Parabolic bundles, elliptic surfaces and $SU(2)$–representation spaces of genus zero Fuchsian groups

by

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The concept of parabolic bundles on curves was introduced by Seshadri as a generalization of bundles on projective curves. Parabolic bundles share many of the properties of ordinary bundles. A sensible notion of stability for example, leads to moduli spaces with nice properties. Similar as in Narasimhan's theorem for ordinary bundles, these moduli have an interpretation, due to Mehta and Seshadri [80], as representation spaces of orbifold fundamental groups. These are groups of orientation preserving discontinuous automorphisms of the sphere, Euclidean plane or the complex upper half plane with orbit space of finite volume. In the hyperbolic case they are called Fuchsian and may be identified as subgroups of $\mathbf{P}Sl(2,\mathbf{R})$. In this identification the parabolic structure corresponds to fixing the weights of the representation for the elliptic, parabolic and hyperbolic elements in the surface group.

In the present paper I want to point out another aspect of parabolic bundles: They essentially describe bundles on elliptic surfaces with certain conditions on the Chern classes. The idea is quite simple. Similar to ordinary bundles one can define a pull back $\pi^{\mathfrak{p}}$ of a parabolic $\mathcal{P}$ bundle along the projection $\pi : X \longrightarrow C$ of the elliptic surface X onto the curve C. In contrast to the usual pull back, the restriction of $\pi^{\mathfrak{p}}\mathcal{P}$ to a multiple fibre is not trivial, but depends on the parabolic structure of $\mathcal{P}$. Of course, in order to define $\pi^{\mathfrak{p}}$, the weights of $\mathcal{P}$ and the multiplicities of the fibres of π have to match.

The definition of a inverse push forward $\pi_{\mathfrak{p}}$ needs some prerequisites. The restriction of a bundle $\mathcal{E}$ to a generic fibre should be trivial. This can be achieved by imposing conditions on $\mathcal{E}$ and on X. The elliptic surface X is supposed to have singular reduced fibres; a sufficient condition for $\mathcal{E}$ then is semistability with respect to a suitable Kähler metric. The main theorem in chapter I asserts that in such a situation there exists an inverse functor $\pi_{\mathfrak{p}}$ to $\pi^{\mathfrak{p}}$. In particular the moduli spaces of semistable parabolic bundles on curves with vanishing parabolic degree are isomorphic to moduli of semistable bundles on elliptic surfaces with numerically trivial Chern classes.

A C^∞–diffeomorphism of the respective moduli spaces can be obtained easily in a different way, using the fact that the fundamental groups of the elliptic surfaces considered are cocompact surface groups. Donaldson's solution to the Kobayashi–Hitchin conjecture [85] implies that the moduli space $M_{X,s}(0,0)$ of stable bundles is diffeomorphic to the space

$$\mathcal{R}(\pi_1(X)) = Hom^*(\pi_1(X), U(r))/ad\ U(r)$$

of irreducible representations of the surface group. But the latter space, as already mentioned above, is isomorphic to a moduli space of parabolic bundles.

The correspondence between bundles on elliptic surfaces and parabolic bundles is used in the second part for the computation of the moduli spaces $\mathcal{U}(\alpha)$ of parabolic rank–2 bundles on the projective line $C = \mathbf{P}^1$ with fixed weights α. Any such $\mathcal{U}(\alpha)$ is obtained by an explicite sequence of blow ups and blow downs, starting from a projective space $\mathbf{P}^{n-3} \cong Sym^{n-3}C$, where $n = \sharp I$ is the finite cardinality of the subset I of C where the parabolic structure is concentrated in. The locus of the monoidal transformations depends on the parabolic structure, both on weights and on I. Also the singularities of $\mathcal{U}(\alpha)$ can be described completely.

Parts of the results in this chapter had been obtained earlier in works of Kirk–Klassen, Bauer–Okonek and Furuta–Steer [89].

A final word on the relation of $\mathcal{U}(\alpha)$ to representation spaces. Suppose $\Gamma \subset \mathbf{P}Sl(2,\mathbf{R})$ is a Fuchsian group of genus zero. This means that the orbit space $\mathbf{H}/\Gamma$ of the fractional linear action on the upper half plane has a compactification diffeomorphic to $\mathbf{P}^1$. The group Γ has a presentation

$$\langle x_1, \ldots, x_n \mid x_i^{m_i} = 1, i \leq l, \prod_{i=1}^{n} x_i = 1 \rangle$$

with elliptic ($i \leq l$), hyperbolic and parabolic elements. For a representation $\rho : \Gamma \longrightarrow SU(2)$ the images

$$\rho(x) \sim \begin{pmatrix} b_x & 0 \\ 0 & b_x^{-1} \end{pmatrix}, \quad b_x = exp(2\pi i a_x), \ 0 \leq a_x < \frac{1}{2}$$

of the generators are diagonalizable with weights a_x. Fixing these weights, one gets a representation space isomorphic to $\mathcal{U}(\alpha)$, where α and (a_x) can be transformed into one another as explained in chapter II.

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I. Parabolic bundles on curves and bundles on elliptic surfaces

A parabolic structure on a bundle over a projective smooth curve C fixes some additional structure — flags and weights — over a finite set I of points on C. In this chapter it is shown that such parabolic bundles essentially define bundles on those elliptic surfaces X over C where the multiple fibres are contained in the preimage of I. In case X has nonnegative arithmetic genus this leads to a natural isomorphism of the associated moduli spaces of stable and semistable objects.

Let $\pi : X \longrightarrow C$ denote a relatively minimal elliptic surface of Kodaira dimension 1 with $\chi(X) > 0$. The fundamental group of X is known to be a cocompact surface group (compare Ue [86], prop. 2). In particular the first Betti number is even and, by a result of Miyaoka [74], X may be equipped with a Kähler metric. X is always assumed to carry a "good" Kähler metric: If X is not projective, any will do. Otherwise one has to specify, depending mainly on the rank r of the bundles considered, a "good" Kähler metric. A multiple of the ample divisor $H_n = H_0 + rnK_X$, where H_0 is an arbitrary ample divisor and K_X a canonical one, will determine an embedding into a projective space and thus induce a metric from the Fubini metric. In order to get a "good" metric, n has to be sufficiently big; for our purposes (except in lemma 1.2) $n \geq 2(K_X \cdot H_0)$ will do.

For a torsion free coherent sheaf $\mathcal{S}$ let $\mu(\mathcal{S})$ denote the slope

$$\mu(\mathcal{S}) = \frac{1}{rank(\mathcal{S})} \int_X c_1(\mathcal{S}) \wedge \Phi,$$

where Φ is the Kähler form on X. In the projective case one may as well set

$$\mu(\mathcal{S}) = \frac{c_1(\mathcal{S}) \cdot H_n}{rank(\mathcal{S})}.$$

Definition: A holomorphic vector bundle $\mathcal{E}$ over X will be called vertical, if there is a sequence of subbundles $0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$ with subquotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ isomorphic to $\mathcal{O}_X(C_i)$ for a vertical divisor C_i satisfying $C_i \cdot C_i = 0$.

Lemma 1.1: Let $\mathcal{E}$ be a semistable bundle of rank r with $c_2(\mathcal{E}) = 0$ and $det\ \mathcal{E} \cong \mathcal{O}_X(C)$ for a vertical divisor C with $C^2 = 0$. Then $\mathcal{E}$ is vertical.

Proof: After tensoring with a suitable power of the canonical bundle $\mathcal{K}_X$ one may assume $\mu(\mathcal{K}_X) < \mu(\mathcal{E}) \le \mu(\mathcal{K}_X^{\otimes 2})$. For $i \in \{0, \ldots, r-1\}$ one can inductively construct surjective homomorphisms $\mathcal{E} \longrightarrow \mathcal{Q}_i$ onto torsion free sheaves $\mathcal{Q}_i$ of rank $r - i$ and kernel $\mathcal{E}_i$. The sheaves $\mathcal{Q}_i$ satisfy:
i) $det(\mathcal{Q}_i)^{\vee\vee} \cong \mathcal{O}_X(D_i)$ for a vertical divisor D_i
ii) $H^2(X; \mathcal{Q}_i) = 0$
iii) $H^0(X; \mathcal{Q}_i) \neq 0$
iv) $\mu(\mathcal{E}_i) \ge 0$.
Note that these conditions are satisfied for $\mathcal{E} = \mathcal{Q}_0$: The semistable bundle $\mathcal{E}^\vee \otimes \mathcal{K}_X$ only admits the trivial section, since $\mu(\mathcal{E}^\vee \otimes \mathcal{K}_X) < 0$. Serre–Poincare duality implies ii) and iii) follows from the Riemann–Roch theorem. The assumption $\chi(X) > 0$ is essential for this last step.
A nontrivial section of $\mathcal{Q}_{i-1}$ leads to a short exact sequence

$$0 \longrightarrow \mathcal{J}_i(C_i) \longrightarrow \mathcal{Q}_{i-1} \longrightarrow \mathcal{Q}_i \longrightarrow 0,$$

where $\mathcal{J}_i(C_i)$ is the ideal sheaf of a 0-dimensional complex subspace of X, twisted by a divisor $C_i \ge 0$. In case X is not projective, any divisor, and in particular C_i, is vertical: Otherwise an irreducible divisor C with $C \cdot K_X > 0$ would yield $(C + lK_X)^2 > 0$ for sufficiently large l, contradicting non–projectivity. So assume X projective. Semistability of $\mathcal{E}$ gives

$$(H_0 + nrK_X) \cdot C_i + (i-1)\mu(\mathcal{E}_{i-1}) = i\mu(\mathcal{E}_i) \le i\mu(\mathcal{E}) \le 2i(H_0 \cdot K_X) \le in$$

and ampleness of H_0

$$H_0 \cdot C_i + (i-1)\mu(\mathcal{E}) \ge 0.$$

Therefor $C_i \cdot K_X \leq 0$ and the divisor C_i has to be vertical.
Conditions ii) and iv) are easily verified and iii) follows from the Riemann–Roch theorem using the estimates $c_2(\mathcal{J}_i) = length(\mathcal{O}_X/\mathcal{J}_i) \geq 0$, $C_i^2 \leq 0$ and thus inductively

$$c_2(\mathcal{Q}_i) = c_2(\mathcal{Q}_{i-1}) - [c_2(\mathcal{J}_i) - C_i^2] \leq 0.$$

The iterated application of this inequality

$$0 = c_2(\mathcal{Q}_0) \geq c_2(\mathcal{Q}_1) \geq \ldots \geq c_2(\mathcal{Q}_{r-1}) = length(\mathcal{Q}_{r-1}^{\vee\vee}/\mathcal{Q}_{r-1}) - (C - \sum_{i=1}^{r-1} C_i)^2 \geq 0$$

forces $\mathcal{J}_i \cong \mathcal{O}_X$ and $C_i^2 = 0$, proving the lemma. ♣

Let $\mathcal{E}$ be a vertical bundle of rank r with $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{O}_X(C_i)$. For the following lemma denote $d_{min} = min\{\mu(\mathcal{O}_X(C_i))\}$ and $d_{max} = max\{\mu(\mathcal{O}_X(C_i))\}$. (In the projective case set $d_{min/max} = min/max\{H_0 \cdot C_i\}$ and choose H_n such that $n \geq d_{max} - d_{min}$)

Lemma 1.2: Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_k = \mathcal{E}$ be a sequence of subsheaves with semistable torsion free subquotients $\mathcal{F}_j/\mathcal{F}_{j-1}$ and suppose $d_{min} \leq \mu(\mathcal{F}_j/\mathcal{F}_{j-1}) \leq d_{max}$. Then each $\mathcal{F}_j$ is a vertical bundle.

Proof: Induction over k.
The filtering $(\mathcal{E}_i)$ of $\mathcal{E}$ induces a filtering $(\mathcal{Q}_i)$ of $\mathcal{Q}_r = \mathcal{E}/\mathcal{F}_1$, where each subquotient is torsion free of rank ≤ 1. If it is nontrivial, $\mathcal{Q}_i/\mathcal{Q}_{i-1} \cong \mathcal{J}_i(C_i + D_i)$ with an effective divisor D_i and the ideal sheaf $\mathcal{J}_i$ of a 0-dimensional subspace of X. In particular $\mu(\mathcal{Q}_i) \geq d_{min}$. The filtering $(\mathcal{F}_j)$ induces a filtering of $\mathcal{Q}_i$. Semistability of the subquotients $\mathcal{F}_j/\mathcal{F}_{j-1}$ and the estimate $\mu(\mathcal{F}_j/\mathcal{F}_{j-1}) \leq d_{max}$ applied to this filtering imply $\mu(\mathcal{Q}_i) \leq d_{max}$. Similarly to the preceeding lemma the divisor D_i is vertical: in the projective case the estimate

$$nr(D_i \cdot K_X) \leq \mu(\mathcal{O}_X(D_i)) = \mu((\mathcal{Q}_i/\mathcal{Q}_{i-1})(-C_i)) \leq rank(\mathcal{Q}_i)(d_{max} - d_{min}) < nr$$

shows $D_i \cdot K_X \leq 0$. With this information one can compute Chern numbers:

$$c_2(\mathcal{E}/\mathcal{F}_1) = \sum_i length(\mathcal{O}_X/\mathcal{J}_i) + \sum_{i<j} D_i \cdot D_j = m + \frac{1}{2}(\sum_i D_i)^2 - \frac{1}{2}(\sum_i D_i^2)$$

$$c_2(\mathcal{F}_1) = -c_2(\mathcal{E}/\mathcal{F}_1) + (\sum_i D_i)^2.$$

Being semistable, $\mathcal{F}_1$ has to satisfy an inequality due to Bogomolov (compare Kobayashi [87], p 114) and in the Kähler case to Lübke [82] ($s = rank\ \mathcal{F}_1$):

$$0 \leq 2sc_2(\mathcal{F}_1) - (s-1)c_1^2(\mathcal{F}_1) = (-2sm) + (-s\sum_i D_1^2) + (\sum_i D_i)^2$$

Any summand in the latter expression is less or equal to zero with equality iff for all i one has $D_i^2 = 0$ and $\mathcal{J}_i = \mathcal{O}_X$. This shows that $\mathcal{E}/\mathcal{F}_1$ is vertical. Because of lemma 1 the sheaf $\mathcal{F}_1$ is a vertical bundle as well. ♣

Corollary 1.3: The Harder–Narasimhan decomposition of the bundle $\mathcal{E}$ above is vertical.

Proof: Suppose $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{E}$ is the HN–filtration. With j maximal such that $\mathcal{E}_j \subset \mathcal{F}_{k-1}$ and l minimal such that $\mathcal{F}_1 \subset \mathcal{E}_l$, one gets the inequalities:

$$d_{min} \leq \mu(\mathcal{O}_X(C_{j+1})) \leq \mu(\mathcal{F}_k/\mathcal{F}_{k-1}) < \ldots < \mu(\mathcal{F}_1) \leq \mu(\mathcal{O}_X(C_l)) \leq d_{max}$$

♣

Let $\hat{z}$ denote the formal neighborhood of a point $z \in C$ with structure sheaf isomorphic to $\mathbf{C}[[t_z]]$. Similarly let $\hat{X}_z$ be the formal neighborhood $X \times_C \hat{z}$ of the fiber $\pi^{-1}(z)$.

Proposition 1.4: Suppose $\mathcal{E}$ is a semistable bundle of rank r over X satisfying $c_2(\mathcal{E}) = 0$ and $det(\mathcal{E}) \cong \mathcal{O}_X(C)$ for a vertical divisor C with $C^2 = 0$. Then the restriction $\hat{\mathcal{E}}_z$ of $\mathcal{E}$ to $\hat{X}_z$ splits as a sum $\hat{\mathcal{E}}_z \cong \oplus_{i=1}^r \mathcal{O}_{\hat{X}_z}(l_{z,i}F_z)$ of line bundles for certain numbers $0 \leq l_{z,i} < m_z$, where m_z denotes the multiplicity of the fiber X_z and $F_z = X_{z,red}$. In particular the natural morphism $\pi^*\pi_*\mathcal{E} \longrightarrow \mathcal{E}$ is injective with quotient sheaf isomorphic to $\oplus_{l_{z,i}>0}\mathcal{O}_{l_{z,i}F_z}(l_{z,i}F_z)$.

Proof: Induction on the rank of $\mathcal{E}$. Since $C^2 = 0$ and the fibers of π are connected, the divisor C is a linear combination of reduced fibers $C = C' + \sum_z l_z F_z$ with $0 \leq l_z < m_z$. Applying $\pi^*\pi_*$ to the sequence

$$0 \longrightarrow \mathcal{O}_X(C') \longrightarrow \mathcal{O}_X(C) \longrightarrow \oplus_z \mathcal{O}_{l_z F_z}(l_z F_z) \longrightarrow 0$$

gives $\pi^*\pi_*\mathcal{O}_X(C) \cong \mathcal{O}_X(C')$, since $\pi_*\mathcal{O}_{l_z F_z}(l_z F_z) = 0$.
In the general case let $\mathcal{E}' \subset \mathcal{E}$ be a subsheaf of lower rank with $\mu(\mathcal{E}')$ maximal and with torsion free quotient $\mathcal{E}'' = \mathcal{E}/\mathcal{E}'$. Both $\mathcal{E}'$ and $\mathcal{E}''$ are semistable and satisfy the conditions of 1.2. Hence they are vertical, too.

(*) Claim : $$H^0(C; \pi_{*1}(\mathcal{H}om(\mathcal{E}'', \mathcal{E}'))) = 0$$

To verify this claim, consider the natural homomorphism $\pi^*\pi_*(\mathcal{F}^\vee) \longrightarrow \mathcal{F}^\vee$, where $\mathcal{F}$ denotes the bundle $\mathcal{H}om(\mathcal{E}'', \mathcal{E}') \otimes \omega_{X/C}^\vee$. This map is an isomorphism away from the multiple fibers, since by induction the same is true for $\mathcal{F}$ replaced by any of the factors $\mathcal{E}'', \mathcal{E}'$ or $\omega_{X/C}^\vee$. Moreover the quotient sheaf $\mathcal{Q}$ is supported on the divisor $M = \sum_z (m_z - 1)F_z$: By induction $\mathcal{Q} \otimes \mathcal{O}_{\hat{X}_z} \cong \oplus_i \mathcal{O}_{(k_i m_z + l_i)F_z}(l_i F_z)$. But $\pi_*(\mathcal{Q})$ injects into the locally free sheaf $\pi_{*1}(\pi^*\pi_*(\mathcal{F}^\vee))_{\hat{z}}$ and thus is trivial. This forces $k_i = 0$.
Relative duality, compare Barth et al. [85], p. 99, provides an isomorphism

$$\pi^*\pi_*(\mathcal{F}^\vee)^\vee \cong \pi^*\pi_{*1}(\mathcal{F} \otimes \omega_{X/C}).$$

A nontrivial section of $\pi_{*1}(\mathcal{F} \otimes \omega_{X/C}) = \pi_{*1}(\mathcal{H}om(\mathcal{E}'', \mathcal{E}'))$ then induces a nontrivial sheaf homomorphism $\mathcal{O}_X(-M) \longrightarrow \mathcal{F}$, or equivalently, a nontrivial homomorphism

$$\psi : \mathcal{E}'' \longrightarrow \mathcal{E}' \otimes \omega_{X/C}^\vee \otimes \mathcal{O}_X(M) \cong \mathcal{E}' \otimes \pi^*\pi_{*1}\mathcal{O}_X.$$

The assumption $\chi(X) > 0$ is equivalent to $deg(\pi_{*1}\mathcal{O}_X) < 0$, compare Barth et al. [85], p. 162. The inequalities $\mu(\mathcal{E}'') \geq \mu(\mathcal{E}') > \mu(\mathcal{E}' \otimes \pi^*\pi_{*1}\mathcal{O}_X)$ show that the existence of ψ contradicts the semistability of $\mathcal{E}''$ and $\mathcal{E}'$. This proves the claim.

As an extension, $\mathcal{E}$ is represented by an element of $Ext^1_{\mathcal{O}_X}(\mathcal{E}'', \mathcal{E}) = H^1(X; \mathcal{H}om(\mathcal{E}'', \mathcal{E}'))$. Let $\mathcal{O}_n$ denote the sheaf $\pi^*(\mathcal{O}_{\hat{z}}/t_z^n)$. The Leray spectral sequence induces a decomposition

$$0 \longrightarrow H^1(C; \pi_*\mathcal{L}) \longrightarrow H^1(X; \mathcal{L}) \longrightarrow H^0(C; \pi_{*1}\mathcal{L}) \longrightarrow 0$$

for any sheaf $\mathcal{L}$ on X. Naturality with respect to sheaf homomorphisms and the vanishing of $H^1(C; \pi_*(\mathcal{H}om(\mathcal{E}'', \mathcal{E}') \otimes \mathcal{O}_n))$ force the natural map

$$r_n : H^1(X; \mathcal{H}om(\mathcal{E}'', \mathcal{E}')) \longrightarrow H^1(X; \mathcal{H}om(\mathcal{E}'', \mathcal{E}') \otimes \mathcal{O}_n)$$

to factor through $H^0(C; \pi_{*1}(\mathcal{H}om(\mathcal{E}'', \mathcal{E}')) = 0$. As a consequence r_n is zero and the same is true for the map $H^1(X; \mathcal{H}om(\mathcal{E}'', \mathcal{E}')) \longrightarrow H^1(\hat{X}_z; \mathcal{H}om(\mathcal{E}'', \mathcal{E}')^{\wedge}_z)$, compare Grothendieck, EGA III,4 [61]. The statement on $\pi^*\pi_*\mathcal{E} \longrightarrow \mathcal{E}$ follows by a base change argument. ♣

Corollary 1.5: Let $\mathcal{E}$ be as in 1.5. Then $H^1(X; \mathcal{E}nd(\mathcal{E})) \cong H^1(C; \mathcal{E}nd(\mathcal{E}))$.

Proof: Apply the claim $(*)$ in the proof above to the case $\mathcal{E}' = \mathcal{E}''$. ♣

The proposition suggests that semistable bundles on the elliptic surface X essentially are "pull backs" of parabolic bundles on C. This will now be made more precise. For the basic definitions and properties of parabolic bundles the reader may consult Seshadri's asterisque volume [82]. To recapitulate the notation: C is a projective curve and $\pi : X \longrightarrow C$ a minimal elliptic surface of Kodaira dimension 1 and $\chi(\mathcal{O}_X) > 0$, equipped with a "good" Kähler metric (i.e. if X is projective, the metric stems from a polarisation by an ample divisor $H_n = H_0 + rnK_X$ for $n \geq 2(H_0K_X)$).
Let $S(\chi, a, d)$ (resp. $S'(\chi, a, d)$) denote the set of isomorphism classes of semistable (resp. stable) parabolic bundles of rank r, parabolic degree d, weights $a = (a_{z,i})_{z \in I; 1 \leq i \leq n_z}$ and multiplicities $\chi = (k_{z,i})_{z \in I; 1 \leq i \leq n_z}$. The functor

$$\mathcal{S}'(\chi, a, d) : \mathcal{AN} \longrightarrow \mathcal{ENS}$$

associates to each complex space T the equivalence classes of families of elements in $S'(\chi, a, d)$ parametrized by T. For this functor there exist a coarse moduli space $U_s(\chi, a, d)$ with natural compactification $U(\chi, a, d)$.
Let on the other hand $R(r, d)$ (resp. $R'(r, d)$) be the set of semistable (resp. stable) bundles $\mathcal{E}$ of rank r over X with $c_2(\mathcal{E}) = 0$ and $det(\mathcal{E}) \cong \mathcal{O}_X(D)$ for a vertical divisor D numerically equivalent to $d \cdot \pi^*(z)$ for a point $z \in C$. The corresponding functor

$$\mathcal{R}'(r, d) : \mathcal{AN} \longrightarrow \mathcal{ENS}$$

has a moduli space $M_{X,s}(d, 0)$ with compactification $M_X(d, 0)$.

Theorem 1.6: There exists a natural transformation

$$\pi^p : \coprod \mathcal{S}'(\chi, a, d) \longrightarrow \mathcal{R}'(r, d).$$

Here the sum is over all possible sets of multiplicities χ and all sets of weights satisfying $a_{z,i} \in (m_z)^{-1}\mathbf{Z}$, where m_z is the multiplicity of the fiber X_z. The functor π^p has an inverse π_p and thus induces identifications $M_{X,s}(d,0) \cong \coprod U_s(\chi, a, d)$ and $M_X(d,0) \cong \coprod U(\chi, a, d)$ of the corresponding moduli spaces.

Remarks 1.7:

i) The theorem asserts the equivalence of moduli spaces as analytic spaces. The moduli spaces $U(\chi, a, d)$ are known to be projective. A result of Miyajima [89] shows $U_{an}(\chi, a, d) \cong (U_{alg})_{an}(\chi, a, d)$. In particular the moduli spaces $M_X(d, 0)$ are projective, even if X is not.

ii) The weights of parabolic structures a priori are real numbers in $[0, 1[$. However, by a result of Mehta–Seshadri [80], Théorème 13, for any set a of weights there exist sets b of rational weights, such that $S(\chi, a, 0)$ and $S(\chi, b, 0)$ are equivalent via the underlying quasiparabolic structures. In particular $U(\chi, a, 0) \cong U(\chi, b, 0)$.

iii) The argument of Mehta–Seshadri shows that there exist only finitely many isomorphism classes of moduli spaces $U(\chi, a, 0)$ for a fixed rank. Applying suitable logarithmic transformations over $I \subset C$, one can construct an elliptic surface X over C, such that any $U(\chi, a, 0)$ is isomorphic to an open and closed subspace of $M_X(0, 0)$.

iv) Actually $\pi^p\mathcal{P}$ is also defined for non-semistable parabolic bundles; one doesn't even need the restricting hypothesis $\chi(X) > 0$ on the elliptic surface X. Furthermore $\pi_p\mathcal{E}$ is always defined, as long as the conclusion of 1.4 holds. As a consequence it is possible to define tensor products of parabolic bundles. In fact, if the weights are all rational, $\mathcal{P}_1 \otimes \mathcal{P}_2 = \pi_p(\pi^p\mathcal{P}_1 \otimes_{\mathcal{O}_X} \pi^p\mathcal{P}_2)$. The weights of the tensor product then are sums of weights of the factors mod 1. To get the underlying quasiparabolic structure for general weights, one has to replace the weights by suitably chosen nearby rational ones. Anyway, after taking tensor product with a parabolic line bundle one can assume $d = 0$.

Proof of the theorem: For $\mathcal{E} \in R(r, d)$ the bundle $\pi_*\mathcal{E}$ carries a natural parabolic structure, denoted by $\pi_p\mathcal{E}$: Proposition 1.4 shows

$$(\pi_*\mathcal{E})_z \cong Hom_{\mathcal{O}_X}(\mathcal{O}_{m_z F_z}, \mathcal{E}_{X_z})$$

and a flag structure

$$(\pi_*\mathcal{E})_z = F_1(\pi_p\mathcal{E})_z \supset \dots F_{m_z}(\pi_p\mathcal{E})_z$$

is given by $F_i(\pi_p\mathcal{E})_z = Hom_{\mathcal{O}_X}(\mathcal{O}_{(m_z-i+1)F_z}, \mathcal{E}_{X_z})$. The weight l/m_z has the multiplicity $k_{z,l} = dim(F_{l+1}(\pi_p\mathcal{E})_z / F_l(\pi_p\mathcal{E})_z)$. The parabolic degree of $\pi_p\mathcal{E}$ is d.
Now let $\mathcal{P}$ denote a parabolic bundle in $\coprod S(\chi, a, d)$. By assumption $l_{z,i} = m_z a_{z,i}$ is a nonnegative integer less than m_z. The bundle $\pi^p\mathcal{P}$ is conveniently described using the dual: Let $|\mathcal{P}|^\vee$ be the dual of the underlying bundle of $\mathcal{P}$. The bundle $(\pi^p\mathcal{P})^\vee$ then is the kernel of a surjective map

$$\psi_{\mathcal{P}} : \pi^*(|\mathcal{P}|^\vee) \longrightarrow \oplus_{z,i}\, k_{z,i} \cdot \mathcal{O}_{l_{z,i} F_z}.$$

Let $F_i(\mathcal{P})_z$ denote the i–th stratum of the flag at the point z and $\varphi_{z,i} : \mathcal{P}_z^\vee \longrightarrow (F_i(\mathcal{P})_z)^\vee$ the map dual to the inclusion. Then $\psi_{\mathcal{P}}$ is characterized by commuting squares

$$\begin{array}{ccc} \pi^*(\mathcal{P}^\vee) & \xrightarrow{\pi^*\varphi_{z,i}} & \pi^*(F_i(\mathcal{P})_z^\vee) \cong \oplus_{j\geq i} k_{z,j} \cdot \mathcal{O}_{m_z F_z} \\ \psi_{\mathcal{P}} \downarrow & & \downarrow \\ \oplus_{z,j} k_{z,j} \cdot \mathcal{O}_{l_{z,j} F_z} & \xrightarrow{pr} & \oplus_{j\geq i} k_{z,j} \cdot \mathcal{O}_{l_{z,j} F_z}. \end{array}$$

The generalization to families $\mathcal{V}$ of parabolic bundles parametrized by T is straightforward using the subbundles $F_i(\mathcal{V})_{z\times T} \subset \mathcal{V}_{z\times T}$.
Dualising the defining short exact sequence for $(\pi^{\mathfrak{p}}\mathcal{P})^\vee$, one gets the short exact sequence

$$0 \longrightarrow \pi^*\mathcal{P} \longrightarrow \pi^{\mathfrak{p}}\mathcal{P} \longrightarrow \oplus_{l_{z,i}>0}\, k_{z,i} \cdot \mathcal{O}_{l_{z,i} F_z}(l_{z,i} F_z) \longrightarrow 0.$$

One easily verifies $\pi_{\mathfrak{p}}\pi^{\mathfrak{p}}\mathcal{P} \cong \mathcal{P}$ for any parabolic bundle on C and $\pi^{\mathfrak{p}}\pi_{\mathfrak{p}}\mathcal{E} \cong \mathcal{E}$ for any semistable bundle $\mathcal{E} \in R(r,d)$ on X. Lemma 1.1 and corollary 1.3 together show that $\pi^{\mathfrak{p}}$ and $\pi_{\mathfrak{p}}$ both preserve semistability and stability.
The definition of $\pi_{\mathfrak{p}}$ for families of stable bundles is immediate from the following lemma, which also concludes the proof of the theorem: ♣

Lemma 1.8: Let $\mathcal{F}$ be a family of stable vector bundles, parametrized by a complex space T. Then $(\pi \times id_T)_*\mathcal{F}$ is locally free and the bundles

$$((\pi \times id_T)_*\mathcal{F})_{z\times T} \cong (\pi \times id)_* \mathcal{H}om(\mathcal{O}_{X_z\times T}, \mathcal{F} \otimes \mathcal{O}_{X_z\times T})$$

have natural subbundles

$$F_i(\mathcal{F})_{z\times T} \cong (\pi \times id)_* \mathcal{H}om(\mathcal{O}_{(m_z - i+1)F_z\times T}, \mathcal{F} \otimes \mathcal{O}_{X_z\times T}).$$

Proof: The lemma certainly holds, if $\mathcal{F} \cong \pi^{\mathfrak{p}}\mathcal{V}$ for a family of parabolic bundles. Since the claim is of local nature, it suffices to prove that the moduli spaces are smooth and that the dimensions of versal deformations of a stable bundle $\mathcal{E}$ over X and of $\pi_{\mathfrak{p}}\mathcal{E}$ are the same. This can be shown using arguments of N. Nitsure [86]: The Zariski tangent space to $\pi_{\mathfrak{p}}\mathcal{E}$ is isomorphic to $H^1(C, \mathcal{P}ar\mathcal{E}nd(\pi_{\mathfrak{p}}\mathcal{E}))$, where $\mathcal{P}ar\mathcal{E}nd(\mathcal{P})$ is the sheaf of germs of endomorphisms of $\mathcal{P}$ preserving the parabolic structure. Using proposition 1.4 one easily sees $\mathcal{P}ar\mathcal{E}nd(\pi_{\mathfrak{p}}\mathcal{E}) \cong \pi_*\mathcal{E}nd(\mathcal{E})$ for a stable bundle $\mathcal{E}$. Corollary 1.5 shows that $\pi_{\mathfrak{p}}$ identifies the Zariski tangent spaces. Finally Nitsure constructs in [86], proposition 1.13 a smooth versal deformation space for any parabolic bundle $\mathcal{P}$. ♣

II. Rank–2 parabolic bundles on $\mathbf{P}^1$

Moduli spaces of bundles over curves are known to be quite complicated objects. Additional parabolic structures tend not to improve the situation. Nevertheless quite a few results are known by the work of Seshadri [77],[82], Mehta–Seshadri [80] and Nitsure [86]. The purpose of this chapter is to explicitely describe the moduli spaces in the easiest case of bundles of rank two over the projective line.

So let $\mathcal{P}$ henceforth denote a parabolic rank–2 bundle over $C = \mathbf{P}^1$. After tensoring with a suitable parabolic line bundle (compare 1.7.iv), one may assume: $pardeg\ \mathcal{P} = 0$ and for each $z \in I$ the weights have multiplicity 1 and add up to 1. Identify the set of such weights $(a_{z,i})$ with elements $\alpha \in W = (0,1)^I$ the following way: If the number of elements n in I is even, pick an arbitrary element z_0. Set

$$a_{z,1} = \begin{cases} \frac{\alpha_z}{2} & \text{for } z = z_0 \\ \frac{1-\alpha_z}{2} & \text{else} \end{cases}$$

and $a_{z,2} = 1 - a_{z,1} > \frac{1}{2}$. Let $\mathcal{U}(\alpha)$ denote the corresponding moduli space of semistable parabolic bundles. The following fact is basically due to Mehta–Seshadri [80]:

Theorem 2.1: For any $\alpha \in W$ the moduli space $\mathcal{U}(\alpha)$ has the structure of a projective variety. The open subvariety $\mathcal{U}_s(\alpha)$ of stable bundles is a smooth quasiprojective variety of dimension $n-3$. In particular $\mathcal{U}(\alpha)$ is normal.

(Note that for some α the variety $\mathcal{U}_s(\alpha)$ may be empty.)

Proof: Theorem 4.1 in Mehta–Seshadri [80], whereas stated only for curves of genus $g \geq 2$, also holds in this case. Using 1.7.iv) and the fact that elliptic surfaces X over $\mathbf{P}^1$ with $\chi(X) = 1$ are projective, one could as well apply Maruyama's theorem [77] to see that $\mathcal{U}(\alpha)$ is projective. Smoothness, already used in the proof of 1.8, follows from proposition 1.13 in Nitsure [86]. The dimension of the moduli space can be computed from the cohomology exact sequence associated to the inclusion of sheaves $\mathcal{P}ar\mathcal{E}nd(\mathcal{P}) \longrightarrow \mathcal{E}nd(\mathcal{P})$:

$$h^1(C; \mathcal{P}ar\mathcal{E}nd(\mathcal{P})) = h^0(C; \mathcal{E}nd(\mathcal{P})/\mathcal{P}ar\mathcal{E}nd(\mathcal{P})) - \chi(\mathcal{E}nd(\mathcal{P})) + h^0(C; \mathcal{P}ar\mathcal{E}nd(\mathcal{P})).$$

The first summand is the sum of dimensions f_z of the flag varieties of type determined by the quasi–parabolic structure at $z \in I$; the last is 1, since the stable bundle $\mathcal{P}$ is simple. Thus one gets with Riemann–Roch the dimension formula of Mehta–Seshadri:

$$dim\ \mathcal{U}_s(\chi, a, 0) = \sum_I f_z + r^2(g-1) + 1.$$

♣

For a more detailed study of the parabolic bundles it is necessary to characterize the parabolic subbundles. To this end let K be the free abelian monoid generated by I with the relations $2z = 2y$ for $y, z \in I$ and let $\varphi : K \longrightarrow \mathbf{Z}$ be the homomorphism defined by $\varphi(z) = 1$ for $z \in I$. The monoid K acts on W by involutions

$$z(\alpha_y) = \begin{cases} \alpha_y & \text{if } y \neq z \\ 1 - \alpha_z & \text{for } y = z. \end{cases}$$

The quasi-parabolic structure of $\mathcal{P}$ consists of distinguished 1-dimensional linear subspaces V_z in the fibres $\mathcal{P}_z$ of the underlying holomorphic bundle $|\mathcal{P}|$. The parabolic structure on a 1-dimensional subbundle $|\mathcal{L}|$ of $|\mathcal{P}|$ is determined by the subset of I for which the fiber $\mathcal{L}_z$ is contained in V_z. The weights associated to $\mathcal{L} \subset \mathcal{P}$ are

$$l_z = \begin{cases} a_{z,1} & \text{if } \mathcal{L}_z \not\subset V_z \\ a_{z,2} & \text{if } \mathcal{L}_z \subset V_z \end{cases}$$

and the parabolic degree of $\mathcal{L}$ is $deg(|\mathcal{L}|) + \sum_{z \in I} l_z$.
The parabolic bundle $\mathcal{P}$ is (semi-) stable, if for any parabolic line bundle $\mathcal{L} \subset \mathcal{P}$ one has $pardeg(\mathcal{L}) < 0$ (resp. ≤ 0). So by fixing the quasi-parabolic structure of $\mathcal{P}$, the subbundles $\mathcal{L}$ of $\mathcal{P}$ define affine linear forms $pardeg(\mathcal{L})$ on W. The parabolic bundle $\mathcal{P}$ then is stable if and only if its weights arein the negative cone of all these forms.
Denote by $\mathcal{L}(0)$ the parabolic bundle with $deg|\mathcal{L}(0)| = -[\frac{n}{2}]$ and with weights

$$l_z = \begin{cases} a_{z,2} & \text{if } n \text{ is even and } z = z_0 \\ a_{z,1} & \text{else.} \end{cases} .$$

For $k \in K$ construct parabolic bundles $\mathcal{O}(k)$ the following way: The underlying holomorphic bundle of $\mathcal{O}(z)$ for $z \in I$ has degree 0 and the parabolic structure with weight α_z is concentrated in z. Set $O(2z) = \mathcal{O}_C(1)$ and $\mathcal{O}(k + y) = \mathcal{O}(k) \otimes \mathcal{O}(y)$, if $k = \sum_{I \setminus \{y\}} b_z z$. The bundle $\mathcal{L}(k)$ then is $\mathcal{L}(0) \otimes \mathcal{O}(k)$.

Lemma 2.2: Let $\mathcal{P}$ be a semistable parabolic bundle for some weight $\alpha \in W$. Then $\mathcal{P}$ is an extension of parabolic line bundles

$$0 \longrightarrow \mathcal{L}(k_0) \longrightarrow \mathcal{P} \longrightarrow \mathcal{L}(k_0)^\vee \longrightarrow 0$$

for a uniquely determined $k_0 \in K$. Any parabolic subbundle different from $\mathcal{L}(k_0)$ is isomorphic to $(\mathcal{L}(k_0 + k))^\vee$ for some $k \in K \setminus \{0\}$. The parabolic degree of $\mathcal{L}(k)$ is given by the formula

$$2pardeg(\mathcal{L}(k)) = \varphi(k) + 1 - \sum(k\alpha),$$

where $\sum(\alpha)$ is the sum $\sum_{z \in I} \alpha_z$ for $\alpha = (\alpha_z)_{z \in I}$.

Proof: A well known theorem of Grothendieck states that $|\mathcal{P}|$ is isomorphic to a sum of line bundles $\mathcal{O}_{\mathbf{P}^1}(-m_1) \oplus \mathcal{O}_{\mathbf{P}^1}(-m_2)$ and one easily verifies $m_1 + m_2 = n$. So there is a uniquely determined parabolic subbundle $\mathcal{L} \subset \mathcal{P}$ with $deg|\mathcal{L}| = -m_1$, and in case $m_1 = m_2 = \frac{n}{2}$ with $\mathcal{L}(z_0) = V_{z_0}$. One immediately verifies that this implies $\mathcal{L} \cong \mathcal{L}(k_0)$ for some $k_0 \in K$. A parabolic subbundle $\mathcal{M}$ different from $\mathcal{L}(k_0)$ admits a nontrivial parabolic map to $\mathcal{P}/\mathcal{L} = \mathcal{L}^\vee$. In particular $\mathcal{M} \otimes \mathcal{O}(k) \cong \mathcal{L}(k_0)^\vee$ for some $k \in K \setminus \{0\}$. ♣

Note that there are only a finite number of hyperplanes $H_k = \{\sum(k\alpha) - \varphi(k) - 1 = 0\}$ for $k \in K$ contained in W.

Corollary 2.3: Suppose the line connecting two elements α, β of the weight space W does not intersect transversely any of the hyperplanes $H_k, k \in K$, then $\mathcal{U}(\alpha) \cong \mathcal{U}(\beta)$. Moreover $\mathcal{U}_s(\alpha) = \mathcal{U}(\alpha)$ for $\alpha \in W \setminus \cup_K H_k$.

Proof: The isomorphism is induced by identifying the underlying quasi–parabolic structures. ♣

Consider $\mathbf{P}^l$ as the l–fold symmetric product $Sym^l C$ of the projective line C and simultaneously as the space $\mathbf{P}roj(S^l V)$ of hyperplanes in $H^0(C; \mathcal{O}_C(l)) = S^l H^0(C; \mathcal{O}_C(1)) = S^l V$. Then C is naturally embedded in $\mathbf{P}^l$ via the diagonal map $C \longrightarrow Sym^l C; x \mapsto x^l$. The hyperplane corresponding to x is the kernel of the form $s_x^\vee \in H^0(C; \mathcal{O}_C(1))^\vee$ dual to a nontrivial section s_x of $\mathcal{O}_C(1)$ vanishing in x. The kernel of the form $(s_x^\vee)^l = (s_{x^l})^\vee$ then is the image of the inclusion $H^0(C; \mathcal{O}_C(l-1)) \longrightarrow H^0(C; \mathcal{O}_C(l))$ obtained by multiplication with s_x.

Let $V(z)$ be the point z and $V(2z)$ the curve $C \subset Sym^l C$. Denoting by $A * B$ the join of two subvarieties $A, B \subset \mathbf{P}^l$, the setting $V(k' + y) = V(k') * V(y)$ for $k = \sum_{I \setminus \{y\}} b_z z$ uniquely associates to each $k \in K$ a subvariety $V(k) \subset \mathbf{P}^l$ of dimension $min(\varphi(k) - 1, l)$. Any such variety is the join $Sec_i(C) * \{z_1\} * \ldots * \{z_j\}$ of a secant variety of the rational norm curve C with a number of points in $I \subset C$.

Applying the results of the first part, the following theorem can now be viewed as a summary of 2.6, 3.4, and 4.1 in Bauer–Okonek [89]:

Theorem 2.4: Let $\alpha \in W$ be a weight.
i) The moduli space $\mathcal{U}(\alpha)$ is connected. It admits a stratification by locally closed subvarieties, each of which is isomorphic to a Zariski open subset of a projective space.
ii) $\mathcal{U}_s(\alpha) = \emptyset$ if and only if $\sum(k\alpha) < 1$ for some $k \in K$ with $\varphi(k)$ even.
iii) $Sym^{n-3}C$ represents quasi–parabolic bundles on C. A Zariski open subset of $Sym^{n-3}C$ can be identified with a Zariski open subset of $\mathcal{U}_s(\alpha)$.
iv) An unstable parabolic bundle $[\mathcal{P}_\alpha] \in Sym^{n-3}C$ is contained in $V(k)$ for some $k \in K$. On the other hand a subvariety $V(k) \subset Sym^{n-3}C$ represents unstable bundles if and only if $\sum(k\alpha) > \varphi(k) + 1$.

Proof: 1) Let $\mathcal{P}$ be semistable, representing an element of $\mathcal{U}(\alpha)$. Because of 2.2 there is a short exact sequence

$$0 \longrightarrow \mathcal{L}(k) \longrightarrow \mathcal{P} \longrightarrow \mathcal{L}(k)^\vee \longrightarrow 0.$$

Since $\mathcal{P}ar\mathcal{H}om(\mathcal{L}(k)^\vee, \mathcal{L}(k)) = 0$, $\mathcal{P}$ is classified by a 1–dimensional linear subspace of $ParExt^1(\mathcal{L}(k)^\vee, \mathcal{L}(k))$. By choosing a suitable elliptic surface X over C one can identify the latter with $Ext^1_{\mathcal{O}_X}(\pi^p\mathcal{L}(k)^\vee, \pi^p\mathcal{L}(k))$, using 1.7.iii). As in the proof of 1.4, this Ext–group is isomorphic to $H^1(C; \pi_*\pi^p\mathcal{L}(k)^{\otimes 2})$. The easily checked isomorphism $\pi_*\pi^p(\mathcal{L}(k)^{\otimes 2}) \cong \mathcal{O}_C(-n + \varphi(k) + 1)$ and Serre duality finally achieve

$$\mathcal{P}ar\mathcal{E}xt^1(\mathcal{L}(k)^\vee, \mathcal{L}(k)) \cong H^0(C; \mathcal{O}_C(n - \varphi(k) - 3))^\vee.$$

Quasi–parabolic bundles admitting semistable parabolic structures are thus uniquely determined by elements of

$$\coprod_{k\in K} Proj(H^0(C;\mathcal{O}_C(n-\varphi(k)-3))) = \coprod_{k\in K} Sym^{n-3-\varphi(k)}C.$$

Stability being an open condition and the fact that there is only one component of maximal dimension $n-3$ imply i) and iii).
2) In what follows, keep in mind that computations with parabolic bundles can always be executed using vertical bundles (in the sense of the preceeding chapter) on an auxiliary elliptic surface. Let $\mathcal{P}$ be represented by an element of $Sym^{n-3}C$. The proof of 2.2. showed the only possible parabolic subbundles to be of the form $\mathcal{L}(k)^\vee$. The parabolic homomorphism $\mathcal{L}(k)^\vee \longrightarrow \mathcal{P} \longrightarrow \mathcal{L}(0)^\vee$ gives a parabolic morphism $g: \mathcal{O}_C \longrightarrow \mathcal{L}(k)\otimes\mathcal{L}(0)^\vee$. In the commuting diagram of parabolic bundles:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}(0)^{\otimes 2} & \longrightarrow & \mathcal{P}\otimes\mathcal{L}(0) & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow g & & \\ 0 & \longrightarrow & \mathcal{L}(0)\otimes\mathcal{L}(k) & \longrightarrow & \mathcal{P}\otimes\mathcal{L}(k) & \longrightarrow & \mathcal{L}(0)^\vee\otimes\mathcal{L}(k) & \longrightarrow & 0 \end{array}$$

obtained by tensoring with g, the map g lifts. So $[\mathcal{P}]$ is in the kernel of the map

$$H^1(\pi_*\pi^p g): H^1(C;\pi_*\pi^p\mathcal{L}(0)^{\otimes 2}) \longrightarrow H^1(C;\pi_*\pi^p(\mathcal{L}(0)\otimes\mathcal{L}(k))).$$

Serre duality identifies the kernel of $H^1(\pi_*\pi^p g)$ with the osculating linear subspace in $Proj(H^0(C;\mathcal{O}_C(n-3)) \cong Sym^{n-3}C$ determined by the zero divisor of $\pi_*\pi^p g$ on $C \subset Sym^{n-3}C$. The variety $V(k)$ is the union over all the possible osculating linear subspaces. This far we have seen that $\mathcal{P}$ admits a subbundle $\mathcal{L}(k)^\vee$ if and only if $[\mathcal{P}] \in V(k) \subset Sym^{n-3}C$. Such a subbundle destabilizes $\mathcal{P}$ iff $\mu(\mathcal{L}(k)^\vee) > 0$. With 2.2 this translates into the claim of iv).
To prove ii) one finally has to examine whether there exists an element $[\mathcal{P}] \in Sym^{n-3}C$ representing a stable bundle. This fails to be true only in two cases: Either $\mu(\mathcal{L}(0)) > 0$ (equivalently $\sum(\alpha) < 1$), or $\mu(\mathcal{L}(k')^\vee) > 0$ for some $k' \in K$ with $dim\ V(k') = \varphi(k') - 1 = n-3$. In the latter case consider an element $k \in K$ with $k + k' \equiv \sum_I z \ \ mod\ 2K$. The condition $\mu(\mathcal{L}(k')^\vee) > 0$ is equivalent to $\sum(k\alpha) < 1$. This follows from the identities $\sum(k'\alpha) + \sum(k\alpha) = n$ and

$\{k \in K \mid k + k' \equiv \sum_I z$ for some $k' \in K$ with $\varphi(k') = n-2\} \equiv \{k \in K \mid \varphi(k) \equiv 0\ mod\ 2\}$.
♣

An easy computation shows that for $\alpha = (\alpha_z)$ with $\alpha_z = \frac{1}{n-1}$ the moduli space is isomorphic to $\mathbf{P}^{n-3} \cong Sym^{n-3}C$.
To analyze the general situation, assume $\alpha, \beta \in W \setminus (\cup_{k\in K} H_k)$ are separated by exactly one hyperplane H_{k_0} and choose a weight $\gamma \in H_{k_0} \setminus (\cup_{K\setminus k_0} H_k)$ on this hyperplane.

Theorem 2.5: There exists a smooth complex manifold $\mathcal{U}(\alpha,\beta)$ and a commuting diagram

$$\begin{array}{ccc} \mathcal{U}(\alpha,\beta) & \longrightarrow & \mathcal{U}(\beta) \\ \downarrow & & \downarrow \\ \mathcal{U}(\alpha) & \longrightarrow & \mathcal{U}(\gamma), \end{array}$$

where the morphisms are birational equivalences. The exceptional loci form a diagram

$$\begin{array}{ccc} \mathbf{P}^{\varphi(k_0)-1} \times \mathbf{P}^{n-\varphi(k_0)-3} & \xrightarrow{pr_2} & \mathbf{P}^{n-\varphi(k_0)-3} \\ pr_1 \downarrow & & \downarrow \\ \mathbf{P}^{\varphi(k_0)-1} & \longrightarrow & \mathbf{P}^0. \end{array}$$

Moreover the exceptional loci are smoothly embedded, except perhaps in U_γ.

Remark 2.6: As an immediate consequence for any weight $\alpha \in W$ the moduli space $\mathcal{U}(\alpha)$ can be constructed from $Sym^{n-3}C$ by a sequence of blow ups and blow downs the following way: After the i–th step the strict transforms of the i–dimensional subvarieties $V(k) \cong Sec_l C * \{z_1\} * \ldots * \{z_j\}$ with $\varphi(k) = i + 1$ representing unstable bundles are isomorphic to projective spaces $\mathbf{P}^i$. After blowing up along these $\mathbf{P}^i$, one can blow down "onto the other factor in the exceptional divisors", completing the (i+1)–st step.

Proof of 2.5: Let $\mathcal{L}(k_0, \alpha)$ be the the parabolic line bundle with weights determined by k_0 and α as in 2.2 and suppose $\mu(\mathcal{L}(k_0, \alpha)) > 0 > \mu(\mathcal{L}(k_0, \beta))$. The nontrivial parabolic extensions

$$0 \longrightarrow \mathcal{L}(k_0, \beta) \longrightarrow \mathcal{P}_\beta \longrightarrow \mathcal{L}(k_0, \beta 0)^\vee \longrightarrow 0$$

are stable: Any subbundle $\mathcal{N}_\beta$ of $\mathcal{P}_\beta$ different from $\mathcal{L}(k_0, \beta)$ maps nontrivially to $\mathcal{L}(k_0, \beta)^\vee$ and therefor $\mu(\mathcal{N}_i) < -\mu(\mathcal{L}(k_0, i))$ for $i \in \{\alpha, \beta\}$. The inequality $\mu(\mathrm{N}_\beta) > 0$ thus would imply that α and β are situated on different sides of the hyperplane $\{pardeg\ \mathcal{N} = 0\} \neq H_{k_0}$, contradicting the assumption. Similarly all extensions

$$0 \longrightarrow \mathcal{L}(k_0, \alpha)^\vee \longrightarrow \mathcal{Q}_\alpha \longrightarrow \mathcal{L}(k_0, \alpha) \longrightarrow 0$$

are stable. The extensions $\mathcal{Q}$ and $\mathcal{P}$ are classified by projective spaces $\mathbf{P}^{\varphi(k_0)-1}$ and $\mathbf{P}^{n-\varphi(k_0)-3}$, respectively. The vector bundle $\mathcal{Q}_\gamma$ with the same quasi–parabolic structure as $\mathcal{Q}_\alpha$ is semistable, as is $\mathcal{P}_\gamma$. These semistable bundles are represented by one element $\mathbf{P}^0 \in \mathcal{U}(\gamma)$.
Now let $\mathcal{U}(\alpha, \beta) \subset \mathcal{U}(\alpha) \times \mathcal{U}(\beta)$ denote the subset of all pairs $(\mathcal{E}_\alpha, \mathcal{F}_\beta)$ of stable parabolic bundles admitting a nontrivial parabolic map $f : \mathcal{E}_\beta \longrightarrow \mathcal{F}_\beta$. Note that such a map only exist if either $\mathcal{E}_\beta \cong \mathcal{F}_\beta$ or $\mathcal{E}_\beta \cong \mathcal{Q}_\beta$ and $\mathcal{F}_\beta \cong \mathcal{P}_\beta$. One may view $\mathcal{E}_\beta$ and $\mathcal{F}_\beta$ as bundles on an auxiliary elliptic surface. If $\mathcal{V}_\mathcal{E}$ and$\mathcal{V}_\mathcal{F}$ are versal deformations of these bundles, then $\mathcal{U}(\alpha, \beta)$ locally is the support of the function $dim\ H^0(X_t, \mathcal{V}_\mathcal{E}^\vee \times \mathcal{V}_\mathcal{F})$. The semicontinuity theorem implies that $\mathcal{U}(\alpha, \beta)$ is a subvariety of $\mathcal{U}(\alpha) \times \mathcal{U}(\beta)$. It remains to show the smoothness of this variety. Dropping the subskript β, one can apply lemma 2.7 below.

By an argument similar to the proof of $(*)$ in 1.4, it remains to compute the map $H^1(\pi_*(f \otimes id - id \otimes f^\vee))$. The bundles in the exact sequence

$$0 \longrightarrow \mathcal{E} \otimes \mathcal{F}^\vee \stackrel{(id \otimes f^\vee, f \otimes id)}{\longrightarrow} \mathcal{E} \otimes \mathcal{E} \oplus \mathcal{F} \otimes \mathcal{F}^\vee \stackrel{f \otimes id - id \otimes f^\vee}{\longrightarrow} \mathcal{F} \otimes \mathcal{E}^\vee \longrightarrow 0$$

are simple, hence the sequence remains exact after application of the functor $H^1(C; \pi_*(.))$. The identities $\mathcal{E} \cong \mathcal{E}^\vee$ and $\mathcal{F} \cong \mathcal{F}^\vee$ show that the Zariski tangent space of $\mathcal{U}(\alpha, \beta)$ has the same dimension as the Zariski tangent space of $\mathcal{U}(\alpha)$ or $\mathcal{U}(\beta)$. This proves the theorem.♣

Lemma 2.7: The Zariski tangent space of $\mathcal{U}(\alpha,\beta)$ at $([\mathcal{E}],[\mathcal{F}])$ is isomorphic to the kernel of the map

$$H^1(f\otimes id - id\otimes f^\vee): H^1(X;\mathcal{E}\otimes\mathcal{E}^\vee\oplus\mathcal{F}\otimes\mathcal{F}^\vee)\longrightarrow H^1(X;\mathcal{F}\otimes\mathcal{E}^\vee).$$

Proof: Apply the method of Forster–Knorr [74], §9: One has to compute the extensions over the double point p (with structure sheaf $\mathbf{C}[\epsilon]/\epsilon^2$) of the pair $(\mathcal{E},\mathcal{F})$ admitting an extension of f. The holomorphic vector bundles $\mathcal{E}$ and $\mathcal{F}$ are described by cocycles (g_{ij}) and $(h_{ij})\in Z^1(\mathcal{U}^*;Gl(2,\mathcal{O}_X))$ for a suitable open covering $\mathcal{U}^*$ of X. The Čech cohomology $H^*(\mathcal{U};\mathcal{H}om(\mathcal{E},\mathcal{F}))$ for an open covering is the homology of the cochain complex $C^*(\mathcal{U};M(2\times 2,\mathcal{O}_X))$ with differential

$$(\delta x)_{i_0\ldots i_{q+1}} = h_{i_0i_1}x_{i_1\ldots i_{q+1}} + \sum_{k=1}^{q}(-1)^k x_{i_0\ldots\hat{i}_k\ldots i_{q+1}} - (-1)^q x_{i_0\ldots i_q}g_{i_qi_{q+1}}.$$

The space of vector bundles $\tilde{\mathcal{E}}$ over $X\times p$ restricting to $\mathcal{E}$ over $X\subset X\times p$ is isomorphic to $H^1(X,\mathcal{E}nd\ \mathcal{E})$. For a cocycle γ_{ij} the corresponding bundle $\tilde{\mathcal{E}}$ is represented by the cocycle $(g_{ij}+\epsilon\gamma_{ij})\in Z^1(\mathcal{U};Gl(2,\mathcal{O}_X[\epsilon]))$. For $\tilde{\mathcal{F}}$ choose a representing cocycle $(h_{ij}+\epsilon\kappa_{ij})$. An extension $\tilde{f}=(f-i+\epsilon\phi_i):\tilde{\mathcal{E}}\longrightarrow\tilde{\mathcal{F}}$ of the map $f=(f_i)\in Z^0(\mathcal{U},\mathcal{H}om(\mathcal{E},\mathcal{F}))$ has to satisfy the cocycle relation

$$(h_{ij}+\epsilon\kappa_{ij})(f_j+\epsilon\phi_j) = (f_i+\epsilon\phi_i)(g_{ij}+\epsilon\gamma_{ij})$$

or equivalently

$$f_i\gamma_{ij}-\kappa_{ij}f_j = h_{ij}\phi_j - \phi_i g_{ij}.$$

But this is just the claim. ♣

The theorem applies to a conjecture of Fintushel–Stern [88], proved by Kirk–Klassen [89], compare also Bauer–Okonek [89] and Furuta–Steer [89].

Corollary 2.8: The representation space $\mathcal{R} = Hom(\Gamma, SU(2))/ad\ SU(2)$ of a perfect cocompact Fuchsian group Γ admits a Morse function with only even indices.

Proof: Any component of $\mathcal{R}$ is diffeomorphic to $\mathcal{U}(\rho)$ for some weight $\rho\in W\setminus\cup_K H_k$. For 1– or 2–dimensional rational manifolds it is easy to directly construct such Morse functions. Otherwise by a theorem of Smale [62] it suffices to prove the vanishing of the odd dimensional integral cohomology. Taking a path in W from a σ with $\mathcal{U}(\sigma)\cong\mathbf{P}^{n-3}$ to the given ρ, which intersects only one hyperplane H_k at a time, one is reduced to computing the cohomology of the blow–ups in 2.5 (compare Griffiths–Harris [78], p. 602ff):

$$H^*(\mathcal{U}(\alpha,\beta))\cong H^*(\mathcal{U}(\alpha))\oplus H^*(\mathbf{P}^{\varphi(k_0)-1}\times\mathbf{P}^{n-\varphi(k_0)-3})/H^*(\mathbf{P}^{\varphi(k_0)-1}).$$

♣

The Betti numbers of a component were in principle computed in Bauer–Okonek [89]. Furuta–Steer [89] gave a formula for the Poincaré polynomial:

Corollary 2.9: The moduli space $\mathcal{U}(\alpha)$ for $\alpha \in W \setminus \cup_K H_k$ has the Poincaré polynomial

$$P_t(\mathcal{U}(\alpha)) = \frac{(1+t^2)^n - \sum_{\bar{k}\in K/2K} t^{2\varphi(k)}}{(1-t^2)(1-t^4)},$$

where $k \in K$ is the unique element in the *mod* $2K$ equivalence class $\bar{k}$ satisfying the inequalities

$$-1+\varphi(k) < \sum(k\alpha) < 1+\varphi(k).$$

Proof: First we check the formula for $\alpha = (\alpha_z)_{z\in I}$ with $\alpha_z = \frac{1}{n-1}$. By theorem 2.4, $\mathcal{U}(\alpha) = \emptyset$, so the formula to verify is $(1+t^2)^n = \sum_{\bar{k}\in K/2K} t^{2\varphi(k)}$. For the element $k_J = \sum_{z\in J} z \in K, J \subset I$, one has

$$-1+\varphi(k_J) < \sum(k_J\alpha) = \varphi(k_J) + \frac{n-2\varphi(k)}{n+1} < 1+\varphi(k_J).$$

Thus the coefficient of t^{2l} in the polynomial $\sum_{\bar{k}\in K/2K} t^{2\varphi(k)}$ is $\binom{n}{l}$, which was to be shown. Now suppose $\alpha, \beta \in W \setminus \cup_K H_k$ are separated by one hyperplane only, say

$$\sum(k_0\alpha) < 1+\varphi(k_0) < \sum(k_0\beta).$$

For the elements k^α and k^β in the *mod* $2K$ equivalence class $\bar{k}$ associated to the weights α and β one easily checks $k^\alpha = k^\beta$ with the exeptions: $\bar{k}_0$ and $\overline{k_0 + \sum_I z}$. For these one has $k_0^\alpha = k_0$; $k_0^\beta = k_0 + 2z$; $\varphi((k_0+\sum_I z)^i) = n - \varphi(k_0^i)$ for $i \in \{\alpha, \beta\}$. A simple computation verifies the claim:

$$\begin{aligned}
&[(1+t^2)^n - \sum_{K/2K} t^{2\varphi(k^\beta)}] - [(1+t^2)^n - \sum_{K/2K} t^{2\varphi(k^\alpha)}] \\
&= t^{2\varphi(k_0)} + t^{2(n-\varphi(k_0))} - t^{2\varphi(k_0)+2} - t^{2(n-\varphi(k_0)-2)} \\
&= \left(\frac{(1-t^{2(n-\varphi(k_0)-2)})}{1-t^2} - \frac{(1-t^{2\varphi(k_0)})}{1-t^2}\right)(1-t^4)(1-t^2) \\
&= [P_t(\mathbf{P}^{n-\varphi(k_0)-3}) - P_t(\mathbf{P}^{\varphi(k_0)-1})](1-t^2)(1-t^4) \\
&= [P_t(\mathcal{U}(\beta)) - P_t(\mathcal{U}(\alpha))](1-t^2)(1-t^4)
\end{aligned}$$

♣

The theorem also describes the singularities in $\mathcal{U}(\alpha)$:

Corollary 2.10: Let $\alpha \in W$ be a weight. The singularities of $\mathcal{U}(\alpha)$ are isolated and in 1–1 correspondence to hyperplanes $H_k \subset W$ containing α. Each singularity is determined by its resolution: The exceptional divisor E is isomorphic to $\mathbf{P}^l \times \mathbf{P}^{n-l-4}$ and the projection to either factor leads to a small resolution. In particular the normal bundle of E is isomorphic to $\mathcal{O}_{\mathbf{P}^l\times\mathbf{P}^{n-l-4}}(-1,-1)$.

Proof: The singularities correspond to equivalence classes of semistable bundles. Two semistable bundles are equivalent, if their stable factors are isomorphic. But the stable factors corresponding to H_k are $\mathcal{L}(k)$ and $\mathcal{L}(k)^\vee$. ♣

The subset of W of weights for which the moduli space $\mathcal{U}(\alpha)$ is isomorphic to a projective space is not connected for $n \geq 5$. Hence the procedure explained in 2.6 will describe Cremona transformations $\mathbf{P}^{n-3} - - \rightarrow \mathbf{P}^{n-3}$ for weights in different components. Two examples will be discussed:

1) The transformation $\phi : \mathbf{P}^{n-3} - - \rightarrow \mathbf{P}^{n-3}$, which in homogeneous coordinates is given by $x_i \mapsto x_0 x_1 \ldots \hat{x}_i \ldots x_n$, is not well defined in any hyperplane $x_i = 0$. ϕ can be described in a sequence of $n-3$ blow ups and blow downs: Take the simplex generated by the $n-2$ points $(0 : \ldots 0 : 1 : 0 \ldots : 0)$. First blow up the vertices. Inductively the strict transforms of the (i-1)-dimensinal linear subspaces spanned by i of the $n-2$ points can be blown up with exceptional divisor $\mathbf{P}^{i-1} \times \mathbf{P}^{n-i-3}$. The projection onto the second factor can be extended to a blow down of the ambient space completing the i–th step.

Let the weights α, β be given by $\alpha_z = \frac{1}{n-1}$ for $z \in I$ and

$$\beta_z = \begin{cases} \frac{1}{2n} & \text{for } z \in I \setminus \{z_1, z_2\} \\ \frac{3(n-1)}{4n} & \text{else.} \end{cases}$$

Then $\mathcal{U}(\alpha) \cong \mathcal{U}(\beta) \cong \mathbf{P}^{n-3}$ and the Cremona transformation described in 2.7 coincides with ϕ.
This follows from the fact that $\mathcal{U}(\alpha) \cong Sym^{n-3}C$ and the subvarieties $V(k)$ representing unstable bundles with weights β are the linear subspaces spanned by the proper subsets of $I \setminus \{z_1, z_2\} \subset C \subset Sym^{n-3}C$. ♣

2) For the second example choose α as above and

$$\beta_z = \begin{cases} \frac{1}{n-1} & \text{if } n \text{ is odd and } z = z_1 \\ \frac{n-2}{n-1} & \text{if } n \text{ is even or } z \in I \setminus \{z_1\}. \end{cases}$$

Lemma 2.11: The moduli space $\mathcal{U}(\beta)$ is isomorphic to $\mathbf{P}^{n-3}$.

Proof: Let $\mathcal{L}$ be the line bundle with parabolic structure concentrated in I, if n is even and else in $I \setminus \{z_1\}$, weights $\frac{1}{2}$ and $deg|\mathcal{L}| = -[\frac{n}{2}]$. One easily verifies that tensoring a parabolic bundle with $\mathcal{L}$ changes weights α into weights β and vice versa. ♣

The resulting Cremona transformation can be described by the procedure indicated in 2.7. The subvariety $V(k) \subset Sym^{n-3}C$ corresponds to unstable bundles with weights β, if and only if

$$k \in \{k \in K \mid \exists k' \in K \text{ and } z \in I \text{ with } k + k' = (n-4)z_1 + z\}.$$

In the 3-dimensional case one blows up six points on a rational norm curve $C \subset \mathbf{P}^3$. Blowing up further the strict transforms of the fifteen connecting lines of these points and of C, one can blow down the exceptional divisors $\mathbf{P}^1 \times \mathbf{P}^1$ onto the second factor. The strict transforms of the varieties $C * \{z\}, z \in I$ now are isomorphic to $\mathbf{P}^2$ and can be blown down . The resulting space again is isomorphic to $\mathbf{P}^3$.

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