Notes on a theorem of Schur: generalization to metric spaces, stability

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INTRODUCTION

A classical theorem of Schur asserts that a Riemannian space of dimension greater than two that is isotropic at all its points (i.e. the sectional curvature of the space at every point does not depend on 2-dimensional directions) is a space of constant curvature (see [Ca]). Further, it is required that the space under consideration be a Riemannian manifold with at least trice continuously differentiable metric tensor. Here we present a purely metric variant of this theorem. We shall not assume that the space is a smooth Riemannian manifold or even a topological manifold. We shall show that in fact the only essential condition is that of existence at every point of the space of *isotropic curvature* what can be expressed in terms of excesses of geodesic triangles according to Aleksandrov's theory [A].

In the classical case the constancy of the curvature of an isotropic space is established on the basis of the second Bianchi identity. That is why the third derivatives of a metric tensor are required. Taking advantage of the concept of *current* [Rh] we prove generalized Bianchi identity in order to bring into consideration the derivatives of a metric tensor of order not greater than two. This and also application of the Smoothness Theorem for metric spaces of bounded curvature [N1,2] are crucial steps of the proof of Generalized Schur's Theorem.

In the second part of the paper we consider the problem of stability in Schur's theorem, that is the problem of stability of the differential structure and also the stability of the metric of almost isotropic Riemannian space (i.e. the space whose curvature is almost constant at every point w.r.t. 2-dimensional directions) w.r.t. the class of spaces of constant curvature.

Loosely speaking we prove that every compact almost isotropic Riemannian space is diffeomorphic to a space form whenever one controls the injectivity radius and curvature bounds. Roughly speaking the stability is a consequence of Generalized Schur's Theorem via Gromov's compactness ([Gr1]) and Cheeger's finiteness ([Ch]) theorems. We also make use of the fact that the "bounded curvature closure" of the set of compact Riemannian spaces coincides with the set of compact spaces of bounded curvature [N4,5].

We prove the weak convergence of curvatures in Gromov's class $\mathfrak{M}(n,d,V,\Lambda)$ and construct example showing that it can not be improved to L_p-convergence in general case. Weak convergence of curvatures bring us that every sequence of almost isotropic Riemannian spaces with anisotropy converging to zero can converge only to a space of constant curvature. This is a crucial step in our work.

The same approach based on Generalized Schur's Theorem and Gromov's theorem is used to prove the stability of the metric of almost isotropic spaces. Namely we are interesting in the following question: is it true that almost isotropic Riemannian space has almost constant curvature? The answer is "yes" provided that the curvature deviation is estimated in L_p-classes for $p\in[1,\infty)$. The application of the compactness theorem requires a stronger convergence than in Gromov's theorem and we made use of ideas of work [Ruh] to improve the convergence of curvatures in our situation. Taking advantage of Generalized Schur's Theorem, Gromov's compactness theorem and the bound of the deviation of curvature we obtain W^2_p-bound for the deviation of a metric tensor from the metric tensor of a space of constant curvature.

We would like to mention that our generalization of Schur's theorem solves a problem posed by A.D.Aleksandrov in 1982 at a symposium on geometry "in the Large" and foundations of the theory of relativity (see also [Me] and Remark 1.6).

E.Ruh [Ruh] has proved the stability of the differential structure of almost isotropic spaces of positive curvature (see also the papers [M-Ruh] and [Hu]). We want to mention the recent paper due to [Ye] in which R.Ye constructed deformations of positively and negatively curved manifolds with L_2-small anisotropy to a space form through the Ricci flow. We do not obtain the explicit bounds in our stability results. However, our Theorem A.1 is applicable to Riemannian manifolds without restrictions on the sign of a curvature and gives bounds w.r.t. L_1-small anisotropy (see also Remark A.1).

The author learned the problem of stability of a metric in Schur's theorem from E.G.Poznyak, Yu.G.Reshetnyak and V.A.Toponogov. This problem was considered earlier by I.V.Gribkov [Gri 1-3] who constructed example showing that in general the problem of metric stability in Schur's theorem is not well posed and found some sufficient conditions ensuring stability, and V.V.Slavskii [Sl] who has estimated the difference between the metric of a conformally flat almost isotropic Riemannian manifold and that of a constant curvature w.r.t. C^0-norm relying on the integral representation due to Yu.G.Reshetnyak.

Results concerning metric generalization of Schur's theorem and stability of differential structure were announced in [N3,6].

1. Metric version of Schur's theorem

1.1. Basic concepts. For more details see original paper by A.D. Aleksandrov [A], [Ri] and surveys [A-B-N] and [B-N].

Let $(\mathcal{M},\rho)$ be a metric space. The metric ρ is called *intrinsic* if for any $X,Y\in\mathcal{M}$ the distance $\rho(X,Y)$ is equal to the greatest lower bound of the lengths of curves (measured in the metric ρ) joining X and Y.

A curve $\mathcal{L}$ in $(\mathcal{M},\rho)$ joining points $X,Y\in\mathcal{M}$ is called a *geodesic segment* if its length is equal to $\rho(X,Y)$. Both $\rho(X,Y)$ and the geodesic segment with ends X and Y are denoted by XY.

A *triangle* $T = XYZ$ in a metric space $(\mathcal{M},\rho)$ (where $X,Y,Z\in\mathcal{M}$) is a set consisting of points of geodesic segments XY, XZ and YZ, called the *sides* of T. The points X,Y,Z are called the vertices of T.

Let $\mathcal{L}$ and $\mathcal{P}$ be two geodesic segments in $(\mathcal{M},\rho)$ that have common starting point O. On $\mathcal{L}$ and $\mathcal{P}$ respectively we choose arbitrary points X and Y ($X,Y \neq O$) and $x = OX$, $y = OY$, $z = XY$. We consider a triangle $T' = X'Y'O'$ in a Euclidean plane with lengths of sides $O'X' = x$, $O'Y' = y$, $X'Y' = z$ and denote by $\gamma_{\mathcal{L}\mathcal{P}}(x,y)$ the angle in T' at the vertex T'. The (*upper*) *angle* between $\mathcal{L}$ and $\mathcal{P}$ is by definition the quantity

$$\alpha(\mathcal{L},\mathcal{P}) = \overline{\lim}\ \gamma_{\mathcal{L}\mathcal{P}}(x,y),\ \ x,y \longrightarrow 0.$$

The *area* $\sigma(T)$ of a triangle $T = OXY$ is understood as the area of the Euclidean triangle $T' = X'O'Y'$.

The excess of the triangle $T = XYO$ is the quantity

$$\delta(T) = \alpha + \beta + \gamma - \pi,$$

where by α, β and γ we denote the angles of T at the vertices O, X, Y.

Define the *upper* and *lower curvatures* $\overline{K}(T)$ and $\underline{K}(T)$ of a triangle T as follows. If $\sigma(T) \neq 0$, then

$$\overline{K}(T) = \underline{K}(T) = \delta(T)/\sigma(T).$$

For a degenerate triangle (i.e. $\sigma(T) = 0$), set

$$\overline{K}(T) = \begin{cases} +\infty \text{ if } \delta(T) > 0 \\ -\infty \text{ if } \delta(T) \leq 0, \end{cases} \qquad \underline{K}(T) = \begin{cases} +\infty \text{ if } \delta(T) \geq 0 \\ -\infty \text{ if } \delta(T) < 0. \end{cases}$$

The *upper* and *lower curvatures* of a locally compact metric space $(\mathcal{M}, \rho)$ with intrinsic metric ρ *at a point* P in $\mathcal{M}$ are introduced as follows:

$$\overline{K}_{\mathcal{M}}(P) = \overline{\lim}\ \overline{K}(T), \ \underline{K}_{\mathcal{M}}(P) = \underline{\lim}\ \underline{K}(T), \ T \longrightarrow P,$$

where triangles T contract arbitrarily to the point P.

The *upper* and *lower curvatures of* $(\mathcal{M}, \rho)$ are defined as

$$\overline{K}(\mathcal{M}) = \sup\ (\overline{K}_{\mathcal{M}}(P)), \quad \underline{K}(\mathcal{M}) = \inf\ (\underline{K}_{\mathcal{M}}(P)), P \in \mathcal{M}.$$

1.2. Spaces of bounded curvature. A locally compact metric space with intrinsic metric is called a *space of bounded curvature* if it satisfies the conditions:

(i) *The condition of local extendability of geodesic segments:* For each point of $\mathcal{M}$ there is a ball of sufficiently small radius with center at this point such that if two points lying inside the ball can be joined by a geodesic, then this can be extended so that these points become interior points of the extended geodesic,

(ii) *The condition of local curvature boundedness:* For each point $P\in\mathcal{M}$ the upper and lower curvatures at P satisfy the inequalities: $\overline{K}_{\mathcal{M}}(P) < +\infty$, $\underline{K}_{\mathcal{M}}(P) > -\infty$.

Smoothness Theorem [N1,N2]. *In a space of bounded curvature $(\mathcal{M},\rho)$ it is possible to introduce the structure of a Riemannian manifold with the help of local harmonic coordinates, which form an atlas $\mathfrak{h}$ of smoothness $C^{3,\alpha}$, and the metric tensor in the harmonic coordinates belongs at least to $W^2_p \cap C^{1,\alpha}$ for each $p\in [1,+\infty)$ and $\alpha\in (0,1)$.*

Here by W^2_p we denote Sobolev's class of functions having second generalized derivatives summable to the power p. As usually we use the notation $C^{1,\alpha}$ for the corresponding Hölder class.

The Smoothness Theorem enables us to define the sectional curvatures $K_f(U\wedge V)$ w.r.t. 2-dimensional directions $U\wedge V$ $(U,V\in\mathcal{M}_P$ and bivector $U\wedge V$ is not equal to zero) which are formally calculated "almost everywhere" in $\mathcal{M}$ by the components (g_{ij}) of a metric tensor w.r.t. atlas $\mathfrak{h}$. We shall also bring into consideration the formal curvature tensor $R_f(U,V)W$ that is defined almost everywhere in $\mathcal{M}$ w.r.t. n-dimensional Hausdorff measure (n=

$\dim \mathcal{M}$). The following theorem shows a geometrical meaning of the formally introduced curvature tensor.

Let U, V be vectors at a point P of a space of bounded curvature $(\mathcal{M}, \rho)$ such that

$$|U| = |V| = 1 \text{ and } \sigma = U \wedge V \neq 0. \qquad (1.1)$$

Let $T = PBC$ be a triangle in $(\mathcal{M}, \rho)$. We introduce the notation

$$U(B) = \exp_P^{-1}(B)/PB, \quad U(C) = \exp_P^{-1}(C)/PC.$$

This notation is meaningful provided that the triangle T is small enough.

Theorem 1.1 [N4]. *Let $(\mathcal{M}, \rho)$ be a space of bounded curvature. Then there is a set $\mathfrak{O} \subseteq \mathcal{M}$ of zero n-dimensional Hausdorff measure ($n = \dim \mathcal{M}$) that includes the set $\mathfrak{O}_1$ of all points in $\mathcal{M}$ at which the metric tensor does not have second derivatives. At each point $P \in \mathcal{M} \setminus \mathfrak{O}$ the following condition is satisfied:*

For arbitrary pairs of vectors $U, V \in \mathcal{M}_P$ satisfying (1.1) there is a sequence $(T_m = PB_mC_m)_{m=1,2,...}$ of non-degenerated triangles that contract to P in the direction of the pair (U,V) (that is, $U(B_m) \longrightarrow U, U(C_m) \longrightarrow V$ and $\sigma_m = U(B_m) \wedge V(C_m) \longrightarrow \sigma = U \wedge V$) such that the limit of the ratios $\delta(T_m)/\sigma(T_m)$ exists and

$$K_f(\sigma) = \lim \delta(T_m)/\sigma(T_m), \quad m \longrightarrow \infty.$$

1.3. Generalized Bianchi identity. Let $x : \mathcal{U} \subseteq \mathcal{M} \longrightarrow \mathcal{G} \subseteq \mathbb{R}^n$ be a harmonic system of coordinates in a neighborhood $\mathcal{U}$ of a point $P\in\mathcal{M}$ of a space of bounded curvature $(\mathcal{M},\rho)$ (here $\mathcal{G}$ is a domain in $\mathbb{R}^n$, $n = \dim\mathcal{M}$, $x = (x^1,x^2,\dots,x^n)$). We keep the notation $(g_{ij})_{i,j=1,2,\dots,n}$ for the components of the metric tensor w.r.t. coordinates x. We also denote by $\Gamma^k_{ij},\Gamma_{ij,l}$ and $R^k_{i,jl},R_{ij,kl}$, $i,j,k,l = 1,2,\dots,n$, the Christoffel symbols and the components of the formal curvature tensor $R_f(U,V)W$. It follows from the Smoothness Theorem that $g_{ij}\in W^2_p(\mathcal{G})$, and hence, $\Gamma^k_{ij},\Gamma_{ij,k}\in W^1_p(\mathcal{G})$ for any $p\in[1,+\infty)$, $i,j,k = 1,2,\dots,n$. Theorem 1.1 ensures that in addition $R^k_{i,lj},R_{ij,kl}\in L_\infty(\mathcal{G})$, provided that $\mathcal{U}$ is small enough. Therefore, the numbers

$$|R^k_{i,lj}|_\infty = \mathrm{ess\ sup}\,(|R^k_{i,lj}(x)|),\ |R_{ij,kl}| = \mathrm{ess\ sup}\,(|R_{ij,kl}(x)|_\infty),$$

$(x\in\mathcal{G};\ i,j,k,l = 1,2,\dots,n)$ are finite.

To explain motives of our generalization of Bianchi identity let us first assume that g_{ij}, $i,j = 1,2,\dots,n$, belong to class C^∞.

We introduce differential forms of degree 1, namely

$$dx^1,dx^2,\dots,dx^n \qquad \text{and} \qquad \omega^i_k = \Gamma^k_{ij}dx^j,\quad i,k = 1,2,\dots,n.$$

Note that we consider collections $(dx^1,dx^2,\dots,dx^n)$ and (ω^i_k) as the tensor differential forms

$$dx = (dx^1,dx^2,\dots,dx^n) \qquad \text{and} \qquad \omega = \left(\omega^i_k\right)$$

(see [Ca]). We want to recall also the concept of Cartan's absolute exterior differential D. By definition the absolute exterior differential DY of a vector field Y is the following tensor differential form of degree 1: $DY(X) = \nabla_X Y$, where ∇ is the Levi-Civita connection in $T\mathcal{M}$. In terms of coordinates this formula can be rewritten as follows:

$$DY^k = dY^k + \omega_i^k Y^i, \quad k = 1,2,\dots,n, \tag{1.2}$$

where d means Cartan's exterior differential.

Let $Z = (Z_1, Z_2, \dots, Z_n)$ be a co-vector field. One can easily show that

$$DZ_k = dZ_k - \omega_k^i Z_i. \tag{1.3}$$

We mention that the curvature form is defined as follows:

$$\Omega_i^j = D\omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j = 2 R^j_{i,rs} \, dx^r \wedge dx^s,$$

where "$\wedge$" means the exterior product of differential forms.

One can easily obtain the formula for the exterior absolute differential of a tensor differential form $\left(\Phi_i^j\right)$ of the second order from (1.2), (1.3) and the following equation

$$D\Phi_i^j X^i Y_j = d\Phi_i^j X^i Y_j + \Phi_i^j dX^i Y_j + \Phi_i^j X^i dY_j,$$

where $X = (X^1, X^2, \dots, X^n)$ and $(Y_1, Y_2, \dots, Y_n)$ are parallel vector fields. For smooth Riemannian manifolds the Bianchi identity means

$$D\Omega_i^j = 0 \quad \text{or} \quad D\Omega^{ij} = 0, i, j = 1, 2, \ldots, n,$$

(see [Ca]), where Ω^{ij} is the form Ω^i_j with raised subscript j.

We recall that

$$D\Omega^{ij} = d\Omega^{ij} + \Omega^{kj} \wedge \omega^i_k + \Omega^{ik} \wedge \omega^j_k = 0. \tag{1.4}$$

We denote by $\Lambda^m_0(\mathscr{G})$ the set of C^∞-smooth scalar differential forms ψ of degree m, which are compactly supported in the domain $\mathscr{G}$. Making use of the obvious equation

$$\int_{\mathscr{G}} d\Omega^{ij} \wedge \psi = - \int_{\mathscr{G}} \Omega^{ij} \wedge d\psi$$

and (1.4) we get the following equation:

$$\int_{\mathscr{G}} d\Omega^{ij} \wedge \psi = \int_{\mathscr{G}} \Big[- \Omega^{ij} \wedge d\psi + \Big(\Omega^{kj} \wedge \omega^i_k + \Omega^{ik} \wedge \omega^j_k \Big) \wedge \psi \Big] \tag{1.5}$$

for any $\psi \in \Lambda^{n-3}_0(\mathscr{G})$.

We can generalize this situation as follows. Let $\left(\Phi^{ij}\right)$ be a tensor differential form of degree 2. We assume that the coefficients of $\Phi^{ij} = \phi^{ij}_{rs}\, dx^r \wedge dx^s$ belong to class $L_{p,loc}(\mathscr{G})$ for every $p \in [1, +\infty)$, that is

$$\Phi^{ij} \in L_{p,loc}(\mathscr{G}), \quad p \in [1, +\infty), \quad i, j, r, s = 1, 2, \ldots, n.$$

Then Φ defines the *tensor current*:

$$(D\Phi,\psi) = \Big\{(D\Phi^{ij},\psi) = \int_{\mathcal{G}}\Big[-\Phi^{ij}\wedge d\psi +$$

$$+ \big[\Phi^{kj}\wedge\omega_k^i + \Phi^{ik}\wedge\omega_k^j\big]\wedge\psi\Big]\Big\},\quad \psi\in\Lambda_0^{n-3}(\mathcal{G}).$$

So, in the case of C^{∞}-Riemannian manifold Bianchi identity can be rewritten as

$$(D\Omega,\psi) = 0 \quad \forall\ \psi\in\Lambda_0^{n-3}(\mathcal{G}). \tag{1.6}$$

Let us now turn to the case of a space of bounded curvature.

Lemma 1.1. *Generalized Bianchi identity (1.6) holds in a space of bounded curvature.*

Proof. □ We construct a collection $(\overset{\varepsilon}{g}_{ij})$ of C^{∞}-Riemannian metrics converging to the metric of the space of bounded curvature under consideration taking advantage of the operation of the Sobolev averaging.

Without loss of generality we may assume that harmonic coordinates x can be extended to chart $x' : \mathcal{U}_1 \longrightarrow \mathcal{G}_1$ such that $x'|_{\mathcal{U}} = x$ $(\mathcal{U} \subseteq \mathcal{U}_1,\ \mathcal{G} \subseteq \mathcal{G}_1)$. The averaged metric $(\overset{\varepsilon}{g}_{ij})_{i,j=1,2,\dots,n}$ is introduced as follows:

$$\overset{\varepsilon}{g}_{ij} = \varepsilon^{-1}\int_{\mathcal{G}} g_{ij}(u)\ \phi\left(\frac{x-u}{\varepsilon}\right)\ du,$$

where $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the *averaging kernel*, that is, a function for which the following conditions are satisfied:

(i) The support of ϕ is contained in the unit ball $B(0,1)\subseteq \mathbb{R}^n$

(ii) $\phi\in C^{\infty}(\mathbb{R}^n)$

(iii) $\int_{\mathbb{R}^n} \phi(u)\, du = 1.$

Components of the averaging metric $(\overset{\varepsilon}{g}_{ij})$ belong to class $C^{\infty}(\mathcal{G})$ and the following bounds hold for any $p\in[1,+\infty)$, $\alpha\in(0,1)$:

$$|\overset{\varepsilon}{g}_{ij} - g_{ij}|_{C^{1,\alpha}(\mathcal{G})},\ |\overset{\varepsilon}{g}_{ij} - g_{ij}|_{W^2_p(\mathcal{G})} \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0+.$$

This follows from the Smoothness Theorem and standard properties of Sobolev averaging [G-R].

We denote by $\overset{\varepsilon}{\omega}{}^j_i$ and $\overset{\varepsilon}{\Omega}{}^j_i$ connection and curvature forms that are calculated by metric tensor $(\overset{\varepsilon}{g}_{ij})$. Relying on the latter bounds one can easily prove the following estimates:

$$|\overset{\varepsilon}{\omega}{}^j_i - \omega^j_i|_{C^{0,\alpha}(\mathcal{G})},\ |\overset{\varepsilon}{\Omega}{}^j_i - \Omega^j_i|_{L_p(\mathcal{G})} \longrightarrow \text{ as } \varepsilon \longrightarrow 0+ \tag{1.7}$$

for every $p\in[1,+\infty)$ and $\alpha\in(0,1)$. Since (1.6) holds for $\overset{\varepsilon}{\Omega}$, the following equation holds (see (1.5)):

$$\int_{\mathcal{G}} \left[-\overset{\varepsilon}{\Omega}{}^{ij}\wedge d\psi + \left(\overset{\varepsilon}{\Omega}{}^{kj}\wedge \overset{\varepsilon}{\omega}{}^i_k + \overset{\varepsilon}{\Omega}{}^{ik}\wedge \overset{\varepsilon}{\omega}{}^j_k \right)\wedge \psi \right] = 0 \ \forall \ \ \psi\in\Lambda^{n-3}_0(\mathcal{G}), \varepsilon>0.$$

The bounds (1.7) ensure that both sides of the latter equation converge to the same expression for $\varepsilon = 0$. So, we conclude that for $\varepsilon = 0$ the same equation holds, what means that Generalized Bianchi identity is true for the metric (g_{ij}). This completes the proof of the lemma. ■

1.4. Isotropic metric spaces. We recall that a neighborhood of a point in a metric space is *linear* if it is isometric to a straight line.

We say that the *isotropic curvature* $K(P)$ exists at a point P of a locally compact metric space $(\mathcal{M},\rho)$ with intrinsic metric ρ if the following conditions are satisfied:

(i) No neighborhood of the point is linear

(ii) For every sequence $(T_m)_{m=1,2,\dots}$ of triangles in $(\mathcal{M},\rho)$, that contract arbitrarily to the point P (i.e. the vertices of the triangles converge to P w.r.t. ρ: notation $T_m \longrightarrow P$) the following limit exists:

$$K(P) = \lim \delta(T_m)/\sigma(T_m), \quad T_m \longrightarrow P \qquad (1.7)$$

and does not depend on the choice of the sequence.

Remark 1.1. In the case when $\sigma(T) = 0$ the expression (1.7) does not make sense. Therefore, the existence of the limit in (1.7) is understood in the sense that for an arbitrary $\varepsilon > 0$ there exists a $\sigma > 0$ such that if T is an arbitrary triangle such that the distance from P to each of its vertices does not exceed σ, then $(K(P) - \varepsilon)\, \sigma(T) \le \delta(T) \le (K(P) - \varepsilon)\, \varepsilon(T)$.

A locally compact metric space $(\mathcal{M},\rho)$ with intrinsic metric is said to be *isotropic* if the isotropic curvature exists at each point of $\mathcal{M}$.

Remark 1.2. It is easy to prove that the curvature $K(P)$ of an isotropic metric space is continuous function.

Remark 1.3. An isotropic metric space in which geodesic segments are locally extendible is a space of bounded curvature. This immediately follows from Remark 1.1.

Remark 1.4. In *distance geometry* [B1] a large role is played by the Wald curvature $K_W(P)$ [W]. The curvature $K_W(P)$ is applicable only in 2-dimensional case. A. Wald modified the definition of $K_W(P)$ in such a way that it is suitable also in the multidimensional case. We recall this definition.

A quadruple of points in a metric space has *embedded curvature* equal to k if it is isometric to some quadruple of points on a surface of constant curvature k. A triple of points is called linear if it is isometric to a triple of points in a straight line.

Let $(\mathcal{M},\rho)$ be a metric space with intrinsic metric in which no neighborhood is linear. Then $(\mathcal{M},\rho)$ has Wald's curvature $K'_W(P)$ at an accumulation point P if for each $\varepsilon > 0$ there is a $\sigma > 0$ such that each quadruple Q of points that contains a linear triple of points and is in the ball of radius σ about P has imbedded curvature $k(Q)$ admitting the estimate

$$|K'_W(P) - k(Q)| < \varepsilon.$$

One could also define the existence of a curvature taking advantage of imbedding of quadruples in a space of constant curvature. This would bring us the curvature $K''_W(P)$.

It follows from Theorem 3.1 in [K] that

$$K(P) = K'_W(P)$$

whenever one of these curvature exists. So, one may replace in the definition of isotropic space the existence of $K(P)$ at each point P with the existence of Wald curvature $K'_W(P)$. The similar statement concerning the curvature $K''_W(P)$ is also true.

1.5. Generalization of Schur's theorem to metric spaces.

Generalized Schur's Theorem. *Suppose that* $(\mathcal{M},\rho)$ *is an isotropic metric space with Urysohn-Menger dimension (see [H-W]) greater than two in which the condition of local extendability of geodesic segments holds. Then* $(\mathcal{M},\rho)$ *is isometric to a Riemannian manifold of constant curvature.*

Corollary 1.1. *Let* $(\mathcal{M},\rho)$ *be a locally compact metric space with intrinsic metric and Urysohn-Menger dimension greater than two. Assume that the Wald curvature* $K'_W(P)$ *exists at each point in* $\mathcal{M}$. *Then* $(\mathcal{M},\rho)$ *is isometric to a Riemannian manifold of constant curvature.*

Remark 1.5. One may replace local compactness and local extendability of geodesic segments with the condition that $(\mathcal{M},\rho)$ be a topological manifold of finite dimension.

Remark 1.6. Corollary 1.1 answers a question posed in [K]: what information about a space gives the existence of the Wald curvature $K'_W(P)$ at each point of the space in the multidimensional case?

Proof of the Generalized Schur's Theorem. □ According to Remark 1.3 $(\mathcal{M},\rho)$ is a space of bounded curvature. In view of Theorem 1.1 the equality

$$K(U\wedge V) = K(P),\ U,V\in\mathcal{M}_P,\ U\wedge V \neq 0,$$

holds for almost all points $P\in\mathcal{M}$. But then it is possible to write out a.e. the identity

$$\Omega^{ij} = 2\ K\ dx^i\wedge dx^j, i,j = 1,2,\ldots,n,$$

for the form Ω^{ij} with "raised" index j (see [Ca],p.193).

By Lemma 1.1 Generalized Bianchi identity can be written for $(\mathcal{M},\rho):\{D\Omega^{ij} = 0;\ i,j = 1,2,\dots,n\}$. This enables us to write the equation

$$\int_{\mathcal{G}} K\left[-dx^i\wedge dx^j\wedge d\psi + \left(dx^k\wedge dx^j\wedge \omega^i_k + dx^i\wedge dx^k\wedge \omega^j_k\right)\wedge \psi\right] = 0 \quad (1.8)$$

for any $\psi \in \Lambda^{n-3}_0(\mathcal{G})$. We specify the form $\psi_0\in \Lambda^{n-3}_0(\mathcal{G})$ as follows:

$$\psi_0(x) = f(x)\ dx^1\wedge dx^2\wedge\dots\wedge \widehat{dx^i}\wedge\dots\wedge \widehat{dx^j}\wedge\dots\wedge \widehat{dx^k}\wedge\dots\wedge dx^n,$$

where $k\in \{1,2,\dots,n\}\setminus\{i,j\}$, $f\in \Lambda^0_0(\mathcal{G})$, the sign " ^ " above x^i means that x^i is missed in this expression.

In consequence of the definition of ω^i_j, i.e. $\omega^i_j = \Gamma^i_{jk}\ dx^k$, and the choice of ψ_0 equation (1.8) can be written

$$\int_{\mathcal{G}} K\left[\frac{\partial f}{\partial x^s}\ dx^i\wedge dx^j\wedge dx^s\wedge \zeta^{ijk} -\right.$$

$$\left. - f\ \left(\Gamma^i_{ls}\ dx^l\wedge dx^j\wedge dx^s + \Gamma^j_{ls}\ dx^i\wedge dx^l\wedge dx^s\right)\wedge \zeta^{ijk}\right] = 0, \qquad (1.9)$$

where we have introduced the notation

$$\zeta^{ijk} = dx^1\wedge dx^2\wedge \dots\wedge \widehat{dx^i}\wedge\dots\wedge \widehat{dx^j}\wedge\dots\wedge \widehat{dx^k}\wedge\dots\wedge dx^n.$$

Since $\Gamma^i_{ls} = \Gamma^i_{sl}$, we conclude that

$$\Gamma^i_{ls}\ dx^l\wedge dx^j\wedge dx^s\wedge \zeta^{ijk} = \left(\Gamma^i_{lk} - \Gamma^i_{kl}\right)\ dx^i\wedge dx^j\wedge dx^k\wedge \zeta^{ijk} = 0$$

(here and in what follows there is no summation w.r.t. i, j and k).

Similarly we obtain

$$\Gamma^{j}_{ls}\, dx^{i}\wedge dx^{l}\wedge dx^{s}\wedge \zeta^{ijk} = 0$$

and (1.9) turns to the equation

$$\int_{\mathcal{G}} K(x)\, \frac{\partial f}{\partial x^{k}}(x)\, dx = 0 \quad \forall\, f\in \Lambda^{0}_{0}(\mathcal{G}),\ k = 1,\dots,n,$$

where we denote by dx the volume element $dx^{1}\wedge dx^{2}\wedge \dots\wedge dx^{n}$.

The latter equation means that a generalized derivative of the distribution

$$f\in \Lambda^{0}_{0}(\mathcal{G}) \longrightarrow (K, f) = \int_{\mathcal{G}} K(x)\, f(x)\, dx$$

w.r.t. k is equal to zero for each $k\in \{1,2,\dots,n\}$. This implies that the distribution $K(x)$ is constant (see [H], Theorem 3.1.4). The latter implies that $K(x)$ is equal to a constant for almost all $x\in \mathcal{G}$. Since the function K is continuous, this means that $K(x)$ coincides with the constant for every x. By a theorem of A.D.Aleksandrov [A] $(\mathcal{M},\rho)$ is isometric to a space of constant curvature.■

Remark 1.7. In fact, one can replace the existence of isotropic curvature at every point with the condition:

$(\mathcal{M},\rho)$ is a space of bounded curvature such that a.e. the formal sectional curvature does not depend on the two-dimensional directions.

To argue the statement observe that by [S-Sh] $(\mathcal{M},\rho)$ is C^{∞}-Riemannian manifold and by hypothesis its curvature is constant.

2. Stability in Schur's theorem

2.1. Weak convergence of curvatures in Gromov's class. We want to recall the statement of Gromov's compactness theorem.

Let us consider class $\mathfrak{M}(n,d,V,\Lambda)$ of n-dimensional compact connected C^{∞}-Riemannian manifolds $\mathcal{M}$ with diameter $diam(\mathcal{M}) \le d$, volume $Vol(\mathcal{M}) \ge V > 0$ and sectional curvatures $|K_{\mathcal{M}}| \le \Lambda$. We now introduce a distance in the space $\mathfrak{M}(n,d,V,\Lambda)$ that specifies a natural convergence.

Let $(\mathcal{M}_1,\rho_1)$, $(\mathcal{M}_2,\rho_2)$ be metric spaces with metrics ρ_1 and ρ_2, $f:(\mathcal{M}_1,\rho_1)\longrightarrow(\mathcal{M}_2,\rho_2)$ a Lipschitz map. Then

$$dil\ f = sup_{X,Y\in\mathcal{M}_1;X\neq Y}(\rho_2(f(X),f(Y))/\rho_1(X,Y))$$

is called *dilatation* of f.

Suppose that $(\mathcal{M}_1,\rho_1)$ and $(\mathcal{M}_2,\rho_2)$ are compact. Then

$$d_L(\mathcal{M}_1,\mathcal{M}_2) = inf_{f\ \text{bi-Lipsch. hom.}}(\ |ln\ dil\ f| + |ln\ dil\ f^{-1}|)$$

is the *Lipschitz distance* between $\mathcal{M}_1$ and $\mathcal{M}_2$ provided that bi-Lipschitz homeomorphisms exist. Otherwise $d_L(\mathcal{M}_1,\mathcal{M}_2) = +\infty$.

Gromov's compactness theorem states (see Theorems 8.25 and 8.27 of [Gr1]):

Given a sequence $(\mathcal{M}_k)_{k=1,2,\dots}$ in $\mathfrak{M}(n,d,V,\Lambda)$ there exists a subsequence $(\mathcal{M}_{k_l})_{l=1,2,\dots}$ and a C^0-Riemannian manifold $\mathcal{M}_\infty$ such that $(\mathcal{M}_{k_l})$ converges to $\mathcal{M}_\infty$ in the Lipschitz distance d_L; that is

$\mathfrak{M}(n,d,V,\Lambda)$ *is precompact in a larger class of non-regular Riemannian manifolds.*

In fact, every limit in Gromov's theorem is compact space of bounded curvature and the Smoothness Theorem can be applied. Hence, the metric tensor g_∞ belongs to class $W^2_p \cap C^{1,\alpha}$ for any $p \in [1,+\infty)$ and $\alpha \in (0,1)$. The precompactness of Gromov's class w.r.t. C^1 or more generally $C^{1,\alpha}$-convergence of metric tensors was proved independently by S.Peters [P], O.Durumeric [D] and R.E.Greene and H.Wu [G-W]. We would like to emphasize that while the $C^{1,\alpha}$-smoothness of the metric tensor of the limit space in compactness theorem is obviously a special case of the earlier Smoothness Theorem, that part of the work of the authors mentioned above that concerns the precompactness of class $\mathfrak{M}(n,d,V,\Lambda)$ w.r.t. the $C^{1,\alpha}$-convergence was not known before.

Due to the Smoothness Theorem one could hope to improve the Lipschitz convergence in Gromov's compactness theorem to the W^2_p-convergence. The following example shows that this is not true in general. Yet we shall formulate in a moment the statement in which our W^2_p-result manifests itself.

Example 2.1. Let us consider the sequence of 2-dimensional C^∞-Riemannian metrics which are introduced as follows.

$$ds_k^2 = e^{\lambda_k(x,y)}(dx^2 + dy^2),\ \lambda_k(x,y) = (\cos \pi kx + \cos \pi ky)/k^2,$$
$$x,y \in [-1,1]\ ;\ k = 1,2,\dots\ .$$

It is not difficult to show directly that the sequence (ds_k^2) converges to $(dx^2 + dy^2)$ in $C^{1,\alpha}$ norm for every $\alpha \in (0,1)$ as $k \to \infty$. So, the diameters and the areas of the Riemannian manifolds

$\langle[-1,1]\times[-1,1],ds_k^2\rangle$ converge to the diameter and the area of $[-1,1]\times[-1,1]$ w.r.t. Euclidean metric. A direct computation shows that the curvature of ds_k^2 is equal to

$$K(ds_k^2) = 2^{-1}(\cos kx + \cos ky)\, e^{-(\cos kx + \cos ky)/k^2}$$

and, hence, is uniformly bounded. In the meantime, the curvature can not converge to zero in L_p-norm, since otherwise a subsequence of $\{K(ds_k^2)\}$ would converge to zero almost everywhere.

We complete the consideration of the example by the observation that the metrics ds_k^2 can be realized on a torus T^2, since $\lambda_k(x,y)$ are periodic functions of x and y. So, on account of the properties of the metrics ds_k^2 the manifolds $\langle T^2,ds_k^2\rangle$ belong to Gromov's class $\mathfrak{M}(2,4,1,1)$ for sufficiently large k.

Now we are going to state the weak convergence of curvatures of a C^1-convergent sequence of Riemannian manifolds in Gromov's class $\mathfrak{M}(n,d,V,\Lambda)$. Since the manifolds $\mathcal{M}_k$ are diffeomorphic to each other for sufficiently large k, we can assume that the metrics g_k are given on a fixed C^∞-manifold.

Let $\{\langle\mathcal{M},g_k\rangle\}_{k=1,2,\dots}$ be a sequence of Riemannian manifolds in $\mathfrak{M}(n,d,V,\Lambda)$ that converges to a space of bounded curvature $\langle\mathcal{M},g\rangle$ in the Lipschitz distance. We shall use the following statement that we call the *C^1-convergence* of Riemannian manifolds (see [D], Theorem 5.5):

There is a subsequence $\{\langle\mathcal{M},g_{k_l}\rangle\}$ of the sequence $\{\langle\mathcal{M},g_k\rangle\}$ such that given any point $P\in\mathcal{M}$, there is an open set $\mathcal{U}\subseteq\mathcal{M}$

containing P and and a local coordinate chart $x : \mathcal{U} \longrightarrow \mathbb{R}^n$ *such that* x *is a* C^1*-limit of harmonic (w.r.t.* g_k*) coordinate charts* x_k *. There is also an open set* $\mathcal{G} \subseteq \cap_{k=1}^{\infty} x_k(\mathcal{U})$ *such that if* g_k *is considered as a metric on* $x_k(\mathcal{U}) \subseteq \mathbb{R}^n$ *as* $\tilde{g}_k = (x_k^{-1})^* g_k$*, then* $\tilde{g}_k \longrightarrow \tilde{g} = (x^{-1})^* g$ *in the* C^1 *sense on* $\mathcal{G}$*, namely*

$$|\tilde{g}_k - \tilde{g}|_{C^1(\mathcal{G})} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \quad (2.1)$$

It goes without saying that $\mathcal{G}$ is bounded domain in $\mathbb{R}^n$ and the estimate (2.1) can be extended to a bounded domain $\mathcal{G}' \supseteq \mathcal{G}$ in $\mathbb{R}^n$.

We want to remark that the chart $x: \mathcal{U} \longrightarrow \mathbb{R}^n$ is harmonic w.r.t. the metric tensor g. This follows from the equations

$$g^{ij}\, \Gamma^l_{ij} = 0, \quad l = 1, 2, \ldots, n,$$

for components (g_{ij}) of the metric tensor g w.r.t. chart $x : \mathcal{U} \longrightarrow \mathbb{R}^n$, which are ensured by (2.1) and the fact that $x_k : \mathcal{U} \longrightarrow \mathbb{R}^n$ is harmonic system of coordinates w.r.t. g_k, $k = 1, 2, \ldots$. In particular, the (formal) sectional curvature $K(g)$ is defined almost everywhere in $\mathcal{U}$.

We denote by $K(g_k)$ the sectional curvature of the manifold $\langle \mathcal{M}, g_k \rangle$. The sequence of sectional curvatures $K(g_k)$ is said to be *weakly convergent* to the sectional curvature $K(g)$ (the notation $K(g_k) \xrightarrow{w} K(g)$) if for any pair of C^1-smooth vector fields X and Y on $\mathcal{G}$ such that $\delta = \inf_{x \in \mathcal{G}} (|X(x) \wedge Y(x)|) > 0$ the sectional curvatures of g_k, $k = 1, 2, \ldots$, satisfy

$$\int_{\mathcal{G}} K_{X\wedge Y}(\tilde{g}_k)(x)\ f(x)\ dx \longrightarrow \int_{\mathcal{G}} K_{X\wedge Y}(\tilde{g})(x)\ f(x)\ dx \quad \forall\ f\in \Lambda^0_0(\mathcal{G}).$$

Proposition 2.1. *Let $(\langle\mathcal{M}_k,g_k\rangle)_{k=1,2,\ldots}$ be a sequence in $\mathfrak{M}(n,d,V,\Lambda)$ which is C^1-convergent to a compact space of bounded curvature $\langle\mathcal{M},g\rangle$. Then*

$$K(g_k)\xrightarrow{w}K(g).$$

<u>Proof.</u>□ Let us denote by $R_k(X,Y)Z$, $k=1,2,\ldots,$ the curvature tensor of the Riemannian space $\langle\mathcal{M}_k,g_k\rangle$. We express the sectional curvature $K_{X\wedge Y}(g_k)$ as follows:

$$K_{X\wedge Y}(g_k) = \langle R_k(X,Y)Y,X\rangle/|X\wedge Y|_k^2\ .$$

where we denote by $|X\wedge Y|_k$ the norm of the bivector $X\wedge Y$ w.r.t. $\tilde{g}_k$.

Taking into account (2.1) we observe that the inequality

$$|X\wedge Y|_k^2 \geq \delta^2/2$$

holds for sufficiently large k.

The latter bound and (2.1) imply that

$$|1/|X\wedge Y|_k^2 - 1/|X\wedge Y|^2|_{C^1(\mathcal{G})} \longrightarrow 0 \text{ as } k\longrightarrow\infty\ .$$

On account of the uniform boundedness of the norm of X and Y w.r.t. the metric tensor $\tilde{g}_k$ (see (2.1)) and $K(g_k)$ we arrive at the equality

$$\lim_{k\longrightarrow\infty} |\langle R_k(X,Y)Y,X\rangle/|X\wedge Y|_k^2 - \langle R_k(X,Y)Y,X\rangle/|X\wedge Y|^2|_{C(\mathcal{G})} = 0.$$

Hence, the desired statement follows from the equality

$$\lim_{k\to\infty} \int_{\mathcal{G}} \left(\langle R_k(X,Y)Y,X\rangle - \langle R(X,Y)Y,X\rangle\right) f / |X\wedge Y|^2 \, dx = 0$$

for any infinitely smooth compactly supported in $\mathcal{G}$ function f.

To simplify the notation let us denote $f/|X\wedge Y|^2$ by ψ. It is obvious that ψ is C^1-smooth and compactly supported function in $\mathcal{G}$.

Let $X_i^{(k)}$ be the k-th basis vector field of the harmonic coordinates x_k. We introduce the quantities $((\alpha^k)_i^j)_{i,j=1,2,\dots,n}$ as follows:

$$X_i^{(k)} = (\alpha^k)_i^j \, X_j \, .$$

Since the sequence $(\langle \mathcal{M}, g_k\rangle)$ is C^1-convergent to $\langle \mathcal{M}, g\rangle$, the functions $(\alpha^k)_i^j$ uniformly converge to δ_i^j . So, the coordinates of the vector fields X and Y w.r.t. the basis $(X_i^{(k)})$ converge uniformly in $\mathcal{G}$ to the corresponding coordinates of these vector fields w.r.t. the basis (X_i). Hence, we only need to prove that

$$\lim_{k\to\infty} \int_{\mathcal{G}} \left(R^{(k)}_{rs,pq} - R_{rs,pq}\right) \psi \, dx = 0 \tag{2.2}$$

for any C^1-smooth compactly supported in $\mathcal{G}$ function ψ, where we have introduced the notation $R^{(k)}_{rs,pq}$, $R_{rs,pq}$ for components of the curvature tensor of $\langle \mathcal{M}, g_k\rangle$ and $\langle \mathcal{M}, g\rangle$ w.r.t. charts x_k and x respectively.

Let us denote by $\Gamma^{(k)}_{sp,q}$, $\Gamma^{(k)l}_{sp}$ and so on corresponding Christoffel symbols w.r.t. metric tensor g_k. Then the integral in the right-hand side of (2.2) by integration by parts is transformed to the integral

$$\int_{\mathcal{G}} \left\{ -\left[\Gamma^{(k)}_{sp,q} - \Gamma_{sp,q} \right] \frac{\partial\psi}{\partial x^r} + \left[\Gamma^{(k)}_{rp,q} - \Gamma_{rp,q} \right] \frac{\partial\psi}{\partial x^s} + \right.$$

$$+ \left[\Gamma^{(k)l}_{sp} \Gamma^{(k)}_{lr,q} - \Gamma^{l}_{sp} \Gamma_{lr,q} \right] \psi -$$

$$\left. - \left[\Gamma^{(k)l}_{rp} \Gamma^{(k)}_{sl,q} - \Gamma^{l}_{rp} \Gamma_{sl,q} \right] \psi \right\} dx$$

which converges to zero as $k\to\infty$ (see (2.1)). This proves (2.2) and completes the proof of the proposition.■

Similarly to the weak convergence of the curvatures one can introduce the concept of the weak convergence of the scalar curvatures $S(g_k)$ to $S(g)$ and so on. The following statement obviously follows from (2.1) and Proposition 2.1.

Corollary 2.1. $S(g_k)\xrightarrow{w}S(g)$, $k\to\infty$.

2.2. Almost isotropic Riemannian spaces. Let $\langle\mathcal{M},g\rangle$ be n-dimensional C^∞-Riemannian manifold ($n\geq 3$). Suppose that the curvature of $\langle\mathcal{M},g\rangle$ is almost isotropic at a point P. Then any sectional curvature at P is almost equal to $S(P)/n(n-1)$, where we denote by $S(P)$ the scalar curvature of $\langle\mathcal{M},g\rangle$. Hence, the following definition is meaningful.

We call the quantity

$$\varepsilon(g,P) = \sup \left(|K_{U\wedge V}(g) - S(P)/n(n-1)| \right);\ U,V\in\mathcal{M}_P,\ \ |U\wedge V| = 1$$

curvature anisotropy of $\langle \mathcal{M}, g\rangle$ *at a point* $P\in\mathcal{M}$.

The curvature anisotropy of $\langle \mathcal{M}, g\rangle$ is defined as follows:

$$\varepsilon(g) = \sup\{\varepsilon(g,P)\};\ P\in\mathcal{M}.$$

Let us make the following trivial remark.

We introduce the notation:

$$K(P) = S(P)/n(n-1),\ \ \varepsilon(U\wedge V) = K_{U\wedge V}(g) - K(P),$$

where $U,V\in\mathcal{M}_p$, $U\wedge V \neq 0$.

In consequence of the formula for sectional curvatures we have that

$$R_{rs,kh}\, X^r\, Y^s\, X^k\, Y^h = -\,[K + \varepsilon(\sigma)]\; g_{rs,kh}\, X^r\, Y^s\, X^k\, Y^h,$$

for any non-zero bivector $\sigma = X\wedge Y$. Here and in what follows we use the notation $g_{rs,kh} = g_{rk}\, g_{sh} - g_{rh}\, g_{sk}$.

Following the arguments of [Ca],sec.172,we obtain that

$$R_{rs,kh} = -\,(K + \varepsilon_{rs,kh})\; g_{rs,kh}\,. \qquad (2.3)$$

where $\varepsilon_{rs,kh} = \varepsilon_{sr,kh} = \varepsilon_{kh,rs}$,$k,h,r,s = 1,2,\ldots,n$ and

$$\varepsilon_{rs,rs} = \varepsilon(X_r\wedge X_s),\qquad \varepsilon_{rs,rh} = \varepsilon(X_r\wedge (X_s+X_h)),$$

$$\varepsilon_{rs,kh} = \varepsilon((X_r+X_s)\wedge (X_k+X_h))$$

provided that r, s, k and h are different.

It follows from the definition of $\varepsilon(g)$ that

$$|\varepsilon_{rs,kh}| \leq \varepsilon(g). \tag{2.4}$$

The above computations enables us to write the bound for the tensor $E(g)$ that can be written in a local system of coordinates as follows:

$$E_{ij} = R_{ij} - \frac{S}{n} g_{ij} .$$

Lemma 2.1. *For any (M, g) in $\mathfrak{M}(n, d, V, \Lambda)$ there is a constant C depending on n, d, V and Λ such that*

$$|E(g)| = sup_{P \in M} \left((E_{ij}(P)\ E^{ij}(P))^{1/2} \right) \leq C\ \varepsilon(g).$$

Proof. □ By (2.3) and (2.4) we only need to know the bounds for $|g_{ij}|$ and $|g^{ij}|$. Harmonic coordinates constructed in [J-K] and estimates of $|g_{ij}|_{C^{1,\alpha}}$ due to [J-K] bring us the desired bound. ■

We shall use the following well-known equation relating $E(g)$ and the scalar curvature S, that is a consequence of the Bianchi identity:

$$\frac{\partial S}{\partial x^i} = (g^{pq}\ E_{pi})_{,q} , \tag{2.5}$$

where we denote by $(g^{pq}\ E_{pi})_{,q}$ the covariant derivative of the tensor field $g^{pq}\ E_{pi}$.

We complete this section with the following statement that will be a crucial step in our applications of Generalized Schur's Theorem.

Proposition 2.2. *Let* $\{\langle \mathcal{M}, g_k \rangle\}_{k=1,2,\ldots}$ *be a sequence of Riemannian spaces in Gromov's class* $\mathfrak{M}(n,d,V,\Lambda)$ *for which the following condition is satisfied:*

$$\lim_{k\to\infty} \varepsilon(g_k) = 0.$$

Then every limit space of the sequence $\{\langle \mathcal{M}, g_k \rangle\}_{k=1,2,\ldots}$ *w.r.t. the Lipschitz distance is isometric to a Riemannian space of constant curvature.*

Proof.□ First we observe that by Gromov's compactness theorem at least one limit space exists. As we have mentioned any limit space in Gromov's theorem is a space of bounded curvature. So, on account of the Generalized Schur's Theorem (see also Remark 1.7) to prove the proposition we have to establish that the formal sectional curvature of the limit space does not depend on two-dimensional directions.

Without loss of generality we may assume that the sequence $\{\langle \mathcal{M}, g_k \rangle\}_{k=1,2,\ldots}$ C^1-converges to the space of bounded curvature $\langle \mathcal{M}, g \rangle$ and, hence, the statements of both Proposition 2.1 and Lemma 2.1 are applicable to the sequence $\{\langle \mathcal{M}, g_k \rangle\}_{k=1,2,\ldots}$.

Let X and Y be a pair of continuous vector fields in $\mathcal{G}$ such that $\inf_{x\in\mathcal{G}} \{|X(x)\wedge Y(x)|\} > 0$. By hypothesis the following bound holds:

$$-\varepsilon(g_k) \le K_{X\wedge Y}(g_k)(u) - \frac{S(g_k)(u)}{n(n-1)} \le \varepsilon(g_k), \quad k = 1,2,\ldots \quad (2.6)$$

Let us consider the function

$$\phi_{\eta,x}(u) = \phi(\frac{x-u}{\eta}) / \eta^n,$$

where $\phi(u)$ is the averaging kernel (see sec.1.3), $\eta > 0$.

On multiplying (2.6) by $\phi_{\eta,x}(u)$, integrating over domain $\mathcal{G}$ and also taking into account that

$$\int_{\mathbb{R}^n} \phi_{\eta,x}(u)\, du = 1$$

we arrive at the inequalities

$$-\varepsilon(g_k) \le \int_{\mathcal{G}} \left[K_{X\wedge Y}(g_k)(u) - \frac{S(g_k)(u)}{n(n-1)}\right] \phi_{\eta,x}(u)\, du \le \varepsilon(g_k).$$

Since $K_{X\wedge Y}(g_k) \xrightarrow{w} K_{X\wedge Y}(g)$, $S(g_k) \xrightarrow{w} S(g)$ (see Proposition 2.1 and Corollary 2.1) and $\varepsilon(g_k) \longrightarrow 0$ as $k \longrightarrow \infty$, we obtain that

$$\overset{\eta}{K}_{X\wedge Y}(g)(x) - \frac{\overset{\eta}{S}(g)(x)}{n(n-1)} =$$

$$= \int_{\mathcal{G}} \left[K_{X\wedge Y}(g)(u) - \frac{S(g)(u)}{n(n-1)}\right] \phi_{\eta,x}(u)\, du = 0,$$

where we denote by $\overset{\eta}{K}_{X\wedge Y}(g)$ and $\overset{\eta}{S}(g)$ the Sobolev averaging of corresponding functions w.r.t. parameter η.

Since $\overset{\eta}{K}_{X\wedge Y}(g) \longrightarrow K_{X\wedge Y}(g)$ and $\overset{\eta}{S}(g) \longrightarrow S(g)$ as $\eta \longrightarrow 0$ a.e. in $\mathcal{G}$, we arrive at the equation

$$K_{X\wedge Y}(g)(x) - \frac{S(g)(x)}{n(n-1)} = 0$$

which holds for almost all $x \in \mathcal{G}$, provided that X and Y are fixed.

Without loss of generality we may assume that the latter equation holds for the following pairs of vector fields: X_r , X_s ; X_r , $(X_s + X_h)$; $(X_r + X_s)$, $(X_k + X_h)$, where r, s, h, k are different and take values from $1, 2, \ldots, n$. Then following the arguments of Lemma 2.1 we conclude that

$$R_{rs,kh} = \frac{S}{n\,(n-1)}\, g_{rs,kh}$$

for almost all $x \in \mathcal{G}$. The latter means that $K(g)$ is isotropic and by the Generalized Schur's Theorem $\langle \mathcal{M}, g\rangle$ is a space of constant curvature.■

2.3. Stability of the differential structure of almost isotropic spaces. A Riemannian manifold $\langle \mathcal{M}, g\rangle$ is said to be *ε-isotropic* if

$$\varepsilon(g) \le \varepsilon.$$

Let us consider Gromov's class of compact Riemannian manifolds $\mathfrak{M}(n,d,V,\Lambda)$. We claim the following statement.

Theorem 2.1. *Let $n \ge 3$. Then there is a positive constant $\varepsilon(n,d,V,\Lambda)$ such that any ε-isotropic Riemannian manifold $\langle \mathcal{M}, g\rangle \in \mathfrak{M}(n,d,V,\Lambda)$ with $0 \le \varepsilon \le \varepsilon(n,d,V,\Lambda)$ is diffeomorphic to a compact hyperbolic, flat or spherical space form.*

Remark 2.1. In the case of compact positively curved manifolds there is stronger result by E.Ruh [Ruh]. Loosely speaking in the theorem of Ruh ε depend only on the dimension. M.Gromov and W.Thurston constructed a sequence $(\langle \mathcal{M}_k, g_k\rangle)$ of closed Riemannian manifolds with curvatures satisfying

$$\lambda_k \le K_{\mathcal{M}_k} \le -1,$$

where $\lambda_k \longrightarrow -1$ as $k\longrightarrow\infty$ and $\langle \mathcal{M}_k, g_k\rangle$ carries no metric of constant negative curvature (see [G-T]). In our theorem the curvature may have arbitrary sign. The example by Gromov-Thurston shows that additional assumptions compared those of the theorem of Ruh are absolutely necessary. In Addendum we shall give more general results.

Remark 2.2. If in addition to hypothesis of Theorem 2.1 $\langle \mathcal{M}, g\rangle$ is simply connected then it is diffeomorphic to a sphere S^n.

Proof. □ On the contrary to the statement of Theorem 2.1 let us assume that there are a sequence $(\varepsilon_k)_{k=1,2,...}$ of positive numbers and a sequence of Riemannian manifolds $(\langle \mathcal{M}_k, g_k\rangle)_{k=1,2,...}$ in $\mathfrak{M}(n,d,V,\Lambda)$ satisfying the following conditions:

(i) $\varepsilon_k \longrightarrow 0$ as $k\longrightarrow\infty$

(ii) $\varepsilon(g_k) \le \varepsilon_k$

(iii) For sufficiently large k the manifold $\langle \mathcal{M}_k, g_k\rangle$ is not diffeomorphic to a compact space of constant curvature.

Making use of Cheeger's finiteness theorem [Ch] we can assume that the metrics $g_k, k = 1,2,...,$ are given on a fixed manifold $\mathcal{M}$. Gromov's compactness theorem and Proposition 2.2 are used to yield the metric g of constant curvature on the manifold $\mathcal{M}$. This obviously contradicts to (iii) and we complete the proof of Theorem 2.1.■

2.4. L_p-bound for the deviation of a curvature from a constant. Let $\langle \mathcal{M}, g\rangle$ be a compact space of bounded curvature and $p\in [1,+\infty)$. The set of measurable functions $f : \mathcal{M}\longrightarrow\mathbb{R}$ (w.r.t. the standard measure on $\mathcal{M}$) satisfying the inequality

$$|f|_{L_p(\mathcal{M})} = \left\{\int_{\mathcal{M}} |f(P)|^p \, d\,Vol(g)\right\}^{1/p} < +\infty$$

will be denoted by $L_p(\mathcal{M})$. The average value of the scalar curvature $S(P)$ of $(\mathcal{M},g)$ is defined as the quantity

$$S_a = \frac{1}{Vol(\mathcal{M})} \int_{\mathcal{M}} S(P) \, d\,Vol(g).$$

In particular, the notation just introduced can be applied to C^{∞}-Riemannian spaces.

Theorem 2.2. *Let $n \geq 3$ and $1 \leq p < +\infty$. Then given $\nu > 0$ there is $\varepsilon(n,d,V,\Lambda,p) > 0$ depending only on n,d,V,Λ and p such that*

$$|S(P) - S_a|_{L_p(\mathcal{M})} \leq \nu$$

for every ε-isotropic Riemannian manifold $(\mathcal{M},g)$ in $\mathfrak{M}(n,d,V,\Lambda)$ with $0 \leq \varepsilon \leq \varepsilon(n,d,V,\Lambda,p)$.

Remark 2.3. I.Gribkov [Gri1] constructed example of (an open) manifold with $\Lambda = p = +\infty$ for which the latter bound does not hold.

Remark 2.4. In the proof of Theorem 2.2 we are going to make use of modifications of Lemmas 1 and 2 in [Ruh]. We would like to emphasis that no new ideas are needed to prove these auxiliary lemmas besides those used by E.Ruh.However, the statements we need are somewhat different from those in [Ruh] and for the reader's convenience we want to give the proof.

In what follows we shall consider a harmonic system of coordinates $x:\mathcal{U}\subseteq\mathcal{M}\longrightarrow\mathcal{B}(4r)\subseteq\mathbb{R}^n$ in a compact Riemannian manifold $(\mathcal{M},g)$ (here $r>0$ and $\mathcal{B}(4r) = \{ x\in\mathbb{R}^n \mid |x| < 4r \}$), such that the bound (2.1) (where $\mathcal{G} = \mathcal{B}(2r)$) holds. We shall assume that

$$dim\ \mathcal{M} \geq 3.$$

We keep the notation $S(P)$ for the scalar curvature of $(\mathcal{M},g)$ and consider the averaging kernel $\phi(z)$ (see (i), (ii) and (iii) of sec.1.3) and a cut-off function $\theta:\mathbb{R}\longrightarrow\mathbb{R}$ with support in $|z| < 2r$ and $\theta(z) = 1$ for $|z| \leq r$. We introduce the notation:

$$\bar{S}(x) = \frac{1}{r^n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{r}\right) S(y)\, dy,$$

$$Q(x) = c_n^{-1}\int_{\mathbb{R}^n} \frac{\theta(|x-y|)}{|x-y|^{n-2}} [\bar{S}(y) - S(y)]\, dy, \quad x\in\mathcal{B}(r).$$

where $c_n = -n(n-2)\varkappa_n$, $\varkappa_n$ is equal to the volume of n-dimensional unit ball in $\mathbb{R}^n$.

Lemma 2.2. *Let $p\in(1,+\infty)$. Then the following bound holds*

$$max_{i,j=1,2,\ldots,n} \left\{ \left| \frac{\partial^2 Q}{\partial x^i \partial x^j} \right|_{L_p(B(r))} \right\} \leq C(n,d,V,\Lambda,p,r)\ \varepsilon(g),$$

where $C(n,d,V,\Lambda)$ is a constant depending on n,d,V,Λ,p and r.

<u>Proof.</u> □ Integration by parts implies

$$\frac{\partial^2 Q}{\partial x^i \partial x^j}(x) = - c_n^{-1} \int_{\mathbb{R}^n} \frac{\partial}{\partial x^i} \left(\frac{\theta(|x-y|)}{|x-y|^{n-2}} \right) \frac{\partial \bar{S}(y)}{\partial x^j}\, dy +$$

$$+ c_n^{-1} \int_{\mathbb{R}^n} \frac{\partial}{\partial x^i} \left(\frac{\theta(|x-y|)}{|x-y|^{n-2}} \right) \frac{\partial S(y)}{\partial x^j}\, dy. \tag{2.7}$$

Taking advantage of the obvious equation

$$\frac{\partial \bar{S}(x)}{\partial x^j} = r^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{y-x}{r}\right) \frac{\partial S(y)}{\partial x^j} dy$$

and (2.5) we arrive at the formula

$$\frac{\partial \bar{S}(x)}{\partial x^j} = r^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{y-x}{r}\right) (g^{kl}\ E_{kj})_{,l}\ (y)\ dy\ . \qquad (2.8)$$

(2.8), (2.1), Lemma 2.1 and integration by parts bring us the estimate

$$\left|\frac{\partial \bar{S}(x)}{\partial x^j}\right| \leq C'(n,d,V,\Lambda,r)\ \varepsilon(g),\ \ x\in\mathcal{B}(r). \qquad (2.9)$$

Inequality (2.9) together with the smoothness property of the kernel yield the the bound of the first integral in (2.7) claimed in Lemma 2.2.

Similar trick is used to yield the bound for the second integral in (2.7). Namely, we replace $\partial S/\partial y^j$ by $(g^{kl}\ E_{kj})_{,l}$. By integrating by parts we arrive at integrals the worst of which has the kernel of the form

$$\Psi_{kl}(x,y) = \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^l} \left(\frac{\theta(|x-y|)}{|x-y|^{n-2}}\right) .$$

We observe that for $|x-y| < r$ this is a singular kernel (see [Ca-Z]) and so, the Calderon-Zygmund inequality is applicable. This bring us the desired estimate of the second integral in (2.7).■

We denote by Δ the Laplace operator, i.e. $\delta^{kl} \dfrac{\partial^2}{\partial x^k \partial x^l}$.

Lemma 2.3. *Let $p \in (1,+\infty)$. Then the following bound holds*

$$|\Delta\, Q - (\bar{S} - S)|_{L_p(\mathcal{B}(r))} \le C(n,d,V,\Lambda,p,r)\ \varepsilon(g),$$

where $C(n,d,V,\Lambda)$ is a constant depending on n,d,V,Λ,p and r.

Corollary 2.2. $|\bar{S} - S|_{L_p(\mathcal{B}(r))} \le 2\ C(n,d,V,\Lambda,p,r)\ \varepsilon(g)$.

Proof of Lemma 2.3.□ Let $\zeta(y)$ be a cut-off function such that $\zeta(y) = 1$ for $|y| \le 3r$ and $\zeta(x) = 0$ for $|y| \ge 7r/2$. Since $|x - y| \ge |y| - |x|$, it follows that for $x \in \mathcal{B}(r)$ and $|y| \ge 3r$ the function $\theta(|x-y|)$ vanishes. Hence, one may replace the function $(\bar{S} - S)$ by the function $\zeta\ (\bar{S} - S)$ which is compactly supported in $\mathbb{R}^n$ and equal to $(\bar{S} - S)(y)$ for $|y| \le 3r$. So, we can express $\bar{S} - S$ as follows:

$$\bar{S}(x) - S(x) = c_n^{-1} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}\ \Delta\ ([\bar{S}(y) - S(y)]\ \zeta(y))\ dy.$$

Taking into account that $\theta(|x-y|) = 1$ for $|x-y| \le r$, one can write the difference $\Delta\ Q - (\bar{S} - S)$ in the form:

$$I(x) = c_n^{-1} \int_{|x-y| \ge r} \left\{ \frac{\theta(|x-y|) - 1}{|x-y|^{n-2}}\ \Delta\ \bar{S}(y)\ \zeta(y) \right\} dy\ -$$

$$- c_n^{-1} \int_{|x-y| \ge r} \left\{ \frac{\theta(|x-y|) - 1}{|x-y|^{n-2}}\ \Delta\ (S(y)\ \zeta(y)) \right\} dy\ , \quad x \in \mathcal{B}(r). \qquad (2.10)$$

Similar to the proof of the estimates for the two integrals in (2.7) after integration by parts and taking advantage of (2.5) and (2.9) we establish the desired bounds for integrals in (2.10). We only want to remark that the function

$$\frac{\theta(|x-y|)-1}{|x-y|^{n-2}}$$

is at least twice-continuously differentiable for $|x-y| \geq r$. So, here we do not need the Calderon-Zygmund inequality.■

Proof of Theorem 2.2. □ On the contrary to the statement of Theorem 2.2 let us assume that there are a sequence $(\varepsilon_k)_{k=1,2,...}$ of positive numbers , a sequence of metric tensors $(g_k)_{k=1,2,...}$ on a fixed manifold $\mathcal{M}$ and a positive number ν satisfying the following conditions:

(i) $\varepsilon_k \longrightarrow 0$ as $k \longrightarrow \infty$

(ii) $\varepsilon(g_k) \leq \varepsilon_k$

(iii) The sequence $(\langle \mathcal{M}, g_k \rangle)_{k=1,2,...}$ lies in $\mathfrak{M}(n,d,V,\Lambda)$

(iv) The following inequality holds

$$|S^{(k)}(P) - S_a^{(k)}|_{L_p(\mathcal{M})} \geq \nu, \quad p \in [1,+\infty),$$

where we denote by $S^{(k)}(P)$ and $S_a^{(k)}$ the scalar and average value of the scalar curvature $S^{(k)}$ w.r.t. the metric tensor g_k , where $k = 1,2,...$.

Let us consider the Sobolev averaging of the function $S^{(k)}(P)$ and the constant function $S_a^{(k)}$ w.r.t. r. We have the identities:

$$\bar{s}^{(k)}(x) = \frac{1}{r^n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{r}\right) s^{(k)}(y)\, dy,$$

$$s_a^{(k)} = \bar{s}_a^{(k)} = \frac{1}{r^n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{r}\right) s_a^{(k)} dy.$$

Gromov's compactness theorem and Proposition 2.2 ensure the existence of a subsequence $\{\langle \mathcal{M}, g_{k_l} \rangle\}_{l=1,2,...}$ of the sequence $\{\langle \mathcal{M}, g_k \rangle\}_{k=1,2,...}$ C^1-converging to a space of constant curvature $\langle \mathcal{M}, g \rangle$. Without loss of generality we may assume that

$$\{\langle \mathcal{M}, g_{k_l} \rangle\}_{l=1,2,...} = \{\langle \mathcal{M}, g_k \rangle\}_{k=1,2,...}.$$

Corollary 2.1 yields that

$$\bar{s}^{(k)}(x) \longrightarrow \bar{s}(x), \; s_a^{(k)} \longrightarrow s_a, \text{ as } k \longrightarrow +\infty. \tag{2.11}$$

Let us first obtain estimates for $p \in (1, +\infty)$.

We observe that due to standard properties of the Sobolev averaging and (iii)

$$\sup_{k=1,2,...} \left(|\bar{s}^{(k)}(x)|_{L_\infty(\mathcal{B}(r))} \right) < +\infty. \tag{2.12}$$

By Lebesgue theorem (see (2.11) and (2.12)) we conclude that

$$|\bar{s}^{(k)}(x) - \bar{s}(x)|_{L_p(\mathcal{B}(r))} \longrightarrow 0, \text{ as } k \longrightarrow +\infty. \tag{2.13}$$

The triangle inequality implies

$$|S^{(k)}(x) - S_a^{(k)}|_{L_p(\mathcal{B}(r))} \le |S^{(k)}(x) - \bar{S}^{(k)}(x)|_{L_p(\mathcal{B}(r))} +$$

$$+ |\bar{S}^{(k)}(x) - \bar{S}(x)|_{L_p(\mathcal{B}(r))} + |\bar{S}(x) - S_a|_{L_p(\mathcal{B}(r))} +$$

$$+ |S_a^{(k)} - S_a|_{L_p(\mathcal{B}(r))} . \tag{2.14}$$

Since $\langle \mathcal{M}, g\rangle$ is a space of constant curvature, $\bar{S}(x)$ is the Sobolev averaging of the constant S_a and, hence

$$|\bar{S}(x) - S_a|_{L_p(\mathcal{B}(r))} = 0 .$$

In the meantime, Corollary 2.2 together with (i) and (ii) yields that the first summand in the right-hand side of (2.14) converges to 0 as $k \longrightarrow +\infty$. On account of (2.13) the second summand in the same expression converges to 0 too. The equality to 0 of the last summand in (2.14) is ensured by (2.11). Thus we arrive at the equality

$$\lim_{k \longrightarrow +\infty} |S^{(k)}(x) - S_a^{(k)}|_{L_p(\mathcal{B}(r))} = 0 \tag{2.15}$$

for $p \in (1, +\infty)$.

Since the imbedding operator $\mathcal{I}_{pq} : L_p(\mathcal{G}) \longrightarrow L_q(\mathcal{G})$ $(p \ge q \ge 1)$ is bounded with norm $\le Vol\ (\mathcal{G})^{1/q - 1/p}$, (2.15) holds also for $p = 1$.

Because of (iii) there is a constant $r_0(n, d, V, \Lambda) > 0$ depending only on n, d, V, and Λ such that r can be chosen no less than $r_0(n, d, V, \Lambda)$ [J-K] . Together with the bound on the diameter this implies that for any $p \in [1, +\infty)$

$$\lim_{k\to+\infty} |S^{(k)}(x) - S^{(k)}_a|_{L_p(\mathcal{M})} = 0.$$

This is a contradiction with (iv) and we complete the proof of the theorem.■

2.5. W^2_p-stability of the metric of almost isotropic space. We start with the following proposition.

Proposition 2.3. *Let $\langle\mathcal{M},g\rangle \in\mathfrak{M}(n,d,V,\Lambda)$ and $p\in[1,+\infty)$. Then there are positive constants $C(n,d,V,\Lambda,p)$ and $r(n,d,V,\Lambda)$ depending on n,d,V,Λ,p and n,d,V,Λ respectively such that for any point $P\in\mathcal{M}$ there is a chart $x:\mathcal{B}(P,r)\subseteq\mathcal{M}\longrightarrow\mathcal{G} = (\ |x| < 1\ ) \subseteq\mathbb{R}^n$ such that components (g_{ij}) of g w.r.t. x satisfy the inequality:*

$$|g_{ij}|_{W^2_p(\mathcal{G})} \le C(n,d,V,\Lambda,p).$$

Proof. □ Let us consider harmonic systems of coordinates constructed in [J-K] by means of almost linear functions. Due to what was done in this work we only have to prove the bound claimed in Proposition 2.3. We observe that components (g_{ij}) of the metric tensor g satisfy elliptic equations

$$g^{sl}\ \frac{\partial^2 g_{ij}}{\partial x^s \partial x^l} = 2R_{ij} + \mathrm{I}(g), \tag{2.16}$$

where $\mathrm{I}(g)$ depends only on components of the metric tensor and its first derivatives.

Let us consider C^∞-smooth cut-off function ζ compactly supported in $|x| < 2$ and equal to 1 in the ball $|x| < 1$. Without loss of generality on may assume that $C^{1,\alpha}$-bounds for g_{ij} holds in the ball $|x| \le 2$. Then applying theorem 15.1 in [L-U] to the function ζg_{ij} and taking into account (2.16) we arrive at the desired bound. ■

We keep the notation $\mathfrak{h}_0(g)$ for the $C^{3,\alpha}$-smooth ($\alpha\in(0,1)$) atlas on $\mathcal{M}$ formed by charts x from Proposition 2.3.

Let us denote by $\mathfrak{M}_\varepsilon(n,d,V,\Lambda)$ the subset in $\mathfrak{M}(n,d,V,\Lambda)$ consisting of ε-isotropic Riemannian spaces.

Theorem 2.3. *Let $n \geq 3$ and $p\in[1,+\infty)$. Then for any $\nu > 0$ there is a positive number $\varepsilon(n,d,V,\Lambda,p,\nu)$ depending only on n,d,V,Λ,p and ν such that for every $\langle\mathcal{M},g\rangle \in\mathfrak{M}_\varepsilon(n,d,V,\Lambda)$ with $\varepsilon = \varepsilon(n,d,V,\Lambda,p,\nu)$ there exists a metric g^c of constant curvature on $\mathcal{M}$ for which the following inequality holds:*

$$|g_{ij} - g^c_{ij}|_{W^2_p(\mathfrak{g})} \leq \nu$$

w.r.t. every chart $x\in\mathfrak{h}_0(g)$.

<u>Proof.</u> □ On the contrary to the statement of Theorem 2.3 let us assume that there are a sequence $(\varepsilon_k)_{k=1,2,\ldots}$ of positive numbers , a sequence of metric tensors $(g_k)_{k=1,2,\ldots}$ on a fixed manifold $\mathcal{M}$ and a positive number ν, satisfying the conditions (i)-(iii) (see the proof of Theorem 2.2) and

(iv) The inequality (w.r.t. any chart $x\in\mathfrak{h}_0(g_k)$)

$$|(g_k)_{ij} - g^c_{ij}|_{W^2_p(\mathfrak{g})} \leq \nu,$$

does not hold for every metric of constant curvature on $\mathcal{M}$.

By Gromov's compactness theorem and Proposition 2.2 we may assume that the sequence $(\langle\mathcal{M},g_k\rangle)_{k=1,2,\ldots}$ is C^1-convergent to a compact space of bounded curvature $\langle\mathcal{M},g^c\rangle$. We want to prove that the metric g^c satisfies the equality

$$\lim_{k\to+\infty} |(\tilde{g}_k)_{ij} - g^c_{ij}|_{W^2_p(\mathcal{G})} = 0,$$

where $\tilde{g}_k$ was defined in sec.2.1.

According to (2.16) we consider the differential operator

$$L_g(u) = g^{sl} \frac{\partial^2 u}{\partial x^s \partial x^l} , \quad u\in C^2(\mathcal{G}).$$

Let $\zeta(x)$ be the cut-off function, defined in Proposition 2.3. On account of (2.16) the function $u(x) = \zeta(x)\, g_{ij}(x)$ satisfies the equation:

$$L_g(u) = 2\,\zeta\, R_{ij} + \zeta\, \mathrm{I}(g) + L_g(\zeta)\, u + L_g(\zeta,u), \qquad (2.17)$$

where

$$L_g(\zeta,u) = g^{sl} \frac{\partial u}{\partial x^s} \frac{\partial \zeta}{\partial x^l} .$$

For the sake of brevity we introduce the notation

$$u_k(x) = \zeta(x)\, (\tilde{g}_k)_{ij}(x), \quad L_{\tilde{g}_k} = L_k\ , \quad L_g = L,$$

Then the difference $u_k - u$ satisfies the elliptic equation

$$L(u_k - u) = L_k(u_k) - L(u) - (L_k - L)(u_k).$$

Proposition 2.3 and the property of C^1-convergence ensure that the L_p-norm of $(L_k - L)(u_k)$ converges to zero as $k \to \infty$. On the other hand, Theorem 2.2 together with the property of C^1-convergence implies that the L_p-norm of $L_k(u_k) - L(u)$ converges to zero too (see (2.17)). Hence, the function $h_k = u_k - u$ satisfies the equation

$$L(h_k) = \Phi_k ,$$

where

$$\lim_{k \to +\infty} |\Phi_k|_{L_p(\mathcal{G}')} = 0, \quad \mathcal{G}' = (x : |x| \le 2),$$

with the boundary condition

$$h_k(x) = 0, \quad |x| = 2.$$

In addition, the property of C^1-convergence implies that

$$\lim_{k \to +\infty} |h_k|_{C(\mathcal{G}')} = 0.$$

These equalities together with (11.8) of chapter III in [L-U] imply for $p > n$ that

$$|(\tilde{g}_k)_{ij} - g^c_{ij}|_{W^2_p(\mathcal{G})} \le |h_k|_{W^2_p(\mathcal{G}')} \longrightarrow 0 \text{ as } k \to +\infty. \qquad (2.18)$$

Boundedness of the imbedding operator $\mathcal{J}_{pq} : L_p(\mathcal{G}) \longrightarrow L_q(\mathcal{G})$ for $p > n \ge q \ge 1$ ($\mathcal{G}$ is bounded domain) implies that (2.18) holds also for $1 \le p \le n$ and hence, for $p \in [1, +\infty)$.

Let $i_k : (\mathcal{U}, g_k) \longrightarrow (\mathcal{U}, g)$ be a local diffeomorphism which is identification via harmonic coordinates $x_k \in \mathfrak{H}_0(g_k)$ and $x \in \mathfrak{H}_0(g)$, namely we assign to a point P with harmonic coordinates $(x^1, x^2, \ldots, x^n)$ w.r.t. the chart x_k the point P' with the same coordinates w.r.t. the chart x. By definition $g_k = i_k^*(\tilde{g}_k)$. We consider the metric $g_k^c = i_k^*(g^c)$. Due to what was done in [G-W] the diffeomorphism i_k can be extended to a diffeomorphism $i_k' \; \mathcal{M} \longrightarrow \mathcal{M}$ so that in fact g_k^c is given on the whole $\mathcal{M}$. On account of (2.18) to complete the proof of the theorem we only have to prove that

$$\lim_{k \to +\infty} |g_{ij}^c - (g_k^c)_{ij}|_{W_p^2(g)} = 0 .$$

To prove the latter statement it is sufficient to establish that

$$\lim_{k \to +\infty} |i_k' - id|_{C^3(g)} = 0 , \qquad (2.19)$$

where $id\,(P) = P$ for $P \in \mathcal{M}$.

Due to [G-W] (2.19) holds for $C^{2,\alpha}$-norms ($\alpha \in (0,1)$). We are going to explain how one can improve this bound to $C^{3,\alpha}$-norms.

Shortly speaking the proof of (2.19) w.r.t. the $C^{2,\alpha}$-norm in [G-W] was as follows. The authors made use of harmonic coordinates. Due to [J-K] one may assume that the bound

$$|(g_k)_{ij}|_{C^{1,\alpha}} < c(n, d, V, \Lambda, \alpha)$$

holds. Since a harmonic function $h : \mathcal{U} \longrightarrow \mathbb{R}$ satisfies the elliptic equation

$$div\ grad\ h = g_k^{sl} \frac{\partial^2 h}{\partial x^s \partial x^l} - g_k^{ij} (\Gamma_k)^l_{ij} \frac{\partial h}{\partial x^l} = 0, \tag{2.20}$$

it is not difficult to establish then that $h \in C^{2,\alpha}$ for any $\alpha \in (0,1)$. In particular, for two harmonic systems of coordinates $x: \mathcal{U} \subseteq \mathcal{M} \longrightarrow \mathcal{G}$ and $y: \mathcal{V} \subseteq \mathcal{M} \longrightarrow \mathcal{G}_1$ on $\langle \mathcal{M}, g \rangle$ the function $x \circ y^{-1}$ belongs to class $C^{2,\alpha}$ provided that $\mathcal{G} \cap \mathcal{G}_1 \neq \emptyset$ and $|x \circ y^{-1}|_{C^{2,\alpha}}$ depends only on n, d, V, Λ and α. This and also the Arcela-Ascoli Theorem bring us that local diffeomorphism i_k converges to id in the $C^{2,\alpha}$-norm as k converges to $+\infty$. For a given k the local diffeomorphisms i_k can be "glue together" to get a global diffeomorphism i_k' by taking advantage of the standard center of mass technique w.r.t. a fixed C^∞-smooth Riemannian metric on $\mathcal{M}$. We observe that in harmonic coordinates the following equalities hold

$$g_k^{ij} (\Gamma_k)^l_{ij} = 0, \quad l = 1, 2, \dots, n,$$

and therefore (2.20) in fact brings us the stronger bound

$$|x \circ y^{-1}|_{C^{3,\alpha}} \le c(n, d, V, \Lambda, \alpha) \ .$$

This completes the proof of (2.19) and hence, Theorem 2.3.■

ADDENDUM: SPACES WITH SMALL INTEGRAL ANISOTROPY

Here we shall state a stronger version of Proposition 2.2 and claim theorems concerning stability of differential structure and metric of spaces with small integral anisotropy. We would like to remark that bounds of the curvature anisotropy $\varepsilon(g,P)$ of $\langle\mathcal{M},g\rangle$ at a point $P\in\mathcal{M}$ are equivalent to bounds of the norm of the tensor

$$\tilde{E}_{ij,kl} = R_{ij,kl} - \frac{S}{n\,(n-1)}\, g_{ij,kl}\;,$$

see sec. 2.2 and c) of Theorem 1 in [Gr3]. So, in our statements $|\varepsilon(g,P)|$ can be replaced with $|\tilde{E}(P)|$.

First of all we generalize Proposition 2.2 as follows.

Proposition A.1. *Let $\{\langle\mathcal{M},g_k\rangle\}_{k=1,2,\ldots}$ be a sequence of Riemannian spaces in Gromov's class $\mathfrak{M}(n,d,V,\Lambda)$ for which the following condition is satisfied:*

$$\lim_{k\to+\infty}\int_{\mathcal{M}} \varepsilon(g_k,P)\;\psi(P)\;d\,Vol(g_k) = 0 \tag{A.1}$$

for every C^{∞}-smooth and compactly supported function ψ on $\mathcal{M}$.

Then every limit space of the sequence $\{\langle\mathcal{M},g_k\rangle\}_{k=1,2,\ldots}$ w.r.t. the Lipschitz distance is isometric to a Riemannian space of constant curvature.

Corollary A.1. *The statement of Proposition A.1 is true provided that (A.1) is replaced with the condition*

$$|\varepsilon(g_k,P)|_{L_1(\mathcal{M})} \longrightarrow 0 \text{ as } k\to\infty\;.$$

We want to remark that convergence of the sequence $\{\varepsilon(g_k,P)\}_{k=1,2,\ldots}$ to zero in L_1-norm implies the existence of a subsequence converging to zero a.e. Since $\varepsilon(g_k,P)$ are uniformly bounded and P belongs to a bounded domain, the Lebesgue Theorem yields that the subsequence converges to zero w.r.t. L_p-norm for any $p\in[1,+\infty)$.

Proof of Proposition A.1. □ Let us keep the notation of Proposition 2.2. On multiplying (2.6) by $\phi_{\eta,x}(u)$ and integrating over domain $\mathcal{G}$ we arrive at the inequalities

$$- \int_{\mathcal{G}} \varepsilon(g_k)\ \phi_{\eta,x}(u)\ |g_k(u)|\ du \ \le$$

$$\le \int_{\mathcal{G}} \left(K_{X\wedge Y}(g_k)(u) - \frac{S(g_k)(u)}{n(n-1)}\right) \phi_{\eta,x}(u)\ |g_k(u)|\ du \ \le$$

$$\le \int_{\mathcal{G}} \varepsilon(g_k)\ \phi_{\eta,x}(u)\ |g_k(u)|\ du\ , \tag{A.2}$$

where we denote by $|g_k(u)|$ the determinant of the matrix $(g_{ij}(u))_{i,j=1,2,\ldots,n}$.

By hypothesis the left-hand side and the right-hand side of (A.2) converge to zero as $k\to+\infty$. Taking into account (2.1) we arrive at somewhat different conclusion compared that of Proposition 2.2: for every $\eta > 0$ the Sobolev averaging of the function

$$K_{X\wedge Y}(g)(u)\ |g(u)|$$

coincides with the Sobolev averaging of the function

$$\frac{S(g)(w)}{n(n-1)}|g(w)|$$

everywhere in $\mathcal{G}$. This implies the equality of these functions a.e. in $\mathcal{G}$. Since the function $|g(w)|$ is strictly positive everywhere in $\mathcal{G}$, we arrive at the equality of $K_{X\wedge Y}(g)(w)$ to $S(g)/n(n-1)$ a.e. in $\mathcal{G}$ and the conclusion that the limit space $\langle \mathcal{M}, g\rangle$ is isotropic. Then we complete the proof as in Proposition 2.2. ■

As a corollary of Proposition A.1 we obtain the following generalization of Theorem 2.1.

Let us introduce the set $\tilde{\mathfrak{M}}(n, V, \varkappa)$ of compact (closed) C^∞-Riemannian manifolds $\langle \mathcal{M}, g\rangle$ with $dim(\mathcal{M}) = n$, $Vol(\mathcal{M}) \geq V > 0$ and $diam^2(\mathcal{M})\ |K(g)|_{C^0(\mathcal{M})} \leq \varkappa$. Here we have introduced the notation $|K(g)|_{C^0(\mathcal{M})}$ for the maximal absolute value of sectional curvatures.

We say that a compact Riemannian space $\langle \mathcal{M}, g\rangle$ *has ε-small integral anisotropy* if

$$|\varepsilon(g,P)|_{L_1(\mathcal{M})} \leq \varepsilon .$$

It is obvious that every Riemannian space $\langle \mathcal{M}, g\rangle$ in $\tilde{\mathfrak{M}}(n, V, \varkappa)$ is conformally equivalent to a space in $\mathfrak{M}(n, \varkappa, V, 1)$. Therefore, one can repeat the proof of Theorem 2.1 and apply Proposition A.1 and Corollary A.1 to get the following statement.

Theorem A.1. *Let $n \geq 3$. Then there is a positive constant $\varepsilon(n,\varkappa,V)$ depending on $n, \varkappa$ and V such that any Riemannian manifold $\langle \mathcal{M}, g\rangle \in \tilde{\mathfrak{M}}(n, \varkappa, V)$ with ε-small integral anisotropy for $0 \leq \varepsilon \leq \varepsilon(n,\varkappa,V)$ is diffeomorphic to a compact hyperbolic, flat or spherical space form.*

Remark A.1. In the case of negatively curved manifolds of dimension greater than 3 the volume of a manifold can be bounded below by diameter (see [Gr2],1.2). Therefore, in this case Theorem A.1 can be stated as follows.

Let $n \geq 4$ and $\langle M,g\rangle$ be a compact negatively curved Riemannian manifolds, i.e. $-|K(g)|_{C^0(\mathcal{M})} \leq K(g) < o$ *. Let us assume that one of the following conditions holds:*

(i) $\operatorname{diam}(\mathcal{M})^2 \ |K(g)|_{C^0(\mathcal{M})} \leq \varkappa$,

(i)' $\operatorname{Vol}(\mathcal{M})^{2/n} \ |K(g)|_{C^0(\mathcal{M})} \leq \varkappa$.

Then there is a positive number $\varepsilon(n,\varkappa)$ depending only on n and $\varkappa$ such that every n-dimensional compact Riemannian manifold with ε-small integral anisotropy for $\varepsilon \in [0,\varepsilon(n,\varkappa)]$ satisfying (i) or (i)' is diffeomorphic to a hyperbolic space form.

The case of positively curved manifolds is contained in Theorem A.1. If one restricts himself by considering *1/4*-pinched manifolds then ε can be chosen depending only on n.

We want to note that while Theorem A.1 does not give explicit bound on $\varepsilon(n,d,\varkappa)$ it gives the stability of differential structure under weaker assumption of integral smallness of anisotropy and is applicable to Riemannian manifolds with curvature of arbitrary sign (compare with [Ye],Theorems 4 and 4', see also introduction in [Ye]).

Now we turn to a stability of a metric of spaces with small integral anisotropy. We observe that $\varepsilon(g)$ in the estimates of Lemmas 2.2 and 2.3 can be replaced with

$$\varepsilon_p(g) = \sup_{P\in\mathcal{M}} \left(|\varepsilon(g,P)|_{L_p(\mathcal{M})} \right)$$

and hence, we arrive at the following statement.

Theorem A.2. *Let $n \geq 3$ and $1 \leq p < +\infty$. Then given $\nu > 0$*

there is $\varepsilon(n,d,V,\Lambda,p) > 0$ depending only on n,d,V,Λ and p such that

$$|S(P) - S_a|_{L_p(\mathcal{M})} \le \nu$$

for every Riemannian manifold $\langle\mathcal{M},g\rangle$ in $\mathfrak{M}(n,d,V,\Lambda)$ with ε-small integral anisotropy for any $\varepsilon \in [0, \varepsilon(n,d,V,\Lambda,p)]$.

As a corollary we claim the W^2_p-stability for spaces with small integral anisotropy.

We say that metric g^c is *ν-close to g w.r.t. W^2_p-norm* if the bound stated in Theorem 2.3 holds.

Theorem A.3. *Let $n \ge 3$ and $p\in[1,+\infty)$. Then for any $\nu > 0$ there is a positive number $\varepsilon(n,d,V,\Lambda,p,\nu)$ depending only on n,d,V,Λ,p and ν such that for every $\langle\mathcal{M},g\rangle \in\mathfrak{M}(n,d,V,\Lambda)$ with ε-small integral anisotropy for $\varepsilon\in[0,\varepsilon(n,d,V,\Lambda,p,\nu)]$ there exists a metric g^c of constant curvature on $\mathcal{M}$ which is ν-close to g w.r.t. W^2_p-norm.*

For the sake of brevity we shall say that metrics of $\mathfrak{M}(n,d,V,\Lambda)$ *are W^2_p-stable in the class of metrics of constant curvature w.r.t. the integral anisotropy.*

Let us denote by $\mathfrak{M}(n,d,V;-\Lambda,0)$ the subset of $\mathfrak{M}(n,d,V,\Lambda)$ consisting of manifolds with curvatures satisfying the bound

$$-\Lambda \le K(g) < 0 .$$

Corollary A.2. *Let $n \ge 4$. Then the metrics of $\mathfrak{M}(n,d,+\infty;-\Lambda,0)$ and $\mathfrak{M}(n,+\infty,V;-\Lambda,0)$ are W^2_p-stable in the class of metrics of constant negative curvature w.r.t. the integral anisotropy for every $p\in[1,+\infty)$.*

Let us denote by $\mathfrak{M}(n,+\infty,+\infty;1/4,1)$ the set of $1/4$-pinched Riemannian manifolds, i.e.

$$1/4 < K(g) \leq 1 \ .$$

Corollary A.3. *Let $n \geq 3$. Then the metrics of $\mathfrak{M}(n,+\infty,+\infty;1/4,1)$ are W^2_p-stable in the class of metrics of constant positive curvature w.r.t. the integral anisotropy for every $p\in[1,+\infty)$.*

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