INERTIA ORBIFOLDS, CONFIGURATION SPACES AND THE GHOST LOOP SPACE.

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ABSTRACT. In this paper we define and study the *ghost loop orbifold* $\mathcal{L}_s\mathsf{X}$ of an orbifold X consisting of those loops that remain constant in the coarse moduli space of X. We construct a configuration space model for $\mathcal{L}_s\mathsf{X}$ using an idea of G. Segal. From this we exhibit the relation between the Hochschild and cyclic homologies of the inertia orbifold of X (that generate the so-called twisted sectors in string theory) and the ordinary and equivariant homologies of $\mathcal{L}_s\mathsf{X}$. We also show how this clarifies the relation between orbifold K-theory, Chen-Ruan orbifold cohomology, Hochschild homology, and periodic cyclic homology.

1. INTRODUCTION

Given an orbifold X, Kawasaki [20] has defined an associated orbifold that is often called the *twisted sector* orbifold $\wedge\mathsf{X}$ [27]. This orbifold is also known as the *inertia orbifold* associated to X. Essentially, this orbifold is made of pairs $(x,(g))$ where $x \in X$ and $(g) \in \mathrm{Conj}(G_x)$ where G_x is the stabilizer of X at x. A full description of $\wedge\mathsf{X}$ needs, of course, a precise definition of the orbifold structure on the space of such pairs. This new orbifold appears naturally from the point of view of many geometric questions, and again, from the perspective of string theory its introduction is well motivated. For example, the orbifold K-theory of X is rationally isomorphic to the cohomology of $\wedge\mathsf{X}$ by the Chern character. The purpose of this paper is to study the topology of this orbifold using standard algebraic topology. In particular we will not use the theory of C^*-algebras.

We should clarify here what we mean by the topology of an orbifold. An orbifold X, let us recall, is a pair $(X,\mathcal{U})$ where X is a topological space and $\mathcal{U}$ is an equivalence class of orbifold atlases. An orbifold atlas $\mathcal{U} = (\pi_i, U_i, G_i, V_i)_i$ is made up of orbifold charts so that $\pi_i\colon U_i \to V_i$ is a quotient map that induces a homeomorphism $U_i/G_i \simeq V_i$. It is important to recall that the groups G_i are all finite. The gluing conditions are subtle. At a first level of approximation the topology of an orbifold X is simply the topology of the topological space X. This naive approach is neglecting altogether the orbifold structure $\mathcal{U}$. To consider the orbifold structure from the point of view of algebraic topology we follow Haefliger [18] and Moerdijk-Pronk [25], and associate a topological category G to the orbifold X (actually the category is a groupoid.) Then we consider the classifying space $B\mathsf{G}$. As Moerdijk [24] has shown the homotopy type of the orbifold X (seen as a topological stack or as a topos, in the sense of étale homotopy theory) is exactly the same thing as the homotopy type of $B\mathsf{G}$. The situation is analogous to that of studying the space BG associated to a finite group G. In fact this situation inspires much of what follows.

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The category G associated to an orbifold is étale (the structure maps are local homeomorphisms) and *stable* (the sets $\mathrm{Hom}_{\mathsf{G}}(x,y)$ are finite.) But in fact the theorems of this paper are valid for a general topological category G. Given a groupoid G its inertia groupoid $\wedge\mathsf{G}$ is again a groupoid. Its set of objects is $\mathrm{Hom}(\mathbb{Z},\mathsf{G})$ (here we mean all functors from the category with one object and a $\mathbb{Z}$ worth of arrows, to the category G.) The morphisms of $\wedge\mathsf{G}$ are the obvious ones (induced by the action of G on $\mathrm{Hom}(\mathbb{Z},\mathsf{G})$.) As we said before the orbifold K-theory of X computes the cohomology of $\wedge\mathsf{X}$ rationally.

For every functor $\mathbb{Z}\to\mathsf{G}$ there is a corresponding map $S^1=B\mathbb{Z}\to B\mathsf{G}$, namely an element in $\mathcal{L}B\mathsf{G}:=\mathrm{Map}(S^1,B\mathsf{G})$. Not every element of $\mathcal{L}B\mathsf{G}$ comes from such a functor. We define then the space of *ghost loops* $\mathcal{L}_sB\mathsf{G}$ of $B\mathsf{G}$ as the subspace of $\mathcal{L}B\mathsf{G}$ consisting of loops so that the composition $S^1\to B\mathsf{G}\to X$ is constant (Here $B\mathsf{G}\to X$ is the natural quotient projection.) In section 4 below we prove the following theorem.

Theorem 1.0.1. *There is an isomorphism*

$$H^*(\mathcal{L}_sB\mathsf{G},\mathbb{Z})\cong H^*(B\wedge\mathsf{G},\mathbb{Z})$$

induced by the natural map $B\wedge\mathsf{G}\longrightarrow\mathcal{L}_sB\mathsf{G}$.

The main idea in the proof of this theorem consist in the introduction of a *configuration space model* for the space $B\wedge\mathsf{G}$. Following a beautiful idea of Segal and Burghelea in the case of a group, we define the space $\mathcal{C}_{S^1}(\mathsf{G})$ to be the configuration space of particles in S^1 labeled by morphisms of G (so that every morphism composes with the ones adjacent to it.)

Theorem 1.0.2. *The is a natural homotopy equivalence* $B\wedge\mathsf{G}\simeq\mathcal{C}_{S^1}(\mathsf{G})$.

For a finite group there is a well known relation between the homology of $\mathcal{L}BG$ and the Hochschild and cyclic homologies of $R[G]$ [8, 17, 19]. Using the previous theorem we prove the following result, that can be seen as a generalization of this relation to the case of topological groupoids (and therefore orbifolds.)

Theorem 1.0.3. *Let* G *be a topological groupoid and* $\wedge\mathsf{G}$ *its inertia groupoid. Then there are canonical isomorphisms*

$$\begin{array}{rcl} HH_*((\wedge\mathsf{G}_*,t_*)) \stackrel{\cong}{\to} H_*(B\wedge\mathsf{G}) & \stackrel{\cong}{\to} & H_*(\mathcal{C}_{S^1}(\mathsf{G})) \stackrel{\cong_{\mathbb{Q}}}{\to} H_*(\mathcal{L}_sB\mathsf{G}) \\ HC_*((\wedge\mathsf{G}_*,t_*)) \stackrel{\cong}{\to} H_*^{S^1}(B\wedge\mathsf{G}) & \stackrel{\cong}{\to} & H_*^{S^1}(\mathcal{C}_{S^1}(\mathsf{G})) \stackrel{\cong_{\mathbb{Q}}}{\to} H_*^{S^1}(\mathcal{L}_sB\mathsf{G}) \end{array}$$

We should point out that these results are related to the theorems of Brylinski-Nistor [6] and Crainic [13] obtained using the theory of C^*-algebras.

Finally as a direct consequence of the previous theorem we show the following isomorphisms involving the Chen-Ruan cohomology of the orbifold $H^*_{\mathrm{orb}}(\mathsf{G};\mathbb{C})$ and also its orbifold K-theory [2, 10, 21].

Theorem 1.0.4. *For* G *a compact complex* SL*-orbifold the following holds,*

$$\prod_{m\in n+2\mathbb{Z}} HH_m((\wedge\mathsf{G}_*,t_*);\mathbb{C})\cong\prod_{m\in n+2\mathbb{Z}} H^m_{\mathrm{orb}}(\mathsf{G};\mathbb{C})$$

and if G *is reduced*

$$K^n_{\mathrm{orb}}(\mathsf{X})\otimes\mathbb{C}\cong\prod_{m\in n+2\mathbb{Z}} H^m_{\mathrm{orb}}(\mathsf{X},\mathbb{C})\cong HP_n(\wedge\mathsf{G},\mathbb{C}).$$

We would like to call the attention of the reader to the recent papers of Baranovsky [3], and Căldăraru-Giaquinto-Witherspoon [9] where relations between orbifold cohomology and Hochschild and periodic cyclic homology are also explored.

We would like to thank conversations with A. Adem, D. Berenstein, M. Karoubi, B. Oliver, J. Robbin, and Y. Ruan on different matters related to this work.

We dedicate this paper to Prof. Graeme B. Segal on the occasion of his 60[th] birthday.

2. Groupoids

The groupoids we will consider are small categories G in which every morphism is invertible. By G_1 and G_0 we will denote the space of morphisms (arrows) and of objects respectively, and the structure maps

$$\mathsf{G}_1 \,{}_{\mathsf{t}}\!\times_{\mathsf{s}} \mathsf{G}_1 \xrightarrow{\mathsf{m}} \mathsf{G}_1 \xrightarrow{\mathsf{i}} \mathsf{G}_1 \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} \mathsf{G}_0 \xrightarrow{\mathsf{e}} \mathsf{G}_1$$

where s and t are the source and the target maps, m is the composition of two arrows, i is the inverse and e gives the identity arrow over every object.

The groupoid will be called *topological (smooth)* if the sets G_1 and G_0 and the structure maps belong to the category of topological spaces (smooth manifolds). In the case of a smooth groupoid we will also require that the maps s and t must be submersions, so that $\mathsf{G}_1 \,{}_{\mathsf{t}}\!\times_{\mathsf{s}} \mathsf{G}_1$ is also a manifold (smooth groupoids are also known by the name of *Lie groupoids* [23]).

A topological (smooth) groupoid is called *étale* if all the structure maps are local homeomorphisms (local diffeomorphisms). For an étale groupoid we will mean a topological étale groupoid. In what follows, sometimes the kind of groupoid will not be specified, but it will be clear from the context to which one we are referring to. We will always denote groupoids by letters of the type $\mathsf{G}, \mathsf{H}, \mathsf{S}$. Orbifolds are an special kind of étale groupoids, they have the peculiarity the inverse imagine of a point under the map $(\mathsf{s}, \mathsf{t}) : \mathsf{G}_1 \to \mathsf{G}_0 \times \mathsf{G}_0$ is always finite. Groupoids with this property are called *stable*[1] Whenever we write orbifold, a stable étale smooth groupoid will be understood.

A morphism of groupoids $\Psi : \mathsf{H} \to \mathsf{G}$ is a pair of maps $\Psi_i : \mathsf{H}_i \to \mathsf{G}_i$ $i = 1, 2$ such that they commute with the structure maps. The maps Ψ_i will be continuous (smooth) depending on which category we are working on.

For a groupoid G, we denote by G_n the space of n-arrows

$$x_0 \overset{g_1}{\to} x_1 \overset{g_2}{\to} \cdots \overset{g_n}{\to} x_n.$$

The spaces G_n $(n \geq 0)$ form a simplicial space $(\mathsf{G}_*, d_n^i, s_n^i)$

$$\cdots \; \mathsf{G}_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathsf{G}_1 \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} \mathsf{G}_0 \tag{2.0.1}$$

together with the face $d_n^i : \mathsf{G}_n \to \mathsf{G}_{n-1}$ and degeneracy maps $s_n^i : \mathsf{G}_n \to \mathsf{G}_{n+1}$ (see 3.0.4) form what is called the *nerve of the groupoid.* Its geometric realization $|\mathsf{G}_*|$ (see Appendix) is the classifying space of G, also denoted by $B\mathsf{G}$.

[1]We owe this terminology to J. Robbin. It is inspired in the stable map compactification for holomorphic maps.

3. Configuration space models for groupoids.

In this section we will apply ideas of G. Segal regarding the configuration spaces on groups to the more general setting of topological groupoids. On what follows the groupoids will be topological unless otherwise stated.

By $S^1 = \mathbb{R}/\mathbb{Z}$ we will denote the circle of radius $\frac{1}{2\pi}$ and by R we will denote the open interval $(0,1)$. Let $\iota : R \to S^1$ be the composition of the projection and inclusion maps $(0,1) \hookrightarrow \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Let A be either R or S^1.

For $\alpha : A \to \mathsf{G}_1$ a map, we will call the support of α the set:

$$\mathrm{supp}(\alpha) := \{a \in A | \alpha(a) \notin \mathsf{G}_0\}$$

where we consider the set G_0 as the image under the identity map of $\mathsf{e}(\mathsf{G}_0) \subset \mathsf{G}_1$.

Definition 3.0.5. A configuration in A with values in G is a map $\alpha : A \to \mathsf{G}_1$ with finite support

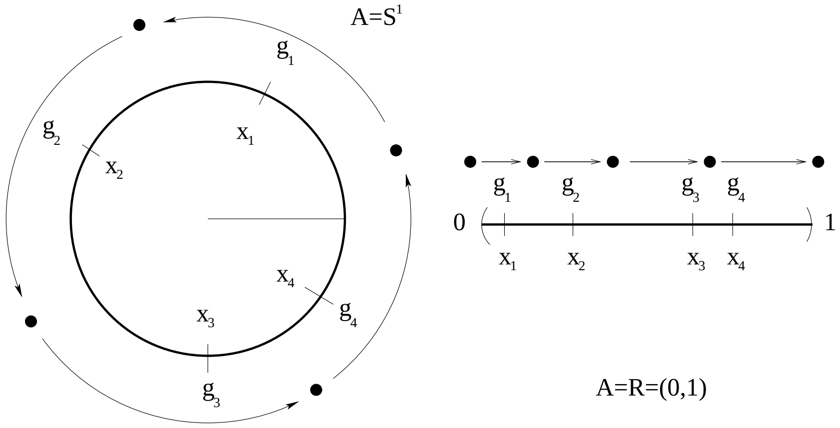
$$\mathrm{supp}(\alpha) := \{x_1, x_2, \ldots, x_n\} \qquad 0 \leq x_i < x_{i+1} < 1$$

such that for $g_i := \alpha(x_i)$

$$\mathsf{t}(g_i) = \mathsf{s}(g_{i+1}) \qquad \text{for} \ \ 1 \leq \mathrm{i} < \mathrm{n}$$

and when $A = S^1$

$$\mathsf{s}(g_1) = \mathsf{t}(g_n).$$



Also we require that the image of α outside its support be determined by the g_i's, as follows, for $x_i < y < x_{i+1}$

$$\alpha(y) = \mathsf{et}(g_i)$$

and $\alpha(y) = \mathsf{es}(g_1)$ for $y < x_1$.

The collection of configurations will be denoted by $\mathcal{C}_A(\mathsf{G})$.

We are going to endow these configuration spaces with natural topologies in such a way that when two morphisms of the groupoid collapse they get composed, and when a morphism gets sent to infinity it disappear (in the case $A = R$).

A base for the topology of $\mathcal{C}_A(\mathsf{G})$ is given by the sets $\mathcal{O}(U_1, U_2, \ldots, U_k; I_1, I_2, \ldots, I_k)$ where U_i are open sets G_1 and I_i are connected, open and disjoint intervals (on the circle the orientation of S^1 provides us with a total order in the I_i's). The set

$\mathcal{O}(U_1, \ldots, U_k; I_1, \ldots, I_k)$ denotes the set of configurations α whose support consists of the points x_i^j with

$$x_1^j < x_2^j < \cdots < x_{l_j}^j, \qquad x_i^j \in I_j \qquad \text{and} \qquad g_1^j g_2^j \cdots g_{l_j}^j \in U_j$$

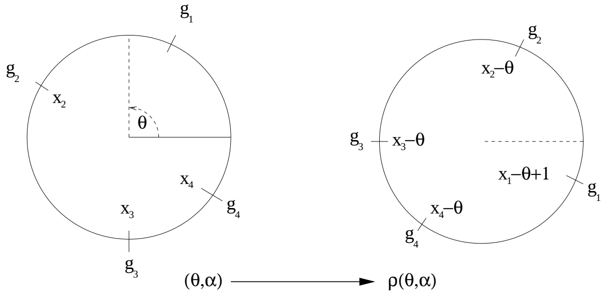
for $g_i^j := \alpha(x_i^j)$.

There are natural maps relating these configuration spaces:

- by rotation of a configuration by an angle θ,

$$\rho : S^1 \times \mathcal{C}_{S^1}(\mathsf{G}) \to \mathcal{C}_{S^1}(\mathsf{G}) \tag{3.0.2}$$

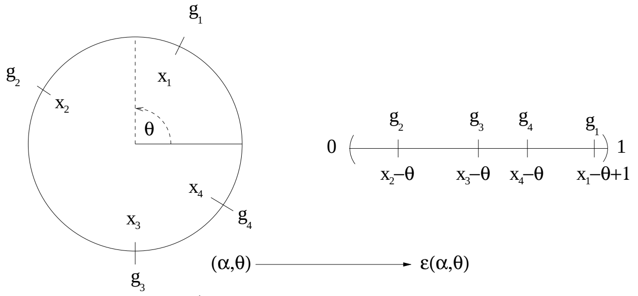
where $\rho(\theta, \alpha)(x) := \alpha(\theta + x)$,

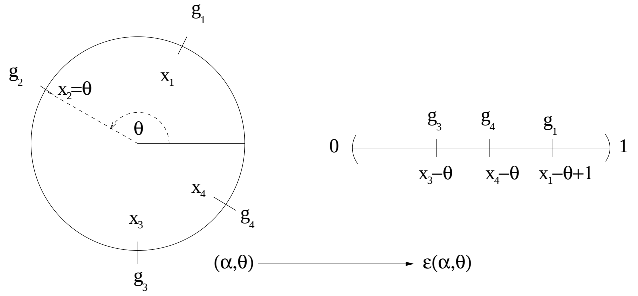


- by sending to infinity the morphism at the angle θ,

$$\varepsilon : \mathcal{C}_{S^1}(\mathsf{G}) \times S^1 \to \mathcal{C}_R(\mathsf{G}) \tag{3.0.3}$$

where $\varepsilon(\alpha, \theta) \in \mathcal{C}_R(\mathsf{G})$ such that for $x \in (0,1)$, $\varepsilon(\alpha, \theta)(x) := \alpha(\theta + \iota(x))$





With the topology previously defined it is clear that the maps ρ and ε are continuous. The set $\mathcal{C}_A(\mathsf{G})$ can be filtered

$$* = \mathcal{C}_A^0(\mathsf{G}) \subset \mathcal{C}_A^1(\mathsf{G}) \cdots \mathcal{C}_A^n(\mathsf{G}) \subset \mathcal{C}_A^{n+1}(\mathsf{G}) \subset \cdots$$

where $\mathcal{C}_A^n(\mathsf{G}) := \{\alpha \in \mathcal{C}_A(\mathsf{G}) \ : \ |\text{supp}(\alpha)| \leq n\}$. It is also clear that the maps ρ and ε are compatible with the filtration.

Proposition 3.0.6. *$\mathcal{C}_R(\mathsf{G})$ is canonically isomorphic to $B\mathsf{G}$.*

Proof. First of all let's recall how the classifying space of a groupoid is built. For a groupoid G we associate a simplicial space $\mathsf{G}_* = (\mathsf{G}_n, d_n^i, s_n^i)$ where

$$\mathsf{G}_n := \underbrace{\mathsf{G}_1 \, {}_{\mathsf{t}}\!\times_{\mathsf{s}} \mathsf{G}_1 \, {}_{\mathsf{t}}\!\times_{\mathsf{s}} \cdots \, {}_{\mathsf{t}}\!\times_{\mathsf{s}} \mathsf{G}_1}_{n} \tag{3.0.4}$$

$$d_n^i(g_1, g_2, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & i = 0 \\ (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) & i \neq 0, n \\ (g_1, \dots, g_{n-1}) & i = n \end{cases}$$

$$s_n^i(g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, \mathsf{es}(g_i), g_i, \dots, g_n).$$

The geometric realization $|\mathsf{G}_*|$ of the simplicial space G_* is what we denote $B\mathsf{G}$, the classifying space of the groupoid. This geometric realization is defined as

$$B\mathsf{G} = |\mathsf{G}_*| := \bigsqcup_{n=0}^{\infty} \mathsf{G}_n \times \Delta_n \; / \sim = \bigsqcup_{n=0}^{\infty} (\mathsf{G}_n - \mathrm{Deg}\ \mathsf{G}_n) \times \Delta_n \; / \sim$$

with $\mathrm{Deg}\ \mathsf{G}_n = \bigcup_{i=0}^{n-1} s_{n-1}^i(\mathsf{G}_{n-1})$ and $\sim$ the equivalence relation generated by the identifications

$$(s_{n-1}^i(x), \vec{t}\,) \sim (x, \sigma_n^i(\vec{t}\,) \quad \text{and} \quad (d_{n+1}^i(y), \vec{t}\,) \sim (y, \delta_n^i(\vec{t}\,)$$

with $x \in \mathsf{G}_{n-1}$, $y \in \mathsf{G}_{n+1}$, $\vec{t} \in \Delta_n$ and $\Delta^* = (\Delta_n, \delta_n^i, \sigma_n^i)$ the cosimplicial set of the n-simplices with their face and degeneracy maps:

$$\Delta_n := \{\vec{t} = (t_0, \dots, t_n) \in \mathbb{R}^{n+1}\}$$

$$\delta_n^i(t_0, \dots, t_n) := (t_0, \dots, t_i, 0, t_{i+1} \dots, t_n)$$

$$\sigma_n^i(t_0, \dots, t_n) := (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_n).$$

Taking out the images of the face maps, in other words, only considering the set

$$\mathrm{Int}\ \Delta_n := \Delta_n - \bigcup_{i=0}^{n-1} \delta_{n-1}^i(\Delta_{n-1})$$

then it is clear that as a set

$$B\mathsf{G} = \bigsqcup (\mathsf{G}_n - \mathrm{Deg}\ \mathsf{G}_n) \times \mathrm{Int}\ \Delta_n.$$

Now we can construct a bijective correspondence between the points in $\mathcal{C}_R(\mathsf{G})$ and the points in $B\mathsf{G}$ in the following way: to the configuration $\alpha : R \to \mathsf{G}_1$ with $\mathrm{supp}(\alpha) = \{x_1, x_2, \dots, x_n\}$ and $\alpha(x_i) = g_i$ we associate the point

$$(g_1, g_2, \dots, g_n; t_0, t_1, \dots, t_n) \in (\mathsf{G}_n - \mathrm{Deg}\ \mathsf{G}_n) \times \mathrm{Int}\ \Delta_n$$

with $t_0 = x_1$, $t_1 = x_2 - x_1$, $\dots t_n = 1 - x_n$. It is easy to see now that this identification is a homeomorphism. $\square$

4. The Inertia Groupoid

The Inertia groupoid $\wedge\mathsf{G}$ of a groupoid G is defined as follows.

An object a ($a \in \wedge\mathsf{G}_0$) is an arrow of G_1 such that its source and target are equal, so

$$\wedge\mathsf{G}_0 = \{a \in \mathsf{G}_1 | \mathsf{s}(a) = \mathsf{t}(a)\}$$

and a morphism $v \in \wedge\mathsf{G}_1$ joining two objects a, b

$$\begin{tikzcd} \circ \arrow[loop above, "a"] \arrow[r, bend left, "v"] & \circ \arrow[loop above, "b"] \end{tikzcd}$$

is an arrow in G_1 such that $a \cdot v = v \cdot b$; b can be seen as the conjugate of a by v because $b = v^{-1} \cdot a \cdot v$. But in order to identify the arrow v we need to keep track of its source a, then we can consider the morphisms of $\wedge\mathsf{G}$ as

$$\wedge\mathsf{G}_1 = \{(a, v) \in \mathsf{G}_2 | a \in \wedge\mathsf{G}_0\}$$

where $\mathsf{s}(a, v) = a$ and $\mathsf{t}(a, v) = v^{-1} \cdot a \cdot v$.

Let $\wedge\mathsf{G}_* = (\wedge\mathsf{G}_n, d_n^i, s_n^i)$ be the simplicial space associated to the inertia groupoid, where

$$\wedge\mathsf{G}_n\{(a, v_1, v_2, \ldots, v_n) \in \mathsf{G}_{n+1} | a \in \wedge\mathsf{G}_0\}$$

$$d_n^i(a, v_1, v_2, \ldots, v_n) = \begin{cases} (v_1^{-1} a v_1, v_2, \ldots, v_n) & i = 0 \\ (a, v_1, \ldots, v_{i-1} v_i, \ldots, v_n) & i \neq 0, n \\ (a, v_1, \ldots, v_{n-1}) & i = n \end{cases}$$

$$s_n^i(a, v_1, v_2, \ldots, v_n) = (a, v_1, \ldots, v_{i-1}, \mathsf{es}(v_i), v_i, \ldots, v_n)$$

It turns out that there is another simplicial space $\mathcal{Y}_*(\mathsf{G})$ associated to a groupoid which is isomorphic to $\wedge\mathsf{G}_*$ and that is related in a more direct way to the configuration space $\mathcal{C}_{S^1}(\mathsf{G})$. $\mathcal{Y}_*(\mathsf{G})$ is defined as follows:

$$\mathcal{Y}_n(\mathsf{G}) := \{(g_0, g_1, \ldots, g_n) \in \mathsf{G}_n \mid \mathsf{s}(g_0) = \mathsf{t}(g_n)\}$$

$$d_n^i(g_0, g_1, \ldots, g_n) = \begin{cases} (g_0, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & i \leq n-1 \\ (g_n g_0, g_1, \ldots, g_{n-1}) & i = n \end{cases}$$

$$s_n^i(g_0, g_1, \ldots, g_n) = (g_0, \ldots, g_i, \mathsf{es}(g_{i+1}), g_{i+1}, \ldots, g_n)$$

Proposition 4.0.7. *The simplicial spaces $\wedge\mathsf{G}_*$ and $\mathcal{Y}_*(\mathsf{G})$ are isomorphic.*

Proof. Define the maps

$$\begin{aligned} f_n : \wedge\mathsf{G}_n &\to \mathcal{Y}_n(\mathsf{G}) \\ (a, v_1, v_2, \ldots, v_n) &\mapsto (v_n^{-1} \cdots v_2^{-1} v_1^{-1} a, v_1, v_2, \ldots, v_n) \end{aligned}$$

$$\begin{aligned} h_n : \mathcal{Y}_n(\mathsf{G}) &\to \wedge\mathsf{G}_n \\ (g_0, g_1, \ldots, g_n) &\mapsto (g_1 g_2 \cdots g_n g_0, g_1, g_2, \ldots, g_n), \end{aligned}$$

clearly $h_n \circ f_n = \mathrm{id}_{\wedge\mathsf{G}_n}$ and $f_n \circ h_n = \mathrm{id}_{\mathcal{Y}_n(\mathsf{G})}$. It is easy to see that the maps f_n and h_n commute with the degeneracy maps s_n^i for $0 \leq i \leq n$ and with the face maps d_n^i for $0 \leq i < n$. For d_n^n we get that:

$$\begin{aligned} f_{n-1} \circ d_n^n(a, v_1, v_2, \ldots, v_n) &= (v_{n-1}^{-1} \cdots v_2^{-1} v_1^{-1} a v_1, v_2, \ldots, v_{n-1}) \\ d_n^n \circ f_n(a, v_1, v_2, \ldots, v_n) &= (v_{n-1}^{-1} \cdots v_2^{-1} v_1^{-1} a v_1, v_2, \ldots, v_{n-1}). \end{aligned}$$

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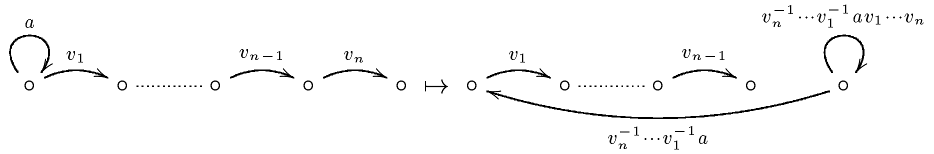
Moreover, both simplicial spaces $\wedge\mathsf{G}_*$ and $\mathcal{Y}_*(\mathsf{G})$ can be given a structure of cyclic space (see 7.2.1) in such a way that they remain isomorphic as cyclic spaces.

Remark 4.0.8. The space $\mathcal{Y}_*(\mathsf{G})$ is also known by the name of *cyclic nerve* of the groupoid G.

The cyclic structure associated to $\wedge\mathsf{G}_*$ is:

$$\begin{array}{rcl} t_n : \wedge\mathsf{G}_n & \to & \wedge\mathsf{G}_n \\ (a, v_1, \dots, v_n) & \mapsto & ((v_n^{-1} \cdots v_1^{-1} a v_1 \dots v_n), (v_n^{-1} \cdots v_1^{-1} a), v_1, \dots, v_{n-1}) \end{array}$$



and the cyclic structure of $\mathcal{Y}_*(\mathsf{G})$ is:

$$\begin{array}{rcl} t_n : \mathcal{Y}_n(\mathsf{G}) & \to & \mathcal{Y}_n(\mathsf{G}) \\ (g_0, g_1, \dots, g_n) & \mapsto & (g_n, g_0, g_1, \dots, g_{n-1}) \end{array}$$

It is easy to verify that the simplicial maps f_* and h_* become cyclic maps, hence

Proposition 4.0.9. *The maps* $f_* : (\wedge\mathsf{G}_*, t_*) \to (\mathcal{Y}_*(\mathsf{G}), t_*)$ *and* $h_* : (\mathcal{Y}_*(\mathsf{G}), t_*) \to (\wedge\mathsf{G}_*, t_*)$ *are isomorphisms of cyclic spaces.*

The cyclic structure on a simplicial space $\mathcal{X}_*$ allows one to define a circle action on the geometrical realization $|\mathcal{X}_*|$ as is shown in [8]. The construction is explained in the Appendix, Prop. 7.2.4.

Going back to the configuration space $\mathcal{C}_{S^1}(\mathsf{G})$ we have

Proposition 4.0.10. *The configuration space* $\mathcal{C}_{S^1}(\mathsf{G})$ *is homeomorphic to the geometrical realization* $|\mathcal{Y}_*(\mathsf{G})|$*. Moreover, the homeomorphism is* S^1*-equivariant.*

Proof. The argument is very similar to the one in Proposition 3.0.6. We define a bijective map $\varphi : \mathcal{C}_{S^1}(\mathsf{G}) \to |\mathcal{Y}_*(\mathsf{G})|$ using the fact that as sets

$$|\mathcal{Y}_*(\mathsf{G})| = \bigsqcup (\mathcal{Y}_n(\mathsf{G}) - \mathrm{Deg}\ \mathcal{Y}_n(\mathsf{G})) \times \mathrm{Int}\ \Delta_n.$$

For a configuration $\alpha \in \mathcal{C}_{S^1}(\mathsf{G})$ with

$$\mathrm{supp}(\alpha) = \{x_0, x_1, \dots, x_n\}, \quad 0 \leq x_0 < x_1 < \cdots < x_n < 1$$

and $g_i := \alpha(x_i)$ we will associate the point

$$\varphi(\alpha) = \begin{cases} (\mathsf{es}(g_0), g_0, g_1, \dots, g_n; x_0, x_1 - x_0, \dots, 1 - x_n) & \text{if} \quad x_0 \neq 0 \\ (g_0, g_1, \dots, g_n; x_1, x_2 - x_1 \dots, 1 - x_n) & \text{if} \quad x_0 = 0. \end{cases}$$

This function is clearly a homeomorphism.

To prove that φ is S^1-equivariant we will mimic to proof given in [7]. Let's recall the rotation map $\rho : S^1 \times \mathcal{C}_{S^1}(\mathsf{G}) \to \mathcal{C}_{S^1}(\mathsf{G})$ defined in 3.0.2 and the circle action $\mu : S^1 \times |\mathcal{Y}_*(\mathsf{G})| \to |\mathcal{Y}_*(\mathsf{G})|$ explained in 7.2.4; we want to verify the commutativity of the following diagram

$$\begin{tikzcd}
 & S^1 \times \mathcal{C}^{S^1}(\mathsf{G}) \arrow[rr, "\rho"] & & \mathcal{C}_{S^1}(\mathsf{G}) & \\
|S^1_*| \times |\mathcal{Y}_*(\mathsf{G})| \arrow[ur, "i\times\varphi^{-1}"] & |S^1_* \times \mathcal{Y}_*(\mathsf{G})| \arrow[l, "j"'] & |S^1_* \overset{\sim}{\times} \mathcal{Y}_*(\mathsf{G})| \arrow[l, "h"'] \arrow[r, "|ev|"] & |\mathcal{Y}_*(\mathsf{G})|. \arrow[u, "\varphi^{-1}"']
\end{tikzcd}$$

We take a point in $|S^1_* \overset{\sim}{\times} \mathcal{Y}_*(\mathsf{G})|$ written as $((\tau^r_n, x); (u_0, \dots, u_n))$ $0 < u_i \leq 1$, $\sum u_i = 1$ and $x = (g_0, \dots, g_n)$ with $g_r \notin \mathsf{e}(\mathsf{G}_0)$ for $r = 1, \dots, n$.

In order to make the following argument more understandable, let us recall what the configuration α associated to the pair $(x; u_0, \dots, u_n) \in \mathcal{Y}_*(\mathsf{G})$ given by the map φ^{-1} is. As the u_i's are positive then $\alpha(0) = g_0$, $\alpha(u_0) = g_1, \dots, \alpha(u_0+\cdots+u_{n-1}) = g_n$.

Going back to the diagram we have following set of equalities

$$\begin{aligned}
&(i \times \varphi^{-1}) \cdot j \cdot h((\tau^r_n, x); (u_0, \dots, u_n)) \\
&= (i \times \varphi^{-1}) \cdot j((\tau^{n-r+1}_n, x); (u_r, u_{r+1}, \dots, u_n, u_0, \dots, u_{r-1})) \\
&= (i \times \varphi^{-1})[(\tau^1_1; s, t), (x; u_r, u_{r+1}, \dots, u_n, u_0, \dots, u_{r-1})] \\
&= \big(i(\tau^1_1; s, t); \varphi^{-1}(x; u_r, u_{r+1}, \dots, u_n, u_0, \dots, u_{r-1})\big) \\
&= (s, \gamma)
\end{aligned}$$

with $s = u_r + u_{r+1} + \cdots + u_n$, $t = u_0 + \cdots + u_{r-1}$, and γ the configuration given by $\gamma(0) = g_0$, $\gamma(u_r) = g_1$, $\gamma(u_r + u_{r+1}) = g_2$, $\gamma(u_0 + \cdots + u_r) = g_{n-r+1}$, $\gamma(u_r + \cdots + u_n + u_0 + \cdots + u_{r-2}) = g_n$.

Then the configuration $\lambda := \rho(s, \gamma)$ is given by $\lambda(u) = \gamma(s + u)$; therefore $\lambda(0) = \gamma(s) = g_{n-r+1}$, $\lambda(u_0+\cdots+u_{r-1}) = \gamma(s+t) = \gamma(0) = g_0$, $\lambda(u_0+\cdots+u_{n-1}) = \gamma(u_r + \cdots + u_{n-1}) = g_{n-r}$.

On the other side of the diagram we have

$$\begin{aligned}
&\varphi^{-1} \cdot |ev|((\tau^r_n, x); (u_0, \dots, u_n)) \\
&= \varphi^{-1}(t^r_n x; u_0, \dots, u_n) \\
&= \varphi^{-1}((g_{n-r+1}, g_{n-r+2}, \dots, g_n, g_0, \dots, g_{n-r}); u_0, \dots, u_n) \\
&= \beta
\end{aligned}$$

with $\beta(0) = g_{n-r+1}$, $\beta(u_0) = g_{n-r+2}$, $\beta(u_0 + \cdots + u_{r-1}) = g_0$, $\beta(u_0 + \cdots + u_{n-1}) = g_{n-r}$. Hence $\beta = \lambda$ and the diagram is commutative. $\square$

We can conclude

Corollary 4.0.11. *The classifying space of the inertia groupoid $B \wedge \mathsf{G} = |\mathcal{Y}_*(\mathsf{G})|$ is S^1-homeomorphic to the configuration space $\mathcal{C}_{S^1}(\mathsf{G})$.*

5. The ghost loop space

We will postpone formal definitions for a while and we will start instead by explaining what a "ghost loop" is.

An object in the inertia groupoid can be seen as a morphism of groupoids

$$\Psi : \bar{\mathbb{Z}} \to \mathsf{G}$$

where we think of $\bar{\mathbb{Z}}$ as the groupoid with one object $\{*\}$ and whose morphisms are the integers (see [22, Corollary 3.6.4]). If we get the geometrical realization of the morphism Ψ, $B\Psi : B\mathbb{Z} \to B\mathsf{G}$ we obtain a map $\psi : S^1 \to B\mathsf{G}$. Now, is any map $S^1 \to B\mathsf{G}$ homotopic to the realization of a morphism Ψ? The answer of course depends on the groupoid G, but in general this is not the case. This section is devoted to give a very down to earth, geometrical description of the realization of such morphisms.

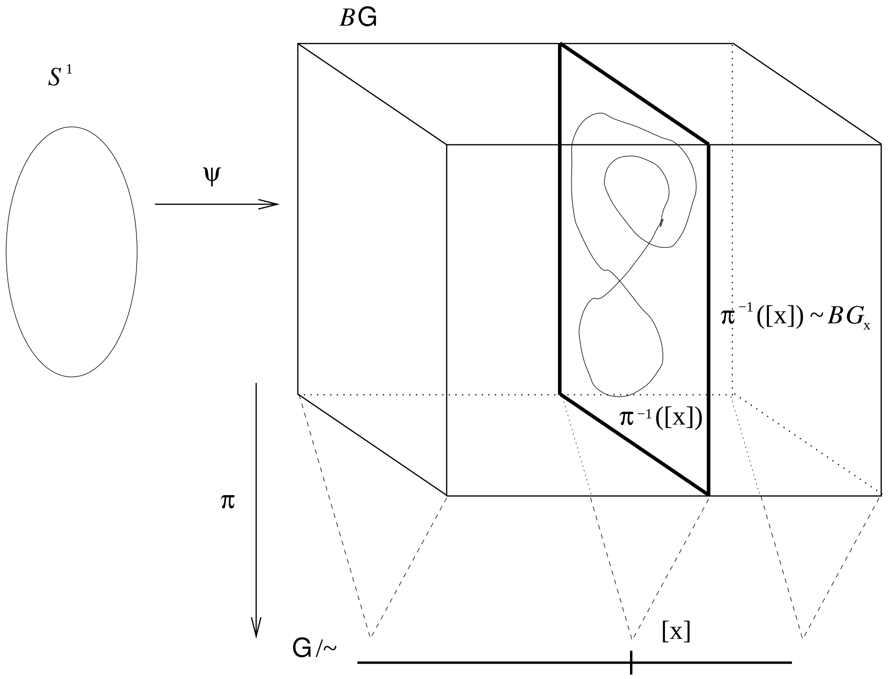
The loop $\psi = B\Psi$ will be *anchored* at the point $x := \Psi(*) \in \mathsf{G}_0$ and its image will live on the fiber $\pi^{-1}([x]) \simeq BG_x$ where

$$\pi : B\mathsf{G} \to \mathsf{G}/\!\sim$$

is the projection of $B\mathsf{G}$ in the coarse moduli space $\mathsf{G}/\!\sim := \mathsf{G}_0/\mathsf{G}_1$ (where G_1 provides the equivalence relation for the elements in G_0), $[x]$ is the equivalence class of x in $\mathsf{G}/\!\sim$ and

$$G_x := \{g \in \mathsf{G}_1 \mid \mathsf{s}(g) = \mathsf{t}(g) = x\}$$

is the isotropy group at x.



These are the loops that we will call "ghost loops" (our terminology is motivated by string theory.) After this brief introduction we are ready to define the spaces we need.

Let $\mathcal{L}B\mathsf{G} := \mathrm{Map}(S^1, B\mathsf{G})$ be the space of loops in $B\mathsf{G}$ with the natural compact-open topology.

Definition 5.0.12. The space $\mathcal{L}_s B\mathsf{G}$ of ghost loops of $B\mathsf{G}$ is the set of maps $f \in \mathcal{L}B\mathsf{G}$ such that the image of f lies in one fiber of π, that is, it exists $x \in \mathsf{G}_0$ with

$$\mathrm{Im}\ f \subset \pi^{-1}([x])$$

and $[x] \in \mathsf{G}/\!\sim$.

It is equivalent to say that the following diagram commutes:

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d] & B\mathsf{G} \arrow[d, "\pi"] \\ \{[x]\} \arrow[r, hook] & \mathsf{G}/\sim \end{tikzcd}$$

From the isomorphism of Proposition 3.0.6 and the evaluation map defined in formula (3.0.3) we obtain a map $\mu : \mathcal{C}_{S^1}(\mathsf{G}) \times S^1 \to B\mathsf{G}$. This map induces a natural S^1 equivariant map

$$\nu : \mathcal{C}_{S^1}(\mathsf{G}) \to \mathcal{L}B\mathsf{G}$$

where the action on the l.h.s. is explained in formula (3.0.2) and the one in the r.h.s is the obvious one. But the image of ν in $\mathcal{L}B\mathsf{G}$ lies in the ghost loops space.

Lemma 5.0.13. $\mathrm{Im}(\nu) \subset \mathcal{L}_s B\mathsf{G}$.

Proof. Let α be a configuration in $\mathcal{C}_{S^1}(\mathsf{G})$ with $y \in \mathrm{supp}(\alpha)$ and $g = \alpha(y)$. If $x = \mathsf{s}(g)$ it is clear that $\pi(\mu(\alpha, \theta)) = [x]$ for any $\theta \in S^1$. Hence Im $\nu(\alpha) \subset \pi^{-1}([x])$. Without confusion we can assume that

$$\nu : \mathcal{C}_{S^1}(\mathsf{G}) \to \mathcal{L}_s B\mathsf{G}.$$

□

Let's consider the following commutative diagram

$$\begin{tikzcd} \mathcal{C}_{S^1}(\mathsf{G}) \arrow[r, "\varepsilon_0"] \arrow[d, "\nu"] & \mathcal{C}_R(\mathsf{G}) \arrow[d, "\cong"] \\ \mathcal{L}_s B\mathsf{G} \arrow[r, "ev_0"] & B\mathsf{G} \end{tikzcd}$$

where ε_0 is the restriction of the map ε to $\mathcal{C}_{S^1}(\mathsf{G}) \times \{0\}$, ev_0 is the evaluation of a loop at 0, i.e. for $f \in \mathcal{L}B\mathsf{G}$, $ev_0(f) := f(0) \in B\mathsf{G}$ and $\cong$ is the S^1-equivariant homeomorphism of Proposition 3.0.6.

Theorem 5.0.14. *The map $\nu : \mathcal{C}_{S^1}(\mathsf{G}) \to \mathcal{L}_s B\mathsf{G}$ is S^1-equivariant and induces a weak homotopy equivalence on the fibers*

$$\nu_\alpha : {\varepsilon_0}^{-1}(\alpha) \xrightarrow{\simeq} {ev_0}^{-1}(\alpha)$$

for every $\alpha \in \mathcal{C}_R(\mathsf{G})$.

Proof. Let α be as in definition 3.0.5; a map $\alpha : R = (0,1) \to \mathsf{G}_1$ with support $\mathrm{supp}(\alpha) := \{t_1, t_2, \ldots, t_n\}$, $g_i := \alpha(t_i)$ and $\mathsf{t}(g_i) = \mathsf{s}(g_{i+1})$. The space ${\varepsilon_0}^{-1}(\alpha)$ will consist of all the configurations on the circle $\beta \in \mathcal{C}_{S^1}(\mathsf{G})$ such that when restricted to the interval $(0,1)$ matches α, i.e $\beta|_{(0,1)} = \alpha$. If we call $x := \mathsf{s}(g_1)$ and $y := \mathsf{t}(g_n)$ it is clear that all the possible values of $\beta(0)$ lie in the set

$$M(x,y) := \{g \in \mathsf{G}_1 | \mathsf{s}(g) = y \ \& \ \mathsf{t}(g) = x\},$$

and is easy to see that ${\varepsilon_0}^{-1}(\alpha) \cong M(x,y)$.

There is a canonical homeomorphism between the space $M(x,y)$ and the isotropy group G_x at x given by the following map

$$\begin{array}{ccc} M(x,y) & \xrightarrow{\simeq} & G_x \\ g & \mapsto & g_1 g_2 \cdots g_n g \end{array}$$

Recall now that the space ${ev_0}^{-1}(\alpha)$ consist of all loops $f : S^1 \to B\mathsf{G}$ in $\mathcal{L}_s B\mathsf{G}$ such that $f(0) = \alpha$. We have already seen that the image of these loops lie in $\pi^{-1}([x])$ which is homotopic to BG_x. Let G_0^x be the orbit of x under the groupoid G and let's construct the groupoid G^x whose morphisms are given by the set G_1^x which makes the following diagram into a cartesian square

$$\begin{tikzcd}
\mathsf{G}_1^x \arrow[r] \arrow[d, "\mathsf{s}\times\mathsf{t}"] & \mathsf{G}_1 \arrow[d, "\mathsf{s}\times\mathsf{t}"] \\
\mathsf{G}_0^x \times \mathsf{G}_0^x \arrow[r] & \mathsf{G}_0 \times \mathsf{G}_0
\end{tikzcd}$$

We can identify G^x as a subgroupoid of G and therefore $B\mathsf{G}^x = \pi^{-1}([x])$. Then ${ev_0}^{-1}(\alpha)$ is the space of loops in $B\mathsf{G}^x$ based at α (i.e. $\Omega B\mathsf{G}^x$).

So we obtain the following maps

$$G_x \xrightarrow{\cong} M(x,y) \xrightarrow{\cong} {\varepsilon_0}^{-1}(\alpha) \xrightarrow{\nu_\alpha} {ev_0}^{-1}(\alpha) \xrightarrow{\cong} \Omega B\mathsf{G}^x \xrightarrow{\simeq} \Omega BG_x$$

where the composition of all of them $G_x \to \Omega BG_x$ is the weak homotopy equivalence induced by by the embedding $\{h\} \times \Delta_1 \subset |\mathcal{Y}_*(\mathsf{G})| \cong \mathcal{C}_{S^1}(G_x) \simeq \mathcal{L}BG_x$ for $h \in G_x$.

The fact that ν is S^1-equivariant follows from its definition. □

It seems that the previous information is enough for ν to induce a weak homotopy equivalence, but we were not able to prove it. Nevertheless it induces equivalences in cohomology and in S^1-equivariant cohomology.

Theorem 5.0.15. *The map $\nu : \mathcal{C}_{S^1}(\mathsf{G}) \to \mathcal{L}_s B\mathsf{G}$ induces an equivalence in cohomology and in S^1-equivariant cohomology.*

Proof. Let $\mathcal{Z}$ be the locally constant sheaf of integral valued functions (the base of the sheaf will be clear from the context). The Leray sheaf[2] $\mathcal{H}^*(\varepsilon_0; \mathcal{Z})$ associated to the map $\varepsilon_0 : \mathcal{C}_{S^1}(\mathsf{G}) \to \mathcal{C}_R(\mathsf{G})$ has by stalks the cohomology of the fibers, i.e. $\mathcal{H}^*(\varepsilon_0, \mathcal{Z})_\alpha \cong H^*({\varepsilon_0}^{-1}(\alpha), \mathcal{Z})$. The same can be said to the Leray sheaf associated to the map $ev_0 : \mathcal{L}_s B\mathsf{G} \to \mathcal{C}_R(\mathsf{G})$, and so the morphism of sheaves induced by ν is an isomorphism,

$$\mathcal{H}^*(ev_0, \mathcal{Z}) \xrightarrow{\cong} \mathcal{H}^*(\varepsilon_0, \mathcal{Z}),$$

(is an isomorphism on the stalks $\nu_\alpha^* : H^*({ev_0}^{-1}(\alpha), \mathcal{Z}) \cong H^*({\varepsilon_0}^{-1}(\alpha), \mathcal{Z})$).

From the Leray spectral sequence of both maps we get that ν induces an isomorphism of cohomologies

$$\nu^* : H^*(\mathcal{L}_s B\mathcal{G}, \mathbb{Z}) \xrightarrow{\cong} H^*(\mathcal{C}_{S^1}(\mathsf{G}), \mathbb{Z}). \tag{5.0.5}$$

For the second equivalence we define the maps $\overline{\varepsilon_0} = \pi \circ \varepsilon_0 : \mathcal{C}^{S^1}(\mathsf{G}) \to \mathsf{G}/\sim$ and $\overline{ev_0} = \pi \circ ev_0 : \mathcal{L}_s B\mathsf{G} \to \mathsf{G}/\sim$, where $\pi : B\mathsf{G} \to \mathsf{G}/\sim$ is the natural projection. For $[x] \in \mathsf{G}/\sim$ the fibers $\overline{\varepsilon_0}^{-1}([x]) \overset{\simeq}{\to} \overline{ev_0}^{-1}([x])$ are S^1-homotopic (as shown in the next lemma); applying the same procedure as before to the Borel constructions $ES^1 \times_{S^1} \mathcal{C}_{S^1}(\mathsf{G})$ and $ES^1 \times_{S^1} \mathcal{L}_s B\mathsf{G}$ and their induced maps to $\mathsf{G}/\sim$, we obtain that ν induces an isomorphism in S^1 equivariant cohomology

$$H^*_{S^1}(\mathcal{L}_s B\mathcal{G}, \mathbb{Z}) \xrightarrow{\cong} H^*_{S^1}(\mathcal{C}_{S^1}(\mathsf{G}), \mathbb{Z}). \tag{5.0.6}$$

□

Lemma 5.0.16. *For $[x] \in \mathsf{G}/\sim$ the fibers $\overline{\varepsilon_0}^{-1}([x]) \overset{\simeq}{\to} \overline{ev_0}^{-1}([x])$ are S^1-homotopic.*

[2]For definitions and notations regarding the Leray spectral sequence of a map see [5]

Proof. From 5.0.14 we know that $\overline{\varepsilon_0}^{-1}([x]) = \mathcal{C}_{S^1}(\mathsf{G}^x)$ and $\overline{ev_0}^{-1}([x]) = \mathcal{L}_s B\mathsf{G}^x$ where G^x is the subgroupoid of G generated by the orbit of $x \in \mathsf{G}_0$. As $B\mathsf{G}^x \simeq BG_x$ then the homotopy type of the fibers

$$G_x \cong \varepsilon_0^{-1}(\alpha) \xrightarrow{\simeq} ev_0^{-1}(\alpha) \simeq \Omega BG_x$$

is constant for every $\alpha \in B\mathsf{G}^x$. Is a result of Dold [14] on quasifibrations that this induces the required homotopy. $\square$

Therefore ν induces isomorphisms in rational homology,

Corollary 5.0.17.

$$H_*(\mathcal{C}_{S^1}(\mathsf{G}), \mathbb{Q}) \xrightarrow{\cong} H_*(\mathcal{L}_s B\mathsf{G}, \mathbb{Q}).$$
$$H_*^{S^1}(\mathcal{C}_{S^1}(\mathsf{G}), \mathbb{Q}) \xrightarrow{\cong} H_*^{S^1}(\mathcal{L}_s B\mathsf{G}, \mathbb{Q}).$$

Now we will relate the previous study to other homology theories.

6. Hochschild and cyclic homology

6.1. **Preliminaries.**

Definition 6.1.1. A cyclic R-chain complex A is a functor from Λ^{op} to R-chain complexes (see Remark 7.2.2); it is determined by the chain complexes $A_n := A(\mathsf{n})$ and the morphisms $d_n^i : A_n \to A_{n-1}$, $S_n^i : A_n \to A_{n+1}$ and $t_n : A_n \to A_n$. One defines the *Hochschild homology* $HH_*(A_\bullet)$ as the total homology of the chain complex of chain complexes

$$A_0 \xleftarrow{b} A_1 \xleftarrow{b} A_2 \xleftarrow{b} \cdots \xleftarrow{b} A_n \xleftarrow{b} \cdots \tag{6.1.1}$$

and the *cyclic homology* $HC_*(A_\bullet)$ as the homology of the chain bicomplex of chain of complexes

(6.1.2)
$$\begin{tikzcd}
\vdots \arrow[d, "b"'] & \vdots \arrow[d, "-b'"'] & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "-b'"'] & \\
A_2 \arrow[d, "b"'] & A_2 \arrow[l, "1-T"'] \arrow[d, "-b'"'] & A_2 \arrow[l, "N"'] \arrow[d, "b"'] & A_2 \arrow[l, "1-T"'] \arrow[d, "-b'"'] & \cdots \arrow[l] \\
A_1 \arrow[d, "b"'] & A_1 \arrow[l, "1-T"'] \arrow[d, "-b'"'] & A_1 \arrow[l, "N"'] \arrow[d, "b"'] & A_1 \arrow[l, "1-T"'] \arrow[d, "-b'"'] & \cdots \arrow[l] \\
A_0 & A_0 \arrow[l, "1-T"'] & A_0 \arrow[l, "N"'] & A_0 \arrow[l, "1-T"'] & \cdots \arrow[l]
\end{tikzcd}$$

with $b, b' : A_n \to A_{n-1}$, $N, T : A_n \to A_n$ being given by

$$b = \sum_{i=0}^{n} (-1)^i d_n^i, \quad b' = \sum_{i=0}^{n-1} (-1)^i d_n^i, \quad T = (-1)^n t_n, \quad N = 1 + T + \cdots + T^n$$

As the complex

$$A_0 \xleftarrow{-b'} A_1 \xleftarrow{-b'} A_2 \xleftarrow{-b'} \cdots \xleftarrow{-b'} A_n \xleftarrow{-b'} \cdots$$

is acyclic one obtains an exact sequence

$$\cdots \longrightarrow HH_*(A_\bullet) \xrightarrow{I} HC_*(A_\bullet) \xrightarrow{S} HC_{*-2}(A_\bullet) \longrightarrow HH_{*-1}(A_\bullet) \longrightarrow \cdots$$

with I induced by the inclusion of 6.1.1 in 6.1.2 as the first column and S induced by the canonical projection of 6.1.2 on itself which send the first two columns to zero and the $(k+2)$-th column to the k-th one. This sequence is called the *Connes sequence* of $A_\bullet$.

Definition 6.1.2. The homology of the inverse limit of complexes induced by S is denoted $HP_*(A_\bullet)$ and is called *periodic homology*; it is a $\mathbb{Z}/2\mathbb{Z}$ graded theory.

Example 6.1.3. If (X_*, t_*) is a cyclic space, we consider $\mathcal{X}$ to be the composition of $(X_*, t_*) : \Lambda^{op} \to Top$ with $C_*(\cdot; R) : Top \to R - chain\ complexes$, where $C_*(X; R)$ is the normalized singular chain complex of X with coefficients in R. One denotes by $HH_*((X_*, t_*); R)$ resp. $HC_*((X_*, t_*); R)$ the Hochschild homology $HH_*(\mathcal{X}_\bullet)$ resp. cyclic homology $HC_*(\mathcal{X}_\bullet)$.

Is a result in [8] that for R a commutative ring with unit

Proposition 6.1.4. *For any cyclic space (X_*, t_*) there exist natural isomorphisms*

$$\begin{aligned} HH_*((X_*, t_*); R) &\xrightarrow{\cong} H_*(\|X_*\|; R) \\ HC_*((X_*, t_*); R) &\xrightarrow{\cong} H_*(\|X_*, t_*\|; R) \end{aligned}$$

which identifies Connes' homomorphism $S : HC_*((X_*, t_*); R) \to HC_{*-2}((X_*, t_*); R)$ with the Gysin homomorphism $S : H_*(\|X_*, t_*\|; R) \to H_{*-2}(\|X_*, t_*\|; R)$ of the fibration $\|X_*\| \to \|X_*, t_*\| \to BS^1$.

6.2. **Groupoids.** Applying the previous formalism to topological groupoids we obtain the following results

Theorem 6.2.1. *Let G be a topological groupoid and $\wedge\mathsf{G}$ its inertia groupoid. Then there are canonical isomorphisms*

$$\begin{aligned} HH_*((\wedge\mathsf{G}_*, t_*)) \stackrel{\cong}{\to} H_*(|\mathcal{Y}_*(\mathsf{G})|) &\stackrel{\cong}{\to} H_*(\mathcal{C}_{S^1}(\mathsf{G})) \stackrel{\cong_{\mathbb{Q}}}{\to} H_*(\mathcal{L}_s B\mathsf{G}) \\ HC_*((\wedge\mathsf{G}_*, t_*)) \stackrel{\cong}{\to} H_*^{S^1}(|\mathcal{Y}_*(\mathsf{G})|) &\stackrel{\cong}{\to} H_*^{S^1}(\mathcal{C}_{S^1}(\mathsf{G})) \stackrel{\cong_{\mathbb{Q}}}{\to} H_*^{S^1}(\mathcal{L}_s B\mathsf{G}) \end{aligned}$$

Proof. It follows directly from 4.0.10, 5.0.17, and 6.1.4. □

6.3. **Chen-Ruan Orbifold cohomology.** Motivated by the theory of quantum cohomology Chen and Ruan [10] defined a remarkable cohomology ring associated to a smooth complex orbifold. One of the peculiarities of the theory is that in certain interesting cases is isomorphic to the cohomology ring of the smooth space which resolves the singularities of the orbifold [27, 29, 15], namely a crepant resolution.

As we mentioned in the introduction, orbifolds can be seen as a special kind of topological groupoids i.e. smooth, proper and étale [21, 23, 25]. The twisted sectors $(X_{(g)}, g \in T_1)$ of an orbifold X [10, 28], on which the definition of the orbifold cohomology relies, are known to be equivalent to the inertia groupoid of the orbifold (seen as a groupoid) [1, 22, 23]. To each connected component of the twisted sectors there is associated a rational number $\iota_{(g)}$ which is called *degree shifting number* [10] (also known as *age* [26]), and the orbifold cohomology is defined as the direct sum of the cohomology of the twisted sectors whose degree is shifted up by twice the degree shifting number, i.e.

$$H^*_{\mathrm{orb}}(X, \mathbb{C}) \stackrel{def}{=} \bigoplus_{(g)} H^{*-2\iota_{(g)}}(X_{(g)}, \mathbb{C}).$$

For smooth SL-orbifolds (Gorenstein orbifolds,) namely orbifolds for which the local action at every point can be seen as an element of $SL(d, \mathbb{C})$, the shifting number is an integer. In the case of compact orbifolds , i.e proper étale groupoids whose coarse moduli space is compact, we get the following immediate consequence of the results of the previous section.

Corollary 6.3.1. *For* G *a compact complex* SL*-orbifold the groups*

$$\prod_{m\in n+2\mathbb{Z}} HH_m((\wedge\mathsf{G}_*,t_*);\mathbb{C}) \cong \prod_{m\in n+2\mathbb{Z}} H^m_{\text{orb}}(\mathsf{G};\mathbb{C})$$

are isomorphic.

Proof. Here we use the fact that the homological dimensions of the connected components of the inertia groupoid are all even (because the orbifold is complex) as well as the shifting on the degree. □

Finally using the Chern character for orbifold K-theory [2, 23] and reduced orbifolds (i.e. for every $x \in \mathsf{G}_0$ the induced homomorphism $G_x \to Aut(\mathsf{G}_0)$ is injective) ,we can obtain the following isomorphism.

Theorem 6.3.2. *Let* X *be a compact complex reduced* SL*-orbifold and* G *a groupoid associated to it. Then the periodic homology of* $\wedge\mathsf{G}$ *is isomorphic to the Chen-Ruan orbifold cohomology of* X *(but* $\mathbb{Z}/2\mathbb{Z}$*-graded)*

$$K^n_{\text{orb}}(\mathsf{X})\otimes\mathbb{C} \cong \prod_{m\in n+2\mathbb{Z}} H^m_{\text{orb}}(\mathsf{X},\mathbb{C}) \cong HP_n((\wedge\mathsf{G}_*,t_*),\mathbb{C}).$$

Proof. The first isomorphism is given by the Chern character map of orbifold K-theory [2, 23], and the second follows from the previous corollary and the fact that for orbifolds there is an isomorphism (Cor. 6.3.4)

$$HP_n((\wedge\mathsf{G}_*,t_*),\mathbb{C}) \cong \prod_{m\in n+2\mathbb{Z}} HH_m((\wedge\mathsf{G}_*,t_*),\mathbb{C}).$$

□

Proposition 6.3.3. *Let* X *be an orbifold and* G *a groupoid associated to it. Then the Connes exact sequence split into the short exact sequences*

$$0 \to HH_k((\wedge\mathsf{G}_*,t_*);\mathbb{Q}) \to HC_k((\wedge\mathsf{G}_*,t_*);\mathbb{Q}) \to HC_{k-2}((\wedge\mathsf{G}_*,t_*);\mathbb{Q}) \to 0$$

for every $k\in\mathbb{Z}$.

Proof. In rational homology the action of the circle S^1 on $|\wedge\mathsf{G}_*|$ is trivial. This because the isotropy is finite and the classifying spaces of finite groups are rationally trivial. Then the spectral sequence associated to the fibration of theorem 7.2.6 reduces to the isomorphism

$$H^{S^1}_*(|\wedge\mathsf{G}_*|;\mathbb{Q}) \cong H_*(|\wedge\mathsf{G}_*|;\mathbb{Q})\otimes H^{S^1}_*(*;\mathbb{Q}).$$

Applying theorem 6.1.4 the result follows. □

In the case of an orbifold the periodic homology is the inverse limit of the maps $S: HC_n \to HC_{n-2}$, therefore we can easily see that

Corollary 6.3.4. *For* X *a* d*-dimensional orbifold,* G *a groupoid associated to it,* $k=0,1$ *and any* l *such that* $2l\geq d$ *then*

$$HP_k((\wedge\mathsf{G}_*,t_*);\mathbb{Q}) \cong HC_{2l+k}((\wedge\mathsf{G}_*,t_*);\mathbb{Q}) \cong \prod_{n=0}^{|\frac{d}{2}|} HH_{2n+k}((\wedge\mathsf{G}_*,t_*);\mathbb{Q}).$$

Remark 6.3.5. The previous isomorphisms are additive. We will return to the study of the multiplicative structures in a future paper.

7. Appendix

This section will be devoted to give a summary of some of the properties of simplicial and cyclic spaces. We will be mostly based on the papers [8, 16] but we recommend the reader the book of Connes [12] for a friendly description of the subject.

7.1. Simplicial spaces.

Let Ord be the category whose objects are the sets $\mathsf{n} = \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}$ and whose morphisms are the order preserving maps $\mathsf{n} \to \mathsf{m}$. The face maps $\partial_i^n : \mathsf{n-1} \to \mathsf{n}$ which skip i, and the degeneracy maps $\lambda_i^n : \mathsf{n+1} \to \mathsf{n}$ which repeat i, generate all the morphism of the category.

Definition 7.1.1. A *simplicial space* X is a contravariant functor from Ord to the category of topological spaces Top.

It can also be described as (X_*, d_n^i, s_n^i) where $X_n := X(\mathsf{n})$, $d_n^i := X(\partial_i^n)$, $s_n^i := X(\lambda_i^n)$. Hence $d_n^i : X_n \to X_{n-1}$ and $s_n^i : X_n \to X_{n+1}$ with

$$\begin{array}{rcll}
d_{n-1}^i d_n^j & = & d_{n-1}^{j-1} d_n^i & i < j \\
s_{n+1}^i s_n^j & = & s_{n+1}^{j+1} s_n^i & i \le j \\
d_{n+1}^i s_n^j & = & \begin{cases} s_{n-1}^{j-1} d_n^i & i < j \\ 1 & i = j, j+1 \\ s_{n-1}^{j} d_n^{i-1} & i > j+1 \end{cases} &
\end{array}$$

A simplicial map between simplicial spaces $f_* : X_* \to Y_*$ is a family of continuous maps $f_n : X_n \to Y_n$ that commute with the face and degeneracy maps.

Now let $\Delta : Ord \to Top$ be the covariant functor such that $\Delta(\mathsf{n}) := \Delta_n$ is the n-simplex and $\Delta(\mathsf{n} \xrightarrow{\theta} \mathsf{m}) : \Delta_n \to \Delta_m$ the linear map induced by θ, then we can define

Definition 7.1.2. The *geometrical realization* $|X_*|$ of the simplicial space X is

$$\bigsqcup_n \Delta_n \times X_n / \sim$$

where $\sim$ is the equivalence relation $(t, X(\theta)x) \sim (\Delta(\theta)t, x)$ generated by all morphisms θ in Ord.

In the case of a groupoid G its nerve G_* form a simplicial space (see 3.0.4) and its realization is what is known as the classifying space of the groupoid $B\mathsf{G}$

7.2. Cyclic spaces.

Definition 7.2.1. A *cyclic space* (X_*, t_*) consists of a simplicial space X_* together with continuous maps $t_n : X_n \to X_n$ which satisfy the following relations with respect to the faces and degeneracies of X_*:

- $d_n^i \circ t_n = t_{n-1} \circ d_n^{i-1}$ and $s_n^i \circ t_n = t_{n+1} \circ s_n^{i-1}$ for $1 \le i \le n$ and
- $t_n^{n+1} = \mathrm{id}_{X_n}$, with t_n^r the r-fold composition of t_n.

The maps t_n will be referred to as the cyclic structure of X_*. A morphism of cyclic spaces $f_* : (X_*, t_*) \to (Y_*, t'_*)$ is a simplicial map $f_* : X_* \to Y_*$ such that $t'_n \circ f_n = f_n \circ t_n$.

Remark 7.2.2. A cyclic space can also be seen as a contravariant functor from Λ, the Connes cyclic category [12], to Top. This category Λ has for objects the sets n for $n \in \mathbb{N}$, the same objects of Ord, with $Hom_\Lambda(\mathsf{n},\mathsf{m})$ the homotopy classes of increasing continuous maps ϕ of degree 1 from S^1 to S^1 which send the $(m+1)$-st roots of unity to the $(n+1)$-st roots of unity. Ord can be seen as a subcategory of Λ.

The automorphism group of n in Λ can naturally be identified with $\mathbb{Z}_{n+1}$, and any morphism $\mathsf{n} \to \mathsf{m}$ can be uniquely written as $\alpha\tau_m^r$ (see [11]), with $\alpha \in Hom_{Ord}(\mathsf{n},\mathsf{m})$ and τ_m the generator of $\mathbb{Z}_{m+1}$.

A cyclic space is a functor $(X_*, t_*) : \Lambda^{op} \to Top$; its underlying simplicial space X_* is given by restriction to Ord^{op}.

One peculiarity of cyclic spaces is that we can endow its realization with a circle action. This is shown in [8, 16] but we will sketch its construction.

Definition 7.2.3. Let X_* be a simplicial space. The free cyclic space on X_* is the cyclic space $(S^1_* \tilde{\times} X_*, t_*)$ defined by:

$$\begin{array}{rcl} & & (S^1_* \tilde{\times} X_*)_n = \mathbb{Z}_{n+1} \times X_n \\ d_n^i(\tau_n^r, x) & = & \begin{cases} (\tau_{n-1}^r, d_n^{i-r}x) & if\ \ r \le i \le n \\ (\tau_{n-1}^{r-1}, d_n^{n-r+i+1}x) & if\ \ 0 \le i \le r-1 \end{cases} \\ s_n^i(\tau_n^r, x) & = & \begin{cases} (\tau_{n+1}^r, s_n^{i-r}x) & if\ \ r \le i \le n \\ (\tau_{n+1}^{r+1}, s_n^{n-r+i+1}x) & if\ \ 0 \le i \le r-1 \end{cases} \\ t_n(\tau_n^r, x) & = & (\tau_n^{r+1}, x). \end{array}$$

Where $0 \le r \le n$ and τ_n denotes the generator of $\mathbb{Z}_{n+1}$. We will write S^1_* for $S^1_* \tilde{\times}\, pt$.

Proposition 7.2.4. [8, Prop 1.4] *If (X_*, t_*) is a cyclic space, the evaluation map $ev_* : S^1_* \tilde{\times} X_* \to X_*$ given by $ev_n(\tau_n^r, x) = t_n^r x$ induces a circle action $\mu : S^1 \times |X_*| \to |X_*|$.*

Proof. The proof can also be found in the Appendix to section 3 of [7]. We recall it because is used in the proof of proposition 4.0.10.

The action μ is the composition $|ev| \circ h^{-1} \circ j^{-1}$

$$S^1 \times |X_*| = |S^1_*| \times |X_*| \xleftarrow{j} |S^1_* \times X_*| \xleftarrow{h} |S^1_* \tilde{\times} X_*| \xrightarrow{|ev|} |X_*|$$

where the homeomorphism h is given by

$$h((\tau_n^r, x); (u_0, \dots, u_n)) = ((\tau_n^{n-r+1}, x); \tau_n^r(u_0, \dots, u_n)), \tag{7.2.1}$$

with $(u_0, \dots, u_n) \in \Delta_n$, $\tau_n(u_0, \dots, u_n) = (u_1, \dots, u_n, u_0)$ and j is the obvious homeomorphism from $|S^1_* \times X_*|$ to $|S^1_*| \times |X_*|$. □

We want to associate a topological space to a cyclic space. This is done using homotopy colimits.

Definition 7.2.5. Let $\mathcal{T}$ be a small category and $C : \mathcal{T} \to Top$ a functor. The homotopy colimit *hocolim* C is the classifying space of the topological category $\mathcal{C}$ whose space of objects is $\sqcup_{i \in Ob(\mathcal{T})} C(i)$, with morphisms from $x \in C(i)$ to $x' \in C(i')$ consisting of all morphism $\alpha : i \to i'$ in $\mathcal{T}$ such that $C(\alpha)(x) = x'$. The morphisms in $\mathcal{C}$ are topologized by identification with $\sqcup_{i,i' \in Ob(\mathcal{T})} C(i) \times Hom(i, i')$. If X_* is a simplicial space we define $\|X_*\| = hocolim_{\ Ord^{op}} X_*$; if (X_*, t_*) is a cyclic space we define $\|X_*, t_*\| = hocolim_{\ \Lambda^{op}} X_*$.

For a constant functor $* : \mathcal{T} \to Top$ to a point $hocolim * = B\mathcal{T}$; so $\|*, t_*\| = B\Lambda$. If (X_*, t_*) is a cyclic space the inclusion $Ord^{op} \subset \Lambda^{op}$ induces an inclusion $\|X_*\| \to \|X_*, t_*\|$ and the constant map $(X_*, t_*) \to (*, t_*)$ induces a map $\|X_*, t_*\| \to B\Lambda$. Then

Theorem 7.2.6. [8, Thm. 1.8] *Let (X_*, t_*) be a cyclic space, then the sequence*

$$\|X_*\| \to \|X_*, t_*\| \to B\Lambda$$

is a fibration up to homotopy, and is equivalent as a fibration to

$$|X_*| \to ES^1 \times_{S^1} |X_*| \to BS^1.$$

In here is used the result of Connes [11]: $B\Lambda \cong BS^1$, and the one of Bousfield and Kan [4]: for X_* a simplicial space, $\|X_*\|$ and $|X_*|$ are naturally equivalent.

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