# ON HYPERGEOMETRIC FUNCTIONS CONNECTED WITH QUANTUM COHOMOLOGY OF FLAG SPACES

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## Introduction

In the Givental's work on the Gromov-Witten invariants for projective complete intersections, [G1], the principal role is played by certain formal power series connected with the quantum cohomology of a manifold. One has a manifold $X$ with a natural torus action, with the finite number of fixed points, and one has a power series $Z_w$ associated with each fixed point $x_w$. The coefficients of these series are certain integrals over the spaces of stable maps of genus 0 curves with two marked points to $X$. These series form a fundamental system of solutions of a certain lisse $\mathcal{D}$-module on a power of the punctured disk. The small quantum cohomology of $X$ coincides with the algebra of functions on its characteristic variety. The series $Z_w$ are uniquely determined by certain **recursion relations** relating $Z_w$ with all $Z_{w'}$ if $x_w$ is connected with $x_{w'}$ by a fixed line.

The present work consists of three parts. The First part contains nothing new. Here we present the Givental's computations from [G1], in the simplest case of a projective space, in more detail than in [G1]. In the Second part, we write down the above recursion relations for the flag spaces $X = G/B$ ($G$ being a simple algebraic group), see **Theorem II.3.8**, which is the main result of this paper. Here $X$ is equipped with the natural action of the maximal torus of $G$.

In the work [G2], Givental gave another beautiful set of relations which also determines the above mentioned series completely. Namely, these are Toda lattice differential equations (more precisely, we need the equivariant version of the results of [G2]). This set of relations has completely different nature, and it is highly non-trivial fact that both sets of relations determine the same series. In the Third part, we check this by a direct computation for $G = SL(3)$. It turns out that in this case the series $Z_w$ admit a nice explicit expression, see III.1.2, III.2.2. (*A posteriori* it is not surprising, since in this case $X$ admits the Plücker embedding into $\mathbb{P}^1 \times \mathbb{P}^1$, and one can use another computation by Givental, dealing with the toric complete intersections).

This paper arose from the author's attempt to make an exercise on the beautiful fixed point technique of Givental and Kontsevich. I am grateful to Yu. I. Manin for many stimulating discussions, to A. Goncharov for the stimulating interest on the first stage of this work, and to V. Batyrev who communicated to me formula III.1.4. This work was done in the highly stimulating atmosphere of Max-Planck-Institut für Mathematik, and I am very grateful to MPI for the hospitality.

## Part I. PROJECTIVE SPACES

### §1. Equivariant cohomology of $\mathbb{P}^n$

**1.1.** Let $X$ denote the $n$-dimensional projective space $\mathbb{P}^n = \{(z_0 : \ldots : z_n)\}$, the space of lines $L \subset V = \mathbb{C}^{n+1}$. The torus $T = (\mathbb{C}^*)^{n+1}$ acts on $X$ by the rule $(\alpha_0, \ldots, \alpha_n) \cdot (z_0 : \ldots : z_n) = (\alpha_0 z_0 : \ldots : \alpha_n z_n)$; this action has $n+1$ fixed points $x_i = (0 : \ldots 0 : 1 : 0 \ldots 0)$ (one on $i$-th place), $i = 0, \ldots, n$.

Let $\mathcal{L}$ denote the line bundle over $X$ whose fiber over $L \subset V$ is $L$; $\mathcal{L}$ has an obvious $T$-equivariant structure.

Let $\mathfrak{t}$ denote the Lie algebra of $T$, $\mathfrak{t}^*_{\mathbb{Z}} := Hom(T, \mathbb{C}^*) \subset \mathfrak{t}^*$ the lattice of characters. For $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$, let $L_\lambda$ be the $T$-equivariant line bundle over the point, equal to $\mathbb{C}$, with $T$ acting by means of the character $\lambda$. Assigning to $\lambda$ the first Chern class $c_1(L_\lambda) \in H^2_T(pt)$, we identify the graded ring $A := H^*_T(pt)$ with $\mathbb{C}[t^*_{\mathbb{Z}}] = \mathbb{C}[\lambda_0, \ldots, \lambda_n]$, $\lambda_i$ being the projection on $i$-th factor.

The graded $A$-algebra $R := H^*_T(X)$ is identified with $\mathbb{C}[p, \lambda_0, \ldots, \lambda_n]/(\prod_i (p - \lambda_i))$ where $p := c_1(\mathcal{L}) \in H^2_T(X)$. It is computed using the Bott's fixed point theorem, [AB].

Since our cohomologies will be even anyway, it is convenient to use the grading of the rings $A$, $R$, etc. by assigning to $p$ and $\lambda_i$ the degree 1.

**1.2.** The Euler classes of the tangent spaces at the fixed points $x_i$ are equal to

$$e_i := e(T_{X;x_i}) = \prod_{b \neq i} (\lambda_i - \lambda_b) \in H^{2n}_T(pt) = A^n \tag{1.1}$$

Let us consider the restriction map $i^*_b : \ H^*_T(X) \longrightarrow H^*_T(x_b)$. We have

$$i^*_b(p) = i^*_b(c_1(\mathcal{L})) = c_1(\mathcal{L}_{x_b})$$

The fiber $\mathcal{L}_{x_b}$ is the line $\mathbb{C} \cdot (0, \ldots, 1, \ldots 0) \subset \mathbb{C}^{n+1}$ (1 on $b$-th place); the Lie algebra $\mathfrak{t}$ acts on this line by means of the character $\lambda_b$. Therefore,

$$i^*_b(p) = \lambda_b,$$

whence

$$i^*_b(f(p)) = f(\lambda_b) \tag{1.2}$$

Let $A'$ be the ring obtained from $A$ by inverting all elements $e_i$, i.e. by inverting all the differences $\lambda_a - \lambda_b$ $(a \neq b)$. Bott's theorem says that the restriction map

$$i^* : \ H^*_T(X) \longrightarrow H^*_T(X^T) = \oplus_{b=1}^n \ A \cdot 1_{x_b}$$

becomes an isomorphism after the base change $A \longrightarrow A'$. We denote $R' = R \otimes_A A'$.

Let us introduce the elements

$$\phi_i(p) := \prod_{b \neq i} (p - \lambda_b) \in R^n \tag{1.3}$$

Obviously,

$$\phi_i(\lambda_j) = e_i\delta_{ij} \tag{1.4}$$

It follows from the Bott's theorem that the set

$$\{\phi_i(p)/e_i^{1/2};\ i = 0, \ldots, b\} \tag{1.5}$$

is the basis of orthonormal idempotents of the algebra $R'$ (to be precise, we should adjoin the square roots of $e_i$ to $R'$).

One can express this in a slightly different way. We have the integration map

$$\int_X :\ R \longrightarrow A$$

of degree $-2n$, given by

$$\int_X f(p) = \sum_i res_{p=\lambda_i}\left(\frac{f(p)}{\prod_b (p-\lambda_b)}dp\right) = \sum_i \frac{f(\lambda_i)}{e_i} \tag{1.6}$$

We have the Poincaré pairing $\langle\cdot,\cdot\rangle :\ R\times R \longrightarrow A$,

$$\langle f, g\rangle = \int_X fg$$

We have

$$\langle f, \phi_i\rangle = f(\lambda_i), \tag{1.7}$$

hence

$$\langle \phi_i, \phi_j\rangle = e_i\delta_{ij} \tag{1.8}$$

For each $f\in R$,

$$f = \sum_i \frac{f(\lambda_i)}{e_i}\phi_i \tag{1.9}$$

Note that the rhs *a proiori* lies in $R'$ but in fact it belongs to $R\subset R'$, since the lhs does.

## §2. Partition function

**2.1.** Let $X_d$ $(d\geq 0)$ denote the stack of stable maps $\{f : (C; y_1, y_2) \longrightarrow X\}$ of genus 0 curves with two marked points, such that $f_*([C]) = d\cdot\beta$ where $\beta\in H_2(X)$ is the generator dual to $p$. It is a Deligne-Mumford stack for $d\geq 1$.

Let $\mathcal{L}_1$ be the line bundle over $X_d$ whose fiber at a point $(f,\ldots)$ is the tangent space $\mathcal{T}_{C;y_1}$; denote $c(d) = c_1(\mathcal{L}_1)\in H^2_T(X_d)$.

We want to calculate the following formal power series

$$Z(q,p) = Z(q,p,\lambda,h) = 1 + \sum_{d\geq 1} e_{1*}\left(\frac{1}{h+c(d)}\right)q^d \tag{2.1}$$

Here

$$e_1 :\ X_d \longrightarrow X$$

is the evaluation map sending $(f, \dots)$ to $f(y_1)$. This is the same as to compute the series

$$Z_i(q) = \langle Z(q), \phi_i\rangle = \sum_d \int_X \phi_i e_{1*}\left(\frac{1}{h+c(d)}\right) q^d = \sum_d \left(\int_{X_d} \frac{e_1^*\phi_i}{h+c(d)}\right) q^d \quad (2.2)$$

$(i = 0, \dots, n)$. First, let us formulate the answer.

**2.2.** Define the series

$$S(q,p) = \sum_{d\ge 0} \frac{1}{\prod_{b=0}^n \prod_{m=1}^d (p-\lambda_b+mh)} \cdot q^d \quad (2.3)$$

and

$$S_i(q) := S(q,\lambda_i) = \sum_{d\ge 0} \frac{1}{d!\prod_{b\ne i} \prod_{m=1}^d (\lambda_i-\lambda_b+mh)} \cdot \frac{q^d}{h^d} \quad (2.4)$$

**2.3. Theorem.** *(a)* $Z(q,p) = S(q,p)$.

*(b) For all* $i = 0, \dots, n$, $Z_i(q) = S_i(q)$.

Of course, (a) and (b) are equivalent, due to the remarks of the previous section. The theorem will be proven in Section 5, after preliminaries in Sections 3, 4.

**2.4. Dimension count.** The dimension of $X_d$ is equal to

$$\dim(X_d) = nd + n + d - 1 \quad (2.5)$$

Indeed, one sees easily that the dimension of the space of maps $\mathbb{P}^1 \longrightarrow \mathbb{P}^n$ of degree $d$ is equal to $(d+1)(n+1)-1$. To get the dimension of $X_d$, we have to subtract from this number 3 (reparametrizations of $\mathbb{P}^1$) and add 2 (marked points).

The theorem says that for each $d \ge 0$,

$$e_{1*}\left(\frac{1}{h+c(d)}\right) = \frac{1}{\prod_{b=0}^n \prod_{m=1}^d (p-\lambda_b+mh)} \quad (2.6)$$

Let us assign to the variable $h$ the degree 1. The map $e_{1*}$ decreases the degree by $\dim(X_d) - \dim(X) = nd + d - 1$. Therefore, the degree of the lhs of (2.6) is equal to

$$-1 - (nd + d - 1) = -(n+1)d,$$

which equals the degree of the rhs. Note that this is true for $d = 0$ as well (the map $e_{1*}$ has a *positive* degree 1 in this case!).

## §3. Recursion relation

In this section we will study the series $S(q,p)$ and $S_i(q)$.

**3.1.** As a first remark, note that the *normalized* series

$$S_i^{norm} := \frac{\langle S, \phi_i\rangle}{\langle \phi_i, \phi_i\rangle} \quad (3.1)$$

may be written as

$$S_i^{norm}(q) = \sum_{d\geq 0} \frac{1}{\prod_{b\neq i}\prod_{m=0}^{d}(\lambda_i - \lambda_b + mh)} \cdot \frac{q^d}{d!h^d} =$$

$$= \sum_{d\geq 0} \frac{1}{\prod_{b=0}^{n}\prod_{m=0;(b,m)\neq(i,0)}^{d}(\lambda_i - \lambda_b + mh)} \cdot q^d \tag{3.2}$$

**3.2.** Define the series

$$s_i(q) := S_i(qh) = \sum_{d\geq 0} b_i(d) q^d \tag{3.3}$$

where

$$b_i(d) = b_i(d, \lambda, h) = \frac{1}{d!\prod_{j\neq i}\prod_{m=1}^{d}(\lambda_i - \lambda_j + mh)} \tag{3.4}$$

The next theorem is the main result about our series.

**3.3. Theorem.** *For each $i = 0, \dots, n$ have*

$$s_i(q, h) = 1 + \sum_{k>0}\sum_{j\neq i} \frac{c_i^j(k)\cdot q^k}{\lambda_i - \lambda_j + kh} \cdot s_j(q; \frac{\lambda_j - \lambda_i}{k}) \tag{3.5}$$

*where*

$$c_i^j(k) = \frac{1}{k!\prod_{b\neq i}\prod_{m=1;\ (b,m)\neq(j,k)}^{k}\left(\lambda_i - \lambda_b + \frac{m(\lambda_j - \lambda_i)}{k}\right)} \tag{3.6}$$

*The relations (3.5), (3.6) uniquely determine the series $s_i$.*

The theorem is a variant of a simple fractions decomposition. We use the following elementary fact.

**3.4. Lemma.** *Let $f(h)$ be a non-constant polynomial with distinct roots $\alpha_1, \dots, \alpha_N$. Then*

$$\frac{1}{f(h)} = \sum_{k=1}^{N} \frac{1}{(h - \alpha_k)f'(\alpha_k)} \tag{3.7}$$

*We have*

$$f'(\alpha_k) = \left(\frac{f(h)}{h - \alpha_k}\right)\bigg|_{h=\alpha_k} \tag{3.8}$$

Indeed, the difference of the rhs and lhs of (3.7) does not have singularities on $h \in \mathbb{P}^1$, hence it is a constant; but the value of both sides at $\infty$ is equal to 0, hence they are equal. The formula (3.8) is evident. $\triangle$

Now let us apply this to the coefficients $b_i(d)$, (3.4); we get

$$b_i(d) = \frac{1}{d!}\sum_{j\neq i}$$

$$\sum_{k=1}^{d} \frac{1}{\lambda_i - \lambda_j + kh} \cdot \frac{1}{\prod_{b\neq i}\prod_{m=1;\ (b,m)\neq(j,k)}^{d}(\lambda_i - \lambda_b + m\cdot\frac{\lambda_j - \lambda_i}{k})} \tag{3.9}$$

Let us split the product in the denominator in the two parts:

$$\prod_{m=1}^{d} = \prod_{m=1}^{k} \cdot \prod_{m=k+1}^{d}$$

The second product is equal to

$$\prod_{m=k+1}^{d} := \prod_{b\neq i} \prod_{m=k+1}^{d} (\lambda_i - \lambda_b + m\frac{\lambda_j - \lambda_i}{k}) =$$

$$= \prod_{b\neq i} \prod_{m'=1}^{d-k} (\lambda_i - \lambda_b + \lambda_j - \lambda_i + m' \cdot \frac{\lambda_j - \lambda_i}{k}) = \prod_{b\neq i} \prod_{m'=1}^{d-k} (\lambda_j - \lambda_b + m' \cdot \frac{\lambda_j - \lambda_i}{k})$$

Hence

$$\frac{1}{\prod_{m=k+1}^{d}} = \frac{d!}{k!} b_j(d-k, \frac{\lambda_j - \lambda_i}{k}) \tag{3.10}$$

Therefore,

$$b_i(d,h) = \sum_{j\neq i} \sum_{k=1}^{d} \frac{1}{\lambda_i - \lambda_j + kh} \cdot \frac{b_j(d-k, \frac{\lambda_j - \lambda_i}{k})}{k! \prod_{b\neq i} \prod_{m=1;\ (b,m)\neq(j,k)}^{k} (\lambda_i - \lambda_b + m \cdot \frac{\lambda_j - \lambda_i}{k})} =$$

$$= \sum_{j\neq i} \sum_{k=1}^{d} \frac{1}{\lambda_i - \lambda_j + kh} \cdot c_i^j(k) \cdot b_j(d-k, \frac{\lambda_j - \lambda_i}{k}) \tag{3.11}$$

Obviously, (3.11) is equivalent to (3.5). This proves (3.5).

The uniqueness is obvious since $s_i(0) = 1$, and the recursion relations determine $s_i(q)$ modulo $q^{k+1}$ once we know $s_j(q)$ modulo $q^k$ for all $j \neq i$.

The theorem is proved. $\triangle$

**3.5.** In order to get a better feeling what is going on, let us consider some examples. First, **the case** $n = 0$ is almost trivial, but gives a nice answer.

We have in this case $A = \mathbb{C}[\lambda]$ $(\lambda := \lambda_0)$; $R = \mathbb{C}[p, \lambda]/(p - \lambda) = A$;

$$S(q) = S_0(q) = e^{q/h} \tag{3.12}$$

There are no recursion relations.

**3.6. The case** $n = 1$**.** We have

$$R = \mathbb{C}[p, \lambda_0, \lambda_1]/((p - \lambda_0)(p - \lambda_1));$$

$$\phi_0 = p - \lambda_1,\ \phi_1 = p - \lambda_0;\ e_0 = \lambda_0 - \lambda_1,\ e_1 = \lambda_1 - \lambda_0.$$

It is convenient to introduce the "root" $\alpha := \lambda_0 - \lambda_1$. We have

$$s_0(q) = s_0(q;\alpha) = \sum_{d\geq 0} \frac{q^d}{d! \prod_{m=1}^{d} (\alpha + mh)}; \tag{3.13}$$

$$s_1(q) = s_1(q;\alpha) = \sum_{d\geq 0} \frac{q^d}{d!\prod_{m=1}^{d}(-\alpha + mh)} = s_0(q;-\alpha) \tag{3.13$'$}$$

The recursion relations look as follows

$$s_0(q;\alpha,h) = 1 + \sum_{k>0} \frac{c(k;\alpha)q^k}{\alpha + kh} \cdot s_1(q;\alpha;-\frac{\alpha}{k}) \tag{3.14}$$

and

$$s_1(q;\alpha,h) = 1 + \sum_{k>0} \frac{c(k;-\alpha)q^k}{-\alpha + kh} \cdot s_0(q;\alpha;\frac{\alpha}{k}) \tag{3.14$'$}$$

where

$$c(k;\alpha) = \frac{k^k}{(k!)^2} \cdot \frac{1}{\alpha^{k-1}} \tag{3.15}$$

The first few values of $c(k;\alpha)$:

$$c(1;\alpha) = 1;\ c(2;\alpha) = \frac{1}{\alpha};\ c(3;\alpha) = \frac{3}{4\alpha^2} \tag{3.16}$$

The relation $(3.14)'$ is obtained from (3.14) by switching $\alpha$ to $-\alpha$.

Now let us make a little computation: start building up the series $s_0, s_1$ using (3.14), $(3.14)'$. We have

$$s_0(h) = 1 + \frac{q}{\alpha + h}s_1(-\alpha) + \frac{q^2}{\alpha + 2h} \cdot \frac{1}{\alpha}s_1(-\frac{\alpha}{2}) + \ldots$$

$$s_1(h) = 1 + \frac{q}{-\alpha + h}s_0(\alpha) + \ldots$$

Thus,

$$s_0 = 1 + \frac{q}{\alpha + h} + \ldots;\ s_1 = 1 + \frac{q}{-\alpha + h} + \ldots$$

hence

$$s_0 = 1 + \frac{q}{\alpha + h} + \left(\frac{1}{\alpha + 2h} \cdot \frac{1}{\alpha} + \frac{1}{\alpha + h} \cdot \frac{1}{-2\alpha}\right) q^2 =$$

$$= 1 + \frac{q}{\alpha + h} + \frac{q^2}{2(\alpha + h)(\alpha + 2h)} + \ldots \tag{3.17}$$

which is the correct answer, up to this order.

**3.7.** As the last example, assume that $n$ is arbitrary and let us check (3.3), (3.4) up to the *first* order(*sic!*) using (3.5), (3.6).

We have

$$c_i^j(1) = \frac{1}{\prod_{b\neq i,j}(\lambda_j - \lambda_b)} \tag{3.18}$$

Therefore,

$$s_i(h) = 1 + \sum_{j\neq i} \frac{q}{\lambda_i - \lambda_j + h} \cdot \frac{1}{\prod_{b\neq i,j}(\lambda_j - \lambda_b)} + \ldots =$$

$$= 1 + \frac{q}{\prod_{j\neq i}(\lambda_i - \lambda_j + h)} + \dots \tag{3.19}$$

where we have used the formula

$$\frac{1}{\prod_{j\neq i}(\lambda_i - \lambda_j + h)} = \sum_{j\neq i} \frac{1}{\prod_{b\neq i,j}(\lambda_j - \lambda_b)} \cdot \frac{1}{\lambda_i - \lambda_j + h} \tag{3.20}$$

## §4. First reduction.

**4.1.** We start proving Theorem 2.3.

Let us define the series $z_i$ by

$$z_i(q) = Z_i(qh) \tag{4.1}$$

(cf. (3.3)). According to Theorem 3.3, in order to prove Theorem 2.3, it suffices to show that $z_i(q)$ satisfy the relations (3.5).

**4.2.** Let us define the coefficients $B_i$ by

$$S_i(q) = \sum_d B_i(d) q^d \tag{4.2}$$

(cf. (2.4)). Thus, we have

$$B_i(d) = \frac{b_i(d)}{h^d} = \frac{1}{\prod_{b=0}^n \prod_{m=1}^d (\lambda_i - \lambda_b + mh)} \tag{4.3}$$

As we have already noted, (3.5) is equivalent to the identities

$$b_i(d, h) = \sum_{j\neq i} \sum_{k=1}^{d} \frac{c_i^j(k)}{\lambda_i - \lambda_j + kh} \cdot b_j(d-k, \frac{\lambda_j - \lambda_i}{k}) \tag{4.4}$$

or

$$b_i(d, h) = \sum_{j\neq i} \sum_{k=1}^{d} \frac{c_i^j(k)}{\lambda_i - \lambda_j + kh} \cdot \left(\frac{\lambda_j - \lambda_i}{k}\right)^{d-k} \cdot B_j(d-k, \frac{\lambda_j - \lambda_i}{k}) \tag{4.5}$$

If we assign to $\lambda_i$ and $h$ the degree 1 then we have

$$deg\ b_i(d) = -dn;\ deg\ B_i(d) = -dn - n;\ deg\ c_i^j(k) = -kn + 1 \tag{4.6}$$

and the identities (4.4), (4.5) are homogeneous, cf. 2.4.

**4.3.** Let us denote

$$U_i(d) := \int_{X_d} \frac{e_1^* \phi_i}{h + c(d)} \tag{4.7}$$

Thus,

$$Z_i(q) = \sum_{d\geq 0} U_i(d) q^d \tag{4.8}$$

We have

$$U_i(d) = \frac{1}{h}\sum_{a\geq 0}\frac{(-1)^a}{h^a}\int_{X_d} e_1^*\phi_i\cdot c(d)^a$$

We have $U_i(0)=1$. Assume now that $d\geq 1$. The degree of the integral $\int_{X_d} e_1^*\phi_i\cdot c(d)^a$ is equal to $n+a-(nd+n+d-1) = -nd+a-d+1$ which is less than zero for $a<d$, hence this integral is zero for these $a$. Therefore,

$$U_i(d) = \frac{1}{h}\sum_{a\geq d}\frac{(-1)^a}{h^a}\int_{X_d} e_1^*\phi_i\cdot c(d)^a = \frac{1}{h^d}\int_{X_d}\frac{e_1^*\phi_i\cdot(-c(d))^d}{h+c(d)}$$

Let us denote

$$u_i(d) := \int_{X_d}\frac{e_1^*\phi_i\cdot(-c(d))^d}{h+c(d)} \tag{4.9}$$

(for $d=0$ we set $u_i(0):=1$). Thus,

$$z_i(q) = \sum_{d\geq 0} u_i(d)q^d;\ u_i(d) = U_i(d)h^d \tag{4.10}$$

We have to prove that $z_i(d)=b_i(d)$. Therefore, Theorem 2.3 is equivalent to

**4.4. Theorem.** *The integrals $u_i(d)$ satisfy the relations*

$$u_i(d,h) = \sum_{j\neq i}\sum_{k=1}^{d}\frac{c_i^j(k)}{\lambda_i-\lambda_j+kh}\cdot u_j(d-k,\frac{\lambda_j-\lambda_i}{k}) \tag{4.10}$$

This is what we are going to prove in the next section, using the Bott's localization theorem.

## §5. Fixed point formula.

**5.1.** We will compute the integrals $u_i(d)$ (see (4.9)) by means of the *Bott's fixed point formula.* It says that

$$u_i(d) = \sum_P u_i(P), \tag{5.1}$$

the summation running over all connected components $P\subset X_d^T$. Here $u_i(P)$ denotes the integral

$$u_i(P) := \int_P\left(\frac{e_1^*\phi_i\cdot(-c(d))^d}{h+c(d)}\right)\Bigg|_P\cdot\frac{1}{e(\mathcal{N}_{P/X_d})} \tag{5.2}$$

Here $\mathcal{N}$ denotes the normal bundle, $e$ the Euler (top Chern) class.

What do the connected components look like (cf. [K])? Let $l_{ij}\subset X$ denote the straight line connecting the points $x_i$ and $x_j$. These are the curves in $X$ stable under the action of $T$.

A point in $X_d^T$ is a stable map

$$f:(C;y_1,y_2)\longrightarrow X \tag{5.3}$$

such that $f(y_i) \in X^T = \{x_0, \dots, x_n\}$ and each irreducible component $C_1 \subset C$ is mapped either to one of the points $x_i$ — in this case we call $C_1$ **vertical**, or to one of the lines $l_{ij}$ — in this case we call $C_1$ **horizontal**. The map

$$f\big|_{C_1} : \ C_1 = \mathbb{P}^1 \longrightarrow l_{ij} \tag{5.4}$$

is a finite covering ramified at points $x_i$ and $x_j$. The sum of the degrees of these coverings over all horizontal $C_1$ should be equal to $d$.

The connected component $P$ to which the point (5.3) belongs, is specified by the combinatorial data: which irreducible components of $C$ are vertical or horizontal, and the degrees of the coverings (5.4) for horizontal components.

**5.2.** Let us consider the integral $u_i(P)$ (5.2). Let $f$ as in (5.3) be a point in $P$. We have

$$e_1^*\phi_i\big|_P = \phi_i(\lambda_j) \tag{5.5}$$

if $f(y_1) = x_j$. Therefore, $u_i(P)$ may be nonzero only if $f(y_1) = x_i$. We will suppose this is the case from now on.

Let $C_1 \subset C$ be the irreducible component containing $y_1$.

**5.3. Claim.** *If $u_i(P)$ is nonzero then $C_1$ is horizontal.*

Indeed, suppose that $C_1$ is vertical. Let us call a *special point* a marked point or a point of intersection of two irreducible components.

The connected component $P$ has the form $\mathcal{M}_{s+1}\times?$ where $\mathcal{M}_{s+1}$ is the Deligne-Mumford moduli stack of genus 0 curves with $s+1$ marked points, this whole component mapping to $x_i$. A generic curve in $\mathcal{M}_{s+1}$ contains the marked point $y_1$, maybe the marked point $y_2$, and $s$ or $s-1$ points of intersection with horizontal curves, depending on whether it contains $y_2$ or not; $s$ special points altogether.

We have $\dim(\mathcal{M}_{s+1}) = s-2$. Since the total degree of $f$ is $d$, the number of horizontal components does not exceed $d$; therefore, $s-1 \le d$, hence

$$\dim(\mathcal{M}_{s+1}) < d \tag{5.6}$$

Consequently, $c(d)^d\big|_P = 0$, i.e. $u_i(P) = 0$. The claim is proven. $\triangle$

**5.4.** Now let us consider the component $P$ containing

a stable curve

$$(f : C = C_1 \cup C_2 \longrightarrow X) \in P \subset X_d^T \tag{5.7}$$

where $C_1$ is the irreducible component containing the marked point $y_1$, which is mapped with multiplicity $k$ onto the line $l_{ij}$, and $C_2$ is all the rest. The map

$$f_1 := f\big|_{C_1} : \ C_1 = \mathbb{P}^1 \longrightarrow l_{ij} \tag{5.8}$$

is the $k$-fold covering ramified only over two points $x_i$ and $x_j$, where $1 \le k \le d$. The map

$$f_2 := f\big|_{C_2} : \ C_2 \longrightarrow X \tag{5.9}$$

belongs to a connected component $P_2$ of the fixed point space $X_{d-k}^T$.

We want to compute the integral (5.2). Let us compute the terms under the integral. As we have already noted,

$$e_1^*\phi_i\big|_P = \phi_i(\lambda_i) = e_i = \prod_{b\neq i}(\lambda_i - \lambda_b) \tag{5.10}$$

We have

$$c(d)\big|_P = \frac{\lambda_i - \lambda_j}{k} \tag{5.11}$$

**5.5. Normal bundle.** The most labourous job is to compute the Euler class of the normal bundle over $P$. We use the *Kontsevich formula*, (cf. [K], 3.3.1): the class of $\mathcal{N}_{P/X_d}$ in the Grothendieck group of $T$-equivariant bundles over $P$ is equal to

$$\begin{aligned}[\mathcal{N}_{P/X_d}] = [H^0(C_1; f_1^*\mathcal{T}_X)_{0\ at\ y}] - [H^0(C_1; \mathcal{T}_{C_1})_{0\ at\ y}] + \\ +[T_{y;C_1}\otimes T_{y;C_2}] + [T_{y_1;C_1}] + [\mathcal{N}_{P_2/X_{d-k}}]\end{aligned} \tag{5.12}$$

(we use the notations describing a bundle by the fiber at a point $f$). We have

$$[H^0(C_1; f_1^*\mathcal{T}_X)_{0\ at\ y}] = [H^0(C_1; f_1^*\mathcal{T}_X)] - [(f_1^*\mathcal{T}_X)_y] \tag{5.13}$$

and

$$[H^0(C_1; \mathcal{T}_{C_1})_{0\ at\ y}] = [H^0(C_1; \mathcal{T}_{C_1})] - [T_{y;C_1}] \tag{5.14}$$

Therefore,

$$\begin{aligned}[\mathcal{N}_{P/X_d}] = ([H^0(C_1; f_1^*\mathcal{T}_X)] - [(f_1^*\mathcal{T}_X)_y]) + \\ +(-[H^0(C_1;\mathcal{T}_{C_1})] + [T_{y_1;C_1}] + [T_{y;C_1}]) + [T_{y;C_1}\otimes T_{y;C_2}] + [\mathcal{N}_{P_2/X_{d-k}}]\end{aligned} \tag{5.15}$$

We have

$$e([T_{y_1;C_1}]) = \frac{\lambda_i - \lambda_j}{k};\ e([T_{y;C_1}]) = \frac{\lambda_j - \lambda_i}{k} \tag{5.16}$$

and

$$e([H^0(C_1; \mathcal{T}_{C_1})]) = \frac{\lambda_i - \lambda_j}{k}\cdot [0] \cdot \frac{\lambda_j - \lambda_i}{k} \tag{5.17}$$

so that the second bracket in (5.15) gives simply $-[0]$. All zeros in this game will cancel out in the final expression for $e(\mathcal{N}_{P/X_d})$ (see (5.22) below)!

**5.6. Lemma.** *We have*

$$e([H^0(C_1; f_1^*\mathcal{T}_X)]) = \prod_{b=0}^{n}\prod_{m=0}^{k}(\frac{k-m}{k}\lambda_i + \frac{m}{k}\lambda_j - \lambda_b)/[0] \tag{5.18}$$

Note that in the product there two factors equal to $[0]$: they correspond to the values $(m,b) = (0,i)$ or $(k,j)$. One of these zeros is cancelled.

**Proof.** We have the exact sequence of vector bundles over $X = \mathbb{P}^n$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow V^*\otimes\mathcal{O}_X(1) \longrightarrow \mathcal{T}_X \longrightarrow 0 \tag{5.19}$$

We have

$$H^0(C_1; f_1^*(V^*\otimes\mathcal{O}_X(1))) = H^0(C_1; f_1^*\mathcal{O}_X(1))\otimes V^*$$

The Lie algebra $\mathfrak{t}$ acts on $H^0(C_1; f_1^*\mathcal{O}_X(1))$ by the characters

$$\frac{k-m}{k}\lambda_i + \frac{m}{k}\lambda_j; \quad m = 0, \ldots, k,$$

and on $V^*$ by the characters $-\lambda_b,\ b = 0, \ldots, n$. This implies the lemma. $\triangle$

The remaining zero in (5.18) will cancel out with the zero from (5.17).

**5.7.** We have

$$e([(f_1^*\mathcal{T}_X)_y]) = e([\mathcal{T}_{x_j;X}]) = e_j = \prod_{b\neq j}(\lambda_j - \lambda_b) \tag{5.20}$$

Finally, we have

$$e([T_{y;C_1}] \otimes [T_{y;C_2}]) = \frac{\lambda_j - \lambda_i}{k} + c(d-k) \tag{5.21}$$

Combining everything together, we have proven

**5.8. Lemma.** *We have*

$$e(\mathcal{N}_{P/X_d}) = \frac{\prod_{b=0}^n \prod_{m=0;\ (b,m)\neq(i,0),(j,k)}^k (\frac{k-m}{k}\lambda_i + \frac{m}{k}\lambda_j - \lambda_b)}{e_j}\cdot$$

$$\cdot(\frac{\lambda_j - \lambda_i}{k} + c(d-k)) \cdot e(\mathcal{N}_{P_2;X_{d-k}}) \quad \triangle \tag{5.22}$$

**5.9.** Now we can plug our computations in (5.2). We get

$$u_i(P) = \frac{1}{k} \cdot \frac{e_i \cdot \left(\frac{\lambda_j - \lambda_i}{k}\right)^d}{h + \frac{\lambda_i - \lambda_j}{k}}\cdot$$

$$\cdot\frac{1}{\prod_{b=0}^n \prod_{m=0;\ (b,m)\neq(i,0),(j,k)}^k (\frac{k-m}{k}\lambda_i + \frac{m}{k}\lambda_j - \lambda_b)}\cdot$$

$$\cdot\int_{P_2} \frac{e_j}{c(d-k) + \frac{\lambda_i - \lambda_j}{k}} \cdot \frac{1}{e(\mathcal{N}_{P_2/X_{d-k}})} \tag{5.23}$$

The overall factor $1/k$ is due to the group of automorphisms of the covering $f_1$, having order $k$. In the big product, the terms with $m = 0$ give the contribution $e_i$, and the terms with $b = i$ give the contribution

$$\prod_{m=1}^k (-\frac{m}{k}\lambda_i + \frac{m}{k}\lambda_j) = k!\left(\frac{\lambda_j - \lambda_i}{k}\right)^k \tag{5.24}$$

The integral is by definition $U_j(P_2)$ (remember that $P_2$ is the connected component of the smaller space $X_{d-k}$!). Thus, we get

$$u_i(P) = \frac{1}{kh + \lambda_i - \lambda_j}\cdot$$

$$\cdot\frac{1}{k!\prod_{b\neq i}\prod_{m=1;\ (b,m)\neq(j,k)}^k \left(\lambda_i - \lambda_b + \frac{m(\lambda_j - \lambda_i)}{k}\right)} \cdot \left(\frac{\lambda_j - \lambda_i}{k}\right)^{d-k} \cdot U_j(P_2) =$$

$$= \frac{c_i^j(k)}{kh + \lambda_i - \lambda_j} \cdot \left(\frac{\lambda_j - \lambda_i}{k}\right)^{d-k} \cdot U_j(P_2) \tag{5.25}$$

This implies formula (4.10) (cf. 4.5). Theorem 4.4, and hence Theorem 2.3 are proven. $\triangle$

## Part II. FLAG SPACES

### §1. Equivariant cohomology of flag spaces

**1.1.** Let $V = \mathbb{C}^{n+1}$. Let $X$ be the variety of complete flags of linear subspaces $0 \subset V_1 \subset \ldots \subset V_n \subset V$, $\dim(V_i) = i$. We set

$$D := \dim(X) = n + (n-1) + \ldots + 1 = \frac{n(n+1)}{2} \tag{1.1}$$

The torus $T = (\mathbb{C}^*)^{n+1}$ acts on $V$. Namely, if $v_0, \ldots, v_n$ is the standard basis of $V$, we put

$$\alpha \circ \left( \sum z_i v_i \right) = \sum \alpha_i z_i v_i, \ \alpha = (\alpha_i) \in T$$

By functoriality, $T$ acts on $X$.

We denote by $\mathfrak{t}$ the Lie algebra of $T$, and denote by $\lambda_i \in \mathfrak{t}^*$ the character which is the projection on the $i$-th component.

**1.2. Fixed points.** Let $F = (V_1 \subset \ldots \subset V_n)$ be a flag fixed under the action of $T$. Since $TV_1 = V_1$, there exists $i_0$ such that $V = \mathbb{C} \cdot e_{i_0}$. Taking the quotient, we get the flag $F' = (V_2/V_1 \subset \ldots \subset V_n/V_1)$ in the space $V' = V/V_1$, fixed under the action of the torus $T' = T/(\mathbb{C}^*)_{i_0}$. By induction on $n$, we conclude that there exists the unique permutation $(i_0, \ldots, i_n)$ of the set $\{0, \ldots, n\}$ such that $V_k$ is spanned by $e_{i_0}, \ldots, e_{i_{k-1}}$ $(k = 1, \ldots, n)$.

We shall denote the group of all bijections $w : \ \{0, \ldots, n\} \xrightarrow{\sim} \{0, \ldots, n\}$ by $W$, and we shall identify such a bijection with the permutation $(w(0), \ldots, w(n))$.

The previous discussion identifies the set of fixed points $X^T$ with $W$: to an element $w \in W$ correponds the flag

$$x_w = (\mathbb{C} \cdot e_{w(0)} \subset \ldots \subset \oplus_{i=0}^{k} \mathbb{C} \cdot e_{w(i)} \subset \ldots \subset V) \in X^T \tag{1.2}$$

**Tangent spaces.** As a $\mathfrak{t}$-module, the tangent space $T_w := T_{X;x_w}$ at the point $x_w$, has the set of characters $\{\lambda_{i_p} - \lambda_{i_q}, \ 0 \leq p < q \leq n\}$. Hence, its Euler class is given by

$$e_w := e(T_w) = \prod_{p<q} (\lambda_{i_p} - \lambda_{i_q}) \tag{1.3}$$

**Fixed lines.** Given a permutation $w = (i_0, \ldots, i_n)$ and an integer $p$, $(1 \leq p \leq n)$, let $s_p w$ be the permutation with $i_p$ and $i_{p-1}$ transposed, the other entries remaining in place.

If $x_w = (V_1 \subset \ldots \subset V_n)$, let $\ell_{w,s_p w}$ be the projective line inside $X$ consisting of all flags of the form

$$V_1 \subset \ldots \subset V_{p-1} \subset V' \subset V_{p+1} \subset \ldots \subset V$$

where all the subspaces $V_i$ are fixed, and $V'$ is varying. This line is fixed under the action of $T$, and contains exactly two fixed points: $x_w$ and $x_{s_p w}$.

However, these are not all fixed lines. In fact, the fixed lines passing through one fixed point, correspond to all *positive* roots (and we have just described the lines corresponding to the *simple* roots), cf. [BGG].

Consider the case $w = id$ (identity permutation); so $x_{id}$ is the standard flag, with $V_i$ spanned by $\{e_0, \ldots, e_{i-1}\}$. Given $p < q$, let $s_{pq} \in W$ be the permutation of $p$ and $q$. For a matrix $A = (a_{ij}) \in GL(2)$, set

$$e_{p;A} = a_{11}e_p + a_{21}e_q; \; e_{q;A} = a_{12}e_p + a_{22}e_q \tag{1.4}$$

Let $x_A$ be the flag with the spaces $V_{i;A}$ spanned by $e_{0;A}, \ldots, e_{i-1;A}$ where $e_{i;A} = e_i$ for $i \neq p, q$. When $A$ runs through $GL(2)$, the flags $x_A$ form the projective line $\ell_{e;s_{pq}} = GL(2)/B$, stable under the action of $T$, and passing through the fixed points $x_e$ and $x_{s_{pq}}$.

In a similar manner, one defines the $T$-stable line $\ell_{w;s_{pq}w}$ passing through $x_w$ and $x_{s_{pq}w}$.

**1.3. Cohomology.** Let $\mathcal{L}_i$ be the line bundle over $X$ whose fiber over a flag $V_1 \subset \ldots \subset V_n$ is equal to $V_{i+1}/V_i$; let $u_i \in H^2(X)$ be its first Chern class. The cohomology algebra of $X$ is equal to

$$H^*(X) = \mathbb{C}[u_0, \ldots, u_n]/(\sigma_0(u), \ldots, \sigma(u)) \tag{1.5}$$

Here $\sigma_i(u)$ are the elementary symmetric functions defined by the rule

$$\prod_{i=0}^{n}(Z + u_i) = Z^{n+1} + \sigma_0(u)Z^n + \ldots + \sigma_n(u) \tag{1.6}$$

The $T$-equivariant cohomology $R = H^*_T(X)$ is an $A = H^*_T(pt) = \mathbb{C}[\lambda_0, \ldots, \lambda_n]$-algebra isomorphic to

$$H^*_T(X) = \mathbb{C}[u_0, \ldots, u_n; \lambda_0, \ldots, \lambda_n]/(\sigma_0(u) - \sigma_0(\lambda), \ldots, \sigma_n(u) - \sigma_n(\lambda)) \tag{1.7}$$

For each $w = (i_0, \ldots, i_n) \in W$, define an element of $H^{2D}_T(X)$ by

$$\phi_w = \prod_{p<q} (u_{i_p} - \lambda_{i_q}) \tag{1.8}$$

Let $i_w$ denote the embedding $\{x_w\} \hookrightarrow X$.

**1.4. Lemma.** *We have* $i^*_w \phi_{w'} = e_w \cdot \delta_{ww'}$.

**Proof.** It follows from definitions, that

$$i^*_w u_p = \lambda_{w(p)} \tag{1.9}$$

On the other hand,

$$\prod_{p<q} (\lambda_{w(p)} - \lambda_q) = 0 \tag{1.10}$$

for $w \neq e$. This implies the lemma. $\triangle$

Set

$$A' = A[e_w^{-1}]_{w\in W} = A[(\lambda_p - \lambda_q)^{-1}]_{p<q};\ R' = R\otimes_A A' \tag{1.11}$$

**1.5. Corollary.** *The elements $\{\phi_w\}_{w\in W}$ form an $A'$-basis of the algebra $R'$. The multiplication in $R'$ is recovered from the rule*

$$\phi_w\phi_{w'} = e_w\delta_{ww'} \tag{1.12}$$

**Proof.** This is a corollary of the Bott's fixed point theorem, cf. I.1.2. $\triangle$

## §2. Partition function

**2.1.** It is convenient to switch to an arbitrary flag space $X = G/B$, $G$ being the simple simply connected algebraic group $G$ associated to a finite root system $R$. The manifold $X$ is acted upon by $T$, the maximal torus of $G$. We identify $H_2(X;\mathbb{Z})$ with the coroot lattice

$$H_2(X;\mathbb{Z}) = Q = \oplus_{i\in I}\ \mathbb{Z}\cdot\alpha_i^\vee, \tag{2.1}$$

where $\alpha_i^\vee$ are the simple coroots; it contains the submonoid of positive coroots

$$Q^+ = \oplus_I\ \mathbb{N}\cdot\alpha_i^\vee \subset Q \tag{2.2}$$

The cohomology group $H^2(X;\mathbb{Z})$ is the dual weight lattice

$$H^2(X;\mathbb{Z}) = P = Char(T) = \oplus_I\ \mathbb{Z}\cdot\omega_i, \tag{2.3}$$

where $\omega_i$ are the fundamental weights; it contains the submonoid of dominant weights

$$P^+ = \oplus_I\ \mathbb{N}\cdot\omega_i \subset P \tag{2.4}$$

The root system $R$ lies inside $P$; we denote by $R^+\subset R$ the subset of positive roots and by $\{\alpha_i\}_{i\in I}$ the set of simple roots. To each $\lambda\in P = Char(T)$ corresponds the line bundle $\mathcal{L}_\lambda$ over $X$, with $c_1(\mathcal{L}_\lambda) = \lambda$.

In the Grothendieck group of $T$-equivariant bundles, the class of the tangent bundle of $X$ is equal to the sum

$$[\mathcal{T}_X] = \oplus_{\alpha\in R^+}\ [\mathcal{L}_\alpha] \tag{2.5}$$

In particular,

$$\dim(X) = card(R^+) \tag{2.6}$$

Let $W$ be the Weyl group of $R$. The torus $T$ acts on $X$ with the finite set of fixed points $\{x_w\}_{w\in W}$, each $x_w$ lying inside the corresponding Schubert cell (cf. [BGG]). The fixed lines pass through the pairs $x_w, x_{s_\alpha w}$ where $s_\alpha$ is the reflection corresonding to a positive root $\alpha$.

We identify the cohomology ring $A = H^*_T(pt)$ with $\mathbb{C}[P] = \mathbb{C}[\alpha_i]_I$. The Euler classes of the tangent spaces are given by

$$e_w = e(T_{x_w;X}) = w(e_{id});\ e_{id} = \prod_{\alpha \in R^+} \alpha \tag{2.7}$$

Set $A' = A[\alpha^{-1}]_{\alpha \in R}$. Bott's Theorem gives the $A'$-base $\{\phi_w\}_{w \in W}$ in $H^*_T(X)_{A'}$, such that

$$i^*_w \phi'_w = \delta_{ww'} e_w \tag{2.8}$$

**2.2.** To each $\beta \in Q^+$ corresponds the space $X_\beta$ of stable maps

$$f : (C; y_1, y_2) \longrightarrow X;\ f_*([C]) = \beta;\ g(C) = 0 \tag{2.9}$$

of curves of genus 0 with two marked points, to $X$. Set $|\beta| = \sum\ \nu_i$ if $\beta = \sum\ \nu_i \alpha^\vee$.

**2.3. Lemma.** $\dim(X_\beta) = 2|\beta| + \dim(X) - 1$.

**Proof.** Let us choose a point (2.9) with $C = \mathbb{P}^1$. We have

$$\dim(X_\beta) = \dim(T_{f;X_\beta}) = \dim(H^0(C; f^*\mathcal{T}_X)) - \dim(H^0(C; \mathcal{T}_C)) +$$

$$+ \dim(T_{y_1;C}) + \dim(T_{y_2;C}) \tag{2.10}$$

Note that $H^1(C; f^*\mathcal{T}_X) = 0$. Therefore,

$$\dim(H^0(C; f^*\mathcal{T}_X)) = \chi(C; f^*\mathcal{T}_X) = \sum_{\alpha \in R^+} \chi(C; f^*\mathcal{L}_\alpha) =$$

$$= \sum_{\alpha \in R^+} (\langle \beta, \alpha \rangle + 1) = \langle \beta, 2\rho \rangle + card(R^+) = 2|\beta| + \dim(X), \tag{2.11}$$

since $\langle \alpha_i^\vee, \rho \rangle = 1$ for each $i$. Here $\rho$ denotes the half-sum of the positive roots. Obviously, $\dim(H^0(C; \mathcal{T}_C)) = 3$. Plugging this into (2.10), we get the lemma. $\triangle$

In (2.11), we have used the equalities

$$\chi(C; f^*\mathcal{L}_\alpha) = \langle \beta, \alpha \rangle + 1 \tag{2.12}$$

This number may be negative if the root system is not simply laced.

**2.4.** Denote

$$c(\beta) = c_1(\mathcal{T}_1) \in H^2_T(X_\beta) \tag{2.13}$$

where $\mathcal{T}_1$ is the line bundle over $X_\beta$ whose fiber over a point (2.9) is equal to $T_{y_1;C}$.

Let $e_1 :\ X_\beta \longrightarrow X$ be the "evaluation at $y_1$" map. Our aim is to calculate the formal power series

$$Z(q) = \sum_{\beta \in Q^+} e_{1*}\left(\frac{1}{h + c(\beta)}\right) \cdot q^\beta \tag{2.14}$$

in variables $q = (q_i)_{i \in I}$, with coefficients in the ring $H^*_T(X)(h)$. Here $q^\beta = \prod\ q_i^{\nu_i}$, for $\beta = \sum\ \nu_i \alpha_i^\vee$.

This amounts to calculating $|W|$ power series

$$Z_w(q) = \langle \phi_w, Z \rangle = \sum_{\beta \in Q^+} \Big( \int_{X_\beta} \frac{e_1^* \phi_w}{h + c(\beta)} \Big) \cdot q^\beta \tag{2.15}$$

**2.5. Example.** For the root system $A_1$, the series $Z(q)$ is given by the expression I, (2.3), with $n = 1$. The Weyl group has order 2; there are two series $Z_w(q)$:

$$Z_\pm(q) = \sum_{d=0}^{\infty} \frac{q^d}{d! h^d \prod_{m=1}^{d} (\pm\alpha + mh)} = \sum_{d=0}^{\infty} \frac{q^d}{d^!_{0;h} d^!_{\pm\alpha;h}} \tag{2.16}$$

Here we have used the notation

$$d^!_{\alpha;h} = \prod_{m=1}^{d} (\alpha + mh) \tag{2.17}$$

The series $Z_+$ (resp. $Z_-$) corresponds to the trivial (resp. non-trivial) element of the Weyl group, cf. I, (2.4) and I.3.6.

## §3. Fixed point computation

**3.1.** Let us return to the series (2.15). Denote

$$I_w(\beta) = \int_{X_\beta} \frac{e_1^* \phi_w}{h + c(\beta)} \tag{3.1}$$

Set

$$J_w(\beta) = \int_{X_\beta} \frac{e_1^* \phi_w \cdot (-c(\beta))^{|\beta|}}{h + c(\beta)} \tag{3.2}$$

The next lemma is proved in the same manner as I, (4.10).

**3.2. Lemma.** *We have*

$$I_w(\beta) = \frac{1}{h^{|\beta|}} J_w(\beta) \ \triangle$$

**3.3.** Now, we want to compute the integral $J_w(\beta)$ using the fixed point formula. We have

$$J_w(\beta) = \sum_P J_w(P) \tag{3.3}$$

where

$$J_w(P) = \int_P \left( \frac{e_1^* \phi_w \cdot (-c(\beta))^{|\beta|}}{h + c(\beta)} \right)\Big|_P \cdot \frac{1}{e(\mathcal{N}_{P/X_\beta})} \tag{3.4}$$

Here $P$ denotes a component of the fixed point space $X_\beta^T$.

Let us compute $J_w(P)$. The picture of a connected component is the same as in I, §5. Thus, let $f:\ (C;y_1,y_2)\longrightarrow X$ be a point in $P\subset X_\beta$. The integral $J_w(P)$ may be non-zero only if $f(y_1)=x_w$, so we will assume this. Let $C=C_1\cup C_2$ where $C_1$ is the connected component containing $y_1$ and $C_2$ is all the rest. As in I.5.3, we prove that $C_1$ is horizontal, i.e. it covers with some multiplicity $k>0$ a fixed line $\ell_{w;s_\alpha w}$, for some $\alpha\in R^+$.

To simplify the notations, assume that $w=id$. We have

$$e_1^*\phi_{id}\big|_P=e_{id}=e(T_{x_{id};X})=\prod_{\gamma\in R^+}\gamma \tag{3.5}$$

and

$$c(\beta)\big|_P=\frac{\alpha}{k} \tag{3.6}$$

**3.4. Lemma.** *We have*

$$e(\mathcal{N}_{P/X_\beta})=(-1)^k(k!)^2\cdot\left(\frac{\alpha}{k}\right)^{2k-1}\cdot\prod_{\gamma\in R^+;\ \gamma\neq\alpha}\left(\frac{\prod_{m\geq1}(\gamma-\frac{m}{k}\alpha)}{\prod_{m\geq1}(s_\alpha\gamma-\frac{m}{k}\alpha)}\right)\cdot$$

$$\cdot\frac{e_{id}}{e_{s_\alpha}}\cdot(-\frac{\alpha}{k}+c(\beta-k\alpha^\vee))$$

**Proof.** We have (cf. I.5.5)

$$[\mathcal{N}_{P/X_\beta}]=[H^0(C_1;f_1^*\mathcal{T}_X)]-[(f_1^*\mathcal{T}_X)_y]-$$

$$-[H^0(C_1;\mathcal{T}_{C_1})]+[T_{y_1;C_1}]+[T_{y;C_1}]+[T_{y;C_1}\otimes T_{y;C_2}]+[\mathcal{N}_{P_2/X_{\beta-k\alpha^\vee}}] \tag{3.7}$$

Note that $P_2$ (containing the map $f|_{C_2}$) is the connected component of $X_{\beta-k\alpha^\vee}$. We have

$$e([(f_1^*\mathcal{T}_X)_y])=e([T_{x_{s_\alpha};X}])=e_{s_\alpha}; \tag{3.8}$$

next,

$$e([T_{y_1;C_1}])=\frac{\alpha}{k};\ e([T_{y;C_1}]=-\frac{\alpha}{k}, \tag{3.9}$$

and

$$e([H^0(C_1;\mathcal{T}_{C_1})])=\frac{\alpha}{k}\cdot[0]\cdot-\frac{\alpha}{k} \tag{3.10}$$

We have

$$e([T_{y;C_1}\otimes T_{y;C_2}])=-\frac{\alpha}{k}+c(\beta-k\alpha^\vee) \tag{3.11}$$

**3.5. Lemma.** *We have*

$$e([H^0(C_1;f_1^*\mathcal{T}_X)])=\prod_{\gamma\in R^+}\left(\frac{\prod_{m\geq0}(\gamma-\frac{m}{k}\alpha)}{\prod_{m\geq1}(s_\alpha\gamma-\frac{m}{k}\alpha)}\right) \tag{3.12}$$

**Proof.** As we have already noted, $[\mathcal{T}_X]=\sum_{\gamma\in R^+}[\mathcal{L}_\gamma]$ in the Grothendieck group. Also, $H^1(C_1;f_1^*\mathcal{T}_X)=0$ (convexity of $X$), and our lemma is the corollary of the following one.

**3.6. Lemma.** *We have*

$$e([\chi(C_1; f_1^*\mathcal{L}_\gamma)]) = \frac{\prod_{m\geq 0} (\gamma - \frac{m}{k}\alpha)}{\prod_{m\geq 1} (s_\alpha\gamma - \frac{m}{k}\alpha)} \tag{3.13}$$

All but finite number of terms in the fraction cancel out, and we are left with a finite product in the numerator (resp. denominator), if $\langle\gamma, \alpha^\vee\rangle$ is positive (resp. negative). $\triangle$

Now, Lemma 3.4 follows from Lemma 3.5. Note that the product (3.12) contains one factor equal to zero: in the numerator, coresponding to $\gamma = \alpha,\ m = k$. This zero cancels out with the zero in (3.10), and we are left with the well defined non-zero product.

This completes the proof of Lemma 3.4. $\triangle$

Substituting this result into (3.4), we get

**3.7. Lemma.** *We have*

$$J_w(P; h) = \frac{C_w(\alpha; k)}{kh + w(\alpha)} \cdot J_{s_\alpha w}(P_2; -\frac{w(\alpha)}{k}) \tag{3.14}$$

*where*

$$C_w(\alpha; k) = w(C_{id}(\alpha; k)) \tag{3.15}$$

*and*

$$C_{id}(\alpha; k) = (-1)^{k(|\alpha|+1)}\alpha^{k|\alpha|-2k+1} \cdot \frac{k^{k(2-|\alpha|)}}{(k!)^2}\cdot$$

$$\cdot \prod_{\gamma\in R^+;\ \gamma\neq\alpha} \left(\frac{\prod_{m\geq 1} (s_\alpha\gamma - \frac{m}{k}\alpha)}{\prod_{m\geq 1} (\gamma - \frac{m}{k}\alpha)}\right) \quad \triangle \tag{3.16}$$

Here we use the notation $|\alpha| = \sum a_i$ for $\alpha = \sum a_i\alpha_i$.

Let us introduce the series

$$z_w(q) = \sum_\beta J_w(\beta)q^\beta = Z_w(hq) \tag{3.17}$$

**3.8. Theorem.** *We have*

$$z_w(q; h) = 1 + \sum_{\alpha\in R^+;\ k>0} \frac{C_w(\alpha; k)q^{k\alpha^\vee}}{kh + \alpha} \cdot z_{s_\alpha w}(q; -\frac{w(\alpha)}{k}) \tag{3.18}$$

*where $C_w(a; k)$ are given by the formulas (3.15), (3.16).*

This is an immediate corollary of Lemma 3.7. $\triangle$

Let us look more attentively at the expression (3.16). Recall

**3.9. Lemma** ([B], Ch. VI, §1, 1.6, Prop. 17, Cor. 2). *Let*

$$w = s_1 \cdot \ldots \cdot s_q \tag{3.19}$$

*be a reduced decomposition of an element of the Weyl group, where $s_i$ is the reflection correposnding to a simple root $\alpha_i$. Then the roots $\gamma_i = s_q s_{q-1} \cdot \ldots \cdot s_{i+1}(\alpha_i)$ $(i = 1, \ldots, q)$ are all positive, distinct, and*

$$R^+ \cap w^{-1}(-R^+) = \{\theta_1, \ldots, \theta_q\} \quad \triangle \tag{3.20}$$

Obviously, if in the product in (3.16) both $\gamma$ and $s_\alpha \gamma$ are positive, then the two factors corresponding to them cancel out. The previous lemma says that we must keep only $l(s_\alpha) - 1$ terms in the product (we have already taken care of the term $\gamma = \alpha$).

**3.10. Corollary.** *If the root $\alpha$ is simple then*

$$C_{id}(\alpha; k) = \frac{k^k}{(k!)^2} \alpha^{-k+1} \tag{3.21}$$

with

In fact, for a simple $\alpha$, the big product disappears at all. The expression (3.21) coincides with I, (3.15).

# Part III. COMPUTATIONS FOR $SL(3)$

## §1. Formula

**1.1.** In this part, $X$ will denote the flag manifold $G/B$, with $G = SL(3)$. According to [G2], the quantum cohomology of $X$ is given by the Fourier transform of the following $\mathcal{D}$-module on the three-torus.

Let $A = (a_{ij})$ be the $3 \times 3$-matrix with $a_{ii} = u_{i-1}$ $(i = 1, 2, 3)$; $a_{i,i-1} = v_i$ $(i = 2, 3)$; $a_{i,i+1} = -1$ $(i = 1, 2)$; $a_{13} = a_{31} = 0$. Consider the characteristic polynomial

$$\det(\lambda + A) = \lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 = \lambda^3 + (u_0 + u_1 + u_2)\lambda^2 +$$

$$+(u_0u_1 + u_0u_2 + u_1u_2 + v_1 + v_2)\lambda + u_0u_1u_2 + u_0v_2 + u_2v_1 \tag{1.1}$$

Thus, the polynomials $P_i = P_i(u; v)$ are the deformed symmetric functions.

Consider the three-dimensional torus $T$, with multiplicative coordinates $q_0, q_1, q_2$. We define the differential operators $D_i$ on $T$, where $D_i$ is obtained from $P_i$ by the substitution $u_j = q_j\partial_{q_j}$, $v_j = q_j/q_{j-1}$. We are interested in the solutions of the system

$$D_1\phi = D_2\phi = D_3\phi = 0, \tag{1.2}$$

$\phi = \phi(q_0, q_1, q_2)$.

First of all, since $D_1\phi = 0$, the function $\phi$ depends in fact only on the quotients $v_1 = q_1/q_0$, $v_2 = q_2/q_1$. It is useful to write up the expressions of the operators $q_i\partial_{q_i}$ acting on such functions, in coordinates $v_1, v_2$:

$$q_0\partial_{q_0} = -v_1\partial_1;\ q_1\partial_{q_1} = v_1\partial_1 - v_2\partial_2;\ q_2\partial_{q_2} = v_2\partial_2 \tag{1.3}$$

(we set for brevity $\partial_i = \partial_{v_i}$).

In these coordinates, the remaining operators look as follows.

$$D_2 = -(v_1\partial_1)^2 + (v_1\partial_1)(v_2\partial_2) - (v_2\partial_2)^2 + v_1 + v_2 \tag{1.4}$$

$$D_3 = -(v_1\partial_1)^2 v_2\partial_2 + v_1\partial_1(v_2\partial_2)^2 - v_2(v_1\partial_1) + v_1(v_2\partial_2) \tag{1.5}$$

**1.2. Theorem.** *There exists a unique, up to a multiplicative constant, solution $\phi(v_1, v_2)$ of the system*

$$D_2\phi = D_3\phi = 0 \tag{1.6}$$

*in the ring of formal power series $\mathbb{Q}[[v_1, v_2]]$.*

*If we normalize $\phi$ by the condition $\phi(0) = 1$, it will be given by the formula*

$$\phi(v_1, v_2) = \sum_{i,j\geq 0} \frac{(i+j)!}{(i!)^3(j!)^3} v_1^i v_2^j \tag{1.7}$$

Here $D_2, D_3$ are given by (1.4), (1.5).

**Proof.** Let us denote the power series (1.7) by $\sum_{i,j\geq 0}\ a_{ij}v_1^iv_2^j,\ a_{00}=1$. Equation $D_2\phi=0$ is equivalent to the recursion

$$(i^2-ij+j^2)a_{ij}=a_{i-1,j}+a_{i,j-1} \tag{1.8}$$

(we imply that $a_{ij}=0$ if either $i$ or $j$ is negative). Equation $D_3\phi=0$ is equivalent to

$$ij(i-j)a_{ij}=-ia_{i,j-1}+ja_{i-1,j} \tag{1.9}$$

Both formulas are checked immediately for $a_{ij}$ given by (1.6). Already (1.8) defines all $a_{ij}$ uniquely from $a_{00}$. △

**1.3. Remarks.** (a) We have $a_{ij}=a_{ji}$.

(b) It follows from (1.8) that

$$a_{i0}=\frac{1}{(i!)^2} \tag{1.10}$$

The function

$$\phi(v,0)=\sum\ \frac{v^i}{(i!)^2} \tag{1.11}$$

conicides with the hypergeometric function associated with $G/B$ for $G=SL(2)$.

(c) The formulas (1.8) and (1.9) imply the identity

$$i^3a_{i,j-1}-j^3a_{i-1,j}=0 \tag{1.12}$$

This, together with (1.10), gives immediately (1.7).

**1.4. Another formula.** V. Batyrev communicated to me another formula for the solution of (1.6): $\tilde{\phi}=\sum_{i,j}\ b_{ij}v_1^iv_2^j$ where

$$b_{ij}=\frac{1}{(i!)^2(j!)^2}\sum_r\ C_i^rC_j^r \tag{1.13}$$

where

$$C_b^a=\frac{b!}{a!(b-a)!} \tag{1.14}$$

for integers $a,b$ such that $0\leq a\leq b$, and 0 if $a<0$ or $a>b$.

**Claim.** *The function $\tilde{\phi}$ is equal to $\phi$.*

Indeed, our claim is equivalent to the identity

$$\sum_r\ C_i^rC_j^r=C_{i+j}^i \tag{1.15}$$

To prove this, one remarks that to choose $i$ elements from a set which is a disjoint union of two sets of cardinalities $j$ and $i$, is the same as to choose some $r$ elements from the first set, and $i-r$ from the second one.

## §2. Equivariant version

Below, we will deform equations (1.6) and the solution (1.7). In terms of quantum cohomology, this deformation corresponds to passing to the equivariant cohomology, with respect to a natural action of a three-torus on $X$.

**2.1.** Introduce differential operators

$$D_{i;h;\lambda} = P_i(hq_0\partial_{q_0}, hq_1\partial_{q_1}, hq_2\partial_{q_2}; q_1/q_0, q_2/q_1) - \sigma_i(\lambda_0, \lambda_1, \lambda_2) \tag{2.1}$$

Here $P_i(u_0, u_1, u_2; v_1, v_2)$ are the polynomials (1.1); $\sigma_i(\lambda) = P_i(\lambda; 0)$ are the elementary symmetric functions.

We are interested in the solutions of the system

$$D_{1;h;\lambda}\psi = D_{2;h;\lambda}\psi = D_{3;h;\lambda}\phi = 0 \tag{2.2}$$

of the form

$$\psi(q_0, q_1, q_2) = q_0^{\lambda_0/h} q_1^{\lambda_1/h} q_2^{\lambda_2/h} \phi(q_0, q_1, q_2);\ \phi(q) = \sum_{i,j,k} b_{ijk} q_0^i q_1^j q_2^k \tag{2.3}$$

The equation $D_{1;h;\lambda}\phi = 0$ implies $i + j + k = 0$, thus, for a solution $\psi$, the factor $\phi(q)$ would depend only on $v_1 = q_1/q_0$ and $v_2 = q_2/q_1$.

To formulate the answer, we introduce the notations $\alpha_i = \lambda_i - \lambda_{i-1}$;

$$p^!_\alpha = (h + \alpha)(2h + \alpha) \cdot \ldots \cdot (ph + \alpha) \tag{2.4}$$

**2.2. Theorem.** *There exists a unique, up to a multiplicative constant, solution of the system (2.2) having the form (2.3), with $\phi \in \mathbb{Q}[[v_1, v_2]]$, $v_1 = q_1/q_0, v_2 = q_2/q_1$. If we normalize it by the condition $\phi(0) = 1$, it will have the form*

$$\phi(v_1, v_2) = \sum_{i,j\geq 0} \frac{(i+j)^!_{\alpha_1+\alpha_2}}{i!j!i^!_{\alpha_1} j^!_{\alpha_2} i^!_{\alpha_1+\alpha_1} j^!_{\alpha_1+\alpha_2}} \frac{v_1^i v_2^j}{h^{i+j}} \tag{2.5}$$

**Proof.** Let us denote by $a_{ij}$ the coefficient at $v_1^i v_2^j$ of the unknown $\phi$. The equation $D_{2;h;\lambda}\phi = 0$ is equivalent to

$$(i^2h^2 - ijh^2 + j^2h^2 + ih\alpha_1 + jh\alpha_2)a_{ij} = a_{i-1,j} + a_{i,j-1} \tag{2.6}$$

(cf. 1.8)). The equation $D_{3;h;\lambda}\phi = 0$ is equivalent to

$$[(ih - \lambda_0)(ih - jh + \lambda_1)(jh + \lambda_2) + \lambda_0\lambda_1\lambda_2]a_{ij} =$$

$$= (-ih + \lambda_0)a_{i,j-1} + (jh + \lambda_2)a_{i-1,j} \tag{2.7}$$

(cf. 1.9)). These identities are checked directly. The uniqueness follows from (2.6). △

**2.3. Remarks.** We have $a_{ij}(\alpha_1,\alpha_2)=a_{ji}(\alpha_2,\alpha_1)$. Equation (2.6) implies

$$a_{i0}=\frac{1}{h^i i! i^!_{\alpha_1}} \tag{2.7}$$

(cf. (1.10)). On the other hand, (2.6) and (2.7) together imply the identity

$$ih(ih+\alpha_1)(ih+\alpha_1+\alpha_2)a_{i,j-1}-jh(jh+\alpha_2)(jh+\alpha_1+\alpha_2)a_{i-1,j}=0 \tag{2.8}$$

(cf. (1.12)). This and (2.7) give the expression (2.5).

## §3. Comparison with the fixed point method

**3.1.** Let us see what does Theorem II.3.8 give for $G=SL(3)$, i.e. for the root system $A_2$. The simple roots are $\alpha_1,\alpha_2$, the positive ones are $\alpha_1,\alpha_2,\alpha:=\alpha_1+\alpha_2$.

The Weyl group $W=\Sigma_3$ has two Coxeter generators $s_1,s_2$ corresponding to the simple roots, and $s_\alpha=s_1s_2s_1$. For example, $s_1$ takes $\alpha_1$ to $-\alpha_1$ and $\alpha_2$ to $\alpha$, etc.

We have six series $z_w$ $(w\in W)$:

$$z_w=z_w(q_1,q_2;\alpha_1,\alpha_2;h)=\sum_{i\geq 0,j\geq 0} a_{ij;w}q^iq^j; \tag{3.1}$$

where

$$a_{ij;w}=w(a_{ij});\quad a_{ij}=a_{ij}(\alpha_1,\alpha_2;h):=a_{ij;id} \tag{3.2}$$

The element $w$ acts only on the arguments $\alpha_1,\alpha_2$.

It is easy to see that the recursion relation of Theorem II.3.8 takes the following form.

**3.2. Theorem.** *We have*

$$z_{id}(q;h)=1+\sum_{k>0}\frac{q_1^k}{kh+\alpha_1}\cdot\frac{k^k}{(k!)^2}\cdot\frac{1}{\alpha_1^{k-1}}\cdot z_{s_1}(q;-\alpha_1/k)+$$

$$+\sum_{k>0}\frac{q_2^k}{kh+\alpha_2}\cdot\frac{k^k}{(k!)^2}\cdot\frac{1}{\alpha_2^{k-1}}\cdot z_{s_2}(q;-\alpha_2/k)-\sum_{k>0}\left(\frac{q_1^kq_2^k}{kh+\alpha_1+\alpha_2}\cdot\frac{k^{2(k-1)}}{(k!)^2}\cdot\right.$$

$$\left.\cdot\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}\cdot\frac{1}{\prod_{m=1}^{k-1}(m\alpha_1-(k-m)\alpha_2)^2}\cdot z_{s_\alpha}(q;-(\alpha_1+\alpha_2)/k)\right)\quad\triangle \tag{3.3}$$

Now, the main result of this section is

**3.3. Theorem.** *The recursion relation* (3.3) *is satisfied, with*

$$a_{ij}=\frac{(i+j)^!_{\alpha_1+\alpha_2}}{i!j!i^!_{\alpha_1}j^!_{\alpha_2}i^!_{\alpha_1+\alpha_2}j^!_{\alpha_1+\alpha_2}} \tag{3.4}$$

**Proof.** Assume that $i \le j$. Let us write $a_{ij}$ in the form

$$a_{ij}(\alpha_1, \alpha_2; h) = \frac{[(i+j)h + \alpha_1 + \alpha_2] \cdot \ldots \cdot [(1+j)h + \alpha_1 + \alpha_2]}{(jh + \alpha_1 + \alpha_2) \cdot \ldots \cdot (h + \alpha_1 + \alpha_2)} \cdot$$

$$\cdot \frac{1}{i!(h+\alpha_1) \cdot \ldots \cdot (ih + \alpha_1) \cdot j!(h+\alpha_2) \cdot \ldots \cdot (jh + \alpha_2)} \tag{3.5}$$

The denominator (as a function of $h$) has the distinct roots:

$$h = -\frac{\alpha_1 + \alpha_2}{k} \ (k = 1, \ldots, i); \ h = -\frac{\alpha_1}{k} \ (k = 1, \ldots, i);$$

$$h = -\frac{\alpha_2}{k} \ (k = 1, \ldots, j). \tag{3.6}$$

Accordingly, we have the simple fraction decomposition

$$a_{ij}(\alpha_1, \alpha_2; h) = \sum_{k=1}^{i} \frac{{}^1b_{ij}^k(\alpha_1, \alpha_2)}{kh + \alpha_1} + \sum_{k=1}^{j} \frac{{}^2b_{ij}^k(\alpha_1, \alpha_2)}{kh + \alpha_2} + \sum_{k=1}^{i} \frac{{}^3b_{ij}^k(\alpha_1, \alpha_2)}{kh + \alpha_1 + \alpha_2} \tag{3.7}$$

The theorem is equivalent to

**3.4. Lemma.** *We have*

(a) *For* $1 \le k \le i$,

$${}^1b_{ij}^k(\alpha_1, \alpha_2) = \frac{k^k}{(k!)^2} \cdot \frac{1}{\alpha_1^{k-1}} \cdot a_{i-k,j}(-\alpha_1, \alpha_1 + \alpha_2; -\alpha_1/k); \tag{3.8}$$

(b) *For* $1 \le k \le j$,

$${}^2b_{ij}^k(\alpha_1, \alpha_2) = \frac{k^k}{(k!)^2} \cdot \frac{1}{\alpha_2^{k-1}} \cdot a_{i,j-k}(\alpha_1 + \alpha_2, -\alpha_2; -\alpha_2/k); \tag{3.9}$$

(c) *For* $1 \le k \le i$,

$${}^3b_{ij}^k = -\frac{k^{2(k-1)}}{(k!)^2} \cdot \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} \cdot \frac{1}{\prod_{m=1}^{k-1} [m\alpha_1 - (k-m)\alpha_2]^2} \cdot$$

$$\cdot a_{i-k,j-k}(-\alpha_2, -\alpha_1; -(\alpha_1 + \alpha_2)/k) \tag{3.10}$$

This lemma is established by a direct computation. The theorem is proved. $\triangle$

**3.5. Corollary.** *The series* $Z_{id}(q_1, q_2) = z_{id}(q_1/h, q_2/h)$ *coincides with the series* $\phi(q_1, q_2)$ *from* (2.5). $\triangle$

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