Short geodesic loops on complete Riemannian manifolds with a finite volume.

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Abstract

In this paper we will show that on any complete noncompact Riemannian manifold with a finite volume there exist uncountably many geodesic loops of arbitrarily small length.

Introduction.

In the paper we will prove the following theorem:

Theorem 0.1 *Let M^n be a complete noncompact Riemannian manifold of a finite volume V. Then, given a point $p \in M^n$ and $\varepsilon > 0$, there exists a set $A \subset (0, \infty)$ of measure $\frac{V}{f(\varepsilon)}$, where $f(\varepsilon) = (\frac{\varepsilon}{12 \cdot 108^{n-1} n!})^{n-1}$, such that for every t not in $A \cup (0, \frac{\varepsilon}{6})$ there exists a geodesic loop on M^n of length $\leq \varepsilon$ based at the distance t from the point p.*

Remarks.
(1) Theorem 0.1 immediately implies that the set of distinct geodesic loops of length $\leq \varepsilon$ on M^n is uncountable.
(2) As it will be seen from the proof, the theorem also applies to closed Riemannian manifolds and $\varepsilon > 0$ providing that $r_p = \max_{q \in M^n} dist(p, q) > \frac{V}{f(\varepsilon)} + \frac{\varepsilon}{3}$ with the conclusion valid for the values of $t \in (\frac{\varepsilon}{6}, r_p - \frac{\varepsilon}{6})$ in the complement of A.

Note that the existence of arbitrarily short geodesic loops on a complete noncompact Riemannian manifold of a finite volume also easily follows from the following theorem proven by S. Sabourau, (see [S2]):

Theorem 0.2 ([S2]) *Let M^n be a complete Riemannian manifold of dimension n. Then there exists $C(n) > 0$, such that the volume of any ball of radius $B(x_0, r)$ of radius $r \leq \frac{sgl(M^n)}{2}$, where $sgl(M^n)$ is the length of a shortest geodesic loop on M^n, is at least $C(n)r^n$.*

It is clear from Sabourau's argument that one can, alternatively, define $sgl(M^n)$ as the the infimum of lengths of geodesic loops on M^n in order to cover the situation when there is no shortest geodesic loop on M^n. In this case the same lower bound for the volume of metric balls will still hold. This will immediately imply that if M^n is complete and noncompact Riemannian manifold with a finite volume, then $sgl(M^n) = 0$, as otherwise, one would have an infinite set of disjoint metric balls with volumes uniformly bounded from below. Their combined volume will be infinite contradicting the fact that M^n has a finite volume.

Yet, it is not clear how one can adapt the approach from [S2] to derive more information about short geodesic loops. In particular, the existence of uncountably many loops of length at most ε is not guaranteed by his method.

In this paper we also prove the following result:

Theorem 0.3 *Let M^n be a complete noncompact Riemannian manifold of a finite volume V. Then given a point $p \in M^n$ there exists $T > 0$, such that for all $t > T$ there exists a geodesic loop of length at most ε at the distance t from p.*

Note that, if desired, one can combine the statements of Theorems 0.1 and 0.3 by demanding that the set A in the text of Theorem 0.1 is bounded.

In view of Theorem 0.3 one can ask if given an $\varepsilon > 0$ there exists $T > 0$ such that for every $t > T$ and every point $q \in M^n$ at the distance $t > T$ from p there exists a geodesic loop of length at most ε based at q. The answer to this question is negative. To see this consider a complete noncompact manifold of a finite volume and attach to it an infinite sequence of long cylindrical "fingers" of exponentially decreasing radii that are capped by hemispheres H_i. Assume that the sequence of the distances from the bases of these fingers to p are unbounded. It is clear that there are no short geodesic loops based at the centers of H_i.

These theorems provide an answer to one of many questions about relationship between the volume of a complete noncompact Riemannian manifold and lengths of various stationary objects.

Previously, questions of a similar nature were investigated by C. B. Croke, who has established a volume upper bound for the length of a shortest periodic geodesic on a surface of a finite volume, (see [C]) and by S. Sabourau, who has indicated how to bound the length of a shortest geodesic loop on a complete Riemannian manifold by its volume, (see [S2]) as well as by M. Gromov, (see [G]), who obtained some estimates for 1-systoles in the case of manifolds that are essential relative to infinity and manifolds that have essential ends.

Note that it is not known whether on any complete Riemannian manifold of finite volume of dimension greater than two there exists a periodic geodesic, (though it was shown by V. Bangert and G. Thorbergsson that there exist infinitely many geodesics on a complete surface of a finite volume, (see [B], [T])).

In the case of a closed Riemannian manifold M^n, there are numerous results that connect the size, (i. e. the length or the area) of various stationary objects, such as geodesic loops, minimal geodesic cycles and nets, minimal surfaces or submanifolds to the size of a manifold as measured either by its volume or the diameter, (see, for instance, [Bl], [NR2], [NR3], [NR4], [R2], [R3], [R4], [S2]).

Presently there are no general curvature-free upper bounds of this nature for the length of a shortest periodic geodesic on a simply connected manifold, except in dimension two, (see [C], [M], [NR1], [R1], [S1]), though many results for manifolds with nontrivial fundamental group are known, (see [BrZ], [CK] for surveys of these results). The most notable is the result of M. Gromov, which gives a volume estimate for the length of a shortest periodic geodesic on closed Riemannian manifolds that are essential, (see [G]).

Our proof will make use of the following definition and result by M. Gromov, (see [G] as well as recent paper by S. Wenger [W]. Wenger provides a short proof of the filling volume versus volume inequality, which is at the core of the Gromov's original proof. His paper also implies some improvements of the original result, in particular, an improvement of the dimensional constant.) We will also use some ideas from Gromov's paper [G] and our approach of constructing "fillings" of cycles in the absence of short geodesic loops used in [R3].

Definition 0.4 *Filling Radius ([G]). Let M^n be an n-dimensional Riemannian manifold in an arbitrary metric space X. Then the filling radius $FillRad(M^n \subset X)$ is the infimum of $\varepsilon > 0$, such that M^n bounds in the*

ε-neighborhood $N_\varepsilon(M^n)$, that is $i_*(H_n(Z_n)) = \{0\}$, where i_* is induced by the inclusion $i : M^n \longrightarrow N_\varepsilon(M^n)$ and, where $H_n(M^n)$ is taken with coefficients in $\mathbf{Z}$, when M^n is orientable, and with coefficients in $\mathbf{Z}_2$, when M^n is nonorientable. The filling radius of an abstract Riemannian manifold is then defined to be $FillRad(M^n \subset L^\infty(M^n))$, where the Kuratowski embedding of M^n into $L^\infty(M^n)$ is a map that to each point p of M^n assigns a distance function $p \longrightarrow f_p = d(p,q)$. Equivalently, $FillRadM^n$ can be defined as the infimum of $FillRad(M^n \subset X)$ over all metric spaces X and isometric embeddings of M^n into X.

Theorem 0.5 *(*[**G**]*)* *Let M^n be an n-dimensional manifold. Then $FillRadM^n \leq k(n)vol(M^n)^{\frac{1}{n}}$, where $k(n)$ is an explicit function of the dimension of a manifold.*

Gromov's dimensional constant $k(n) = (n+1)n^n\sqrt{(n+1)!}$ can be improved to $\tilde{k}(n) = 27^n(n+1)!$ by combining the result by Wenger in [W] with the inequality (2.6) in [G].

Note that L. Guth has recently announced an important improvement of the above result by showing that a complete Riemannian manifold with the filling radius R contains a ball of radius R of volume bounded from below by $c(n)R^n$, (see [Gt1]).

1 Three simple lemmas

We will begin the proof of the main results with the following three lemmas:

Lemma 1.1 *Let M^n be a complete noncompact Riemannian manifold of a finite volume V, $p \in M^n$. Let $\sigma(t)$ be a geodesic ray, starting at a point p. Then given $\tilde{\varepsilon} > 0$ there exists a set $A = A(\tilde{\varepsilon}) \subset (0,\infty)$ of measure at most $\frac{16V}{\tilde{\varepsilon}}$, such that for every t^* in A^c, (the complement of A in $(0,\infty)$), and for every $0 < \delta < \min\{1, \frac{\tilde{\varepsilon}}{2}\}$ there exists an $(n-1)$-dimensional submanifold $Z^\delta_{\tilde{\varepsilon}}$ of M^n with the following properties:*
(1) $vol_{n-1}(Z^\delta_{\tilde{\varepsilon}}) < \tilde{\varepsilon}$;
(2) $Z^\delta_{\tilde{\varepsilon}}$ does not bound in $M^n \setminus \{p\}$;
(3) the distance between $Z^\delta_{\tilde{\varepsilon}}$ and the geodesic sphere $\tilde{S}_{t^*}(p) = \{x \in M^n | dist(x,p) = t^*\}$ is at most δ.

Proof. Let $\tilde{S}_t(p)$ be a family of geodesic spheres centered at the point p of radius $t \in (0,\infty)$. By the coarea formula $\int_0^\infty vol_{n-1}(\tilde{S}_t(p))dt = V$.

Therefore there exists a set $A = A(\tilde{\varepsilon})$ of measure at most $\frac{16V}{\tilde{\varepsilon}}$ such that for any $t^* \in A^c$ the $(n-1)$-dimensional Hausdorff measure $vol_{n-1}(\tilde{S}_{t^*})$ is at most $\frac{\tilde{\varepsilon}}{16}$. Moreover, there exists a small neighborhood $(t^* - \tau, t^* + \tau)$ of t^* such that for any $t \in (t^* - \tau, t^* + \tau)$, $vol_{n-1}(\tilde{S}_t(p)) < \frac{\tilde{\varepsilon}}{8}$. Thus, $\int_{t^*-\tau}^{t^*+\tau} \tilde{S}_t(p)dt < \frac{\tilde{\varepsilon}\tau}{4}$.

Let $0 < \delta < \min\{1, \frac{\tilde{\varepsilon}}{2}\}$ be given.

Let $\varrho_\sigma : M^n \longrightarrow \mathbf{R}, \sigma < \delta$ be a function that is smooth on $M^n \backslash \{p\}$ and that approximates a distance function ϱ_p from the point p in the following way: (1) $\varrho_\sigma = \varrho$ on a geodesic ball centered at p of radius smaller than the injectivity radius of M^n at p; (2) $|\varrho_p - \varrho_\sigma| \le \sigma$ and (3) $|\mathrm{grad}\varrho_\sigma| \le 1 + \sigma$. The details of constructing such a function can be found in M. P. Gaffney's work [Ga].

Let us now consider the sublevel sets $S_t(p)$ of ϱ_σ. For some small values of $t \in \mathbf{R}$, they will be geodesic spheres, because ϱ_σ agrees with the distance function in some neighborhood of p. Let $\tilde{S}_r(p)$ be a geodesic sphere centered at p with radius r smaller than the injectivity radius at the point p. Then $\tilde{S}_r(p)$ is homeomorphic to S^{n-1}.

By the virtue of Mayer-Vietoris exact sequence it follows that $\tilde{S}_r(p)$ does not bound in $M^n \backslash \{p\}$. Otherwise, $H_n(M^n) \neq \{0\}$, which would contradict the assumption that M^n is not compact. The sublevel sets $S_t(p), t \in (0, \infty)$ are homologous for all t. Thus, for all t, $S_t(p)$ does not bound in $M^n \backslash \{p\}$.

Without any loss of generality we can assume that $\tau < \delta$. Let $\sigma = \frac{\tau}{2}$. Consider $\int_{t^*-\frac{\tau}{2}}^{t^*+\frac{\tau}{2}} vol_{n-1}(S_t(p))dt$, which, by coarea formula is at most $\frac{\tilde{\varepsilon}\tau(1+\sigma)}{4} \le \frac{\tilde{\varepsilon}\tau}{2}$. Thus, there exists a set $B \subset (t^* - \tau + \sigma, t^* + \tau - \sigma)$ of measure at least $\frac{\tau}{3}$, such that $vol_{n-1}(S_t(p)) < \tilde{\varepsilon}$ for every $t \in B$.

Furthermore, when $t \in B$ the distance between $\tilde{S}_{t^*}(p)$ and $S_t(p)$ is at most δ. Finally, note that by Sard's theorem, $S_t(p)$ is an $(n-1)$-dimensional submanifold of M^n for almost all t. Thus, we can select $t \in B$ so that this $S_t(p)$ is a submanifold. We will denote it $Z^\sigma_{\tilde{\varepsilon}}$.

□

The following two lemmas were used in [R3]. We will present them here for the sake of completeness.

The first is a Morse-theoretic type lemma asserting that the space of loops based at a fixed point q of length smaller than the length of a minimal geodesic loop at q is contractible.

Lemma 1.2 *Let M^n be a complete Riemannian manifold. Let $q \in M^n$. Suppose that the length of a shortest geodesic loop $l_q(M^n)$ based at q is greater* *than L. Then given any piecewise differentiable loop $\gamma : [0,1] \longrightarrow M^n$ of length $\leq L$ such that $\gamma(0) = \gamma(1) = q$ there exists a length decreasing path homotopy connecting this curve with q that depends continuously on initial loop γ.*

Proof. There is a standard explicit length shortening deformation of the space of loops based at q of length $\leq L$ to the constant loop via the Birkhoff Curve Shortening Process, (see [C] for the detailed description of the Birkhoff Curve Shortening Process (BCSP) for closed curves. The only difference between the BCSP for closed curves and the BCSP for loops is that one fixes a base point during the homotopies in the latter case.)

□

The third lemma can be viewed as an effective version of an elementary assertion that two curves γ_1, γ_2 connecting points q_1, q_2 are path homotopic if and only if the loop $\gamma_2 * -\gamma_1$ is path homotopic to a point. Lemma 1.3 is analogous to a similar statement in [C], namely, Lemma 3.1.

Lemma 1.3 *Let γ_1, γ_2 be two curves with $\gamma_1(0) = \gamma_2(0) = q_1$ and $\gamma_1(1) = \gamma_2(1) = q_2$ on a complete Riemannian manifold M^n of length l_1, l_2 respectively.*

Let $\gamma_2 * -\gamma_1$ be the product of γ_2 and $-\gamma_1$ based at q_1. If this curve is contractible to q_1 as a loop along the curves of length $\leq l_1 + l_2$ then there is a path homotopy, (i.e. a homotopy that fixes the end points), $h_\tau(t), \tau \in [0,1]$, such that $h_0(t) = \gamma_1(t), h_1(t) = \gamma_2(t)$ and the length of curves during this homotopy is bounded above by $2l_1 + l_2$. Alternatively there exists a path homotopy with the same properties, such that the length of curves in it is bounded by $l_1 + 2l_2$. Moreover, when M^n has no geodesic loops of length $\leq l_1 + l_2$, this path homotopy can be made to continuously depend on the digon formed by γ_1 and γ_2, see (1.2).

Proof. Let $\tilde{h}_\tau(t)$ be a homotopy that connects $\gamma_2 * -\gamma_1$ with a point p, (see fig. 1 (a) and (b)). Then let us consider the following homotopy $\gamma_1 \sim \tilde{h}_{1-\tau} * \gamma_1 \sim \gamma_2 * -\gamma_1 * \gamma_1 \sim \gamma_2$, (see fig. 1 (a)-(g)). The length of curves during this homotopy is $\leq 2l_1 + l_2$.

Note that, assuming there are no geodesic loops of length $\leq l_1 + l_2$, one can contract $\gamma_2 * -\gamma_1$ via the BCSP, which continuously depends on the initial curve, (see Lemma 1.2). Thus, the path homotopy between $\gamma_1(t)$ and $\gamma_2(t)$ will also continuously depend on the initial digon.

Also, one can reverse the role of γ_1 and γ_2 and construct a path homotopy between γ_2 and γ_1 passing through curves of length l_1+2l_2. Then we reverse the direction of this path homotopy obtaining a path homotopy from l_1 to l_2 with the required properties.

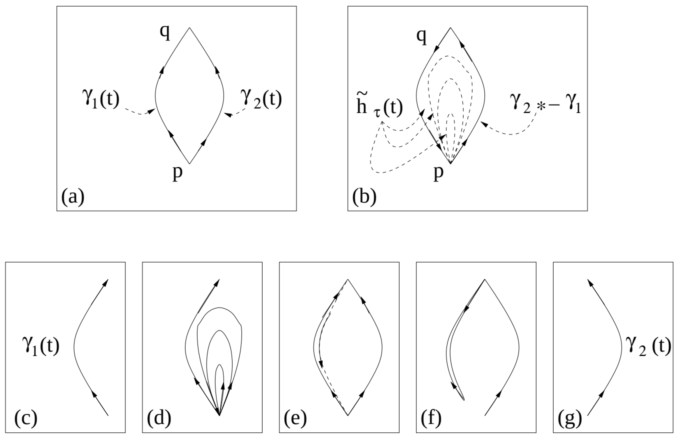

Figure 1: Illustration of the proof of Lemma 1.3.

□

2 Proof of Theorems 0.1 and 0.3.

In the following definition, (Definition 3.1 in [R3]), we let σ^{m+1} denote the standard $(m+1)$-dimensional simplex, and $C(X,Y)$ denote the space of continuous maps from X into Y.

Definition 2.1 *Given $l > 0$ and a positive integer m, let $K_{m,l}$ be a space of piecewise smooth maps of the complete graph with $(m+2)$ vertices into M^n, such that each edge is mapped into a curve of length $\leq l$. We define an N-filling of $K_{*,l}$ as a a collection of continuous maps $\phi_m : K_{m,l} \longrightarrow C(\sigma^{m+1}, M^n)$,$m = 1, 2, \ldots, N$, satisfying the following properties:*
(1) For every $k \in K_{m,l}$ the restriction of $\phi_m(k)$ to the 1-skeleton of σ^{m+1} coincides with k, that is, each $\phi_m(k)$ extends k.
(2) For every $k \in K_{m,l}$, $(m \leq N)$, the restriction of $\phi_m(k)$ to an m-dimensional face of σ^{m+1} coincides with ϕ_{m-1} evaluated on the element of $K_{m-1,l}$ obtained from k by restricting k to the set of all 1-dimensional simplices in the 1-skeleton of this face of σ^m.

Here is an informal description of the above definition: we are "filling" graphs with "short" edges, (i.e. of length $\leq l$) that correspond to the immersed 1-skeleton of a simplex of dimension $m+1$ by discs of dimension $m+1$, so that the map of the disc extends the map from 1-skeleton. Moreover, this extension is done in a coherent way, that is, if we consider the restriction of this map to a face of the simplex, it will be a "filling" of the 1-skeleton of the face. In particular, that means that each N-filling agrees with its subfillings and depends continuously on its 1-skeleton.

Lemma 2.2 *Suppose that the length of a shortest geodesic loop on M^n is greater than $3 \cdot 4^{n-1}l$. Then there exists an n-filling of $K_{*,l}$. Moreover, if $k \in K_{m,l}$, $(m \leq n)$, the disc that fills k will lie in the $6 \cdot 4^{n-2}l$-neighborhood of the set of vertices of k, that is, the maximal distance between points of the disc and the set of vertices of k is at most $6 \cdot 4^{n-2}l$.*

Proof. We will prove the existence of the i-fillings of $K_{*,l}$ for every $i \leq n$. The proof will be by induction with respect to i. The base step corresponds to $i = 1$. Let $k_1 \in K_{1,l}$. By Definition 2.1 it is an immersion of a full graph that consists of three vertices and three edges. Let v_0, v_1, v_2 be the vertices of this immersed graph. The three edges form a loop based at v_0. Since we have assumed that there are no "short" geodesic loops, (and, in particular, no geodesic loops of length $\leq 3l$), this loop is contractible to v_0 via shorter loops based at v_0. This homotopy generates a disc that "fills" k^1.

At each subsequent step, to construct ϕ_j we consider its restriction to $\partial\sigma^{j+1}$. This restriction is uniquely determined by the definition of N-fillings and, if $i > 1$, by the previous steps of the induction. That is, the previous step of the induction results in a filling of elements of $K_{j-1,l}$ obtained from elements of $K_{j,l}$ by deleting a vertex. Consider $k \in K_{j,l}$. Then the fillings of

$j+2$ elements of $K_{j-1,l}$ that are obtained from k by deleting a vertex are discs of dimension j provided by the previous step of the construction. They together form a j-dimensional sphere, which, according to our definition, is a restriction of ϕ_j to $\partial\sigma^{j+1}$. The required disc is then generated by a homotopy that contracts this sphere to a point. To construct this homotopy we begin by constructing a 1-parameter family k_t of immersed graphs connecting $k = k_0$ with a complete graph k_1 with $(j+2)$ vertices immersed in M^n such that all of its edges are mapped to some paths in a tree. This path k_t should continuously depend on the initial graph k. Next, we construct a 1-parameter family of spheres S^j_t by filling all k_ts. This result in a homotopy between the sphere $\phi_j(\partial\sigma^{j+1})$ and the degenerate sphere S^j_1 that lives in a tree and is, therefore, contractible. (In order to contract this degenerate sphere we contract k_1 over itself and fill it by the n-sphere at each moment of the homotopy).

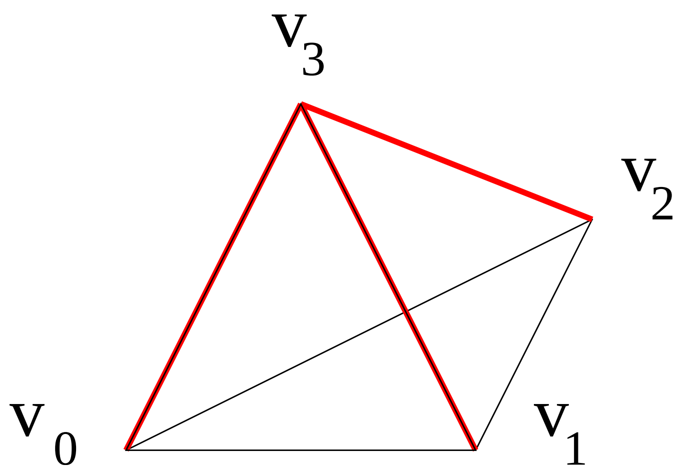

Figure 2: Collapsing triangles

For every t k_t is constructed by several applications of an operation of a *collapsing of a triangle*: Let k_a, k_b, k_c be any of the three edges of k. As there are no geodesic loops of length $\leq$ length k_a+ length k_b+ length k_c in M^n, we can apply Lemma 1.3 to construct a path homotopy between k_a and $k_b * k_c$. This homotopy passes through paths of length $\leq 2length(k_a) + length(k_b) + length(k_c) \leq 4l$. This homotopy induces a homotopy of triangles $(k_a)_t, k_b, k_c, t \in [0,1]$ that we call a collapsing of the triangle k_a, k_b, k_c. At the end of this homotopy k_a is being replaced by another edge that goes along $k_b * k_c$, and the considered triangle becomes thin. Note that when one is given a triangle with the sides k_a, k_b, k_c there is a freedom of what side is being deformed and which vertex is used as a base point for contracting a loop. To avoid the ambiguity we can assume that the side k_a that is being deformed is the one that connects vertices with the smallest indices, and the loops are always being contracted to a vertex with the smallest index.

After collapsing finitly many triangles, we can obtain an element of $K_{j,4l}$, where all edges run along the tree-shaped union k_1 of edges of k adjacent to one vertex of k, let's say the vertex with the highest number, (see fig. 2 , which illustrates that the edge $[v_0, v_1]$ is being collapsed to the path $[v_0, v_3, v_1]$, the edge $[v_1, v_2]$ is being collapsed to $[v_1, v_3, v_2]$, and the edge $[v_0, v_2]$ is being collapsed to $[v_0, v_3, v_2]$).

Now we can continue the homotopy of complete graphs by contracting all edges of k_1 to a point, (to v_3 on fig. 2) along the tree by a length non-increasing homotopy.

The resulting graphs are filled by j-spheres using the induction assumption on $K_{m,4l}$, since the length of edges that result in the process of collapsing of triangles is bounded above by $4l$.

Let $k \in K_{m,l}$. Then k is a (map of) the complete graph with $m+2$ vertices $v_0, v_1, ..., v_{m+1}$. Let k_{t_1} denote a one parameter family of graphs obtained from k by collapsing triangles. We define $k_{t_1}^{i_1}$ to be a subgraph of k_{t_1} obtained from it by removing a vertex v_{i_1}. In general, let $k_{t_1,...,t_j}^{i_1,...,i_{j-1}}$ be a family of complete graphs with $m+3-j$ vertices obtained from $k_{t_1,...,t_{j-1}}^{i_1,...,i_{j-1}}$ by collapsing triangles and let $k_{t_1,...,t_j}^{i_1,...,i_j}$ be complete graph with $m+2-k$ vertices obtained from $k_{t_1,...,t_j}^{i_1,...,i_{j-1}}$ by removing a vertex v_j. Let $a(j)$ be the maximal possible length of an edge of $k_{t_1,...,t_j}^{i_1,...,i_j}$. Note that $a(1) \leq 4l$ and that $a(j+1) \leq 4a(j)$. Thus, $a(m-1) \leq 4^{m-1}l$. So, the length of loops that one contracts in the recursive process described above is at most $3 \times 4^{n-1}l$.

Note also, that as all the homotopies are constructed by contracting loops to one of the vertices of k, the maximal distance from the points of the resulting disc to one of the vertices is at most half the maximal length of such loops. □

Remark 2.3. Assume that we are applying the above proof to fill an individual $k \in K_{m,l}$. In the course of the construction we need to contract the loops that are based at the vertices of k by path homotopies that pass through loops that are short. Moreover, two vertices with the highest order are never used. Therefore to fill k only the absence of "short" geodesic loops based at all vertices of k but the two with the highest indices is required.

Here is the informal description of the above proof when $m = 2$. We would like to show that in the case when the length of a shortest geodesic loop is $> 12l$ we can fill $K_{2,l}$. Let us recall that $K_{2,l}$ is the space of immersed 1-skeleta of simplices of dimension 3, such that the length of each edge does not exceed l. We would like to extend each of the immersions to a 3-simplex, so that these extensions are continuous with respect to the original graph, and so that they are coherent. The last requirement means that if we consider a restriction of the immersion to a subcomplex, which is a 1-skeleton of a 2-face, it will agree with the earlier extension. Thus, the procedure is inductive and we will begin by filling $K_{1,4l}$. In this case, if

$k \in K_{1,4l}$ then its total length is at most $12l$. However, since the length of a shortest geodesic loop is greater than $12l$ each such curve is contractible via the BCSP as a loop to any of the vertices of k. Let us, however, choose to contract to vertices with the smallest index. Here we use Lemma 1.3 to construct the required path homotopy between one side of k and its two other sides. Next let us consider $k^2_{v_0,v_1,v_2,v_3} \in K_{2,l}$. Note that, as we know how to extend each $k^1_{v_0,,,,\hat{v}_i,\ldots,v_3}$ we, as the result of these extensions and a natural identifications of the four 2-discs have a map of the 2-sphere into M^n naturally assigned to $k^2_{v_0,v_1,v_2,v_3}$. Let us denote this (map of the) 2-sphere by S^2_0. We would like to construct a map of a 3-disc that fills 2-sphere. It will be constructed as a 1-parameter family of 2-spheres S^2_τ that begins with the original sphere obtained in the previous step S^2_0 and ends with a point. Here is how we will construct S^2_τ. Let us begin by constructing a 1-parameter family of graphs $k^2_\tau, \tau \in [0,2]$. We will let $k^2_0 = k^2_{v_0,v_1,v_2,v_3}$. Next, by Lemma 1.3 there is a homotopy that moves edges $[v_i, v_{(i+1) \bmod 3}], 0 \leq i \leq 2$ to $[v_i, v_3] + [v_3, v_{(i+1) \bmod 3}]$. This path homotopy passes through curves of length $\leq 4l$. Let us denote the curves in these homotopies by e^i_τ. So, we will continuously replace edges $e^i = [v_i, v_{(i+1) \bmod 3}]$ by the edges e^i_τ respectively, thus forming k^2_τ. Let us now consider all the subcomplexes of k^2_τ that correspond to elements of $K_{1,4l}$. By the previous step they can all be "filled" by 2-discs. Gluing these discs together results in a 2-sphere S^2_τ. When $\tau = 1$ this sphere will degenerate to (a map of the 2-sphere into) a tree with root at v_3 and three edges connecting v_3 with v_0, v_1, v_2. This sphere fills a degenerate element of $K_{2,2l}$ where all edges are mapped into this tree. This element can be contracted over itself to the constant map of the complete graph into v_3. Filling the resulting homotopy by spheres we obtain a family of 2-spheres $S^2_\tau, \tau \in [1,2]$ that connects S^2_1 and $S^2_2 = \{v_3\}$. Thus, we obtain a 3-disc that "fills" any $k^2_{v_0,\ldots,v_3} \in K_{2,l}$.

Proof of Theorem 0.1. Let $p \in M^n$ and $\varepsilon > 0$ be given. Let $\tilde{\varepsilon} = (\frac{\varepsilon}{6 \cdot 4^n k(n-1)})^{n-1}$, where $k(n-1) = 27^{n-1} n!$.

Then by Lemma 1.1 there exists a set $A(\tilde{\varepsilon})$ satisfying the following: at each t^* in the complement of $A(\tilde{\varepsilon})$ in $(0, \infty)$, there exists a geodesic sphere $\tilde{S}_{t^*}(p)$ of radius t^* centered at the point p such that for any $\delta > 0$ there exists a smooth submanifold $Z^\delta_{\tilde{\varepsilon}}$ that has $vol_{n-1}(Z^\delta_{\tilde{\varepsilon}}) < \tilde{\varepsilon}$ and that is within distance δ from the sphere. Moreover, $Z^\delta_{\tilde{\varepsilon}}$ does not bound in $M^n - p$.

Take $t^* > T = 4^n k(n-1) \tilde{\varepsilon}^{\frac{1}{n-1}} = \frac{\varepsilon}{6}$.

Let $i_S = \inf_{q \in \tilde{S}_{t^*}(p)} inj_q M^n$, where $inj_q M^n$ is the injectivity radius of M^n at q. We will consider $\delta < \frac{i_S}{100^n}$ and we will let it eventually go to 0.

Let $X = L^\infty(Z^\delta_{\tilde\varepsilon})$. By Definition 0.4, $Z^\delta_{\tilde\varepsilon}$ isometrically embedds into X and for every $\delta > 0$ there exists a singular chain c in the $(FillRad(Z^\delta_{\tilde\varepsilon}) + \delta)$-neighborhood of $Z^\delta_{\tilde\varepsilon}$ in X, such that $Z^\delta_{\tilde\varepsilon}$ bounds c. Without loss of generality we can take c to be an n-dimensional polyhedron, (see Statement 1.2.C on page 10 in [G].) Also, recall that the Kuratowski embedding of $Z^\delta_{\tilde\varepsilon}$ in X is an isometry.

Assume that lengths of all nontrivial geodesic loops in M^n based at the points of $Z^\delta_{\tilde\varepsilon}$ are greater than

$$\varepsilon = 6 \cdot 4^{n-1} k(n-1)\tilde\varepsilon^{\frac{1}{n-1}} > 6 \cdot 4^{n-1} k(n-1) vol_{n-1}(Z^\delta_{\tilde\varepsilon})^{\frac{1}{n-1}} \ (*).$$

Gromov's Theorem 0.5 further implies that $\varepsilon > 6 \cdot 4^{n-1}\ FillRad(Z^\delta_{\tilde\varepsilon})$.

We will construct an n-chain in $M^n - p$ that has $Z^\delta_{\tilde\varepsilon}$ as its boundary, thus obtaining a contradiction. This chain will be constructed in two steps.

During the first step we will construct a simplicial map $f : Z^\delta_{\tilde\varepsilon} \longrightarrow M^n - p$ the image of which will lie in a small neighbourhood of $\tilde S_{t^*}(p)$. This neighborhood is so small that $f_*([Z^\delta_{\tilde\varepsilon}]) = i_*([Z^\delta_{\tilde\varepsilon}])$, where $[Z^\delta_{\tilde\varepsilon}]$ is the fundamental homology class of $Z^\delta_{\tilde\varepsilon}$ and i_* is the inclusion homomorphism from $H_{n-1}(Z^\delta_{\tilde\varepsilon})$ into $H_{n-1}(M^n - p)$. During the second step we will extend the constructed map from $Z^\delta_{\tilde\varepsilon} = \partial c$ into $M^n - p$ to a map $\tau : c \longrightarrow M^n - p$, which would imply $Z^\delta_{\tilde\varepsilon}$ bounds in $M^n - p$.

We will begin by triangulating $Z^\delta_{\tilde\varepsilon}$ into simplices of diameter at most δ. We will define the map on the 0-skeleton, by mapping each vertex $v_i \in Z^\delta_{\tilde\varepsilon}$ to a closest point $w_i \in \tilde S_{t^*}(p)$. Next, we will map 1-edges. The boundary of each edge $[v_i, v_j]$ consists of two vertices v_i, v_j. The image of each such vertex was established during the previous step. The distance between them is at most 3δ. We will connect corresponding images by a minimal geodesic segment in M^n.

To extend to the 2-skeleton we proceed as follows. Consider a 2-simplex $[v_{i_1}, v_{i_2}, v_{i_3}]$. Its boundary is mapped to a closed curve of length at most 9δ. As this curve is much smaller than the injectivity radius of M^n on $\tilde S_{t^*}(p)$, it is contractible as a loop without the length increase to any of the vertices. The 2-simplex is mapped to the disc generated by the homotopy of contracting the loop.

Now suppose we have extended to the $(i-1)$-skeleton of $Z^\delta_{\tilde\varepsilon}$. We will next extend to the i-skeleton, which is accomplished by filling the image of its 1-skeleton as described in Lemma 2.2.

First enumerate all the vertices of the chosen triangulation of $Z^\delta_{\tilde\varepsilon}$ by increasing successive integers. Next apply Lemma 2.2 to previously constructed images of 1-skeleta of all i-dimensional simplices of $Z^\delta_{\tilde\varepsilon}$.

In order to do that we need to renumber vertices of every i-dimensional simplex of c by numbers $0, 1, ..., i$. To do this we take the numbering of all of the vertices of c and then renumbering the vertices of every i-simplex by $\{0, 1, ..., n\}$ in the increasing order. Now apply Lemma 2.2. This will be possible, because our choice of δ insures that there is no geodesic loops based at points of $\tilde{S}_{t^*}(p)$ of length smaller than $9 \cdot 4^{i-1}\delta$ for all $i \leq n-1$.

Note that after we finish the construction of the map, the image of each $(n-1)$-simplex will lie in the $18 \cdot 4^{n-3}\delta$-neighborhood of one of w_is. This also follows from Lemma 2.2.

The second step of extending the map to c is almost identical to the first one. Let us triangulate c into simplices of the diameter at most $\delta > 0$, that will eventually approach zero. The extension will be done by induction on the skeletal dimension of c.

Each of the 0-simplices of c, (excluding those in $Z^{\delta}_{\tilde{\varepsilon}}$), will be mapped to one of the closest vertices in $Z^{\delta}_{\tilde{\varepsilon}}$ (in the metric of the ambient space X). It will then be mapped to a closest point in $\tilde{S}_{t^*}(p)$.

Each of the 1-simplices of $c \setminus Z^{\delta}_{\tilde{\varepsilon}}$ will be mapped into a minimizing geodesic in M^n between the images w_{i_1}, w_{i_2} of the vertices $\tilde{v}_{i_1}, \tilde{v}_{i_2}$ respectively of this simplex. Then by applying the triangle inequality twice $dist(w_{i_1}, w_{i_2}) \leq dist(w_{i_1}, v_{i_1}) + dist(v_{i_1}, v_{i_2}) + dist(v_{i_2}, w_{i_2}) \leq 2\delta + dist(v_{i_1}, \tilde{v}_{i_1}) + dist(\tilde{v}_{i_1}, v_{i_2}) + dist(\tilde{v}_{i_1}, \tilde{v}_{i_2}) \leq 2FillRad + \tilde{\delta}$, where $\tilde{\tilde{\delta}} = 3\delta$. Thus, the length of the image of each 1-simplex of $c \setminus Z_{\tilde{\varepsilon}}$ is at most $2FillRad(Z^{\delta}_{\tilde{\varepsilon}}) + \tilde{\delta}$, where $\tilde{\delta}$ can be made arbitrarily small by selecting a sufficiently small δ and by refining the chosen triangulation of c.

Now suppose we have extended the map to the $(i-1)$-skeleton of c. The extension to the i-skeleton is likewise accomplished by an application of Lemma 2.2.

For instance, the desired extension of τ to any closed n-dimensional simplex of $c \setminus Z^{\delta}_{\tilde{\varepsilon}}$ is accomplished by filling the image of its 1-skeleton described in Lemma 2.2. One, however, has to take care to fill every k-simplex in the same way, when it is considered as a k-face of different n-simplices. To achieve this we just need to number them identically by $0, 1, ..., k$ every time when we fill this k-simplex.

Let us begin by enumerating all the vertices of the chosen triangulation of c by increasing successive integers. We will apply Lemma 2.2 to previously constructed images of 1-skeleta of all n-dimensional simplices of c. In order to do that we need to number vertices of every n-dimensional simplex of c by numbers $0, 1, ..., n$. To do this we take the numbering of all of the vertices of c and then renumbering the vertices of every n-simplex by $\{0, 1, ..., n\}$ in the increasing order with respect to their number in the list of all vertices of c. Next apply Lemma 2.2 using $(*)$ and taking $\tilde{\delta}$ to be sufficiently small. As the result, we obtain an extension to the n-skeleton of c. Note that the resulting image does not pass through the point p, because the distance between p and the image is sufficiently large. Thus, we have reached a contradiction refuting the assumption in $(*)$, and so there must be a geodesic loop of length at most ε based at some point of $\tilde{S}_{t^*}(p)$.

□

Next we will present a proof of Theorem 0.3, which is very similar to that of Theorem 0.1.

Proof of Theorem 0.3. Let $\varepsilon > 0$ be given. Without any loss of generality we can assume that $\varepsilon \leq 1$. Let $\tilde{\varepsilon} = (\frac{\varepsilon}{12 \cdot 4^{n-2} 27^{n-1} n!})^{n-1}$. First we will show that there exists $T > 0$, such that for all $t > T$ there exists a $\tilde{t}$, such that $|\tilde{t} - t| < \frac{\tilde{\varepsilon}}{100^n}$ and $vol_{n-1}(\tilde{S}_{\tilde{t}}(p)) < \frac{\tilde{\varepsilon}}{16}$. This is easily seen from the fact that the number of intervals of size $\frac{2\tilde{\varepsilon}}{100^n}$, such that the measure of every geodesic sphere with the radius in one of these intervals is greater than $\frac{\tilde{\varepsilon}}{16}$ is finite. Let $\{(a_i, b_i), i = 1, ..., N\}$ be the collection of all such intervals and take $T = \max\{b_i, i = 1, ..., N\}$.

Now suppose $t > T$ is given. Then there exists $\tilde{t}$ satisfying the above property. Moreover, by Lemma 1.1 for every δ that is small enough there exists a submanifold $Z^{\delta}_{\tilde{\varepsilon}}$ such that $vol_{n-1}(Z^{\delta}_{\tilde{\varepsilon}}) < \tilde{\varepsilon}$ within distance δ from $\tilde{S}_{\tilde{t}}(p)$ that does not bound in $M^n - p$.

The proof of Theorem 0.3 now goes by contradiction. Suppose there is no geodesic loop of length smaller than ε based at some point of $\tilde{S}_t(p)$. We will show that then we can construct an n-dimensional chain on $M^n - p$ that has $Z^{\delta}_{\tilde{\varepsilon}}$ as its boundary, thus reaching a contradiction. The construction of this chain will be done in three steps. In the first step we construct a simplicial map $f : Z^{\delta}_{\tilde{\varepsilon}} \longrightarrow M^n - p$.

During the second step we extend the map f from $Z^{\delta}_{\tilde{\varepsilon}}$ to the n-dimensional chain c that fills $Z^{\delta}_{\tilde{\varepsilon}}$ in $L^{\infty}(Z^{\delta}_{\tilde{\varepsilon}})$.

Finally we show that the image of the fundamental class $[Z^{\delta}_{\tilde{\varepsilon}}]$ under the inclusion homomorphism to $H_{n-1}(M^n - p)$ and $f_*([Z^{\delta}_{\tilde{\varepsilon}}])$ are equal.

The first and the second steps are similar to the constructions in the proof of Theorem 0.1.

Step 1. Triangulate $Z^{\delta}_{\tilde{\varepsilon}}$ into simplices of size at most δ. We will first define a map on 0-skeleton of $Z^{\delta}_{\tilde{\varepsilon}}$. Each vertex $\tilde{v}_i \in Z^{\delta}_{\tilde{\varepsilon}}$ is mapped to a closest point in $\tilde{S}_t(p)$, which is located within distance $\delta + \frac{\tilde{\varepsilon}}{100^n}$. Next we define a map on the 1-skeleton. Let $[\tilde{v}_{i-1}, \tilde{v}_{i-2}]$ be an arbitrary 1-simplex. Its boundary is mapped to a pair of points within distance $\frac{2\tilde{\varepsilon}}{100^n} + 3\delta$. We connect them by a minimal geodesic segment. We can now, by induction, extend to an i-skeleton for $i \leq n-1$ by applying Lemma 2.2, since, by our assumption there are no short loops based at $\tilde{S}_t(p)$.

Step 2. Next let c be a chain that fills $Z^\delta_{\tilde{\varepsilon}}$ in the $FillRadZ^\delta_{\tilde{\varepsilon}}+\delta$-neighborhood of $Z^\delta_{\tilde{\varepsilon}}$ in the $L^\infty(Z^\delta_{\tilde{\varepsilon}})$. Let us triangulate it, so that the diameter of each simplex is smaller than δ. We extend to the 0 skeleton by mapping each vertex $\tilde{w}_i$ to a closest vertex $\tilde{v}_i$ in $Z^\delta_{\tilde{\varepsilon}}$ and, subsequently, mapping it to a closest point v_i in $\tilde{S}_t(p)$. To extend to the 1-skeleton, consider an arbitrary edge $[\tilde{w}_{i_1}, \tilde{w}_{i_2}]$. Its boundary is mapped to a pair of points, distance between them being at most $2FillRadZ^\delta_{\tilde{\varepsilon}} + 3\delta + \frac{2\tilde{\varepsilon}}{100^n}$, which is at most $27^{n-1}n!\tilde{\varepsilon}^{\frac{1}{n-1}} + 3\delta + \frac{2\tilde{\varepsilon}}{100^n}$. Next we once again apply Lemma 2.2 to extend to the remaining skeleta. This extension is possible, because of our assumption on the length of short geodesic loops with vertices at the points of $\tilde{S}_t(p)$.

Step 3. This is the only step that was skipped in the constructions of the proof of the previous theorem. We will now construct an n dimensional polyhedron that has as its boundary $Z^\delta_{\tilde{\varepsilon}}$ and $f(Z^\delta_{\tilde{\varepsilon}})$. This step was almost trivial in the proof of Theorem 0.1. There we have chosen $\tilde{S}_{t^*}(p)$ and then were able to choose $Z^\delta_{\tilde{\varepsilon}}$ that was $\frac{i_S}{100^n}$-close to $\tilde{S}_{t^*}(p)$, (recall that $i_S = \min_{q \in \tilde{S}_{t^*}(p)} inj_q M^n$). However, in the present situation we have no means to ensure that $dist(\tilde{S}_t(p), Z^\delta_{\tilde{\varepsilon}})$ is sufficiently small compared to the $\min_{q \in \tilde{S}_t(p)} inj_q M^n$.

The procedure will be an induction on the dimension of the skeleta that will go as follows: for each pair of simplices of dimension $i-1$, i.e. $\tilde{\sigma}^{i-1}_j$ of Z^δ_ε and its image $\sigma^{i-1}_j = f(\tilde{\sigma}^{i-1}_j)$ we will construct a cell τ^i_j of the dimension i that "connects" them.

It will be done by the filling technique similar to the one described in Lemma 2.2.

We will begin with the 1-skeleton. The 1-skeleton will consist of minimal geodesic segments connecting a vertex of $Z^\delta_{\tilde{\varepsilon}}$ with its image under the map f. Next we will construct the 2-cells. Consider the closed curves composed of a simplex $[\tilde{v}_{i_1}, \tilde{v}_{i_2}]$, its image $[v_{i_1}, v_{i_2}]$ and two minimal geodesic segments joining the corresponding vertices. This closed curve can be contracted as a loop to either one of the vertices v_{i_1} or v_{i_2} because by our assumption there are no geodesic loops of sufficiently small length based at the points of $\tilde{S}_t(p)$. This homotopy generated a 2-cell. These 2-cells will comprise the 2-skeleton of the polyhedron we are constructing.

To construct an i-skeleton of the polyhedron we use the procedure similar to the one described in Lemma 2.2. Let us consider "prisms" P_j^{i-1} that consist of the two simplices $\tilde{\sigma}_j^{i-1}$ of $Z_{\tilde{\varepsilon}}^{\delta}$ and its image σ_j^{i-1} together with the "walls", that is cells of dimension $i-1$ that connect simplices of dimension $i-2$ in the boundaries of $\tilde{\sigma}_j^{i-1}$ and σ_j^{i-1}. We will construct a "filling" of this "prism", thus obtaining cells of dimension i. The filling is obtained by first, regarding the simplex $\tilde{\sigma}_j^{i-1}$ as a point. (A formal argument, similar to the Remark on pg. 504 in [R3] allows one to treat it in such a way. In this argument, one essentially constructs a homotopy between P_j^{i-1} and a polyhedron in which $\tilde{\sigma}_j^{i-1}$ is replaced by a point. It is done by gradually decreasing the size of $\tilde{\sigma}_j^{i-1}$). One then enumerates vertices of $Z_{\tilde{\varepsilon}}^{\delta}$ and its image, starting from the vertices in the image and then applies Lemma 2.2. The main idea of this lemma is that one can fill the 1-skeleta by discs in the absence of short geodesic loops. In our case, the absence of short geodesic loops based at points of the sphere $\tilde{S}_t(p)$ is sufficient, (see Remark 2,3).

□

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