Non-Glimm-Effros equivalence relations at second projective level

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Abstract

A model is presented, in which the Σ^1_2 equivalence relation $x \mathsf{C} y$ iff $\mathrm{L}[x] = \mathrm{L}[y]$ on reals does not admit a reasonable form of the Glimm – Effros theorem. The model is a kind of iterated Sacks generic extension of the constructible model, but with an "ill"founded "length" of the iteration. In another model of this type, we get an example of a Π^1_2 non–Glimm–Effros equivalence relation on reals.

As a more elementary applications of the technique of "ill"founded Sacks iterations, we obtain a simple cardinal invariant which distinguishes product and iterated Sacks extensions, and a model in which every nonconstructible real codes a collapse of a given cardinal $\kappa \geq \aleph_2^{\mathrm{old}}$ to $\aleph_1^{\mathrm{old}}$.

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Introduction

Theorems of the following type are quite usual in mathematics:

every object in some domain is either "regular" in some specified sense, or, if it is "singular" then it includes a certain distinguished "singular" object.

For instance by an old Souslin theorem a Borel, or, more generally, $\mathbf{\Sigma}^1_1$ set of reals either is countable (= "regular") or contains a perfect subset (= the distinguished type of uncountable sets).

The question: how more complicated sets behave with respect to this particular "dichotomy", was completely solved in the early era of forcing: first, a $\mathbf{\Sigma}^1_2$ set is either of cardinality $\leq \aleph_1$ or contains a perfect subset; second, nothing like this can be proved for $\mathbf{\Pi}^1_2$ sets unless we use special strong axioms (like the axiom of determinacy) or work in special "regular" models (for example the Solovay model).

It is a related but more general and much more difficult problem to investigate, in this manner, the number of equivalence classes of an equivalence relation on reals. This problem can be traced (at least) back to the origins of descriptive set theory. [1]

It was in 1970'es that Silver [17] proved that a $\mathbf{\Pi}^1_1$ equivalence relation on reals either has countably many equivalence classes or admits a perfect set of pairwise inequivalent reals. (The Souslin theorem is an easy corollary: indeed if X is a $\mathbf{\Sigma}^1_1$ set of reals then the equivalence E defined as equality on X and $x \,\mathsf{E}\, y$ for all $x, y \notin X$, is $\mathbf{\Pi}^1_1$.)

Moreover, it was recently recognized that equivalence relations allow a different type of investigation, related to enumeration of classes by *sets of ordinals* (e. g. reals) rather than ordinals themselves. Harrington, Kechris, and Louveau [5] proved that each Borel equivalence relation E on reals satisfies one and only one of the following conditions:

(I) E admits a Borel enumeration of the equivalence classes by reals.

(II) E continuously embeds E_0, the Vitali equivalence. [2]

Some notation. An *enumeration of classes* for an equivalence E on reals is a function U defined on reals and satisfying $x \,\mathsf{E}\, y$ iff $U(x) = U(y)$ for all x, y. E_0 is the *Vitali equivalence* on the *Cantor space* $\mathcal{D} = 2^\omega$, defined by: $x \,\mathsf{E}_0\, y$ iff $x(n) = y(n)$ for all but finite $n \in \omega$. An *embedding* of E_0 into E is a $1-1$ function $U : \mathcal{D} \longrightarrow$ reals such that $x \,\mathsf{E}_0\, y \longleftrightarrow U(x) \,\mathsf{E}\, U(y)$ for all $x, y \in \mathcal{D}$.

[1] Luzin noted in [15], section 64, that, although it looks natural that the Vitali equivalence on reals has continuum-many equivalence classes, a concrete enumeration of the equivalence classes by reals had not been known. (If the axiom of choice is not assumed, the Vitali equivalence can have strictly more equivalence classes than the cardinal of continuum, see Kanovei [10].) Even earlier Sierpinski [16] demonstrated that if the set of all Vitali classes can be *linearly* ordered then there exists a nonmeasurable set of reals, having approximately the same projective class as the linear order, provided it is projective. We shall see that the Vitali equivalence in general plays a distinguished role in modern investigations.

[2] Relations satisfying (I) are called *smooth*. Take notice that E_0 is *not* smooth.

The dichotomy of type (I) vs. (II) was called the *Glimm – Effros* dichotomy in [5]. (We refer the reader to [5] as the basic sourse of information on the history of this type of theorems, to Hjorth and Kechris [9] and Kechris [13] as a review of further development, to all the three mentioned in concern of applications and related topics, and to Kechris [14] as a broad reference in the subject.)

Theorems of this type, but with a weaker condition (I) [3] are known for $\mathbf{\Sigma}^1_1$ equivalence relations, provided either the universe satisfies the sharps hypothesis (Hjorth and Kechris [9]) or each real belongs to a generic extension of L (Kanovei [12]). [4]

However we prove that classes Σ^1_2 and Π^1_2 contain counterexamples, equivalence relations which do not admit a theorem of the Glimm–Effros type in **ZFC**, at least in the domain of real–ordinal definable (R-OD, in brief) enumerations and embeddings.

Theorem 1 *It is consistent with* **ZFC** *that the* Σ^1_2 *equivalence relation* C, *defined on reals by* $x \mathsf{C} y$ *iff* $\mathrm{L}[x] = \mathrm{L}[y]$, *has* $\mathfrak{c}$*-many equivalence classes, and*:

- *neither has a R-OD enumeration of the equivalence classes by sets of ordinals,*
- *nor admits a R-OD pairwise* C*-inequivalent set of cardinality* $\mathfrak{c}$,

and in addition either of the following two cardinal equalities can be modeled in the universe: $\mathfrak{c} = \aleph_1 = \aleph_1^{\mathrm{L}}$ *or* $\mathfrak{c} = \aleph_2 = \aleph_2^{\mathrm{L}}$.

Theorem 2 *It is consistent with* **ZFC** *that some* Π^1_2 *equivalence relation on reals has* $\mathfrak{c}$*-many equivalence classes, and*:

- *neither has a R-OD enumeration of the equivalence classes by sets of ordinals,*
- *nor embeds* E_0, *the Vitali equivalence, via an R-OD embedding,*

and in addition either of the following two cardinal equalities can be modeled in the universe: $\mathfrak{c} = \aleph_1 = \aleph_1^{\mathrm{L}}$ *or* $\mathfrak{c} = \aleph_2 = \aleph_2^{\mathrm{L}}$.

Remarks

The "nor" assertion of Theorem 1 implies the "nor" assertion of Theorem 2, because obviously there exists a perfect set of pairwise E_0-inequivalent points. It is not clear whether one can strengthen the "nor" assertion of Theorem 2 to the form of Theorem 1.

It makes no sense to look for non-R-OD enumerations, assuming we work in **ZFC** (with Choice). Equally it would be silly to look for enumerations by *sets of sets* of ordinals (the next level) because each equivalence class is an object of this type.

[3] $\mathbf{\Delta}^{\mathrm{HC}}_1$ enumeration of the equivalence classes by countable (of any length $< \omega_1$) binary sequences.

[4] Friedman [2], Hjorth [7, 8], Kanovei [11] obtained partial results of this type for $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, and more complicated relations, and different relevant theorems on equivalence relations, which we do not intend to discuss in detail.

The theorems are close to a possible optimal counterexample. Indeed Hjorth [8] proved that every $\mathbf{\Delta}^1_2$ relation (more generally, a relation which is both ω_1-Šouslin and ω_1-co-Souslin), which satisfies the property that the equivalence of the $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$ definitions is preserved in Cohen generic extensions, admits a Glimm – Effros theorem, with an enumeration of classes by ω_1-long binary sequences in (I).

It is a very interesting problem at the moment to figure out whether <u>all</u> $\mathbf{\Delta}^1_2$ relations admit a reasonable Glimm – Effros dichotomy. (Since the models we construct for the theorems are very special, it would be reasonable to expect that even classes $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$ admit a Glimm – Effros dichotomy under certain reasonably weak assumptions.)

The models

The models for theorems 1 and 2 we propose, are iterated Sacks extensions of the constructible model, having a nonwellordered set as the "length" of iteration. Therefore, this is not a kind of iterated generic models in the usual setting (see Baumgartner and Laver [1] on iterations of Sacks forcing), where the length of the iteration is, by definition, an ordinal. We use "ill"ordered Sacks iterations to prove the theorems.

An idea as how to define iterated Sacks generic extensions, having inverse ordinals as the "length" of iteration, was developed by Groszek [3]. We make different technical arrangements to obtain "ill"ordered and even "ill"founded Sacks iterations. (The model for Theorem 2 is an example of an "ill"founded and not linear iteration; a model for Theorem 1 can be obtained in two ways: as a linear "ill"ordered Sacks iteration, and as a nonlinear wellfounded Sacks iteration; the latter version is equivalent to the usual countable support iteration of Sacks $\times$ Sacks forcing, of length ω_1 or ω_2.)

Let $\mathbf{I}$ be a partially ordered set in $\mathfrak{M}$, the ground model, – the intended "length" of the iteration. A typical forcing condition is, in $\mathfrak{M}$, a set $X \subseteq \mathcal{D}^\zeta$, where $\zeta \subseteq \mathbf{I}$ is countable, of the form $X = H\,"\mathcal{D}^\zeta$, where H is a $1-1$ continuous function such that

$$x \restriction \xi = y \restriction \xi \;\longleftrightarrow\; H(x) \restriction \xi = H(y) \restriction \xi$$

for all $x,\, y \in \mathcal{D}^\zeta$ and any initial segment ξ of ζ. Section 1 contains the definition and several basic lemmas related to the conditions.

Section 2 shows how one splits the forcing conditions into smaller ones, and gathers forcing conditions via a kind of fusion technique, common for the Sacks forcing.

Section 3 ends the study of the forcing notion by a theorem which describes the behaviour of continuous functions mapping the conditions into reals.

Sections 4 and 5 define and study the extensions. We prove that the forcing notion associated with a partially ordered set $\mathbf{I}$ in the ground model $\mathfrak{M}$ produces a generic model of the form $\mathfrak{N} = \mathfrak{M}[\langle \mathbf{a}_i : i \in \mathbf{I}\rangle]$, where each $\mathbf{a}_i \in \mathcal{D}$ is Sacks generic over the model $\mathfrak{M}[\langle \mathbf{a}_j : j < i\rangle]$, – the property which witnesses that $\mathfrak{N}$ is a kind of iterated Sacks extension of $\mathfrak{M}$ despite $\mathbf{I}$ can be not wellordered.

We prove a cardinal preservation theorem, and a very important theorem which says that each real in $\mathfrak{N}$ can be obtained applying a continuous function coded in $\mathfrak{M}$ to a countable sequence of generic reals. This theorem allows to convert properties of continuous functions in the ground model to properties of reals in the extension.

In particular it occurs (Section 5) that, if every initial segment of $\mathbf{I}$ belongs to $\mathfrak{M}$ then the degrees of $\mathfrak{M}$-constructibility of reals in the extension are in $1-1$ correspondence with the countably cofinal initial segments of $\mathbf{I}$.

Section 6 presents the proof of Theorem 1. The proof utilizes a particular property of the sets $\mathbf{I} = (\omega_1 \text{ or } \omega_2) \times \mathbb{Z}$, where $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ – the integers: each copy of $\mathbb{Z}$ admits nontrivial order isomorphisms – shiftings. This does not allow a real in the extension to "know" definitely the exact place, say $\langle \alpha, z \rangle \in \omega_1 \times \mathbb{Z}$, of its degree of constructibility. One more possibility is $\mathbf{I} = (\omega_1 \text{ or } \omega_2) \times (\text{unordered } \{0,1\})$, which is equivalent to the usual iteration of the forcing Sacks $\times$ Sacks, of length ω_1 or ω_2. We do not know whether an iteration of the Sacks forcing of an ordinal length can prove Theorem 1.

A modification, $\mathbf{I} = (\omega_1 \text{ or } \omega_2) \times (\mathbb{Z} \times \{0,1\})$, is used to prove Theorem 2. We do not know how to prove this theorem not using ill-founded Sacks iterations.

Two more applications

Two easier applications of the technique of Sacks iterations are obtained in Section 5.

Theorem 3 *Let $\kappa > \aleph_1^{\mathfrak{M}}$ be a cardinal in a countable model $\mathfrak{M}$. There exists a generic extension $\mathfrak{N}$ of $\mathfrak{M}$ in which $\aleph_1^{\mathfrak{M}}$ is still a cardinal, but every nonconstructible real collapses κ to $\aleph_1^{\mathfrak{M}}$.*

(Clearly the result is impossible for $\aleph_0$ instead of $\aleph_1$ since a collapse of an uncountable cardinal to $\aleph_0$ provides nonconstructible reals which do not collapse cardinals.)

Of course different forcing notions produce reals that code much more sophisticated things, but the model for Theorem 3 is somewhat exceptional because first it is really simple (we use inverse κ Sacks iteration and exploit the known phenomena that each next Sacks real "knows" the previous steps, which compels *every* nonconstructible real to code the collapse) and does not involve a complicated coding technique.

The second application is devoted to cardinal invariants which distinguish "long" product and iterated Sacks extensions. Steprāns gave some invariants in a talk on this matter at Haifa Logic Colloquium (August 1995). We present a simpler invariant.

Every collection $\mathcal{F}$ of continuous functions f : reals $\longrightarrow$ reals determines a partial order $\le_{\mathcal{F}}$ on reals as follows: $x \le_{\mathcal{F}} y$ iff $x = f_1(f_2(\dots f_n(y)\dots))$ for some functions $f_1, f_2, ..., f_n \in \mathcal{F}$. Let $\mathfrak{l}$ (the *linear order cardinal*) denote the least cardinality of a family $\mathcal{F}$ such that $\le_{\mathcal{F}}$ linearly orders the reals. Obviously $\mathfrak{l} \le \mathfrak{c}$.

Theorem 4 *Let $\mathfrak{M}$ be a countable transitive model of* **ZFC**. *Then $\mathfrak{l} = \mathfrak{c}$ in each countable support product Sacks extension of $\mathfrak{M}$ but $\mathfrak{l} = \operatorname{card} \mathfrak{c}^{\mathfrak{M}}$ in each countable support iterated Sacks extension of $\mathfrak{M}$.*

1 The forcing

Let **CPO** be the class of all countable (including finite) partially ordered sets. Greek letters ξ, η, ζ, ϑ will denote sets in **CPO**. Characters i, j are used to denote elements of sets in **CPO**. For any $\zeta \in \mathbf{CPO}$, $\mathbf{IS}_\zeta$ is the collection of all initial segments of ζ. For instance $\emptyset$ and ζ itself belong to $\mathbf{IS}_\zeta$.

We shall usually have fixed a "basic" p. o. set $\zeta \in \mathbf{CPO}$, so that all other p. o. sets actually involved in the reasoning are subsets of ζ and even members of $\mathbf{IS}_\zeta$. In this case, for any $i \in \zeta$ we shall consider special initial segments $[<i] = \{j \in \zeta : j < i\}$ and $[\not\geq i] = \{j \in \zeta : j \not\geq i\}$, and $[\leq i]$, $[\not> i]$ defined in the same way.

As usual, $\mathcal{N} = \omega^\omega$ is the *Baire space*; points of $\mathcal{N}$ will be called *reals*. .

$\mathcal{D} = 2^\omega$ is the *Cantor space*. For any countable set ξ, $\mathcal{D}^\xi$ is the product of ξ-many copies of $\mathcal{D}$ with the product topology. Then every $\mathcal{D}^\xi$ is a compact space, homeomorphic to $\mathcal{D}$ itself unless $\xi = \emptyset$.

Assume that $\eta \subseteq \xi$. If $x \in \mathcal{D}^\xi$ then let $x \restriction \eta \in \mathcal{D}^\eta$ denote the usual restriction. If $X \subseteq \mathcal{D}^\xi$ then let $X \restriction \eta = \{x \restriction \eta : x \in X\}$.

But if $Y \subseteq \mathcal{D}^\eta$ then we set $Y \restriction^{-1} \xi = \{x \in \mathcal{D}^\xi : x \restriction \eta \in Y\}$.

To save space, let $X \restriction_{<i}$ mean $X \restriction [<i]$, $X \restriction_{\not\geq i}$ mean $X \restriction [\not\geq i]$, etc.

Definition [The forcing]
For any set $\zeta \in \mathbf{CPO}$, Perf_ζ is the collection of all sets $X \subseteq \mathcal{D}^\zeta$ such that there exists a homeomorphism $H : \mathcal{D}^\zeta$ onto X satisfying

$$x_0 \restriction \xi = x_1 \restriction \xi \longleftrightarrow H(x_0) \restriction \xi = H(x_1) \restriction \xi \quad \text{— for all } x_0, x_1 \in \operatorname{dom} H \text{ and } \xi \in \mathbf{IS}_\zeta .$$

Homeomorphisms H satisfying this requirement will be called *projection–keeping*. So, sets in Perf_ζ are images of $\mathcal{D}^\zeta$ via projection–keeping homeomorphisms. □

If $H : \mathcal{D}^\zeta$ onto X is a projection–keeping homeomorphism then we define, for any $\xi \in \mathbf{IS}_\zeta$, an associated projection–keeping homeomorphism $H_\xi : \mathcal{D}^\xi$ onto $X \restriction \xi$ by $H_\xi(x \restriction \xi) = H(x) \restriction \xi$ for all $x \in \mathcal{D}^\zeta$.

Proposition 5 *Every set $X \in \mathrm{Perf}_\zeta$ is closed and satisfies the following properties:*

P-1. *If $i \in \zeta$ and $z \in X \restriction_{<i}$ then the set $D_{Xz}(i) = \{x(i) : x \in X \ \&\ z = x \restriction_{<i}\}$ is a perfect set in $\mathcal{D}$.*

P-2. *If $\xi \in \mathbf{IS}_\zeta$ and a set $X' \subseteq X$ is open in X (in the relative topology) then the projection $X' \restriction \xi$ is open in $X \restriction \xi$.* [5]

P-3. *If $\xi, \eta \in \mathbf{IS}_\zeta$, $x \in X \restriction \xi$, $y \in X \restriction \eta$, and $x \restriction (\xi \cap \eta) = y \restriction (\xi \cap \eta)$, then $x \cup y \in X \restriction (\xi \cup \eta)$.*

[5] In other words, it is required that the projection from $X \restriction_{\leq i}$ to $X \restriction_{<i}$ is an open map.

This proposition can be taken as the base for an independent definition of the forcing; however it it not true that the requirements P-1, P-2, P-3 fully characterize Perf_ζ.

Proof Obviously $\mathfrak{D}^\zeta$ satisfies the requirements. On the other hand, one easily proves that projection–keeping homeomorphisms keep the requirements. □

Let us prove several simple lemmas on forcing conditions.

The following lemma shows how P-3 works.

Lemma 6 *Suppose that* $X \in \mathrm{Perf}_\zeta$, $\xi, \eta \in \mathbf{IS}_\zeta$, $Y \subseteq X \restriction \eta$, *and* $Z = X \cap (Y \restriction^{-1} \zeta)$. *Then* $Z \restriction \xi = (X \restriction \xi) \cap (Y \restriction (\xi \cap \eta) \restriction^{-1} \xi)$.

Proof The inclusion $\subseteq$ is quite easy. To prove the opposite direction let x belong to the right–hand side. Then in particular $x \restriction (\xi \cap \eta) = y \restriction (\xi \cap \eta)$ for some $y \in Y$. On the other hand, $x \in X \restriction \xi$ and $y \in X \restriction \eta$. Property P-3 of X (see Proposition 5) implies $x \cup y \in X \restriction (\xi \cup \eta)$. Thus $x \cup y \in Z \restriction (\xi \cup \eta)$ since $y \in Y \subseteq X \restriction \eta$, so $x \in Z \restriction \xi$. □

Lemma 7 *If* $X \in \mathrm{Perf}_\zeta$ *and* $\xi \in \mathbf{IS}_\zeta$ *then* $X \restriction \xi \in \mathrm{Perf}_\xi$. □

Proof If H witnesses that $X \in \mathrm{Perf}_\zeta$ then H_ξ witnesses that $X \restriction \xi \in \mathrm{Perf}_\xi$. □

Lemma 8 *Suppose that* H *is a projection–keeping homeomorphism, defined on some* $X \in \mathrm{Perf}_\zeta$. *Then the image* $H"X = \{H(x) : x \in X\}$ *belongs to* Perf_ζ.

Proof It is easy to see that a superposition of projection–keeping homeomorphisms is a projection–keeping homeomorphism. □

Lemma 9 *Assume that* $X \in \mathrm{Perf}_\zeta$, $X' \subseteq X$ *is open in* X, *and* $x_0 \in X'$. *There exists a clopen in* X *set* $X'' \in \mathrm{Perf}_\zeta$, $X'' \subseteq X'$, *containing* x_0.

Proof By the previous lemma, it suffices to prove the result provided $X = \mathfrak{D}^\zeta$. We observe that if $x_0 \in X' \subseteq \mathfrak{D}^\zeta$ and X' is open in $\mathfrak{D}^\zeta$ then there exists a basic clopen set $C \subseteq X'$ containing x_0. (By *basic clopen sets* we understand sets of the form

$$C = \{x \in \mathfrak{D}^\zeta : u_1 \subseteq x(i_1) \ \& \ \dots \ \& \ u_m \subseteq x(i_m)\},$$

where $m \in \omega$, $i_1, \dots, i_m \in \zeta$ are pairwise different, and $u_1, \dots, u_m \in 2^{<\omega}$.) One can easily prove that every set C of this type actually belongs to Perf_ζ. □

Lemma 10 *Let* $X, Y \in \mathrm{Perf}_\zeta$ *and* $\eta \in \mathbf{IS}_\zeta$, $X \restriction \eta = Y \restriction \eta$. *There exists a projection–keeping homeomorphism* $H : X$ *onto* Y *such that* $H(x) \restriction \eta = x \restriction \eta$ *for all* $x \in X$.

Proof Let $F : \mathfrak{D}^\zeta$ onto X and $G : \mathfrak{D}^\zeta$ onto Y witness that resp. X and Y belong to Perf_ζ. We put $H(x) = G(G_\eta^{-1}(x \restriction \eta) \cup F^{-1}(x) \restriction (\zeta \setminus \eta))$ for all $x \in X$.

Then $H"X \subseteq Y$ by the choice of G. Let us prove that $H"X = Y$. Let $y \in Y$. We put $x = F(F_\eta^{-1}(y \restriction \eta) \cup G^{-1}(y) \restriction (\zeta \setminus \eta))$ (the dual transform). Then $x \restriction \eta = y \restriction \eta$ while $F^{-1}(x) \restriction (\zeta \setminus \eta) = G^{-1}(y) \restriction (\zeta \setminus \eta))$, so that $H(x) = G(G^{-1}(y)) = y$, as required.

Take notice that $H(x) \restriction \eta = G_\eta(G_\eta^{-1}(x \restriction \eta)) = x \restriction \eta$ by definition.

To prove that H is projection–keeping, let $x_0, x_1 \in X$. Assume that $\xi \in \mathbf{IS}_\zeta$ and $x_0 \restriction \xi = x_1 \restriction \xi$; we have to prove that $H(x_0) \restriction \xi = H(x_1) \restriction \xi$. Since G is projection–keeping, it would be enough to prove that the points

$$z_l = G^{-1}(H(x_l)) = G_\eta^{-1}(x_l \restriction \eta) \cup F^{-1}(x_l) \restriction (\zeta \setminus \eta), \quad l = 0, 1,$$

satisfy $z_0 \restriction \xi = z_1 \restriction \xi$. We observe that $z_l \restriction \xi = G_{\xi'}^{-1}(x_l \restriction \xi') \cup F^{-1}(x_l) \restriction \xi''$, where $\xi' = \xi \cap \eta$ and $\xi'' = \xi \setminus \eta$, so that $z_0 \restriction \xi = z_1 \restriction \xi$ because $x_0 \restriction \xi = x_1 \restriction \xi$ and both F and G are projection–keeping. The converse is proved similarly. □

Lemma 11 *Suppose that* $X \in \mathrm{Perf}_\zeta$, $\eta \in \mathbf{IS}_\zeta$, $Y \in \mathrm{Perf}_\eta$, *and* $Y \subseteq X \restriction \eta$. *Then* $Z = X \cap (Y \restriction^{-1} \zeta)$ *belongs to* Perf_ζ.

Proof Let $F : \mathfrak{D}^\zeta$ onto X and $G : \mathfrak{D}^\eta$ onto Y witness that resp. $X \in \mathrm{Perf}_\zeta$ and $Y \in \mathrm{Perf}_\eta$. We define a projection–keeping homeomorphism $H : \mathfrak{D}^\zeta \longrightarrow Z$ by

$$H(z) = F\left(F_\eta^{-1}(G(z \restriction \eta)) \cup z \restriction (\zeta \setminus \eta)\right)$$

for all $z \in \mathfrak{D}^\zeta$. We check that H maps $\mathfrak{D}^\zeta$ onto Z. Let $z \in \mathfrak{D}^\zeta$. Then $H(z) \in X$ by the choice of F. Furthermore $H(z) \restriction \eta = F_\eta\left(F_\eta^{-1}(G(z \restriction \eta))\right) = G(z \restriction \eta) \in Y$, so $H(z) \in Z$. Let conversely $z' \in Z$, so that $z' = F(x)$ for some $x \in \mathfrak{D}^\zeta$. We define $z \in \mathfrak{D}^\zeta$ by: $z \restriction (\zeta \setminus \eta) = x \restriction (\zeta \setminus \eta)$, but $z \restriction \eta = G^{-1}(F_\eta(x \restriction \eta))$. (To be sure that G^{-1} is applicable in the last equality, note that $F_\eta(x \restriction \eta) = F(x) \restriction \eta = z' \restriction \eta \in Z \restriction \eta = Y$.) Then by definition $H(z) = F(x) = z'$.

We prove that H is projection–keeping. Let $z_0, z_1 \in \mathfrak{D}^\zeta$ and $\xi \in \mathbf{IS}_\zeta$. Suppose that $z_0 \restriction \xi = z_1 \restriction \xi$, and prove $H(z_0) \restriction \xi = H(z_1) \restriction \xi$. Let us define $x_l \in \mathfrak{D}^\zeta$ $(l = 0, 1)$ so that $x_l \restriction (\zeta \setminus \eta) = z_l \restriction (\zeta \setminus \eta)$, but $x_l \restriction \eta = F_\eta^{-1}(G(z_l \restriction \eta))$. Then, first, $H(z_l) = F(x_l)$, second, since both F and G are projection–keeping, we have $x_0 \restriction \xi = x_1 \restriction \xi$ and finally $F(x_0) \restriction \xi = F(x_1) \restriction \xi$, as required. The converse is proved in the same way. □

Lemma 12 *Assume that* $\zeta \subseteq \vartheta \in \mathbf{CPO}$, $X, Y \in \mathrm{Perf}_\zeta$, H *is a projection–keeping homeomorphism* X *onto* Y. *Then the sets* $X' = X \restriction^{-1} \vartheta$ *and* $Y' = Y \restriction^{-1} \vartheta$ *belong to* Perf_ϑ *and the function* H', *defined on* X' *by* $H'(x') \restriction (\vartheta \setminus \zeta) = x' \restriction (\vartheta \setminus \zeta)$ *and* $H'(x') \restriction \zeta = H(x' \restriction \zeta)$, *is a projection–keeping homeomorphism* X' *onto* Y'.

Proof If a projection–keeping homeomorphism $F : \mathfrak{D}^\zeta$ onto X witnesses that $X \in \mathrm{Perf}_\zeta$ then the homeomorphism F', defined on $\mathfrak{D}^\vartheta$ by $F'(x') \restriction (\vartheta \setminus \zeta) = x' \restriction (\vartheta \setminus \zeta)$ and $F'(x') \restriction \zeta = F(x' \restriction \zeta)$ for all $x' \in \mathfrak{D}^\vartheta$, witnesses that $X' \in \mathrm{Perf}_\vartheta$. The rest of the proof is equally simple. □

2 Fusion technique

We shall exploit later the construction of sets in Perf_ζ as $X = \bigcap_{m\in\omega}\bigcup_{u\in 2^m} X_u$, where every X_u belongs to Perf_ζ. This section introduces the technique.

First of all we have to specify requirements which would imply an appropriate behaviour of the sets $X_u \in \mathrm{Perf}_\zeta$ with respect to projections. We need to determine, for any pair of finite binary sequences $u, v \in 2^m$ $(m \in \omega)$, the largest initial segment $\xi = \zeta[u,v]$ of ζ such that the projections $X_u \restriction \xi$ and $X_v \restriction \xi$ have to be equal, to run the construction in proper way.

Let us fix $\zeta \in \mathbf{CPO}$ and an arbitrary function $\Phi : \omega \longrightarrow \zeta$.

We define, for a pair of finite sequences $u, v \in 2^m$ $(m \in \omega)$, an initial segment

$$\zeta_\Phi[u,v] = \bigcap_{l<m,\, u(l)\neq v(l)} [\not\geq \Phi(l)] = \{j \in \zeta : \neg\, \exists\, l < m\, [u(l) \neq v(l)\ \&\ j \geq \Phi(l)]\} \in \mathbf{IS}_\zeta .$$

Definition A Φ-*splitting system* of order m in Perf_ζ is a family $\langle X_u : u \in 2^m\rangle$ of sets $X_u \in \mathrm{Perf}_\zeta$ such that

S-1. $X_u \restriction \zeta_\Phi[u,v] = X_v \restriction \zeta_\Phi[u,v]$ for all $u, v \in 2^m$, and

S-2. If $i \in \zeta \setminus \zeta_\Phi[u,v]$, then $X_u \restriction_{\leq i} \cap X_v \restriction_{\leq i} = \emptyset$ — for all $u, v \in 2^m$.

A splitting system $\langle X_{u'} : u' \in 2^{m+1}\rangle$ is an *expansion* of a splitting system $\langle X_u : u \in 2^m\rangle$ iff $X_{u^\wedge e} \subseteq X_u$ for all $u \in 2^m$ and $e = 0, 1$. [6] □

We consider two ways how an existing splitting system can be transformed to another splitting system. One of them treats the case when we have to change one of the sets to a smaller set in Perf_ζ, the other one expands to the next level.

Lemma 13 *Assume that* $\langle X_u : u \in 2^m\rangle$ *is a* Φ*-splitting system in* Perf_ζ, $u_0 \in 2^m$, *and* $X \in \mathrm{Perf}_\zeta$, $X \subseteq X_{u_0}$. *We re-define* X_u *by* $X'_u = X_u \cap (X \restriction \zeta_\Phi[u,u_0] \restriction^{-1} \zeta)$ *for all* $u \in 2^m$. *Then the re-defined* [7] *family is again a* Φ*-splitting system.*

Proof Each set X'_u belongs to Perf_ζ by lemmas 7 and 11. We have to check only requirement S-1. Thus let $u, v \in 2^m$, $\xi = \zeta_\Phi[u,v]$. We prove that $X'_u \restriction \xi = X'_v \restriction \xi$. Let in addition $\zeta_u = \zeta_\Phi[u,u_0]$ and $\zeta_v = \zeta_\Phi[v,u_0]$. Then

$$X'_u \restriction \xi = (X_u \restriction \xi) \cap (X_0 \restriction (\xi \cap \zeta_u) \restriction^{-1} \xi) \quad \text{and} \quad X'_v \restriction \xi = (X_v \restriction \xi) \cap (X_0 \restriction (\xi \cap \zeta_v) \restriction^{-1} \xi)$$

by Lemma 6. Thus it remains to prove that $\xi \cap \zeta_u = \xi \cap \zeta_v$ (the "triangle" equality). Assume on the contrary that $i \in \xi \cap \zeta_u$ but $i \notin \zeta_v$. The latter means that $i \geq \Phi(l)$ in ζ for some $l < m$ such that $v(l) \neq u_0(l)$. But then either $u(l) \neq u_0(l)$ – so $i \notin \zeta_u$, or $v(l) \neq u(l)$ – so $i \notin \xi$, contradiction. □

[6] Characters e, d will always denote numbers 0 and 1 .

[7] Notice that $X'_{u_0} = X$.

We are going to prove that each splitting system has an expansion. This needs to define first a special construction of the expansion.

Let $i \in \zeta$, $X \in \mathrm{Perf}_\zeta$. A pair of sets $X_0, X_1 \in \mathrm{Perf}_\zeta$ will be called an *i-splitting* of X if $X_0 \cup X_1 \subseteq X$, $X_0 \restriction_{\not\geq i} = X_1 \restriction_{\not\geq i}$, and $X_0 \restriction_{\leq i} \cap X_1 \restriction_{\leq i} = \emptyset$. The splitting will be called *complete* if $X_0 \cup X_1 = X$ – in this case we have $X_0 \restriction_{\not\geq i} = X_1 \restriction_{\not\geq i} = X \restriction_{\not\geq i}$.

Assertion 14 *If $i \in \zeta$ and $X \in \mathrm{Perf}_\zeta$ then there exists a complete i-splitting of X.*

Proof If $X = \mathcal{D}^\zeta$ then we define $X_e = \{x \in X : x(i)(0) = e\}$, $e = 0, 1$. Lemma 8 extends the result on the general case. □

Lemma 15 *Every Φ-splitting system $\langle X_u : u \in 2^m \rangle$ in Perf_ζ can be expanded to a Φ-splitting system $\langle X_{u'} : u' \in 2^{m+1} \rangle$ in Perf_ζ so that for each $u \in 2^m$, $X_{u^\wedge 0}, X_{u^\wedge 1}$ is a complete i-splitting of X_u, where $i = \Phi(m)$.*

Proof We shall write $\zeta[u,v]$ instead of $\zeta_\Phi[u,v]$, since Φ is fixed. Let us consider, one by one in an arbitrary but fixed order, all sequences $u \in 2^m$. At each step u, we shall i-split X_u in one of two different ways.

Case A. Suppose that there does <u>not</u> exist $w \in 2^m$, considered earlier than u, such that $i \in \zeta[u,w]$. Let $X_{u^\wedge 0}, X_{u^\wedge 1}$ be an arbitrary complete i-splitting of X_u.

Case B. Otherwise, let w be the one which was considered first among all sequences w of the mentioned type. We put $X_{u^\wedge e} = X_u \cap (X_{w^\wedge e} \restriction_{\leq i} \restriction^{-1} \zeta)$ for $e = 0, 1$.

Let us prove that $X_{u^\wedge 0}, X_{u^\wedge 1}$ is a complete i-splitting of X_u in this case. First of all, $X_u \restriction \zeta[u,w] = X_w \restriction \zeta[u,w]$ by S-1; it follows that $X_{w^\wedge e} \restriction_{\leq i} \subseteq X_w \restriction_{\leq i} = X_u \restriction_{\leq i}$, so that the sets $X_{u^\wedge e}$ belong to Perf_ζ by lemmas 7 and 11.

By the choice of w, we had Case A at step w. (Indeed, if otherwise $i \in \zeta[w,w']$ for some $w' \in 2^m$ considered even earlier, then $i \in \zeta[u,w']$ – by the "triangle" equality in the proof of Lemma 13 – contradiction with the choice of w.) Therefore for sure $X_{w^\wedge 0}, X_{w^\wedge 1}$ is a complete i-splitting of X_w. In particular, $X_{w^\wedge e} \restriction_{<i} = X_w \restriction_{<i}$. On the other hand, Lemma 6 implies $X_{u^\wedge e} \restriction_{\not\geq i} = X_u \restriction_{\not\geq i} \cap (X_{w^\wedge e} \restriction_{<i} \restriction^{-1} [\not\geq i])$ for $e = 0, 1$, since $[\not\geq i] \cap [\leq i] = [<i]$ – so we get $X_{u^\wedge 0} \restriction_{\not\geq i} = X_{u^\wedge 1} \restriction_{\not\geq i}$.

By definition, $X_{u^\wedge e} \restriction_{\leq i} = X_{w^\wedge e} \restriction_{\leq i}$ for $e = 0, 1$, so that $X_{u^\wedge 0} \restriction_{\leq i} \cap X_{u^\wedge 1} \restriction_{\leq i} = \emptyset$ because $X_{w^\wedge 0}, X_{w^\wedge 1}$ is a splitting of X_w. Finally, since $X_{w^\wedge 0}, X_{w^\wedge 1}$ is a complete i-splitting of X_w, and $X_w \restriction_{\leq i} = X_u \restriction_{\leq i}$, we have $X_{u^\wedge 0} \cup X_{u^\wedge 1} = X_u$, as required.

Thus $X_{u^\wedge 0}, X_{u^\wedge 1}$ is a complete i-splitting of X_u for all $u \in 2^m$. It remains to prove that $\langle X_{u'} : u' \in 2^{m+1} \rangle$ is a splitting system.

To prove S-1 and S-2, let $u' = u^\wedge d$ and $v' = v^\wedge e$ belong to 2^{m+1}; $d, e \in \{0,1\}$; $\xi = \zeta[u,v]$, $\xi' = \zeta[u',v']$, and $Y = X_u \restriction \xi = X_v \restriction \xi$. We consider three cases.

Case 1: $i \notin \xi$. Then by definition $\xi = \xi' \subseteq [\not\geq i]$. We have $X_{u'} \restriction \xi = Y = X_{v'} \restriction \xi$. This proves S-1 for the sets $X_{u'}, X_{v'}$, while S-2 is inherited from the pair X_u, X_v because $\xi = \xi'$ and $X_{u'} \subseteq X_u$, $X_{v'} \subseteq X_v$.

Case 2: $i \in \xi$ and $d = e$, say $d = e = 0$. Then again $\xi = \xi'$ by definition, so S-2 is clear, but $i \in \xi'$. To prove S-1, let $w \in 2^m$ be the first (in the order fixed at the beginning of the proof) sequence in 2^m such that $i \in \zeta[u,w] \cup \zeta[v,w]$ (e. g. w can be one of u, v). Then, since $i \in \xi = \zeta[u,v]$, we have $i \in \zeta[u,w] \cap \zeta[v,w]$ by the "triangle" equality. Finally it follows from the construction (Case B) that

$$X_{u^\wedge 0} \restriction \xi = (X_u \restriction \xi) \cap (X_{w^\wedge 0} \restriction_{\leq i} \restriction^{-1} \xi) \quad \text{and} \quad X_{v^\wedge 0} \restriction \xi = (X_v \restriction \xi) \cap (X_{w^\wedge 0} \restriction_{\leq i} \restriction^{-1} \xi).$$

However $X_u \restriction \xi = X_v \restriction \xi = Y$; this proves $X_{u^\wedge 0} \restriction \xi' = X_{v^\wedge 0} \restriction \xi'$. (Note that $\xi' = \xi$.)

Case 3: $i \in \xi$ but $d \neq e$, say $d = 0$, $e = 1$. Now $\xi' = \xi \cap [\not\geq i]$, a proper subset of ξ. Let w be introduced as in Case 2. We observe that $\xi' \cap [\leq i] = [< i]$, so

$$X_{u^\wedge 0} \restriction \xi' = (X_u \restriction \xi') \cap (X_{w^\wedge 0} \restriction_{< i} \restriction^{-1} \xi') \quad \text{and} \quad X_{v^\wedge 1} \restriction \xi' = (X_v \restriction \xi') \cap (X_{w^\wedge 1} \restriction_{< i} \restriction^{-1} \xi')$$

by the construction and Lemma 6. However $X_{w^\wedge 0} \restriction_{< i} = X_{w^\wedge 1} \restriction_{< i}$ because the pair $X_{w^\wedge 0}$, $X_{w^\wedge 1}$ is an i-splitting of X_w. Furthermore, $X_u \restriction \xi' = X_v \restriction \xi' = Y \restriction \xi'$ because $X_u \restriction \xi = X_v \restriction \xi = Y$. We conclude that $X_{u^\wedge 0} \restriction \xi' = X_{v^\wedge 0} \restriction \xi'$, as required.

Let us prove S-2 for some $i' \in \zeta \setminus \xi'$. If $i' \notin \xi$ then already $X_u \restriction_{\leq i'} \cap X_v \restriction_{\leq i'} = \emptyset$. If $i' \in \xi \setminus \xi'$ then $i' \geq i$, so that it suffices to prove S-2 only for $i' = i = \Phi(m)$. To prove S-2 in this case, note that $X_{u^\wedge 0} \restriction_{\leq i} = X_{w^\wedge 0} \restriction_{\leq i}$ and $X_{v^\wedge 1} \restriction_{\leq i} = X_{w^\wedge 1} \restriction_{\leq i}$ by the construction. But $X_{w^\wedge 0} \restriction_{\leq i} \cap X_{w^\wedge 1} \restriction_{\leq i} = \emptyset$ because the pair $X_{w^\wedge 0}$, $X_{w^\wedge 1}$ is an i-splitting, so $X_{u^\wedge 0} \restriction_{\leq i} \cap X_{v^\wedge 1} \restriction_{\leq i} = \emptyset$. □

To formulate the fusion lemma we need a couple more definitions.

Definition An indexed family of sets $X_u \in \text{Perf}_\zeta$, $u \in 2^{<\omega}$, is a Φ-*fusion sequence* in Perf_ζ if for every $m \in \omega$ the subfamily $\langle X_u : u \in 2^m \rangle$ is a Φ-splitting system, expanded by $\langle X_u : u \in 2^{m+1} \rangle$ to the next level, and

S-3. For any $\epsilon > 0$ there exists $m \in \omega$ such that $\text{diam} X_u < \epsilon$ for all $u \in 2^m$. (A Polish metric on $\mathfrak{D}^\zeta$ is assumed to be fixed.) □

Definition A function $\Phi : \omega \longrightarrow \zeta$ is ζ-*complete* iff it takes each value $i \in \zeta$ infinitely many times. □

Theorem 16 [Fusion lemma]

Let Φ be a ζ-complete function. Suppose that $\langle X_u : u \in 2^{<\omega} \rangle$ is a Φ-fusion sequence in Perf_ζ. *Then the set $X = \bigcap_{m \in \omega} \bigcup_{u \in 2^m} X_u$ belongs to* Perf_ζ .

Proof The idea of the proof is to obtain a parallel presentation of the set $D = \mathfrak{D}^\zeta$ as the "limit" of a Φ-fusion sequence, and associate the points in D and X which are generated by one and the same branch in $2^{<\omega}$. So first of all we have to define a fusion sequence of sets $D_u \in \text{Perf}_\zeta$ such that $\mathfrak{D}^\zeta = D = \bigcap_{m \in \omega} \bigcup_{u \in 2^m} D_u$.

Lemma 15 cannot be used: we would face problems with requirement S-3. We rather maintain a direct construction. For $m \in \omega$, we put $\zeta_m = \{\Phi(l) : l < m\}$. Let $i \in \zeta_m$, and $\{l < m : \Phi(l) = i\} = \{l^i_0, ..., l^i_{s(i)-1}\}$ in the increasing order. If $u \in 2^m$ then we define $u_i \in 2^{s(i)}$ by $u_i(s) = u(l^i_s)$ for all $s < s(i)$, and put

$$D_u = \{y \in D = \mathfrak{D}^\zeta : \forall i \in \zeta_m\ (u_i \subset y(i))\},$$

so that D_u is a basic clopen set in $\mathfrak{D}^\zeta$. (Note that $y(i) \in \mathfrak{D}$ whenever $y \in \mathfrak{D}^\zeta$ and $i \in \zeta$.) We omit a routine verification of the fact that the sets D_u form a Φ-fusion sequence (S-3 follows from the ζ-completeness of Φ) and $\bigcup_{u \in 2^m} D_u = \mathfrak{D}^\zeta$ for all m.

We observe that for each $a \in 2^\omega = \mathfrak{D}$ the intersections $\bigcap_m X_{a \restriction m}$ and $\bigcap_m D_{a \restriction m}$ contain single points, say $x_a \in X$ and $d_a \in D$ respectively, by S-3, and the maps $a \longmapsto x_a$, $a \longmapsto d_a$ are continuous. Let us define $\zeta_\Phi[a,b] = \bigcap_{m \in \omega} \zeta_\Phi[a \restriction m, b \restriction m]$. In particular $\zeta_\Phi[a,b] = \zeta$ iff $a = b$. It follows from S-1 and S-2 that

$$(*)\begin{cases} x_a \restriction \zeta_\Phi[a,b] = x_b \restriction \zeta_\Phi[a,b] \quad \text{and} \quad d_a \restriction \zeta_\Phi[a,b] = d_b \restriction \zeta_\Phi[a,b] \quad \text{for all } a, b \in 2^\omega \\ x_a \restriction_{\le i} \ne x_b \restriction_{\le i} \quad \text{and} \quad d_a \restriction_{\le i} \ne d_b \restriction_{\le i} \quad \text{whenever } i \notin \zeta_\Phi[a,b] \end{cases}$$

This allows to define a homeomorphism $H : D = \mathfrak{D}^\zeta$ onto X by $F(d_a) = x_a$ for all $a \in 2^\omega$. To see that H is a projection–keeping homeomorphism, let $\xi \in \mathbf{IS}_\zeta$ and, for instance, $d_a, d_b \in \mathfrak{D}^\zeta$ and $d_a \restriction \xi = d_b \restriction \xi$. Then $\xi \subseteq \zeta_\Phi[a,b]$ by the second line in $(*)$, so we get $x_a \restriction \xi = x_b \restriction \xi$ by the first line, as required. □

Corollary 17 *Suppose that* $X \in \mathrm{Perf}_\zeta$, *and* $C_m \subseteq \mathfrak{D}^\zeta$ *is closed for each* $m \in \omega$. *There exists* $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$ *such that* $C_m \cap Y$ *is clopen in* Y *for every* m.

Proof It follows from Proposition 9 that for any m and any $X' \in \mathrm{Perf}_\zeta$ there exists $Y' \in \mathrm{Perf}_\zeta$, $Y' \subseteq X'$, such that either $Y' \subseteq C_m$ or $Y' \cap C_m = \emptyset$. Therefore we can define, using lemmas 13 and 15, a fusion sequence $\langle X_u : u \in 2^{<\omega}\rangle$ of sets $X_u \in \mathrm{Perf}_\zeta$ such that $X_\Lambda = X$ and either $X_u \subseteq C_m$ or $X_u \cap C_m = \emptyset$ whenever $u \in 2^m$ — for all $m \in \omega$. The set $Y = \bigcap_{m \in \omega} \bigcup_{u \in 2^m} X_u$ is as required. □

Corollary 18 *Assume that* $X \in \mathrm{Perf}_\zeta$, *and* $B \subseteq \mathfrak{D}^\zeta$ *is a set of a finite Borel level. There exists* $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$ *such that either* $Y \subseteq B$ *or* $Y \cap B = \emptyset$.

Proof [8] Let B be defined by a finite level Borel scheme (countable unions plus countable intersections) from closed sets C_m, $m \in \omega$. The preceding corollary shows that there exists $X' \in \mathrm{Perf}_\zeta$, $X' \subseteq X$ such that every $X' \cap C_m$ is clopen in X'. Thus the Borel level can be reduced. When one finally achieves the level of closed or open sets, the previous corollary is applied. □

[8] In fact this is true for all Borel sets B but needs a more elaborate reasoning.

3 Reducibility of continuous functions

This section studies the behaviour of continuous functions defined on sets in Perf_ζ, $\zeta \in \mathbf{CPO}$, from the point of view of a certain reducibility.

Definition For each set ζ, Cont_ζ will denote the set of all continuous functions $F : \mathcal{D}^\zeta \longrightarrow$ reals. (As usual, reals $= \mathcal{N} = \omega^\omega$.) Let $F \in \mathrm{Cont}_\zeta$, $\xi \subseteq \zeta$, $X \subseteq \mathcal{D}^\zeta$.

1. F is *reducible* to ξ on X iff $x \restriction \xi = y \restriction \xi$ implies $F(x) = F(y)$ for all $x, y \in X$.

2. F *captures* $i \in \zeta$ on X iff $F(x) = F(y)$ implies $x(i) = y(i)$ for all $x, y \in X$. □

Remark 19 It follows from the compactness of the spaces we consider, that if X is closed then in item 1 there exists a function $F' \in \mathrm{Cont}_\xi$ such that $F(x) = F'(x \restriction \xi)$ for all $x \in X$, while in item 2 there exists a continuous function $H : \mathcal{N} \longrightarrow \mathcal{D}$ such that $x(i) = H(F(x))$ for all $x \in X$. □

The following theorem contains several statements related to the notion of reducibility. These statements will later be transformed to properties of constructibility of reals in the related generic extensions.

Theorem 20 *Assume that* $X \in \mathrm{Perf}_\zeta$, $\xi \in \mathbf{IS}_\zeta$, *and* $F \in \mathrm{Cont}_\zeta$. *Then*

1. *If* $i, j \in \zeta$ *and* $i < j$ *then there exists* $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$, *such that the co-ordinate function* C_j, *defined on* $\mathcal{D}^\zeta$ *by* $C_j(x) = x(j)$, *captures* i *on* Y.
2. *If* $i \in \zeta \setminus \xi$ *and* F *is reducible to* ξ *on* X *then* F *does not capture* i *on* X.
3. *Suppose that for each* $X' \in \mathrm{Perf}_\zeta$, $X' \subseteq X$, *and each* $i \in \xi$ *there exists a set* $X'' \in \mathrm{Perf}_\zeta$, $X'' \subseteq X'$ *such that* F *captures* i *on* X''. *Then there exists* $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$ *such that* F *captures* <u>*every*</u> $i \in \xi$ *on* Y.
4. *If* $i \in \zeta$, *then there exists* $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$ *such that either* F *is reducible to* $[\not\geq i]$ *on* Y, *or* F *captures* i *on* Y.
5. *There exists a set* $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$ *satisfying one from the following two requirements:* (a) F *is reducible to* ξ *on* Y, *or*
   (b) F *captures some* $i \in \zeta \setminus \xi$ *on* Y.

Proof We begin with a couple of technical lemmas, then come to the theorem.

Lemma 21 *Let* $\xi, \eta \in \mathbf{IS}_\zeta$. *If* F *is reducible to both* ξ *and* η *on* $X \in \mathrm{Perf}_\zeta$ *then* F *is reducible to* $\vartheta = \xi \cap \eta$ *on* X.

Proof Let, on the contrary, $x, y \in X$ satisfy $x \restriction \vartheta = y \restriction \vartheta$ but $F(x) \neq F(y)$. Then by property P-3 of X (see Proposition 5) there exists $z \in X$ such that $z \restriction \xi = x \restriction \xi$ and $z \restriction \eta = y \restriction \eta$. We obtain $F(x) = F(z) = F(y)$, contradiction. □

Lemma 22 *Suppose that* $\xi \in \mathbf{IS}_\zeta$, *the sets* X_1 *and* X_2 *belong to* Perf_ζ, *and* $X_1 \restriction \xi = X_2 \restriction \xi$. *Then either* F *is reducible to* ξ *on* $X_1 \cup X_2$ *— and then obviously* $F"X_1 = F"X_2$, *— or there exist sets* $X_1', X_2' \in \mathrm{Perf}_\zeta$, $X_1' \subseteq X_1$ *and* $X_2' \subseteq X_2$, *such that still* $X_1' \restriction \xi = X_2' \restriction \xi$, *but* $F"X_1' \cap F"X_2' = \emptyset$. [9]

Proof We assume that the function F is not reducible to ξ on $X_1 \cup X_2$, and prove the "or" alternative. By the assumption, there exist points $x_1, x_2 \in X_1 \cup X_2$ satisfying $x_1 \restriction \xi = x_2 \restriction \xi$ and $F(x_1) \neq F(x_2)$. It may be supposed that $x_1 \in X_1$ and $x_2 \in X_2$, because $X_1 \restriction \xi = X_2 \restriction \xi$. By the continuity of F there exist clopen neighbourhoods U_1 and U_2 of x_1 and x_2 respectively such that $F"U_1 \cap F"U_2 = \emptyset$. Lemma 9 provides a set $X_1'' \in \mathrm{Perf}_\zeta$, $X_1'' \subseteq X_1 \cap U_1$, containing x_1.

By Lemma 11 the set $X_2'' = X_2 \cap (X_1'' \restriction \xi \restriction^{-1} \zeta)$ belongs to Perf_ζ, and contains x_2 since $x_1 \restriction \xi = x_2 \restriction \xi$. By Lemma 9 again, there exists a set $X_2' \in \mathrm{Perf}_\zeta$, $X_2' \subseteq X_2'' \cap U_2$. Now putting $X_1' = X_1'' \cap (X_2' \restriction \xi \restriction^{-1} \zeta)$, we get the required sets X_1' and X_2'. □

We are already equipped enough to handle different items of Theorem 20.

Item 2. Suppose that F is reducible to ξ on X and, on the contrary, F does capture some $i \in \zeta \setminus \xi$ on X. Then the co-ordinate function $C_i(x) = x(i)$ is itself reducible to ξ on X. Since i does not belong to ξ, and on the other hand C_i is obviously reducible to $[\leq i]$, we conclude that C_i is reducible to $[<i]$ on X by Lemma 21. But this clearly contradicts property P-1 of X (see Proposition 5).

Items 3 and 4 are carried out by one and the same construction.

Let us fix a ζ-complete function Φ and define the initial segments $\zeta[u,v] = \zeta_\Phi[u,v]$ (as in Section 2) for every pair of finite sequences $u, v \in 2^{<\omega}$ of equal length. The notions of splitting system and fusion sequence are understood in the sense of Φ.

We define a fusion sequence $\langle X_u : u \in 2^{<\omega}\rangle$ satisfying $X_\Lambda = X$ and the property

($\star$) If $m \in \omega$ and $u, v \in 2^m$ then either (1) F is reducible to $\zeta[u,v]$ on the set $X_u \cup X_v$, or (2) $F"X_u \cap F"X_v = \emptyset$.

First we put $X_\Lambda = X$, as indicated.

Assume that sets X_u $(u \in 2^m)$ have been defined for some m. We use Lemma 15 to get a splitting system $\langle W_u : u \in 2^{m+1}\rangle$ which expands the already obtained splitting system $\langle X_u : u \in 2^m\rangle$ to the next level $m+1$. It follows from lemmas 9 and 13 (applied consecutively 2^{m+1} times) that there exists a splitting system of sets $Z_u \subseteq W_u$ which satisfies $\operatorname{diam} Z_u \leq m^{-1}$ for all $u \in 2^{m+1}$. (We need this to provide requirement S-3.)

We now consider consecutively all pairs $u, v \in 2^{m+1}$. For every such a pair we first apply Lemma 22, getting sets $S_u, S_v \in \mathrm{Perf}_\zeta$ such that $S_u \subseteq Z_u$, $S_v \subseteq Z_v$, $S_u \restriction \zeta[u,v] = S_v \restriction \zeta[u,v]$, and either the function F is reducible to $\zeta[u,v]$ on $S_u \cup S_v$ or $F"S_u \cap F"S_v = \emptyset$.

[9] We recall that $F"X$ is the image of X via F.

We set $S'_w = Z_w \cap (S_u \restriction \zeta[w,u] \restriction^{-1} \zeta)$ for all $w \in 2^{m+1}$; $\langle S'_w : w \in 2^{m+1} \rangle$ is a splitting system by Lemma 13. Note that $S'_v = S_v$ since $S_u \restriction \zeta[u,v] = S_v \restriction \zeta[u,v]$. This allows to repeat the operation: putting $Z'_w = S'_w \cap (S_v \restriction \zeta[w,v] \restriction^{-1} \zeta)$ for all $w \in 2^{m+1}$, we obtain a new splitting system of sets $Z'_w \subseteq S'_w$ $(w \in 2^{m+1})$ such that $Z'_u = S_u$ and $Z'_v = S_v$. This ends the consideration of the particular pair of $u, v \in 2^{m+1}$, and one comes to the next pair.

Let $X_u \subseteq Z_u$ $(u \in 2^{m+1})$ be the sets obtained after 2^{m+2} steps of this construction (the number of pairs u, v to consider). One sees easily that this is a splitting system in Perf_ζ satisfying $(\star)$ for $m+1$.

After the construction is accomplished for all m, we obtain a fusion sequence of sets X_u $(u \in 2^{<\omega})$ satisfying $(\star)$. The set $Y = \bigcap_m \bigcup_{u \in 2^m} X_u$ belongs to Perf_ζ by Theorem 16.

Item 4. Let us assume that a set $Y' \subseteq X$ of the "either" type does not exist. We prove that the set $Y = \bigcap_m \bigcup_{u \in 2^m} X_u$ is of the "or" type, that is, F captures i on Y. Assume that, on the contrary, there exists a pair of points $x, y \in Y$ such that $F(x) = F(y)$ but $x(i) \neq y(i)$. Let $x = x_a$ and $y = x_b$, where $a, b \in 2^\omega$, that is, $\{x\} = \bigcap_{m \in \omega} X_{a \restriction m}$ and $\{y\} = \bigcap_{m \in \omega} X_{b \restriction m}$, see the proof of Theorem 16. Then $i \notin \zeta[a,b] = \bigcap_m \zeta[a \restriction m, b \restriction m]$ (see assertion $(*)$ in the proof of Theorem 16).

Let m be the least among those satisfying $i \notin \xi = \zeta[a \restriction m, b \restriction m]$. Then $\xi \subseteq [\not\geq i]$, so that the case (1) in $(\star)$ is impossible for $u = a \restriction m$ and $v = b \restriction m$ by the assumption of the "either" nonexistence above. (Otherwise F would be reducible to $[\not\geq i]$ on each of X_u and X_v !). Therefore $F"X_u \cap F"X_v = \emptyset$, contradiction with the choice of x and y because $x \in X_u$ and $y \in X_v$.

Item 3. We show that the set $Y = \bigcap_n \bigcup_{u \in 2^m} X_u$ proves this item, too. Suppose that $x, y \in Y$ satisfy $F(x) = F(y)$; we have to verify that $x \restriction \xi = y \restriction \xi$. As above, $x = x_a$ and $y = x_b$ for some $a, b \in 2^\omega$. It suffices to check that $\xi \subseteq \zeta[a \restriction m, b \restriction m]$ for all m.

Assume on the contrary that $\xi \not\subseteq \zeta[u,v]$, where $u = a \restriction m$ and $v = b \restriction m$ for some m. We assert that the case (1) of $(\star)$ is impossible for this pair u, v. (Indeed otherwise in particular F is reducible to $\zeta[u,v]$ on a set $X' = X_u \subseteq X$. Take an arbitrary $i \in \xi \setminus \zeta[u,v]$. Then F captures i on a set $X'' \in \mathrm{Perf}_\zeta$, $X'' \subseteq X'$, by the assumption of item 3. Thus the co-ordinate function C_i is reduced to $\zeta[u,v]$ on X'' – contradiction with the already proved item 2.) Thus we have case (2) of $(\star)$, that is, $F"X_u \cap F"X_v = \emptyset$. But this contradicts the assumption $F(x) = F(y)$.

Item 1. Otherwise, by item 4 C_j would be reducible to $\xi = [\not\geq i]$ on some $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$, contradiction with the already proved item 2.

Item 5. Assume that a set $Y \in \mathrm{Perf}_\zeta$ of type (b) of item 5 does not exist. Then by item 4, if $i \in \zeta \setminus \xi$ then every set $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$ contains a subset $Z \in \mathrm{Perf}_\zeta$ such that F is reducible to $[\not\geq i]$ on Z. Arguing as above, we obtain a fusion sequence $\langle X_u : u \in 2^{<\omega} \rangle$ such that $X_\Lambda \subseteq X$ and F is reducible to $[\not\geq \Phi(m)]$ on X_u whenever

$u \in 2^m$ and $\Phi(m) \notin \xi$. Then $Y = \bigcap_m \bigcup_{u\in 2^m} X_u \in \mathrm{Perf}_\zeta$.

We prove that Y is a set of type (a), that is, F is reducible to ξ on Y.

Let us define, for every $m \in \omega$, an initial segment $\zeta_m \subseteq \zeta$ by

$$\zeta_m = \bigcap_{l<m,\, \Phi(l)\notin\xi} [\not\geq \Phi(l)] = \{j \in \zeta : \neg\, \exists\, l < m\, (j \geq \Phi(l) \notin \xi)\}$$

Then obviously $\xi \subseteq \zeta_m$ for all m. Furthermore $\zeta_m \subseteq \zeta[u,v]$ whenever $u, v \in 2^m$ satisfy $\xi \subseteq \zeta[u,v]$.

Assertion *For any* m, F *is reducible to* ζ_m *on* $X_m = \bigcup_{u\in 2^m} X_u$.

Proof of the assertion. We argue by induction on m. The case $m = 0$ is trivial: we have $\zeta_0 = \zeta$ by definition. Let us carry out the step from m to $m+1$. Let $i = \Phi(m)$. If $i \in \xi$ then $\zeta_{m+1} = \zeta_m$ and the statement is obvious. Therefore one can assume that $i = \Phi(m) \notin \xi$. Then F is reducible to $[\not\geq i]$ on each set $X_{u'}$ ($u' \in 2^m$) by the construction of the fusion sequence.

Suppose that u, $v \in 2^{m+1}$, and points $x \in X_u$, $y \in X_v$ satisfy $x \restriction \zeta_{m+1} = y \restriction \zeta_{m+1}$, and prove $F(x) = F(y)$. We put $u' = u \restriction m$, $v' = v \restriction m$; then $u', v' \in 2^m$.

We have $\zeta_{m+1} \subseteq \zeta[u,v]$ (otherwise $X_u \restriction \zeta_{m+1} \cap X_v \restriction \zeta_{m+1} = \emptyset$ by S-2, but $x \restriction \zeta_{m+1} = y \restriction \zeta_{m+1}$), therefore $\xi \subseteq \zeta[u,v]$ because every set ζ_n includes ξ. This implies $\xi \subseteq \zeta[u',v']$. It follows (see above) that $\zeta_m \subseteq \zeta[u',v']$. Therefore $X_{u'} \restriction \zeta_m = X_{v'} \restriction \zeta_m$ by S-1, so $y \restriction \zeta_m \in X_{u'} \restriction \zeta_m$. We choose some $x' \in X_{u'}$ satisfying $x' \restriction \zeta_m = y \restriction \zeta_m$. Then $F(x') = F(y)$ by the induction hypothesis, so it remains to verify that $F(x) = F(x')$.

Take notice that x and x' belong to $X_{u'}$ and $x \restriction \zeta_{m+1} = x' \restriction \zeta_{m+1}$ by the choice of x'. Thus it suffices to prove that F is reducible to ζ_{m+1} on $X_{u'}$. We observe that, since $i = \Phi(m) \notin \xi$, F is reducible to $[\not\geq i]$ on $X_{u'}$, see above. Moreover F is reducible to ζ_m on $X_{u'}$ by the induction hypothesis. Therefore F is reducible to $[\not\geq i] \cap \zeta_m$ on $X_{u'}$ by Lemma 21. Finally, we have $\zeta_{m+1} = [\not\geq i] \cap \zeta_m$ by definition. □

We end the proof of item 5 of Theorem 20.

It follows from the assertion that F is reducible to every ζ_m on Y. This allows to conclude that F is also reducible to ξ on Y. Indeed assume on the contrary that $x, y \in Y$ and $x \restriction \xi = y \restriction \xi$ but $F(x) \neq F(y)$. By the continuity of F there exist $m \in \omega$ and $u, v \in 2^m$ such that $x \in X_u$, $y \in X_v$, and $F''X_u \cap F''X_v = \emptyset$. On the other hand, we have $X_u \restriction \xi \cap X_v \restriction \xi \neq \emptyset$, therefore $\xi \subseteq \zeta[u,v]$ by S-2. This implies $\xi \subseteq \zeta_m \subseteq \zeta[u,v]$, as above. Therefore F is reducible to $\zeta[u,v]$ on Y, contradiction with the equality $F''X_u \cap F''X_v = \emptyset$, because $X_u \restriction \zeta[u,v] = X_v \restriction \zeta[u,v]$ by S-1. □

4 Introduction to generic models

This section gives an introduction to generic models obtained by forcing conditions in different sets Perf_ζ. This approach will then be detalized for particular applications.

Ground model. Let $\mathfrak{M}$ be a countable transitive model of **ZFC**, $\mathbf{I} \in \mathfrak{M}$ be a partially ordered set (generally speaking, uncountable in $\mathfrak{M}$) – the intended "length" of the planned Sacks iteration.

We let $\Xi = \mathbf{CPO}^{\mathfrak{M}}(\mathbf{I}) \in \mathfrak{M}$ be the collection of all finite and $\mathfrak{M}$-countable sets $\xi \in \mathfrak{M}$, $\xi \subseteq \mathbf{I}$ [10], therefore $\Xi \subseteq \mathbf{CPO}$ in $\mathfrak{M}$.

The forcing. For any $\zeta \in \Xi$, let $\mathbb{P}_\zeta = (\mathrm{Perf}_\zeta)^{\mathfrak{M}}$. The set $\mathbb{P} = \bigcup_{\zeta \in \Xi} \mathbb{P}_\zeta$ will be the forcing notion. To define the order, we first put $\|X\| = \zeta$ whenever $X \in \mathbb{P}_\zeta$. Now we define $X \leq Y$ (i. e. X is *stronger* than Y) iff $\zeta = \|Y\| \subseteq \|X\|$ and $X \restriction \zeta \subseteq Y$.

Notice that every set in $\mathbb{P}_\zeta$ is then a countable subset of $\mathcal{D}^\zeta$ in the universe. However we can transform it to a perfect set in the universe by the closure operation: the topological closure $X^\#$ of a set $X \in \mathbb{P}_\zeta$ is a set in Perf_ζ from the point of view of the universe.

The extension. Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic ultrafilter over $\mathfrak{M}$. It easily follows from Lemma 9 that there exists unique indexed set $\mathbf{x} = \langle \mathbf{a}_i : i \in \mathbf{I} \rangle$, all $\mathbf{a}_i$ being elements of $\mathcal{D}$, such that $\mathbf{x} \restriction \xi \in X^\#$ whenever $X \in G$ and $\|X\| = \xi \in \Xi$. Then $\mathfrak{M}[G] = \mathfrak{M}[\mathbf{x}] = \mathfrak{M}[\langle \mathbf{a}_i : i \in \mathbf{I} \rangle]$.

In this section, we prove a cardinal preservation theorem for the extension $\mathfrak{N} = \mathfrak{M}[G]$, and an important technical theorem which will allow to study reals in $\mathfrak{N}$ using continuous functions in the ground model $\mathfrak{M}$. We also prove that the model $\mathfrak{N}$ is in fact a sort of iterated Sacks extensions of $\mathfrak{M}$.

The next section will contain a more detailed study of reals in the extension.

Theorem 23 *$\aleph_1^{\mathfrak{M}}$ remains a cardinal in $\mathfrak{N}$. If $2^{\aleph_0} = \aleph_1$ in $\mathfrak{M}$ and every proper initial segment $J \in \mathfrak{M}$, $J \subseteq \mathbf{I}$ has cardinality $\mathrm{card}\, J \leq \aleph_1^{\mathfrak{M}}$ in $\mathfrak{M}$ then $\aleph_2^{\mathfrak{M}}$ remains a cardinal in $\mathfrak{N}$.* [11]

Proof We prove the 1st assertion. Let $\underset{\sim}{f}$ be a name of a function mapping ω to $\omega_1^{\mathfrak{M}}$. We fix $X_0 \in \mathbb{P}$. The aim is to obtain a condition $X \in \mathbb{P}$, stronger than X_0, and a countable in $\mathfrak{M}$ set R such that X forces that the range of $\underset{\sim}{f}$ is included in R.

We argue in $\mathfrak{M}$.

Let $\xi_0 = \|X_0\|$. We define the following objects:

[10] In the case when all initial segments of $\mathbf{I}$ with perhaps the exception of $\mathbf{I}$ itself are countable in $\mathfrak{M}$, it might be technically more convenient to define Ξ to be the set of all $\mathfrak{M}$-countable initial segments of $\mathbf{I}$ in $\mathfrak{M}$.

[11] The behaviour of other cardinals depends on the cardinal structure in $\mathfrak{M}$, the cardinality of $\mathbf{I}$, and the cardinality of chains in $\mathbf{I}$. It is not our intension here to investigate this matter.

1) *a sequence* $\zeta_0 \subseteq \zeta_1 \subseteq \zeta_2 \subseteq \dots$ *of sets* $\zeta_m \in \Xi$ *such that* $\xi_0 \subseteq \zeta_0$;

2) *the set* $\zeta = \bigcup_{m\in\omega} \zeta_m \in \Xi$, *and a* ζ*-complete function* $\Phi : \omega \longrightarrow \zeta$, *such that* $\Phi(m) \in \zeta_m$ *for all* m;

3) *for any* m, *a* Φ*-splitting system* $\langle X_u : u \in 2^m \rangle$ *of sets* $X_u \in \mathrm{Perf}_{\zeta_m}$ *such that* $X_\Lambda \subseteq X_0 \restriction^{-1} \zeta_0$ *and*

   (a) $X_{u^\wedge e} \subseteq X_u \restriction^{-1} \zeta_{m+1}$ *for all* $u \in 2^m$ *and* $e = 0, 1$;

   (b) *every set* X_u $(u \in 2^m)$ *has* $\mathrm{diam}\, X_u \le m^{-1}$;

   (c) *every condition* X_u $(u \in 2^m)$ *forces* $\underline{f}(m) = \rho_u$ *for a certain ordinal* ρ_u .

This solves the problem. Indeed, the family of sets $Y_u = X_u \restriction^{-1} \zeta$ is a Φ-fusion sequence, [12] therefore $X = \bigcap_{m\in\omega} \bigcup_{u\in 2^m} Y_u \in \mathrm{Perf}_\zeta$ by Theorem 16, and X is stronger than X_0 by the construction. Finally, X forces that the range of $\underline{f}$ is a subset of a countable in $\mathfrak{M}$ set $R = \{\rho_u : u \in 2^{<\omega}\}$. So let us concentrate on the construction.

To begin with, we find a condition X_Λ, stronger than the given X_0, which decides the value $\underline{f}(0)$, and put $\zeta_0 = \|X_\Lambda\|$.

Suppose that $\Phi \restriction m$, ζ_m, and the sets X_u $(u \in 2^m)$ have been defined. Let $u_0 \in 2^m$. There exists a condition $Z_{u_0} \in \mathrm{Perf}_{\zeta'}$ for some $\zeta' \in \Xi$, $\zeta' \supseteq \zeta_m$, which is stronger than X_{u_0}, decides the value $\underline{f}(m+1)$, and has $\mathrm{diam}\, Z_{u_0} \le (m+1)^{-1}$. (We use Lemma 9 to provide the last inequality.) Let $Y'_u = X_u \restriction^{-1} \zeta'$ for all $u \in 2^m$; then $\langle Y'_u : u \in 2^m \rangle$ is a splitting system in $\mathrm{Perf}_{\zeta'}$ and $Z_{u_0} \subseteq Y'_{u_0}$. Using Lemma 13, we obtain a splitting system $\langle X'_u : u \in 2^m \rangle$ in $\mathrm{Perf}_{\zeta'}$ such that $X'_u \subseteq Y'_u = X_u \restriction^{-1} \zeta'$ for all $u \in 2^m$ and the condition $X'_{u_0} = Z_{u_0}$ decides the value $\underline{f}(m+1)$.

Running this procedure 2^m times, we finally get a set $\zeta_{m+1} \in \Xi$, $\zeta_{m+1} \supseteq \zeta_m$, and an auxiliary splitting system $\langle X'_u : u \in 2^m \rangle$ in $\mathrm{Perf}_{\zeta_{m+1}}$ such that $X'_u \subseteq X_u \restriction^{-1} \zeta_{m+1}$, $\mathrm{diam}\, X'_u \le (m+1)^{-1}$, and X'_u decides the value $\underline{f}(m+1)$ for all $u \in 2^m$.

At this moment, we define $\Phi(m) \in \zeta_m$ appropriately, with the purpose to provide the final ζ-completeness of Φ, and use Lemma 15 to get a splitting system $\langle X_{u'} : u' \in 2^{m+1} \rangle$ in $\mathrm{Perf}_{\zeta_{m+1}}$ such that $X_{u^\wedge e} \subseteq X'_u \subseteq X_u \restriction^{-1} \zeta_{m+1}$ for all $u \in 2^m$ and $e = 0, 1$. This ends the recursive step of the construction.

Thus the equality $\aleph_1^{\mathfrak{N}} = \aleph_1^{\mathfrak{M}}$ has been verified.

To prove that $\aleph_2^{\mathfrak{N}} = \aleph_2^{\mathfrak{M}}$, it suffices to show that, in $\mathfrak{M}$, $\mathbb{P}$ does not have an antichain of cardinality $> \aleph_1$.

We argue in $\mathfrak{M}$. In particular, $\mathfrak{c} = \aleph_1$.

Let $A \subseteq \mathbb{P}$ be a maximal antichain. The set $\mathbb{P}_J = \bigcup_{\zeta\in\Xi, \zeta\subseteq J} \mathrm{Perf}_\zeta$ has cardinality $\mathrm{card}\, \mathbb{P}_J \le \aleph_1$ (in fact $=$, of course) for any proper (i.e. other than $\mathbf{I}$ itself) initial segment $J \subsetneq \mathbf{I}$ by the assumptions of the theorem. Therefore there exists an initial

[12] We assume that $\mathrm{diam}(Z \restriction^{-1} \zeta) \le \mathrm{diam}\, Z$ whenever $Z \subseteq \mathcal{D}^\xi$ and $\xi \subseteq \zeta$. This suffices to prove requirement S-3 for the sets X_u by $\mathrm{diam}\, Y_u \le \mathrm{diam}\, X_u \le m^{-1}$ for $u \in 2^m$.

segment $J \subseteq \mathbf{I}$ of cardinality $\operatorname{card} J \leq \aleph_1$ such that $A' = A \cap \mathbb{P}_J$ is a maximal antichain in $\mathbb{P}_J$. It remains to check that $A = A'$.

Suppose on the contrary that $X \in A \setminus A'$. Let $\zeta = \|X\|$, $\eta = \zeta \cap J$. Then $X \in \mathrm{Perf}_\zeta$ and $Y = X \restriction \eta \in \mathrm{Perf}_\eta$ and $\in \mathbb{P}_J$. Therefore there exist sets $Z' \in A'$ and $Z \in \mathbb{P}_J$ such that Z is stronger than both Z' and Y. We come to contradiction if prove that Z and X are compatible in $\mathbb{P}$.

Let $\xi = \|Z\|$, so that $\eta \subseteq \xi \subseteq J$, and $\vartheta = \xi \cup \zeta$. Then $X' = X \restriction^{-1} \vartheta \in \mathrm{Perf}_\vartheta$ by Lemma 12. The set $\xi = \vartheta \cap J$ is an initial segment in ϑ and obviously $X' \restriction \xi = Y \restriction^{-1} \xi$; therefore $Z \subseteq X' \restriction \xi$. Now $X'' = X' \cap (Z \restriction^{-1} \vartheta) \in \mathrm{Perf}_\vartheta$ by Lemma 11. But X'' is stronger than both Z and X. □

Continuous functions

We put $\mathbb{F}_\zeta = (\mathrm{Cont}_\zeta)^{\mathfrak{M}}$ for $\zeta \in \Xi$. It is a principal property of several forcing notions (including Sacks forcing and for instance random forcing) that reals in the generic extensions can be obtained by application of continuous functions (having a code) in the ground model, to generic sequences of reals. As we shall prove, this is also a property of the generic models considered here.

Obviously every $F \in \mathbb{F}_\zeta$ is a countable subset of $\mathcal{D}^\zeta \times \omega^\omega$ in the universe, but since the domain of F in $\mathfrak{M}$ is the compact set $\mathcal{D}^\zeta$, the topological closure $F^\#$ is a continuous function mapping $\mathcal{D}^\zeta$ into the reals in the universe.

By "reals" we understand elements of the set $\mathcal{N} = \omega^\omega$, as usual.

Theorem 24 *Let $J \in \mathfrak{M}$ be an initial segment of $\mathbf{I}$ and r a real in $\mathfrak{M}[\mathbf{x} \restriction J]$. There exists $\zeta \in \Xi$, $\zeta \subseteq J$, and a function $F \in \mathbb{F}_\zeta$ such that $r = F^\#(\mathbf{x} \restriction \zeta)$.*

(It is clear that the equality is absolute for any model containing r, $\mathbf{x} \restriction \zeta$, and F.)

Proof Let $\underset{\sim}{r}$ be a name for the real r containing an explicit absolute construction of r from $\mathbf{x} \restriction J$ and some parameter $p \in \mathfrak{M}$. Let $X_0 \in \mathbb{P}$, $\xi_0 = \|X_0\|$.

We argue in $\mathfrak{M}$.

First of all, we observe that by lemmas 7 and 11 the forcing of statements about $\underset{\sim}{r}$ can be reduced to J in the following sense: if $X \in \mathrm{Perf}_\zeta$ forces $\underset{\sim}{r}(m) = k$ then $X \restriction (\zeta \cap J)$ also forces $\underset{\sim}{r}(m) = k$. (The usual "restriction" argument.)

Having this in mind and arguing as in the proof of Theorem 23, one gets a system of objects satisfying 1), 2), 3), with the following corrections: in 1), additionally, $\zeta_m \subseteq J$ – therefore $\zeta \subseteq J$, and in 3)(c), every condition X_u, $u \in 2^m$, forces $\underset{\sim}{r}(m) = k_u$ for some $k_u \in \omega$. We set $Y_u = X_u \restriction^{-1} \zeta$ for all $u \in 2^{<\omega}$.

Let us define a continuous function F' on the set $X = \bigcap_m \bigcup_{u \in 2^m} Y_u \in \mathrm{Perf}_\zeta$ as follows. Let $x \in X$, $m \in \omega$. There exists unique $u \in 2^m$ such that $x \in Y_u$. We put $F'(x)(m) = k_u$. The function F' can be expanded to a function $F \in \mathrm{Cont}_\zeta$ (that is, defined on $\mathcal{D}^\zeta$). Then X forces $\underset{\sim}{r} = F'^\#(\mathbf{x} \restriction \zeta) = F^\#(\mathbf{x} \restriction \zeta)$. □

The "Sacksness"

We are going to prove that the model $\mathfrak{N}$ is a sort of iterated Sacks generic extension of $\mathfrak{M}$, i. e. every real $\mathbf{a}_i$ is Sacks generic over the model $\mathfrak{M}[\langle \mathbf{a}_j : j < i\rangle]$. (To be more exact, we shall not actually prove that, in the case when $\mathbf{I}$ is an ordinal in the ground model $\mathfrak{M}$, the extension $\mathfrak{N}$ is equal to a "conventional" countable support iterated Sacks generic extension of $\mathfrak{M}$. This more substantial characterization also true, but would need much more efforts.)

Theorem 25 *Every* $\mathbf{a}_i$ *is Sacks generic over* $\mathfrak{M}[\mathbf{x}\restriction_{<i}] = \mathfrak{M}[\langle \mathbf{a}_j : j < i\rangle]$.

Before the proof starts, we have to present one more construction of forcing conditions. Perhaps, Section 1 would be a more appropriate place, but we decide to put it here because it is used only to prove Theorem 25.

We consider trees $T \subseteq 2^{<\omega}$. Let a $2^{<\omega}$*-like tree* be any (nonempty) tree $T \subseteq 2^{<\omega}$ such that the set $B(T) = \{t \in T : t^\wedge 0 \in T \;\&\; t^\wedge 1 \in T\}$ of all splitting points of T is cofinal in T. Suppose T is such a tree. We define the following objects.

- $[T] = \{a \in 2^\omega : \forall m\, (a \restriction m \in T)\}$, a perfect set is $\mathcal{D} = 2^\omega$. (Conversely if $P \subseteq \mathcal{D}$ is a perfect set then $T = \{a \restriction m : a \in P \;\&\; m \in \omega\}$ is a $2^{<\omega}$-like tree satisfying $X = [T]$.)
- An order isomorphism $\beta_T : 2^{<\omega}$ onto $B(T)$. We define $\beta_T(u) \in B(T)$ for every $u \in 2^{<\omega}$ by induction on $\operatorname{dom} u$, putting $\beta_T(u^\wedge e)$ to be the least $s \in B(T)$ such that $\beta_T(u)^\wedge e \subseteq s$, for $e = 0, 1$.
- A homeomorphism $H_T : \mathcal{D}$ onto $[T]$ by $H_T(a) = \bigcap_{m\in\omega} \beta_T(a \restriction m)$ for all $a \in \mathcal{D}$.

Lemma 26 *Assume that* i *is the largest element of* $\zeta \in \Xi$, $\eta = \zeta \setminus \{i\}$, $Y \in \mathrm{Perf}_\eta$, $y \longmapsto T(y)$ *is a continuous map* Y *into* $\mathcal{P}(2^{<\omega})$, *and* $T(y)$ *is a* $2^{<\omega}$*-like tree for all* $y \in Y$. *Then the set* $X = \{x \in \mathcal{D}^\zeta : x \restriction \eta \in Y \;\&\; x(i) \in [T(x \restriction \eta)]\}$ *belongs to* Perf_ζ .

Proof of the lemma. The set $Z = Y \restriction^{-1} \zeta$ belongs to Perf_ζ by Lemma 12, so it suffices to define a projection–keeping homeomorphism $H : Z$ onto X, by Lemma 8. Let $z \in Z$. Then $y = z \restriction \eta \in Y$ while $a = z(i) \in \mathcal{D}$ is arbitrary. We define $x = H(z) \in \mathcal{D}^\zeta$ so that $x \restriction \eta = y$ and $x(i) = H_{T(y)}(a)$. Then H maps Z onto X because every $H_{T(y)}$ maps $\mathcal{D}$ onto $[T(y)] = \{x(i) : x \in X \;\&\; x \restriction \eta = y\}$. H is $1-1$ since each H_T is $1-1$, and H is continuous since so is the map $y \longmapsto T(y)$. It remains to prove that H is projection–keeping, i. e. $z_0 \restriction \xi = z_1 \restriction \xi \;\longleftrightarrow\; H(z_0) \restriction \xi = H(z_1) \restriction \xi$ for all $z_0, z_1 \in Z$ and $\xi \in \mathbf{IS}_\zeta$. If $i \notin \xi$ then $\xi \subseteq \eta$ and $z \restriction \xi = H(z) \restriction \xi$ by definition. If $i \in \xi$ then $\xi = \zeta$, so the result is obvious as well. □

Proof of Theorem 25. Suppose that $S \in \mathfrak{M}[\mathbf{x}\restriction_{<i}]$ is, in $\mathfrak{M}[\mathbf{x}\restriction_{<i}]$, a dense subset in the collection of all perfect subsets of $\mathcal{D} = 2^\omega$; we have to prove that $\mathbf{a}_i \in P^\#$ for

some $P \in S$. Assume on the contrary that a condition $X_0 \in G \cap \mathbb{P}_\zeta$ ($\zeta \in \Xi$) forces the opposite. Since the forced statement is relativized to $\mathfrak{M}[\mathbf{x}\restriction_{\le i}]$, we may assume that $\zeta \subseteq [\le i]$. We can also suppose that $i \in \zeta$, so that i is a maximal element in ζ. We put $\eta = \zeta \cap [<i] = \zeta \setminus \{i\}$; η is an initial segment in ζ.

We argue in $\mathfrak{M}$.

Note that the set $D(y) = D_{X_0 y}(i) = \{x(i) : x \in X_0 \ \& \ x \restriction \eta = y\}$ is a perfect subset of $\mathcal{D} = 2^\omega$ for all $y \in Y_0 = X_0 \restriction \eta$ by property P-1 of X_0 (see Proposition 5).

We argue in $\mathfrak{M}[\mathbf{x}\restriction_{<i}]$.

Take notice that $\mathbf{y} = \mathbf{x} \restriction \eta$ belongs to $Y_0^{\#}$. Therefore $D^{\#}(\mathbf{y})$ is a perfect set. Thus there exists a set $P \in S$ such that $P \subseteq D^{\#}(\mathbf{y})$.

By the assumption, $\mathbf{a}_i = \mathbf{x}(i) \notin P$.

We put $\tau = \{p \restriction m : p \in P \ \& \ m \in \omega\}$. Then τ is a $2^{<\omega}$-like tree and $P = [\tau]$. By Theorem 24, there exist: $\xi \in \Xi$ and a continuous map $y \longmapsto T^{\#}(y)$: $\mathcal{D}^\xi$ into $\mathcal{P}(2^{<\omega})$, coded in $\mathfrak{M}$, such that $\xi \subseteq [<i]$ and $\tau = T^{\#}(\mathbf{x} \restriction \xi)$. We can assume that $\xi \subseteq \zeta$ (otherwise put $\zeta' = \zeta \cup \xi$ and $X_0' = X_0 \restriction^{-1} \zeta'$ in $\mathfrak{M}$, etc.). Then $\xi \subseteq \eta$, so it can be assumed that simply $\xi = \eta$. Then $\tau = T^{\#}(\mathbf{y})$, so that $[T^{\#}(\mathbf{y})] = P \subseteq D^{\#}(\mathbf{y})$.

The statement " $T^{\#}(\mathbf{y})$ is a $2^{<\omega}$-like tree, $[T^{\#}(\mathbf{y})] \in S$, and $[T^{\#}(\mathbf{y})] \subseteq D^{\#}(\mathbf{y})$ " is relativized to $\mathfrak{M}[\mathbf{y}] = \mathfrak{M}[\mathbf{x} \restriction \eta]$; therefore it is forced by a condition $Y_1 \in G$ stronger than Y_0 and such that $\xi = \|Y_1\| \subseteq [<i]$. As above, we can assume that in fact $\xi = \eta$, so that $Y_1 \subseteq Y_0$.

We argue in $\mathfrak{M}$.

The set $U = \{y \in Y_1 : T(y)$ is a $2^{<\omega}$-like tree and $[T(y)] \subseteq D(y)\}$ is a subset of Y_1 of a finite Borel level because T is continuous. Therefore, by Corollary 18, there exists a set $Y \in \mathrm{Perf}_\eta$ such that either $Y \subseteq U$ or $Y \cap U = \emptyset$.

Suppose that $Y \cap U = \emptyset$. Then by Shoenfield Y would force that either $T^{\#}(\mathbf{y})$ is not a $2^{<\omega}$-like tree or $[T^{\#}(\mathbf{y})] \not\subseteq D^{\#}(\mathbf{y})$ – contradiction with the choice of Y_1. Therefore in fact $Y \subseteq U$. In particular $T(y)$ is a $2^{<\omega}$-like tree for all $y \in Y$.

It follows that the set $X = \{x \in \mathcal{D}^\zeta : x \restriction \eta \in Y$ and $x(i) \in [T(x \restriction \eta)]\}$ belongs to Perf_ζ by Lemma 26 and, since $Y \subseteq U$, we have $[T(y)] \subseteq D(y) = D_{X_0 y}(i)$ for all $y \in Y$, so that $X \subseteq X_0$.

Since X is also stronger than Y_1, X forces everything which is forced by X_0 and/or Y_1, and everything which logically follows from the mentioned.

In particular, since X_0 forces that $\mathbf{a}_i$ does not belong to a set in S while Y_1 forces that $[T^{\#}(\mathbf{y})] \in S$, we conclude that X forces $\mathbf{a}_i \notin [T^{\#}(\mathbf{y})]$. It follows that X forces $\mathbf{a}_i \notin D_{X^{\#}\mathbf{y}}(i)$ because by definition $D_{Xy}(i) = [T(y)]$. This means that X forces $\mathbf{x} \restriction \zeta \notin X^{\#}$, contradiction. □

5 Reals in the extension

Theorem 24 practically reduces properties of reals in $\mathbb{P}$-generic extensions to properties of continuous functions in the ground model.

To demonstrate how Theorem 24 works we prove several lemmas on reals in a $\mathbb{P}$-generic model $\mathfrak{N} = \mathfrak{M}[G]$. Theorem 20 will be taken as a sourse of different properties of continuous functions in the ground model.

We keep the notation of the previous section.

Lemma 27 *If* $i, j \in \mathbf{I}$ *and* $i < j$ *then* $\mathbf{a}_i \in \mathfrak{M}[\mathbf{a}_j]$.

Proof Theorem 20 (item 1) implies the existence of a condition $X \in G$ such that, in $\mathfrak{M}$, the co-ordinate function C_j captures i on X. In other words, in $\mathfrak{M}$ there exists a continuous function $H : \mathcal{D} \longrightarrow \mathcal{D}$ such that $x(i) = H(x(j))$ for all $x \in X$. It follows that $x(i) = H^{\#}(x(j))$ for all $x \in X^{\#}$ is true in $\mathfrak{N}$. ($H^{\#}$ is the topological closure of H as a subset of $\mathcal{D}^2$.) Therefore $\mathbf{a}_i = H^{\#}(\mathbf{a}_j) \in \mathfrak{M}[\mathbf{a}_j]$. □

Lemma 28 *Suppose that* $J \in \mathfrak{M}$ *is an initial segment in* $\mathbf{I}$ *and* $i \in \mathbf{I} \setminus J$. *Then* $\mathbf{a}_i \notin \mathfrak{M}[\mathbf{x} \restriction J]$.

Proof Assume on the contrary that $\mathbf{a}_i \in \mathfrak{M}[\mathbf{x} \restriction J]$. Applying Theorem 24, we obtain in $\mathfrak{M}$ a set $\xi \in \Xi$, $\xi \subseteq J$, a function $F \in \mathrm{Cont}_\xi$, and a condition $X \in \mathbb{P}$ which forces $\mathbf{a}_i = F^{\#}(\mathbf{x} \restriction \xi)$. Let $\zeta = \|X\|$, so that $X \in \mathrm{Perf}_\zeta$ in $\mathfrak{M}$. We can assume that $\xi \in \mathbf{IS}_\zeta$ and $i \in \zeta$. (Otherwise put $\zeta' = \zeta \cup \xi \cup \{i\}$, $\xi' = \{j' \in \zeta' : \exists j \in \xi\, (j' \le j)\}$, define $F'(x') = F(x' \restriction \xi)$ for $x' \in \mathcal{D}^{\zeta'}$, and consider $X' = X \restriction^{-1} \zeta'$.)

We argue in $\mathfrak{M}$.

We have $x(i) = F(x \restriction \xi)$ for all $x \in X$, because $\mathbf{a}_i = F^{\#}(\mathbf{x} \restriction \xi)$ is forced by X. (Indeed otherwise there exist $m \in \omega$ and a condition $Y \subseteq X$, $Y \in \mathrm{Perf}_\zeta$ such that $x(i)(m) = 0$ but $F(x \restriction \xi)(m) = 1$, or vice versa, for all $x \in Y$, by Lemma 9, a contradiction with the choice of X .) Thus the co-ordinate function C_i is reducible to ξ on X, a contradiction with Theorem 20 (item 2) because $i \notin \xi$. □

Lemma 29 *Suppose that* $\xi \in \Xi$ *and* r *is a real in* $\mathfrak{N}$ *such that* $\mathbf{a}_i \in \mathfrak{M}[r]$ *for all* $i \in \xi$. *Then the indexed set* $\mathbf{x} \restriction \xi = \langle \mathbf{a}_i : i \in \xi \rangle$ *belongs to* $\mathfrak{M}[r]$.

Proof Otherwise by Theorem 24 there exist: a set $\zeta \in \Xi$ such that $\xi \subseteq \zeta$, a function $F \in \mathbb{F}_\zeta$, and a condition $X \in \mathbb{P}_\zeta$ which forces that $\mathbf{a}_i \in \mathfrak{M}[F^{\#}(\mathbf{x} \restriction \zeta)]$ for each $i \in \xi$, but also forces $\mathbf{x} \restriction \xi \notin \mathfrak{M}[F^{\#}(\mathbf{x} \restriction \zeta)]$. One can assume, by Lemma 27, that ξ is an initial segment of ζ .

We argue in $\mathfrak{M}$.

We assert that if $i \in \xi$ then for any set $X' \in \mathrm{Perf}_\zeta$, $X' \subseteq X$, F captures i on a condition $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X'$. (Indeed, otherwise by Theorem 20 – item 4, there exists a condition $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$ such that F is reducible to $\eta = \zeta \cap [\not\geq i]$ on Y. Then Y forces $F^\#(\mathbf{x} \restriction \zeta) \in \mathfrak{M}[\mathbf{x} \restriction \eta]$, hence forces $\mathbf{a}_i \in \mathfrak{M}[\mathbf{x} \restriction \eta]$ by the choice of X, contradiction with Lemma 28 since $i \notin \eta$.)

Now, using Theorem 20 (item 3), we obtain a condition $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$ such that F captures each $i \in \xi$ on Y. This implies the existence of a continuous function $H : \mathcal{N} \longrightarrow \mathcal{D}^\xi$ such that $\mathbf{x} \restriction \xi = H(F(x))$ for all $x \in Y$. We conclude that Y forces $\mathbf{x} \restriction \xi \in \mathfrak{M}[F^\#(\mathbf{x} \restriction \zeta)]$, contradiction. □

Lemma 30 *Suppose that* $J \in \mathfrak{M}$ *is an initial segment of* $\mathbf{I}$, *and* r *is a real in* $\mathfrak{N}$. *Then either* $r \in \mathfrak{M}[\mathbf{x} \restriction J]$ *or there exists* $i \notin J$ *such that* $\mathbf{a}_i \in \mathfrak{M}[r]$.

Proof It follows from Theorem 24 that $r = F^\#(\mathbf{x} \restriction \zeta)$ for some $\zeta \in \Xi$ and a function $F \in \mathrm{Cont}_\zeta$ in $\mathfrak{M}$. Let this be forced by some $X \in \mathrm{Perf}_\zeta$. We assume on the contrary that r *does not* satisfy the requirements of the lemma, and this also is forced by X.

We argue in $\mathfrak{M}$.

We put $\xi = \zeta \cap J$. Then ξ is an initial segment of ζ. It is implied by Theorem 20 (item 5) that there exists $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$, such that either F is reducible to ξ on Y or F captures some $i \in \zeta \setminus \xi$ on Y.

Consider the "either" case. There exists (see Remark 19) a function $F' \in \mathrm{Cont}_\xi$ such that $F(x) = F'(x \restriction \xi)$ for all $x \in Y$. In this case Y forces that $r \in \mathfrak{M}[\mathbf{x} \restriction \xi]$, contradiction with the choice of X and Y because $\xi \subseteq J$.

Consider the "or" case. There exists a continuous $H : \mathcal{N} \longrightarrow \mathcal{D}$ such that $x(i) = H(F(x))$ for all $x \in Y$. Then Y forces $\mathbf{a}_i = \mathbf{x}(i) = H^\#(F^\#(\mathbf{x} \restriction \zeta)) \in \mathfrak{M}[F^\#(\mathbf{x} \restriction \zeta)]$, again a contradiction with the choice of X and Y because $i \notin J$. □

The "discrete" case and the degrees of constructibility

In this subsection we consider a special but quite important class of sets $\mathbf{I}$ which admit a complete description of the degrees of $\mathfrak{M}$-constructibility of reals in the extension. As a rather simple corollaries, we shall prove theorems 3 and 4.

We keep the notation introduced above.

Definition A (partially ordered) set $\mathbf{I} \in \mathfrak{M}$ is called $\mathfrak{M}$-*discrete* iff all initial segments of $\mathbf{I}$ belong to $\mathfrak{M}$. □

For instance $\mathbb{Z}$ (the integers), ordinals, and inverse ordinals are discrete. Rationals and reals in $\mathfrak{M}$ are not discrete. An infinite set with the empty order is not discrete.

For a real $r \in \mathfrak{N}$, we set $\mathbf{I}_r = \{i \in \mathbf{I} : \mathbf{a}_i \in \mathfrak{M}[r]\}$, then $\mathbf{I}_r$ is an initial segment of $\mathbf{I}$ by Lemma 27. The following theorem shows, in particular, that in the case of a discrete set $\mathbf{I}$ the $\mathfrak{M}$-degrees of reals in $\mathfrak{M}$ are in a 1–1 correspondence with initial segments of $\mathbf{I}$ having countable cofinality in $\mathfrak{M}$.

Theorem 31 *Suppose that* $\mathbf{I}$ *is* $\mathfrak{M}$*-discrete. Then*

1. *For each real* $r \in \mathfrak{N}$, $\mathbf{I}_r$ *belongs to* $\mathfrak{M}$ *and has countable cofinality* [13] *in* $\mathfrak{M}$. *Conversely each initial segment* $J \in \mathfrak{M}$, $J \subseteq \mathbf{I}$, *of countable cofinality in* $\mathfrak{M}$, *has the form* $J = \mathbf{I}_r$ *for a real* $r \in \mathfrak{N}$.
2. *If* $\zeta \in \Xi$ *is cofinal in* $\mathbf{I}_r$ *then* $\mathfrak{M}[r] = \mathfrak{M}[\mathbf{x} \restriction \zeta]$.
3. *For all reals* $r, r' \in \mathfrak{N}$, $r \in \mathfrak{M}[r']$ *iff* $\mathbf{I}_r \subseteq \mathbf{I}_{r'}$.
4. *For all reals* $r, r' \in \mathfrak{N}$, *if* $r \in \mathfrak{M}[r']$ *then there exists* $H \in \mathfrak{M}$, *a continuous map* reals $\longrightarrow$ reals *from the* $\mathfrak{M}$*'s point of view, such that* $r = H^{\#}(r')$. [14]

Proof *Item 1.* First of all, $\mathbf{I}_r \in \mathfrak{M}$ since $\mathbf{I}$ is $\mathfrak{M}$-discrete. We have $r \in \mathfrak{M}[\mathbf{x} \restriction \mathbf{I}_r]$ by Lemma 30. Hence $r \in \mathfrak{M}[\mathbf{x} \restriction \xi]$ for some $\xi \in \Xi$, $\xi \subseteq \mathbf{I}_r$ by Theorem 24. It follows that $\mathbf{a}_i \in \mathfrak{M}[\mathbf{x} \restriction \xi]$ whenever $i \in \mathbf{I}_r$. Therefore ξ is cofinal in $\mathbf{I}_r$ by Lemma 28.

Conversely, suppose that $J \in \mathfrak{M}$ is an initial segment of $\mathbf{I}$ of countable cofinality in $\mathfrak{M}$. Let $\xi \in \mathfrak{M}$ be a countable in $\mathfrak{M}$ cofinal subset of J. Obviously there exists a real $r \in \mathfrak{N}$ such that $\mathfrak{M}[r] = \mathfrak{M}[\mathbf{x} \restriction \xi]$. One easily proves that $J = \mathbf{I}_r$ using Lemma 27.

Item 2. Let $\zeta \in \Xi$ be cofinal in $\mathbf{I}_r$. Then $\mathbf{x} \restriction \zeta \in \mathfrak{M}[r]$ by Lemma 29. As above, $r \in \mathfrak{M}[\mathbf{x} \restriction \xi]$ for some $\xi \in \Xi$, $\xi \subseteq \mathbf{I}_r$. Let $r' \in \mathfrak{N}$ be a real which codes $\mathbf{x} \restriction \zeta$ in the sense that $\mathfrak{M}[r'] = \mathfrak{M}[\mathbf{x} \restriction \zeta]$. Then, since every $i \in \xi$ is $\leq$ than some $j \in \zeta$ by the cofinality of ζ, we have $\mathbf{a}_i \in \mathfrak{M}[r']$ for all $i \in \xi$ by Lemma 27. Therefore $\mathbf{x} \restriction \xi \in \mathfrak{M}[r']$ by Lemma 29. We conclude that $r \in \mathfrak{M}[\mathbf{x} \restriction \zeta]$, as required.

Item 3. Suppose that $\mathbf{I}_r \subseteq \mathbf{I}_{r'}$. As above there exists $\xi \in \Xi$, $\xi \subseteq \mathbf{I}_r$, such that $r \in \mathfrak{M}[\mathbf{x} \restriction \xi]$. Then we have $\xi \subseteq \mathbf{I}_{r'}$ as well, hence $\mathbf{x} \restriction \xi \in \mathfrak{M}[r']$ by Lemma 29.

Item 4. Let $J = \mathbf{I}_{r'}$. Assume on the contrary that such a function H does not exist. Arguing as above, we find $\zeta \in \Xi$ and functions $F, F' \in \mathbb{F}_\zeta$ such that $r = F^{\#}(\mathbf{x} \restriction \zeta)$ and $r' = F'^{\#}(\mathbf{x} \restriction \zeta)$, and a condition $X \in \mathbb{P}_\zeta$ which forces the assumption as a property of ζ, F, F', and also forces that $J = \mathbf{I}_{F'(\mathbf{x} \restriction \zeta)}$ and $F^{\#}(\mathbf{x} \restriction \zeta) \in \mathfrak{M}[F'^{\#}(\mathbf{x} \restriction \zeta)]$.

We argue in $\mathfrak{M}$.

Then $X \in \mathrm{Perf}_\zeta$ and $F, F' \in \mathrm{Cont}_\zeta$. Let $\xi = \zeta \cap J$.

Fact 1. There exists $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$ such that F' captures *every* $i \in \xi$ on Y.

Indeed we observe that for all $i \in \xi$ and $X' \in \mathrm{Perf}_\zeta$, $X' \subseteq X$, there exists $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X'$ such that F' captures i on Y. (Otherwise by Theorem 20 – item 4 there would exist a condition $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X'$ such that F' is reducible to $\eta = \{j \in \zeta : j \not\geq i\}$ on Y for some $i \in \xi$. Such a condition Y forces that

[13] We understand countable cofinality so that it includes in particular sets having the largest element.

[14] This item should be true independently of the assumption that $\mathbf{I}$ is discrete. In fact it should be true that, given a pair of functions $F, F' \in \mathrm{Cont}_\zeta$ and a set $X \in \mathrm{Perf}_\zeta$, there exists $Y \in \mathrm{Perf}_\zeta$, $Y \subseteq X$, such that either for a continuous H : reals $\longrightarrow$ reals we have $F(x) = H(F'(x))$ for all $x \in Y$, or, for some $i \in \zeta$, F captures i on Y but F' is reducible to $[\not\geq i]$ on Y.

$F'^{\#}(\mathbf{x}\restriction\zeta)$ belongs to $\mathfrak{M}[\mathbf{x}\restriction\eta]$, therefore that $\mathbf{a}_i$ belongs to $\mathfrak{M}[\mathbf{x}\restriction\eta]$, contradiction with Lemma 28.) It remains to apply Theorem 20 (item 3).

Fact 2. There exists $Z \in \mathrm{Perf}_\zeta$, $Z \subseteq Y$ such that F is reducible to ξ on Z.

Indeed otherwise by Theorem 20 (item 5) F would capture some $i \in \zeta\setminus\xi$ – therefore $i \notin J$ – on some $Y' \in \mathrm{Perf}_\zeta$, $Y' \subseteq Y$. Such a condition Y' forces $\mathbf{a}_i \in \mathfrak{M}[\mathbf{x}\restriction J]$, contradiction because $i \notin J$.

To end the proof of item 4, we observe that by the choice of Y and Z one has $F(x) = H(F'(x))$ for all $x \in Z$, for a certain continuous function H : reals $\longrightarrow$ reals, one and the same for all x. Then Z forces $F^{\#}(\mathbf{x}\restriction\zeta) = H^{\#}(F'^{\#}(\mathbf{x}\restriction\zeta))$, contradiction with the choice of X. □

A model in which all nonconstructible reals collapse κ to $\aleph_1$

To get a model for Theorem 3 we suppose that κ is an uncountable cardinal in $\mathfrak{M}$. Let $\mathbf{I} = \kappa^*$ (i. e. κ with the inverse order). Obviously $\mathbf{I}$ is $\mathfrak{M}$-discrete.

Take notice that every (nonempty) initial segment of $\mathbf{I}$ has cardinality κ in $\mathfrak{M}$. In this case, since by Lemma 31 $\mathbf{I}_r$ is nonempty for any real $r \notin \mathfrak{M}$, and all reals $\mathbf{a}_i$ are pairwise different by Lemma 28, $\mathfrak{M}[r]$ contains at least κ different reals. However $\aleph_1^{\mathfrak{M}}$ is preserved by Theorem 23. This proves Theorem 3. □

A cardinal invariant to distinguish iterated and product Sacks forcing

We prove Theorem 4. We recall that the cardinal $\mathfrak{l}$ was defined in Introduction.

Proposition 32 *In any countable support* <u>*product*</u> *Sacks extension of* $\mathfrak{M}$ *with at least* $\mathfrak{c}^{\mathfrak{M}}$*-many factors,* $\mathfrak{l} = \mathfrak{c}$.

Proof In such an extension, $\mathfrak{c}$ is equal to the number κ of factors. Indeed since the reals $\mathbf{a}_\alpha$ are pairwise incompatible in the sense of the $\mathfrak{M}$-constructibility, there cannot exist (in the extension) a family $\mathcal{F}$ of less than κ functions f : reals $\longrightarrow$ reals such that $\leq_{\mathcal{F}}$ linearly orders the reals. □

Proposition 33 *In any* <u>*iterated*</u> *Sacks extension* $\mathfrak{N}$ *of* $\mathfrak{M}$, *of the type we introduced in Section 4, via a* $\mathfrak{M}$*-discrete p. o. set* $\mathbf{I} \in \mathfrak{M}$, $\mathfrak{l} \leq \mathrm{card}\, \mathfrak{c}^{\mathfrak{M}}$.

Proof The order $<_{\mathcal{F}}$ determined by family $\mathcal{F}$ of all continuous functions coded in $\mathfrak{M}$ — is a linear ordering on the reals in $\mathfrak{N}$ by Theorem 31 (items 1 and 4). □

We observe that in the case of "long" products and iterations (strictly more than $\mathfrak{c}^{\mathfrak{M}}$ factors or iteration steps), the invariant $\mathfrak{l}$ really makes a distinction between the product and iterated models. □

6 Non–Glimm–Effros equivalence relations

This section presents the proof of theorems 1 and 2. The proofs differ in some detail, but also have much in common, in particular are based on several facts of general nature. Therefore we start with those general properties of the iterated Sacks models, and then detalize the reasoning at the appropriate splitting point.

We keep the notation ($\mathbb{P}$, $\mathbb{P}_\zeta$ for $\zeta \in \Xi$, $\mathbf{x}$ etc.) of the preceding sections, but assume the following in addition:

(i) $\mathfrak{M}$, the ground model, satisfies the axiom of constructibility $\mathrm{V} = \mathrm{L}$.

(ii) $\mathbf{I}$ is an $\mathfrak{M}$-*discrete* set, that is, all initial segments of $\mathbf{I}$ belong to $\mathfrak{M}$.

Let us fix a $\mathbb{P}$-generic over $\mathfrak{M}$ set $G \subseteq \mathbb{P}$ and consider the equivalence relation C, defined on reals by

$$x \mathrel{\mathsf{C}} y \quad \text{iff} \quad \mathrm{L}[x] = \mathrm{L}[y]\,,$$

in the model $\mathfrak{N} = \mathfrak{M}[G] = \mathfrak{M}[\mathbf{x}] = \mathfrak{M}[\langle \mathbf{a}_i : i \in \mathbf{I}\rangle]$. Take notice that C-equivalence classes, degrees of constructibility of reals, and degrees of $\mathfrak{M}$-constructibility of reals — is one and the same in $\mathfrak{N}$ since $\mathfrak{M}$ models $\mathrm{V} = \mathrm{L}$.

We say that a set S of reals is C-*invariant* if $x \mathrel{\mathsf{C}} y$ implies $x \in S \longleftrightarrow y \in S$ for any two reals x, y. We say that a variable v is C-*invariant* in a formula $\varphi(v)$ if it enters the formula only through the expression $\mathrm{L}[v]$.

Applications of uniformity of the forcing

In this subsection, we exploit the uniformity of the forcing, to obtain some definability results.

For a set $I \in \mathfrak{M}$, $I \subseteq \mathbf{I}$, we let $\underline{\mathbf{x} \restriction I}$ be the name for $\mathbf{x} \restriction I = \langle \mathbf{a}_i : i \in I\rangle$ in the forcing language associated with $\mathbb{P}$, to avoid ambiguities.

Proposition 34 *Suppose that* $\xi \in \Xi$, J *is an initial segment of* $\mathbf{I}$, *and the variable* v *is* C-*invariant in* $\varphi(\mathbf{x} \restriction J, v)$, *a formula containing ordinals and* $\mathbf{x} \restriction J$ *as parameters. Assume that* $\vartheta \in \Xi$, $\vartheta' = \vartheta \cap J$, *and a condition* $Z \in \mathbb{P}_\vartheta$ *forces* $\varphi(\underline{\mathbf{x} \restriction J}, \underline{\mathbf{x} \restriction \xi})$. *Then the weaker condition* $Z' = Z \restriction \vartheta'$ *forces* $\varphi(\underline{\mathbf{x} \restriction J}, \underline{\mathbf{x} \restriction \xi})$ *as well.*

Note that the assertion is not merely an example of the usual "restriction" argument because it is not excluded that $\xi \not\subseteq J$. However $\underline{\mathbf{x} \restriction \xi}$ enters the formula in quite a specific way: in fact only the L-degree of $\underline{\mathbf{x} \restriction \xi}$ rather than $\underline{\mathbf{x} \restriction \xi}$ itself participates in the formula. This makes it possible to use the homeomorphisms included in the definition of forcing conditions in Section 1.

Proof Assume that this is not the case. We assert that there exist: $\zeta \in \Xi$ and a pair of conditions $X, Y \in \mathbb{P}_\zeta$ such that $X \restriction \zeta' = Y \restriction \zeta'$, where $\zeta' = \zeta \cap J$, X forces $\varphi(\underline{\mathbf{x} \restriction J}, \underline{\mathbf{x} \restriction \xi})$, but Y forces $\neg\, \varphi(\underline{\mathbf{x} \restriction J}, \underline{\mathbf{x} \restriction \xi})$.

(Indeed, let us argue in $\mathfrak{M}$. There exists a condition $Z^* \in \mathrm{Perf}_{\vartheta^*}$ for some $\vartheta^* \supseteq \vartheta$, stronger than Z', which forces $\neg\, \varphi(\underline{\mathbf{x} \restriction J}, \underline{\mathbf{x} \restriction \xi})$. We define $\zeta = \vartheta^* \cup \vartheta$ and $Y = Z^* \restriction^{-1} \zeta$, $X^* = Z \restriction^{-1} \zeta$; then $Y, X^* \in \mathrm{Perf}_\zeta$ by Lemma 12, X^* forces $\varphi(\underline{\mathbf{x} \restriction J}, \underline{\mathbf{x} \restriction \xi})$ and Y forces $\neg\, \varphi(\underline{\mathbf{x} \restriction J}, \underline{\mathbf{x} \restriction \xi})$. To obtain X, let $\zeta' = \zeta \cap J$; then $\zeta' = \vartheta^* \cap J$. Then $X^* \restriction \zeta' = Z' \restriction^{-1} \zeta'$ while $Y \restriction \zeta' = Z^* \restriction \zeta'$. Therefore $Y \restriction \zeta' \subseteq X^* \restriction \zeta'$ because Z^* is stronger than Z'. We conclude that $X = X^* \cap (Y \restriction \zeta' \restriction^{-1} \zeta) \in \mathrm{Perf}_\zeta$ by Lemma 11, and $X \restriction \zeta' = Y' = Y \restriction \zeta'$. Finally, $X \subseteq X^*$, therefore X forces $\varphi(\underline{\mathbf{x} \restriction J}, \underline{\mathbf{x} \restriction \xi})$.)

In $\mathfrak{M}$, both X and Y are members of Perf_ζ. Lemma 10 asserts that there exists a projection–keeping homeomorphism $H : X$ onto Y, satisfying $x \restriction \zeta' = H(x) \restriction \zeta'$ for all $x \in X$, because $X \restriction \zeta' = Y \restriction \zeta'$.

The homeomorphism H induces a total order automorphism of the part of $\mathbb{P}$ stronger than X onto the part of $\mathbb{P}$ stronger than Y, by lemmas 8 and 12. Take notice that this automorphism does not change projections outside of $\zeta \setminus \zeta'$, therefore does not change the projection on J because $\zeta' = \zeta \cap J$.

Applying the automorphism to the given generic set G, we obtain a $\mathbb{P}$-generic over $\mathfrak{M}$ set $G' \subseteq \mathbb{P}$ and the corresponding $\mathbf{x}' \in \mathfrak{D}^{\mathbf{I}}$ such that $Y \in G'$, $\mathfrak{M}[G] = \mathfrak{M}[G']$, $\mathbf{x} \restriction J = \mathbf{x}' \restriction J$ (by the "does not change projections" property above), and finally $\mathfrak{M}[\mathbf{x} \restriction \xi] = \mathfrak{M}[\mathbf{x}' \restriction \xi]$ for all $\xi \in \Xi$ because $H \in \mathfrak{M}$. Thus one and the same generic extension $\mathfrak{N} = \mathfrak{M}[G] = \mathfrak{M}[G']$ is defined using two different generic sets.

We observe that the statement $\varphi(\mathbf{x} \restriction J, \mathbf{x} \restriction \xi)$ is true in $\mathfrak{N} = \mathfrak{M}[G]$ while the statement $\varphi(\mathbf{x} \restriction J, \mathbf{x}' \restriction \xi)$ is false in $\mathfrak{N} = \mathfrak{M}[G']$ by the choice of X and Y, contradiction since the variable v is C-invariant in the formula $\varphi(\mathbf{x} \restriction J, v)$. □

Corollary 35 *Suppose that* $\xi \in \Xi$, J *is an initial segment of* $\mathbf{I}$, *and the variable* v *is* C*-invariant in* $\varphi(\mathbf{x} \restriction J, v, \alpha)$, *a formula containing ordinals and* $\mathbf{x} \restriction J$ *as parameters. Then the set* $\Omega_\lambda = \{\alpha < \lambda : \mathfrak{N} \models \varphi(\mathbf{x} \restriction J, \mathbf{x} \restriction \xi, \alpha)\}$ *belongs to* $\mathfrak{M}[\mathbf{x} \restriction J]$ *for every ordinal* $\lambda \in \mathfrak{M}$.

Proof We have $\Omega_\lambda = \{\alpha < \lambda : \exists\, X \in G\, [\, \|X\| \subseteq J \;\&\; X \text{ forces } \varphi(\underline{\mathbf{x} \restriction J}, \underline{\mathbf{x} \restriction \xi}, \alpha)]\}$ by Proposition 34. Therefore it suffices to prove that

$$\{X \in G : \|X\| \subseteq J\} = \{X \in \mathbb{P} : \|X\| \subseteq J \;\&\; \mathbf{x} \restriction \|X\| \in X^\#\}.$$

The nontrivial direction is $\supseteq$, so assume that $\vartheta \subseteq J$, $X \in \mathbb{P}_\vartheta$, and $\mathbf{x} \restriction \vartheta \in X^\#$, and prove that $X \in G$. Suppose on the contrary that some $Z \in G$ forces the opposite, and also forces that $\mathbf{x} \restriction \vartheta \in X^\#$. One may assume that $\vartheta \subseteq \zeta = \|Z\|$.

Then, in $\mathfrak{M}$, $X \in \mathrm{Perf}_\vartheta$ and $Z \in \mathrm{Perf}_\zeta$. Lemma 9 implies the existence of a set $Z' \in \mathrm{Perf}_\zeta$, $Z' \subseteq Z$, such that either $Z' \restriction \vartheta \subseteq X$ or $(Z' \restriction \vartheta) \cap X = \emptyset$. In the first case Z' is stronger than X, so Z' forces that X belongs to G, contradiction. In the second case, Z' forces that $\mathbf{x} \restriction \vartheta \notin X^\#$, contradiction as well. □

Applications of order automorphisms

An ordinal does not admit a nontrivial order automorphism. However both nonlinear wellfounded order relations and nonwellordered linear orders do admit. We consider the effects available in the case when $\mathbf{I}$, the intended "length" of the Sacks iteration, has a nontrivial order automorphism.

Proposition 36 *Suppose that J is an initial segment in $\mathbf{I}$, $h \in \mathfrak{M}$ is an order automorphism of $\mathbf{I}$, $h \restriction J$ is the identity, $i \in \mathbf{I}$, $h(i) = i' \neq i$, $A \in \mathfrak{N}$ is a set of reals, definable in $\mathfrak{N}$ by a formula containing only $\mathbf{x} \restriction J$ and ordinals as parameters. Then, in $\mathfrak{N}$, $A \cap [\mathbf{a}_i]_{\mathsf{C}} = \emptyset$ iff $A \cap [\mathbf{a}_{i'}]_{\mathsf{C}} = \emptyset$.*

Proof Let $A = \{r : \psi(\mathbf{x} \restriction J, r)\}$ in $\mathfrak{N}$, where ψ contains only $\mathbf{x} \restriction J$ and ordinals as parameters. Let $\varphi(\mathbf{x} \restriction J, r)$ be the formula $\exists r' (r \mathsf{C} r' \ \& \ \psi(\mathbf{x} \restriction J, r'))$.

Assume on the contrary that e.g. $A \cap [\mathbf{a}_i]_{\mathsf{C}} \neq \emptyset$ but $A \cap [\mathbf{a}_{i'}]_{\mathsf{C}} = \emptyset$ in $\mathfrak{N}$. This means that, in $\mathfrak{N}$, $\varphi(\mathbf{x} \restriction J, r)$ is true for any $r \in [\mathbf{a}_i]_{\mathsf{C}}$ and false for any $r \in [\mathbf{a}_{i'}]_{\mathsf{C}}$. Therefore a condition $X \in G$ forces

$$\forall r \, [r \mathsf{C} \mathbf{a}_i \longrightarrow \varphi(\mathbf{x} \restriction J, r) \text{ and } r \mathsf{C} \mathbf{a}_{i'} \longrightarrow \neg \varphi(\mathbf{x} \restriction J, r)]. \qquad (*)$$

Let $\vartheta = \|X\|$ and $\vartheta' = \vartheta \cap J$. It is implied by Proposition 34 (take $\xi = \{i\}$ and $\xi = \{i'\}$ independently) that even the weaker condition $Y = X \restriction \vartheta' \in G$ forces $(*)$.

The automorphism h obviously generates an order automorphism $Z \longmapsto Z'$: $\mathbb{P}$ onto $\mathbb{P}$. We observe that $Y' = Y$ because h is assumed to be the identity on the set $J \supseteq \vartheta' = \|Y\|$.

We set $G' = \{Z' : Z \in G\}$. Then $Y \in G'$, G' is $\mathbb{P}$-generic over $\mathfrak{M}$, and moreover, $\mathfrak{M}[G'] = \mathfrak{M}[G]$ because $h \in \mathfrak{M}$. Let $\mathbf{x}' = \langle \mathbf{a}'_j : j \in \mathbf{I} \rangle$ be defined from G' as $\mathbf{x}$ was defined from G. Then we have $\mathbf{a}'_{h(j)} = \mathbf{a}_j$ for all j; in particular (a) $\mathbf{a}'_{i'} = \mathbf{a}_i$, and (b) $\mathbf{x}' \restriction J = \mathbf{x} \restriction J$.

Since Y forces $(*)$, (b) implies $\neg \varphi(\mathbf{x} \restriction J, r)$ in $\mathfrak{N} = \mathfrak{M}[G']$ for any real $r \in \mathfrak{N}$ satisfying $r \mathsf{C} \mathbf{a}'_{i'}$ in $\mathfrak{N}$. On the other hand, the same property of Y directly implies $\varphi(\mathbf{x} \restriction J, r)$ in $\mathfrak{N} = \mathfrak{M}[G]$ provided $r \mathsf{C} \mathbf{a}_i$ in $\mathfrak{N}$, contradiction by (a). □

Proof of Theorem 1

We prove Theorem 1 in this subsection. In principle, a special choice of a model where C, the equiconstructibility on reals, neither admits a R-OD enumeration of the equivalence clesses by sets of ordinals, nor admits a R-OD pairwise C-inequivalent set of cardinality $\mathfrak{c}$, is not necessary. It occurs that everything what we need in addition to requirements (i) and (ii) (see the beginning of this section) is the three more requirements:

(iii) In $\mathfrak{M}$, $\mathbf{I}$ is not countably cofinal and has cardinality either $\aleph_1^{\mathfrak{M}}$ or $\aleph_2^{\mathfrak{M}}$.

(iv) Every proper (i. e. $J \neq \mathbf{I}$) initial segment $J \subseteq \mathbf{I}$ (belongs to $\mathfrak{M}$ by (ii) and) satisfies $\operatorname{card} J < \operatorname{card} \mathbf{I}$ in $\mathfrak{M}$.

(v) If J be a proper initial segment of $\mathbf{I}$ then there exists an order automorphism $h \in \mathfrak{M} : \mathbf{I}$ onto $\mathbf{I}$, equal to the identity on J but not equal to the identity on $\mathbf{I}$.

Surely a wellordered set $\mathbf{I}$ cannot satisfy (v), but we have both nonlinear wellfounded order relations and nonwellordered linear orders $\mathbf{I} \in \mathfrak{M}$ which do satisfy (ii) through (v), see the examples below.

Theorem 37 *Suppose that conditions* (i) *through* (v) *are satisfied. Then it is true in* $\mathfrak{N}$ *that* C *has* $\mathfrak{c}$*-many equivalence classes and*:

- *neither admits a R-OD enumeration of the equivalence classes by sets of ordinals*;
- *nor admits a R-OD pairwise* C*-inequivalent set of cardinality* $\mathfrak{c}$.

In addition, $\mathfrak{c} = \aleph_1^{\mathfrak{M}} = \aleph_1^{\mathfrak{N}}$ *provided* $\operatorname{card} \mathbf{I} = \aleph_1^{\mathfrak{M}}$ *in* $\mathfrak{M}$, *and* $\mathfrak{c} = \aleph_2^{\mathfrak{M}} = \aleph_2^{\mathfrak{N}}$ *provided* $\operatorname{card} \mathbf{I} = \aleph_2^{\mathfrak{M}}$ *in* $\mathfrak{M}$.

This theorem obviously implies Theorem 1, provided we are able to realize requirements (ii) through (v) on a partial order $\mathbf{I}$ in a countable model $\mathfrak{M} \models \mathrm{V} = \mathrm{L}$.

Proof *We prove the "additional" part of the theorem.* The cardinals $\aleph_1$ and $\aleph_2$ are preserved by Theorem 23. The reals $\mathbf{a}_i$ $(i \in \mathbf{I})$ are pairwise different by Lemma 27, therefore $\mathfrak{c} \geq \operatorname{card} \mathbf{I}$ in $\mathfrak{N}$. On the other hand, $\mathfrak{c} \leq \operatorname{card} \Xi \times \aleph_1^{\mathfrak{M}}$ in $\mathfrak{N}$ by Theorem 24, therefore $\mathfrak{c} \leq \operatorname{card} \mathbf{I}$ in $\mathfrak{N}$, whichever cardinality, $\aleph_1^{\mathfrak{M}}$ or $\aleph_2^{\mathfrak{M}}$, $\mathbf{I}$ has in $\mathfrak{M}$.

This reasoning also proves that C has $\mathfrak{c}$-many equivalence classes in $\mathfrak{M}$, because different $\mathbf{a}_i$ are C-nonequivalent, not merely different. Therefore it remains to prove the "neither" and "nor" statements.

We prove the "nor" part of the theorem. Let a pairwise C-inequivalent set S of reals be defined in $\mathfrak{N}$ by a formula containing ordinals and a real $p \in \mathfrak{N}$ as parameters. It follows from Theorem 31 (items 1 and 2) that $J = \mathbf{I}_p$ is an initial segment in $\mathbf{I}$ of countable cofinality in $\mathfrak{M}$; furthermore $p \in \mathfrak{M}[J]$ by Lemma 30. Then S is definable in $\mathfrak{N}$ by a formula containing $\mathbf{x} \restriction J$ and ordinals as parameters.

We assert that $S \subseteq \mathfrak{M}[\mathbf{x} \restriction J]$. Indeed, let $r \in S$. We have $\mathfrak{M}[r] = \mathfrak{M}[\mathbf{x} \restriction \xi]$ for some $\xi \in \Xi$ by Theorem 31 (items 1 and 2). Therefore r is definable in $\mathfrak{N}$ as the unique real $r \in S$ which satisfies the equality $\mathrm{L}[r] = \mathrm{L}[\mathbf{x} \restriction \xi]$. Then $r \in \mathfrak{M}[\mathbf{x} \restriction J]$ by Corollary 35, as required.

It remains to prove that reals in $\mathfrak{M}[\mathbf{x} \restriction J]$ generate less than $\mathfrak{c}$-many $\mathfrak{M}$-degrees in $\mathfrak{N}$. It suffices to check that J has $(< \operatorname{card} \mathbf{I})$-many initial segments in $\mathfrak{M}$, by Theorem 31 (item 3).

Since J is countably cofinal in $\mathfrak{M}$, it follows from (iii) that $J \neq \mathbf{I}$, therefore $\operatorname{card} J < \operatorname{card} \mathbf{I}$ in $\mathfrak{M}$ by (iv). We have two cases, by (iii).

Case 1: $\operatorname{card} \mathbf{I} = \aleph_1$ in $\mathfrak{M}$. Then J is countable in $\mathfrak{M}$. The collection $\mathbf{IS}_J$ of all initial segments of J is a Borel subset of 2^J, hence either $\mathbf{IS}_J$ belongs to $\mathfrak{M}$ and is countable in $\mathfrak{M}$, or $\mathbf{IS}_J \not\subseteq \mathfrak{M}$. However the "or" case is incompatible with (ii).

Case 2: $\operatorname{card} \mathbf{I} = \aleph_2$ in $\mathfrak{M}$. Then $\operatorname{card} J \leq \aleph_1$ in $\mathfrak{M}$, so that J has at most $\aleph_1$ countable subsets in $\mathfrak{M}$, i. e. less than $\operatorname{card} \mathbf{I} = \aleph_2$.

We prove the "neither" part of the theorem. It follows from Theorem 31 that for each real $r \in \mathfrak{N}$ there exists unique initial segment $\mathbf{I}_r \in \mathbf{IS}$ such that $\mathfrak{M}[r] = \mathfrak{M}[r']$ iff $\mathbf{I}_r = \mathbf{I}_{r'}$. Thus the map $r \longmapsto \mathbf{I}_r$ enumerates the C-classes by initial segments of $\mathbf{I}$ (all of them belong to $\mathfrak{M}$ by (ii), therefore we can extract even an enumeration by ordinals) in $\mathfrak{N}$, but we shall see that such an enumeration cannot be R-OD in $\mathfrak{N}$!

Suppose that on the contrary U is a R-OD enumeration of C-equivalence classes in $\mathfrak{N}$ by sets of ordinals. Then, as in the proof of the "nor" part, U is definable in $\mathfrak{N}$ by a formula containing ordinals and some $\mathbf{x} \upharpoonright J$ – where $J \in \mathfrak{M}$ is an initial segment of $\mathbf{I}$, $J \neq \mathbf{I}$, – as parameters.

We assert that $U(r) \in \mathfrak{M}[\mathbf{x} \upharpoonright J]$ for each real $r \in \mathfrak{N}$. Indeed, there exists $\xi \in \Xi$ such that $\mathfrak{M}[r] = \mathfrak{M}[\mathbf{x} \upharpoonright \xi]$. Then $U(r)$ is definable in $\mathfrak{N}$ as the set of ordinals equal to the value $U(r')$ for an arbitrary real r' such that $\mathrm{L}[r'] = \mathrm{L}[\mathbf{x} \upharpoonright \xi]$, hence $U(r) \in \mathfrak{M}[\mathbf{x} \upharpoonright J]$ by Corollary 35.

Thus each C-class is definable in $\mathfrak{N}$ by a formula containing only ordinals and $\mathbf{x} \upharpoonright J$ as parameters. In particular, $\mathbf{x} \upharpoonright J$ plus ordinals is enough to distinguish all C-degrees from each other. This leads to a contradiction.

It is provided by condition (v) that there exist $i \in \mathbf{I}$ and an order automorphism $h \in \mathfrak{M}$ of $\mathbf{I}$ such that $h \upharpoonright J$ is the identity but $h(i) = i' \neq i$. The C-classes $[\mathbf{a}_i]_{\mathrm{C}}$ and $[\mathbf{a}_{i'}]_{\mathrm{C}}$ are different (by Lemma 27, since $i \neq i'$) in $\mathfrak{N}$. Moreover, as we demonstrated above, each of them is definable in $\mathfrak{N}$ by a formula containing only $\mathbf{x} \upharpoonright J$ and ordinals as parameters. But, this contradicts Proposition 36: for take A to be any of the two sets, $[\mathbf{a}_i]_{\mathrm{C}}$ or $[\mathbf{a}_{i'}]_{\mathrm{C}}$. □

Particular models

Let $\mathfrak{M}$ be a countable transitive model satisfying the axiom of constructibility, so that (i) is provided. The following examples of the p. o. set $\mathbf{I}$, the "length" of the iteration, demonstrate different possibilities of realization of conditions (ii) through (v).

Example 1: $I_1 = \omega_1^{\mathfrak{M}} \times \{0,1\}$ ($\omega_1^{\mathfrak{M}}$ copies of the unordered two-element set $\{0,1\}$), ordered lexicographically. "Simmetries" $\langle \alpha, 0 \rangle \longleftrightarrow \langle \alpha, 1 \rangle$ for big enough ordinals α prove (v). In the extension, $\mathfrak{c} = \aleph_1^{\mathfrak{M}} = \aleph_1^{\mathfrak{N}}$. (In this case the extension $\mathfrak{N} = \mathfrak{M}[G]$ is in fact the ordinary Sacks $\times$ Sacks countable support iteration of length $\omega_1^{\mathfrak{M}}$.)

Example 2: $I_2 = \omega_2^{\mathfrak{M}} \times \{0,1\}$. Quite similar to the previous one, however we have $\mathfrak{c} = \aleph_2^{\mathfrak{M}} = \aleph_2^{\mathfrak{N}}$ in the extension. (One gets nothing new taking say $\omega_3^{\mathfrak{M}}$, because in this case $\aleph_2^{\mathfrak{M}}$ collapses to $\aleph_1^{\mathfrak{M}}$ in the extension.)

Example 3: $I_3 = \omega_1^{\mathfrak{M}} \times \mathbb{Z}$ ($\omega_1^{\mathfrak{M}}$ copies of the integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$), ordered lexicographically. This is a linearly ordered but not wellordered set, so the model cannot be defined as an ordinary Sacks iteration. (v) is provided by shiftings inside a far enough $\mathbb{Z}$-group. We have $\mathfrak{c} = \aleph_1^{\mathfrak{M}} = \aleph_1^{\mathfrak{N}}$ in the extension.

Example 4: $I_4 = \omega_2^{\mathfrak{M}} \times \mathbb{Z}$. Similar to the previous example, but $\mathfrak{c} = \aleph_2^{\mathfrak{M}} = \aleph_2^{\mathfrak{N}}$ in the extension.

Example 5: $I_5 = \omega_1^{\mathfrak{M}} \times (\mathbb{Z} \times \{0,1\})$, ordered lexicographically. (As above, the two-element set $\{0,1\}$ is assumed to be unordered, i. e. ordered by the empty order).

Thus, from the point of view of $\mathfrak{M}$, I_5 is the set of all triples $i = \langle \alpha, z, d \rangle$, where $\alpha < \omega_1$, $z \in \mathbb{Z}$, and $d = 0, 1$, partially ordered lexicographically, but of course not wellfounded and not linear. To avoid any ambiguity, we stress that $\langle \alpha, z, d \rangle < \langle \alpha', z', d' \rangle$ in $\mathbf{I}$ iff either $\alpha < \alpha'$ or $\alpha = \alpha'$ & $z < z'$, independently on the values of d, d'.

Example 6: $I_6 = \omega_2^{\mathfrak{M}} \times (\mathbb{Z} \times \{0,1\})$.

This ends the proof of Theorem 1. □

Proof of Theorem 2

Let $\mathbf{I}$ be one of the sets I_5, I_6 henceforth. (The difference between the two possibilities will be essential only for the computation of $\mathfrak{c}$ in the extension.) The requirements (ii) through (v) are obviously satisfied.

Take notice that the pairs of the form $\{\langle \alpha, z, 0 \rangle, \langle \alpha, z, 1 \rangle\}$, and only them, are order–incomparable in $\mathbf{I}$.

We keep the notation introduced above. Let us fix a $\mathbb{P}$-generic over $\mathfrak{M}$ set $G \subseteq \mathbb{P}$ and consider the generic extension $\mathfrak{N} = \mathfrak{M}[G] = \mathfrak{M}[\mathbf{x}] = \mathfrak{M}[\langle \mathbf{a}_i : i \in \mathbf{I} \rangle]$.

The plan is to define, in $\mathfrak{N}$, an uncountable Π^1_2 set W such that the relation of equiconstructibility C restricted to W also belongs to Π^1_2, prove that $\mathsf{C} \restriction W$ behaves in $\mathfrak{N}$ similarly to the unrestricted C in the models of the preceding subsection, and finally expand $\mathsf{C} \restriction W$ to all reals in $\mathfrak{N}$, putting the expanded relation to be the equality outside W.

Theorem 38 *It is true in $\mathfrak{N}$ that there exists a Π^1_2 set of reals W such that the restricted relation $\mathsf{C} \restriction W$ is Π^1_2, has $\mathfrak{c}$-many equivalence classes, and*:

– *neither admits a R-OD enumeration of the equivalence classes by sets of ordinals*;

– *nor admits a R-OD pairwise inequivalent set of cardinality $\mathfrak{c}$.*

In addition, $\mathfrak{c} = \aleph_1^{\mathfrak{M}} = \aleph_1^{\mathfrak{N}}$ provided $\mathbf{I} = I_5$, and $\mathfrak{c} = \aleph_2^{\mathfrak{M}} = \aleph_2^{\mathfrak{N}}$ provided $\mathbf{I} = I_6$.

First of all we demonstrate that this theorem implies Theorem 2. We have to expand the relation $\mathsf{C}\restriction W$ onto all reals. Let us define the relation C' on reals in $\mathfrak{N}$ as follows:

$$x\,\mathsf{C}'\,y \quad \text{iff} \quad (x,y\in W \;\&\; x\,\mathsf{C}\,y) \;\bigvee\; x=y\,.$$

The expanded relation is a Π^1_2 equivalence relation on reals in $\mathfrak{N}$. The C'-classes are the old C-classes of reals in W plus the singletons $\{x\}$, $x\notin W$. Therefore C' cannot admit a R-OD enumeration of the equivalence classes by sets of ordinals since otherwise such an enumeration would be available for $\mathsf{C}\restriction W$, contradiction with Theorem 38.

Finally, C' does not embed E_0 via a R-OD embedding. Indeed, since E_0-classes are countable while the newly added C'-classes are singletons, the embedding must embed E_0 in $\mathsf{C}\restriction W$; this implies the existence of an uncountable R-OD pairwise inequivalent subset of W simply because E_0 admits pairwise inequivalent perfect sets of reals — again contradiction with Theorem 38.

Proof of Theorem 38. It will be technically more convenient to define W as a set of *pairs* of reals rather than reals themselves, but essentially this does not make a big difference.

Definition In $\mathfrak{N}$, W is the set of all pairs of reals $\langle x,y\rangle$ such that, for some ordinal α ($\alpha<\omega_1^{\mathfrak{M}}$ in the case $\mathbf{I}=I_5$ and $\alpha<\omega_2^{\mathfrak{M}}$ in the case $\mathbf{I}=I_6$) and $z\in\mathbb{Z}$, either $x\,\mathsf{C}\,\mathbf{a}_{\alpha z0}$ and $y\,\mathsf{C}\,\mathbf{a}_{\alpha z1}$, or vice versa $x\,\mathsf{C}\,\mathbf{a}_{\alpha z1}$ and $y\,\mathsf{C}\,\mathbf{a}_{\alpha z0}$. □

Lemma 39 *In $\mathfrak{N}$, W is a Π^1_2 set and the restriction $\mathsf{C}\restriction W$ is a Π^1_2 relation.*

(We understand that $\langle x,y\rangle\,\mathsf{C}\,\langle x',y'\rangle$ iff $\mathrm{L}[x,y]=\mathrm{L}[x',y']$. In particular it is always true that $\langle x,y\rangle\,\mathsf{C}\,\langle y,x\rangle$, but $\langle x,y\rangle\,\mathsf{C}\,\langle x',y'\rangle$ does not imply $x\,\mathsf{C}\,x'$ or $y\,\mathsf{C}\,y'$.)

Proof of the lemma. We observe that by Theorem 31, W coincides with the set of all pairs of reals $\langle x,y\rangle$ such that x and y are $\le_L$-incomparable (that is, neither $x\in\mathrm{L}[y]$ nor $y\in\mathrm{L}[x]$), which is Π^1_2, in $\mathfrak{N}$.

We further assert that, given pairs $\langle x,y\rangle$ and $\langle x',y'\rangle$ in W, it is true in $\mathfrak{N}$ that $\langle x,y\rangle\in\mathrm{L}[x',y']$ iff $\langle x',y'\rangle\notin\mathrm{L}[x]$. Indeed, let on the contrary $\langle x,y\rangle\in\mathrm{L}[x',y']$ and $\langle x',y'\rangle\in\mathrm{L}[x]$, so that $y\in\mathrm{L}[x]$ — contradiction because x and y are incomparable.

For the converse, suppose that $\langle x,y\rangle\notin\mathrm{L}[x',y']$. Take notice that since the pairs belong to W, one can assume that $\langle x,y\rangle=\langle\mathbf{a}_{\alpha z0},\mathbf{a}_{\alpha z1}\rangle$ and $\langle x',y'\rangle=\langle\mathbf{a}_{\alpha'z'0},\mathbf{a}_{\alpha'z'1}\rangle$ for some ordinals α,α' and integers $z,z'\in\mathbb{Z}$. Since $\langle x,y\rangle\notin\mathrm{L}[x',y']$, we have $\langle\alpha',z'\rangle<\langle\alpha,z\rangle$ lexicographically, therefore $\langle\alpha',z',d'\rangle<\langle\alpha,z,d\rangle$ in $\mathbf{I}$ for any choice of d, $d'\in\{0,1\}$. Therefore $\langle x',y'\rangle\in\mathrm{L}[x]$ in $\mathfrak{N}$ by Lemma 27, as required. □

After we have established the class Π^1_2 of both the set W and the relation $\mathsf{C}\restriction W$, the remainder of the proof of Theorem 38 can be carried out similarly to the proof of Theorem 37 above.

For instance, practically the same reasoning proves the "additional" assertion, as well as the fact that C has $\mathfrak{c}$-many classes on W. But the "neither" and "nor" assertions need some care.

We prove the "nor" part. Suppose on the contrary that, in $\mathfrak{N}$, $S \subseteq W$ is a pairwise C-inequivalent R-OD subset of W of cardinality $\mathfrak{c}$. We recall that W consists of pairs of reals. Let us consider the set $S' = \{x : \exists y (\langle x, y\rangle \in S)\}$. Then, in $\mathfrak{N}$, S' is a pairwise C-inequivalent R-OD set of reals of cardinality $\mathfrak{c}$ — contradiction with Theorem 37.

We prove the "neither" part. Suppose on the contrary that, in $\mathfrak{N}$, U is an enumeration of the collection of all $(\mathsf{C}\restriction W)$-equivalence classes by subsets of an ordinal γ. In other words, U maps W into $\mathcal{P}(\gamma)$ so that $U(x,y) = U(x',y')$ iff $\langle x,y\rangle \,\mathsf{C}\, \langle x',y'\rangle$.

It is easy to see that if both $\langle x, y\rangle$ and $\langle x, y'\rangle$ belong to W then $y \,\mathsf{C}\, y'$, so we have $U(x,y) = U(x,y')$. Thus one can define, for each real $x \in W' = \{x : \exists y (\langle x,y\rangle \in W)\}$, $U'(x) = U(x,y)$ for any y satisfying $\langle x,y\rangle \in W$.

Take notice that W' is the set of all reals $x \in \mathfrak{N}$ such that $x \,\mathsf{C}\, \mathbf{a}_i$ in $\mathfrak{N}$ for some $i = \langle \alpha, z, d\rangle \in \mathbf{I}$, in particular, W' is a C-invariant set.

It is not completely true that U' enumerates C-classes of reals in W'. Of course $x \,\mathsf{C}\, x'$ still implies $U'(x) = U'(x')$, but now not conversely. But the following is true: if $U'(x) = U'(x')$ then there exist α ($\alpha < \omega_1^{\mathfrak{M}}$ in the case $\mathbf{I} = I_5$ and $\alpha < \omega_2^{\mathfrak{M}}$ in the case $\mathbf{I} = I_6$) and $z \in \mathbb{Z}$ such that each of the reals x, x' is C-equivalent to one of $\mathbf{a}_{\alpha z 0}$, $\mathbf{a}_{\alpha z 1}$, independently of each other.

(One may say that U' is an enumeration of the C^+-equivalence classes, where the equivalence C^+ is defined so that, in addition to C, it glues each pair $\mathbf{a}_{\alpha z 0}$, $\mathbf{a}_{\alpha z 1}$ in one class. This "amalgamation" of classes makes the simmetries $\langle \alpha, z, 0\rangle \longleftrightarrow \langle \alpha, z, 1\rangle$ useless, but fortunately we still have the other type: shiftings inside $\mathbb{Z}$-groups. This allows to run the reasoning in the proof of Theorem 37.)

We first notice that U' is definable in $\mathfrak{N}$ by a formula containing ordinals and some $\mathbf{x}\restriction J$, where $J \in \mathfrak{M}$ is an initial segment of $\mathbf{I}$ not equal to $\mathbf{I}$, as parameters – see the proof of Theorem 37 above. Then $U'(x) \in \mathrm{L}[\mathbf{x}\restriction J]$ in $\mathfrak{N}$ for all reals $x \in W'$, again as in the proof of Theorem 37.

Since $J \neq \mathbf{I}$, there exists an ordinal α ($\alpha < \omega_1^{\mathfrak{M}}$ in the case $\mathbf{I} = I_5$ and $\alpha < \omega_2^{\mathfrak{M}}$ in the case $\mathbf{I} = I_6$) such that $\langle \alpha, z, d\rangle$ does not belong to J for all z and d. In particular both $i = \langle \alpha, 7, 0\rangle$ and $i' = \langle \alpha, 8, 0\rangle$ are not members of J.

Let us define an order automorphism h of $\mathbf{I}$ by $h(\langle \alpha, z, d\rangle) = \langle \alpha, z+1, d\rangle$ for all $z \in \mathbb{Z}$ and $d = 0, 1$, and this particular α, and $h(\langle \alpha', z, d\rangle) = \langle \alpha', z, d\rangle$ whenever $\alpha' \neq \alpha$. Then $h \in \mathfrak{M}$, $h(i) = i'$, but $h\restriction J$ is the identity.

To end the proof of the "neither" part (and Theorem 38 as a whole), it now suffices to reproduce the very end in the proof of Theorem 37. □

This also ends the proof of Theorem 2. □

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