L^2-Topological Invariants of 3-manifolds

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Abstract : We give results on the L^2-Betti numbers and Novikov-Shubin invariants of compact manifolds, especially 3-manifolds. We first study the Betti numbers and Novikov-Shubin invariants of a chain complex of Hilbert modules over a finite von Neumann algebra. We establish inequalities among the Novikov-Shubin invariants of the terms in a short exact sequence of chain complexes. Our algebraic results, along with some analytic results on geometric 3-manifolds, are used to compute the L^2-Betti numbers of compact 3-manifolds which satisfy a weak form of the geometrization conjecture, and to compute or estimate their Novikov-Shubin invariants.

0. Introduction

The L^2-Betti numbers of a smooth closed manifold M, introduced by Atiyah [2], are invariants of M which are defined in terms of the universal cover $\widetilde{M}$. Roughly speaking, if M is Riemannian then the p-th L^2-Betti number $b_p(M)$ measures the size of the space of harmonic L^2 p-forms on $\widetilde{M}$, relative to the action of the fundamental group π on $\widetilde{M}$. We give the precise definition later. The L^2-Betti numbers are homotopy invariants of M (Dodziuk [13]), and can be extended to become Γ-homotopy invariants of topological spaces upon which a countable group Γ acts (Cheeger-Gromov [11]).

By means of a Laplace transform, there is an interpretation of the L^2-Betti numbers in terms of the large-time asymptotics of heat flow on $\widetilde{M}$. Let $e^{-t\widetilde{\Delta_p}}(x,y)$ be the Schwartz kernel of the heat operator acting on L^2 p-forms on $\widetilde{M}$. The von Neumann trace of the heat operator is given by

$$\mathrm{tr}_{N(\pi)}\left(e^{-t\widetilde{\Delta_p}}\right) = \int_{\mathcal{F}} \mathrm{tr}\left(e^{-t\widetilde{\Delta_p}}(x,x)\right) dvol(x),$$

where $\mathcal{F}$ is a fundamental domain for the π-action on $\widetilde{M}$ and the trace on the right-hand-side is the ordinary trace on $End(\wedge^p(T_x^*\widetilde{M}))$. The L^2-Betti numbers of M can be expressed by

$$b_p(M) = \lim_{t\to\infty} \mathrm{tr}_{N(\pi)}\left(e^{-t\widetilde{\Delta_p}}\right).$$

In many examples one finds that $\mathrm{tr}_{N(\pi)}\left(e^{-t\widetilde{\Delta_p}}\right) - b_p(M)$ approaches zero with an exponential or power decay as $t \to \infty$. Novikov and Shubin [33] introduced invariants which quantify this phenomenon. If there is an exponential decay, put $\widetilde{\alpha}_p(M) = \infty^+$. Otherwise, put

$$\widetilde{\alpha}_p(M) = \sup\left\{\beta_p \,:\, \mathrm{tr}_{N(\pi)}\left(e^{-t\widetilde{\Delta_p}}\right) - b_p(M) \text{ is } O(t^{-\beta_p/2}) \text{ as } t \to \infty\right\} \in [0,\infty].$$

Roughly speaking, $\widetilde{\alpha}_p(M)$ measures the thickness of the spectrum of $\widetilde{\Delta_p}$ near 0; the larger $\widetilde{\alpha}_p(M)$, the thinner the spectrum near 0. Novikov and Shubin stated that these invariants are independent of the choice of Riemannian metric on M, and hence are smooth invariants of M. The first author showed that they are defined for all topological manifolds and depend only on the homeomorphism type, and computed them in certain cases [24]. Gromov and Shubin [18] proved that the Novikov-Shubin invariants are homotopy invariants of M. A combinatorial Novikov-Shubin invariant was defined by Efremov in [15] and shown to be the same as the analytically defined invariant, again under the assumption that M is closed.

In this paper we give some results on the L^2-Betti numbers and Novikov-Shubin invariants of compact manifolds (possibly with boundary), especially 3-manifolds. Our interest in these invariants comes from our work on related L^2-invariants, the L^2-Reidemeister and analytic torsions [7, 24, 27, 28]. In particular, one wishes to know that the Novikov-Shubin invariants of a manifold are all positive, in order for the L^2-torsions to be defined. We make some remarks on the L^2-torsions in section 9.

We define an invariant $\alpha_p(M)$ in terms of the boundary operator acting on p-chains on $\widetilde{M}$. The relationship with $\widetilde{\alpha_p}(M)$ is that $\widetilde{\alpha_p}(M) = min(\alpha_p(M), \alpha_{p+1}(M))$, where the left-hand-side is defined using p-forms on $\widetilde{M}$ which satisfy absolute boundary conditions if M has boundary. Let us say that a prime 3-manifold is *exceptional* if it is closed and no finite cover of it is homotopy equivalent to a Haken, Seifert or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known, and standard conjectures (Thurston geometrization conjecture, Waldhausen conjecture) imply that there are none. The main results of this paper are given in the following theorem:

Theorem Let M be the connected sum $M_1\sharp\ldots\sharp M_r$ of (compact connected orientable) nonexceptional prime 3-manifolds M_j. Assume that $\pi_1(M)$ is infinite. Then

1. The L^2-Betti numbers of M are given by:

$$\begin{aligned}
b_0(M) &= 0 \\
b_1(M) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} - \chi(M) + \mid \{C \in \pi_0(\partial M) \; s.t. \; C \cong S^2\} \mid \\
b_2(M) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} + \mid \{C \in \pi_0(\partial M) \; s.t. \; C \cong S^2\} \mid \\
b_3(M) &= 0.
\end{aligned}$$

Equivalently, if $\chi(\pi_1(M))$ denotes the rational-valued group Euler characteristic then $b_1(M) = -\chi(\pi_1(M))$ and $b_2(M) = \chi(M) - \chi(\pi_1(M))$.

In particular, M has vanishing L^2-cohomology iff M is homotopy equivalent to $S^1 \times S^2$, $RP^3 \sharp RP^3$ or an irreducible 3-manifold with infinite fundamental group whose boundary is empty or a union of tori.

2. Let the Poincaré associate $P(M)$ be the connected sum of the M_j's which are not 3-disks or homotopy 3-spheres. Then $\alpha_p(P(M)) = \alpha_p(M)$ for $p \leq 2$. We have $\alpha_1(M) = \infty^+$ except for the following cases:

   (a) $\alpha_1(M) = 1$ if $P(M)$ is $S^1 \times D^2$, a closed $S^2 \times R$-manifold or homotopy equivalent to $RP^3 \sharp RP^3$.

   (b) $\alpha_1(M) = 2$ if $P(M)$ is $T^2 \times I$ or a twisted I-bundle over the Klein bottle K.

   (c) $\alpha_1(M) = 3$ if $P(M)$ is a closed R^3-manifold.

   (d) $\alpha_1(M) = 4$ if $P(M)$ is a closed Nil-manifold.

   (e) $\alpha_1(M) = \infty$ if $P(M)$ is a closed Sol-manifold.

3. $\alpha_2(M) > 0$.

4. If M is a closed hyperbolic 3-manifold then $\alpha_2(M) = 1$. If M is a closed Seifert 3-manifold then $\alpha_2(M)$ is given in terms of the Euler class e of the bundle and the Euler characteristic χ of the base orbifold by:

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	∞^+	3	1
$e \neq 0$	∞^+	2	1

   If M is a Seifert 3-manifold with boundary then $\alpha_2(M)$ is ∞^+ if $M = S^1 \times D^2$, 2 if M is $T^2 \times I$ or a twisted I-bundle over K, and 1 otherwise. If M is a closed Sol-manifold then $\alpha_2(M) \geq 1$.

5. If ∂M contains an incompressible torus then $\alpha_2(M) \leq 2$. If one of the M_j's is closed with infinite fundamental group and does not admit an R^3, $S^2 \times R$ or Sol-structure, then $\alpha_2(M) \leq 2$.

6. If M is closed then $\alpha_3(M) = \alpha_1(M)$. If M is not closed then $\alpha_3(M) = \infty^+$. ■

Let us briefly indicate how we prove that $\alpha_2(M)$ is positive. The important case is when M is an irreducible Haken 3-manifold with infinite fundamental group whose boundary is empty or consists of incompressible tori; the general case follows by further arguments. The Jaco-Shalen-Johannson splitting of M, together with the work of Thurston, gives a family of embedded incompressible tori which cut the manifold into pieces that are either Seifert manifolds or whose interiors admit complete finite-volume hyperbolic metrics. The α_2-invariants of the Seifert pieces can be computed explicitly. By analytic means we derive a lower bound for the α_2-invariants of the (compact) hyperbolic pieces. We then face the problem of understanding what happens to the Novikov-Shubin invariants when one glues along incompressible tori. This is done algebraically by means of inequalities among the Novikov-Shubin invariants of the terms in a short exact sequence.

A description of the contents of the paper is as follows. The natural algebraic setting for our work is that of Hilbert $\mathcal{A}$-modules, where $\mathcal{A}$ is a finite von Neumann algebra. In Section 1 we define the Betti numbers and Novikov-Shubin invariants of a morphism of finitely generated Hilbert $\mathcal{A}$-modules, and derive some useful inequalities on the Novikov-Shubin invariants. In Section 2 we define the Betti numbers and Novikov-Shubin invariants of a finite Hilbert $\mathcal{A}$-chain complex. If one has a short exact sequence of finite Hilbert $\mathcal{A}$-chain complexes then there is an induced long weakly exact homology sequence, with which one can relate the Betti numbers of the chain complexes (Cheeger-Gromov [10]). We show that in addition, the Novikov-Shubin invariants of the chain complexes are related. We prove

Theorem 2.2: *Let $0 \longrightarrow C \xrightarrow{j} D \xrightarrow{k} E \longrightarrow 0$ be an exact sequence of finite Hilbert $\mathcal{A}$-chain complexes. Denote the boundary operator in the long weakly exact homology sequence [10, Theorem 2.1] by $\delta_p : H_p(E) \longrightarrow H_{p-1}(C)$. Then*

1. $\dfrac{1}{\alpha_p(D)} \leq \dfrac{1}{\alpha_p(C)} + \dfrac{1}{\alpha_p(E)} + \dfrac{1}{\alpha(\delta_p)}$.
2. $\dfrac{1}{\alpha_p(E)} \leq \dfrac{1}{\alpha_{p-1}(C)} + \dfrac{1}{\alpha_p(D)} + \dfrac{1}{\alpha(H_{p-1}(j))}$.
3. $\dfrac{1}{\alpha_p(C)} \leq \dfrac{1}{\alpha_p(D)} + \dfrac{1}{\alpha_{p+1}(E)} + \dfrac{1}{\alpha(H_p(k))}$. ∎

In Section 3 we give examples to show that these inequalities are sharp.

In Section 4 we specialize to the case of manifolds, in which $\mathcal{A}$ is the group von Neumann algebra $N(\pi)$ of the fundamental group π. Proposition 4.2 gives the relations on the L^2-Betti numbers and Novikov-Shubin invariants due to Poincaré duality, and Proposition 4.7 computes the L^2-Betti numbers and Novikov-Shubin invariants of connected sums. In Theorem 4.8 we show that if M admits a homotopically nontrivial S^1-action then the L^2-Betti numbers vanish and the Novikov-Shubin invariants are bounded below by 1. In Corollary 4.4 we show that the Novikov-Shubin invariants of closed manifolds of dimension less than or equal to 4 depend only on the fundamental group. In Section 5 we compute the

L^2-Betti numbers and Novikov-Shubin invariants of Seifert 3-manifolds (Theorems 5.1 and 5.4).

Section 6 first extends the results of [13, 15] on the equality of combinatorial and analytic L^2-topological invariants from the case of closed manifolds to that of manifolds with boundary. We then consider the Novikov-Shubin invariants of a compact 3-manifold M whose interior admits a complete finite-volume hyperbolic structure. If M is closed, the Novikov-Shubin invariants were computed in [24]. If M is not closed then we use a Mayer-Vietoris construction in the analytic setting, along with Theorem 2.2, to derive needed inequalities on the Novikov-Shubin invariants of the compact manifold, defined with absolute boundary conditions.

The results on 3-manifolds, Theorems 7.1 and 7.8, are proven in Section 7. Section 8 gives some applications of our results to the question of whether a covering space can have an invertible differential-form Laplacian. Section 9 has some remarks and gives some conjectures that are supported by the results of this paper. In the appendix we compute the L^2-Betti numbers and Novikov-Shubin invariants of infinite cyclic covers in terms of the homology of the cover.

The sections of the paper are:

1. L^2-Betti numbers and Novikov-Shubin invariants for morphisms of Hilbert $\mathcal{A}$-modules
2. L^2-Betti numbers and Novikov-Shubin invariants for Hilbert $\mathcal{A}$-chain complexes
3. Examples proving sharpness of various inequalities
4. L^2-Betti numbers and Novikov-Shubin invariants for manifolds
5. Seifert 3-manifolds
6. Analytic L^2-Betti numbers and Novikov-Shubin invariants for manifolds with boundary, and hyperbolic 3-manifolds
7. L^2-Betti numbers and Novikov-Shubin invariants for 3-manifolds
8. L^2-contractibility
9. Remarks and conjectures

A. Infinite cyclic coverings
References

To understand the statements of Sections 4-9, it suffices to understand Definitions 1.2, 1.7 and 2.1.

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1. L^2-Betti numbers and Novikov-Shubin invariants for morphisms of Hilbert $\mathcal{A}$-modules

In this section we introduce the Betti numbers and Novikov-Shubin invariants for morphisms of finitely generated Hilbert $\mathcal{A}$-modules over a finite von Neumann algebra $\mathcal{A}$. We study their behaviour under composition and exact sequences. For background material on finite von Neumann algebras and their Hilbert modules, we refer to [2, 9, 12, 27].

Let $\mathcal{A}$ be a von Neumann algebra with finite faithful normal trace $\mathrm{tr}_{\mathcal{A}}$. Let $l^2(\mathcal{A})$ denote the Hilbert completion of $\mathcal{A}$ with respect to the inner product given by $\mathrm{tr}_{\mathcal{A}}(a^*b)$ for $a, b \in \mathcal{A}$. A *Hilbert $\mathcal{A}$-module* is a Hilbert space V with a continuous left $\mathcal{A}$-module structure such that there exists an isometric $\mathcal{A}$-embedding of V into $l^2(\mathcal{A}) \otimes H$ for some Hilbert space H. A morphism of Hilbert $\mathcal{A}$-modules $f : U \longrightarrow V$ is a bounded operator which is compatible with the $\mathcal{A}$-multiplication. A Hilbert $\mathcal{A}$-module V is *finitely generated* if there is a surjective morphism $\oplus_{i=1}^n l^2(\mathcal{A}) \longrightarrow V$ for some positive integer n. The *dimension* $\dim_{\mathcal{A}}(V)$ of a finitely generated Hilbert $\mathcal{A}$-module is the trace of any projection $pr : \oplus_{i=1}^n l^2(\mathcal{A}) \longrightarrow \oplus_{i=1}^n l^2(\mathcal{A})$ whose image is isometrically $\mathcal{A}$-isomorphic to V. A morphism $f : U \longrightarrow V$ is a *weak isomorphism* if its kernel is trivial and its image is dense. A sequence of Hilbert $\mathcal{A}$-modules $0 \longrightarrow U \xrightarrow{j} V \xrightarrow{q} W \longrightarrow 0$ is *weakly exact* if j is injective, $\mathrm{clos}(\mathrm{im}(j)) = \ker(q)$ and q has dense image.

Lemma 1.1

1. *If $0 \longrightarrow U \xrightarrow{j} V \xrightarrow{q} W \longrightarrow 0$ is weakly exact then*
$$\dim_{\mathcal{A}}(U) - \dim_{\mathcal{A}}(V) + \dim_{\mathcal{A}}(W) = 0.$$
2. *Let $f : U \longrightarrow V$ be a weak isomorphism and $L \subset V$ be a Hilbert $\mathcal{A}$-submodule. Then*
$$\dim_{\mathcal{A}}(f^{-1}(L)) = \dim_{\mathcal{A}}(L).$$

Proof : 1.) We have the exact sequence $0 \longrightarrow \ker(q) \longrightarrow V \longrightarrow \ker(q)^{\perp} \longrightarrow 0$ and the assertion is well known in this case. If $f : W_1 \longrightarrow W_2$ is a weak isomorphism, the polar decomposition theorem yields an isometric $\mathcal{A}$-isomorphism $W_1 \longrightarrow W_2$ and so

$$\dim_{\mathcal{A}}(W_1) = \dim_{\mathcal{A}}(W_2).$$

There are canonical weak isomorphisms $U \longrightarrow \ker(q)$ and $\ker(q)^{\perp} \longrightarrow W$, and the claim follows.

2.) We decompose f as

$$\begin{pmatrix} k & g \\ 0 & h \end{pmatrix} : f^{-1}(L) \oplus \left(f^{-1}(L)\right)^{\perp} \longrightarrow L \oplus L^{\perp}.$$

If $u \in (f^{-1}(L))^{\perp}$ is in the kernel of h then, thinking of u as an element of U, $f(u)$ lies in L, and so u belongs to $f^{-1}(L)$. Thus $u = 0$. This shows that h is injective. Since k is also injective, we conclude from assertion 1.)

$$\begin{aligned}
&\dim_{\mathcal{A}}(f^{-1}(L)) = \dim_{\mathcal{A}}(\operatorname{clos}(f(f^{-1}(L)))) \leq \dim_{\mathcal{A}}(L) \\
&\dim_{\mathcal{A}}\left(f^{-1}(L)^{\perp}\right) = \dim_{\mathcal{A}}\left(\operatorname{clos}(f(f^{-1}(L)^{\perp}))\right) \leq \dim_{\mathcal{A}}(L^{\perp}) \\
&\dim_{\mathcal{A}}(L) + \dim_{\mathcal{A}}(L^{\perp}) = \dim_{\mathcal{A}}(V) \\
&\dim_{\mathcal{A}}(f^{-1}(L)) + \dim_{\mathcal{A}}\left(f^{-1}(L)^{\perp}\right) = \dim_{\mathcal{A}}(U) \\
&\dim_{\mathcal{A}}(U) = \dim_{\mathcal{A}}(V)
\end{aligned}$$

Now the claim follows. ■

Let $f : U \longrightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{A}$-modules. Let $\{E_{\lambda}^{f^*f} : \lambda \in R\}$ denote the (right-continuous) spectral family of the self-adjoint non-negative operator f^*f. In what follows, $|x|$ will denote the norm of an element in a Hilbert $\mathcal{A}$-module and $\| f \|$ will denote an operator norm.

Definition 1.2 *Define the* spectral density function *of f by*

$$F(f, \lambda) = dim_{\mathcal{A}}(im(E_{\lambda^2}^{f^*f}))$$

for $\lambda \in [0, \infty)$. ■

Lemma 1.3 . *If $x \in U$, $E_{\lambda^2}^{f^*f}(x) = 0$ and $x \neq 0$ then $|f(x)| > \lambda \cdot |x|$.*
If $E_{\lambda^2}^{f^*f}(x) = x$ then $|f(x)| \leq \lambda \cdot |x|$.

<u>Proof</u> : From the definition of the spectral family, we have

$$\langle f^* f(x), x\rangle = \int_0^{\infty} \lambda \, d\langle E_{\lambda}^{f^*f}(x), x\rangle.$$

Since $\langle f^* f(x), x\rangle = | f(x) |^2$, the claim follows. ■

Let $\mathcal{L}(f, \lambda)$ denote the set of all Hilbert $\mathcal{A}$-submodules L of U with the property that $|f(x)| \leq \lambda \cdot |x|$ holds for all $x \in L$.

Lemma 1.4 $F(f,\lambda) = \sup\{\dim_{\mathcal{A}}(L) \,:\, L \in \mathcal{L}(f,\lambda)\}$.

Proof : From Lemma 1.3, the image of $E_{\lambda^2}^{f^*f}$ belongs to $\mathcal{L}(f,\lambda)$. Hence

$$F(f,\lambda) \leq \sup\{\dim_{\mathcal{A}}(L) \,:\, L \in \mathcal{L}(f,\lambda)\},$$

and it remains to show that for all $L \in \mathcal{L}(f,\lambda)$,

$$\dim_{\mathcal{A}}(L) \leq \dim_{\mathcal{A}}(\mathrm{im}(E_{\lambda^2}^{f^*f})).$$

Lemma 1.3 implies that $\ker(E_{\lambda^2}^{f^*f}\,|_L)$ is trivial. Hence $E_{\lambda^2}^{f^*f}$ induces a weak isomorphism $L \longrightarrow \mathrm{clos}(E_{\lambda^2}^{f^*f}(L))$ and the claim follows. ■

Proposition 1.5 *Let $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be morphisms of finitely generated Hilbert $\mathcal{A}$-modules. Suppose that neither f nor g is the zero map. Then*

1. $F(f,\frac{\lambda}{\|g\|}) \leq F(gf,\lambda)$.
2. $F(g,\frac{\lambda}{\|f\|}) \leq F(gf,\lambda)$ *if f has dense image.*
3. $F(gf,\lambda) \leq F(g,\lambda^{1-r}) + F(f,\lambda^r)$ *for all $r \in (0,1)$.*

Proof : 1.) Consider $L \in \mathcal{L}(f,\frac{\lambda}{\|g\|})$. For all $x \in L$, we have

$$|gf(x)| \leq \|g\| \cdot |f(x)| \leq \|g\| \cdot \frac{\lambda}{\|g\|} \cdot |x| = \lambda \cdot |x|.$$

This implies $L \in \mathcal{L}(gf,\lambda)$ and the claim follows.

2.) Consider $L \in \mathcal{L}(g,\frac{\lambda}{\|f\|})$. For all $x \in f^{-1}(L)$, we have

$$|gf(x)| \leq \frac{\lambda}{\|f\|} \cdot |f(x)| \leq \frac{\lambda}{\|f\|} \cdot \|f\| \cdot |x| = \lambda \cdot |x|,$$

implying $f^{-1}(L) \in \mathcal{L}(gf,\lambda)$. Hence it suffices to show $\dim_{\mathcal{A}}(L) \leq \dim_{\mathcal{A}}(f^{-1}(L))$.

Let $\overline{f} : U/ker(f) \longrightarrow V$ be the map induced by f and $p : U \longrightarrow U/\ker f$ be the projection. Since p is surjective and $\overline{f}$ is a weak isomorphism, Lemma 1.1 implies that

$$\dim_{\mathcal{A}}(f^{-1}(L)) \geq \dim_{\mathcal{A}} p(f^{-1}(L)) = \dim_{\mathcal{A}}(\overline{f}^{-1}(L)) = \dim_{\mathcal{A}}(L).$$

Assertion 2.) follows.

3.) Consider $L \in \mathcal{L}(gf, \lambda)$. Let L_0 be the kernel of $E^{f^*f}_{\lambda^{2r}}|_L$. We have a weakly exact sequence $0 \longrightarrow L_0 \longrightarrow L \longrightarrow \mathrm{clos}(E^{f^*f}_{\lambda^{2r}}(L)) \longrightarrow 0$. From Lemma 1.3, we have that $|f(x)| > \lambda^r \cdot |x|$ for all nonzero $x \in L_0$. In particular, $f|_{L_0}: L_0 \longrightarrow clos(f(L_0))$ is a weak isomorphism, and so $\dim_{\mathcal{A}}(L_0) = \dim_{\mathcal{A}}(\mathrm{clos}(f(L_0)))$. For $x \in L_0$, we compute:

$$|gf(x)| \leq \lambda \cdot |x| \leq \frac{\lambda}{\lambda^r} \cdot |f(x)| = \lambda^{1-r} \cdot |f(x)|.$$

Hence $\mathrm{clos}(f(L_0)) \in \mathcal{L}(g, \lambda^{1-r})$. This shows that

$$\dim_{\mathcal{A}}(L_0) \leq F(g, \lambda^{1-r}).$$

We also have that

$$\dim_{\mathcal{A}}(\mathrm{clos}(E^{f^*f}_{\lambda^{2r}}(L))) \leq \dim_{\mathcal{A}}(\mathrm{im}(E^{f^*f}_{\lambda^{2r}})) = F(f, \lambda^r).$$

Since Lemma 1.1 implies that $\dim_{\mathcal{A}}(L) = \dim_{\mathcal{A}}(L_0) + \dim_{\mathcal{A}}(\mathrm{clos}(E^{f^*f}_{\lambda^r}(L)))$, we get

$$\dim_{\mathcal{A}}(L) \leq F(g, \lambda^{1-r}) + F(f, \lambda^r). \qquad \blacksquare$$

Definition 1.6 *We say that a function $F : [0, \infty) \longrightarrow [0, \infty]$ is a* density function *if F is monotone non-decreasing and right-continuous and $F(\lambda) < \infty$ for some $\lambda > 0$. Let $\mathcal{D}$ be the set of density functions. We write $F \preceq G$ for $F, G \in \mathcal{D}$ if there is a constant $C > 0$ such that $F(\lambda) \leq G(C \cdot \lambda)$ holds for all $\lambda \in [0, \infty)$. As in [33], we say that F and G are* dilatationally equivalent *(in signs $F \simeq G$) if $F \preceq G$ and $G \preceq F$ is true.* $\blacksquare$

Of course, the spectral density function $F(f, \lambda)$ is a density function. We introduce the following invariants of a density function F:

Definition 1.7 *The* Betti number *of F is*

$$b(F) = F(0).$$

Its Novikov-Shubin invariant *is*

$$\alpha(F) = \liminf_{\lambda \to 0+} \frac{\ln(F(\lambda) - b(F))}{\ln(\lambda)} \in [0, \infty],$$

provided that $F(\lambda) > b(F)$ holds for all $\lambda > 0$. Otherwise, we put $\alpha(F) = \infty^+$.

If f is a morphism of finitely generated Hilbert $\mathcal{A}$-modules, we write $b(f) = b(F(f, \lambda))$ and $\alpha(f) = \alpha(F(f, \lambda))$. $\blacksquare$

Here ∞^+ is a new formal symbol which should not be confused with ∞. We have $\alpha(F) = \infty^+$ if and only if there is an $\epsilon > 0$ such that $F(\lambda) = b(F)$ for $\lambda < \epsilon$.

Example 1.8 The following are examples of Novikov-Shubin invariants of density functions:

$F(\lambda) = \lambda^r$	$\alpha(F) = r$
$F(\lambda) = \exp((\ln(\lambda))^{1/3})$	$\alpha(F) = 0$
$F(\lambda) = \exp(-\lambda^{-1})$	$\alpha(F) = \infty$
$F(\lambda) = 0$	$\alpha(F) = \infty^+$

■

We make the following conventions:

Convention 1.9 The ordering on $[0,\infty] \cup \{\infty^+\}$ is given by the standard ordering on R and $r < \infty < \infty^+$ for all $r \in R$. For all $\alpha, \beta \in [0,\infty] \cup \{\infty^+\}$ we define

$$\frac{1}{\alpha} \leq \frac{1}{\beta} \Leftrightarrow \alpha \geq \beta.$$

Given $\alpha, \beta \in [0,\infty] \cup \{\infty^+\}$, we give meaning to γ in the expression

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\gamma}$$

as follows: If $\alpha, \beta \in R$, let γ be the real number for which this arithmetic expression of real numbers is true. If $\alpha \in R$ and $\beta \in \{\infty, \infty^+\}$, put γ to be α. If $\beta \in R$ and $\alpha \in \{\infty, \infty^+\}$, put γ to be β. If α and β belong to $\{\infty, \infty^+\}$ and are not both ∞^+, put $\gamma = \infty$. If both α and β are ∞^+, put $\gamma = \infty^+$.

For example,

$\frac{1}{\infty} + \frac{1}{\pi} = \frac{1}{\pi}$

$\frac{1}{\infty^+} + \frac{1}{\pi} = \frac{1}{\pi}$

$\frac{1}{\infty} + \frac{1}{\infty^+} = \frac{1}{\infty}$

$\frac{1}{\infty^+} + \frac{1}{\infty^+} = \frac{1}{\infty^+}$

$\frac{1}{\alpha} \leq \frac{1}{\infty} + \frac{1}{4} + \frac{1}{2} \Leftrightarrow \alpha \geq 4/3$

$\frac{1}{\alpha} \leq \frac{1}{\infty} + \frac{1}{\infty^+} + \frac{1}{\infty} \Leftrightarrow \alpha \geq \infty.$

Given $r \in (0,\infty)$ and $\alpha \in [0,\infty)$, we define $r\alpha \in [0,\infty)$ to be the ordinary product of real numbers, and we put $r\infty = \infty$ and $r\infty^+ = \infty^+$. ■

Here are the basic properties of these invariants.

Lemma 1.10 *Given $F, G \in \mathcal{D}$ and f a morphism of finitely generated $\mathcal{A}$-Hilbert modules,*

1. *If $F \preceq G$ then $b(F) \leq b(G)$.*
2. *If $F \preceq G$ and $b(F) = b(G)$ then $\alpha(F) \geq \alpha(G)$.*
3. *If $F \simeq G$ then $b(F) = b(G)$ and $\alpha(F) = \alpha(G)$.*
4. *$\alpha(F(\lambda^r)) = r \cdot \alpha(F(\lambda))$ for $r \in (0, \infty)$.*
5. *$\alpha(F) = \alpha(F - b(F))$.*
6. *$b(f) = \dim_{\mathcal{A}}(\ker(f^* f)) = \dim_{\mathcal{A}}(\ker(f))$.*
7. *If f is an isomorphism or zero then $\alpha(f) = \infty^+$.*
8. *An endomorphism f is an automorphism iff $b(f) = 0$ and $\alpha(f) = \infty^+$.*
9. *If i is injective with closed image and p is surjective then $\alpha(i \circ f \circ p) = \alpha(f)$.*
10. *$\alpha(F + G) = \min\{\alpha(F), \alpha(G)\}$.*

Proof : The assertions 1.) to 5.) follow directly from the definitions.

6.) By definition, $b(f)$ is the von Neumann dimension of $im(E_0^{f^* f}) = ker(f^* f)$. As $|f(x)|^2 = \langle f^* f(x), x \rangle$, f and $f^* f$ have the same kernel.

7.) If f is an isomorphism or zero then $F(f, \lambda)$ is constant for small λ.

8.) By the polar decomposition theorem, we may assume that f is self-adjoint and non-negative. Suppose that $b(f) = 0$ and $\alpha(f) = \infty^+$. Then the spectrum of f is contained in $[a, b]$ for positive real numbers $a \leq b$. An inverse of f is given by $\int_a^b \lambda^{-1} dE_\lambda$. The other implication follows from assertions 6.) and 7.).

9.) By the open mapping theorem, there is a positive constant C such that for all x,

$$C^{-1} \cdot |x| \leq |i(x)| \leq C \cdot |x|.$$

Hence $F(f \circ p, \lambda)$ and $F(i \circ f \circ p, \lambda)$ are dilatationally equivalent. Assertion 3.) implies that

$$\alpha(i \circ f \circ p) = \alpha(f \circ p).$$

We may write p as the composition $j \circ \mathrm{pr}$ of an isomorphism and a projection pr. Now one easily checks that $F(f \circ j, \lambda)$ and $F(f, \lambda)$ are dilatationally equivalent and that

$$F(f \circ j, \lambda) + \dim_{\mathcal{A}}(\ker(\mathrm{pr})) = F(f \circ p, \lambda)$$

holds for $\lambda \geq 0$. Then assertions 3.) and 5.) prove the claim.

10.) As $b(F+G) = b(F) + b(G)$, by assertion 5.) we may assume without loss of generality that $b(F) = b(G) = b(F+G) = 0$. Because $F, G \leq F+G$, assertion 2.) implies that $\alpha(F+G) \leq \min\{\alpha(F), \alpha(G)\}$. To verify the reverse inequality, we may assume without loss of generality that $\alpha(F) \leq \alpha(G)$. The cases $\alpha(F) = 0$ and $\alpha(F) = \infty^+$ are trivial, and so we assume that $0 < \alpha(F) \leq \infty$. Consider any real number α satisfying $0 < \alpha < \alpha(F)$. Then there exists a constant $K > 0$ such that for small positive λ we have $F(\lambda), G(\lambda) \leq K\lambda^\alpha$, and so $F(\lambda) + G(\lambda) \leq 2K \cdot \lambda^\alpha$, implying that $\alpha \leq \alpha(F+G)$. The assertion follows. ■

Proposition 1.11 *Let $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be morphisms of finitely generated Hilbert $\mathcal{A}$-modules. Then*

1. *If f has dense image then $\alpha(g) \geq \alpha(gf)$.*
2. *If $\ker(g) \cap im(f) = \{0\}$ then $\alpha(f) \geq \alpha(gf)$.*
3. *If $\ker(g) \subset clos(im(f))$ then*

$$\frac{1}{\alpha(gf)} \leq \frac{1}{\alpha(f)} + \frac{1}{\alpha(g)}.$$

Proof : First, f factorizes over the projection $U \longrightarrow U/\ker(f)$ into an injective morphism $\overline{f} : U/\ker(f) \longrightarrow V$. From Lemma 1.10.9, $\alpha(\overline{f}) = \alpha(f)$ and $\alpha(g\overline{f}) = \alpha(gf)$, and so we may assume without loss of generality that f is injective.

1.) Then f induces an injection $\ker(gf) \longrightarrow \ker(g)$, and so $b(gf) \leq b(g)$. From Proposition 1.5,

$$F(g, \frac{\lambda}{\|f\|}) - b(g) \leq F(gf, \lambda) - b(gf).$$

Now the claim follows from Lemma 1.10.2.

2.) Since $\ker(g) \cap \mathrm{im}(f) = \{0\}$ holds by assumption, we have that $\ker(gf) = \ker(f)$ and hence $b(gf) = b(f)$. Now the assertion follows from Proposition 1.5 and Lemma 1.10.2.

3.) By assumption, $\ker(g) \subset \mathrm{clos}(\mathrm{im}(f))$. As $f : U \longrightarrow \mathrm{clos}(\mathrm{im}(f))$ is assumed to be a weak isomorphism, Lemma 1.1 implies that $b(gf) = b(g) = b(f) + b(g)$. From Proposition 1.5 we have that for $0 < r < 1$,

$$F(gf,\lambda) - b(gf) \leq F(f,\lambda^r) - b(f) + F(g,\lambda^{1-r}) - b(g).$$

Lemma 1.10.10 shows that

$$\alpha(gf) \geq \min\{r \cdot \alpha(f), (1-r)\cdot\alpha(g)\}.$$

Taking inverses gives

$$\frac{1}{\alpha(gf)} \leq \max\left\{\frac{1}{r\cdot\alpha(f)}, \frac{1}{(1-r)\cdot\alpha(g)}\right\}.$$

We only consider the case $\alpha(f), \alpha(g) \in (0,\infty)$, the other cases being now trivial. Since $\frac{1}{r\cdot\alpha(f)}$ (resp. $\frac{1}{(1-r)\cdot\alpha(g)}$) is a strictly monotonically decreasing (resp. increasing) function in r, the maximum on the right side, viewed as a function of r, obtains its minimum precisely if the two functions of r have the same value. One easily checks that this is the case if and only if $r = \frac{\alpha(g)}{\alpha(f)+\alpha(g)}$, and the claim follows. ∎

Lemma 1.12 *Let $f : U_1 \longrightarrow V_1$, $g : U_2 \longrightarrow V_1$, and $h : U_2 \longrightarrow V_2$ be morphisms of finitely generated Hilbert $\mathcal{A}$-modules. Then*

1. $\alpha\begin{pmatrix} f & 0 \\ 0 & h \end{pmatrix} = \min\{\alpha(f), \alpha(h)\}$.

2. *If f is invertible then* $\alpha\begin{pmatrix} f & g \\ 0 & h \end{pmatrix} = \alpha(h)$.

3. *If h is injective then*

$$\alpha(f) \geq \alpha\begin{pmatrix} f & g \\ 0 & h \end{pmatrix}$$

$$\left(\alpha\begin{pmatrix} f & g \\ 0 & h \end{pmatrix}\right)^{-1} \leq \frac{1}{\alpha(f)} + \frac{1}{\alpha(h)}.$$

4. *If f has dense image then*

$$\alpha(h) \geq \alpha\begin{pmatrix} f & g \\ 0 & h \end{pmatrix}$$

$$\left(\alpha\begin{pmatrix} f & g \\ 0 & h \end{pmatrix}\right)^{-1} \leq \frac{1}{\alpha(f)} + \frac{1}{\alpha(h)}.$$

5. $F(f,\lambda) - b(f) = F(f^*,\lambda) - b(f^*)$ *and* $\alpha(f) = \alpha(f^*)$.

Proof : 1.) We have

$$F(\begin{pmatrix} f & 0 \\ 0 & h \end{pmatrix}, \lambda) = F(f,\lambda) + F(h,\lambda).$$

Now apply Lemma 1.10.10.

2.) Apply Lemma 1.10.9 and assertion 1.) to

$$\begin{pmatrix} f & g \\ 0 & h \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & h \end{pmatrix} \cdot \begin{pmatrix} 1 & f^{-1}g \\ 0 & 1 \end{pmatrix}.$$

3. and 4.) We have

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \cdot \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & h \end{pmatrix}.$$

Lemma 1.10.7 and assertions 1.) and 2.) imply

$$\alpha(h) = \alpha \begin{pmatrix} 1 & g \\ 0 & h \end{pmatrix}$$

and

$$\alpha(f) = \alpha \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}.$$

We have

$$\begin{pmatrix} f & g \\ 0 & h \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & h \end{pmatrix} \cdot \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}.$$

If h is injective then $\begin{pmatrix} 1 & g \\ 0 & h \end{pmatrix}$ has trivial kernel. If f has dense image then $\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}$ has dense image. The claim now follows from Proposition 1.11.

5.) As $f(f^*f) = (ff^*)f$ and $f^*(ff^*) = (f^*f)f^*$, f and f^* induce morphisms

$$\overline{f} : E_\lambda^{f^*f} / \ker(f) \longrightarrow E_\lambda^{ff^*} / \ker(f^*)$$

and

$$\overline{f^*} : E_\lambda^{ff^*} / \ker(f^*) \longrightarrow E_\lambda^{f^*f} / \ker(f).$$

As $\ker(f) = \ker(f^*f)$ and $\ker(f^*) = \ker(ff^*)$, the compositions $\overline{f^*} \circ \overline{f}$ and $\overline{f} \circ \overline{f^*}$ are injective endomorphisms, and hence are weak isomorphisms by Lemma 1.1. It follows that $\overline{f}$ is a weak isomorphism. Lemma 1.1 now implies that the dimensions of $E_\lambda^{f^*f} / \ker(f)$ and $E_\lambda^{ff^*} / \ker(f^*)$ are the same, and so $F(f,\lambda) - b(f) = F(f^*,\lambda) - b(f^*)$. ■

2. L^2-Betti numbers and Novikov-Shubin invariants for Hilbert $\mathcal{A}$-chain complexes

In this section we introduce and study the Betti numbers and Novikov-Shubin invariants for chain complexes, and investigate their behaviour with respect to exact sequences and homotopy equivalences.

A Hilbert $\mathcal{A}$-chain complex is said to be *finite* if C_n is a finitely generated Hilbert $\mathcal{A}$-module for all integers n and is zero for all but a finite number of integers n. The homology of C is defined to be $H_p(C) = \ker(c_p)/\text{clos}(\text{im}(c_p))$ where c_p denotes the differential. Note that we have to quotient by the closure of the image of c_p if we want to ensure that the homology is a Hilbert space.

Definition 2.1 *Let C be a finite Hilbert $\mathcal{A}$-chain complex with p-th differential c_p. Its p-th* Betti-number is

$$b_p(C) = \dim_{\mathcal{A}}(H_p(C)).$$

Its p-th *Novikov-Shubin invariant* is

$$\alpha_p(C) = \alpha(c_p).$$

Put

$$\widetilde{\alpha}_p(C) = \min\{\alpha(c_{p+1}), \alpha(c_p)\}. \qquad \blacksquare$$

Note that $\widetilde{\alpha}_p(C)$ correspond to the notion of Novikov-Shubin invariants as introduced in [33]. However, it turns out to be easier and more efficient to deal with the numbers $\alpha_p(C)$.

Theorem 2.2 (Additivity inequalities for the Novikov-Shubin invariants)
Let $0 \longrightarrow C \xrightarrow{j} D \xrightarrow{q} E \longrightarrow 0$ be an exact sequence of finite Hilbert $\mathcal{A}$-chain complexes. Let $\delta : H_p(E) \longrightarrow H_{p-1}(C)$ denote the boundary operator in the long weakly exact homology sequence given in [10, Theorem 2.1 on page 10]. Then

1. $\dfrac{1}{\alpha_p(D)} \leq \dfrac{1}{\alpha_p(C)} + \dfrac{1}{\alpha_p(E)} + \dfrac{1}{\alpha(\delta_p)}.$
2. $\dfrac{1}{\alpha_p(E)} \leq \dfrac{1}{\alpha_{p-1}(C)} + \dfrac{1}{\alpha_p(D)} + \dfrac{1}{\alpha(H_{p-1}(j))}.$
3. $\dfrac{1}{\alpha_p(C)} \leq \dfrac{1}{\alpha_p(D)} + \dfrac{1}{\alpha_{p+1}(E)} + \dfrac{1}{\alpha(H_p(q))}.$

Proof : 1.) The exact sequence $0 \longrightarrow C \xrightarrow{j} D \xrightarrow{q} E \longrightarrow 0$ induces the following commutative diagram with exact rows, where $\overline{q_p}$, $\overline{d_p}$ and $\overline{e_p}$ are canonical homomorphisms induced by q_p, d_p and e_p and i is the inclusion:

$$\begin{tikzcd}
\ker\overline{q_p} \arrow[r, "i"] \arrow[d, "\partial_p"] & D_p/\ker(d_p) \arrow[r, "\overline{q_p}"] \arrow[d, "\overline{d_p}"] & E_p/\ker(e_p) \arrow[d, "\overline{e_p}"] \\
C_{p-1} \arrow[r, "j_{p-1}"] & D_{p-1} \arrow[r, "q_{p-1}"] & E_{p-1}.
\end{tikzcd}$$

To define ∂_p in the above diagram, let $x \in ker(e_p q_p)$ represent $[x] \in ker(\overline{q_p})$. Then $d_p(x) = j_{p-1}(y)$ for a unique $y \in C_{p-1}$. We put $\partial_p([x]) = y$. (In fact, $y \in ker(c_{p-1})$.) Since $\overline{e_p}$ is injective, Lemma 1.12.3 gives that

$$\frac{1}{\alpha(\overline{d_p})} \leq \frac{1}{\alpha(\partial_p)} + \frac{1}{\alpha(\overline{e_p})}.$$

From Lemma 1.10.9 we conclude that $\alpha(\overline{d_p}) = \alpha(d_p)$ and $\alpha(\overline{e_p}) = \alpha(e_p)$. This implies that

$$\frac{1}{\alpha(d_p)} \leq \frac{1}{\alpha(\partial_p)} + \frac{1}{\alpha(e_p)}.$$

It remains to show that

$$\frac{1}{\alpha(\partial_p)} \leq \frac{1}{\alpha(\delta_p)} + \frac{1}{\alpha(c_p)}.$$

We construct a short weakly exact sequence

$$0 \longrightarrow C_p/\ker(c_p) \xrightarrow{\overline{j_p}} \ker(\overline{q_p}) \xrightarrow{\widehat{q_p}} H_p(E)/\mathrm{clos}(\mathrm{im}(H_p(q))) \longrightarrow 0$$

The map $\overline{j_p}$ is induced by j_p in the obvious way. To define $\widehat{q_p}$, consider $x \in D_p$ whose class $[x] \in D_p/\ker(d_p)$ lies in $\ker(\overline{q_p})$. Then $q_p(x)$ is in the kernel of e_p and determines a class $[q_p(x)]$ in $H_p(E)/\mathrm{clos}(\mathrm{im}(H_p(q)))$. Define $\widehat{q_p}([x])$ to be $[q_p(x)]$. One easily checks that $\overline{j_p}$ is injective, $\widehat{q_p} \circ \overline{j_p}$ is zero and $\widehat{q_p}$ is surjective. We will show that $\ker(\widehat{q_p})$ is contained in $\mathrm{clos}(\mathrm{im}(\overline{j_p}))$. We must show that if $x \in D_p$ is such that $q_p(x) \in clos(im(e_{p+1})) \oplus q_p(ker(d_p))$ then $x \in im(j_p) \oplus ker(d_p)$, or equivalently, that $q_p^{-1}(clos(im(e_{p+1}))) \subset im(j_p) \oplus ker(d_p)$. Suppose that $x \in q_p^{-1}(clos(im(e_{p+1})))$. Then there is a sequence $\{y_n\}_{n=1}^{\infty}$ in E_{p+1} such that

$q_p(x) = \lim_{n\to\infty} e_{p+1}(y_n)$. As q_{p+1} is surjective, there is a sequence $\{u_n\}_{n=1}^{\infty}$ in D_{p+1} such that $y_n = q_{p+1}(u_n)$. Thus $\lim_{n\to\infty} q_p(x - d_{p+1}(u_n)) = 0$. Write $x - d_{p+1}(u_n) = j_p(w_n) + r_n$ with $w_n \in C_p$ and $r_n \in im(j_p)^{\perp}$. Then $\lim_{n\to\infty} q_p(r_n) = 0$. As the restriction of q_p to $im(j_p)^{\perp}$ is an isomorphism, it follows that $\lim_{n\to\infty} r_n = 0$. Thus $x = \lim_{n\to\infty}(j_p(w_n) + d_{p+1}(u_n))$ lies in the closed subspace $im(j_p) \oplus ker(d_p)$. This finishes the proof of weak exactness.

Next we construct a commutative diagram with exact rows

$$\begin{array}{ccccc}
\ker(\widehat{q_p}) & \xrightarrow{i_1} & \ker(\overline{q_p}) & \xrightarrow{\widehat{q_p}} & H_p(E)/\mathrm{clos}(\mathrm{im}(H_p(q))) \\
\Big\downarrow{\scriptstyle \overline{\partial_p}} & & \Big\downarrow{\scriptstyle \partial_p} & & \Big\downarrow{\scriptstyle \overline{\delta_p}} \\
\mathrm{clos}(\mathrm{im}(c_p)) & \xrightarrow{i_2} & \ker(c_{p-1}) & \xrightarrow{\mathrm{pr}} & H_{p-1}(C).
\end{array}$$

The maps i_1 and i_2 are the canonical inclusions and the map pr is the canonical projection. Recall that the boundary operator $\delta_p : H_p(E) \longrightarrow H_{p-1}(C)$ is defined as follows: Let $x \in \ker(e_p)$ represent $[x]$ in $H_p(E)$. Choose $y \in D_p$ and $z \in \ker(c_{p-1})$ such that $q_p(y) = x$ and $j_{q-1}(z) = d_p(y)$. Then $\delta_p([x])$ is defined to be the class $[z] \in H_{p-1}(C)$. The map $\overline{\delta_p}$ is induced from δ_p. The map $\overline{\partial_p}$ is induced from ∂_p. One easily verifies that the diagram commutes. The rows are clearly exact. Since $\overline{\delta_p}$ is injective, we conclude from Lemma 1.12 that

$$\frac{1}{\alpha(\partial_p)} \leq \frac{1}{\alpha(\overline{\partial_p})} + \frac{1}{\alpha(\overline{\delta_p})}.$$

Since Lemma 1.10.5 implies that $\alpha(\overline{\delta_p}) = \alpha(\delta_p)$, it remains to show that

$$\alpha(c_p) \leq \alpha(\overline{\partial_p}).$$

Let $\tilde{\overline{j_p}} : C_p/\ker(c_p) \longrightarrow \ker(\widehat{q_p})$ be the weak isomorphism induced by $\overline{j_p}$. The map c_p induces a morphism $\overline{c_p} : C_p/\ker(c_p) \longrightarrow \mathrm{clos}(\mathrm{im}(c_p))$. One easily checks that $\overline{\partial_p} \circ \tilde{\overline{j_p}} = \overline{c_p}$. Proposition 1.11.1 implies that

$$\alpha(\overline{\partial_p}) \geq \alpha(\overline{c_p}).$$

From Lemma 1.10.9 we obtain that $\alpha(c_p) = \alpha(\overline{c_p})$. This finishes the proof of the first assertion of Theorem 2.2.

2.) Recall that in general [26, p. 213], the n-th differential of the mapping cylinder of a chain map $g : C \longrightarrow D$ is defined by

$$\begin{pmatrix} -c_{n-1} & 0 & 0 \\ -id & c_n & 0 \\ g_{n-1} & 0 & d_n \end{pmatrix} : C_{n-1} \oplus C_n \oplus D_n \longrightarrow C_{n-2} \oplus C_{n-1} \oplus D_{n-1}.$$

There is a canonical map $i : C \longrightarrow cyl(g)$ and $cone(g)$ is defined to be the cokernel of i. That is, the n-th differential of $cone(g)$ is

$$\begin{pmatrix} -c_{n-1} & 0 \\ g_{n-1} & d_n \end{pmatrix} : C_{n-1} \oplus D_n \longrightarrow C_{n-2} \oplus D_{n-1}.$$

We define $cone(C)$ to be the mapping cone of the identity map on C, and the suspension ΣC to be the mapping cone of the 0-map on C i.e. $(\Sigma C)_n = C_{n-1}$.

In our case there is a canonical exact sequence $0 \longrightarrow D \longrightarrow cyl(q) \longrightarrow cone(q) \longrightarrow 0$ and chain homotopy equivalences $E \longrightarrow cyl(q)$ and $\Sigma C \longrightarrow cone(q)$. We will show later using only assertion 1.) that the numbers $\alpha(c_p)$ are homotopy invariants. So we may assume the existence of an exact sequence $0 \longrightarrow D \longrightarrow E \longrightarrow \Sigma C \longrightarrow 0$. Moreover, the connecting map from $H_p(\Sigma C)$ to $H_{p-1}(D)$ agrees under these identifications with the map $H_{p-1}(j) : H_{p-1}(C) \longrightarrow H_{p-1}(D)$. The claim now follows from assertion 1.).

3.) Repeat the argument in the proof of assertion 2.), yielding a short exact sequence $0 \longrightarrow E \longrightarrow \Sigma C \longrightarrow \Sigma D \longrightarrow 0$. ■

The dual chain complex C^* is the cochain complex with the same chain modules as C and the adjoints of the differentials of C as codifferentials. The definitions of the Betti numbers and the Novikov-Shubin invariants carry over directly to cochain complexes. The *Laplace operator* $\Delta_p : C_p \longrightarrow C_p$ is defined to be $c_{p+1}c_{p+1}^* + c_p^*c_p$.

Lemma 2.3 *Let C be a finite Hilbert $\mathcal{A}$-chain complex.*

1. $2 \cdot \tilde{\alpha}_p(C) = \alpha(\Delta_p)$ *and* $b_p(C) = b_p(\Delta_p)$.
2. $\alpha_p(C) = \alpha_p(C^*)$ *and* $b_p(C) = b_p(C^*)$.
3. $\alpha_p(C \oplus D) = \min\{\alpha_p(C), \alpha_p(D)\}$ *and* $b_p(C \oplus D) = b_p(C) + b_p(D)$.

1.) The Hodge decomposition theorem (see e.g. [27, Theorem 3.7]) gives the claim for the Betti numbers. Moreover, we have the following commutative square with isomorphisms as horizontal morphisms:

$$\begin{array}{ccc} & \cong & \\ \ker(c_p)^{\perp} \oplus \ker(c_{p+1}^*)^{\perp} \oplus \ker(\Delta_p) & \longrightarrow & C_p \\ c_p^*c_p \oplus c_{p+1}c_{p+1}^* \oplus 0 \downarrow & & \downarrow \Delta_p \\ & \cong & \\ \ker(c_p)^{\perp} \oplus \ker(c_{p+1}^*)^{\perp} \oplus \ker(\Delta_p) & \longrightarrow & C_p \end{array}$$

Lemmas 1.10.7 and 1.12.1 imply that

$$\alpha(\Delta_p) = \min\left\{\alpha(c_p^*c_p), \alpha(c_{p+1}c_{p+1}^*)\right\}.$$

Since $E_\lambda^{f^*f} = E_{\lambda^2}^{(f^*f)^2}$, Lemma 1.10.4 implies that $\alpha(f^*f) = 2\cdot\alpha(f)$. We have shown in Lemma 1.12.5 that $\alpha(f) = \alpha(f^*)$. This implies that $2\cdot\alpha(c_p) = \alpha(c_p^*c_p)$ and $2\cdot\alpha(c_{p+1}) = \alpha(c_{p+1}c_{p+1}^*)$, and the claim follows.

2.) follows from assertion 1.)

3.) is a consequence of Lemma 1.12.1. ■

We recall that C is said to be contractible if C has a chain contraction γ, i.e a collection of morphisms $\gamma_p : C_p \longrightarrow C_{p+1}$ such that $\gamma_{p-1}c_p + c_{p+1}\gamma_p = \text{id.}$ for all p.

Lemma 2.4 *The following assertions are equivalent for a finite Hilbert $\mathcal{A}$-chain complex C:*

1. *C is contractible.*
2. *Δ_p is invertible for all p.*
3. *$b_p(C) = 0$ and $\alpha_p(C) = \infty^+$ for all p.*

<u>**Proof**</u> : 1.) $\Rightarrow$ 3.)We use induction on the length l of C, i.e. the difference $n-m$, where n (resp. m) is the smallest (resp. largest) number for which $C_i = \{0\}$ holds for $i > n$ (resp. $i < m$). The initial step $l \leq 1$ is trivial since all of the differentials in a contractible chain complex of length $l \leq 1$ are zero or isomorphisms, and hence have Novikov-Shubin invariant ∞^+ by Lemma 1.10.7. In the induction step one constructs a short exact sequence $0 \longrightarrow D \xrightarrow{j} C \xrightarrow{q} E \longrightarrow 0$ of contractible chain complexes where D is concentrated in dimensions n and $n-1$ and is given there by $D_n = C_n \xrightarrow{\text{id}} D_{n-1} = C_n$, and E is concentrated in dimensions less than n and is given by $E_{n-1} = ker(c_{n-1})^{\perp}$, $E_i = C_i$ for $i < n-1$. Take $j_n = \text{id}$, $j_{n-1} = c_n$ and q_{n-1} to be orthogonal projection. One easily checks that D and E are contractible and the sequence is exact. As E has a smaller length than C, the induction

hypothesis applies to D and E and the claim now follows from Theorem 2.2.1.

3.) $\Rightarrow$ 2.) From Lemma 2.3, $b(\Delta_p) = 0$ and $\alpha(\Delta_p) = \infty^+$ for all p. Now apply Lemma 1.10.8.

2.) $\Rightarrow$ 1.) Suppose that Δ_* is invertible. Then $\Delta_{p+1}^{-1} \circ c_{p+1}^*$ is a chain contraction for C. ■

We now reprove the homotopy invariance of the L^2-Betti numbers and the Novikov-Shubin invariants [13, 15, 18].

Theorem 2.5 (Homotopy invariance) *If $f : C \longrightarrow D$ is a chain homotopy equivalence then for all $p \in Z$ we have*

$$
\begin{aligned}
&b_p(C) = b_p(D)\\
&F(c_p) \simeq F(d_p)\\
&\alpha_p(C) = \alpha_p(D)\\
&\tilde{\alpha}_p(C) = \tilde{\alpha}_p(D).
\end{aligned}
$$

Proof : There are exact sequences of chain complexes

$$0 \longrightarrow C \longrightarrow cyl(f) \longrightarrow cone(f) \longrightarrow 0$$

and

$$0 \longrightarrow D \longrightarrow cyl(f) \longrightarrow cone(C) \longrightarrow 0$$

with $cone(f)$ and $cone(C)$ being contractible. We obtain chain isomorphisms

$$
\begin{aligned}
&C \oplus cone(f) \longrightarrow cyl(f)\\
&D \oplus cone(C) \longrightarrow cyl(f)
\end{aligned}
$$

by the following general construction for an exact sequence $0 \longrightarrow C \xrightarrow{j} D \xrightarrow{q} E \longrightarrow 0$ with contractible E. Choose a chain contraction ϵ for E and for each p choose a morphism $t_p : E_p \longrightarrow D_p$ such that $q_p \circ t_p = \mathrm{id}$. Put

$$s_p = d_{p+1} \circ t_{p+1} \circ \epsilon_p + t_p \circ \epsilon_{p-1} \circ e_p.$$

This defines a chain map $s : E \longrightarrow D$ satisfying $q \circ s = \mathrm{id}$. Define a chain map $u : D \to C$ by putting $u_p(x)$, for $x \in D_p$, to be the unique $y \in C_p$ such that $x = s_p q_p(x) + j_p(y)$. Then $j + s$ is a chain isomorphism $C \oplus E \longrightarrow D$, with inverse $u \oplus q$.

Since $C \oplus cone(f)$ and $D \oplus cone(C)$ are isomorphic and $cone(f)$ and $cone(C)$ are contractible, we conclude that $F(c_p) \simeq F(d_p)$. The other claims now follow from Lemma 1.10. ■

3. Examples proving sharpness of various inequalities

We give examples which show that the inequalities of the preceeding sections are sharp. A trusting reader can skip this section. Throughout this section $\mathcal{A}$ will be the von Neumann algebra $N(Z)$ of the integers. Note that $N(Z)$ can be identified with the space $L^\infty(S^1)$ of essentially bounded complex functions on S^1. The space $l^2(N(Z))$ is isomorphic to the space $L^2(S^1)$ of L^2-functions on S^1, and the regular representation $L^\infty(S^1) \longrightarrow B(L^2(S^1), L^2(S^1))^Z$ sends f to the operator m_f of pointwise multiplication by f. These identifications are based on elementary Fourier analysis.

Lemma 3.1

1. *Let μ be the Lebesgue measure on S^1. Given $f \in L^\infty(S^1)$, the spectral density function of m_f is*
$$F(m_f, \lambda) = \mu\left\{x \in S^1 \ : \ |f(x)| \leq \lambda\right\}.$$
2. *Let $p(z) = az^r \cdot \prod_{i=1}^n (z - a_i)^{r_i}$ be an element in $C[Z]$, with $r \in Z$, $r_i \in Z^{\geq 1}$, $a \in C - \{0\}$ and the nonzero complex numbers a_i pairwise disjoint. Then*
$$\alpha(m_p) = \min\left\{\frac{1}{r_i} \ : \ 1 \leq i \leq n \text{ and } |a_i| = 1\right\}.$$
(The minimum over the empty set is taken to be ∞^+.)

Proof : 1.) follows directly from the definition of a spectral family.

2.) From 1.), we have that for small λ,

$$F(m_p, \lambda) \simeq \sum_{i=1}^n F(m_{p_i}, \lambda),$$

where $p_i = (z - a_i)^{r_i}$. Lemma 1.10.10 implies that

$$\alpha(m_p) = \min\left\{\alpha(m_{p_i}) : 1 \leq i \leq n\right\}.$$

If $|a_i| \neq 1$ then m_{p_i} is an isomorphism and Lemma 1.10.7 implies that $\alpha(m_{p_i}) = \infty^+$. Since the group of isometries acts transitively on S^1, it is now enough to show that for $r \geq 1$,

$$\alpha(m_{(z-1)^r}) = \frac{1}{r}.$$

Writing $z = \cos(\phi) + i \cdot \sin(\phi)$, we have

$$|z-1| = \sqrt{2 - 2\cos(\phi)}.$$

This implies that

$$F(m_{(z-1)^r}, \lambda) = \mu\left\{\phi \in [-\pi, \pi) \ : \ |2 - 2\cos(\phi)|^{r/2} \leq \lambda\right\}.$$

Because

$$\lim_{\phi \to 0} \frac{2 - 2\cos(\phi)}{\phi^2} = 1,$$

the claim follows. ■

Example 3.2 (examples for Proposition 1.11)
Put $f = g = z - 1 \in L^\infty(S^1)$. Then

$$\frac{1}{\alpha(gf)} = \frac{1}{\alpha(f)} + \frac{1}{\alpha(g)}.$$

Hence the inequality Proposition 1.11.3 is sharp. The condition $\ker(g) \subset \operatorname{clos}(\operatorname{im}(f))$ is necessary, as the following example shows: Let $f : U \longrightarrow U \oplus U$ be the inclusion onto the first factor and $g : U \oplus U \longrightarrow U$ be given by $(m_{z-1} \oplus 1)$. We have $\alpha(f) = \alpha(g) = \infty^+$. On the other hand, $\alpha(gf) = \alpha(m_{z-1}) = 1$.

The first two inequalities of Proposition 1.11 are clearly sharp; take e.g. $f = 1$ or $g = 1$. The conditions in the first two inequalities are necessary; take e.g. $f = 0$ and $g = z - 1$ and vice versa. ■

Example 3.3 (examples for Lemma 1.12)
Put $f = h = z - 1 \in L^\infty(S^1)$. We have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} hf & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} = \begin{pmatrix} f & 1 \\ 0 & h \end{pmatrix}.$$

Lemmas 1.10 and 1.12 imply that

$$\alpha\begin{pmatrix} f & 1 \\ 0 & h \end{pmatrix} = \alpha(hf) = \left(\frac{1}{\alpha(f)} + \frac{1}{\alpha(h)}\right)^{-1}.$$

This shows that the second inequality in item 3.) (resp. 4.)) of Lemma 1.12 is sharp. The condition that h be injective (resp. that f have dense image) is necessary, as the example $f = 0$, $g = m_{z-1}$ and $h = 0$ shows. Namely, in this case

$$\alpha\begin{pmatrix} f & g \\ 0 & h \end{pmatrix} = 1 \text{ and } \alpha(f) = \alpha(h) = \infty^+.$$

The first inequality in Lemma 1.12.3 is sharp, as the example $g = h = 0$ shows. Again the condition that h be injective is necessary. For example, put $f = m_{z-1}$, $g = 1$ and $h = 0$. Then

$$\alpha \begin{pmatrix} f & g \\ 0 & h \end{pmatrix} = \infty^+ \text{ and } \alpha(f) = 1.$$

The first inequality in of Lemma 1.12.4 is clearly sharp. The condition that f have dense image is also necessary. Put $f = 0$, $g = 1$ and $h = m_{z-1}$. Then $\alpha(h) = 1$. Since $\begin{pmatrix} f & g \\ 0 & h \end{pmatrix}$ is the composition of the injection (with closed image) $\begin{pmatrix} 1 \\ m_{(z-1)} \end{pmatrix}$ and the projection $(0,1)$, Lemma 1.10.9 implies that

$$\alpha \begin{pmatrix} f & g \\ 0 & h \end{pmatrix} = \infty^+. \qquad \blacksquare$$

Example 3.4 (Examples for Theorem 2.2)
Upon interpreting the morphisms in the first part of Example 3.3 as weakly acyclic 1-dimensional chain complexes, we obtain an example where the first inequality in Theorem 2.2 is sharp.

Let C be a weakly acyclic chain complex such that $\alpha_p(C) < \alpha_{p-1}(C)$. We have the canonical exact sequence $0 \longrightarrow C \longrightarrow cone(C) \longrightarrow \Sigma C \longrightarrow 0$. Since $cone(C)$ is contractible, Theorem 2.5 implies that $\alpha_p(cone(C)) = \infty^+$ for all $p \in Z$. It follows from the definition of suspension that $\alpha_{p+1}(\Sigma C) = \alpha_p(C)$. We have now constructed an example where the second and third inequalities are sharp.

Here is an example in which the boundary operator of the long weakly exact homology sequence enters. Consider a short exact sequence $0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$ such that C is concentrated in dimension 0 and E is concentrated in dimension 1, and the Novikov-Shubin invariant of the first differential of D is not ∞^+. The boundary operator $\delta_1 : H_1(E) \longrightarrow H_0(C)$ is, up to composition with isomorphisms, the same as the differential of D and so $\alpha(\delta_1)$ equals $\alpha(d_1)$. Clearly $\alpha_p(C)$ and $\alpha_p(E)$ are ∞^+ for all p. This example shows that the term depending on δ_p has to appear in the first inequality of Theorem 2.2.
$\blacksquare$

4. L^2-Betti numbers and Novikov-Shubin invariants for manifolds

In this section we analyse the L^2-Betti numbers and the Novikov-Shubin invariants of compact manifolds.

Throughout this section we will use the following setup: Let M be a compact connected orientable smooth manifold of dimension m with fundamental group π and universal cover $\widetilde{M}$. Suppose that ∂M is the union of two submanifolds $\partial_0 M$ and $\partial_1 M$ such that $\partial(\partial_0 M) = \partial_0 M \cap \partial_1 M = \partial(\partial_1 M)$. We allow that $\partial_0 M$ or $\partial_1 M$ are empty. Let $\widetilde{\partial_0 M}$ denote the preimage of $\partial_0 M$ under the projection $\widetilde{M} \longrightarrow M$. Let $\mathcal{A}$ be a finite von Neumann algebra, V be a finitely generated Hilbert $\mathcal{A}$-module and $\mu : \pi \longrightarrow Iso_{\mathcal{A}}(V)^{op}$ be a right unitary representation of π. In most applications $\mathcal{A}$ will be the von Neumann algebra $N(\pi)$ of π, V will be $l^2(\pi)$ and μ will be the right regular representation.

Let $C(\widetilde{M}, \widetilde{\partial_0 M})$ be the simplicial $Z\pi$-chain complex of the lift of any triangulation of M to a π-equivariant triangulation of $\widetilde{M}$. Note that π acts on the left on $C(\widetilde{M}, \widetilde{\partial_0 M})$, and on the right on V. Let $C(M, \partial_0 M; V)$ denote the cellular Hilbert $\mathcal{A}$-chain complex given by the tensor product $V \otimes_{Z\pi} C(\widetilde{M}, \widetilde{\partial_0 M})$. If c denotes the differential of $C(M, \partial_0 M; V)$, define the L^2-homology $H_p(M, \partial_0 M; V)$ with coefficients in V to be the Hilbert $\mathcal{A}$-module $\ker(c_p)/\text{clos}(\text{im}(c_p))$. In this section we will only deal with homology. We note that the corresponding cohomology groups are isometrically isomorphic to the homology groups. Recall that we have defined the L^2-Betti numbers and Novikov-Shubin invariants for chain complexes in Definition 2.1. Since they are homotopy invariants (see Theorem 2.5), the following definition is independent of the choice of triangulation:

Definition 4.1 *Define the p-th L^2* Betti-number *of $(M, \partial_0 M)$, with coefficients in V, to be*

$$b_p(M, \partial_0 M; V) = b_p(C(M, \partial_0 M; V)) = dim_{\mathcal{A}}(H_p(M, \partial_0 M; V)).$$

Define the p-th Novikov-Shubin invariant *to be*

$$\alpha_p(M, \partial_0 M; V) = \alpha_p(C(M, \partial_0 M; V))$$

and put

$$\tilde{\alpha}_p(M, \partial_0 M; V) = \tilde{\alpha}_p(C(M, \partial_0 M; V)).$$

If $V = l^2(\pi)$ then we abbreviate:

$$b_p(M, \partial_0 M) = b_p(M, \partial_0 M; l^2(\pi));$$
$$\alpha_p(M, \partial_0 M) = \alpha_p(M, \partial_0 M; l^2(\pi));$$

$\tilde{\alpha}_p(M, \partial_0 M) = \tilde{\alpha}_p(M, \partial_0 M; l^2(\pi))$.

We abbreviate $b_p(M, \emptyset)$ *by* $b_p(M)$, $\alpha_p(M, \emptyset)$ *by* $\alpha_p(M)$ *and* $\tilde{\alpha}_p(M, \emptyset)$ *by* $\tilde{\alpha}_p(M)$. ■

We refer to $\alpha_p(M, \partial_0 M; V)$ as the Novikov-Shubin invariant, whereas in the previous literature $\tilde{\alpha}_p(M, \partial_0 M; V)$ is called the Novikov-Shubin invariant. Also, in previous articles the values ∞ and ∞^+ are not distinguished. Moreover, we use the normalization of [24], which differs by a factor of 2 from that used in [15, 18, 33].

We start with Poincaré duality. It gives a $Z\pi$-chain homotopy equivalence

$$\cap [M] : C^{m-*}(\widetilde{M}, \widetilde{\partial_1 M}) \longrightarrow C_*(\widetilde{M}, \widetilde{\partial_0 M}).$$

Tensoring over $Z\pi$ with V then gives a chain homotopy equivalence of Hilbert $\mathcal{A}$-chain complexes. From Theorem 2.5 and Lemma 2.3 we derive

Proposition 4.2 (Poincaré duality) *1.* $b_{m-p}(M, \partial_1 M; V) = b_p(M, \partial_0 M; V)$.

2. $\alpha_{m+1-p}(M, \partial_1 M; V) = \alpha_p(M, \partial_0 M; V)$.

3. $\tilde{\alpha}_{m-p}(M, \partial_1 M; V) = \tilde{\alpha}_p(M, \partial_0 M; V)$. ■

Lemma 4.3 *Let* $(f, f_0) : (M, \partial_0 M) \longrightarrow (N, \partial_0 N)$ *be a map between pairs such that* f *and* f_0 *are* n*-connected for some* $n \geq 2$. *Then*

1. $b_p(M, \partial_0 M; V) = b_p(N, \partial_0 N; V)$ *for* $p \leq n-1$ *and* $b_n(M, \partial_0 M; V) \geq b_n(N, \partial_0 N; V)$.
2. $\alpha_p(M, \partial_0 M; V) = \alpha_p(N, \partial_0 N; V)$ *for* $p \leq n$.

Proof : Let $C(\tilde{f}) : C(\widetilde{M}, \widetilde{\partial_0 M}) \longrightarrow C(\widetilde{N}, \widetilde{\partial_0 N})$ be the $Z\pi$-chain map induced by f. We will abbreviate $cyl(C(\tilde{f}))$ by cyl and $cone(C(\tilde{f}))$ by $cone$. We have the exact sequence

$$0 \longrightarrow C(\widetilde{M}, \widetilde{\partial_0 M}) \xrightarrow{i} cyl \xrightarrow{\mathrm{pr}} cone \longrightarrow 0$$

Let P be the subcomplex of $cone$ such that $P_i = \{0\}$ for $i \leq n$, P_{n+1} is the kernel of the $(n+1)$-differential of $cone$ and $P_i = cone_i$ for $i > n+1$. As $cone$ is n-connected by the

Hurewicz theorem, P_{n+1} is finitely-generated stably free, and the inclusion of P into *cone* is a homotopy equivalence. A chain complex C is *elementary* if it is concentrated in two adjacent dimensions n and $n+1$ and is given there by the same module $C_{n+1} = C_n$, with the identity as the $n+1$-th differential. By possibly adding a finitely-generated free elementary chain complex concentrated in dimensions $n+1$ and $n+2$ to P, we obtain a finite free $Z\pi$-chain complex Q together with a chain homotopy equivalence $g : Q \longrightarrow cone$. Let D be the pullback chain complex of $g : Q \longrightarrow cone$ and the canonical projection $cyl \longrightarrow cone$, i.e. the kernel of $g \oplus \mathrm{pr} : Q \oplus cyl \longrightarrow cone$. Then we obtain a short exact sequence

$$0 \longrightarrow C(\widetilde{M}, \widetilde{\partial_0 M}) \longrightarrow D \longrightarrow Q \longrightarrow 0$$

of finitely-generated free $Z\pi$-chain complexes such that D is chain homotopy equivalent to $C(\widetilde{N}, \widetilde{\partial_0 N})$ and $Q_i = \{0\}$ for $i \leq n$. By Theorem 2.5, it suffices to prove the claim for $l^2(\pi) \otimes_{Z\pi} C(\widetilde{M}, \widetilde{\partial_0 M})$ and $l^2(\pi) \otimes_{Z\pi} D$. Since these chain complexes have the same chain modules and differentials in dimensions less than or equal to n, the claim follows. ■

Corollary 4.4 *1. The L^2-Betti numbers $b_p(M)$ (respectively the Novikov-Shubin invariants $\alpha_p(M)$) of a compact connected manifold depend only on the fundamental group provided that $p \leq 1$ (respectively $p \leq 2$).*

2. *The L^2-Betti numbers $b_p(M)$ and the Novikov-Shubin invariants $\alpha_p(M)$ of a closed connected 3-manifold depend only on the fundamental group.*

3. *The Novikov-Shubin invariants $\alpha_p(M)$ of a closed connected 4-manifold depend only on the fundamental group.*

Proof : The classifying map $M \longrightarrow B\pi$ for $\pi = \pi_1(M)$ is 2-connected, and $B\pi$ can be chosen to be a CW-complex whose 2-skeleton $B\pi^2$ is finite. Hence Lemma 4.3 implies that $\alpha_p(M) = \alpha_p(B\pi^2)$ (respectively $b_p(M) = b_p(B\pi^2)$) depends only on π provided that $p \leq 2$ (respectively $p \leq 1$). (Note that in the proof of Lemma 4.3, one only needs that $C_p(\widetilde{N}, \partial_0 \widetilde{N})$ be a finitely generated $Z\pi_1(N)$-module for $p \leq n$.) The other claims follow from Theorem 4.2 on Poincaré duality. ■

Note that the second L^2-Betti number of a closed 4-manifold depends on more than just the fundamental group. For example, by taking repeated connected sums with CP^2 one can increase b_2 by any positive integer.

In the top and bottom dimensions the invariants can be computed completely. We recall that a finitely generated group Γ is said to be *amenable* if there is a π-invariant bounded linear operator $\mu : L^\infty(\Gamma) \longrightarrow R$ such that

$$\inf\{f(\gamma) : \gamma \in \Gamma\} \leq \mu(f) \leq \sup\{f(\gamma) : \gamma \in \Gamma\}.$$

Note that any finitely generated abelian group is amenable and any finite group is amenable. A subgroup and a quotient group of an amenable group are amenable. An extension of an amenable group by an amenable group is amenable. A group containing a free group on two generators is not amenable. A finitely generated group Γ is *nilpotent* if Γ possesses a finite lower central series

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_s = \{1\} \qquad \Gamma_{k+1} = [\Gamma, \Gamma_k].$$

If $\overline{\Gamma}$ contains a nilpotent subgroup Γ of finite index then $\overline{\Gamma}$ is said to be *virtually nilpotent.* Let d_i be the rank of the quotient Γ_i/Γ_{i+1} and let d be the integer $\sum_{i\geq 1} id_i$. Then $\overline{\Gamma}$ has polynomial growth of degree d [4]. Note that a group has polynomial growth if and only if it is virtually nilpotent [16].

Lemma 4.5

1. *$\alpha_1(M) = \tilde{\alpha}_0(M)$ is finite if and only if π is infinite and virtually nilpotent. In this case, $\alpha_1(M)$ is the growth rate of π.*
2. *$\alpha_1(M) = \tilde{\alpha}_0(M)$ is ∞^+ if and only if π is finite or nonamenable.*
3. *$\alpha_1(M) = \tilde{\alpha}_0(M)$ is ∞ if and only if π is nonamenable and not virtually nilpotent.*
4. *$b_0(M) = 0$ if π is infinite and $1/\,|\pi|$ otherwise.*
5. *If $\partial_0 M$ is not empty then $\alpha_1(M, \partial_0 M; V)$ and $\alpha_m(M, \partial_1 M; V)$ are equal to ∞^+ and $b_0(M, \partial_0 M; V)$ and $b_m(M, \partial_1 M; V)$ are zero.*
6. *If $\partial_0 M$ is empty then $\alpha_m(M; V) = \alpha_1(M; V)$ and $b_m(M; V) = b_0(M; V)$.*

Proof : 1.) to 3.) Since $\alpha_1(M)$ depends only on the fundamental group and there is a closed manifold with π as its fundamental group, we may assume that M is closed. Efremov [15] shows that $\alpha_1(M)$ equals its analytic counterpart. For the analytic counterpart, assertion 1.) is proven in [41] and assertion 2.) is proven in [5]. Assertion 3.) is a direct consequence of 1.) and 2.)

4.) is proven in [11, Proposition 2.4].

5.) and 6.) If $\partial_0 M$ is nonempty then the pair $(M, \partial_0 M)$ is homotopy equivalent to a pair of finite CW-complexes (X, A) such that all of the 0-cells of X lie in A. Hence the cellular $Z\pi_1(M)$-chain complex $C(\widetilde{M}, \widetilde{\partial_0 M}; V)$ is $Z\pi_1(M)$-chain homotopy equivalent to a $Z\pi_1(M)$-chain complex which is trivial in dimension 0. Now apply Theorems 2.5 and 4.2. ■

For later purposes we will need the following result:

Lemma 4.6 *Let $j : \pi_1(M) \longrightarrow \Gamma$ be an inclusion of discrete groups. Let $j^* l^2(\Gamma)$ be the unitary representation $\pi_1(M) \longrightarrow Iso_{N(\Gamma)}(l^2(\Gamma))^{op}$ obtained from the right regular representation of Γ by composing with j. Then for all p, we have*

1. $b_p(M, \partial_0 M) = b_p(M, \partial_0 M; j^* l^2(\Gamma))$.
2. $\alpha_p(M, \partial_0 M) = \alpha_p(M, \partial_0 M; j^* l^2(\Gamma))$.

Proof : Let $f : \oplus_{i=1}^n Z\pi_1(M) \longrightarrow \oplus_{i=1}^n Z\pi_1(M)$ be a $Z\pi_1(M)$-linear map. By tensoring with $l^2(\pi_1(M))$ (resp. $j^* l^2(\Gamma)$), we get a morphism of Hilbert $N(\pi_1(M))$ (resp. $N(\Gamma)$)-modules denoted by f_1 (resp. f_2). Let $\{E_\lambda^{f_2^* f_2} : \lambda \in R\}$ denote the spectral family of the self-adjoint operator $f_2^* f_2 : \oplus_{i=1}^n l^2(\Gamma) \longrightarrow \oplus_{i=1}^n l^2(\Gamma)$ and $\{E_\lambda^{f_1^* f_1} : \lambda \in R\}$ denote the spectral family of $f_1^* f_1 : \oplus_{i=1}^n l^2(\pi_1(M)) \longrightarrow \oplus_{i=1}^n l^2(\pi_1(M))$. Then $E_\lambda^{f_2^* f_2}$ maps $\oplus_{i=1}^n l^2(\pi_1(M))$ into itself and the restriction of $E_\lambda^{f_2^* f_2}$ to $\oplus_{i=1}^n l^2(\pi_1(M))$ is just $E_\lambda^{f_1^* f_1}$. By [12, Theorem 1, p. 97], this implies

$$F(f_1, \lambda) = \mathrm{tr}_{N(\pi_1(M))}\left(E_{\lambda^2}^{f_1^* f_1}\right) = \langle E_{\lambda^2}^{f_1^* f_1}(1), 1\rangle_{l^2(\pi_1(M))} =$$
$$\langle E_{\lambda^2}^{f_2^* f_2}(1), 1\rangle_{l^2(\Gamma)} = \mathrm{tr}_{N(\Gamma)}\left(E_{\lambda^2}^{f_2^* f_2}\right) = F(f_2, \lambda),$$

and the claim follows. ■

We now investigate the behaviour with respect to connected sums.

Proposition 4.7 *Let M_1, M_2, $\ldots M_r$ be compact connected m-dimensional manifolds, with $m \geq 3$. Let M be their connected sum $M_1 \sharp \ldots \sharp M_r$. Then*

1. $b_1(M) - b_0(M) = r - 1 + \sum_{j=1}^r (b_1(M_j) - b_0(M_j))$.
2. $b_p(M) = \sum_{j=1}^r b_p(M_j)$ *for* $2 \leq p \leq m - 2$.
3. $\alpha_p(M) = \min\{\alpha_p(M_j) : 1 \leq j \leq r\}$ *for* $2 \leq p \leq m - 1$.

4. *If $\pi_1(M_i)$ is trivial for all i except for $i = i_0$ then $\alpha_1(M) = \alpha_1(M_{i_0})$. Suppose $\pi_1(M_i)$ is trivial for all i except for $i \in \{i_0, i_1\}$, $i_0 \neq i_1$, and that $\pi_1(M_{i_0}) = \pi_1(M_{i_1}) = Z/2$. Then $\alpha_1(M) = 1$. In all other cases $\alpha_1(M) = \infty^+$.*

Proof : We may assume without loss of generality that $r = 2$. The connected sum $M_1 \sharp M_2$ is obtained by glueing $M_1 \backslash \mathrm{int}(D^m)$ and $M_2 \backslash \mathrm{int}(D^m)$ together along ∂D^m. Since $\partial D^m \longrightarrow D^m$ is $(m-1)$-connected, the inclusion of $M_j \backslash \mathrm{int}(D^m)$ into M_j is $(m-1)$-connected. Hence the inclusion

$$M_1 \backslash \mathrm{int}(D^m) \cup_{\partial D^m} M_2 \backslash \mathrm{int}(D^m) \longrightarrow M_1 \cup_{D^m} M_2$$

is $(m-1)$-connected. Since $M_1 \cup_{D^m} M_2$ is homotopy equivalent to the wedge $M_1 \vee M_2$, from Lemma 4.3 it suffices to prove the claims for $M_1 \vee M_2$.

1.) to 3.) Let π denote $\pi_1(M_1 \vee M_2) = \pi_1(M_1) * \pi_1(M_2)$. If $*$ denotes the base point, we obtain an exact sequence

$$0 \longrightarrow C(*; l^2(\pi)) \longrightarrow C(M_1; l^2(\pi)) \oplus C(M_2; l^2(\pi)) \longrightarrow C(M_1 \vee M_2; l^2(\pi)) \longrightarrow 0$$

The long weakly exact Mayer-Vietoris sequence reduces to weak isomorphisms

$$H_p(M_1; l^2(\pi)) \oplus H_p(M_2; l^2(\pi)) \longrightarrow H_p(M_1 \bigvee M_2; l^2(\pi)) \text{ , } p \geq 2,$$

and the weakly exact sequence

$$\begin{aligned} 0 \longrightarrow H_1(M_1; l^2(\pi)) \oplus H_1(M_2; l^2(\pi)) &\longrightarrow H_1(M_1 \vee M_2; l^2(\pi)) \longrightarrow l^2(\pi) \\ \longrightarrow H_0(M_1; l^2(\pi)) \oplus H_0(M_2; l^2(\pi)) &\longrightarrow H_0(M_1 \vee M_2; l^2(\pi)) \longrightarrow 0 \end{aligned}$$

We conclude from Lemmas 1.1 and 4.6 that

$$\begin{aligned} &b_1(M_1) + b_1(M_2) - b_1(M_1 \vee M_2) + 1 - b_0(M_1) - b_0(M_0) + b_0(M_1 \vee M_2) = 0 \\ &b_p(M_1) + b_p(M_2) = b_p(M_1 \vee M_2) \text{ for } p \geq 2, \end{aligned}$$

from which assertions 1.) and 2.) follow. We obtain assertion 3.) from Theorem 2.2.

4.) Since $\alpha_1(M)$ only depends on the fundamental group and $\pi_1(M) = \pi_1(M_1)$ if $\pi_1(M_2)$ is trivial, the first part of the assertion follows. It remains to consider the case when $\pi_1(M_1)$ and $\pi_1(M_2)$ are nontrivial. From Lemma 4.5.2, $\alpha_1(M)$ is ∞^+ if and only if $\pi_1(M)$ is nonamenable. We claim that $\pi_1(M)$ is amenable if and only if $\pi_1(M_1) = \pi_1(M_2) = Z/2$, in which case $\alpha_1(M) = 1$. Namely, suppose that $\pi_1(M)$ is amenable. Then it follows from [11, Theorem 0.2] that $b_1(M) = b_0(M) = 0$. But then assertion 1.) and Lemma 4.5 imply that $\mid \pi_1(M_i) \mid = 2$

for $i = 1, 2$. As $Z/2 * Z/2$ is an extension of Z by $Z/2$, it is amenable. Also, there is a two-fold covering of M with the fundamental group of a circle. Hence $\alpha_1(M) = \alpha_1(S^1)$, which is 1 by Lemma 3.1. ■

Next we study manifolds with an S^1-action. Let $(M; \partial_0 M)$ be as above. Suppose that S^1 acts smoothly on M. Let $\phi : \pi_1(M) \longrightarrow \Gamma$ be an homomorphism such that for one orbit (and hence all orbits) S^1/H in M, the composition of ϕ with the map induced by the inclusion $\pi_1(S^1/H) \longrightarrow \pi_1(M)$ has infinite image. In particular, the S^1-action has no fixed points. Choose $\mathcal{A}$ to be $N(\Gamma)$ and the representation $\phi^* l^2(\Gamma)$ to be the composition of the regular representation $\Gamma \longrightarrow Iso_{N(\Gamma)}(l^2(\Gamma))$ with ϕ. In other words, we are looking at the cover $\overline{M} \longrightarrow M$ of M associated with ϕ.

Theorem 4.8 **(S^1-manifolds)** *With the above conditions on the S^1-manifold M, for all $p \geq 0$ we have:*

1. $b_p(M, \partial_0 M; \phi^* l^2(\Gamma)) = 0$.
2. $\alpha_p(M, \partial_0 M; \phi^* l^2(\Gamma)) \geq 1$.

<u>**Proof**</u> : The first assertion was proven in [27, Theorem 3.20].

In what follows we will write $l^2(\Gamma)$ instead of $\phi^* l^2(\Gamma)$, or $j^* \phi^* l^2(\Gamma)$ for j an inclusion. Since we have a smooth S^1-action, M carries a S^1-equivariant CW-structure. This means that we have a filtration

$$\emptyset = M_{-1} \subset M_0 \subset M_1 \subset \ldots M_{m-1} = M$$

such that M_i is obtained from M_{i-1} by attaching a finite number of S^1-equivariant cells $S^1/H \times D^i$ with attaching maps $S^1/H \times S^{i-1} \longrightarrow M_{i-1}$. Since the S^1-action has no fixed points, the subgroups $H \subset S^1$ are all finite cyclic groups. We will show that

$$\alpha_p(M_i, \partial_0 M \cap M_i; l^2(\Gamma)) \geq 1 \text{ for } p \leq i + 1$$
$$\alpha_p(M_i, \partial_0 M \cap M_i; l^2(\Gamma)) = \infty^+ \text{ for } p > i + 1$$

by induction over i, where the representation of $\pi_1(M_i)$ is induced from the inclusion $\pi_1(M_i) \longrightarrow \pi_1(M)$. The initial step $i = -1$ is trivial. The induction step from $i - 1$ to i is done as follows:

There is an exact sequence of chain complexes

$$0 \longrightarrow C(M_{i-1}, \partial_0 M \cap M_{i-1}; l^2(\Gamma)) \longrightarrow C(M_i, \partial_0 M \cap M_i; l^2(\Gamma)) \longrightarrow$$
$$C(M_i, M_{i-1} \cup (\partial_0 M \cap M_i); l^2(\Gamma)) \longrightarrow 0$$

The last chain complex is isomorphic to a direct sum of chain complexes of the form $C(S^1/H \times D^i, S^1/H \times S^{i-1}; l^2(\Gamma))$. Since all isotropy groups H must be finite, such a chain complex looks like $\Sigma^i C(S^1; l^2(\Gamma))$, where $l^2(\Gamma)$ is viewed as a representation space of $\pi_1(S^1)$ by means of an injection $\pi_1(S^1) \longrightarrow \Gamma$. Lemmas 4.6 and 3.1 imply that $\alpha_p(\Sigma^i C(S^1; l^2(\Gamma)))$ is 1 if $p = i+1$ and ∞^+ otherwise. Lemma 2.3.3 implies that $\alpha_p(C(M_i, M_{i-1} \cup (\partial_0 M \cap M_i); l^2(\Gamma))$ is also 1 for $p = i+1$ and ∞^+ otherwise. Upon applying Theorem 2.2.1 to the short exact sequence of weakly acyclic chain complexes above and using the induction hypothesis on M_{i-1}, the claim follows. ■

We now consider manifolds which fiber over S^1.

Theorem 4.9 *Let $(M, \partial_0 M)$ fiber over S^1 with fiber $(F, \partial_0 F)$. Suppose that $(F, \partial_0 F)$ has vanishing L^2-cohomology. Then*

1. *$b_p(M, \partial_0 M) = 0$ for all p.*
2. $\dfrac{1}{\alpha_p(M, \partial_0 M)} \leq \dfrac{1}{\alpha_{p-1}(F, \partial_0 F)} + \dfrac{1}{\alpha_p(F, \partial_0 F)}.$

Proof : We have a short exact (Wang) sequence of Hilbert chain complexes:

$$\begin{aligned} 0 \longrightarrow \quad & C(F, \partial_0 F; l^2(\pi_1(M))) && \xrightarrow{j} C(F \times I, \partial_0 F \times I; l^2(\pi_1(M))) \xrightarrow{q} \\ & C(M, \partial_0 M; l^2(\pi_1(M))) && \longrightarrow 0. \end{aligned}$$

Since $H_p(F, \partial_0 F, l^2(\pi_1(F))$ vanishes for all p by assumption, Lemma 4.6 implies that the same is true for $H_p(F, \partial_0 F; l^2(\pi_1(M)))$ and $H_p(F \times I, \partial_0 F \times I; l^2(\pi_1(M)))$. Consideration of the long weakly exact homology sequence associated to the Wang sequence gives assertion 1. Assertion 2. follows from Theorem 2.2.2. ■

Remark 4.10 Let $g : M \longrightarrow N$ be an n-fold finite covering. Then $b_p(M) = n \cdot b_p(N)$ and $\alpha_p(M) = \alpha_p(N)$ for all $p \geq 0$. Note that the ordinary Betti numbers of a manifold are not multiplicative under finite coverings. ■

Example 4.11 We state the values of the L^2-Betti numbers and Novikov-Shubin invariants for all compact connected 1- and 2-manifolds. In dimension 1 there are only S^1 and the unit interval I. From Theorem 3.1, we have that $b_0(S^1) = b_1(S^1) = 0$ and $\alpha_1(S^1) = 1$. As I is contractible, we have that $b_0(I) = 1$, $b_1(I) = 0$ and $\alpha_1(I) = \infty^+$.

Let F_g^d be the orientable closed surface of genus g with d embedded 2-disks removed. (As any nonorientable compact surface is finitely-covered by an orientable surface, Example 4.10 shows that it is enough to handle the orientable case.) Using the general formula for the Euler characteristic in terms of L^2-Betti numbers [9] :

$$\chi(M) = \sum_p (-1)^p b_p(M),$$

Lemma 4.5 and the fact that a compact surface with boundary is homotopy-equivalent to a bouquet of circles, one derives:

$$b_0(F_g^d) = \begin{cases} 1 & g = 0, d = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

$$b_1(F_g^d) = \begin{cases} 0 & g = 0, d = 0,1 \\ d + 2(g-1) & \text{otherwise.} \end{cases}$$

$$b_2(F_g^d) = \begin{cases} 1 & g = 0, d = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\alpha_1(F_g^d) = \tilde{\alpha}_0(F_g^d) = \tilde{\alpha}_1(F_g^d) = \begin{cases} 1 & g = 0, d = 2 \\ 2 & g = 1, d = 0 \\ \infty^+ & \text{otherwise.} \end{cases}$$

$$\alpha_2(F_g^d) = \tilde{\alpha}_2(F_g^d) = \begin{cases} 2 & g = 1, d = 0 \\ \infty^+ & \text{otherwise.} \end{cases} \qquad \blacksquare$$

Example 4.12 Suppose that M is a compact connected orientable 3-manifold with finite fundamental group π. We have that $\alpha_p(M) = \infty^+$ for all p. If M is closed then $\widetilde{M}$ is a homotopy sphere, and Remark 4.10 implies that $b_0(M) = b_3(M) = \frac{1}{|\pi|}$ and $b_1(M) = b_2(M) = 0$. If ∂M is nonempty then $\widetilde{M}$ is a connected sum of a homotopy sphere and k 3-disks, for some positive integer k [20]. Then $b_0(M) = \frac{1}{|\pi|}$, $b_2(M) = \frac{k-1}{|\pi|}$ and $b_1(M) = b_3(M) = 0$. $\blacksquare$

5. Seifert 3-Manifolds

In this section we compute the L^2-Betti numbers and Novikov-Shubin invariants of Seifert 3-manifolds. We also discuss *Sol* manifolds. We use the definition of *Seifert fibred 3-manifold* , or briefly *Seifert manifold* , given in [36], which we will use as a reference on Seifert manifolds. Recall that a *geometry* on a 3-manifold M is a complete locally homogeneous Riemanian metric on its interior. The universal cover of the interior has a complete homogeneous Riemannian metric, meaning that the isometry group acts transitively [37]. Thurston has shown that there are precisely eight maximal simply-connected 3-dimensional geometries having compact quotients, namely S^3, R^3, $S^2 \times R$, $H^2 \times R$, Nil, $\widetilde{SL_2}(R)$, Sol and H^3. If a closed 3-manifold admits a geometric structure modelled on one of these eight geometries then the geometry involved is unique.

In the case of the L^2-Betti numbers, the following result was already given in [7].

Theorem 5.1 *Let M be a closed Seifert 3-manifold. If its fundamental group is infinite then it has vanishing L^2-cohomology. In terms of the Euler class e of the bundle and the Euler characteristic χ of the base orbifold, $\alpha_1(M) = \alpha_3(M)$ is given by*

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	1	3	∞^+
$e \neq 0$	∞^+	4	∞^+

and $\alpha_2(M)$ is given by

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	∞^+	3	1
$e \neq 0$	∞^+	2	$1.$

Proof : The geometric structure of M is determined as follows: [36, Theorem 5.3]:

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times R$	R^3	$H^2 \times R$
$e \neq 0$	S^3	Nil	$\widetilde{SL_2}(R).$

If M has a S^3-structure then $\pi_1(M)$ is finite and we can apply Example 4.12.

In all other cases M is finitely covered by the total space $\overline{M}$ of an S^1-principal bundle over an orientable closed surface F. Moreover, $e(M) = 0$ iff $e(\overline{M}) = 0$, and the Euler characteristic χ of the orbifold base of M is negative, zero or positive according to the same condition for $\chi(\overline{M}/S^1)$ [36, p. 426, 427 and 436]. From Remark 4.10, in what follows we may assume without loss of generality that M is $\overline{M}$. Theorem 4.8 implies that $b_p(M) = 0$. If $\chi(F)$ is negative then $\pi_1(F)$ is non-amenable since it contains a free subgroup of rank 2. As $\pi_1(F)$ is a quotient of $\pi_1(M)$, $\pi_1(M)$ is also non-amenable and so $\alpha_1(M) = \infty^+$ by Lemma 4.5. Next, we verify the remaining claims for α_1 and α_2.

R^3: We may assume that $M = T^3$. A direct computation by Fourier analysis gives that $\alpha_p(T^3) = 3$ for all $1 \leq p \leq 3$.

$S^2 \times R$: We may assume that $M = S^1 \times S^2$. Now apply Lemma 5.2.

$H^2 \times R$: We may assume that $M = S^1 \times F_g$ for $g \geq 2$. Now apply Lemma 5.2.

Nil: From [24] we have that $\tilde{\alpha}_0(M) = 4$ and $\tilde{\alpha}_1(M) = 2$, and so the claim for α_1 and α_2 follows.

$\widetilde{SL_2(R)}$: A computation using harmonic analysis on $\widetilde{SL_2(R)}$, which we will not reproduce here, gives $\alpha_2(M) = 1$.

The next lemma will finish the proof of Theorem 5.1. ■

Lemma 5.2 *Let F_g^d be the (orientable compact connected) surface of genus g with d boundary components. Then*

1. $b_p(S^1 \times F_g^d) = 0$ *for all* p.

2. $$\alpha_1(S^1 \times F_g^d) = \begin{cases} 1 & g=0, d=0,1 \\ 2 & g=0, d=2 \\ 3 & g=1, d=0 \\ \infty^+ & \textit{otherwise} \end{cases}$$

3. $$\alpha_2(S^1 \times F_g^d) = \begin{cases} \infty^+ & g=0, d=0,1 \\ 3 & g=1, d=0 \\ 2 & g=0, d=2 \\ 1 & \textit{otherwise} \end{cases}$$

4. $$\alpha_3(S^1 \times F_g^d) = \begin{cases} 1 & g=0, d=0 \\ 3 & g=1, d=0 \\ \infty^+ & \textit{otherwise} \end{cases}$$

Proof : The claim for the L^2-Betti numbers follows from Theorem 4.8. In the cases $g = 0, d = 0, 1, 2$ and $g = 1, d = 0$, i.e. $S^1 \times S^2$, $S^1 \times D^2$, $S^1 \times S^1 \times I$ and T^3, the claim follows from earlier computations for S^1, T^2 and T^3 (see Example 4.11 and Theorem 5.1). In the remaining cases Example 4.11 gives that $\alpha_p(F_g^d) = \infty^+$ for all p and $b_p(F_g^d) = 0$ for $p \neq 1$. We abbreviate $F = F_g^d$. Let H be the Hilbert chain complex over the von Neumann algebra of $\pi_1(F)$ which is concentrated in dimension 1, and is given there by $\ker(\Delta_1)$, where $\Delta_1 : C_1(F; l^2(\pi_1(F))) \longrightarrow C_1(F; l^2(\pi_1(F)))$ is the Laplace operator. There is a natural split inclusion $i : H \longrightarrow C(F; l^2(\pi_1(F)))$. From Lemma 2.4, i is a homotopy equivalence. We have that $C(S^1 \times F, l^2(\pi_1(S^1 \times F)))$ is the Hilbert tensor product of $C(F; l^2(\pi_1(F)))$ and $C(S^1; l^2(\pi_1(S^1)))$, and so is homotopy equivalent to the Hilbert tensor product of H and $C(S^1; l^2(\pi_1(S^1)))$. As the part of H in dimension one is isomorphic to $\oplus_{i=1}^{-\chi(F)} l^2(\pi_1(F))$, this Hilbert tensor product is isometrically isomorphic to the suspension of the direct sum of $-\chi(F)$ copies of $C(S^1; l^2(\pi_1(S^1 \times F)))$. From Lemma 2.3, Theorem 2.5 and Lemma 4.6, the Novikov-Shubin invariants of M are the same as those of the suspension of $C(S^1; l^2(\pi_1(S^1)))$. The claim now follows from Example 4.11. ■

Remark 5.3 The fact that the Novikov-Shubin invariants are the same for closed $H^2 \times R$-manifolds and $\widetilde{SL_2(R)}$-manifolds is probably related to the fact that the universal covers of such manifolds are quasi-isometric. This latter statement, which is due independently to D. Epstein and S. Gersten and was communicated to us by M. Gromov, follows easily from the fact that the fundamental class of a closed orientable hyperbolic surface, considered as an element of the group cohomology of the fundamental group, can be represented by a bounded group cocycle. ■

Theorem 5.4 *Let M be a Seifert manifold with nonempty boundary. Then all L^2-Betti numbers vanish. We have that $\alpha_3(M) = \infty^+$, and the other Novikov-Shubin invariants are given by:*

α_1	α_2	
1	∞^+	*M is a solid torus or Klein bottle*
2	2	*M is an I-bundle over T^2 or over a Klein bottle K*
∞^+	1	*otherwise.*

Proof : We have that the boundary of M is compressible iff M is homeomorphic to a solid torus or Klein bottle [36, Corollary 3.3]. The theorem follows in this case from Remark 4.10 and Lemma 5.2, and so we may assume that M has incompressible boundary. As any 2-dimensional orbifold with boundary is finitely covered by a 2-dimensional surface with boundary, we can find a finite cover $\overline{M}$ of M which is homeomorphic to some $S^1 \times F_g^d$, with $d \geq 1$. From Remark 4.10 and Lemma 5.2, we have to know that M is an I-bundle over T^2 or K iff $F_g^d = S^1 \times I$. This follows from [20, Theorem 10.5]. ■

Proposition 5.5 *If M is a closed Sol-manifold then M has vanishing L^2-Betti numbers, $\alpha_1(M) = \infty$ and $\alpha_2(M) \geq 1$.*

Proof : By taking a finite cover, we may assume that our *Sol*-manifold is a torus bundle over S^1 with hyperbolic glueing map ϕ [36, Theorem 5.3]. Hence $\pi_1(M)$ is a semi-direct product of Z^2 and Z where the action of Z on Z^2 is given by a hyperbolic automorphism of Z^2. Then $\pi_1(M)$ is amenable, as it is an extension of amenable groups. It is easy to see that $\pi_1(M)$ is not virtually nilpotent. Lemma 4.5.3 implies that $\alpha_1(M) = \infty$.

By Example 4.11, $b_p(T^2) = 0$ for all p and $\alpha_p(T^2) = 2$ for $p \in \{1, 2\}$. Then Theorem 4.9 implies that the L^2-Betti numbers of M vanish, and that

$$\frac{1}{\alpha_2(M)} \leq \frac{1}{2} + \frac{1}{2}. \quad \blacksquare$$

6. Analytic L^2-Betti numbers and Novikov-Shubin invariants for manifolds with boundary, and hyperbolic 3-manifolds

In this section we define analytic Novikov-Shubin invariants and L^2-Betti numbers for manifolds with boundary, and show the equivalences between the analytic invariants and the combinatorial invariants of the previous section. As an application, we give a lower bound for the Novikov-Shubin invariants of a compact 3-manifold whose interior admits a complete finite-volume hyperbolic metric.

As the Hilbert spaces with which we deal in this section will have infinite von Neumann dimension, we must first discuss the notion of $\mathcal{A}$-Fredholmness of morphisms. A related discussion appears in [10]. Let $f : U \to V$ be a morphism of (possibly infinite dimensional) $\mathcal{A}$-Hilbert modules. Our morphisms are still bounded operators. We again have the spectral density function $F(f, \lambda)$, although it may now take infinite values.

Definition 6.1 *The morphism f is $\mathcal{A}$-Fredholm if there exists a $\lambda > 0$ such that $F(f, \lambda) < \infty$ and $F(f^*, \lambda) < \infty$.*

Note that if $\mathcal{A} = C$ then we recover the usual notion of a bounded Fredholm operator.

Definition 6.2 *Let C be a Hilbert $\mathcal{A}$-chain complex with differentials c_p. Let*

$$\overline{c}_p : C_p/clos(im(c_{p+1})) \to C_{p-1}$$

be the quotient map. Then C is an $\mathcal{A}$-Fredholm complex if for all p, there exists a $\lambda_p > 0$ such that $F(\overline{c}_p, \lambda_p) < \infty$.

Note that an $\mathcal{A}$-Fredholm complex has finitely generated homology groups. The relationship between Definitions 6.1 and 6.2 is given by the following proposition.

Proposition 6.3 *If $f : U \to V$ is a morphism of (possibly infinitely dimensional) $\mathcal{A}$-Hilbert modules and C is the Hilbert $\mathcal{A}$-chain complex*

$$0 \to U \xrightarrow{f} V \to 0$$

then f is an $\mathcal{A}$-Fredholm morphism if and only if C is an $\mathcal{A}$-Fredholm complex.

Proof : The condition that C be an $\mathcal{A}$-Fredholm complex is equivalent to requiring that there exist a $\lambda > 0$ such that $F(f, \lambda) < \infty$, and that $b(f^*) = dim_{\mathcal{A}}(ker(f^*)) < \infty$. It is clear that if f is an $\mathcal{A}$-Fredholm morphism then C is an $\mathcal{A}$-Fredholm complex. Suppose that C is an $\mathcal{A}$-Fredholm complex. We showed in Lemma 1.12.5 that $F(f, \lambda) - b(f) = F(f^*, \lambda) - b(f^*)$. If $F(f, \lambda) < \infty$ then $F(f^*, \lambda) < \infty$, which shows that f is an $\mathcal{A}$-Fredholm morphism. ■

If C is an $\mathcal{A}$-Fredholm complex, we define its Betti numbers as in Definition 2.1 and we define its Novikov-Shubin invariants by $\alpha_p(C) = \alpha(\overline{c}_p)$. One can check that all of the results of Sections 1-3 hold when one replaces morphisms of finitely generated Hilbert $\mathcal{A}$-modules by $\mathcal{A}$-Fredholm morphisms, and finite Hilbert $\mathcal{A}$-chain complexes by $\mathcal{A}$-Fredholm chain complexes.

For closed manifolds, the facts that the analytic L^2-Betti numbers and Novikov-Shubin invariants equal their combinatorial counterparts were proven in [13] and [15]. In order to make the comparisons between the analytic and combinatorial invariants for a compact manifold M with boundary, it will be convenient for us to think of the combinatorial invariants as defined by simplicial cochains, instead of simplicial chains. In this section, except where otherwise stated, the Novikov-Shubin invariants will be those of the coboundary operator. The smooth forms on $\widetilde{M}$ will be denoted by $C^\infty(\wedge^*(\widetilde{M}))$. Those with compact support will be denoted by $C_0^\infty(\wedge^*(\widetilde{M}))$. Note that the elements of $C_0^\infty(\wedge^*(\widetilde{M}))$ do not necessarily vanish on $\widetilde{\partial M}$.

We assume that M has a smooth Riemannian metric. We give $\widetilde{M}$ the induced Riemannian metric. Let d denote the exterior derivative, δ denote its formal adjoint, $\triangle$ denote the Laplacian $d\delta + \delta d$, $*$ denote the Hodge duality operator and $b : \widetilde{\partial M} \to \widetilde{M}$ denote the boundary inclusion in $\widetilde{M}$. As before, π denotes the fundamental group of M.

Definition 6.4 *Define norms $\| \bullet \|_s$ on $C_0^\infty(\wedge^*(\widetilde{M}))$ for nonnegative integers s inductively by saying that $\| \bullet \|_0$ is the L^2-norm and $\| \omega \|_{s+1}^2 = \| \omega \|_s^2 + \| d\omega \|_s^2 + \| \delta\omega \|_s^2$. Let $\mathcal{H}_s^*(M; l^2(\pi))$ be the Hilbert space completion of $C_0^\infty(\wedge^*(\widetilde{M}))$ under the norm $\| \bullet \|_s$.* ■

Put $\mathcal{A} = N(\pi)$. We have a Hilbert $\mathcal{A}$-cochain complex

$$\ldots \xrightarrow{d_{p-2}} \mathcal{H}_2^{p-1}(M; l^2(\pi)) \xrightarrow{d_{p-1}} \mathcal{H}_1^p(M; l^2(\pi)) \xrightarrow{d_p} \mathcal{H}_0^{p+1}(M; l^2(\pi)). \tag{1}$$

We will show in Lemma 6.8 that the complex (1) is $\mathcal{A}$-Fredholm at d_p, which will be all that we need.

Definition 6.5 *The analytic p-th L^2-cohomology group is*

$$H^p(M; l^2(\pi)) = ker(d_p)/clos(im(d_{p-1})),$$

the analytic p-th L^2-Betti number is

$$b_p(M; l^2(\pi)) = dim_{\mathcal{A}}(H^p(M; l^2(\pi)))$$

and the analytic p-th Novikov-Shubin invariant is

$$\alpha_p(M; l^2(\pi)) = \alpha(\overline{d}_p). \qquad \blacksquare$$

If we put $\tilde{\alpha}_p(M) = \min(\alpha_p(M), \alpha_{p-1}(M))$ then the application of a Laplace transform to the spectral density function shows that the analytic invariants of the introduction, defined using heat kernels, are the same as those defined here [18, Appendix].

As a topological vector space, $\mathcal{H}^*_s(M; l^2(\pi))$ is independent of the Riemannian metric, as the norms $\| \bullet \|_s$ on $C_0^\infty(\wedge^*(\widetilde{M}))$ coming from two Riemannian metrics are relatively bounded. Given two Riemannian metrics, the identity map between the corresponding complexes (1) is a bounded cochain homotopy equivalence, and so Theorem 2.5 implies that the analytic L^2-Betti numbers and Novikov-Shubin invariants are independent of the Riemannian metric.

We note that $\mathcal{H}^p_1(M; l^2(\pi))/clos(im(d_{p-1}))$ and $im(d_{p-1})^\perp \subset \mathcal{H}^p_1(M; l^2(\pi))$ are isometrically isomorphic.

Proposition 6.6

$$im(d_{p-1})^\perp = \{\omega \in \mathcal{H}^p_1(M; l^2(\pi)) : \delta\omega = b^*(*\omega) = 0\}. \tag{2}$$

<u>**Proof**</u> : Given $\omega \in im(d_{p-1})^\perp$, for all $\eta \in C^\infty(\wedge^*(\widetilde{M})) \cap \mathcal{H}^{p-1}_2(M; l^2(\pi))$ we have

$$\begin{aligned} 0 &= < d\eta, \omega >_1 = < d\eta, \omega >_0 + < \delta d\eta, \delta\omega >_0 = \int_{\widetilde{M}} [d\eta \wedge *\omega + \delta d\eta \wedge *\delta\omega] \\ &= \int_{\widetilde{M}} (I + \delta d)\eta \wedge *\delta\omega + \int_{\partial\widetilde{M}} b^*\eta \wedge b^*(*\omega). \end{aligned} \tag{3}$$

Let $\rho \in C_0^\infty(\wedge^*(\widetilde{M}))$ have support within the interior of $\widetilde{M}$. We claim that there exists an $\eta \in C^\infty(\wedge^*(\widetilde{M})) \cap \mathcal{H}^{p-1}_2(M; l^2(\pi))$ such that $(I + \delta d)\eta = \rho$ and $b^*\eta = 0$. To see this, consider the elliptic equation on $\widetilde{M}$

$$(I + \triangle)\eta' = \rho + d\delta\rho, \tag{4}$$

with the (relative) boundary conditions $b^*\eta' = b^*(\delta\eta') = 0$. By standard elliptic theory, this system has a solution $\eta' \in C^\infty(\Lambda^*(\widetilde{M})) \cap \mathcal{H}_2^{p-1}(M; l^2(\pi))$. Applying δ to both sides of (4) and putting $\sigma = \delta\eta' - \delta\rho$, we obtain $(I + \delta d)\sigma = 0$, with $b^*\sigma = 0$. Then

$$\begin{aligned} 0 &= \int_{\widetilde{M}} (\sigma + \delta d\sigma) \wedge *\sigma = \int_{\widetilde{M}} \sigma \wedge *\sigma + \int_{\widetilde{M}} d\sigma \wedge *d\sigma \pm \int_{\widetilde{\partial M}} b^*(*d\sigma) \wedge b^*\sigma \\ &= \int_{\widetilde{M}} \sigma \wedge *\sigma + \int_{\widetilde{M}} d\sigma \wedge *d\sigma = \| \sigma \|_1^2 . \end{aligned} \tag{5}$$

Thus $\sigma = 0$ and equation (4) becomes $(I + \delta d)\eta' = \rho$. So we can take η' for η.

As ρ was arbitrary, it follows from (3) that $\delta\omega = 0$. Then considering η's which do not vanish on $\widetilde{\partial M}$, it follows from (3) that $b^*(*\omega) = 0$. That is, we have shown

$$im(d_{p-1})^\perp \subset \{\omega \in \mathcal{H}_1^p(M; l^2(\pi)) : \delta\omega = b^*(*\omega) = 0\}.$$

Conversely, given $\omega \in \mathcal{H}_1^p(M; l^2(\pi))$ such that $\delta\omega = b^*(*\omega) = 0$, equation (3) implies that $0 = < d\eta, \omega >_1$ for all $\eta \in C^\infty(\Lambda^*(\widetilde{M})) \cap \mathcal{H}_2^{p-1}(M; l^2(\pi))$. The density of $C^\infty(\Lambda^*(\widetilde{M})) \cap \mathcal{H}_2^{p-1}(M; l^2(\pi))$ in $\mathcal{H}_2^{p-1}(M; l^2(\pi))$ gives

$$\{\omega \in \mathcal{H}_1^p(M; l^2(\pi)) : \delta\omega = b^*(*\omega) = 0\} \subset im(d_{p-1})^\perp. \qquad \blacksquare$$

For a moment, let us take the Riemannian metric on M to be a product near ∂M. Then there is an induced Riemannian metric on the double DM, upon which Z_2 acts by isometries. With π still denoting $\pi_1(M)$, there is a π-normal cover of DM, namely the double $\widetilde{DM}$ of $\widetilde{M}$, and it is easy to see that $im(d_{p-1})^\perp \subset \mathcal{H}_1^p(M; l^2(\pi))$ is isomorphic to $(ker(\delta) \subset \mathcal{H}_1^p(DM; l^2(\pi)))^{Z_2}$, the subspace of $ker(\delta) \subset \mathcal{H}_1^p(DM; l^2(\pi))$ which is invariant under the induced Z_2 action. The papers [13] and [15] imply the equality of the analytic and combinatorial invariants as defined for the π-cover on DM. One can go through their proofs making everything equivariant with respect to the Z_2 action, in order to show that the same is true when one restricts to the Z_2-invariant subspaces. (As in [13] and [15], one first deals with Sobolev spaces of a high enough order that the de Rham map is well-defined. One then shows the analytic invariants are independent of order of the Sobolev space. In our case, we are finally interested in the Sobolev space $\mathcal{H}_1^p$. All of these steps will go through equivariantly.) Now the combinatorial invariants defined with Z_2-invariant cochains on DM can be identified with the absolute invariants of M. Putting all this together, we have shown

Proposition 6.7 *The analytic L^2-Betti numbers and Novikov-Shubin invariants of Definition 6.5 are equal to the combinatorial invariants of Section 4, with $\partial_0 M = \emptyset$.*

Since we know that the analytic invariants are independent of the Riemannian metric on M, we may now say that the combinatorial invariants equal the analytic invariants as computed using the quadratic form $q(\omega,\omega') = < d\omega, d\omega' >_0$ with domain $Dom(q) = im(d_{p-1})^{\perp} \subset \mathcal{H}_1^p(M; l^2(\pi))$, for any Riemannian metric on M. That is, $b_p(M; l^2(\pi))$ is the Betti number and $\alpha_p(M; l^2(\pi))$ is the Novikov-Shubin invariant of the density function

$$\begin{aligned} F(\lambda) = \quad & \sup_L\{dim_{\mathcal{A}}(L) : L\, is\, a\, Hilbert\, \mathcal{A}-submodule\, of\, Dom(q)\, s.t. \\ & \forall\omega \in L, q(\omega,\omega) \leq \lambda^2 \parallel \omega \parallel_1^2\}. \end{aligned}$$

In particular, $H^p(M; l^2(\pi))$ is isomorphic to

$$\begin{gathered} \{\omega \in \mathcal{H}_1^p(M; l^2(\pi)) : d\omega = \delta\omega = b^*(*\omega) = 0\} = \\ \{\omega \in C^\infty(\wedge^*(\widetilde{M})) \cap L^2(\wedge^*(\widetilde{M})) : d\omega = \delta\omega = b^*(*\omega) = 0\}. \end{gathered}$$

Lemma 6.8 *The complex (1) is $\mathcal{A}$-Fredholm at d_p.*

<u>**Proof**</u> : Suppose first that the Riemannian metric g on M is a product near ∂M. As above, $im(d_{p-1})^{\perp} \subset \mathcal{H}_1^p(M; l^2(\pi))$ is isomorphic to $(ker(\delta) \subset \mathcal{H}_1^p(DM; l^2(\pi)))^{Z_2}$, and the differential d_p on $im(d_{p-1})^{\perp} \subset \mathcal{H}_1^p(M; l^2(\pi))$ comes from the differential on $\mathcal{H}_1^p(DM; l^2(\pi))$. As the latter differential is $\mathcal{A}$-Fredholm [18], it follows that the differential d_p of complex (1) is, too. As the Hilbert spaces defined using two Riemannian metrics on M are relatively bounded, the differential d_p of the complex (1) is $\mathcal{A}$-Fredholm regardless of the Riemannian metric on M. ■

Although it is not necessary for this paper, there is a description of the analytic invariants in terms of differential forms with boundary conditions. Let us define Sobolev spaces of differential forms with absolute boundary conditions by

$$\begin{aligned} \mathcal{H}_{2,abs}^p(M; l^2(\pi)) &= \{\omega \in \mathcal{H}_2^p(M; l^2(\pi)) : b^*(*\omega) = b^*(*d\omega) = 0\}, \\ \mathcal{H}_{1,abs}^p(M; l^2(\pi)) &= \{\omega \in \mathcal{H}_1^p(M; l^2(\pi)) : b^*(*\omega) = 0\}, \\ \mathcal{H}_{0,abs}^p(M; l^2(\pi)) &= \mathcal{H}_0^p(M; l^2(\pi)). \end{aligned} \tag{6}$$

Proposition 6.9 *$H^p(M; l^2(\pi))$ is isomorphic to the homology of the sequence*

$$\mathcal{H}_{2,abs}^{p-1}(M; l^2(\pi)) \stackrel{d_{p-1}^{abs}}{\rightarrow} \mathcal{H}_{1,abs}^p(M; l^2(\pi)) \stackrel{d_p^{abs}}{\rightarrow} \mathcal{H}_{0,abs}^{p+1}(M; l^2(\pi)). \tag{7}$$

<u>**Proof**</u> : The same argument as in the proof of Proposition 6.6 gives that $im(d_{p-1}^{abs})^{\perp} = ker(\delta) \subset \mathcal{H}_{1,abs}^p(M; l^2(\pi))$. (The only difference is that now there is no boundary term in the analog of equation (3)). Then the homology of the sequence (7) is isomorphic to

$ker(d_p^{abs}) \cap im(d_{p-1}^{abs})^{\perp} = ker(d_p^{abs}) \cap ker(\delta) \subset \mathcal{H}^p_{1,abs}(M; l^2(\pi))$. But we showed above that this is isomorphic to $H^p(M; l^2(\pi))$. ∎

Let the Hilbert space $\mathcal{H}$ be the closure of $ker(\delta) \subset \mathcal{H}^p_{1,abs}(M; l^2(\pi))$ in $\mathcal{H}^p_{0,abs}(M; l^2(\pi))$. The domain $Dom(q)$ of the quadratic form q defined above is dense in $\mathcal{H}$. It follows immediately that q is closed in the sense of [34, Chapter 8]. Theorem VIII.15 of [34] implies that there is a unique self-adjoint operator A, densely defined on $\mathcal{H}$, to which q is the associated quadratic form. The next proposition identifies the domain and action of A, at least when M is isometrically a product near the boundary.

Proposition 6.10 *If M is isometrically a product near the boundary then $Dom(A) = ker(\delta) \subset \mathcal{H}^p_{2,abs}(M; l^2(\pi))$ and $A = \delta d$.*

Proof : As above, we have

$$\begin{aligned}
\left(ker(\delta) \subset \mathcal{H}^p_{2,abs}(M; l^2(\pi))\right) &\cong \left(ker(\delta) \subset \mathcal{H}^p_2(DM; l^2(\pi))^{Z_2}\right) \\
\left(ker(\delta) \subset \mathcal{H}^p_{1,abs}(M; l^2(\pi))\right) &\cong \left(ker(\delta) \subset \mathcal{H}^p_1(DM; l^2(\pi))^{Z_2}\right) \\
\mathcal{H} &\cong \left(ker(\delta) \subset \mathcal{H}^p_0(DM; l^2(\pi))^{Z_2}\right).
\end{aligned} \tag{8}$$

Here $ker(\delta) \subset \mathcal{H}^p_0(DM; l^2(\pi))^{Z_2}$ is understood to be in the sense of the Hodge decomposition. Thus it is enough to work on $D\widetilde{M}$ and only consider Z_2-invariant differential forms. In particular, as $D\widetilde{M}$ is closed there is no need to worry about boundary terms. By definition,

$$Dom(A) = \{\omega \in Dom(q) : \exists \eta \in \mathcal{H} \; s.t. \, \forall \omega' \in Dom(q), q(\omega, \omega') = <\eta, \omega'>_{\mathcal{H}}\}.$$

Then $A\omega = \eta$.

Suppose that ω lies in $Dom(A)$. Then for all smooth compactly-supported Z_2-invariant $(p+1)$-forms σ on $D\widetilde{M}$,

$$q(\omega, \delta\sigma) = <d\omega, d\delta\sigma>_0 = <d\delta d\omega, \sigma> = <\eta, \delta\sigma>_0 = <d\eta, \sigma>,$$

where $d\delta d\omega$ and $d\eta$ are taken in the distributional sense. It follows that $d(\delta d\omega - \eta) = 0$. As $\delta(\delta d\omega - \eta) = 0$, $\delta d\omega - \eta$ is harmonic. Elliptic theory then implies that it is smooth and lies in all Sobolev spaces. Taking ω' to be a arbitrary harmonic Z_2-invariant p-form on $D\widetilde{M}$, we have

$$0 = <d\omega, d\omega'>_0 = q(\omega, \omega') = <\eta, \omega'>_0 .$$

Thus η is perpendicular to such harmonic forms, and from the Hodge decomposition on $D\widetilde{M}$ we conclude that $\eta = \delta d\omega$. As $\delta\omega = 0$, it follows that ω lies in $\mathcal{H}^p_2(DM; l^2(\pi))^{Z_2}$, and so

$Dom(A) \subset \left(ker(\delta) \subset \mathcal{H}^p_{2,abs}(M;l^2(\pi))\right)$. Conversely, given $\omega \in ker(\delta) \subset \mathcal{H}^p_{2,abs}(M;l^2(\pi))$, for all $\omega' \in Dom(q)$ we have $q(\omega,\omega') = <\delta d\omega, \omega'>_0$. Thus we can take $\eta = \delta d\omega$, and so $\left(ker(\delta) \subset \mathcal{H}^p_{2,abs}(M;l^2(\pi))\right) \subset Dom(A)$, with $A\omega = \delta d\omega$. ∎

Now let M be a compact 3-manifold whose interior admits a complete finite-volume hyperbolic metric. If M is closed then we have that $b_*(M;l^2(\pi)) = 0$ [14] and the Novikov-Shubin invariants of the exterior derivative operator are computed in [24] as

$$\alpha_0(M;l^2(\pi)) = \alpha_2(M;l^2(\pi)) = \infty^+, \alpha_1(M;l^2(\pi)) = 1.$$

Suppose M is not closed. Then it has incompressible torus boundary and the interior M' of M is the union of a compact core and a finite number of hyperbolic cusps (see [40] or [31, pages 52 and 54]). Let $i : M \to M'$ be an embedding of M in M' obtained by smoothly truncating the cusps of M' and let M have the induced Riemannian metric. Let $i_1 : M_1 \to M'$ be the embedding of a submanifold (with boundary) M_1 of M' obtained by attaching a collar to M, and let $i_2 : M_2 \to M'$ be the embedding of a submanifold (with boundary) M_2 of M' obtained by attaching a collar to $M' - M$. Then $M_3 = M_1 \cap M_2$ is diffeomorphic to a disjoint union of $I \times T^2$'s (where we take $\{0\} \times T^2$ to be contained in the interior of M_1 and $\{1\} \times T^2$ to be contained in the interior of M_2) and is embedded in M' by a map $i_3 : M_3 \to M'$. Let $i_4 : M_3 \to M_1$ and $i_5 : M_3 \to M_2$ be the obvious embeddings. Put $\pi = \pi_1(M)$.

For each $p \in \{0,1,2,3\}$, define the Hilbert cochain complexes

$$\begin{aligned} C^*_{(p)} &= \mathcal{H}^*_{p+1-*}(M';l^2(\pi)) \\ D^*_{(p)} &= \mathcal{H}^*_{p+1-*}(M_1;l^2(\pi)) \oplus \mathcal{H}^*_{p+1-*}(M_2;i_2^*l^2(\pi)) \\ E^*_{(p)} &= \mathcal{H}^*_{p+1-*}(M_3;i_3^*l^2(\pi)), \end{aligned}$$

with differentials c, d and e given by exterior differentiation. (Although M' is noncompact, the Sobolev space $\mathcal{H}^*_s(M';l^2(\pi))$ can be defined as in Definition 6.4, and is in fact a Sobolev space of differential forms on H^3, the hyperbolic 3-space.)

Lemma 6.11 *There is an exact sequence of Hilbert cochain complexes*

$$0 \to C_{(p)} \xrightarrow{j} D_{(p)} \xrightarrow{k} E_{(p)} \to 0, \tag{9}$$

with $j(\omega) = i_1^*(\omega) \oplus i_2^*(\omega)$ *and* $k(\omega_1 \oplus \omega_2) = i_4^*(\omega_1) - i_5^*(\omega_2)$.

Proof : It follows from the definitions that $ker(j) = 0$, and it is easy to check that $ker(k) = im(j)$. To see that k is onto, let $\phi : I \to R$ be a bump function which is identically zero near 0 and identically one near 1. Let $\tilde{\phi} : \widetilde{M_3} \to R$ denote the composition of the pullbacks of ϕ to M_3 and then to $\widetilde{M_3}$, the preimage of M_3 in H^3. We can think of an element η of $E^*_{(p)}$ as a differential form $\tilde{\eta}$ on $\widetilde{M_3}$. Then $\tilde{\phi}\tilde{\eta}$ extends by zero to a differential form on $\widetilde{M_1}$, which comes from an element ω_1 of $\mathcal{H}^*_{p+1-*}(M_1; l^2(\pi))$. Similarly, we can extend $(\tilde{\phi}-1)\tilde{\eta}$ by zero to a differential form on $\widetilde{M_2}$, which comes from an element ω_2 of $\mathcal{H}^*_*(M_2; i_2^* l^2(\pi))$. Then $k(\omega_1 \oplus \omega_2) = \eta$. ■

Proposition 6.12 $b_p(E_{(p)}) = 0$, $\alpha_0(E_{(0)}) = \alpha_1(E_{(1)}) = 2$ *and* $\alpha_2(E_{(2)}) = \infty^+$.

Proof : As the map $Z^2 = \pi_1(M_3) \to \pi$ is an inclusion, the proof of Lemma 4.6 goes through for the analytic invariants to give that $b_p(E_{(p)}) = b_p(I \times T^2; l^2(Z^2))$ and $\alpha_p(E_{(p)}) = \alpha_p(I \times T^2; l^2(Z^2))$, where the right-hand-sides are defined by Definition 6.5. By the equivalence of the analytic and combinatorial invariants and the homotopy invariance of the combinatorial invariants (Theorem 2.5), these are the same as the invariants of T^2, which were given in Example 4.11. ■

Proposition 6.13 $b_p(C_{(p)}) = 0$, $\alpha_0(C_{(0)}) = \alpha_2(C_{(2)}) = \infty^+$ *and* $\alpha_1(C_{(1)}) = 1$.

Proof : As the universal cover of M' is isometrically H^3, this follows from the same calculation in [24] as was cited above for the case of closed hyperbolic 3-manifolds. ■

Theorem 6.14 $\alpha_1(M; l^2(\pi)) \geq 2/3$.

Proof : We apply Theorem 2.2 to the exact sequence (9) with $p = 1$. As $H_1(E_{(1)}) = 0$, $\alpha(\delta_1) = \infty^+$. From Proposition 6.13, $\alpha_1(C_{(1)}) = 1$ and from Proposition 6.12, $\alpha_1(E_{(1)}) = 2$. Then Theorem 2.2 gives $\alpha_1(D_{(1)}) \geq 2/3$. From Lemma 1.10,

$$\alpha_1(D_{(1)}) = min(\alpha_1(M_1; l^2(\pi)), \alpha_1(M_2; i_2^* l^2(\pi)),$$

from which the assertion of the theorem follows. ■

Theorem 6.15 $b_p(M; l^2(\pi)) = 0$ *for all* p.

Proof : We can exhaust $M' = int(M)$ by a sequence of compact manifolds (with boundary) $\{M_k\}$ which are all diffeomorphic to M. From [10, Theorem 1.1], $b_p(M; l^2(\pi)) = b_p(M_k; l^2(\pi))$

is the von Neumann dimension of the space of L^2 harmonic p-forms on $\widetilde{M'}$. As $\widetilde{M'}$ is H^3, there are no such forms [14]. ■

We now revert to letting the $\alpha_p(M)$-invariants refer to boundaries, as in the previous sections, as opposed to coboundaries. The translation is that $\alpha_p(M)$, defined using coboundaries, equals $\alpha_{p+1}(M)$, defined using boundaries.

Theorem 6.16 $\alpha_1(M) = \alpha_3(M) = \infty^+$.

Proof : It follows from [43, Proposition 4.1.11] that $\pi_1(M)$ is nonamenable. We derive from Lemma 4.5.2 that $\alpha_1(M) = \infty^+$. As M has nonempty boundary, Lemma 4.5.5 gives that $\alpha_3(M) = \infty^+$. ■

In summary, we have shown

Theorem 6.17 *If M is a compact 3-manifold whose interior admits a complete finite-volume hyperbolic structure then M has vanishing L^2-cohomology and $\alpha_1(M) = \alpha_3(M) = \infty^+$. If M is closed then $\alpha_2(M) = 1$ and if M is not closed then $\alpha_2(M) \geq 2/3$.* ■

It will follow from Theorem 7.8 that if M is not closed then $\alpha_2(M) \leq 2$.

7. L^2-Betti numbers and Novikov-Shubin invariants for 3-manifolds

In this section we analyse the L^2-Betti numbers and Novikov-Shubin invariants of compact connected orientable 3-manifolds. It is easy to extend the results to the nonorientable case by means of the orientation covering.

We recall some basic facts about (compact connected orientable) 3-manifolds [20, 36]. A 3-manifold M is *prime* if for any decomposition of M as a connected sum $M_1 \sharp M_2$, M_1 or M_2 is homeomorphic to S^3. It is *irreducible* if every embedded 2-sphere bounds an embedded 3-disk. Any prime 3-manifold is irreducible or is homeomorphic to $S^1 \times S^2$ [20, Lemma 3.13]. One can write M as a connected sum

$$M = M_1 \sharp M_2 \sharp \ldots M_r$$

where each M_j is prime, and this prime decomposition is unique up to renumbering [20, Theorems 3.15, 3.21]. By the sphere theorem [20, Theorem 4.3], an irreducible 3-manifold is a $K(\pi, 1)$ Eilenberg-MacLane space if and only if it is a 3-disk or has infinite fundamental group.

A properly-embedded orientable connected surface in a 3-manifold is *incompressible* if it is not a 2-sphere and the inclusion induces a injection on the fundamental groups. One says that ∂M is *incompressible in M* if and only if ∂M is empty or any component C of ∂M is incompressible in the sense above. An irreducible 3-manifold is *Haken* if it contains an embedded orientable incompressible surface. If M is irreducible and in addition $H_1(M)$ is infinite, which is implied if ∂M contains a surface other than S^2, then M is Haken [20, Lemma 6.6 and 6.7]. (With our definitions, any properly embedded 2-disk is incompressible, and the 3-disk is Haken.)

Before we state the main theorem of this section, we must mention what is known about Thurston's geometrization conjecture for irreducible 3-manifolds with infinite fundamental groups. (Again, our 3-manifolds are understood to be compact, connected and orientable.) Johannson [22] and Jaco and Shalen [21] have shown that given an irreducible 3-manifold M with incompressible boundary, there is a finite family of disjoint, pairwise-nonisotopic incompressible tori in M which are not isotopic to boundary components and which split M into pieces that are Seifert manifolds or are geometrically atoroidal, meaning that they admit no embedded incompressible torus (except possibly parallel to the boundary). A minimal family of such tori is unique up to isotopy, and we will say that it gives a *toral splitting* of M. We will say that the toral splitting is a *geometric toral splitting* if the geometrically atoroidal pieces which do not admit a Seifert structure have complete hyperbolic metrics on their interiors. Thurston's geometrization conjecture for irreducible 3-manifolds with infinite fundamental groups states that such manifolds have geometric toral splittings.

Suppose that M is Haken. The pieces in its toral splitting are certainly Haken. Let N be a geometrically atoroidal piece. The torus theorem says that N is a special Seifert manifold or is homotopically atoroidal i.e. any subgroup of $\pi_1(N)$ which is isomorphic to $Z \times Z$ is conjugate into the fundamental group of a boundary component. Thurston has shown that a homotopically atoroidal Haken manifold is a twisted I-bundle over the Klein bottle (which is Seifert), or admits a complete hyperbolic metric on its interior.

Thus the case in which Thurston's geometrization conjecture for an irreducible 3-manifold M with infinite fundamental group is still open is when M is a closed non-Haken irreducible 3-manifold with infinite fundamental group which is not Seifert. The conjecture states that such a manifold is hyperbolic.

Our goal is to make general statements about the L^2-Betti numbers and Novikov-Shubin invariants of a 3-manifold. We have already treated the case when the fundamental group is finite in Example 4.12. We will confine ourselves in the sequel to the case when $\pi_1(M)$ is infinite. We will compute the invariants using the putative geometric decomposition of M. As we are studying homotopy invariants which have a simple behaviour with respect to finite coverings, it is enough to assume a weaker condition than that M have a geometric decomposition. Recall from the introduction that we say that a prime 3-manifold is *exceptional* if it is closed and no finite cover of it is homotopy-equivalent to a Haken, Seifert or hyperbolic 3-manifold.

Theorem 7.1 *Let M be the connected sum $M_1 \sharp \ldots \sharp M_r$ of (compact connected orientable) nonexceptional prime 3-manifolds M_j. Assume that $\pi_1(M)$ is infinite. Then*

1. *The L^2-Betti numbers of M are given by:*

$$\begin{aligned} b_0(M) &= 0 \\ b_1(M) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} - \chi(M) + \mid \{C \in \pi_0(\partial M) \, s.t. \, C \cong S^2\} \mid \\ b_2(M) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} + \mid \{C \in \pi_0(\partial M) \, s.t. \, C \cong S^2\} \mid \\ b_3(M) &= 0. \end{aligned}$$

2. *Let the Poincaré associate $P(M)$ be the connected sum of the M_j's which are not 3-disks or homotopy 3-spheres. Then $\alpha_p(P(M)) = \alpha_p(M)$ for $p \le 2$. We have $\alpha_1(M) = \infty^+$ except for the following cases:*

   (a) *$\alpha_1(M) = 1$ if $P(M)$ is $S^1 \times D^2$, a closed $S^2 \times R$-manifold or homotopy equivalent to $RP^3 \sharp RP^3$.*

(b) $\alpha_1(M) = 2$ *if* $P(M)$ *is* $T^2 \times I$ *or a twisted* I*-bundle over the Klein bottle* K.

(c) $\alpha_1(M) = 3$ *if* $P(M)$ *is a closed* R^3*-manifold.*

(d) $\alpha_1(M) = 4$ *if* $P(M)$ *is a closed Nil-manifold.*

(e) $\alpha_1(M) = \infty$ *if* $P(M)$ *is a closed Sol-manifold.*

3. $\alpha_2(M) > 0$.

We will prove Theorem 7.1 by a succession of lemmas. In order to prove the statement about $\alpha_1(M)$, we will show that if $\alpha_1(M) < \infty^+$ then M is one of the special cases listed in the statement of the theorem. The values of $\alpha_1(M)$ in these special cases follow from previous calculations.

Lemma 7.2 *If M is an irreducible Haken manifold with incompressible torus boundary then M has vanishing L^2-cohomology and $\alpha_2(M) > 0$. If $\alpha_1(M) < \infty^+$ then M is one of the special cases listed in Theorem 7.1.2.*

Proof : We know that M has a geometric toral splitting. As a compact connected orientable 3-manifold with torus boundary whose interior has a complete hyperbolic metric is either $T^2 \times I$ or has a complete finite-volume hyperbolic metric [31, p. 52], the pieces in the toral splitting either admit a Seifert structure or have a complete finite-volume hyperbolic metric on their interior. Let s be the number of tori in such a minimal splitting. We will use induction over s. To begin the induction, if $s = 0$ then M is Seifert or hyperbolic and the claim follows from Theorems 5.1, 5.4 and 6.17. The induction step from $s - 1$ to s is done as follows:

Let T^2 be a torus in a minimal family of splitting tori. Depending on whether T^2 is separating or not, we get decompositions $M = M_1 \cup_{T^2} M_2$ or $M = M_1 \cup_{T^2 \times \partial I} T^2 \times I$ by cutting M open along T^2. We have the short exact sequences

$$0 \longrightarrow C(T^2) \longrightarrow C(M_1) \oplus C(M_2) \longrightarrow C(M) \longrightarrow 0$$

or

$$0 \longrightarrow C(T^2 \times \partial I) \longrightarrow C(M_1) \oplus C(T^2 \times I) \longrightarrow C(M) \longrightarrow 0$$

with coefficients in $l^2(\pi_1(M))$. Note that each M_j satisfies the induction hypothesis. Hence $b_p(M_j) = 0$ for all p and $\alpha_2(M_j) > 0$. From Lemma 4.6 and Example 4.11 we have that $b_p(T^2) = 0$ for all p and $\alpha_p(T^2) = 2$ for $p \in \{1,2\}$. The weakly exact Mayer-Vietoris sequence gives that M has vanishing L^2-cohomology, and Theorem 2.2.2 and Lemma 2.3.3 give the inequalities

$$\frac{1}{\alpha_2(M)} \leq \frac{1}{\alpha_1(T^2)} + \frac{1}{\min\{\alpha_2(M_1), \alpha_2(M_2)\}} \text{ or}$$
$$\frac{1}{\alpha_2(M)} \leq \frac{1}{\alpha_1(T^2 \times \partial I)} + \frac{1}{\min\{\alpha_1(M_1), \alpha_2(T^2 \times I)\}}.$$

Thus $\alpha_2(M) > 0$.

We also have the exact sequences

$$0 \longrightarrow C(M_1) \longrightarrow C(M) \longrightarrow C(M_2, T^2) \longrightarrow 0$$

or

$$0 \longrightarrow C(M_1) \longrightarrow C(M) \longrightarrow C(T^2 \times I, T^2 \times \partial I) \longrightarrow 0$$

with $l^2(\pi_1(M))$ as coefficients. As M_1 has vanishing L^2-cohomology, Theorem 2.2.1 gives that

$$\frac{1}{\alpha_1(M)} \leq \frac{1}{\alpha_1(M_1)} + \frac{1}{\alpha_1(M_2, T^2)}$$

or

$$\frac{1}{\alpha_1(M)} \leq \frac{1}{\alpha_1(M_1)} + \frac{1}{\alpha_1(T^2 \times I, T^2 \times \partial I)}.$$

From Lemma 4.5 we have that $\alpha_1(M_2, T^2) = \alpha_1(T^2 \times I, T^2 \times \partial I) = \infty^+$. This implies in both cases that $\alpha_1(M_1) \leq \alpha_1(M)$. Hence $\alpha_1(M_1) < \infty^+$, and by symmetry $\alpha_1(M_2) < \infty^+$ in the first case. By the induction hypothesis, M_j must be $T^2 \times I$ or a twisted I-bundle over K. Thus M is either the gluing of two twisted I-bundles over K along their boundaries, or a T^2-bundle over S^1. If M is the gluing of two twisted I-bundles over K over their boundaries then M is double-covered by a T^2-bundle over S^1. In either case, Lemma 7.3 will give that M has the geometric type of some T^2-bundle over S^1. (For later purposes, Lemma 7.3 is stated in greater generality than is needed here.) Then [36, Theorem 5.5] implies that M has a Sol, Nil or R^3-structure, and is one of the special cases listed. ■

Lemma 7.3 *Let $\overline{M}$ be a finite cover of an irreducible closed oriented 3-manifold M with infinite fundamental group. If $\overline{M}$ is homotopy-equivalent to a closed 3-manifold N with a Seifert or Sol-structure then M has the same geometric type as N.*

Proof : From [29, Theorem 3] we have that $\overline{M}$ is irreducible. If N has a Seifert structure then [35, pages 35 and 36] gives that $\overline{M}$ is homeomorphic to N and that M is also a Seifert manifold of the same geometric type. If N has a Sol-structure then $\overline{M}$ and N are Haken, and so $\overline{M}$ is homeomorphic to N [20, Theorem 13.6]. It follows from [36, Theorem 5.3] that M has a Sol-structure. ■

Lemma 7.4 *If M is an irreducible Haken manifold with incompressible boundary then $b_p(M) = 0$ for $p \neq 1$, $b_1(M) = -\chi(M)$ and $\alpha_2(M) > 0$. If $\alpha_1(M) < \infty^+$ then M is one of the special cases listed in Theorem 7.1.2.*

Proof : Because of Lemma 7.2, we may assume that ∂M is nonempty. Let N be $M \cup_{\partial M} M$. Then [42, Satz 1.8] implies that N is irreducible. Clearly N is a closed Haken manifold. From Lemma 7.2 we have that N has vanishing L^2-cohomology and $\alpha_2(N) > 0$. We have the exact sequence

$$0 \longrightarrow C(\partial M) \longrightarrow C(M) \oplus C(M) \longrightarrow C(N) \longrightarrow 0$$

with coefficients in $l^2(\pi_1(N))$. From Example 4.11 we have that $b_p(\partial M) = 0$ for $p \neq 1$ and $\alpha_p(\partial M) > 0$ for all p. Then we get from the weakly exact Mayer-Vietoris sequence that $b_p(M) = 0$ for $p \neq 1$. From the Euler characteristic formula we derive that $b_1(M) = -\chi(M)$. Theorem 2.2.1 and Lemma 2.3.3 imply that

$$\frac{1}{\alpha_2(M)} \leq \frac{1}{\alpha_2(\partial M)} + \frac{1}{\alpha_2(N)}$$

and hence $\alpha_2(M) > 0$.

Next we prove the claim for $\alpha_1(M)$. Suppose that M does not have a toral boundary. Then ∂M contains a component F_g for $g \geq 2$. As $\pi_1(F_g)$ is nonamenable and is a subgroup of $\pi_1(M)$, $\pi_1(M)$ is nonamenable and Lemma 4.5.2 implies that $\alpha_1(M) = \infty^+$. Hence the claim follows already from Lemma 7.2. ■

Lemma 7.5 *If M is an irreducible Haken manifold and is not a 3-disk, then $b_p(M) = 0$ for $p \neq 1$, $b_1(M) = -\chi(M)$ and $\alpha_2(M) > 0$. If $\alpha_1(M) < \infty^+$ then M is one of the special cases listed in Theorem 7.1.2.*

Proof : Because of Lemma 7.4, we may assume that ∂M is compressible. The loop theorem [20, Theorem 4.2] gives an embedded disk D^2 in M such that D^2 meets ∂M transversally, and $\partial D^2 = D^2 \cap \partial M$ is an essential curve on ∂M. Depending on whether the disk D^2 is separating or not, we get the following two cases:

If D^2 is separating then there are 3-manifolds M_1 and M_2 and embedded disks $D^2 \subset \partial M_1$ and $D^2 \subset \partial M_2$ such that $M = M_1 \cup_{D^2} M_2$. In particular, M is homotopy equivalent to $M_1 \vee M_2$. Since M is prime, M_1 and M_2 are prime. As M_1 and M_2 have nonempty boundary, they are not $S^1 \times S^2$, and so are irreducible. As M is irreducible with infinite fundamental group, it is a $K(\pi, 1)$ Eilenberg-Maclane space. Then the same must be true for M_1 and M_2. If M_i were a 3-disk then the boundary of the embedded 2-disk would not be an essential curve on ∂M. Thus M_1 and M_2 have infinite fundamental groups.

If D^2 is nonseparating then there is a 3-manifold M_1 with embedded $S^0 \times D^2 \subset \partial M_1$ such that $M = M_1 \cup_{S^0 \times D^2} D^1 \times D^2$. The same argument as above shows that M_1 is an irreducible 3-manifold which is a 3-disk or has infinite fundamental group. If it were a 3-disk then M would be $S^1 \times D^2$, which satisfies the claim of the Lemma. So we may assume that M_1 has infinite fundamental group.

We will prove the Lemma using the fact that M is homotopy equivalent to $M_1 \vee M_2$ (respectively $M_1 \vee S^1$). It suffices to verify the claim for M_1 and M_2 (respectively M_1), since the claim for M then follows from the proof of Proposition 4.7. If M_1 and M_2 (respectively M_1) have incompressible boundary then we are done by Lemma 7.4. Otherwise, we repeat the process of cutting along 2-disks described above. This process must stop after finitely many steps. ■

Proof of Theorem 7.1: We have the prime decomposition

$$M = M_1 \sharp M_2 \sharp \ldots M_r.$$

By assumption, each M_j in the decomposition is nonexceptional. We claim first that if $\pi_1(M_j)$ is finite then $b_1(M_j) = 0$, if $\pi_1(M_j)$ is infinite then $b_1(M_j) = -\chi(M_j)$, and that $\alpha_2(M_j) > 0$. The case of finite fundamental group follows from Example 4.12. From Theorem 2.5 and Remark 4.10 we may assume that if M_j is closed then M_j is Seifert, hyperbolic or Haken. If M_j is closed and Seifert then the result follows from Theorem 5.1. If M_j is closed and hyperbolic then the result follows from Theorem 6.17. If M_j is closed and Haken then the result follows from Lemma 7.2. If M_j has a boundary component which is a 2-sphere then M_j is a 3-disk and the result follows from Example 4.12. If M_j has a nonempty boundary with no 2-spheres then it is Haken and the result follows from Lemma 7.5.

From Lemma 4.5 we have that $b_0(M) = b_3(M) = 0$. From Proposition 4.7.1 we have that

$$b_1(M) = r - 1 + \sum_{j=1}^{r} \left(b_1(M_j) - \frac{1}{|\pi_1(M_j)|} \right).$$

As we have shown that $b_1(M_j) = -\chi(M_j) + \{1 \text{ if } M_j \cong D^3\}$, the claim of Theorem 7.1 for $b_1(M)$ follows. The claim for $b_2(M)$ now follows from the Euler characteristic equation. From Proposition 4.7.3 we have

$$\alpha_2(M) = \min\{\alpha_2(M_j) : j = 1, \ldots r\} > 0.$$

From Corollary 4.4.1 we have that $\alpha_1(M) = \alpha_1(P(M))$. Thus, by removing the simply-connected factors, we may assume that $M = P(M)$. Suppose that $\alpha_1(M) < \infty^+$. From Proposition 4.7, we have the possibilities that $r = 1$, or that $r = 2$ and $\pi_1(M_1) = \pi_1(M_2) = Z/2$. If $r = 1$ then $M \cong S^1 \times S^2$ and is one of the special cases listed, or M is irreducible. If M is not closed then it is Haken and Lemma 7.5 implies that it is one of the special cases listed. If M is closed then by assumption a finite cover $\overline{M}$ of M is homotopy equivalent to a Seifert, hyperbolic or Haken manifold N, which must also be closed and orientable. If N is Seifert or hyperbolic then Theorems 5.1, 5.4 and 6.17 imply that N is a closed $S^2 \times R$, R^3, or Nil manifold. If N is Haken then Lemma 7.5 implies that N is a closed $S^2 \times R$, R^3, Nil or Sol manifold. Lemma 7.3 gives that M is of the same geometric type as N, and so is one of the special cases listed.

If $r = 2$, it remains to show that a 3-manifold M with $\pi_1(M) = Z/2$ is homotopy equivalent to RP^3. This follows from [39, Theorem 1.8]. ■

Corollary 7.6 *If M satisfies the hypotheses of Theorem 7.1 and $\chi(\pi_1(M))$ denotes the rational-valued group Euler characteristic [6, Section IX.7] then $b_1(M) = -\chi(\pi_1(M))$ and $b_2(M) = \chi(M) - \chi(\pi_1(M))$.*

Proof : First, for the group Euler characteristic to be defined we must show that $\pi_1(M)$ is virtually torsion-free and of finite homological type. Let $\{M_j\}_{j=1}^{s}$ be the prime factors of M with finite fundamental group. Put $\Gamma_1 = \pi_1(M_1) * \ldots * \pi_1(M_s)$ and $\Gamma_2 = \pi_1(M_{s+1}) * \ldots * \pi_1(M_r)$. It is known that Γ_1 has a finite-index free subgroup F and that Γ_2 is torsion-free. Let $\phi : \Gamma_1 * \Gamma_2 \to \Gamma_1$ be the natural homomorphism. Then $\phi^{-1}(F)$ is finite-index in $\pi_1(M)$, and the Kurosh subgroup theorem [20, Theorem 8.3] implies that it is torsion-free. As Γ_1 and Γ_2 have finite homological type, [6, Proposition IX.7.3.e] implies that $\pi_1(M)$ is of finite homological type and that:

$$\chi(\pi_1(M)) = r - 1 + \sum_{j=1}^{r} \chi(\pi_1(M_j)).$$

Thus in order to show that $b_1(M) = -\chi(\pi_1(M))$, it is enough to verify that for each j,

$$-\frac{1}{|\pi_1(M_j)|} - \chi(M_j) + \{1 \text{ if } M_j \cong D^3\} = -\chi(\pi_1(M_j)).$$

As M_j is either a $K(\pi, 1)$ Eilenberg-Maclane space, a 3-disk or a closed manifold with finite fundamental group, the equation is easy to verify.

The statement for $b_2(M)$ now follows from the Euler characteristic equation. ■

Corollary 7.7 *Let M be a (compact connected orientable) 3-manifold. If all L^2-Betti numbers of M vanish then M satisfies one of the following conditions:*

1. *M is homotopy equivalent to an irreducible 3-manifold N with infinite fundamental group whose boundary is empty or a disjoint union of tori.*
2. *M is homotopy equivalent to $S^1 \times S^2$ or $RP^3 \sharp RP^3$.*

If condition 2.) holds, or if condition 1.) holds and N is nonexceptional, then all of the L^2-Betti numbers of M vanish. ■

Proof : Suppose that M has vanishing L^2-cohomology. From Example 4.12, $\pi_1(M)$ must be infinite. From Proposition 4.7.1 we have that

$$r - 1 + \sum_{j=1}^{r} \left(b_1(M_j) - \frac{1}{|\pi(M_j)|} \right) = 0.$$

Equivalently,

$$\sum_{j=1}^{r} \left(b_1(M_j) - \frac{1}{|\pi(M_j)|} + 1 \right) = 1.$$

It follows that the prime decomposition of M must consist of homotopy 3-spheres, 3-disks and either

A. A prime manifold M' with infinite fundamental group and vanishing b_1 or

B. Two prime manifolds M^1 and M^2 with fundamental group $Z/2$.

In case A, M' is $S^1 \times S^2$ or is irreducible. If M' is irreducible and has nonempty boundary then Lemma 7.5 implies that its boundary components must be tori. From the Euler characteristic equation we have that $\chi(M) = 0$, and so no 3-disks can occur in the prime decomposition of M. In case B, we have already shown that M^1 and M^2 are homotopy-equivalent to RP^3. Again, because $\chi(M) = 0$, no 3-disks can occur in the prime decomposition of M. Thus we have shown that if M has vanishing L^2-cohomology then M satisfies one of the two conditions of the corollary.

If M satisfies condition 2. of the corollary then Theorems 2.5 and 5.1 imply that M has vanishing L^2-cohomology. If M satisfies condition 1. of the corollary, from Theorem 2.5 we may assume without loss of generality that $M = N$. We have that its Euler characteristic vanishes. If M has nonempty boundary then Lemma 7.5 implies that it has vanishing L^2-cohomology. If M is closed and nonexceptional then by passing to a finite cover and using Theorem 2.5, we may assume that M is Seifert, hyperbolic or Haken. Theorems 5.1, 5.4 and 6.17 imply that M has vanishing L^2-cohomology. ■

Theorem 7.8 *If ∂M contains an incompressible torus then $\alpha_2(M) \leq 2$. If one of the M_j's is closed and nonexceptional with infinite fundamental group, and does not admit an R^3, $S^2 \times R$ or Sol-structure, then $\alpha_2(M) \leq 2$.*

Again, we will build up to the theorem by lemmas.

Lemma 7.9 *If M is irreducible and ∂M contains an incompressible torus then $\alpha_2(M) \leq 2$.*

Proof : From Lemma 7.4 we get $b_2(M) = 0$. As T^2 has vanishing L^2-cohomology, the long weakly exact homology sequence of the pair (M, T^2) implies that $H_2(M, T^2; l^2(\pi_1(M)))$ vanishes. We have a short exact sequence of chain complexes

$$0 \longrightarrow C(T^2) \longrightarrow C(M) \longrightarrow C(M, T^2) \longrightarrow 0$$

and so from Theorem 2.2.3,

$$\frac{1}{\alpha_2(T^2)} \leq \frac{1}{\alpha_2(M)} + \frac{1}{\alpha_3(M, T^2)}.$$

Proposition 4.2 implies that $\alpha_3(M, T^2) = \alpha_1(M, \partial M - T^2)$. If this is ∞^+ then $\alpha_2(M) \leq \alpha_2(T^2) = 2$ and we are done. If $\partial M - T^2 \neq \emptyset$ then Lemma 4.5.5 implies that $\alpha_1(M, \partial M - T^2) = \infty^+$. If $\partial M - T^2 = \emptyset$ then Theorem 7.1 gives the possible cases in which $\alpha_1(M, \partial M - T^2) < \infty^+$. The only case in which ∂M is a single incompressible torus is when M is a twisted I-bundle over K, and in this case Theorem 5.4 gives that $\alpha_2(M) = 2$. ■

Lemma 7.10 *If M is a closed Haken manifold and does not admit an R^3 or Sol structure then $\alpha_2(M) \leq 2$.*

Proof : If M is Seifert or hyperbolic then the proposition follows from Theorems 5.1 and 6.17. Otherwise, consider the nonempty minimal family of splitting tori. Let T^2 be a member of the minimal family. Cutting M open along T^2 yields decompositions $M = M_1 \cup_{T^2} M_2$ or $M = M_1 \cup_{T^2 \times \partial T^2} T^2 \times I$, depending on whether T^2 is separating or not. We get the exact sequences

$$0 \longrightarrow \quad C(M_1) \longrightarrow \quad C(M) \longrightarrow \quad C(M_2, T^2) \longrightarrow 0$$

or

$$0 \longrightarrow \quad C(M_1) \longrightarrow \quad C(M) \longrightarrow \quad C(T^2 \times I, T^2 \times \partial I) \longrightarrow 0$$

with coefficients in $l^2(\pi_1(M))$. Since $b_1(M) = 0$ (Lemma 7.2), we derive from Theorem 2.2.2 that

$$\frac{1}{\alpha_2(M_2, T^2)} \leq \frac{1}{\alpha_1(M_1)} + \frac{1}{\alpha_2(M)}$$

or

$$\frac{1}{\alpha_2(T^2 \times I, T^2 \times \partial I)} \leq \frac{1}{\alpha_1(M_1)} + \frac{1}{\alpha_2(M)}$$

Suppose that $\alpha_1(M_1) \geq \infty$. Then we have that $\alpha_2(M) \leq \alpha_2(M_2, T^2)$ (respectively $\alpha_2(M) \leq \alpha_2(T^2 \times I, T^2 \times \partial I) = 2$). Proposition 4.2 gives that $\alpha_2(M_2, T^2) = \alpha_2(M_2)$, and we have already proven that this is less than or equal to two. By symmetry, it remains to treat the case when $\alpha_1(M_1), \alpha_1(M_2) < \infty$, (respectively $\alpha_1(M_1) < \infty$). From Theorem 7.1, M_1 and M_2 must be I-bundles over K (respectively M_1 must be $I \times T^2$). As before, in either case M carries a *Sol*, *Nil* or R^3-structure. Since $\alpha_2(M) = 2$ in the *Nil* case (Theorem 5.1), the lemma follows. ■

Proof of Theorem 7.8: From Proposition 4.7.3 we have that

$$\alpha_2(M) = \min\{\alpha_2(M_j) : j = 1, \ldots r\}.$$

Clearly, it is enough to verify the theorem under the assumption that M is prime. As $S^1 \times S^2$ has an $S^2 \times R$-structure, we may assume that M is irreducible. If ∂M contains an incompressible torus then we are done by Lemma 7.9. Suppose that M is closed, has infinite fundamental group and is nonexceptional. Then a finite cover $\overline{M}$, which is closed, orientable and irreducible, is homotopy equivalent to a manifold N which is Seifert, hyperbolic or Haken. If $\alpha_2(M) > 2$ then Theorems 5.1 and 6.17 and Lemma 7.10 imply that N has an R^3, $S^2 \times R$ or Sol structure. By Lemma 7.3, M also has such a structure. ■

8. L^2-Contractibility

Let Γ be a finitely-presented discrete group. Let $\overline{M}$ be a normal Γ-covering of a compact Riemannian manifold M, possibly with boundary. Give $\overline{M}$ the induced Riemannian metric. Let $\overline{\Delta}$ denote the self-adjoint extension of the Laplacian acting on all compactly-supported smooth forms on $\overline{M}$ which satisfy absolute boundary conditions. We will say that $\overline{M}$ is L^2-contractible if $\overline{\Delta}$ has a bounded L^2-inverse. (In [17] this is called L^2-acyclicity, but we think that our terminology may be less confusing.) By Lemma 2.4, this is equivalent to requiring that $b_p(M; l^2(\Gamma)) = 0$ and $\alpha_p(M; l^2(\Gamma)) = \infty^+$ for all $0 \leq p \leq dim(M)$. It is an open question as to whether L^2-contractible manifolds exist. There are some sufficient conditions to rule out L^2-contractibility. For example, it follows easily from higher index theory that if M is closed, Γ satisfies the Strong Novikov Conjecture [23] and the image of the fundamental class $[M]$ under the classifying map $M \rightarrow B\Gamma$ is nonzero in $H_m(B\Gamma; Q)$ then $\overline{M}$ is not L^2-contractible. (In fact, the Laplacian is noninvertible in dimension $\frac{m}{2}$ if m is even and in dimensions $\frac{m\pm1}{2}$ if m is odd.)

One can similarly consider the question of L^2-contractibility for any CW-complex $\overline{K}$ which is a Γ-covering of a finite CW-complex K. As a small step toward answering these questions, we have the following result:

Proposition 8.1 *Let K be a finite CW-complex whose fundamental group is isomorphic to the fundamental group of a 3-manifold N satisfying the hypothesis of Theorem 7.1. Then the universal cover $\widetilde{K}$ is not L^2-contractible.*

Proof : Suppose that $\widetilde{K}$ is L^2-contractible. By passing to the Poincaré associate of N, we may assume that ∂N contains no 2-spheres. From Theorem 2.5 and Corollary 4.4 we conclude that $b_p(N) = 0$ for $p \leq 1$ and $\alpha_p(N) = \infty^+$ for $p \leq 2$. From Lemma 4.5 we have that $b_3(N) = 0$. As $\chi(\partial N) = 2 \cdot \chi(N)$, we have that $\chi(N)$ is less than or equal to zero. But in this case $\chi(N) = b_2(N)$, so we conclude that $b_p(N) = 0$ for all p. From Corollary 7.7 we may assume that N is an irreducible 3-manifold with infinite fundamental group whose boundary is empty or a disjoint union of incompressible tori, or that N is $S^1 \times S^2$ or $RP^3 \sharp RP^3$. In the first case, Theorem 7.8 and the fact that $\alpha_2(N)$ is ∞^+ imply that N could only be a closed manifold with a *Sol* structure. However, this would then imply that $\alpha_1(N) < \infty^+$ (Theorem 7.1). In the second case we have that $\alpha_1(N) = 1$. In either case we get a contradiction. ■

One can extend the notion of L^2-contractibility (i.e. L^2-invertibility of the differential-form Laplacian) from covering spaces of closed manifolds to general complete Riemannian manifolds. (One may want to consider a condition of bounded geometry). Similarly, one can ask the question of L^2-contractibility for general simplicial complexes, possibly with a uniform local finiteness condition [17]. We do not know of any L^2-contractible complete Riemannian manifold. It follows from equivariant index theory [23] that if G is a connected Lie group and K is a maximal compact subgroup then G/K, with a left-G-invariant metric, is not L^2-contractible. The case of surfaces is considered in the next proposition.

Proposition 8.2 *A complete orientable surface is not L^2-contractible.*

Proof : (The following proof, which is simpler than our original proof, is due to J. Dodziuk.) We use facts from [1, 38]. Suppose that the surface M is L^2-contractible. The Riemannian metric gives a complex structure on M. The condition of having a nonzero L^2 harmonic 1-form is conformally invariant for surfaces, and so only depends on the complex structure. It is known that nonzero L^2 harmonic 1-forms exist on nonplanar surfaces, and so M must be planar. It is also known that a planar surface has nonzero L^2 harmonic 1-forms if and only if it is nonparabolic, so M must be parabolic. As the Laplacian $\triangle_0$ acting on functions is invertible, the infimum λ_0 of its spectrum is strictly positive. If $0 < \lambda < \lambda_0$ then there is a positive superharmonic (non-L^2) eigenfunction of $\triangle_0$ with eigenvalue λ. This contradicts the definition of parabolicity. ■

For further discussion of some of the topics of this section, see [17, Section 8].

9. Remarks and Conjectures

Conjecture 9.1 *Let M be a compact connected manifold, possibly with boundary. Then*

1. *The L^2-Betti numbers of M are rational. If $\pi_1(M)$ is torsion-free then the L^2-Betti numbers of M are integers.*
2. *The Novikov-Shubin invariants of M are positive and rational.* ∎

In the case of the L^2-Betti numbers, this seems to be a well-known conjecture. The question of the rationality of the L^2-Betti numbers, for closed manifolds, appears in [2]. Theorem 7.1 shows that Conjecture 9.1.1 is true for the class of 3-manifolds considered there. By Lemma 4.5.1, Conjecture 9.1.2 is trivially true for $\alpha_1(M)$. Theorems 5.1 and 5.4 give that it is true for $\alpha_2(M)$ if M is a Seifert 3-manifold. Note that for any positive integer k there are examples of closed manifolds in higher dimensions with $\pi_1(M) = Z$ such that $\alpha_3(M) = \frac{1}{k}$ [24].

We claim that Conjecture 9.1 is equivalent to the following conjecture:

Conjecture 9.2 *Let π be a finitely presented group and let $f : \oplus_{i=1}^r Z\pi \longrightarrow \oplus_{i=1}^r Z\pi$ be a $Z\pi$-module homomorphism. Tensor by $l^2(\pi)$ to get a bounded π-equivariant operator*

$$\overline{f} : \oplus_{i=1}^r l^2(\pi) \longrightarrow \oplus_{i=1}^r l^2(\pi).$$

Then

1. *The von Neumann dimension of $ker(\overline{f})$ is rational. If π is torsion-free then it is an integer.*
2. *The Novikov-Shubin invariant of $\overline{f}$ is a positive rational number.* ∎

If Conjecture 9.2 is true then upon triangulating a compact manifold, we obtain that Conjecture 9.1 is true. It remains to show that Conjecture 9.1 implies Conjecture 9.2. Let X be a finite CW-complex with fundamental group π. Let $f : \oplus_{i=1}^r Z\pi \longrightarrow \oplus_{i=1}^r Z\pi$ be any $Z\pi$-module homomorphism. For any given $n \geq 2$, one can attach cells to X in dimensions n and $n+1$ in such a way that the resulting finite CW-complex Y has the same fundamental group as X, and the relative chain complex $C(\widetilde{Y}, \widetilde{X})$ is concentrated in dimensions n and $n+1$ and given there by f [25, Theorem 2.1]. If we choose $n > dim(X)$

then $\alpha_{n+1}(Y) = \alpha(f)$. Moreover, there is a compact manifold, possibly with boundary, which is homotopy equivalent to Y. Since the L^2-Betti numbers and the Novikov-Shubin invariants are homotopy invariants, we obtain that $\alpha_{n+1}(M) = \alpha(f)$. This shows that Conjecture 9.1 is equivalent to Conjecture 9.2.

Conjecture 9.2 in this form implies a well-known conjecture of algebra.

Conjecture 9.3 *Let π be a finitely-presented group. Then the group ring $Q\pi$ has no zero-divisors if and only if π is torsion-free.* ∎

The only-if statement is trivial. The if statement would follow from the second conjecture as follows: Let $u \in Q\pi$ be a zero-divisor. We want to show that $u = 0$. We may assume that u lies in $Z\pi$. Let $f : Z\pi \longrightarrow Z\pi$ be given by right multiplication with u. Since u is a zero-divisor, $\overline{f}$ has a non-trivial kernel. Since the dimension of the kernel of $\overline{f}$ must be a positive number less or equal to the dimension of $l^2(\pi)$, which is 1, it can only be an integer if it is 1. Hence the kernel of $\overline{f}$ is $l^2(\pi)$. This implies that $u = 0$.

Conjecture 9.4 *The second L^2-Betti number of a compact prime 3-manifold vanishes.*

We have shown in Theorem 7.1 that the second L^2-Betti number of a nonexceptional compact prime 3-manifold vanishes. However, there may be a reason why it should vanish which is independent of any geometric decomposition theorem.

Conjecture 9.5 *If M is a closed $K(\pi, 1)$ manifold then its L^2-Betti numbers vanish outside of the middle dimension.*

Corollary 7.7 implies that a closed $K(\pi, 1)$ 3-manifold of the type considered there has vanishing L^2-Betti numbers, thereby verifying the conjecture. Conjecture 9.5 includes the unproven conjecture of Singer which states the same for nonpositively-curved manifolds. If $\pi_1(M)$ contains an infinite normal amenable subgroup then the truth of the conjecture follows immediately from [11, Theorem 0.2]. Conjecture 9.5 was emphasized in the case of 4-manifolds in [17, p. 154]. A consequence would be that if $dim(M) = 4k+2$ then $\chi(M) \leq 0$, and if $dim(M) = 4k$ then $\chi(M) \geq | \sigma(M) |$.

Conjecture 9.6 *Let Γ be a finitely-presented group. Let $b_*(\Gamma)$ and $\alpha_*(\Gamma)$ denote the L^2-Betti numbers and Novikov-Shubin invariants of a $K(\Gamma, 1)$ complex. Suppose that Γ is non-amenable, $b_1(\Gamma) = 0$ and $\alpha_2(\Gamma) = \infty^+$. Then any closed 4-manifold M with fundamental group Γ satisfies $\chi(M) > 0$.*

This conjecture would be a consequence of non-L^2-contractibility of M, as the hypotheses imply that $b_p(M) = 0$ for all $p \neq 2$, and $\alpha_p(M) = \infty^+$ for all p. Thus the only way that M could be non-L^2-contractible would be if $b_2(M) \neq 0$, which then implies that $\chi(M) > 0$. Examples of groups Γ satisfying the hypotheses of the conjecture are given by the fundamental groups of closed irreducible nonpositively-curved locally symmetric spaces of dimension greater than three, and the product of the fundamental groups of two compact surfaces of negative Euler characteristic.

If π is the fundamental group of a closed 4-manifold M then an L^2-extension of [19, Théorème 1] gives that $\chi(M) \geq 2b_0(\pi) - 2b_1(\pi) + b_2(\pi)$. It follows that Conjecture 9.6 is true if Γ is the fundamental group of a closed real or complex hyperbolic 4-manifold, or the product of the fundamental groups of two compact surfaces of negative Euler characteristic.

As mentioned in the introduction, our motivation to study L^2-Betti numbers and Novikov-Shubin invariants comes from our work on the L^2-Reidemeister and analytic torsions [7, 24, 27, 28]. These are L^2-generalizations of the Reidemeister and analytic torsions of manifolds. One needs positivity of the Novikov-Shubin invariants in order to define the L^2-torsion invariants. Thus our results show that if M is of the type considered in Theorem 7.1 then the L^2-torsions are well-defined invariants. If in addition the L^2-cohomology vanishes then the L^2-Reidemeister torsion is a simple homotopy invariant (and in particular a homeomorphism invariant) and the L^2-analytic torsion is a diffeomorphism invariant. Sufficient conditions for this are given in Corollary 7.7.

Conjecture 9.7 *If M is a compact manifold then its Novikov-Shubin invariants are positive and its L^2-Reidemeister torsion equals its L^2-analytic torsion.*

This is the L^2-analog of the Cheeger-Müller theorem for the ordinary Reidemeister and analytic torsions [8, 32]. A proof of Conjecture 9.7 in the case of closed manifolds with positive Novikov-Shubin invariants and vanishing L^2-Betti numbers has been announced by Carey, Mathai and Phillips.

If M is a Seifert 3-manifold with infinite fundamental group then its L^2-Reidemeister torsion vanishes [27]. If M is a closed 3-manifold which admits a hyperbolic structure then its L^2-analytic torsion is $-\frac{1}{3\pi}Vol(M, g_{hyp})$, where g_{hyp} is the unique (up to isometry) hyperbolic metric on M [24, 28].

Conjecture 9.8 *If M is a compact connected 3-manifold with a Thurston geometric decomposition which satisfies one of the conditions of Corollary 7.7 then its L^2-torsion is $-\frac{1}{3\pi}$ times the sum of the (finite) volumes of its hyperbolic pieces.*

As one has a formula for the relationship between the L^2-Reidemeister torsions of the terms in a short exact sequence [27], Conjecture 9.8 would follow from Conjecture 9.7 if one knew that the L^2 torsion of a compact 3-manifold whose interior admitted a complete finite-volume hyperbolic metric were equal to $-\frac{1}{3\pi}$ times the hyperbolic volume of the interior. We note Conjecture 9.8 would imply that for the manifolds it considers, the L^2-torsion is a universal constant times the simplicial volume discussed in [40].

A. Infinite cyclic coverings

In this appendix we discuss infinite cyclic coverings. The Novikov-Shubin invariants in this case were computed in [24] in terms of Massey products. We will show that they can also be computed in terms of the homology of the cover.

Given an epimorphism $\phi : \pi_1(M) \longrightarrow Z$, we take $\mathcal{A} = N(Z)$ and the representation $\phi^* l^2(Z)$ to be the composition of the regular representation $Z \longrightarrow Iso_{N(Z)}(l^2(Z))$ with ϕ. In other words we are looking at the infinite cyclic cover $\overline{M} \longrightarrow M$ of M associated to ϕ. Let $C(\overline{M})$ denote the simplicial $C[Z]$-chain complex of $\overline{M}$ and $H(\overline{M})$ its (ordinary) $C[Z]$-homology $\ker(c)/\mathrm{im}(c)$, where c denotes the differential. We will show how to read off the Novikov-Shubin invariants of M, with coefficients in the representation $\phi^* l^2(Z)$, in terms of $H(\overline{M})$.

The main simplification comes from the fact that the complex group ring of the integers $C[Z]$ is a principal ideal domain [3, Proposition 5.8 and Corollary 8.7]. Given an element $p \in C[Z]$, we computed the Novikov-Shubin invariant of $m_p : l^2(Z) \longrightarrow l^2(Z)$ in Lemma 3.1. We now deal with a $C[Z]$-endomorphism $f : \oplus_{i=1}^{k} C[Z] \longrightarrow \oplus_{i=1}^{k'} C[Z]$. Let A be the (k,k)-matrix over $C[Z]$ satisfying $f(v) = vA$. We derive from the fundamental theorem for principal ideal domains (see Auslander-Buchsbaum [3], chapter 11 Theorem 1.1) that there is an integer l satisfying $0 \leq l \leq k, k'$, a sequence of non-zero elements $p_1, p_2, \ldots p_l$, a invertible (k,k)-matrix U and an invertible (k',k')-matrix V over $C[Z]$ such that p_i divides p_{i+1} and the product UAV is the matrix D whose (i,i)-th entry is p_i for $1 \leq i \leq l$ and whose other entires are zero. The determinant of any (i,i)-submatrix of A is called a (i,i)-minor of A. Let e_i denote the greatest common divisor of all (i,i)-minors of A. If all the (i,i)-minors are zero, put e_i to be zero. Tensoring with $l^2(Z)$ yields a morphism of Hilbert $N(Z)$-modules denoted by $f \otimes_{C[Z]} l^2(Z) : \oplus_{i=1}^{k} l^2(Z) \longrightarrow \oplus_{i=1}^{k} l^2(Z)$.

Lemma A.1 *Under the above conditions, if $f \neq 0$ then*

1. $b(f \otimes_{C[Z]} l^2(Z)) = k - l$.
2. $\alpha(f \otimes_{C[Z]} l^2(Z)) = \alpha(m_{p_l})$.
3. *l is the largest integer i for which e_i is different from zero.*
4. *$e_i = u \cdot \prod_{j=1}^{i} p_j$ for some unit $u \in C[Z]$ and $i \leq l$.*
5. *$p_l = u \cdot e_l / e_{l-1}$ for some unit $u \in C[Z]$.*

Proof : We derive from Lemma 1.10 that $b(f \otimes_{C[Z]} l^2(Z))$ and $\alpha(f \otimes_{C[Z]} l^2(Z))$ agree with the corresponding invariants for the endomorphism given by the diagonal matrix D. Now assertion 1.) and 2.) follow from Lemma 1.12.

One easily verifies that the numbers e_i are the same for A and D and then verifies assertions 3.) and 4.) directly for D. Claim 5.) follows from claim 4.) ■

Note that Lemmas 3.1 and A.1 allow us to compute the Novikov-Shubin invariants of $C \otimes_{C[Z]} l^2(Z)$ for any finite free $C[Z]$-chain complex C. Next, we show that it suffices to know the homology groups of C. Given any finitely-generated $C[Z]$-module P, there are non-negative integers r and l and a sequence of non-zero elements $p_1, p_2, \ldots, p_l$ of $C[Z]$ such that $p_i \mid p_{i+1}$ and

$$P = (\oplus_{i=1}^r C[Z]) \oplus \left(\oplus_{i=1}^l C[Z]/(p_i)\right),$$

where (p_i) is the ideal generated by p_i [3, Chapter 10, Theorem 5.7]. The numbers r and l and the elements p_i, up to multiplication by a unit, are uniquely determined by the isomorphism type of P.

Definition A.2 *Define the rank of P to be*

$$rk(P) = r$$

and the Novikov-Shubin invariant of P to be

$$\alpha(P) = \alpha(m_{p_l})$$

if $l \geq 1$ and $\alpha(P) = \infty^+$ otherwise. ■

Lemma A.3 *Let C be a finite free $C[Z]$-chain complex. Then*

$$b_q(C) = rk(H_q(C))$$

and

$$\alpha_q(C) = \alpha(H_{q-1}(C)).$$

Proof : Let $0 \longrightarrow \oplus_{i=1}^l C[Z] \xrightarrow{f_q} \oplus_{i=1}^{r+l} C[Z] \longrightarrow H_q(C) \longrightarrow 0$ be the finite free resolution of $H_q(C)$ given by a matrix whose (i,i)-th entry is p_i for $1 \leq i \leq l$ and whose entries vanish otherwise. Let F_q denote the 1-dimensional finite free $C[Z]$-chain complex given by f_q. Lemma A.1 gives that $rk(H_q(C)) = b(f_q \otimes_{C[Z]} l^2(Z))$ and $\alpha(H_q(C)) = \alpha(f_q \otimes_{C[Z]} l^2(Z))$. One easily constructs a $C[Z]$-chain map $g : \oplus_{q \geq 0} \Sigma^q F_q \longrightarrow C$ which induces an isomorphism on homology and is hence a chain homotopy equivalence. Now the claim follows from Lemma 2.3 and Theorem 2.5. ■

Example A.4 (mapping torus) Let M be a closed manifold and $f : M \longrightarrow M$ be a diffeomorphism. Let T_f be the mapping torus, the manifold obtained from the cylinder $M \times [0,1]$ by identifying the boundary components by f. There is an epimorphism $\phi : \pi_1(T_f) \longrightarrow Z$. The map f induces an automorphism $H_q(f)$ on $H_q(M; C)$. The Jordan normal form of $H_q(f)$ consists of blocks $B(j, \lambda)$ of j by j matrices of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda \end{pmatrix}.$$

Let $\alpha(H_q(f))$ be the minimum of the numbers $\frac{1}{j}$ over all blocks $B(j, \lambda)$ for which $|\lambda| = 1$. If there are no such blocks, put $\alpha(H_q(f))$ to be ∞^+. Since the $C[Z]$-module $H_q(M)$, with Z-action generated by $H_q(f)^{-1}$, is isomorphic to $H_q(T_f)$, we conclude from Lemma A.3 that

$$b_q(T_f; \phi^* l^2(Z)) = 0$$

and

$$\alpha_q(T_f; \phi^* l^2(Z)) = \alpha(H_{q-1}(f)). \qquad \blacksquare$$

Example A.5 Let us look at the special case in which f is a self-diffeomorphism of the 2-torus. We call $H_1(f)$ *periodic* if $H_1(f)^k = \mathrm{id}$ for some $k \neq 0$, *hyperbolic* if no eigenvalue of $H_1(f)$ has unit norm and *parabolic* otherwise. From [36, Theorem 5.5], we have that $H_1(f)$ is periodic if and only if T_f has a R^3-structure, hyperbolic if and only if T_f has a Sol-structure and parabolic if and only if T_f has a Nil-structure. One easily checks that

$$\begin{array}{lcl} \alpha_2(T_f; \phi^* l^2(Z)) = 1 & \Leftrightarrow & T_f \text{ has a } R^3\text{-structure} \\ \alpha_2(T_f; \phi^* l^2(Z)) = \infty^+ & \Leftrightarrow & T_f \text{ has a } Sol\text{-structure} \\ \alpha_2(T_f; \phi^* l^2(Z)) = \frac{1}{2} & \Leftrightarrow & T_f \text{ has a } Nil\text{-structure} \end{array} \qquad \blacksquare$$

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