Geodesics on a regular dodecahedron

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Abstract

We classify all non-self-intersecting closed geodesics on regular dodecahedra and give some results and conjectures regarding the self-intersecting case. Geodesics on regular polyhedra of all other types were studied in [2].

1 Introduction.

Let S be a polyhedron in space. A *geodesic* on S is a locally shortest curve not passing through the vertices (the second condition follows from the first one if the surface is convex). Obviously, within any face, any geodesic is straight, and at the point of intersection with an edge, a geodesic forms equal angles with the edge in the two adjacent faces. (Conversely, a polygonal line with these properties is a geodesic.)

It is convenient to study geodesics on polyhedra using their *planar developments*. Let a geodesic on a polyhedron S start at a point X of an edge AB and then go into a face $ABC\dots$ of the polyhedron. Let it arrive at some point Y at another edge of the same face, DE, and then pass to a new face, $DEF\dots$. Draw the faces $ABC\dots$ and $DEF\dots$ sharing the edge DE in a plane, and continue doing this along our geodesic (Figure 1).

If the polyhedron is convex, we can visualize this process as a rolling of the polyhedron along the plane in such a way that the geodesic always touches the plane. The *development* of the polyhedron shown in Figure 1 consists of the traces of the faces of the polyhedron and the geodesic. The geodesic becomes a straight line, and it is closed if this straight line arrives at the initial face $ABC\dots$, at the point X' at the same position as X on the edge AB, and the new face $ABC\dots$ is the translation image of the old one.

Remark It is clear (and also shown in Figure 1) that every closed geodesic on a polyhedron belongs to a family of "parallel" geodesics which travel through the same faces and have the same length.

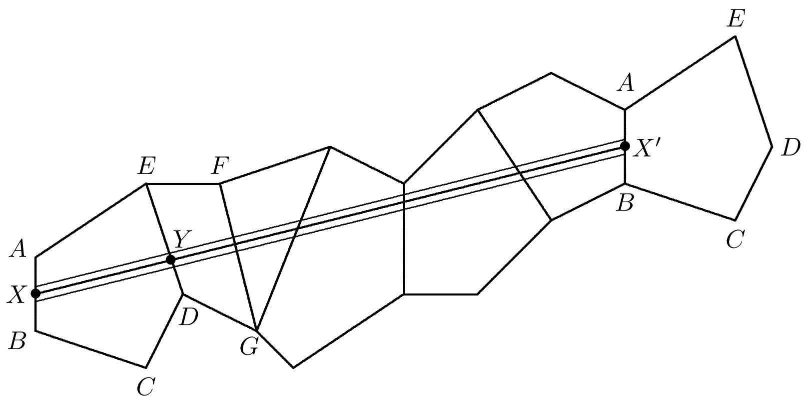

Figure 1: The development of a geodesic

It seems plausible that a polyhedron should be sufficiently symmetric to have a rich family of closed geodesics (see [2] for explanations). Hence, the first case to be studied is that of regular polyhedra. The cases of cubes and regular tetrahedra, octahedra and icosahedra was studied in detail in article [2] (some of results were obtained in an earlier article [1]). In these four cases, the development of geodesics, both closed and non-closed, follow the squares or equilateral triangles from the most classical tilings of the plane, which essentially reduces the problems of classification of geodesics to algebraic problems concerning finite groups.

Here are the results of [2]. In the case of a regular tetrahedron all (non-multiple) closed geodesics are non-self-intersecting, their lengths are unbounded. In the case of a cube, a full description of non-self-intersecting closed geodesics and an almost full description of all closed geodesics are given (with one natural question remaining open). For regular octahedra, [2] contains a full description of both self-intersecting and non-self-intersecting geodesics. For regular icosahedra only partial results including a description of non-self-intersecting geodesics are obtained. It should be mentioned also that (up to rotations and parallelism), there are three closed geodesics on a cube (the squares of their lengths are 16, 18, and 20), two closed geodesics on a regular octahedron (the squares of the lengths are 9 and 12), and three closed geodesics on a regular icosahedron (the squares of the lengths are 25, 27, and 28) (in all cases, we assume that the edge of the regular polyhedron has the length 1.

The case of a dodecahedron is very different. There is no such a thing as a tiling of the plane by regular pentagons. The developments of geodesics (see Figures 2, 6, and 8 below) involve chains of regular pentagons in which even-numbered and odd-numbered members are translationally equivalent. This shows, among other things, that within a face, a geodesic (closed or not) can have 5 directions (differing by multiples of 72°). Informally speaking, the variety of closed geodesics on a regular tetrahedron is more rich than a similar variety for other types of regular polyhedra. We prove below that there are precisely 5 essentially different non-self-intersecting closed geodesics on a regular dodecahedron (Section 3) and present some results. both proved and experimental, about all closed geodesics. In particular, we prove that there are infinitely many non-equivalent simple closed geodesics (Section 4).

This work was all done during my stay at the Max Planck Institute at Bonn, and I take this opportunity to express my deep gratitude to the Institute for its hospitality. It is worth mentioning that the regular icosahedron is the logo of the Max Planck Institute, so it seems the most appropriate to publish this study of geodesics on regular dodecahedra in the MPIM preprint series.

In what follows, a geodesic always means a geodesic on a regular dodecahedron, whether it is stated explicitly or not.

2 Lengths and directions of closed geodesics.

2.1 Lengths.

Proposition 2.1 *The square of the length of any closed geodesic on a regular dodecahedron with edges of length 1 has the form $a + b\tau$ where a, b are integers and τ is the golden ratio.*

Proof Let AB be the development of a closed geodesic on the plane (see Fig. 1). We assume that the endpoints A, B (representing the same point of the geodesic) lie on horizontal sides of two parallel faces with the centers P, Q. Then $|AB| = |PQ|$.

The vector $\overrightarrow{PQ}$ is the sum of even number, $2n$, of vectors of the same length, $\cot\dfrac{\pi}{5}$ and each of them forms with the horizontal direction the angle of the form $\dfrac{k\pi}{10}$ with k odd. Moreover, the numbers $k_1, k_2, \ldots, k_{2n}$ corresponding to these $2n$ vectors have alternating residues modulo 4 (and $k_{2n} \equiv 5 \bmod 20$, but this is not important for us at the moment). For $k = -9, -7, -5, \ldots, 7, 9$, let $n_k = \#\{m \mid k_m \equiv k \bmod 20\}$; the alternating

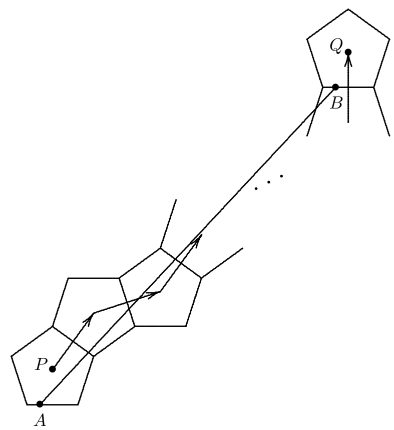

Figure 2: To Proof of Proposition 1

property above impliues that

$$n_{-9} - n_{-7} + n_{-5} - \cdots - n_9 = 0.$$

Thus,

$$\overrightarrow{PQ} = \left(\sum_k n_k \cos \frac{k\pi}{10}, \sum_k n_k \sin \frac{k\pi}{10} \right) \cdot \cot \frac{\pi}{5}$$

and the square of the length of this vector is

$$\left(\sum_k n_k^2 + 2 \sum_{m=1}^{9} N_m \cos \frac{m\pi}{5} \right) \cdot \cot^2 \frac{\pi}{5}$$

where $N_m = \sum\limits_{k-\ell=2m} n_k n_\ell$. Using the equalities

$$2\cos\frac{\pi}{5} = \tau,\ 2\cos\frac{2\pi}{5} = \tau - 1,\ \cot^2\frac{\pi}{5} = \frac{3+4\tau}{5},$$
$$\tau(3+4\tau) = 4 + 7\tau,\ (\tau-1)(3+4\tau) = 1 + 3\tau$$

we get

$$\begin{aligned} 5\|\overrightarrow{PQ}\|^2 = (3+4\tau)\sum n_k^2 \quad & +(4+7\tau)(N_1 - N_4 - N_6 + N_9) \\ & +(1+3\tau)(N_2 - N_3 - N_7 + N_8) - (6+8\tau)N_5 \end{aligned}$$

.Subtracting

$$\begin{aligned} 0 \quad &= (3+4\tau)(n_{-9} - n_{-7} + n_{-5} - \cdots - n_9)^2 \\ &= (3+4\tau)\sum n_k^2 + 2(3+4\tau)(-N_1 + N_2 - \cdots - N_9), \end{aligned}$$

we get

$$5\|\overrightarrow{PQ}\|^2 = (10+15\tau)(N_1 - N_4 - N_6 + N_9) - (5+5\tau)(N_2 - N_3 - N_7 + N_8),$$

which implies our statement.

Remark It is not hard to prove that the integers a, b in Proposition 1 are both non-negative. We do not need this statement and do not prove it.

2.2 Directions.

Let γ be an oriented geodesic on a regular dodecahedron (closed or not). When γ enters a face, it crosses an edge, and we can compute the angle from the edge (edges have canonical orientations within every face) to the geodesic. Along the geodesic, these angles form a sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$, periodic if the geodesic is closed. The following fact is obvious:

$$\forall n,\ \alpha_{n+1} = \alpha_n + \frac{k\pi}{5},\ \text{where } k = \pm 1 \text{ or } \pm 3.$$

(By the way, this formula implies the fact which is obvious anyhow: *the number of edges of a closed geodesic is always even.*)

Let ε be the primitive 20-th root of unity, $\varepsilon = \cos\dfrac{\pi}{10} + i\sin\dfrac{\pi}{10}$ and let ω be a non-zero complex number of the form

$$p\varepsilon + q\varepsilon^3 + r\varepsilon^5 + s\varepsilon^7 + t\varepsilon^9$$

where p, q, r, s, t are integers with $p - q + r - s + t = 0$. We call *preferred angles* the arguments of such numbers ω.

Proposition 2.2 *All the angles formed by a closed geodesic with edges are preferred angles. The squares of cotangents of preferred angles are integral fractional linear functions of* τ.

The first part of Proposition is obvious, the second part (which will not be used by us) is proved by a computation similar to that in the proof of Proposition 1.1.

Conjecture 2.3 *If a geodesic forms a preferred angle with some edge (equivalently: all the angles of a geodesic with edges are preferred), then the geodesic is closed.*

This conjecture which is confirmed by a huge amount of computer experiments, remains unproved. The relations between the angles and the lengths of closed geodesics remain unclear even at the experimental level.

3 Non-self-intersecting closed geodesics.

3.1 The number of edges.

Lemma 3.1 *If a geodesic hits some face at least twice then it is self-intersecting.*

Proof Consider two non-empty components of the intersection of a geodesic with a face. As we have noticed before, the angle between them is a multiple of $\frac{\pi}{10}$. Let us continue these intervals in the direction in which they do not diverge. If they end up on the same edge and are not parallel, then they pass to an adjacent face and cross either there or in the next face (since the angle between them is at least $\frac{\pi}{10}$; see Figure 2a). If they are parallel, then we continue them until there appears a vertex between them (if it never appears, then the development of our geodesic contains two infinite lines which means that this is not a connected geodesic). In all cases, we arrive at the following situation (Figure 2b).

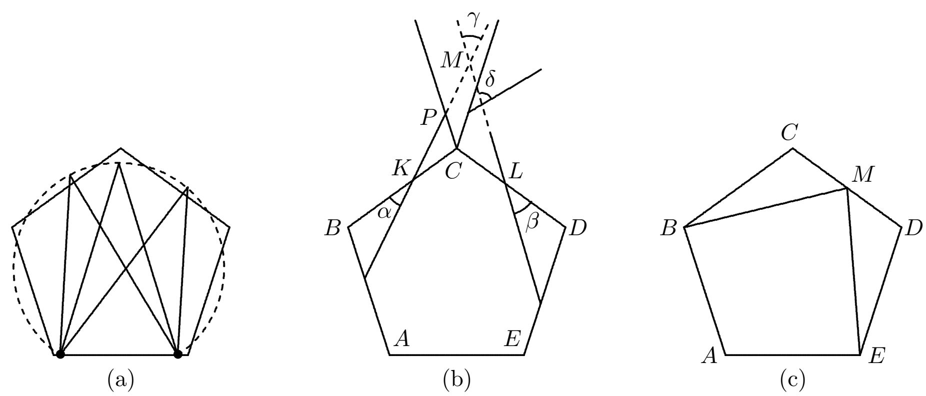

Figure 3: To Proof of Lemma 2

Two parallel or converging segments arrive at points, K, L, of two edges of a pentagon with precisely one vertex, C, between them. They form angles α, β with the edges BC, CD and their continuations meet at some point M (maybe, infinitely distant) at the angle γ. Counting the sum of the angles of the quadrilateral $KMLC$, we find that $\alpha+\beta+\gamma+\frac{7\pi}{5}=2\pi$, which shows that $\alpha+\beta=\frac{3\pi}{5}-\gamma\leq\frac{3\pi}{5}$ which shows, in turn, that α or β, let it be α is $\leq\frac{3\pi}{10}$. Hence, in $\triangle KCP$, $\angle K<\angle P$, and hence $|CP|<|CK|$.

3.2 Classification of closed non-self-intersecting geodesics.

Lemma 2 shows that a closed non-self-intersecting geodesic has at most 12 edges. All geodesics of this length may be constructed by a simple computer program. This computation results in the following statement.

Theorem 3.2 *Up to parallelism and automorphisms of the dodecahedron, there are* 5 *classes of closed non-self-intersecting geodesics; they have, correspondingly,* 6, 8, 8, 8, *and* 10 *edges, and their lengths are*

$$3 + 3\tau, 2\sqrt{5 + 7\tau}, \sqrt{18 + 29\tau}, \sqrt{19 + 29\tau}, 5\tau.$$

The 6-gonal and 10-gonal geodesics are especially simple: they are (parallel to) sections of the dodecahedron by two symmetry planes: one is parallel to two opposite faces and the other one is parpendicular to a longest diagonal (see Figure 3).

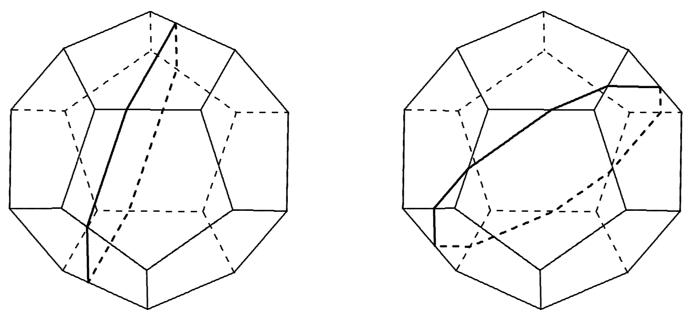

Figure 4: Two planar closed geodesics

The three 8-gonal geodesics are shown in Figure 4.

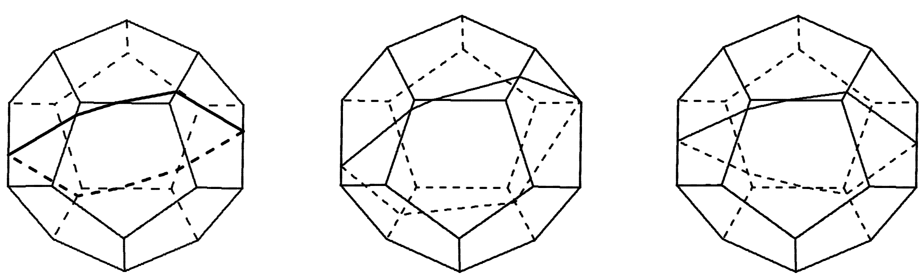

Figure 5: Three 8-gonal closed geodesics

Their developments are shown in Figure 5. The first of the three geodesics shown is orthogonal to two edges. We will return to this case in the next section.

4 Self-intersecting closed geodesics

4.1 Weakly parallel geodesics. Examples.

We will call two geodesics *weakly parallel*, if, maybe, after applying a dodecahedron automorphism to one of them, they have have parallel segments in at least one face.

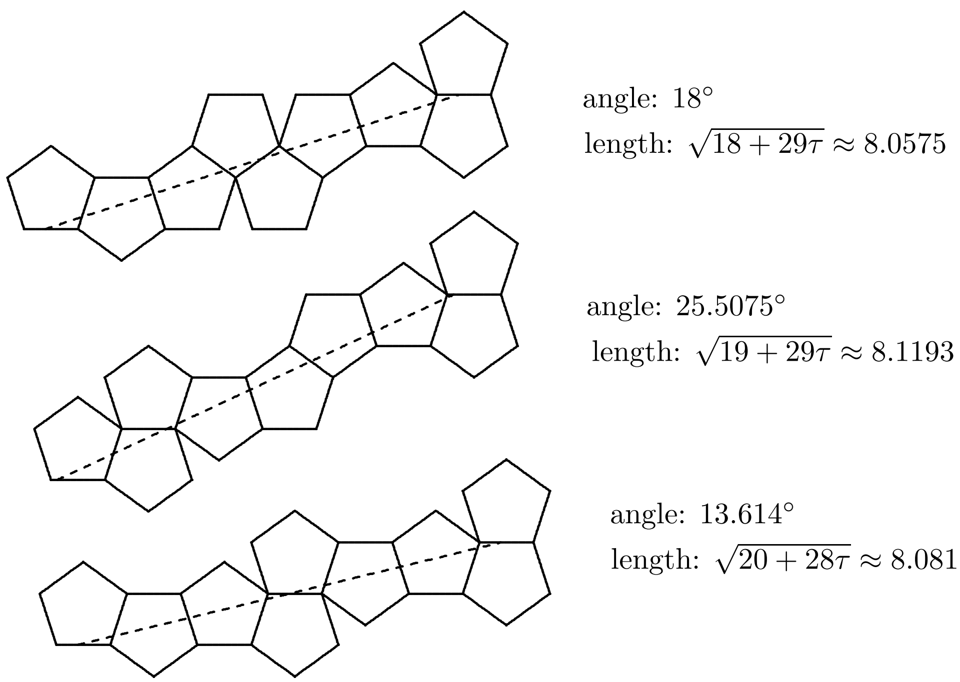

Figure 6: Developments of the 8-gonal geodesics

Proposition 4.1 *The weak parallelism is an equivalence relation.*

Proof. We need only to check that this relation is transitive. The transitivity is implied by the following obvious fact. Let intervals $\gamma_1, \gamma_2, \dots$ be consecutive intersections of a geodesic with faces, and $F_1, F_2, \dots$ be these faces. If $i \equiv j \bmod 2$, then there exists an automorphism of the dodecahedron which takes F_j into F_i and takes γ_j into an interval parallel to γ_i.

Conjecture 4.2 *A geodesic weakly parallel to a closed geodesic is closed.*

This statement is a weaker version of Conjecture 1.3 and it has the same status. Even it is true, it does not establish any relation between the number of edges of weakly parallel closed geodesics (this makes the case of a dodecahedron different from the cases of other regular polyhedra, since for all of those weakly parallel closed geodesics have the same number of edges). To begin with, let us notice that the two geodesics in Figure 3 are weakly parallel, but they have different numbers of edges: 6 and 10. A more interesting example: the first of the three 8-gonal geodesics shown in Figure 4 is parallel to a self-intersecting 20-gonal geodesic shown in Figure 6 (together with the image on the standard planar development of the regular dodecahedron). Notice that every edge of this 20-gonal geodesic is perpendicular to one if the sides of the containing it face. The length of this 20-gonal geodesic is $5\sqrt{7+11\tau} \approx 24.899$.

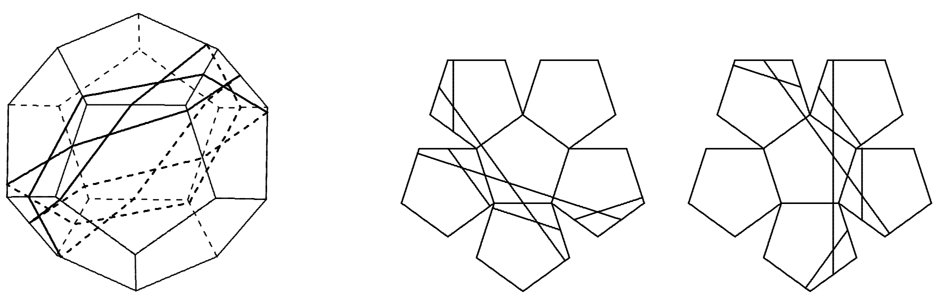

Figure 7: The 20-gonal geodesic

To illustrate in a more convincing way the parallelism of the two geodesics we show in the next drawing their developments.

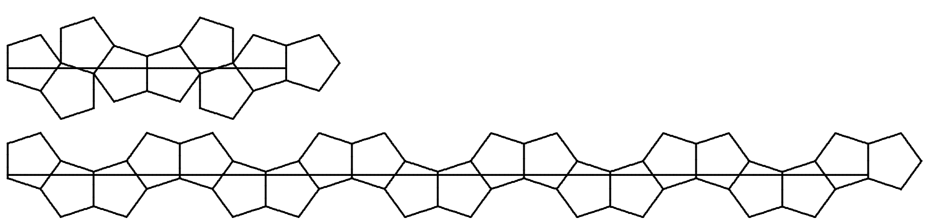

Figure 8: The developments of two parallel geodesics

We conclude this section with drawings of three longer geodesics. The two (weakly parallel) geodesics shown in Figure 8 have, correspondingly 60 and 170 edges.

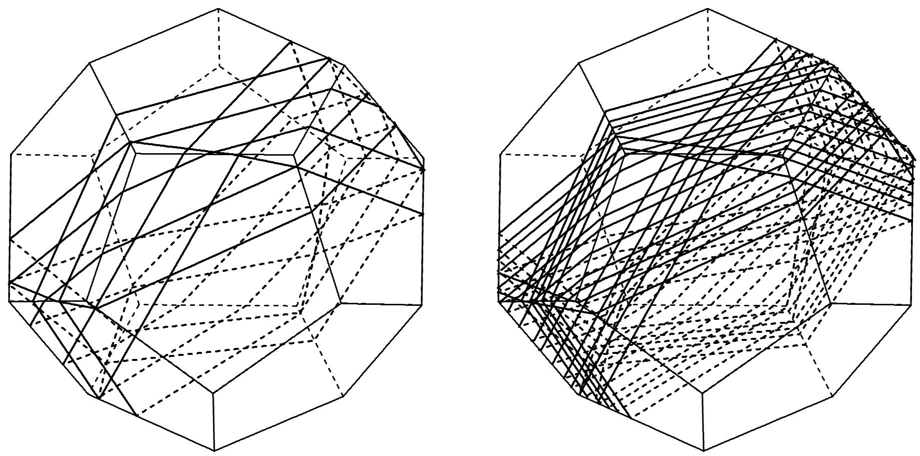

Figure 9: Two longer weakly parallel geodesics

A still longer geodesics is shown in Figure 9. It has 410 edges. To make the picture less messy, we do not show the parts on the invisible side of the dodecahedron (thus, on our picture, the dodecahedron is not transparent).

For a more clear image, we show also this 410-gonal geodesic on the planar development of the dodecahedron (Figure 10).

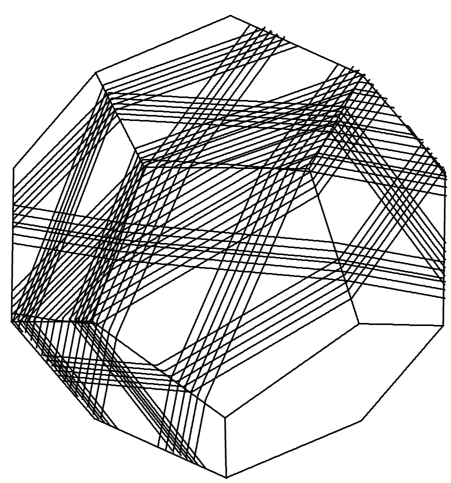

Figure 10: A 410-gonal geodesic

By the way, this 410-gonal geodesic is weakly parallel to some 52-gonal geodesic.

The following statements appear interesting, but their proofs are long and do not deserve the attention of the reader. (1) Every self-intersecting geodesic has at least 20 edges. (2) Twelve shortest non-equivalent geodesics have, correspondingly, 20, 20, 22, 22, 24, 24, 30, 30, 30, 30, 30, and 32 edges. (3) There are 47 equivalence classes of geodesics with the number of edges between 20 and 60. The equivalence meant in statements (2) and (3) is the combination of rotations and parallel translations.

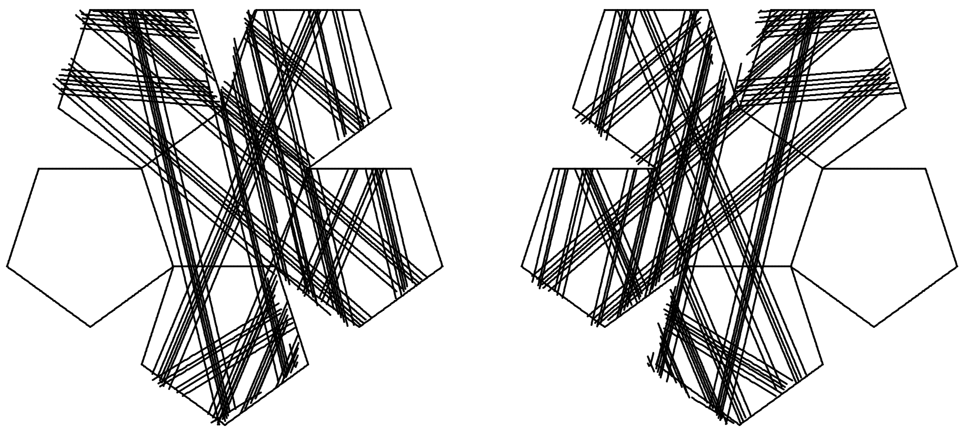

Figure 11: The image of the previous geodesic on the planar development of the dodecahedron

For a final remark, observe that in all our examples the geodesics fill the surface of the dodecahedron very unevenly. In particular, there are two opposite faces never visited by the 410-gonal geodesic above. I do not know not only how to prove, but even how to formulate this statement.

4.2 There are infinitely many geodesics.

The fact in the title follows from conjecture 1.3. But since this conjecture has not been proved yet, I prefer to construct explicitly an infinite sequence of different non-self-repeating geodesics. The geodesics in the sequence will be denoted as $\gamma_k, k = 0, 1, 2, \ldots$. The geodesic γ_k has $8(2k+1)$ edges and $2k(k+1)$ self-intersection points. The geodesic γ_0 is the (non-self-intersecting) 8-gonal geodesic of the length $2\sqrt{5\tau+7}$ (the left geodesic in Figure 4 and also the upper one in the Figure 5.) The structure of the geodesic γ_k is clear from Figure 11 which presents γ_4 on the development of the dodecahedron.

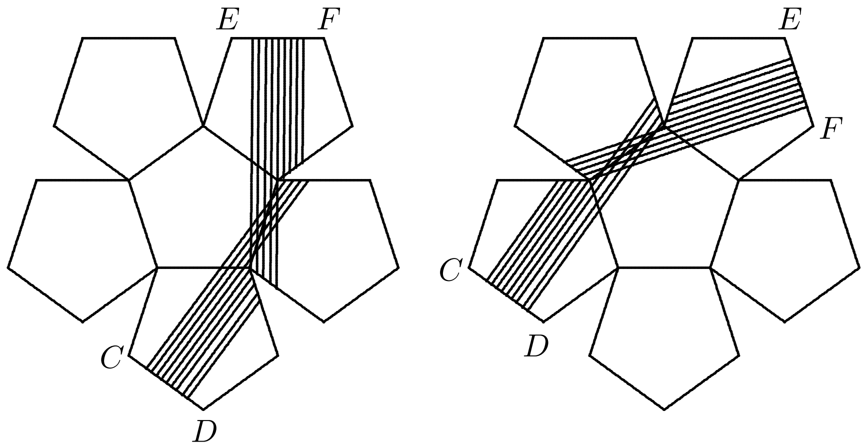

Figure 12: The geodesic γ_4

Here are the details of the construction. The two diagram in Figure 11 are obviously symmetric to each other, so it is sufficient to describe the left one. The geodesic γ_k intersects the edge CD in $2k+1$ points $A_{-k}, \ldots, A_{-1}, A_0, A_1, \ldots, A_k$ and intersects the edge EF in $2k+1$ points $B_{-k}, \ldots, B_{-1}, B_0, B_1, \ldots, B_k$ (see Figure 12). These points are determined by the formulas

$$\begin{aligned} A_s &= \left(\frac{1}{2} - \frac{s}{(2k+1)\tau}\right) C + \left(\frac{1}{2} + \frac{s}{(2k+1)\tau}\right) D, \\ B_s &= \left(\frac{1}{2} - \frac{s}{(2k+1)\tau}\right) E + \left(\frac{1}{2} + \frac{s}{(2k+1)\tau}\right) F \end{aligned}$$

where $-k \leq s \leq k$ (and τ is the golden ratio). The (acute) angle between the geodesic γ_k and the edge CD (the same at all $2k+1$ intersection point) is $\arctan((2k+1)\lambda)$ where λ is a constant,

$$\lambda = \tau\left(3\sin\frac{\pi}{5} + 2\sin\frac{\pi}{10}\right) = \sqrt{47+76\tau} \approx 13.037$$

(for big k this angle is very close to $90°$; in particular, for $k=4$, it is $\approx 89.51°$, and this is why the geodesic seems perpendicular to the edge CD in Figure 11). The length of the geodesic γ_k is

$$\sqrt{(18k^2+18k+5)+(29k^2+29k+7)\tau}.$$

Remark in conclusion that, according to our experimentation, the geodesic γ_k is weakly parallel to a unique (up to the equivalence) geodesic Γ_k not equivalent to γ_k, and Γ_k has $30(2k+1)$ edges and the length

$$5\sqrt{(47k^2+47k+12)+(76k^2+76k+19)\tau}$$

(compare with the formula for λ above).

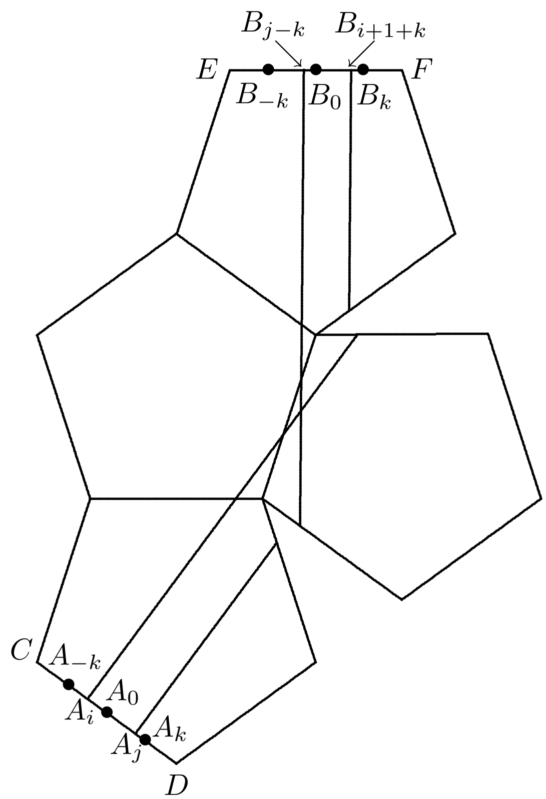

Figure 13: Construction of γ_k

References

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[2] D.Fuchs, E. Fuchs. *Closed geodesics on regular polyhedra.* Moscow Math. J., 2007, **7**, 265–280.