigma) = 0 \text{ and } l_b(\xi^k)(\sigma) = 0 \text{ for } 0 \le k < n \text{ and for all } b \in \bigcup_{i<m} \mathscr{B}_i\}.$$

If $b \in \mathscr{B}_m$ and $\sigma \in H_m$ then $l_b(z)_\gamma(\sigma) = li_b(z)_\gamma(\sigma)$.

Proposition 1. *Let* $\sigma \in H_m(V, z, \overrightarrow{01})$. *Then*

$$(\log\psi_\gamma(\sigma))(1) \equiv \log\Lambda_\gamma(\sigma) \equiv \Lambda_\gamma(\sigma) - 1 \mod \Gamma^{m+1}\mathrm{L}(X, Y_0, \ldots, Y_{n-1}). \tag{5}$$

Proof. The first congruence follows from the formula $\Psi_\gamma = L_{\Lambda_\gamma(\sigma)} \circ \varphi_{\overrightarrow{01}}$ (see [W1], Lemma 1.0.2) after taking logarithm and applying the Baker-Campbell-Hausdorff formula. The second congruence is clear. □

Let us set

$$\gamma_k := ((r_k)_*(\gamma)) \cdot s_k. \tag{6}$$

Our next result is a consequence of the formula (3).

Proposition 2. *Let $\sigma \in H_m(V, z, \overrightarrow{01})$. Then*

$$\log(\Lambda_{\gamma_k}(\sigma)(X, Y_0, \ldots, Y_{n-1})) \equiv \log(\Lambda_\gamma(\sigma)(X, Y_k, \ldots, Y_{n-1}, Y_0, \ldots, Y_{k-1})) \mod \Gamma^{m+1}\mathrm{L}(X, Y_0, \ldots, Y_{n-1}).$$

Proof. The proof is the same as the proof of Lemma 15.2.1 in [W3]. □

Corollary 1. *Let $m > 1$ and let $\sigma \in H_m(V, z, \overrightarrow{01})$. Then we have*

$$\log(\Lambda_\gamma(\sigma)(X, Y_0, \ldots, Y_{n-1})) \equiv \sum_{k=0}^{n-1} l_m(\xi^{-k}z)(\sigma)[Y_k, X^{(m-1)}] \mod \Gamma^{m+1}\mathrm{L}(X, Y_0, \ldots, Y_{n-1}) + I_2$$

for $m > 1$. Let $\sigma \in G_{K(\mu_{l^\infty})}$. Then we have

$$\log(\Lambda_\gamma(\sigma)(X, Y_0, \ldots, Y_{n-1})) \equiv \sum_{k=0}^{n-1} l(1 - \xi^{-k}z) Y_k \mod \Gamma^2\mathrm{L}(X, Y_0, \ldots, Y_{n-1}).$$

Proof. The corollary follows from the very definition of l-adic polylogarithms (see [W2], Definition 11.0.1) and from Proposition 2. □

Now we shall define polylogarithmic quotients of the representations $\varphi_{\overrightarrow{01}}$ and ψ_γ.

Let $\mathscr{I}$ be a closed ideal of $Q_l\{\{X, Y_0, \ldots, Y_{n-1}\}\}$ generated by monomials with any two Y's and by monomials $Y_k X$ for $0 \leq k \leq n-1$. We set

$$Pol(X, Y_0, \ldots, Y_{n-1}) := Q_l\{\{X, Y_0, \ldots, Y_{n-1}\}\}/\mathscr{I}$$

Observe that the classes of $1, X, \ldots, X^m, \ldots, Y_k, XY_k, \ldots, X^{m-1}Y_k, \ldots$ for $m = 1, 2, \ldots$ and $0 \leq k \leq n-1$ form a topological base of $Pol(X, Y_0, \ldots, Y_{n-1})$.

The image of the power series $\Lambda_\gamma(\sigma) \in Q_l\{\{X, Y_0, \ldots, Y_{n-1}\}\}$ in $Pol(X, Y_0, \ldots, Y_{n-1})$ we denote by $\Omega_\gamma(\sigma)$.

Proposition 3. *i) The representation $\varphi_{\overrightarrow{01}}$ (resp. ψ_γ) induces the representation*

$$\bar{\varphi}_{\overrightarrow{01}} : G_K \to \mathrm{Aut}(Pol(X, Y_0, \ldots, Y_{n-1}))$$

.

$$(\text{resp.} \, \bar{\psi}_\gamma : G_K \to \mathrm{GL}(Pol(X, Y_0, \ldots, Y_{n-1})) \,).$$

ii) The representation $\bar{\varphi}_{\overrightarrow{01}}$ *is given by*

$$\bar{\varphi}_{\overrightarrow{01}}(\sigma)(X) = \chi(\sigma)X$$

and

$$\bar{\varphi}_{\overrightarrow{01}}(\sigma)(Y_k) = \chi(\sigma)Y_k + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \chi(\sigma)\big(\frac{k}{n}(\chi(\sigma)-1)\big)^i X^i Y_k$$

for $k = 0, 1, \ldots, n-1$.

iii) The representation $\bar{\psi}_\gamma$ *is given by the formula*

$$\bar{\psi}_\gamma(\sigma) = L_{\Omega_\gamma(\sigma)} \circ \bar{\varphi}_{\overrightarrow{01}}(\sigma).$$

iv) If $n = 1$ *then*

$$\log \Omega_\gamma(\sigma) = l(z)_\gamma(\sigma)X + \sum_{i=1}^{\infty} (-1)^{i-1} l_i(z)_\gamma(\sigma) X^{i-1} Y_0.$$

Proof. It follows from [W3], Proposition 15.1.7 that $\varphi_{\overrightarrow{01}}(\mathscr{I}) \subset \mathscr{I}$. Hence $\varphi_{\overrightarrow{01}}$ induces a representation on the quotient space. The point ii) follows from [W3], Proposition 15.1.7 too.

We recall that $\psi_\gamma(\sigma) = L_{\Lambda_\gamma(\sigma)} \circ \varphi_{\overrightarrow{01}}(\sigma)$ (see [W1], section 4). Hence we get the point i) for ψ_γ and the point iii). The point iv) follows from the definition of l-adic polylogarithms given in [W2]. □

Let $\alpha \in Q_l^\times$. We denote by $\tau(\alpha)$ the automorphism of the Q_l-algebra $Pol(X,Y)$ such that $\tau(\alpha)(X) = \alpha \cdot X$ and $\tau(\alpha)(Y) = \alpha \cdot Y$ and continuous with respect to the topology defined by the powers of the augmentation ideal.

For $n = 1$ we have a very simple description of $\varphi_{\overrightarrow{01}}$.

Corollary 2. *If* $n = 1$ *then*

$$\bar{\varphi}_{\overrightarrow{01}}(\sigma) = \tau(\chi(\sigma)).$$

3 Linear independence over Q_l of l-adic iterated integrals

In this section we shall prove linear independence of l-adic polylogarithms in generic situation. We use the notation of section 2.

If $a_1, \ldots, a_k$ belong to $K^\times$ we denote by $\langle a_1, \ldots, a_k \rangle$ or $\langle a_i \mid 1 \leq i \leq n \rangle$ the subgroup of $K^\times$ generated by $a_1, \ldots, a_k$.

Theorem 4. *Let* $z \in K$. *Suppose that* z *is not a root of any equation of the form* $z^p \cdot \prod_{k=0}^{n-1} (z - \xi^k)^{q_k} = 1$, *where* p *and* q_k *are integers not all equal zero. Suppose that* $\langle z, 1 - \xi^{-k} z \mid 0 \leq k \leq n-1 \rangle \cap \langle 1 - \xi^{-k} \mid 1 \leq k \leq n-1 \rangle \subset \mu_n$. *Then the homomorphisms*

$$l_b(z) : H_m(V,z,\overrightarrow{01})/H_{m+1}(V,z,\overrightarrow{01}) \to Q_l$$

for $b \in \mathscr{B}_m$ are linearly independent over Q_l.

Proof. The morphism

$$\psi_\gamma : G_K \to \mathrm{GL}(Q_l\{\{X,Y_0,\dots,Y_{n-1}\}\})$$

induces the morphism of associated graded Lie algebras

$$\Psi_{z,\overrightarrow{01}} : \bigoplus_{m=1}^{\infty}(H_m(V,z,\overrightarrow{01})/H_{m+1}(V,z,\overrightarrow{01}))\otimes Q \to \mathrm{Lie}(X,Y_0,\dots,Y_{n-1})\tilde{\times}\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})_{\{\}}.$$

(The Lie algebra $\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})_{\{\}}$ and the semi-direct product $\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})\tilde{\times}\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})_{\{\}}$ are defined in [W1], section 5.) The morphism $\Psi_{z,\overrightarrow{01}}$ in degree 1 is given by

$$\Psi_{z,\overrightarrow{01}}(\sigma) = \big(l(z)(\sigma)X + \sum_{k=0}^{n-1} l(1-\xi^{-k}z)(\sigma)Y_k, \sum_{k=1}^{n-1} l(1-\xi^{-k})(\sigma)Y_k\big).$$

Numbers z and $1-\xi^{-k}z$, $0 \le k < n$ are linearly independent in $K^\times \otimes Q$. The intersection of subgroups $\langle 1-\xi^{-k} \mid 1 \le k \le n-1\rangle$ and $\langle z, 1-\xi^{-k}z \mid 0 \le k \le n-1\rangle$ is contained in μ_n. Hence it follows from the Kummer theory that we can find $\tau \in H_1 = K(\mu_{l^\infty})$ and $\sigma_k \in H_1$ for $0 \le k < n$ such that $\Psi_{z,\overrightarrow{01}}(\tau) = (X,0)$ and $\Psi_{z,\overrightarrow{01}}(\sigma_k) = (Y_k,0)$ for $0 \le k < n$. The Lie subalgebra of $Image(\Psi_{z,\overrightarrow{01}})$ generated by these elements is the first factor of the semi-direct product $\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})\tilde{\times}\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})_{\{\}}$, hence it is the free Lie algebra $\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})$. For $\sigma \in H_m(V,z,\overrightarrow{01})$ the morphism $\Psi_{z,\overrightarrow{01}}$ is given by the formulas

$$\Psi_{z,\overrightarrow{01}}(\sigma) = (\log\Lambda_\gamma(\sigma), \log\Lambda_{\pi_0}(\sigma)) \mod \Gamma^{m+1}\big(\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})\tilde{\times}\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})_{\{\}}\big)$$

and

$$\log\Lambda_\gamma(\sigma) \equiv \sum_{b\in\mathscr{B}_m} l_b(z)(\sigma)b \mod \Gamma^{m+1}\mathrm{L}(X,Y_0,\dots,Y_{n-1}).$$

Hence it follows that the functions

$$l_b(z) : H_m(V_K,z,\overrightarrow{01}) \to Q_l$$

are linearly independent over Q_l. □

Theorem 1 of Introduction follows immediately from Theorem 4.

Corollary 3. *The l-adic polylogarithms*

$$l_m(\xi^k z) : H_m(V_K,z,\overrightarrow{01})/H_{m+1}(V_K,z,\overrightarrow{01}) \to Q_l$$

are linearly independent over Q_l.

Proof. The corollary follows immediately from Theorem 4 and Corollary 1 of section 2. □

Remark 1. Theorem 4 is an analogue of the statement - as far as we know unproven - that the iterated integrals indexed by elements of $\mathscr{B}_m$ as in [W6] of sequences of length m of one forms $\frac{dz}{z}$ and $\frac{dz}{z-\xi^k}$ for $0 \leq k \leq n-1$ from $\overrightarrow{01}$ to z satisfying the assumption of Theorem 4, are linearly independent over Q.

4 Ramification properties of l-adic polylogarithms

Let K be a number field. Let $z \in K \setminus \{0,1\}$ or let z be a tangential point of $P^1_{\bar{K}} \setminus \{0,1,\infty\}$ defined over K. Let γ be an l-adic path from $\overrightarrow{01}$ to z.

If L is an algebraic extension of K and $z \in K$, we denote by $M(L)_{l,z}$ (resp. $M(L)^{ab}_{l,z}$) a maximal, pro-l, unramified outside l and z (resp. and abelian) extension of L.

The triple $(P^1_K \setminus \{0,1,\infty\}, z, \overrightarrow{01})$ has good reduction outside the prime ideals dividing z or $1-z$. Therefore the action of G_K on the torsor of l-adic paths $\pi(P^1_{\bar{K}} \setminus \{0,1,\infty\}; z, \overrightarrow{01})$ from $\overrightarrow{01}$ to z factors through $Gal(M(K(\mu_{l^\infty}))_{l,z(1-z)}/K)$. Hence the l-adic polylogarithm

$$l_m(z)_\gamma : G_K \to Q_l$$

factors through $Gal\big(M(K(\mu_{l^\infty}))_{l,z(1-z)}/K\big)$ and we get

$$l_m(z)_\gamma : Gal\big(M(K(\mu_{l^\infty}))_{l,z(1-z)}/K\big) \to Q_l.$$

Let us consider a tower of fields

$$K \hookrightarrow K(\mu^{l^\infty}) \hookrightarrow K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}}).$$

Proposition 4. *The l-adic polylogarithm $l_n(z)_\gamma$ factors through $Gal\big(M(K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}}))^{ab}_{l,1-z}/K\big)$.*

Proof. Let us consider polylogarithmic quotient of the representation $\psi_\gamma : G_K \to \mathrm{GL}(Q_l\{\{X,Y\}\})$, i.e. the representation $\bar{\psi}_\gamma : G_K \to \mathrm{GL}(Pol(X,Y))$ given by

$$G_K \ni \sigma \to L_{\Omega_\gamma(\sigma)} \circ \bar{\varphi}_{\overrightarrow{01}}(\sigma) \in \mathrm{GL}(Pol(X,Y)),$$

where $\log\Omega_\gamma(\sigma) = l(z)_\gamma(\sigma)X + \sum_{n=1}^\infty (-1)^{n-1} l_n(z)_\gamma(\sigma)X^{n-1}Y$ (see Proposition 3). After the restriction to $G_{K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}})}$ we get an abelian representation

$$G_{K(\mu_{l^\infty}, z^{\frac{1}{l^\infty}})} \ni \sigma \to L_{1+\sum_{n=1}^\infty (-1)^{n-1} l_n(z)_\gamma(\sigma)X^{n-1}Y} \in \mathrm{GL}(Pol(X,Y)).$$

Therefore the l-adic polylogarithm $l_n(z)_\gamma$ factors through $Gal(M(K(\mu_{l^\infty},z^{\frac{1}{l^\infty}}))^{ab}_{l,z(1-z)}/K)$. The functions $l_m(z)_\gamma$ are given explicitely by Kummer characters associated to $\prod_{i=0}^{n-1}(1-\xi^i_{l^n}z^{\frac{1}{l^n}})^{\frac{i^{m-1}}{l^n}}$ (see [NW]). Observe that $1-\xi^i_{l^n}z^{\frac{1}{l^n}}\equiv 1$ modulo any prime ideal lying over prime divisors of the principal ideal (z). Hence $l_n(z)_\gamma$ factors through $Gal(M(K(\mu_{l^\infty},z^{\frac{1}{l^\infty}}))^{ab}_{l,1-z}/K)$. □

Corollary 4. *The l-adic polylogarithm $l_n(z)_\gamma$ restricted to the Galois group $Gal\big(M(K(\mu_{l^\infty},z^{\frac{1}{l^\infty}}))^{ab}_{l,1-z}/K(\mu_{l^\infty},z^{\frac{1}{l^\infty}})\big)$ is a homomorphism.*

Proof. In the proof of Proposition 4 we have already seen that the representation $\bar{\psi}_\gamma$ restricted to $G_{K(\mu_{l^\infty},z^{\frac{1}{l^\infty}})}$ is abelian. □

5 Action of $Z_l[[Gal\big(K(\mu_{l^\infty},z^{\frac{1}{l^\infty}})/K\big)]]$ on l-adic polylogarithms

The notation in this section is the same as in the section 4. Let us consider a tower of fields

$$\begin{array}{c} \mathrm{M}(\mathrm{K}(\mu_{l^\infty},z^{\frac{1}{l^\infty}}))^{ab}_{l,1-z} \\ \mathscr{G}\ \ | \\ K(\mu_{l^\infty},z^{\frac{1}{l^\infty}}) \\ Z_l(1)\ \ | \\ K(\mu_{l^\infty}) \\ \Gamma\ \ | \\ K \end{array}$$

where $\Gamma := Gal(K(\mu_{l^\infty})/K)$. Observe that $Gal\big(K(\mu_{l^\infty},z^{\frac{1}{l^\infty}})/K(\mu_{l^\infty})\big)=Z_l(1)$ as a Γ-module.

Let $\Phi := Gal\big(K(\mu_{l^\infty},z^{\frac{1}{l^\infty}})/K\big)$. We want to understand $\mathscr{G}$ as a Φ-module and as a $Z_l[[\Phi]]$-module. The l-adic polylogarithms $l_n(z)_\gamma$, restricted to $\mathscr{G}$, belong to $\mathrm{Hom}(\mathscr{G},Q_l)$. As our first step to understand $\mathscr{G}$ we shall study a $Z_l[[\Phi]]$-module generated by $l_n(z)_\gamma$ in $\mathrm{Hom}(\mathscr{G},Q_l)$.

We recall that Φ acts on $\mathscr{G}$ on the left in the following way. Let $\sigma\in\Phi$ and $\tau\in\mathscr{G}$. Let $\tilde{\sigma}\in Gal\big(M(K(\mu_{l^\infty},z^{\frac{1}{l^\infty}}))^{ab}_{l,1-z}/K\big)$ be a lifting of σ. Then the formula ${}^{\sigma}\tau := \tilde{\sigma}\cdot\tau\cdot\tilde{\sigma}^{-1}$ defines a left action of Φ on $\mathscr{G}$. Hence the right action of Φ on $\mathrm{Hom}(\mathscr{G},Q_l)$ is given by

$$(f^\sigma)(\tau) := f(\tilde{\sigma}\cdot\tau\cdot\tilde{\sigma}^{-1}).$$

To study the action of Φ on $l_n(z)_\gamma$ first we need to calculate $\Lambda_\gamma(\tilde{\sigma}\cdot\tau\cdot\tilde{\sigma}^{-1})$.

Lemma 1. *For any $\alpha,\tau\in G_K$ we have*

$$\Lambda_\gamma(\alpha\cdot\tau\cdot\alpha^{-1})=\Lambda_\gamma(\alpha)\cdot\varphi_{\overrightarrow{01}}(\alpha)(\Lambda_\gamma(\tau))\cdot\varphi_{\overrightarrow{01}}(\alpha\cdot\tau\cdot\alpha^{-1})(\Lambda_\gamma(\alpha)^{-1})$$

in $Q_l\{\{X,Y\}\}$.

Proof. The formula of the lemma follows from [W1], Proposition 1.0.7 and Corollary 1.0.8. □

We define the product $\bigcirc$ by the Baker-Campbell-Hausdorff formula

$$X \bigcirc Y := \log(e^X \cdot e^Y).$$

Proposition 5. *The action of* $\sigma \in \Phi$ *on* $l_m(z)_\gamma \in \mathrm{Hom}(\mathscr{G}, Q_l)$ *is given by the formula*

$$(l_m(z)_\gamma)^\sigma = \chi(\sigma)^m \cdot l_m(z)_\gamma + \sum_{k=1}^{m-1} \frac{(-l(z)_\gamma(\sigma))^k}{k!} \cdot \chi(\sigma)^{m-k} \cdot l_{m-k}(z)_\gamma.$$

Proof. Let $\tau \in \mathscr{G}$ and let $\bar{\sigma}$ and $\bar{\tau}$ be liftings of σ and τ to $Gal(\bar{K}/K)$. It follows from Lemma 1 that

$$\log\Lambda_\gamma(\bar{\sigma}\cdot\bar{\tau}\cdot\bar{\sigma}^{-1}) = \log\Lambda_\gamma(\bar{\sigma}) \bigcirc \varphi_{\overrightarrow{01}}(\bar{\sigma})(\log\Lambda_\gamma(\bar{\tau})) \bigcirc \big(\varphi_{\overrightarrow{01}}(\bar{\sigma}\cdot\bar{\tau}\cdot\bar{\sigma}^{-1})(-\log\Lambda_\gamma(\bar{\sigma}))\big).$$

Hence we get

$$\sum_{n=1}^{\infty} l_n(z)({}^{\sigma}\tau)[Y,X^{(n-1)}] \equiv \big(l(z)(\bar{\sigma})X + \sum_{n=1}^{\infty} l_n(z)(\bar{\sigma})[Y,X^{(n-1)}]\big) \bigcirc \big(\chi(\bar{\sigma})l(z)(\tau)X +$$

$$\sum_{n=1}^{\infty} \chi(\bar{\sigma})^n \cdot l_n(z)(\tau)[Y,X^{(n-1)}]\big) \bigcirc \big(-l(z)(\bar{\sigma})X - \sum_{n=1}^{\infty} l_n(z)(\bar{\sigma})[Y,X^{(n-1)}]\big) \mod I_2.$$

Observe that $l(z)(\bar{\sigma})$ and $\chi(\bar{\sigma})$ depend only on σ. Hence we replace them by $l(z)(\sigma)$ and $\chi(\sigma)$.
We get the formula of the proposition calculating the right hand side of the congruence and comparing coefficients at $[Y,X^{(n-1)}]$. □

Generalization to the action of $Z_l[[\Phi]]$ is straightforward.

Corollary 5. *Let* $\mu \in Z_l[[\Phi]]$. *Then*

$$(l_m(z)_\gamma)^\mu = \big(\int_\Phi \chi(x)^m d\mu(x)\big) l_m(z)_\gamma + \sum_{k=1}^{m-1} \big(\int_\Phi \frac{(-l(z)_\gamma(x))^k}{k!} \cdot \chi(x)^{m-k} d\mu(x)\big) \cdot l_{m-k}(z)_\gamma.$$

6 l-adic sheaves

The l-adic polylogarithms and l-adic iterated integrals studied in [W1], [W2], [W3] and in [NW] arise from actions of Galois groups on the set of homotopy classes of l-adic paths from v to z on $P^1_{\bar{Q}}$ minus a finite number of points.

On the other side in [BD], [BL] and in various other papers there are studied motivic polylogarithmic sheaves. Their l-adic realizations are inverse systems of locally constant sheaves of Z/l^n-modules in étale topology. Each stalk is equipped with a Galois representation. The relation between the parallel transport and the Galois representations in stalks is given by the formula

$$\sigma_t \circ p_* = \sigma(p)_* \circ \sigma_s, \tag{7}$$

where p_* (resp. $\sigma(p)_*$) is the parallel transport along the path p (resp. $\sigma(p)$) from s to t, σ_s (resp. σ_t) is the action of $\sigma \in G_K$ in the stalk over s (resp. over t) and $\sigma(p)$ is the image of p by σ in the torsor of paths from s to t.

The formula (7) is fundamental to relate l-adic polylogarithms introduced in [W2] with polylogarithmic sheaves.

If V is a smooth quasi-projective algebraic variety we denote by $(V)_{\rm et}$ the étale site associated to V.

Example 1. Let $p : X \to S$ be a smooth morphism between smooth quasi-projective algebraic schemes over K. Let $\bar{p} : X_{\bar{K}} \to S_{\bar{K}}$ be obtained from $p : X \to S$ by the extension of scalars to $\bar{K}$. Let $(Z/l^n)_{(X_{\bar{K}})_{\rm et}}$ be the constant sheaf on $(X_{\bar{K}})_{\rm et}$. The sheaves of Z/l^n-modules $R^i(\bar{p})_*\big((Z/l^n)_{(X_{\bar{K}})_{\rm et}}\big)$ on $(S_{\bar{K}})_{\rm et}$ are locally constant in the étale topology. The projective system of sheaves

$$\{R^i(\bar{p})_*((Z/l^n)_{(X_{\bar{K}})_{\rm et}})\}_{n\in N}$$

defines an l-adic sheaf on $(S_{\bar{K}})_{\rm et}$. The stalk over $s \in S(\bar{K})$ is $H^i_{\rm et}((X_s)_{\bar{K}}; Z_l) :=$ projlim$_n H^i_{\rm et}((X_s)_{\bar{K}}; Z/l^n)$. If $s \in S(K)$ then G_K acts on $H^i_{\rm et}((X_s)_{\bar{K}}; Z_l)$. If $s,t \in S(K)$ and γ is an l-adic path from s to t then the parallel transport induces $\gamma_* : H^i_{\rm et}((X_s)_{\bar{K}}; Z_l) \to H^i_{\rm et}((X_t)_{\bar{K}}; Z_l)$ satisfying (7).

The example given above motivates the following definition.

Definition 1. Let S be a smooth quasi-projective algebraic variety defined over K. A profinite sheaf $\mathscr{F}$ on $S_{\bar{K}}$ is an inverse system

$$\{\varphi_{n+1} : \mathscr{F}_{n+1} \to \mathscr{F}_n\}_{n\in N}$$

of sheaves on $(S_{\bar{K}})_{\rm et}$ such that :

1. for each n, $\mathscr{F}_n$ is a sheaf of finite sets, locally constant on $(S_{\bar{K}})_{\rm et}$;
2. each sheaf $\mathscr{F}_n$ is equipped with a continuous action of G_K on $\oplus_{t\in Gal(L/K)s}(\mathscr{F}_n)_t$, if $s \in S(L)$, where L is a finite extension of K and $Gal(L/K)s$ is the $Gal(L/K)$-orbit of s;
3. the structure maps $\varphi_{n+1} : \mathscr{F}_{n+1} \to \mathscr{F}_n$ are surjective and compatible with the Galois actions in the stalks;
4. if s and t are in $S(L)$ (L is a finite extension of K), p is a profinite path from s to t and $\sigma \in G_K$ then
$$\sigma_t \circ p_* = \sigma(p)_* \circ \sigma_s, \tag{8}$$
where $\sigma_s : (\mathscr{F}_n)_s \to (\mathscr{F}_n)_{\sigma(s)}$ and $\sigma_t : (\mathscr{F}_n)_t \to (\mathscr{F}_n)_{\sigma(t)}$ are maps induced by σ and p_* (resp. $\sigma(p)_*$) is a parallel transport along p (resp. $\sigma(p)$).

If each sheaf $\mathscr{F}_n$ is a sheaf of finite l-groups and the maps φ_n are homomorphisms then the profinite sheaf $\mathscr{F} = \{\varphi_{n+1} : \mathscr{F}_{n+1} \to \mathscr{F}_n\}_{n\in N}$ we shall call an l-adic sheaf.

Let $s \in S(\bar{K})$. We shall call

$$\mathscr{F}_s := \mathrm{projlim}_n(\mathscr{F}_n)_s$$

the stalk of the profinite sheaf $\mathscr{F}$ over s. Parallel transports along profinite paths and actions of Galois groups are defined on stalks of a profinite sheaf and they satisfy the equality (8).

We recall that $\pi_1^{\mathrm{et}}(S_{\bar{K}};s)$ is the étale fundamental group of $S_{\bar{K}}$ based at s. It is a profinite group. We define the monodromy representatiom

$$\rho_s : \pi_1^{\mathrm{et}}(S_{\bar{K}};s) \to \mathrm{Aut}(\mathscr{F}_s)$$

of the profinite sheaf $\mathscr{F}$ by the formula

$$\rho_s(T)(w) := T_*(w),$$

where $w \in \mathscr{F}_s$.

Let us observe the following elementary facts about profinite sheaves.

Proposition 6. *Let S be a smooth quasi-projective algebraic variety defined over K and let $s_0 \in S(K)$. Let $\mathscr{F}$ be a profinite sheaf on $S_{\bar{K}}$. Then the representation of G_K in the stalk $\mathscr{F}_{s_0}$ determines the Galois representation in any other stalk.*

Proof. Let p be a path from s_0 to s. Then it follows from the formula (8) that

$$\sigma_s = \sigma(p)_* \circ \sigma_{s_0} \circ (p_*)^{-1}.$$

Hence the Galois action in the stalk over s is uniquely determined by the action of G_K in the stalk over s_0. □

Let us define

$$f_{\pi_1^{\mathrm{et}}(S_{\bar{K}};s)}\big(Gal(\bar{K}/K)\big) := \{T^{-1} \cdot \sigma(T) \in \pi_1^{\mathrm{et}}(S_{\bar{K}};s) \mid T \in \pi_1^{\mathrm{et}}(S_{\bar{K}};s),\ \sigma \in Gal(\bar{K}/K)\}.$$

Proposition 7. *Let $\mathscr{F}$ be a profinite sheaf on $S_{\bar{K}}$. Let us assume that the subset $f_{\pi_1^{\mathrm{et}}(S_{\bar{K}};s)}\big(Gal(\bar{K}/K)\big)$ is dense in $\pi_1^{\mathrm{et}}(S_{\bar{K}};s)$. If the monodromy representation $\rho_s : \pi_1^{\mathrm{et}}(S_{\bar{K}};s) \to \mathrm{Aut}(\mathscr{F}_s)$ is non-trivial then the Galois representation in the stalk $\mathscr{F}_s$*

$$G_K \to \mathrm{Aut}(\mathscr{F}_s)$$

is also non-trivial.

Proof. It follows from the formula (8) that

$$T_*^{-1} \circ \sigma_s \circ T_* \circ (\sigma_s)^{-1} = (T^{-1} \cdot \sigma(T))_*$$

for any $T \in \pi_1^{\mathrm{et}}(S_{\bar{K}};s)$ and any $\sigma \in G_K$. The elements of the form $T^{-1} \cdot \sigma(T)$ are dense in $\pi_1^{\mathrm{et}}(S_{\bar{K}};s)$. Hence σ_s cannot be the identity for all $\sigma \in G_K$. □

Let π and G be profinite groups and let $\varphi : G \to \mathrm{Aut}(\pi)$ be a continuous homomorphism. We denote by $\mathrm{REP}_\varphi(\pi, G)$ the category of pairs of continuous representations $f_V : \pi \to \mathrm{Aut}(V)$ and $\rho_V : G \to \mathrm{Aut}(V)$ in finitely generated Z_l-modules satysfying

$$\rho_V(\sigma) \circ f_V(T) = f_V(\varphi(\sigma)(T)) \circ \rho_V(\sigma)$$

for any $T \in \pi$ and $\sigma \in G$.

Proposition 8. *Let S be a smooth quasi-projective algebraic variety defined over K and let $s \in S(K)$. Let $\varphi_s : G_K \to \mathrm{Aut}(\pi_1^{\mathrm{et}}(S_{\bar{K}}; s))$ be the homomorphism of the action of G_K on the étale fundamental group. The category of l-adic sheaves on $S_{\bar{K}}$ whose stalks are finitely generated Z_l-modules and the category $\mathrm{REP}_{\varphi_s}(\pi_1^{\mathrm{et}}(S_{\bar{K}}; s), G_K)$ are equivalent.*

Proof. It is clear that an l-adic sheaf $\mathscr{F}$ determines an object of the category $\mathrm{REP}_{\varphi_s}(\pi_1^{\mathrm{et}}(S_{\bar{K}}; s), G_K)$ by taking the stalk of $\mathscr{F}$ over s equipped with the monodromy representation and the action of G_K.

Let V be a finitely generated Z_l-module. Let us assume that we have two continuous representations $f : \pi_1^{\mathrm{et}}(S_{\bar{K}}; s) \to \mathrm{Aut}(V)$ and $\rho : G_K \to \mathrm{Aut}(V)$ satisfying $\rho(\sigma) \circ f(T) = f(\varphi_s(\sigma)(T)) \circ \rho(\sigma)$. The continuous representation $f : \pi_1^{\mathrm{et}}(S_{\bar{K}}; s) \to \mathrm{Aut}(V)$ determines the compatible family of continuous representations

$$\{f^{(n)} : \pi_1^{\mathrm{et}}(S_{\bar{K}}; s) \to \mathrm{Aut}(V/l^n V)\}_{n \in N}.$$

For each n there exists a locally constant sheaf $\mathscr{F}_n$ on $(S_{\bar{K}})_{\mathrm{et}}$, whose stalk over s is $V/l^n V$ and whose monodromy representation is $f^{(n)} : \pi_1^{\mathrm{et}}(S_{\bar{K}}; s) \to \mathrm{Aut}(V/l^n V)$. The representation of G_K in the stalk over s is the composition of $\rho : G_K \to \mathrm{Aut}(V)$ with the homomorphism $\mathrm{Aut}(V) \to \mathrm{Aut}(V/l^n V)$. The Galois action in any other stalk is then defined by the formula (8). □

7 *l*-adic sheaves related to bundles of fundamental groups

In this section we shall study examples of l-adic sheaves for which the monodromy representation determines Galois representations in the stalks.

Let S be a smooth quasi-projective algebraic variety defined over K and let s be a K-point of S. If $\sigma \in G_K$ we denote by σ the automorphisms of $\pi_1^{\mathrm{et}}(S_{\bar{K}}; s)$ and of $\pi_1(S_{\bar{K}}; s)$ induced by σ. We denote by σ_s the automorphism induced by σ in the stalk over s of an l-adic sheaf on $S_{\bar{K}}$. If p is a path we denote by p_* the parallel transport along p. We have the surjective map $\pi_1^{\mathrm{et}}(S_{\bar{K}}; s) \to \pi_1(S_{\bar{K}}; s)$. If $T \in \pi_1^{\mathrm{et}}(S_{\bar{K}}; s)$ we denote also by T its image in $\pi_1(S_{\bar{K}}; s)$.

Proposition 9. *Let S and s be as above. We assume that $\pi_1(S_{\bar{K}}; s)$ is a free noncommutative pro-l group. Let Π_1 be an l-adic sheaf on $S_{\bar{K}}$ whose stalk over s is $\pi_1(S_{\bar{K}}; s)$. We assume that the monodromy representation*

$$\rho : \pi_1^{\mathrm{et}}(S_{\bar{K}};s) \to \mathrm{Aut}(\pi_1(S_{\bar{K}};s))$$

is given by $\rho(T)(w) = T^{-1} \cdot w \cdot T$. *We assume also that for any* $\sigma \in G_K$, σ_s *acts on* $\pi_1(S_{\bar{K}};s)$ *by a group homomorphism. Then for any* $\sigma \in G_K$ *and any* $w \in \pi_1(S_{\bar{K}};s)$ *we have*

$$\sigma_s(w) = \sigma(w).$$

Proof. Let $\sigma \in G_K$, $T \in \pi_1^{\mathrm{et}}(S_{\bar{K}};s)$ and $w \in \pi_1(S_{\bar{K}};s)$. The formula (8) implies

$$\sigma_s(T^{-1} \cdot w \cdot T) = \sigma(T)^{-1} \cdot \sigma_s(w) \cdot \sigma(T).$$

Let us take T such that its image in $\pi_1(S_{\bar{K}};s)$ is w. Then

$$\sigma_s(w) = \sigma(w)^{-1} \cdot \sigma_s(w) \cdot \sigma(w).$$

The assumption that $\pi_1(S_{\bar{K}};s)$ is a free pro-l group implies that $\sigma_s(w) = \sigma(w)^{\eta(\sigma,w)}$, where $\eta(\sigma,w) \in Z_l$.

Let $w_1, w_2 \in \pi_1(S_{\bar{K}};s)$ be two arbitrary noncommuting elements. Then

$$\sigma_s(w_1 \cdot w_2) = \sigma(w_1 \cdot w_2)^{\eta(\sigma,w_1 \cdot w_2)} = (\sigma(w_1) \cdot \sigma(w_2))^{\eta(\sigma,w_1 \cdot w_2)}$$

and

$$\sigma_s(w_1) \cdot \sigma_s(w_2) = \sigma(w_1)^{\eta(\sigma,)} \cdot \sigma(w_2)^{\eta(\sigma,w_2)}.$$

Hence we get

$$(\sigma(w_1) \cdot \sigma(w_2))^{\eta(\sigma,w_1 \cdot w_2)} = \sigma(w_1)^{\eta(\sigma,w_1)} \cdot \sigma(w_2)^{\eta(\sigma,w_2)}$$

for two noncommuting elements $\sigma(w_1)$, $\sigma(w_2)$ in the free pro-l group $\pi_1(S_{\bar{K}};s)$ and for $\eta(\sigma,w_1 \cdot w_2) \neq 0$, $\eta(\sigma,w_1) \neq 0$ and $\eta(\sigma,w_2) \neq 0$. This implies that $\eta(\sigma,w) = 1$ for all σ and w. □

Proposition 10. *Let* S *and* s *be as above. Let* Π *be a profinite sheaf on* $S_{\bar{K}} \times S_{\bar{K}}$ *whose stalk over* (s,s) *is* $\pi_1(S_{\bar{K}};s)$ *We assume that the monodromy representation*

$$\rho : \pi_1^{\mathrm{et}}(S_{\bar{K}};s) \times \pi_1^{\mathrm{et}}(S_{\bar{K}};s) \to Bijections(\pi_1(S_{\bar{K}};s))$$

is given by $\rho(T_1,T_2)(w) = T_1^{-1} \cdot w \cdot T_2$. *We assume also that the centrum of the group* $\pi_1(S_{\bar{K}};s)$ *is* 1. *Then for any* $\sigma \in G_K$ *and any* $w \in \pi_1(S_{\bar{K}};s)$ *we have*

$$\sigma_{(s,s)}(w) = \sigma(w).$$

Proof. The formula (8) implies

$$\sigma(T_1)^{-1} \cdot \sigma_{(s,s)}(w) \cdot \sigma(T_2) = \sigma_{(s,s)}(T_1^{-1} \cdot w \cdot T_2). \tag{9}$$

Let us take $T_1 = T_2 = T$ and $w = 1$. Then we get $\sigma(T)^{-1} \cdot \sigma_{(s,s)}(1) \cdot \sigma(T) = \sigma_{(s,s)}(1)$. Hence $\sigma_{(s,s)}(1)$ commutes with every element of $\pi_1(S_{\bar{K}};s)$. The centrum of $\pi_1(S_{\bar{K}};s)$

is 1. Therefore we get that $\sigma_{(s,s)}(1)=1$. Let us take $T_1=w=1$ in formula (9). Then we get $\sigma(T_2)=\sigma_{(s,s)}(T_2)$ for any $T_2\in\pi_1(S_{\bar{K}};s)$. □

8 Polylogarithmic l-adic sheaves and l-adic polylogarithms

We shall show that if an l-adic sheaf on $P^1_{\bar{K}}\setminus\{0,1,\infty\}$ has the same monodromy representation as the classical complex polylogarithms then the Galois representation in the stalk over a K-point z of $P^1_K\setminus\{0,1,\infty\}$ is given by the l-adic polylogarithms evaluated at z.

We start by recalling a result about the monodromy of classical complex polylogarithms. We equip the vector bundle

$$P^1(C)\setminus\{0,1,\infty\}\times Pol(X,Y)\to P^1(C)\setminus\{0,1,\infty\}$$

with the connection given by the one-form

$$\frac{1}{2\pi i}\frac{dz}{z}\otimes X+\frac{1}{2\pi i}\frac{dz}{z-1}\otimes Y.$$

(The algebra $Pol(X,Y)$ is the quotient of $C\{\{X,Y\}\}$ by the ideal $\mathscr{I}$.) Horizontal sections satisfy the equation

$$d\Lambda(z)-\Big(\frac{1}{2\pi i}\frac{dz}{z}\otimes X+\frac{1}{2\pi i}\frac{dz}{z-1}\otimes Y\Big)\cdot\Lambda(z)=0.$$

One checks that

$$\Lambda_{\overrightarrow{01}}(z):=e^{\frac{1}{2\pi i}\log z X}+\frac{1}{2\pi i}\log(1-z)Y+\sum_{k=2}^{\infty}\frac{-1}{(2\pi i)^k}Li_k(z)X^{k-1}Y$$

is a horizontal section. The functions $\log z$, $\log(1-z)$ and $Li_k(z)$ are calculated along a path α from $\overrightarrow{01}$ to z. Let x and y be the standard generators of $\pi_1(P^1(C)\setminus\{0,1,\infty\};\overrightarrow{01})$. To calculate the monodromy of $\Lambda_{\overrightarrow{01}}(z)$ we integrate along the paths $\alpha\cdot x$ and $\alpha\cdot y$.

The monodromy transformation of $\Lambda_{\overrightarrow{01}}(z)$ is given by

$$x:\Lambda_{\overrightarrow{01}}(z)\to\Lambda_{\overrightarrow{01}}(z)\cdot e^X$$

and

$$y:\Lambda_{\overrightarrow{01}}(z)\to\Lambda_{\overrightarrow{01}}(z)\cdot e^Y.$$

The elements $\alpha\cdot x\cdot\alpha^{-1}$ and $\alpha\cdot y\cdot\alpha^{-1}$ are free generators of $\pi_1(P^1(C)\setminus\{0,1,\infty\};z)$. Let $w(\alpha\cdot x\cdot\alpha^{-1},\alpha\cdot y\cdot\alpha^{-1})\in\pi_1(P^1(C)\setminus\{0,1,\infty\};z)$ be a word in $\alpha\cdot x\cdot\alpha^{-1}$ and $\alpha\cdot y\cdot\alpha^{-1}$. The monodromy representation is given by

$\rho_z : \pi_1(P^1(C)\setminus\{0,1,\infty\};z) \to \mathrm{GL}(Pol(X,Y))$; $\rho_z(\alpha\cdot x\cdot\alpha^{-1}) = R_{e^X}$ and $\rho_z(\alpha\cdot x\cdot\alpha^{-1}) = R_{e^Y}$.

Hence $\rho_z(w(\alpha\cdot x\cdot\alpha^{-1},\alpha\cdot y\cdot\alpha^{-1})) = R_{w(e^X,e^Y)}$.

Now we shall study l-adic situation. Let z_0 be a K-point of $P^1_K\setminus\{0,1,\infty\}$. We start with the description of the action of G_K on $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\};z_0)$,

Let γ be a path from z_0 to $\overrightarrow{01}$ and let p be the standard path from $\overrightarrow{01}$ to $\overrightarrow{10}$. We recall that x and y are the standard generators of $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\};\overrightarrow{01})$. Then

$$x_{z_0} := \gamma^{-1}\cdot x\cdot\gamma \text{ and } y_{z_0} := \gamma^{-1}\cdot y\cdot\gamma$$

are free generators of $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\};z_0)$. Let $\sigma\in G_K$. We recall that

$$f_\gamma(\sigma) := \gamma^{-1}\cdot\sigma(\gamma).$$

The following lemma is a standard exercice.

Lemma 2. *The action of G_K on $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\};z_0)$ is given by the formulas*

$$\sigma(x_{z_0}) = f_\gamma(\sigma)^{-1}\cdot {x_{z_0}}^{\chi(\sigma)}\cdot f_\gamma(\sigma)$$

and

$$\sigma(y_{z_0}) = f_\gamma(\sigma)^{-1}\cdot(\gamma^{-1}\cdot f_p(\sigma)^{-1}\cdot\gamma)\cdot {y_{z_0}}^{\chi(\sigma)}\cdot(\gamma^{-1}\cdot f_p(\sigma)\cdot\gamma)\cdot f_\gamma(\sigma).$$

Let z be another K-point of $P^1_K\setminus\{0,1,\infty\}$. Let δ be a path from z to z_0. Let us set

$$\gamma_z := \gamma\cdot\delta.$$

It follows from [W1] that we have the following equalities:

$$f_{\gamma\cdot\delta}(\sigma) = \delta^{-1}\cdot f_\gamma(\sigma)\cdot\delta\cdot f_\delta(\sigma) \text{ and } f_{\delta^{-1}}(\sigma)^{-1} = \delta\cdot f_\delta(\sigma)\cdot\delta^{-1}. \tag{10}$$

Hence we get

$$\delta\cdot f_{\gamma\cdot\delta}(\sigma)\cdot\delta^{-1} = f_\gamma(\sigma)\cdot f_{\delta^{-1}}(\sigma)^{-1}. \tag{11}$$

The elements $x_z := \gamma_z^{-1}\cdot x\cdot\gamma_z$ and $y_z := \gamma_z^{-1}\cdot x\cdot\gamma_z$ are generators of $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\};z)$. We embed the groups $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\};\overrightarrow{01})$, $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\};z_0)$ and $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\};z)$ into the Q_l-algebra $Q\{\{X,Y\}\}$ by setting

$k_{\overrightarrow{01}}(x) := e^X$, $k_{\overrightarrow{01}}(y) := e^Y$ for the first group;

$k_{z_0}(x_{z_0}) := e^X$, $k_{z_0}(y_{z_0}) := e^Y$ for the second group;

and

$k_z(x_z) := e^X$, $k_z(y_z) := e^Y$ for the third group.

In other words we have trivialized the bundle of fundamental groups along the path γ_z. The action of G_K on $Q\{\{X,Y\}\}$ considered over a K-point s is deduced from the action of G_K on $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\};s)$ so it depends over which point we take a stalk.

Using embeddings k_a, $a \in \{\overrightarrow{01}, z_0, z\}$ we can define Λ-series, for example $\Lambda_\delta(\sigma) := k_z(f_\delta(\sigma))$ and $\Lambda_\gamma(\sigma) := k_{z_0}(f_\gamma(\sigma))$. Because of the trivialization of the bundle of fundamental groups we can compare various Λ-series. It follows from (10) and (11) that

$$\Lambda_{\gamma\cdot\delta}(\sigma) = \Lambda_\gamma(\sigma)\cdot\Lambda_\delta(\sigma), \ \ \big(\Lambda_{\delta^{-1}}(\sigma)\big)^{-1} = \Lambda_\delta(\sigma) \tag{12}$$

and

$$\Lambda_{\gamma\cdot\delta}(\sigma) = \Lambda_\gamma(\sigma)\cdot\big(\Lambda_{\delta^{-1}}(\sigma)\big)^{-1}. \tag{13}$$

Theorem 5. *Let z_0 be a K-point of $P^1_K \setminus \{0,1,\infty\}$. Let $\mathscr{P}$ be an l-adic sheaf of Z_l-algebras over $P^1_{\bar{K}} \setminus \{0,1,\infty\}$ such that*

i) the stalk $\mathscr{P}_{z_0}$ tensored with Q is $Pol(X,Y)$;
ii) the monodromy representation after tensoring the stalk over z_0 by Q

$$\rho_{z_0} : \pi_1^{\mathrm{et}}(P^1_{\bar{K}} \setminus \{0,1,\infty\}; z_0) \to \mathrm{GL}(Pol(X,Y))$$

is given by the formula $\rho_{z_0}\big(w(x_{z_0},y_{z_0})\big)(F(X,Y)) = F(X,Y)\cdot w(e^X,e^Y)^{-1}$.

Let z be another K-point of $P^1_K \setminus \{0,1,\infty\}$. Let δ be a path from z to z_0 and let α be a path from $\overrightarrow{01}$ to z. Then

$$\delta_* \circ \sigma_z \circ (\delta_*)^{-1} = L_{B(\sigma)} \circ R_{\Omega_\alpha(\sigma)^{-1}} \circ \tau(\chi(\sigma)),$$

where $B : G_K \to Pol(X,Y)$ is a cocycle and

$$\log\Omega_\alpha(\sigma) = l(z)_\alpha(\sigma)X + \sum_{i=1}^{\infty}(-1)^{i-1} l_i(z)_\alpha(\sigma)X^{i-1}Y.$$

Proof. Let us set $\gamma = (\delta\cdot\alpha)^{-1}$. Then γ is a path from z_0 to $\overrightarrow{01}$. It follows from Lemma 2 that for any $\sigma \in G_K$ and any $w(x_{z_0},y_{z_0}) \in \pi_1(P^1_{\bar{K}} \setminus \{0,1,\infty\}; z_0)$ we have

$$\rho_{z_0}\big(\sigma(w(x_{z_0},y_{z_0}))\big)(1) = (\Omega_\gamma(\sigma))^{-1}\cdot w(e^{\chi(\sigma)X}, e^{\chi(\sigma)Y})^{-1}\cdot\Omega_\gamma(\sigma). \tag{14}$$

Let $F(X,Y) \in Pol(X,Y)$ be in the stalk tensored by Q of $\mathscr{P}$ over z_0. It follows from the formula (8) and the formula (14) that

$$\sigma_{z_0}(F(X,Y)\cdot w(e^X,e^Y)^{-1}) = \sigma_{z_0}(F(X,Y))\cdot\Omega_\gamma(\sigma)^{-1}\cdot w(e^{\chi(\sigma)X}, e^{\chi(\sigma)Y})^{-1}\cdot\Omega_\gamma(\sigma).$$

Setting $F(X,Y) = 1$ we get

$$\sigma_{z_0}(w(e^X,e^Y)^{-1}) = \sigma_{z_0}(1)\cdot(\Omega_\gamma(\sigma))^{-1}\cdot(w(e^{\chi(\sigma)X}, e^{\chi(\sigma)Y}))^{-1}\cdot\Omega_\gamma(\sigma). \tag{15}$$

The action of G_K on the stalk of $\mathscr{P}$ over z_0 is continuous with respect to the topology of $Pol(X,Y)$ defined by the powers of the augmentation ideal. Hence it follows from (15) that for any $W(X,Y) \in Pol(X,Y)$ we have

$$\sigma_{z_0}(W(X,Y)) = \sigma_{z_0}(1)\cdot(\Omega_\gamma(\sigma))^{-1}\cdot W(\chi(\sigma)X,\chi(\sigma)Y)\cdot\Omega_\gamma(\sigma). \tag{16}$$

We recall from the assumptions of the theorem that z is another K-point of $P^1_K \setminus \{0,1,\infty\}$, δ is a path from z to z_0 and α is a path from $\overrightarrow{01}$ to z.

We shall calculate the representation of G_K in the stalk of $\mathscr{P}$ over z. It follows from the fundamental formula (8) that

$$\delta_* \circ \sigma_z \circ \delta_*^{-1} = \delta_* \circ \sigma(\delta)_*^{-1} \circ \sigma_{z_0}.$$

Observe that

$$\delta_* \circ \sigma(\delta)_*^{-1} = (\delta \circ \sigma(\delta^{-1}))_* = (f_{\delta^{-1}}(\sigma))_* = \rho_{z_0}(f_{\delta^{-1}}(\sigma)) = R_{(\Omega_{\delta^{-1}}(\sigma))^{-1}}.$$

Hence we get

$$\delta_* \circ \sigma_z \circ \delta_*^{-1} = R_{(\Omega_{\delta^{-1}}(\sigma))^{-1}} \circ \sigma_{z_0}.$$

The formula (16) implies that $R_{(\Omega_{\delta^{-1}}(\sigma))^{-1}} \circ \sigma_{z_0} = R_{(\Omega_{\delta^{-1}}(\sigma))^{-1}} \circ L_{\sigma_{z_0}(1)\cdot(\Omega_\gamma(\sigma))^{-1}} \circ R_{\Omega_\gamma(\sigma)} \circ \tau(\chi(\sigma)) = L_{\sigma_{z_0}(1)\cdot(\Omega_\gamma(\sigma))^{-1}} \circ R_{\Omega_\gamma(\sigma)\cdot(\Omega_{\delta^{-1}}(\sigma))^{-1}} \circ \tau(\chi(\sigma))$.

We recall that $\alpha^{-1} = \gamma\cdot\delta$. Hence it follows from (13) that $\Omega_\gamma(\sigma)\cdot(\Omega_{\delta^{-1}}(\sigma))^{-1} = \Omega_{\alpha^{-1}}(\sigma) = (\Omega_\alpha(\sigma))^{-1}$. Let us set $B(\sigma) = \sigma_{z_0}(1)\cdot(\Omega_\gamma(\sigma))^{-1}$. Therefore we finally get

$$\delta_* \circ \sigma_z \circ \delta_*^{-1} = L_{B(\sigma)} \circ R_{(\Omega_\alpha(\sigma))^{-1}} \circ \tau(\chi(\sigma)).$$

It follows from the equality $(\tau\cdot\sigma)_z = \tau_z \circ \sigma_z$ that $B : G_K \to Pol(X,Y)$ is a cocycle.

The path α is from $\overrightarrow{01}$ to z. Hence the formula for $\log\Omega_\alpha(\sigma)$ follows from the very definition of l-adic polylogarithms in [W2]. □

9 Cosimplicial spaces and Galois actions

Let V be a smooth algebraic variety over K and let v be a K-point of V. The étale fundamental group $\pi_1^{\mathrm{et}}(V_{\bar K};v)$ and its maximal pro-l quotient $\pi_1(V_{\bar K};v)$ are equipped with the action of G_K.

On the other side, given an algebraic variety V and a K-point v there is a cosimplicial algebraic variety, which we provisionally denote by $V^\bullet$, which is a model in algebraic geometry for the loop space based at v (see [W4] and [W5]). Let us assume that $K \subset C$ and let $V(C)$ be the set of C-points of V. $V(C)$ is a complex variety. The de Rham cohomology group $H^0_{DR}(V^\bullet)\otimes_k C$ is the algebra of polynomial complex valued functions on the Malcev Q-completion $\pi_1(V(C);v)\otimes Q$.

The étale cohomology group $H^0_{\mathrm{et}}(V^\bullet_{\bar K};Q_l)$ can be interpreted as the algebra of Q_l-valued functions on $\pi_1(V(C);v)\otimes Q_l$. The Galois group G_K acts on $H^0_{\mathrm{et}}(V^\bullet_{\bar K};Q_l)$.

In this section we shall compare these two actions of G_K. The first action is the action of G_K on $\pi_1^{\mathrm{et}}(V_{\bar K};v)$, which is defined through étale coverings. The second action is the action of G_K on the 0-th étale cohomology group $H^0_{\mathrm{et}}(V^\bullet_{\bar K};Q_l)$ of the cosimplicial algebraic variety $V^\bullet_{\bar K}$. The cohomology group $H^0_{\mathrm{et}}(V^\bullet_{\bar K};Q_l)$ has a natural interpretation as an algebra of Q_l-valued polynomial functions on on $\pi_1(V_{\bar K};v)\otimes Q$.

We fix the notation we shall use in this section.
X_{et} is the étale site associated to an algebraic variety X;
$A_{X_{\text{et}}}$ (resp. $A_{X(C)}$) is the constant sheaf on X_{et} (resp. $X(C)$) with values in A;
$\Delta[1]$ is the standard simplicial model of the one simplex;
$\partial\Delta[1]$ is the boundary of $\Delta[1]$. It is a constant simplicial set.
$X^{\bullet}_{[n]}$ is the n-th truncation of a cosimplicial object $X^{\bullet}$.

Let X be a smooth quasi-projective algebraic variety over an algebraically closed field k. The inclusion of simplicial sets

$$\partial\Delta[1] \hookrightarrow \Delta[1]$$

induces the morphism of cosimplicial algebraic varieties

$$p^{\bullet} : X^{\Delta[1]} \to X^{\partial\Delta[1]}.$$

Therefore for each n we get the morphism between their n-th truncations

$$p^{\bullet}_{[n]} : X^{\Delta[1]}_{[n]} \longrightarrow X^{\partial\Delta[1]}_{[n]}.$$

For each k,

$$p^k : X^{\Delta[1]_k} = X \times X^k \times X \to X^{\partial\Delta[1]_k} = X \times X$$

is the projection map on the first and the last factors. Let us set

$$TotR(p^{\bullet}_{[n]})_*\big((Z/l^m)_{(X^{\Delta[1]}_{[n]})_{\text{et}}}\big) := \oplus_{i=0}^{n} R(p^i)_*\big((Z/l^m)_{(X^{\Delta[1]_i})_{\text{et}}}\big),$$

where Tot is the total complex of a bicomplex. Let us define

$$R^i(p^{\bullet}_{[n]})_*\big((Z/l^m)_{(X^{\Delta[1]}_{[n]})_{\text{et}}}\big) := H^i(TotR(p^{\bullet}_{[n]})_*\big((Z/l^m)_{(X^{\Delta[1]}_{[n]})_{\text{et}}}\big).$$

Lemma 3. *The cohomology sheaves $R^i(p^{\bullet}_{[n]})_*\big((Z/l^m)_{(X^{\Delta[1]}_{[n]})_{\text{et}}}\big)$ are sheaves of finitely generated Z/l^m-modules on $(X \times X)_{\text{et}}$.*

Proof. The spectral sequence of the bicomplex $\oplus_{i=0}^{n} R(p^i)_*\big((Z/l^m)_{(X^{\Delta[1]_i})_{\text{et}}}\big)$ converges to cohomology sheaves $R^i(p^{\bullet}_{[n]})_*\big((Z/l^m)_{(X^{\Delta[1]}_{[n]})_{\text{et}}}\big)$. The E_1-term $E_1^{j,k} = R^j(p^k)_*\big((Z/l^m)_{(X^{\Delta[1]_k})_{\text{et}}}\big)$ is the constant sheaf on $(X \times X)_{\text{et}}$, whose stalk is a finitely generated Z/l^m-module. There are only finitely many E_1-terms different from zero. Hence the lemma follows. □

We need to know if the sheaves $R^i(p^{\bullet}_{[n]})_*\big((Z/l^m)_{(X^{\Delta[1]}_{[n]})_{\text{et}}}\big)$ are locally constant and we need to calculate their monodromy representations. Therefore we shall study the Gauss-Manin connection associated to the morphism $p^{\bullet} : X^{\Delta[1]} \to X^{\partial\Delta[1]}$. We review briefly the results from [W4] in the form suitable to study the sheaves $R^i(p^{\bullet}_{[n]})_*\big((Z/l^m)_{(X^{\Delta[1]}_{[n]})_{\text{et}}}\big)$.

We apply to the map between the n-th truncations

$$p^{\bullet}_{[n]} : X^{\Delta[1]}_{[n]} \to X^{\partial\Delta[1]}_{[n]}$$

the standard construction of the Gauss-Manin connection (see [W4]). For each $0 \leq i \leq n$ the complex of sheaves $\Omega^*_{X^{\Delta[1]_i}}$ is equipped with a canonical filtration

$$F^j\Omega^*_{X^{\Delta[1]_i}} := Image\big(\Omega^{*-i}_{X^{\Delta[1]_i}/X^{\partial\Delta[1]_i}} \otimes_{\mathscr{O}_{X^{\Delta[1]_i}}} (p^i)^*\Omega^j_{X^{\partial\Delta[1]_i}} \to \Omega^*_{X^{\Delta[1]_i}}\big).$$

Hence on $X^{\partial\Delta[1]_i} = X \times X$ we have a filtered complex $R(p^i)_*\big(\Omega^*_{X^{\Delta[1]_i}}\big)$. We form the total complex

$$TotR(p^{\bullet}_{[n]})_*\big(\Omega^*_{X^{\Delta[1]}_{[n]}}\big) := \oplus^n_{i=0} R(p^i)_*\big(\Omega^*_{X^{\Delta[1]_i}}\big).$$

The filtration on each $R(p^i)_*\big(\Omega^*_{X^{\Delta[1]_i}}\big)$ induces a filtration on $TotR(p^{\bullet}_{[n]})_*\big(\Omega^*_{X^{\Delta[1]}_{[n]}}\big)$. Applying the spectral sequence of a finitely filtered object to the complex $TotR(p^{\bullet}_{[n]})_*\big(\Omega^*_{X^{\Delta[1]}_{[n]}}\big)$, we get a spectral sequence converging to the cohomology sheaves $H^j\big(TotR(p^{\bullet}_{[n]})_*\big(\Omega^*_{X^{\Delta[1]}_{[n]}}\big)\big)$ on $X \times X$. The E_1-terms are equal

$$E^{p,q}_1 = \Omega^p_{X\times X} \otimes_{\mathscr{O}_{X\times X}} H^q\big(TotR(p^{\bullet}_{[n]})_*\big(\Omega^*_{X^{\Delta[1]}_{[n]}/X^{\partial\Delta[1]}_{[n]}}\big)\big).$$

Farther we denote the relative de Rham complex $\Omega^*_{X^{\Delta[1]}_{[n]}/X^{\partial\Delta[1]}_{[n]}}$ by Ω^* in the algebraic case, by Ω^*_{hol} in the holomorphic case and by $\Omega^*_{\mathscr{C}^\infty}$ in the smooth complex case.

The differential $d^{0,q}_1 : E^{0,q}_1 \to E^{1,q}_1$ is the integrable connection on the relative de Rham cohomology sheaves $H^q(TotR(p^{\bullet}_{[n]})_*\Omega^*)$. The fiber of $H^q(TotR(p^{\bullet}_{[n]})_*\Omega^*)$ over a point $(x,y) \in X \times X$ is $H^q_{DR}((p^{\bullet}_{[n]})^{-1}(x,y))$. (If $x = y$ then $(p^{\bullet}_{[n]})^{-1}(x,x)$ is the n-th truncation of the cosimplicial alebraic variety denoted by $X^{\bullet}$ at the very beginning of the section.)

Let us assume that $k \subset C$. Then we get the morphism of cosimlicial complex varieties

$$p(C)^{\bullet} : X(C)^{\Delta[1]} \longrightarrow X(C)^{\partial\Delta[1]}$$

and the maps between the n-th truncations

$$p(C)^{\bullet}_{[n]} : X(C)^{\Delta[1]}_{[n]} \longrightarrow X(C)^{\partial\Delta[1]}_{[n]}.$$

We do the same construction for holomorphic differentials. The holomorphic de Rham sheaf $\Omega^*_{X(C)^{\Delta[1]}_{[n]}}$ is the resolution of the constant sheaf $C_{X(C)^{\Delta[1]}_{[n]}}$ on $X(C)^{\Delta[1]}_{[n]}$. Hence we get that $H^q(TotR(p(C)^{\bullet}_{[n]})_*(C_{X(C)^{\Delta[1]}_{[n]}}))$ is the sheaf of the flat sections of the holomorphic Gauss-Manin connection

$$(d_1^{0,q})_{hol} : H^q(TotR(p(C)^\bullet_{[n]})_*\Omega^*_{hol}) \to \Omega^1_{X(C)\times X(C)} \otimes_{\mathscr{O}_{X(C)\times X(C)}} H^q(TotR(p(C)^\bullet_{[n]})_*\Omega^*_{hol}).$$

We shall calculate the monodromy representation of the locally constant sheaf $H^0(TotR(p(C)^\bullet_{[n]})_*(C_{X(C)^{\Delta[1]}_{[n]}}))$. The de Rham complexes of smooth differentials are acyclic for direct image functors. Hence the complexes $TotR(p(C)^\bullet_{[n]})_*\Omega^*_{hol}$ and $Tot(p(C)^\bullet_{[n]})_*\Omega^*_{\mathscr{C}^\infty}$ are quasi-isomorphic.

Let $\omega_1,\ldots,\omega_n \in \Omega^1_{\mathscr{C}^\infty}(X(C))$ be closed one-forms on $X(C)$. Let us assume that $\omega_i \wedge \omega_{i+1} = 0$ for all i. Then $1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$ defines a global section of $H^0(Tot(p(C)^\bullet_{[n]})_*\Omega^*_{\mathscr{C}^\infty})$. We shall calculate the action of $d^0 := (d_1^{0,0})_{\mathscr{C}^\infty}$ on the section $1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$. The connection d^0 is the boundary homomorphism of the long exact sequence associated to the short exact sequence

$$0 \to F^1/F^2 \to F^0/F^2 \to F^0/F^1 \to 0\,.$$

We recall that the coface maps

$$\delta^i : X\times X^{n-1}\times X \to X\times X^n\times X$$

are given by

$$\delta^i(x_0,x_1,\ldots,x_n) = (x_0,\ldots,x_{i-1},x_i,x_i,\ldots,x_n)$$

for $0\le i\le n$. We set $\delta_n := \sum_{i=0}^n (-1)^{n-i}(\delta^i)^*$. The boundary operator of the total complex is given by $D = \delta_n + (-1)^n d$, where d is the exterior differential of the de Rham complex.

We denote by $\int_a \omega_1,\ldots,\omega_i$ a function defined on a contractible subset of $X(C)$ containing a and sending z to the iterared integral $\int_a^z \omega_1,\ldots,\omega_i$ along any path contained in this contractible subset. After calculations we get the following result.

Lemma 4. *Let* $(a,b)\in X(C)\times X(C)$. *We have*

$$D\Big(\sum_{0\le i\le j\le n}\int_a \omega_1,\ldots,\omega_i\otimes\omega_{i+1}\otimes\ldots\otimes\omega_j\otimes(-1)^{n-j}\int_b \omega_n,\ldots,\omega_{j+1}\Big) = 0\,.$$

We denote by $\pi(X(C);b,a)$ the $\pi_1(X(C);a)$-torsor of paths from a to b on $X(C)$ and by $\pi(X(C);b,a)\otimes Q$, the deduced $\pi_1(X(C);a)\otimes Q$-torsor.
We denote by $Algebra_C(\pi(X(C);b,a)\otimes Q)$ the algebra of complex valued polynomial functions on $\pi(X(C);b,a)\otimes Q$.

The shuffle product defines a multiplication on $H^0_{DR}((p(C)^\bullet)^{-1}(a,b))$, hence the 0-th cohomology group is a C-algebra and if $a=b$ it is a Hopf algebra.

The element $1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$ in the stalk over a point (a,b) determines a polynomial complex valued function on the rational completion of the torsor of paths $\pi(X(C);b,a)\otimes Q$, which to a path γ from a to b associates the iterated integral $\int_\gamma \omega_1\ldots,\omega_n$. Hence we get an isomorphism of C-algebras

$$H^0_{DR}\big((p(C)^\bullet)^{-1}(a,b)\big) \approx Algebra_C(\pi(X(C);b,a)\otimes Q)$$

and if $a=b$ we get an isomorphism of Hopf algebras, which follows from works of Chen.

Observe that $\mathrm{injlim}_n H^0_{DR}((p(C)^\bullet_{[n]})^{-1}(a,b)) = H^0_{DR}((p(C)^\bullet)^{-1}(a,b))$. The same holds also for cohomology sheaves , considered by us, on $X(C)\times X(C)$ and for the connections d^0. Hence we shall calculate the monodromy representation in the fiber of $p(C)^\bullet$.

Proposition 11. *Let X be a smooth affine algebraic curve over a field $k\subset C$. The monodromy representation of the bundle of flat sections of the Gauss-Manin connection d^0 at a point $(a,b)\in X(C)\times X(C)$*

$$\rho_{a,b}:\pi_1(X(C);a)\times\pi_1(X(C);b)\to \mathrm{Aut}(Algebra_C(\pi(X(C);b,a)\otimes Q))$$

is given by the formula

$$((\rho_{a,b}(\alpha,\beta))(f))(\gamma) = f(\beta^{-1}\cdot\gamma\cdot\alpha), \tag{17}$$

where $(\alpha,\beta)\in\pi_1(X(C);a)\times\pi_1(X(C);b)$, $\gamma\in\pi(X(C);b,a)\otimes Q$ and where $f\in Algebra_C(\pi(X(C);b,a)\otimes Q)$.

Proof. We can find smooth closed one-forms $\eta_1,\dots,\eta_r\in\Omega^1_{\mathscr{C}^\infty}(X(C))$ such that their classes form a base of $H^1_{DR}(X(C))$ and $\eta_i\wedge\eta_j=0$ for $1\le i,j\le r$. Then all possible tensor products $1\otimes\eta_{i_1}\otimes\dots\otimes\eta_{i_k}\otimes 1$ form a base of $H^0_{DR}((p(C)^\bullet)^{-1}(a,b))$.

Let $1\otimes\omega_1\otimes\dots\otimes\omega_n\otimes 1$ be one of such products. The stalk of the locally constant sheaf $H^0\big(TotR(p(C)^\bullet_{[n]})_*(C_{X(C)^{\Delta[1]}_{[n]}})\big)$ over the point (a,b) is equal $H^0((p(C)^\bullet_{[n]})^{-1}(a,b))$.

To calculate $H^0((p(C)^\bullet_{[n]})^{-1}(a,b))$ we use complexes of smooth differential forms. Hence the element $1\otimes\omega_1\otimes\dots\otimes\omega_n\otimes 1$ we consider in the stalk of the sheaf $H^0\big(TotR(p(C)^\bullet_{[n]})_*(C_{X(C)^{\Delta[1]}_{[n]}})\big)$ over the point (a,b). We prolongate $1\otimes\omega_1\otimes\dots\otimes\omega_n\otimes 1$ to a continuous section s of the locally constant sheaf $H^0\big(TotR(p(C)^\bullet_{[n]})_*(C_{X(C)^{\Delta[1]}_{[n]}})\big)$ along $(\alpha,\beta)\in\pi_1(X(C);a)\times\pi_1(X(C);b)$. We have $s(0)=1\otimes\omega_1\otimes\dots\otimes\omega_n\otimes 1$. It follows from Lemma 4 that

$$s(1)=\sum_{0\le i\le j\le n}(\int_\alpha\omega_1,\dots,\omega_i)\otimes\omega_{i+1}\otimes\dots\otimes\omega_j\otimes(-1)^{n-j}(\int_b\omega_n,\dots,\omega_{j+1}).$$

The element $s(1)\in Algebra_C(\pi(X(C);b,a)\otimes Q)$ and for any path γ from a to b we have

$$s(1)(\gamma)=\sum_{0\le i\le j\le n}(\int_\alpha\omega_1,\dots,\omega_i)\cdot(\int_\gamma\omega_{i+1},\dots,\omega_j)\cdot(-1)^{n-j}(\int_\beta\omega_n,\dots,\omega_{j+1}). \tag{18}$$

It follows from the Chen formulas (see [Ch]) that the right hand side of (18) is equal $\int_{\beta^{-1}\cdot\gamma\cdot\alpha}\omega_1,\dots,\omega_n$. Hence the monodromy transformation along (α,β) maps the function $f(-):=s(0)\in Algebra_C(\pi(X(C);b,a)\otimes Q)$ into the function $f(\beta^{-1}\cdot - \cdot\alpha)\in Algebra_C(\pi(X(C);b,a)\otimes Q)$. □

Corollary 6. *Let X be a smooth quasi-projective algebraic variety over an algebraically closed field $k \subset C$. Let us assume that there is an affine smooth algebraic curve S over k and a smooth morphism $f : S \to X$ over k such that the induced map $f_* : H_1(S(C);Q) \to H_1(X(C);Q)$ is surjective. Then the monodromy representation of the bundle of flat sections of the Gauss-Manin connection d^0 at a point (a,b) is given by the formula (17).*

Proof. The morphism f induces a morphism of locally constant sheaves

$$H^0\big(TotR(p(C)^\bullet_{[n]})_*(C_{X(C)^{\Delta[1]}_{[n]}})\big) \longrightarrow H^0\big(TotR(p(C)^\bullet_{[n]})_*(C_{S(C)^{\Delta[1]}_{[n]}})\big).$$

Let us assume that $(a,b) \in X(C) \times X(C)$ is the image of a point $(s,t) \in S(C) \times S(C)$. Then $H^0((p(C)^\bullet_{[n]})^{-1}(a,b))$ is the subalgebra of $H^0((p(C)^\bullet_{[n]})^{-1}(s,t))$. Hence it follows from Proposition 11 that the monodromy representation of the sheaf $H^0\big(TotR(p(C)^\bullet_{[n]})_*(C_{X(C)^{\Delta[1]}_{[n]}})\big)$ at the point (a,b) is given by the formula (17). But then it is given by the formula (17) at any point of $X(C) \times X(C)$. □

Let Y be a topological space. We denote by Y_{lh} the site of local homeomorphisms on Y. We have the comparison isomorphisms

$$R^i(p^\bullet_{[n]})_*(Z/l^m)_{(X^{\Delta[1]}_{[n]})_{\text{et}}} \approx R^i(p(C)^\bullet_{[n]})_*(Z/l^m)_{(X(C)^{\Delta[1]}_{[n]})_{lh}} \approx R^i(p(C)^\bullet_{[n]})_*(Z/l^m)_{(X(C)^{\Delta[1]}_{[n]})}\,. \tag{19}$$

We do not know how to show that the sheaves in (19) are locally constant. However

$$\big(\text{projlim}_m R^i(p(C)^\bullet_{[n]})_*(Z/l^m)_{(X(C)^{\Delta[1]}_{[n]})_{lh}}\big) \otimes Q \approx (R^i(p(C)^\bullet_{[n]})_*(Z_{(X(C)^{\Delta[1]}_{[n]})_{lh}})) \otimes Q_l.$$

The sheaf $R^i(p(C)^\bullet_{[n]})_*(C_{(X(C)^{\Delta[1]}_{[n]})_{lh}})$ is locally constant as the sheaf of flat sections of the integrable connection d^0. Hence the sheaf $(R^i(p(C)^\bullet_{[n]})_*(Z_{(X(C)^{\Delta[1]}_{[n]})_{lh}})) \otimes Q$ is locally constant. Therefore the sheaf $(R^i(p(C)^\bullet_{[n]})_*(Z_{(X(C)^{\Delta[1]}_{[n]})_{lh}}))/Torsion$ is also locally constant on $(X(C) \times X(C))_{lh}$. Hence to calculate the stalk of the sheaf

$$\big(\text{projlim}_m R^i(p(C)^\bullet_{[n]})_*(Z/l^m)_{(X(C)^{\Delta[1]}_{[n]})_{lh}}\big) \otimes Q \approx (R^i(p(C)^\bullet_{[n]})_*(Z_{(X(C)^{\Delta[1]}_{[n]})_{lh}})) \otimes Q_l$$

over $(a,b) \in X(C) \times X(C)$, it is sufficient to consider only the family of finite covering spaces $\bar{X}(C) \to X(C) \times X(C)$. By the comparison isomorphism (19) the same is true for the projective system of sheaves

$$\{R^i(p^\bullet_{[n]})_*(Z/l^m)_{(X^{\Delta[1]}_{[n]})_{\text{et}}}\}_{m\in N}\,. \tag{20}$$

If $\bar{X}(C) \to X(C) \times X(C)$ is a Galois covering space then the finite quotient of $\pi_1(X(C) \times X(C);(a,b))$ acts on $\bar{X}(C)$, hence we get an action of $\pi_1^{\text{et}}(X \times X;(a,b))$ on the projective limit tensored with Q of stalks over (a,b) of the projective system of sheaves (20). This projective limit tensored with Q is $H^0_{\text{et}}((p^\bullet_{[n]})^{-1}(a,b);Q_l)$.

It follows from the works of Chen that

$$H^0_{DR}\big((p(C)^\bullet)^{-1}(a,b)\big) \approx Algebra_C(\pi(X(C);b,a)\otimes Q)\ .$$

We shall use Sullivan polynomial differential forms with Q-coefficients (see [Su] page 297). We shall use subscript SDR to denote the corresponding cohomology groups. We get the corresponding isomorphism of Q-algebras

$$H^0_{SDR}\big((p(C)^\bullet)^{-1}(a,b)\big) \approx Algebra_Q(\pi(X(C);b,a)\otimes Q)\ .$$

If $a=b$ then we get an isomorphism of Hopf algebras.

It follows from the comparison isomorphisms

$$H^0_{\mathrm{et}}\big((p^\bullet)^{-1}(a,b);Q_l\big) \approx H^0\big((p(C)^\bullet)^{-1}(a,b);Q\big)\otimes Q_l \approx H^0_{SDR}\big((p(C)^\bullet)^{-1}(a,b)\big)\otimes Q_l$$

between étale and singular cohomology and between singular and de Rham cohomology - the last one calculated using Sullivan polynomial differential forms - that

$$H^0_{\mathrm{et}}\big((p^\bullet)^{-1}(a,b);Q_l\big) \approx Algebra_{Q_l}(\pi(X(C);b,a)\otimes Q)\ .$$

On the other side we have an isomorphisms of torsors

$$\pi(X(C);b,a)\otimes Q_l \approx \pi(X;b,a)\otimes Q\ .$$

deduced from the fact that the finite completion of $\pi_1(X(C);a)$ is isomorphic to $\pi_1^{\mathrm{et}}(X;a)$.

Therefore we get an isomorphism of Q_l-vector spaces

$$H^0_{\mathrm{et}}\big((p^\bullet)^{-1}(a,b);Q_l\big) \approx Algebra_{Q_l}(\pi(X;b,a)\otimes Q)\ . \tag{21}$$

The shuffle product in H^0_{DR} is defined using codegeneracies hence it can be defined in H^0_{et}. The Hopf algebra structure on $H^0_{DR}((p(C)^\bullet)^{-1}(a,a))$ is defined by the maps

$$1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1 \to \sum_{i=0}^{n}(1\otimes\omega_1\otimes\ldots\otimes\omega_i\otimes 1)\otimes(1\otimes\omega_{i+1}\otimes\ldots\otimes\omega_n\otimes 1),$$

hence one can use maps $X^n\to X^i\times X^{n-i}$ to define it. Therefore the isomorphism (21) is an isomorphism of Q_l-algebras and if $a=b$ it is an isomorphism of Hopf algebras.

Hence we get that the monodromy representation associated to the projective system (20) on $(X\times X)_{\mathrm{et}}$, in the projective limit of stalks over (a,b) after tensoring by Q and passing to the inductive limit as $n\to\infty$,

$$\rho_{(a,b)}:\pi_1^{\mathrm{et}}(X,a)\times\pi_1^{\mathrm{et}}(X,b)\longrightarrow \mathrm{Aut}\big(Algebra_{Q_l}(\pi(X;b,a)\otimes Q)\big)$$

is given by the formula

$$((\rho_{(a,b)}(\alpha,\beta))(f))(\gamma) = f(\beta^{-1}\cdot\gamma\cdot\alpha).$$

If X is defined over a number field K contained in k and if a and b are two K-points of X then G_K acts on $H^0_{\mathrm{et}}\big((p^\bullet)^{-1}(a,b);Q_l\big)$. The Galois group G_K acts also on the $\pi_1(X;a)\otimes Q$-torsor $\pi(X;b,a)\otimes Q$. The next result compares these two actions.

Proposition 12. *Let X be an algebraic curve over an algebraically closed field $k\subset C$. Suppose that X is defined over a number field K contained in k. Let a and b be two K-points of X. Then the isomorphism of Q_l-algebras*

$$H^0_{\mathrm{et}}((p^\bullet)^{-1}(a,b);Q_l)\approx Algebra_{Q_l}(\pi(X;b,a)\otimes Q)$$

is an isomorphism of G_K-modules.

Proof. Let $(\alpha,\beta)\in\pi_1^{\mathrm{et}}(X,a)\times\pi_1^{\mathrm{et}}(X,a)$, let $\sigma\in G_K$ and let $f\in H^0_{\mathrm{et}}((p^\bullet)^{-1}(a,a);Q_l)$. Then

$$\sigma_{(a,a)}\big((\alpha,\beta)_*(f)\big) = (\sigma(\alpha),\sigma(\beta))_*(\sigma_{(a,a)}(f)) \tag{22}$$

by the formula (8). Observe that for any $\gamma\in\pi_1(X,a)\otimes Q$ we have

$$((\alpha,\beta)_*(f))(\gamma) = f(\beta^{-1}\cdot\gamma\cdot\alpha).$$

The function $\gamma\to f(\beta^{-1}\cdot\gamma\cdot\alpha)$ is calculated using the Hopf algebra structure on $H^0_{\mathrm{et}}((p^\bullet)^{-1}(a,a);Q_l)$. Therefore after applying $\sigma_{(a,a)}$ and setting $\beta=1$ and $\gamma=1$ we get that the left hand side of (22) is equal $f(\alpha)$.

Applying $(\sigma(\alpha),\sigma(\beta))_*\circ\sigma_{(a,a)}$ to f we get the function $\gamma\to(\sigma_{(a,a)}(f))(\sigma(\beta)^{-1}\cdot\gamma\cdot\sigma(\alpha))$. Hence for $\beta=1$ and $\gamma=1$ we get $(\sigma_{(a,a)}(f))(\sigma(\alpha))$. Hence for any $\sigma\in G_K$ and any $\alpha\in\pi_1(X,a)$ we have

$$(\sigma_{(a,a)}(f))(\alpha) = f(\sigma^{-1}(\alpha)).$$

Therefore the G_K-modules $H^0_{\mathrm{et}}((p^\bullet)^{-1}(a,a);Q_l)$ and $Algebra_{Q_l}(\pi_1(X;a)\otimes Q)$ are isomorphic. Hence for any pair (a,b) the G_K modules $H^0_{\mathrm{et}}((p^\bullet)^{-1}(a,b);Q_l)$ and $Algebra_{Q_l}(\pi(X;b,a)\otimes Q)$ are isomorphic. □

Corollary 7. *Let X be a smooth quasi-projective algebraic variety over a number field $K\subset C$. Let us assume that there is an affine smooth algebraic curve S over K and a smooth morphism $f:S\to X$ over K such that the induced map $f_*:H_1(S(C);Q)\to H_1(X(C);Q)$ is surjective. Let us assume that S has a K-point. Let a and b be any two K-points of X. Then the isomorphism of Q_l-algebras*

$$H^0_{\mathrm{et}}((p^\bullet_{\bar K})^{-1}(a,b);Q_l)\approx Algebra_{Q_l}(\pi(X_{\bar K};b,a)\otimes Q),$$

where $p^\bullet_{\bar K}:X_{\bar K}^{\Delta[1]}\to X_{\bar K}^{\partial\Delta[1]}$, is an isomorphism of G_K-modules.

Proof. The corollary follows from Corollary 6 and Proposition 12.

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