A Hadamard–Cartan Theorem for Metric Spaces

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A corollary of the Rauch Comparison Theorem ([CE]) is that, in a Riemannian manifold M of sectional curvature $\leq K$, the exponential map at any point p has maximal rank on $B(0, \pi/\sqrt{K}) \subset T_p$. One can then "lift" the metric of M to $B(0, \pi/\sqrt{K})$, so that $\exp_p$ becomes a local isometry. For $K \leq 0$ this procedure yields a covering map from $\mathbb{R}^n$ to M and proves the Hadamard-Cartan Theorem.

The main result of this paper is that the existence of a "lift" of the distance function can be proved using only general distance-angle comparisons, which allows the Hadamard-Cartan Theorem to be generalized to the class of inner metric spaces of locally bounded curvature. This lifting of the metric is also applied in [P1] to construct a local action in the tangent space similar to Gromov's "fundamental pseudogroup" ([GLP]).

An *inner metric space* (X,d) is a metric space X with distance d such that for all x, $y \in X$, d(x,y) is the infimum of the lengths of curves α joining x and y in X. Such spaces occur, in particular, as Gromov-Hausdorff limits ([G]) of Riemannian manifolds, and understanding their geometry can be useful in convergence and finiteness problems. For example, the Convergence Theorem for Riemannian manifolds of bounded sectional curvature (above and below), volume (below) and diameter (above) ([GW], [Pe]) follows easily from the results of ([N]) (cf. comments in [P2] and [PD]). The main theorem in the present paper implies that the Gromov-Hausdorff limit of Riemannian n-manifolds M_i having a) sectional curvature $\geq k$ and $\leq K_i$, with $K_i \to 0$, b) volume $\geq v$, and c) diameter $\leq D$, is covered by $\mathbb{R}^n$.

A few definitions and will be recalled below. For more details, and examples, see [P2]. All curves are assumed parameterized proportional to arclength. Let X be a locally compact, metrically complete inner metric space. Every pair of points $x, y \in X$ can be connected by a *minimal curve* whose length realizes $d(x, y)$; a curve which is locally minimizing is called a *geodesic*. The notation γ_{ab} is reserved for a geodesic from a to b. In a space with locally bounded curvature ([P2]) there exists an *angle* $\alpha(\gamma, \beta)$ between any two geodesics γ, β starting at a common point, and every point $x \in X$ is contained in a strictly convex ball B, that is, every $y, z \in B$ can be joined by a *unique* minimal curve lying in B. For any K, the *upper (resp. lower) comparison radius* $c^K(x)$ *(resp.* $c_K(x)$*)* is defined to be the largest r such that the following hold for any curves γ_{ab}, γ_{ac} in $B(x, r)$ (S_K is the simply connected two dimensional space form of constant sectional curvature M):

A1. *There is a uniquely determined (up to congruence) triangle ABC in* S_K *(resp.* S_K*) with* $AB = d(a, b)$, $BC = d(b, c)$, *and* $AC = d(a, c)$, *and this triangle satisfies* $\alpha(\gamma_{ab}, \gamma_{ac}) \leq$ *(resp* $\geq$*)* $\alpha(\gamma_{AB}, \gamma_{AC})$.

A2. *There is a uniquely determined (up to congruence) triangle* $A'B'C'$ *in* S_K *(resp.* S_K*) with side* $\overline{B'C'}$ *of minimal length, such that* $A'B' = d(a,b)$, $A'C' = d(a, c)$, *and* $\alpha(\gamma_{ab}, \gamma_{ac}) = \alpha(\gamma_{AB}, \gamma_{AC})$, *and this triangle satisfies* $d(b, c) \geq$ *(resp.* $\leq$*)* $B'C'$.

For any fixed K, the upper and lower comparison radii are either everywhere infinite, or continuous into $[0, \infty)$. At each point x in a space of locally bounded curvature there exist K, k such that $c^K(x) > 0$ and $c_k(x) > 0$.

The following lemma is well known ([CE]); it is stated here because it is used several times later.

Lemma 1. *For a triangle ABC in S_K having side lengths $< \pi \cdot K^{-1/2}$, the distance BC is a monotone increasing function of the angle at A, on $[0,\pi]$.*

A *(geodesic) triangle* is a set of three geodesics $(\gamma_{ab}, \gamma_{ac}, \gamma_{bc})$. By A1, every triangle T of minimal geodesics in a region of curvature $\le$ K satisfies the following property:

Property A. *There exists a triangle T' in S_K whose sides are geodesics having the same lengths as the sides of T, such that the angles of T' are $\ge$ the corresponding angles in T.*

Definition. *Let $c^K > 0$ on $\bar{U}$ (assumed compact), and let ε be the Lebesque number for the cover of $\bar{U}$ by (open) balls which are regions of curvature $\le$ K of maximal radius. A triangle $(\gamma_1, \gamma_2, \gamma_3)$ in U such that $\ell_1 = \ell(\gamma_1)$ and $\ell_2 = \ell(\gamma_2)$ are both $< \pi/\sqrt{K}$ is called* thin *if for all t, $d(\gamma_1(t), \gamma_2(t)) \le \delta = \min(\varepsilon/4, 2\pi/\sqrt{K} - (\ell_1 + \ell_2))$.*

Any triangle in U whose sides are minimal curves of length $< 2\delta$ is contained in a region of curvature $\le$ K. Furthermore, if T is a thin triangle, then there exists a triangle T′ in S_K whose sides are minimal geodesics having the same lengths as the sides of T.

Thin Triangles Lemma. *Let U be a relatively compact subset of a metrically complete inner metric space of locally bounded curvature. If $c^K > 0$ on the closure of U then every thin triangle $(\gamma_1, \gamma_2, \gamma_3)$ in U such that γ_3 is minimal satisfies Property A.*

Proof. For each $k = 1,2$ partition γ_k into m minimal segments γ_{ki} of length $< \delta$, where δ is as in the definition of "thin triangle," with end points denoted by a_{ki} and $a_{k(i+1)}$. Let α_{ij} denote a minimal geodesic from a_{1i} to a_{2j}, which is, for $|i - j| \le 1$, unique and of length $< 2\delta$. The angles satisfy ([R], [P2]) $\alpha(\gamma_{1i}, \alpha_{ij}) + \alpha(\alpha_{ij}, \gamma_{1(i+1)}) = \alpha(\gamma_{2j}, \alpha_{ij}) + \alpha(\alpha_{ij},\gamma_{2(j+1)}) = \pi$. Finally, let β_{ki} denote the segment of γ_k from p to a_{ki}, and T_{ij} denote the triangle with sides β_{1i}, β_{2j}, and α_{ij}. T_{11} lies in a region of curvature $\le K$ and therefore has Property A. Assume T_{ii} has Property A for some $i \ge 1$, and denote the objects on T'_{ii} by adding primes to the notation for the corresponding objects on T_{ii}. By the observation preceding the lemma, $T_{(i+1)i}$ has a representative $T'_{(i+1)i}$ in S_K having the required side lengths; only the three angle conditions need be established.

Extend β'_{1i} past a'_{1i} to a minimal curve of length $\ell(\beta_{1(i+1)})$, denoting by γ'_{1i} the added segment and $a_{1(i+1)}$ the new end point. By Property A, $\alpha(\beta'_{1i},\alpha'_{ii}) \ge \alpha(\beta_{1i},\alpha_{ii})$, which implies $\alpha(\gamma'_{1i},\alpha'_{ii}) \le \alpha(\gamma_{1i},\alpha_{ii})$; by A2 and Lemma 1, $d(a'_{1(i+1)}, a'_{2i}) \le d(a_{1(i+1)}, a_{2i})$. It now follows from Lemma 1 that the angle condition holds for $\alpha(\beta_{1(i+1)},\beta_{2i})$ of $T_{(i+1)i}$. A similar argument proves the angle condition for $\alpha(\beta_{1(i+1)},\alpha_{(i+1)i})$. For the last angle, choose geodesics β'_{2i}, α'_{ii}, $\alpha'_{(i+1)i}$ in S_K having a common end point a'_{21} having the lengths of their unprimed counterparts and such that $\alpha(\beta_{21}, \alpha_{11}) = \alpha(\beta'_{21}, \alpha'_{11})$ and $\alpha(\alpha_{11}, \alpha_{(1+1)1}) = \alpha(\alpha'_{11}, \alpha'_{(1+1)1})$. Then Property A and Lemma 1 imply that $d(p',a'_{11}) \le \ell(\beta_{11})$ and $d(a'_{11},a'_{1(1+1)}) \le \ell(\gamma_1)$. Since

$$\begin{aligned}\alpha(\beta_{21}, \alpha_{(1+1)1}) &\le \alpha(\beta_{21}, \alpha_{11}) + \alpha(\alpha_{11}, \alpha_{(1+1)1})\\ &= \alpha(\beta'_{21}, \alpha'_{11}) + \alpha(\alpha'_{11}, \alpha'_{(1+1)1})\\ &= \alpha(\beta'_{21}, \alpha'_{(1+1)1}),\end{aligned}$$

Property A is now established for $T_{(1+1)1}$ by the triangle inequality and Lemma 1. An analogous argument shows that Property A for $T_{(1+1)1}$ implies Property A for $T_{(1+1)(1+1)}$, and the induction step is finished.

A space X of locally bounded curvature is said to have *curvature* $\le K$ if every point is contained in a region of curvature $\le$ K. X is said to be *geodesically complete* if every geodesic in X is defined on all of $\mathbb{R}$.

Theorem 1. *Suppose (X,d) is a geodesically complete inner metric space with curvature $\le K$ and locally bounded below, and let $r = \pi/\sqrt{K}$ if $K > 0$, or $r = \infty$ otherwise. Then there exists an n such that for each $p \in X$, there is a mapping $exp_p : \mathbb{R}^n \to X$ with the following properties:*

a) $exp_p(tv)$ is an arclength parameterization of a geodesic γ_v starting at p, for all $v \in \mathbb{R}^n$ with $\|v\| = 1$,

b) for every $v,w \ne 0$, the angle between v and w is $\alpha(\gamma_v,\gamma_w)$,

c) there exists an equivalent metric d' on $B(0,r) \subset \mathbb{R}^n$ such that $exp_p|_{B(0,r)}$ is a local isometry.

Proof. Different proofs of the existence of $\exp_p$ with properties a) and b) can be found in [Be], [P2], and [PD]. The domain of $\exp_p$ will be denoted by T_p.

Suppose $\exp_p$ is not locally 1-1 on $B(0, r) \subset T_p$. Then there exist distinct vectors $v_0, v_1, v_2, \ldots \in T_p$ such that $v_i \to v_0$ and $\exp_p(v_i) = \exp_p(v_j)$ for all i, j. But then for sufficiently large i, the geodesics $\exp_p(tv_i))$ and $\exp_p(tv_0)$, together with a minimal geodesic from $\exp_p((1-\epsilon)v_i)$ to $\exp_p((1-\epsilon)v_0)$ form thin triangles $T_{i,\epsilon}$ which, for small ϵ, violate the Thin Triangles Lemma.

Define a new metric on $B(0,r)$ as follows: For any $x, y \in B(0,r)$, let $d'(x,y) = \inf \{\ell(\exp_p(\alpha))\}$, where the infimum is taken over all curves in $B(0,r)$ from x to y. The fact that $\exp_p$ is a local homeomorphism implies that d' is a metric, and is equivalent to the Euclidean metric on $B(0,r)$. Moreover, very short minimal curves can always be lifted via $\exp_p$, and so $\exp_p|_{B(0,r)}$ is a local isometry.

Letting $K = 0$ in Theorem 1 the exponential map is a local isometry on all of $(\mathbb{R}^n,d')$; [R], Satz 7, §27, implies that such a map is a covering map. In other words:

Corollary 1. *The universal cover of any geodesically complete inner metric space with curvature ≤ 0 and locally bounded below is homeomorphic to Euclidean space.*

References

[Be] Berestovskii, V.N. Introduction of a Riemann structure into certain metric spaces, *Siberian Math. J.*, 16 (1975), 499-507.

[CE] Cheeger, J., and Ebin, D. G. *Comparison Theorems in Riemannian Geometry*, North Holland, Amsterdam, 1975.

[G] Gromov, M. Groups of polynomial growth and expanding maps, *Publ. Math. I.H.E.S.* 53 (1981), 53-78.

[GLP] Gromov, M., Lafontaine, J., and Pansu, P. *Structure Métrique pour les Variétés Riemanniennes*, Cedic/Fernant Nathan, Paris, 1981.

[GW] Greene, R.E. and Wu, H. Lipschitz convergence of Riemannian manifolds, *Pacific Math. J.* 131 (1988), 119-141.

[N] Nikolaev, I. G. Smoothness of the metric in spaces with bilaterally bounded curvature in the sense of A. P. Aleksandrov, *Siberian Math J.* 24 (1983) 247-263.

[P1] Plaut, C. Extending local actions to global actions, with applications to geometry, preprint.

[P2] Plaut, C. A metric characterization of manifolds with boundary, preprint.

[PD] Plaut, C. *Riemannian Geometry of Non-Riemannian Spaces*, dissertation, University of Maryland, 1989.

[Pe] Peters, S. Convergence of Riemannian manifolds, *Compositio Math.* 62 (1987), 3-16.

[R] Rinow, W. *Die Innere Geometrie der Metrischen Räume*, Springer-Verlag, Berlin, 1961.

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