Mellin Quantization in the Cone Calculus for Boutet de Monvel's Algebra

Elmar Schrohe
B.-W. Schulze

Max-Planck-Arbeitsgruppe
"Partielle Differentialgleichungen und komplexe Analysis"
Insitut für Mathematik,
Universität Potsdam
Postfach 60 15 53
D-14415 Potsdam

Germany

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-53225 Bonn

Germany

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Max-Planck-Arbeitsgruppe
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14415 POTSDAM

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Abstract

We present a pseudodifferential calculus for boundary value problems on manifolds with conical singularities. We then show how to associate to each totally characteristic (Fuchs type) pseudodifferential symbol with values in Boutet de Monvel's algebra an operator-valued Mellin symbol is such a way that the difference between the two corresponding operators is smoothing in the interior. This allows us to extend the action of the operators to weighted Mellin-Sobolev spaces.

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Key Words: Manifolds with conical singularities, boundary value problems, Boutet de Monvel's calculus, Mellin calculus

Contents

Introduction

Following the work of Kondrat'ev, Plamenevskij, and Schulze it is now a widely accepted idea that the analysis on manifolds with conical singularities should be based on (i) a pseudodifferential calculus using totally characteristic (or Fuchs type) symbols near the singularities and (ii) weighted Sobolev spaces.

A basic motivation is the interest in an index theory within a pseudodifferential algebra that contains what one considers the typical differential operators: Near a conical singularity we identify the manifold with the cylinder $X \times \mathbf{R}_+$ where X is a smooth compact manifold with boundary. Then the typical differential operators are those of the form $A = t^{-\mu}\sum_{j=0}^{\mu} A_j(t)(t\partial_t)^j$ with families $A_j(\cdot)$ of differential operators of order $\mu - j$ on X which are smooth up to $t = 0$. One reason for this point of view is that any Laplace-Beltrami operator associated with the Riemannian metric $t^2 h_X(t) + dt^2$ of a warped cone provides an example for an operator of this kind whenever $h_X(\cdot)$ is a family of Riemannian metrics on X which is smooth up to $t = 0$. Another justification stems from the observation that one can introduce an 'artificial' conical point on a smooth manifold by introducing polar coordinates and that under this operation any differential operator assumes this form; for details see [14]. In particular, suppose we are given a differential operator and a conical domain D in Euclidean space; then the operator will have the above type in the natural polar coordinates for D.

Operators of this kind are called totally characteristic or Fuchs type operators. Correspondingly, a totally characteristic pseudodifferential operator of order μ on the cylinder $X \times \mathbf{R}_+$ is one whose symbol, up to the weight factor $t^{-\mu}$, is of the form $a(x,t,\xi,\tau) = b(x,t,\xi,t\tau)$, where b is a usual pseudodifferential symbol which is smooth up to $t = 0$. The weighted Mellin-Sobolev space $\mathcal{H}^{s,\gamma}$ are best described for $s \in \mathbf{N}$, when they consist of all functions u such that $t^{n/2-\gamma}(t\partial_t)^k D_x^\alpha u \in L^2(X \times \mathbf{R}_+)$ for all $k + |\alpha| \leq s$; for general s one can use duality and interpolation. Here, x and t are the variables on X and $\mathbf{R}_+$, respectively, ξ and τ are the corresponding covariables; n is the dimension of X.

Outside a neighborhood of the singularities, Boutet de Monvel's calculus in its standard form is the natural choice.

The principal analytical problem then is to define a quantization that associates with a totally characteristic pseudodifferential symbol a continuous operator on the weighted Sobolev spaces. This plays a crucial role for example in the construction of parametrices to totally characteristic differential operators. The Leibniz inversion of the symbol yields a totally characteristic pseudodifferential symbol. A priori it is by no means clear how to define from this symbol a continuous action on the weighted Sobolev spaces.

This is the question we address in this note. Our answer is what we call 'Mellin quantization'. We show that, for every weight γ and each totally characteristic pseudodifferential symbol a with values in Boutet de Monvel's algebra on X which is smooth up to $t = 0$, we find a Mellin symbol f such that the difference $\operatorname{op} a - \operatorname{op}_M^\gamma f$ is a regularizing operator in Boutet de Monvel's calculus on the nonsingular part of the manifold. Here, $\operatorname{op}_M^\gamma f$ is the weighted Mellin pseudodifferential operator associated with the operator-valued Mellin symbol f. It naturally acts on the spaces $\mathcal{H}^{s,\gamma}$ and therefore solves the problem.

There are two more satisfying aspects to this solution. For one thing, the Mellin calculus allows an intrinsic description of the totally characteristic pseudodifferential operators in the sense that it respects the natural $\mathbf{R}_+$ structure of the space near the singularities. Secondly, it turns out that there is a notion of ellipticity that allows us to characterize the Fredholm property between the spaces $\mathcal{H}^{s,\gamma}$ while the degeneracy of the symbols at $t = 0$

in general prevents these operators from being Fredholm on the usual Sobolev spaces.

The case of manifolds *without* boundary is automatically included in the analysis, since the operators on the boundary also belong to the calculus. The methods to treat the latter case have been developed by Schulze [16], [18].

It should be mentioned that the analysis of [14], [15] can be viewed as part of a more general concept pointed out by Schulze [16]: Given a parameter-dependent pseudodifferential calculus on a space X it should be possible to construct a pseudodifferential calculus also for the 'cone' $X \times \mathbf{R}_+$. In the present case the space X is a manifold with boundary, and the calculus is a parameter-dependent version of Boutet de Monvel's calculus on X. Since Boutet de Monvel's calculus in its standard form is already rather complex, the parameter-dependent version has been established in [14] in a new efficient way; the central idea is to use the concept of parameter-dependent pseudodifferential operators based on group actions and wedge Sobolev spaces.

Mellin quantization is also relevant for pseudodifferential boundary value problems without the transmission property, such as those considered by Vishik&Eskin [19] and Eskin [5]. The interior normal direction then plays the role of the cylinder axis $\mathbf{R}_+$, and the asymptotics of solutions can be described in terms of Mellin-Sobolev spaces; the asymptotics for problems with the transmission property then correspond to Taylor asymptotics near $t = 0$, while much more general asymptotics are possible, cf. [18].

1 Parameter-Dependent and Fuchs Type Operators in Boutet de Monvel's Calculus

1.1 Manifolds with Conical Singularities

An $n-$dimensional manifold with boundary is a topologigal (second countable) Hausdorff space M such that each point $m \in M$ has a neighborhood which is diffeomorphic to either $\mathbf{R}^n$ or the closed half-space $\overline{\mathbf{R}}^n_+$. The former points are called the interior points of M, the latter the boundary points. We will use the standard notation $\operatorname{int} M$ and ∂M.

1.1 Definition. A *manifold with boundary and conical singularities D of dimension $n+1$* is a topological (second countable) Hausdorff space with a finite subset $\Sigma \subset D$ ('singularities') such that $D \backslash \Sigma$ is an $n+1$-dimensional manifold with boundary, and for every $v \in \Sigma$ there is an open neighborhood U of v, a compact manifold with boundary X of dimension n, and a system $\mathcal{F} \neq \emptyset$ of mappings with the following properties

(1) For all $\phi \in \mathcal{F}$, $\phi : U \to X \times [0,1)/X \times \{0\}$ is a homeomorphism with $\phi(v) = X \times \{0\}/X \times \{0\}$.

(2) Given $\phi_1, \phi_2 \in \mathcal{F}$, the restriction $\phi_1\phi_2^{-1} : X \times (0,1) \to X \times (0,1)$ extends to a *diffeomorphism* $X \times (-1,1) \to X \times (-1,1)$.

(3) The charts $\phi \in \mathcal{F}$ are compatible with the charts for the manifold for $D\backslash\Sigma$: The restriction $\phi : U\backslash\{v\} \to X \times (0,1)$ is a diffeomorphism.

We can and will assume that for each singularity $v \in \Sigma$, the system $\mathcal{F}$ is maximal with respect to the properties (1), (2), and (3).

1.2 Definition and Remark. By assumption, $D\backslash\Sigma$ is a manifold with boundary. Properties 1.1(1) and (2) imply that any neighborhood of a point $v \in \Sigma$ contains points of the topological boundary of $D\backslash\Sigma$, namely of $\partial X \times (0,1)$.

A point $x \in D$ is an *interior point of* D if there is an open neighborhood of x which is homeomorphic to an open ball in $\mathbf{R}^{n+1}$, and $\operatorname{int} D$ is the collection of all interior points; $\partial D = D\backslash\operatorname{int} D$ is the boundary of D. We always have $\Sigma \subset \partial D$.

1.3 Remark. Let D be a manifold with boundary and conical singularities. Then the topological boundary ∂D of D is a (boundaryless) manifold with conical singularities in the sense of [18, Definition 1.1.15].

1.4 Notation and Assumptions. In a neighborhood of one of the singularities, X will denote the cross-section as in 1.1; it is a manifold with boundary of dimension n, in particular, X contains its boundary. For practical purposes, this is often inconvenient. We shall therefore agree to denote by X the open interior, and by $\overline{X}$ the manifold including the boundary. We let $X^\wedge = X \times \mathbf{R}_+; \overline{X}^\wedge = \overline{X} \times \mathbf{R}_+$, $Y = \partial X$ is the topological boundary of $X; Y$ is a closed manifold of dimension $n-1$. We let $Y^\wedge = Y \times \mathbf{R}_+$.

It is on the cylinder $X^\wedge$ that the analysis in this paper is performed. We assume that X is endowed with a Riemannian metric and embedded in a closed Riemannian manifold Ω and that $X^\wedge$ carries the canonical (cylindrical) metric.

1.2 Parameter-Dependent Symbols and Sobolev Spaces

In a collar neighborhood of the boundary Y of X we introduce normal coordinates. A point there can be written $x = (y, r)$ with $y \in Y, r \geq 0$. If U is an open subset of $\mathbf{R}^{n-1}$, then coordinates in $U \times \mathbf{R}$ will also be written in the form $x = (x', r)$ or likewise $x = (x', x_n)$, with $x' \in U$ and $r, x_n \in \mathbf{R}$.

1.5 Sobolev Spaces on $\mathbf{R}^n$ and $\mathbf{R}^n_+$. Let U be an open subset of $\mathbf{R}^{n-1}$. For a function or distribution u on $U \times \mathbf{R}$ let r^+u denote its restriction to $U \times \mathbf{R}_+$. We shall also use the operator r^+ to indicate the restriction of functions or distributions on Ω to X.

$H^s(\mathbf{R}^n), s \in \mathbf{R}$, is the usual Sobolev space over $\mathbf{R}^n$. We let $H^s(\mathbf{R}^n_+) = \mathrm{r}^+H^s(\mathbf{R}^n)$ and $H^s_0(\mathbf{R}^n_+) = \{u \in H^s(\mathbf{R}^n) : \operatorname{supp} u \subseteq \overline{\mathbf{R}}^n_+\}$. Equivalently, $H^s_0(\mathbf{R}^n_+)$ is the closure of $C^\infty_0(\mathbf{R}^n_+)$ in the topology of $H^s(\mathbf{R}^n)$.

The notation extends to the case of compact manifolds via a partition of unity. This yields the spaces $H^s(\Omega), H^s(X)$, and $H^s_0(X)$. We shall also employ the notation $H^s(X \times \mathbf{R}_+), H^s_0(X \times \mathbf{R}_+)$, etc, understanding that we use the canonical choice of $L^2(X \times \mathbf{R}_+)$.

For functions on $U \times \mathbf{R}_+$ or distributions in $H^s(\mathbf{R}^n_+), s > -\frac{1}{2}$, we let e^+ denote the operator of extension (by zero) to $U \times \mathbf{R}$ and $H^s(\mathbf{R}^n)$. Again this carries over to the manifold case and yields a bounded map $\mathrm{e}^+ : H^s(X) \to H^s(\Omega), -\frac{1}{2} < s < \frac{1}{2}$.

$\mathcal{S}(\mathbf{R}^n)$ denotes the space of all rapidly decreasing functions on $\mathbf{R}^n$, and $\mathcal{S}(\mathbf{R}^n_+)$ is the space of all restrictions of functions in $\mathcal{S}(\mathbf{R}^n)$ to $\mathbf{R}^n_+$. $\mathcal{S}'(\mathbf{R}^n_+)$ is the dual space of $\mathcal{S}(\mathbf{R}^n_+)$. Note that $\mathcal{S}'(\mathbf{R}^n_+)$ contains distributions with support in $\partial\mathbf{R}^n_+$. If we define the weighted Sobolev space $H^{s,\sigma}(\mathbf{R}^n_+)$ and $H^{s,\sigma}_0(\mathbf{R}^n_+)$ as the sets of all $\langle x\rangle^{-\sigma} u$, where u is an element of $H^s(\mathbf{R}^n_+)$ and $H^s_0(\mathbf{R}^n_+)$, respectively, then $\mathcal{S}(\mathbf{R}^n_+) = \operatorname{proj} - \lim_{s,\sigma\to\infty} H^{s,\sigma}(\mathbf{R}^n_+)$ and $\mathcal{S}'(\mathbf{R}^n_+) = \operatorname{ind} - \lim_{s,\sigma\to-\infty} H^{s,\sigma}_0(\mathbf{R}^n_+)$.

It was an important point in [14] to develop a new approach to Boutet de Monvel's calculus based on group actions and operator valued symbols. Since this connection is going to play a role also in this paper we repeat the essential notions.

1.6 Group Actions and Operator-Valued Symbols. Let E, F be Banach spaces with strongly continuous group actions $\{\kappa_\lambda : \lambda \in \mathbf{R}_+\}$ and $\{\tilde{\kappa}_\lambda : \lambda \in \mathbf{R}_+\}$. By definition this means that

(i) $\lambda \mapsto \kappa_\lambda \in C(\mathbf{R}_+, \mathcal{L}_\sigma(E))$, $\lambda \mapsto \tilde{\kappa}_\lambda \in C(\mathbf{R}_+, \mathcal{L}_\sigma(F))$ (strong continuity of κ and $\tilde{\kappa}$); and

(ii) $\kappa_\lambda \kappa_\mu = \kappa_{\lambda\mu}, \tilde{\kappa}_\lambda \tilde{\kappa}_\mu = \tilde{\kappa}_{\lambda\mu}$.

Here $\mathcal{L}_\sigma(\cdot)$ refers to the space $\mathcal{L}(\cdot)$ endowed with the strong topology.

Let $U \subseteq \mathbf{R}^k$ and $p \in C^\infty(U \times \mathbf{R}^n, \mathcal{L}(E,F)), \mu \in \mathbf{R}$. We shall write $p \in S^\mu(U, \mathbf{R}^n; E, F)$ provided that for every $K \subseteq U$ and all multi-indices α, β, there is a constant $C = C(K, \alpha, \beta)$ with

$$\|\tilde{\kappa}_{\langle\eta\rangle^{-1}}\{D_\eta^\alpha D_y^\beta p(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)} \le C\,\langle\eta\rangle^{\mu-|\alpha|}, \quad y \in K, \tag{1}$$

cf. [16, 3.2.1, Definition 1]. The space $S^\mu(U, \mathbf{R}^n; E, F)$ is a Fréchet space topologized by the choice of the best constants C.

For the usual or weighted Sobolev spaces on $\mathbf{R}_+$, we will always employ the group action

$$[\kappa_\lambda f](r) = \lambda^{\frac{1}{2}} f(\lambda r). \tag{2}$$

On $E = \mathbf{C}$ we use the trivial group action $\kappa_\lambda \equiv id$. For $E = F = \mathbf{C}$ we shall write $S^\mu(U, \mathbf{R}^n)$ instead of $S^\mu(U, \mathbf{R}^n; \mathbf{C}, \mathbf{C})$. The above definition then coincides with the standard symbol class notation.

If $F_1 \hookleftarrow F_2 \hookleftarrow \ldots$ is a sequence of Banach spaces with the same group action, and F is the Fréchet space given as the projective limit of the F_k, then let

$$S^\mu(U, \mathbf{R}^n; E, F) = \text{proj} - \lim_k S^\mu(U, \mathbf{R}^n; E, F_k). \tag{3}$$

Vice versa, if E is the inductive limit of the Banach spaces $E_1 \hookrightarrow E_2 \hookrightarrow \ldots$ with the same group action, then

$$S^\mu(U, \mathbf{R}^n; E, F) = \text{ind} - \lim_k S^\mu(U, \mathbf{R}^n; E_k, F), \tag{4}$$

Finally, a symbol p belongs to $S^\mu(U, \mathbf{R}^n; E, F), E = \text{ind} - \lim E_k, F = \text{proj} - \lim F_l$, if the group actions coincide on the E_k and F_l, respectively, and $p \in S^\mu(U, \mathbf{R}^n; E_k, F_l)$ for all k and l. We give it the topology induced by all the topologies of the spaces $S^\mu(U, \mathbf{R}^n; E_k, F_l)$.

We will, in particular, deal with the spaces $S^\mu(U, \mathbf{R}^n; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$. For the inductive and projective limit constructions we shall then use the representation of $\mathcal{S}'(\mathbf{R}_+)$ and $\mathcal{S}(\mathbf{R}_+)$, respectively, as limits of weighted Sobolev spaces over $\mathbf{R}_+$.

In view of the nuclearity of $C^\infty(U)$ we have

$$S^\mu(U, \mathbf{R}^n; E, F) = C^\infty(U) \hat{\otimes}_\pi S^\mu(\mathbf{R}^0, \mathbf{R}^n; E, F), \tag{5}$$

the functions in the last space on the right hand side being independent of y.

1.7 Definition. Let $U = U_1 \times U_2 \subseteq \mathbf{R}^n \times \mathbf{R}^n$ be open and $p \in S^\mu(U, \mathbf{R}^n; E, F)$ an operator-valued symbol. Then the pseudodifferential operator $\operatorname{op} p$ is defined by

$$[\operatorname{op} p(f)](y) = (2\pi)^{-n} \int \int_{U_2} e^{i(y-y')\eta} p(y, y', \eta) f(y') dy' d\eta \tag{1}$$

for $f \in C_0^\infty(U_2, E), y \in U_1$. This reduces to

$$[\operatorname{op} p(f)](y) = (2\pi)^{-\frac{n}{2}} \int e^{iy\eta} p(y,\eta) \hat{f}(\eta) d\eta, \tag{2}$$

for 'simple' symbols, i.e. those that are independent of y'. Here, $\hat{f}(\eta) = (2\pi)^{-\frac{n}{2}} \int e^{-iy\eta} f(y) dy$ is the vector-valued Fourier transform of f.

We may also consider the case where a part of the covariables serves as parameters: For $U \subseteq \mathbf{R}^n$ open, $p \in S^\mu(U_y, \mathbf{R}^n_\eta \times \mathbf{R}^l_\lambda; E, F)$ then defines a parameter-dependent operator $\operatorname{op} p(\lambda)$ by

$$[\operatorname{op} p(\lambda) f](y) = (2\pi)^{-n/2} \int e^{iy\eta} p(y,\eta,\lambda) \hat{f}(\eta) d\eta, \tag{3}$$

$f \in C_0^\infty(U, E)$, similarly for 'double' symbols $p(y, y', \eta, \lambda)$.

1.8 The Manifold Case. Let Ω be a smooth manifold, V_1, V_2 vector bundles over Ω, and E, F Banach spaces with strongly continuous group actions. Moreover, let $P : C_0^\infty(\Omega, E) \to C^\infty(\Omega, F)$ be a continuous operator. We shall say that $P \in \operatorname{op}S^\mu(\Omega, \mathbf{R}^n; E, F)$ if the following holds:

(i) For all C_0^∞ functions ϕ, ψ, supported in the same coordinate neighborhood, the operator $(\phi P \psi)_* : C_0^\infty(U, E) \to C^\infty(U, F)$ induced on $U \subseteq \mathbf{R}^n$ by $\phi P \psi$ and the coordinate maps has the form $(\phi P \psi)_* = \operatorname{op} p$ for some $p \in S^\mu(U, \mathbf{R}^n; E, F)$.

(ii) For all C_0^∞ functions ϕ, ψ, with disjoint supports, the operator $\phi P \psi$ is given as an integral operator with a kernel in $C^\infty(\Omega \times \Omega, \mathcal{L}(E, F))$ (more precisely a kernel section, see [3, Section 23.4]).

If P depends on a parameter $\lambda \in \mathbf{R}^l$, then (i) carries over, while in (ii) we ask that the integral kernel belongs to $\mathcal{S}(\mathbf{R}^l, C^\infty(\Omega \times \Omega, \mathcal{L}(E, F)))$.

Suppose we are given a locally finite covering of the manifold by relatively compact coordinate neighborhoods $\{\Omega_j\}$ with associated coordinate maps $\chi_j : \Omega_j \to U_j$. Then (i) allows us to find $p_j \in S^\mu(U, \mathbf{R}^n; E, F)$ such that $P(f \circ \chi_j)(x) = \operatorname{op} p_j(f)(\chi_j(x))$ for all $f \in C_0^\infty(U_j, E)$. We shall call the tuple $\{p_j\}$ the symbol of P.

Let now $\Omega_j \cap \Omega_k \neq \emptyset$, and suppose that both ϕ and ψ are supported in the intersection. Denote by P_j and P_k the operators on $C_0^\infty(\Omega, E)$ induced by $(\phi \circ \chi_j^{-1}) \operatorname{op} p_j (\psi \circ \chi_j^{-1})$ and $(\phi \circ \chi_k^{-1}) \operatorname{op} p_k (\psi \circ \chi_k^{-1})$. Then $P_j - P_k$ is an integral operator with a kernel in $C^\infty(\Omega \times \Omega, \mathcal{L}(E, F))$. Vice versa, given a tuple $\{p_j\}$ with this property, we can define an operator $P : C_0^\infty(\Omega, E) \to C_0^\infty(\Omega, F)$ whose symbol is $\{p_j\}$. Hence the notion $S^\mu(\Omega, \mathbf{R}^n; E, F)$ makes sense.

Given a parameter-dependent operator $P(\lambda) : C_0^\infty(\Omega, E) \to C^\infty(\Omega, F)$, we define the operator $P_+(\lambda) : C_0^\infty(\overline{X}, E) \to C^\infty(X, F)$ by

$$P_+(\lambda) = \mathrm{r}^+ P(\lambda) \mathrm{e}^+. \tag{1}$$

Just like before, the distributions have to be sufficiently smooth in order to allow an extension by zero.

1.3 Boutet de Monvel's Calculus

1.9 Definition. Let $\mu \in \mathbf{R}, d \in \mathbf{N}$ and $U \subseteq \mathbf{R}^{n-1}$ open. In the following definition the parameter-dependence will always refer to the parameter $\lambda \in \mathbf{R}^l$.

(a) A regularizing parameter-dependent singular Green operator (s.G.o.) on $U \times \mathbf{R}_+$ of type 0 is a family of integral operators

$$G_0(\lambda) : C_0^\infty(U, \mathcal{S}(\mathbf{R}_+)) \to C^\infty(U, C^\infty(\overline{\mathbf{R}}_+))$$

given by a kernel in $\mathcal{S}(\mathbf{R}^l, C^\infty(U \times \overline{\mathbf{R}}_+ \times U \times \overline{\mathbf{R}}_+))$. Here we identify $C_0^\infty(U, \mathcal{S}(\mathbf{R}_+))$ and $C^\infty(U, C^\infty(\overline{\mathbf{R}}_+))$ with subsets of $C^\infty(U \times \overline{\mathbf{R}}_+)$. A regularizing s.G.o. G_0 of type d is a parameter-dependent operator of the form $G_0(\lambda) = \sum_{j=0}^d G_{0j}(\lambda)\partial_r^j$ with regularizing parameter-dependent s.G.o's G_{0j} of type zero and the derivative ∂_r on $\mathbf{R}_+$.

A parameter-dependent s.G.o. of order μ and type d on U is an operator

$$G(\lambda) : C_0^\infty(U, \mathcal{S}(\mathbf{R}_+)) \to C^\infty(U, \mathcal{S}(\mathbf{R}_+))$$

that can be written $G = \sum_{j=0}^d \operatorname{op} g_j \partial_r^j + G_0$, where each g_j is a (parameter-dependent and operator-valued) symbol g_j in $S^{\mu-j}(U, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$ and G_0 is a regularizing parameter-dependent s.G.o..

(b) A regularizing parameter-dependent trace operator of type 0 is an operator

$$T_0(\lambda) : C_0^\infty(U, \mathcal{S}(\mathbf{R}_+)) \to C^\infty(U)$$

with an integral kernel in $\mathcal{S}(\mathbf{R}^l, C^\infty(U \times U \times \overline{\mathbf{R}}_+))$. A regularizing trace operator T_0 of type d is a sum $T_0(\lambda) = \sum_{j=0}^d T_{0j}\partial_r^j$; each T_{0j} being regularizing of type 0.

A parameter-dependent trace operator T of order μ and type d on U is an operator that can be written $T = \sum_{j=0}^d \operatorname{op} t_j \partial_r^j + T_0$, with t_j in $S^{\mu-j}(U, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathcal{S}'(\mathbf{R}_+), \mathbf{C})$ and a regularizing trace operator T_0 of type d.

(c) A regularizing parameter-dependent potential operator on U is an operator

$$K_0(\lambda) : C_0^\infty(U) \to C^\infty(U, C^\infty(\overline{\mathbf{R}}_+))$$

given by an integral kernel in $\mathcal{S}(\mathbf{R}^l, C^\infty(U \times \overline{\mathbf{R}}_+ \times U))$; a parameter-dependent potential operator K of order μ is a sum $K = \operatorname{op} k + K_0$ with a pseudodifferential symbol k in $S^\mu(U, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathbf{C}, \mathcal{S}(\mathbf{R}_+))$ and a regularizing parameter-dependent potential operator K_0.

(d) All these spaces of operators carry Fréchet topologies in a natural way: We use the topology of non-direct sums of Fréchet spaces in connection with the natural topologies on the symbol spaces and on the spaces $\mathcal{S}(\mathbf{R}^l, \ldots)$ for the integral kernels.

1.10 Remark. (Non-direct sums of Fréchet spaces) Let E, F be Fréchet spaces and suppose both are continuously embedded in the same Hausdorff vector space.

The exterior direct sum $E \oplus F$ is Fréchet and has the closed subspace $\Delta = \{(a, -a) : a \in E \cap F\}$. The non-direct sum of E and F then is the Fréchet space $E + F := E \oplus F/\Delta$.

1.11 Parameter-Dependent Operators in Boutet de Monvel's Calculus. Let $U \subseteq \mathbf{R}^{n-1}$ be open. A *parameter-dependent operator of order* $\mu \in \mathbf{R}$ *and type* $d \in \mathbf{N}$ *in Boutet de Monvel's calculus* on U is a family $\{A(\lambda) : \lambda \in \mathbf{R}^l\}$ of operators

$$A(\lambda) = \begin{bmatrix} P_+(\lambda) + G(\lambda) & K(\lambda) \\ T(\lambda) & S(\lambda) \end{bmatrix} : \begin{matrix} C_0^\infty(U \times \overline{\mathbf{R}}_+) \\ \oplus \\ C_0^\infty(U) \end{matrix} \rightarrow \begin{matrix} C^\infty(U \times \overline{\mathbf{R}}_+) \\ \oplus \\ C^\infty(U) \end{matrix}, \tag{1}$$

where

$P(\cdot)$ $= \mathrm{op}\, p(\cdot)$ with $p \in S^\mu_{tr}(U \times \overline{\mathbf{R}}_+ \times U \times \overline{\mathbf{R}}_+, \mathbf{R}^n; \mathbf{R}^l), P_+ = \mathrm{r}^+ P \mathrm{e}^+$,
$G(\cdot)$ is a parameter-dependent singular Green operator of order μ and type d,
$K(\cdot)$ is a parameter-dependent potential operator of order μ,
$T(\cdot)$ is a parameter-dependent trace operator of order μ and type d,
$S(\cdot)$ is a parameter-dependent pseudodifferential operator of order μ on U.

The subscript 'tr' indicates that the symbol p satisfies the transmission condition (see [13, Section 2.2.2.1]) at the boundary $U \times \{0\}$. Note that the decomposition $P_+ + G$ is not unique; the regularizing pseudodifferential operators provide examples for operators that belong to both classes. We shall write $A \in \mathcal{B}^{\mu,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$. The topology on this space is that of a non-direct sum of Fréchet spaces induced by (1) and the topologies on the spaces of pseudodifferential, singular Green, trace, and potential operators.

A parameter-dependent *regularizing* operator A of type d in Boutet de Monvel's calculus on U is one that can be written in the form (1) with all entries being regularizing operators. We shall write $A \in \mathcal{B}^{-\infty,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$, and give this space the obvious Fréchet topology.

It is a consequence of 1.9 that the operators in (1) indeed have the desired mapping properties.

Given an operator $A \in \mathcal{B}^{\mu,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$ we have a symbol a for A, namely the quintuple $a = \{p, g, k, t, s\}$ of the symbols for the operators P, G, K, T, and S, respectively. As pointed out before, there is a certain ambiguity in the choice of the symbols; we understand them as equivalence classes of tuples inducing the same operator modulo $\mathcal{B}^{-\infty,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$.

1.12 Boutet de Monvel's Algebra on a Manifold. Let X be an n-dimensional C^∞ manifold with boundary Y, embedded in an n-dimensional manifold Ω without boundary, all not necessarily compact. Let V_1, V_2 be vector bundles over Ω and W_1, W_2 be vector bundles over Y.

Let $\{\Omega_j\}$ denote a locally finite open covering of Ω and suppose that the coordinate charts map $X \cap \Omega_j$ to $U_j \times \mathbf{R}_+ \subset \mathbf{R}^n_+$ and $Y \cap \Omega_j$ to $U_j \times \{0\}$ for a suitable open set $U_j \subseteq \mathbf{R}^{n-1}$, unless $\Omega_j \cap Y = \emptyset$.

For a smooth function ϕ on Ω write M_ϕ for the multiplication operator with the diagonal matrix $\mathrm{diag}\{\phi, \phi|_Y\}$. We will say that $A \in \mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$, if

$$A(\lambda) : \begin{matrix} C_0^\infty(\overline{X}, V_1) \\ \oplus \\ C_0^\infty(Y, W_1) \end{matrix} \rightarrow \begin{matrix} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, W_2) \end{matrix}, \tag{1}$$

is an operator with the following properties:

(i) For all C_0^∞ functions ϕ, ψ, supported in one and the same coordinate neighborhood Ω_j intersecting the boundary, the operator

$$(M_\phi A(\lambda) M_\psi)_* : \begin{matrix} C_0^\infty(U_j \times \overline{\mathbf{R}}_+, V_1) \\ \oplus \\ C_0^\infty(U_j, W_1) \end{matrix} \rightarrow \begin{matrix} C^\infty(U_j \times \overline{\mathbf{R}}_+, V_2) \\ \oplus \\ C^\infty(U_j, W_2) \end{matrix},$$

induced on $U_j \times \mathbf{R}_+$ by $M_\phi A(\lambda) M_\psi$ and the coordinate maps, is an operator in the class $\mathcal{B}^{\mu,d}(U_j \times \mathbf{R}_+; \mathbf{R}^l)$ of Boutet de Monvel's calculus on $\mathbf{R}^n_+$ in the sense of 1.11.

(ii) If ϕ, ψ are as before, but the coordinate chart does not intersect the boundary, then all entries in the matrix $(M_\phi A(\lambda) M_\psi)_*$ – except for the pseudodifferential part – are regularizing.

(ii) If the supports of the functions $\phi, \psi \in C_0^\infty(\Omega)$ are disjoint, then $(M_\phi A(\lambda) M_\psi)_*$ is an integral operator whose kernel density is C^∞ and a rapidly decreasing function of λ in all semi-norms defining the Fréchet topology of the smooth densities.

In each coordinate patch Ω_j intersecting the boundary we may associate a symbol tuple with A by asking that the operator A_j which is locally induced by A and the coordinate maps has a symbol tuple $a_j = \{p_j, g_j, k_j, t_j, s_j\}$ as in 1.8 and 1.11. In an interior chart, only the pseudodifferential part in the matrix for A is non-regularizing; it has a symbol p_j in the sense of an equivalence class of symbols. We shall call the tuple $\{a_j\}$ a symbol for A.

1.4 Sobolev Spaces Based on the Mellin Transform

1.13 Parameter-Dependent Order Reductions on Ω. For $\mu \in \mathbf{R}$ there is a parameter-elliptic pseudodifferential operator $\Lambda^\mu \in \operatorname{op} S^\mu(\Omega, \mathbf{R}^n; \mathbf{R})$, depending on the parameter $\tau \in \mathbf{R}$ such that

$$\Lambda^\mu(\tau) : H^s(\Omega, V) \to H^{s-\mu}(\Omega, V)$$

is an isomorphism for all τ. *Parameter-ellipticity* simply means that there is a symbol $q \in S^{-\mu}(\Omega, \mathbf{R}^n; \mathbf{R})$ such that $\lambda^\mu q - 1$ and $q\lambda^\mu - 1$ both are elements of $S^{-1}(\Omega, \mathbf{R}^n; \mathbf{R})$.

In order to construct such an operator one can e.g. start with symbols of the form $\langle \xi, (\tau, C) \rangle^\mu \in S^\mu(\mathbf{R}^n, \mathbf{R}^n_\xi; \mathbf{R}_\tau)$ with a large constant $C > 0$ and patch them together to an operator on the manifold Ω with the help of a partition of unity and cut-off functions.

Alternatively, one can choose a Hermitean connection on V and consider the operator $(C + |\tau|^2 - \Delta)^{\frac{\mu}{2}}$, where Δ denotes the connection Laplacian and C is a large positive constant.

1.14 Weighted Mellin-Sobolev Spaces. (a) Let $\{\Lambda^\mu : \mu \in \mathbf{R}\}$ be a family of parameter-dependent pseudodifferential operators as in 1.13. For $s, \gamma \in \mathbf{R}$, the space $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$ is the closure of $C_0^\infty(\Omega^\wedge)$ in the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\Omega^\wedge)} = \left\{ \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|\Lambda^s(\operatorname{Im} z) Mu(z)\|^2_{L^2(\Omega)}\, |dz| \right\}^{\frac{1}{2}}. \tag{1}$$

Recall that n is the dimension of X and Ω and that $\Gamma_\beta = \{z \in \mathbf{C} : \operatorname{Re} z = \beta\}$.

(b) We let $\mathcal{H}^{s,\gamma}(X^\wedge) = \{\mathrm{r}^+ f : f \in \mathcal{H}^{s,\gamma}(\Omega^\wedge)\}$. The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ carries the quotient norm:

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \inf\{\|f\|_{\mathcal{H}^{s,\gamma}(\Omega^\wedge)} : f \in \mathcal{H}^{s,\gamma}(\Omega^\wedge), \mathrm{r}^+ f = u\}.$$

(c) For $s = l \in \mathbf{N}$ we obtain the alternative description

$$u \in \mathcal{H}^{l,\gamma}(\Omega^\wedge) \quad \text{iff} \quad t^{\frac{n}{2}-\gamma}(t\partial_t)^k Du(x,t) \in L^2(\Omega^\wedge)$$

for all $k \leq l$ and all differential operators D of order $\leq l-k$ on Ω, cf. [16, Section 2.1.1, Proposition 2].

(d) The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ is independent of the particular choice of the order-reducing family.

(e) $\mathcal{H}^{s,\gamma}(X^\wedge) \subseteq H^s_{loc}(X^\wedge)$; $\mathcal{H}^{s,\gamma}(X^\wedge) = t^\gamma \mathcal{H}^{s,0}(X^\wedge)$; $\mathcal{H}^{0,0}(X^\wedge) = t^{-n/2}L^2(X^\wedge)$.

(f) If ϕ is the restriction to $X^\wedge$ of a function in $C_0^\infty(\Omega \times \mathbf{R})$, then the operator M_ϕ of multiplication by ϕ

$$M_\phi : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge)$$

is bounded for all $s, \gamma \in \mathbf{R}$, and the mapping $\phi \mapsto M_\phi$ is continuous in the corresponding topology.

Notice that (d) is a simple consequence of the fact that if $\{\Lambda^\mu : \mu \in \mathbf{R}\}$ and $\{\tilde{\Lambda}^\mu : \mu \in \mathbf{R}\}$ are two order-reducing families, then for each μ, the operator $\Lambda^\mu \tilde{\Lambda}^{-\mu}$ is parameter-elliptic of order zero. (f) is immediate from (c) and interpolation.

2 Mellin Quantization

2.1 Mellin Symbols

In Section 4 of [14] we considered Mellin symbols with asymptotics; they are meromorphic functions on $\mathbf{C}$ with values in Boutet de Monvel's algebra. For the definition of the Mellin operator $\mathrm{op}_M^\gamma a$ associated with the Mellin symbol a, we only need to know a on the line $\Gamma_{\frac{1}{2}-\gamma}$, and we certainly do not need analyticity. We can extend the calculus to larger classes of Mellin symbols by considering the case where the symbols additionally depend on the space variables t and t' – comparable to studying pseudodifferential 'double' symbols $p(x, y, \xi)$ after having treated Fourier multipliers $p(\xi)$.

2.1 Mellin Transforms. For $\beta \in \mathbf{R}$, Γ_β denotes the vertical line $\{z \in \mathbf{C} : \mathrm{Re}\, z = \beta\}$. We recall that the classical Mellin transform Mu of a complex-valued $C_0^\infty(\mathbf{R}_+)$-function u is given by

$$(Mu)(z) = \int_0^\infty t^{z-1} u(t)\, dt. \tag{1}$$

M extends to an isomorphism $M : L^2(\mathbf{R}_+) \to L^2(\Gamma_{\frac{1}{2}})$. Of course, (1) also makes sense for functions with values in a Fréchet space E. The fact that $Mu|_{\Gamma_{\frac{1}{2}-\gamma}}(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma)$ motivates the following definition of the *weighted Mellin transform* M_γ:

$$M_\gamma u(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma), \quad u \in C_0^\infty(\mathbf{R}_+, E).$$

2.2 Notation. In the following let $\mu \in \mathbf{Z}$ and $d \in \mathbf{N}$ be fixed. Given $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{\frac{1}{2}-\gamma}))$ we shall write $f = f(t, t', z)$, where z indicates the variable in $\Gamma_{\frac{1}{2}-\gamma}$. For t, t', z fixed, $f(t, t', z)$ is a boundary value problem in Boutet de Monvel's calculus, so it acts on sections of vector bundles over $\overline{X}$ and Y. In order to fix the notation, assume that

$$f(t,t',z) : \begin{array}{c} C^\infty(\overline{X}, V_1) \\ \oplus \\ C^\infty(Y, W_1) \end{array} \to \begin{array}{c} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, W_2) \end{array} \tag{1}$$

with smooth vector bundles V_1, V_2, over $\overline{X}$ and W_1, W_2, over Y.

2.3 Definition. Let $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{\frac{1}{2}-\gamma}))$. For $u \in C_0^\infty(\overline{X}^\wedge, V_1) \oplus C_0^\infty(Y^\wedge, W_1) = C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1))$ let

$$[\mathrm{op}_M^\gamma(f)u](t) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}-\gamma}} \int_0^\infty (t/t')^{-z} f(t,t',z) u(t') \frac{dt'}{t'} dz \quad . \tag{1}$$

The right hand side of (1) is to be understood as an iterated integral. If f is independent of t', or equivalently $f \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{\frac{1}{2}-\gamma}))$, then (1) reduces to

$$[\mathrm{op}_M^\gamma(f)u](t) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}-\gamma}} t^{-z} f(t,z)[M_\gamma u](z) dz. \tag{2}$$

We did not specify the variable x in (1) or (2), understanding that for fixed $t', u(t') = u(\cdot, t')$ is in $C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1)$ and that $f(t,t',z)$ acts as an operator in Boutet de Monvel's calculus with respect to the x-variables.

Like pseudodifferential double symbols, Mellin double symbols are not uniquely determined. It is immediate from integration by parts in (1) that

$$\mathrm{op}_M^{\frac{1}{2}}[\ln^k(t/t') f(t,t',z)] = \mathrm{op}_M^{\frac{1}{2}}[\partial_z^k f(t,t',z)]. \tag{3}$$

For $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{\frac{1}{2}-\gamma}))$ or $f \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{\frac{1}{2}-\gamma}))$ we will have a continuous map

$$\mathrm{op}_M^\gamma f : \begin{array}{c} C_0^\infty(\overline{X}^\wedge, V_1) \\ \oplus \\ C_0^\infty(Y^\wedge, W_1) \end{array} \to \begin{array}{c} C^\infty(\overline{X}^\wedge, V_2) \\ \oplus \\ C^\infty(Y^\wedge, W_2) \end{array} . \tag{4}$$

Smoothness of f up to zero yields continuity of $\mathrm{op}_M^\gamma f$ on the weighted Mellin-Sobolev spaces, cf. Theorem 2.4; the preceding relation (3), however, shows that smoothness is not necessary.

2.4 Theorem. *Let $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{\frac{1}{2}-\gamma})), s > d-\frac{1}{2}$. Given $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$, there is a bounded extension*

$$\omega_1[\mathrm{op}_M^\gamma f]\omega_2 : \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array} . \tag{1}$$

A proof can be found in [15]. We will also need the following results. They show that, just as in the case of pseudodifferential operators, one has asymptotic summation of symbols and the possibility of switching from operators with 'double' symbols to those with 'simple' ones.

2.5 Asymptotic Summation. Let $d \in \mathbf{N}$ be fixed, $\mu_1, \mu_2, \ldots$ a sequence in $\mathbf{Z}$ tending to $-\infty$, $f_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu_j,d}(X;\Gamma_{\frac{1}{2}-\gamma}))$, and $\mu = \max \mu_j$. Then there is an

$$f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{\frac{1}{2}-\gamma}))$$

such that for any $N \in \mathbf{N}$ there is a J with

$$f - \sum_{j=1}^{J} f_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-N,d}(X; \Gamma_{\frac{1}{2}-\gamma})). \tag{1}$$

This f is unique modulo $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X; \Gamma_{\frac{1}{2}-\gamma}))$. We shall write $f \sim \sum_{j=0}^\infty f_j$. The same result is true with $\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+$ replaced by $\overline{\mathbf{R}}_+, \mathbf{R}_+ \times \mathbf{R}_+$, or $\mathbf{R}_+$.

2.6 Theorem. *For $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, \mathcal{B}^{\mu,d}(X; \Gamma_0))$ there is a $g \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X; \Gamma_0))$ such that*

$$\mathrm{op}_M^{\frac{1}{2}} f(t, t', z) - \mathrm{op}_M^{\frac{1}{2}} g(t, z) \in \mathcal{B}^{-\infty,d}(X^\wedge) \ . \tag{1}$$

In particular, relation (1) *holds for any symbol g with the asymptotic expansion*

$$g(t, z) \sim \sum_{j=0}^{\infty} \frac{1}{j!} (-t'\partial_{t'})^j \partial_z^j f(t, t', z)|_{t'=t} \ . \tag{2}$$

2.2 Pseudodifferential Action and Mellin Quantization

2.7 Pushforward of Pseudodifferential Operators. Let U, V be open sets in $\mathbf{R}^n$, $\chi : U \to V$ a diffeomorphism. Moreover, let E, F be Banach spaces with group action. Given an operator

$$P : C_0^\infty(U, E) \to C^\infty(U, F) \ ,$$

the pushforward $\chi_* P : C_0^\infty(V, E) \to C^\infty(V, F)$ is defined by

$$(\chi_* P) f(x) = [P(f \circ \chi)][\chi^{-1}(x)] \ .$$

If $P = \mathrm{op}\, p$ for some $p \in S^\mu(U, \mathbf{R}^n; E, F)$ then there is a symbol $q \in S^\mu(V, \mathbf{R}^n; E, F)$ with $\mathrm{op}\, q = \chi_* P$ modulo regularizing operators, and q is unique up to symbols in $S^{-\infty}(V, \mathbf{R}^n; E, F)$. In this sense χ defines a pushforward also on the symbol level:

$$\chi_* : S^\mu(U, \mathbf{R}^n; E, F)/S^{-\infty}(U, \mathbf{R}^n; E, F) \to S^\mu(V, \mathbf{R}^n; E, F)/S^{-\infty}(V, \mathbf{R}^n; E, F) \ .$$

The mapping is an isomorphism; the inverse is induced by the pushforward via χ^{-1}. The same statements are true for symbols with the transmission property.

One way of proving this is to first convert the symbol p to a 'double' symbol $p_1(y, y', \eta)$ by multiplying p with a cut-off function $\phi = \phi(y, y')$ near the diagonal $\{y = y'\}$; $\mathrm{op}\, p$ and $\mathrm{op}\, p_1$ only differ by a regularizing operator. Then one can compute a 'double' symbol $q_1 \in S^\mu(V \times V, \mathbf{R}^n; E, F)$ with $\chi_* \mathrm{op}\, p_1 = \mathrm{op}\, q_1$ and finally switch to a y'-independent symbol q with $\mathrm{op}\, q_1 \equiv \mathrm{op}\, q$ modulo regularizing operators.

In what follows it will often be possible to find a 'double' symbol q_1 with $\chi_* \mathrm{op}\, p = \mathrm{op}\, q_1$ by a straightforward substitution in oscillatory integrals. We will then also write $q_1 = \chi_* p$.

2.8 Corollary. Let $\chi : U \to V$ be a diffeomorphism of open sets in $\mathbf{R}$, and let $a \in C^\infty(U, \mathcal{B}^{\mu,d}(X; \mathbf{R}))$ induce a pseudodifferential action by

$$\mathrm{op}\, a(u)(y) = \frac{1}{2\pi} \int \int_U e^{i(y-y')\eta} a(y, \eta) u(y') dy' d\eta \tag{1}$$

for $u \in C_0^\infty(U, C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1))$. For the pushforward $\chi_* \mathrm{op}\, a$ we then have

$$\chi_* \mathrm{op}\, a = \mathrm{op}\, b + G \ , \tag{2}$$

where

(i) the symbol b belongs to $C^\infty(V, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$. It is determined via the symbol pushforward of the various local symbols for a. In this sense we shall use the notation $b = \chi_* a$.

(ii) The operator G belongs to $\mathcal{B}^{-\infty,d}(X^\wedge)$. In other words, we can write

$$G = \sum_{j=0}^{\infty} G_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix} ; \tag{3}$$

here ∂_r is the normal derivative on X, and each G_j is a matrix of integral operators with kernel functions which are smooth up to the boundary of X.

Proof. We have $C^\infty(U, \mathcal{B}^{\mu,d}(X;\mathbf{R})) = C^\infty(U)\hat{\otimes}_\pi \mathcal{B}^{\mu,d}(X;\mathbf{R})$. Since convergence of the symbols implies convergence of the associated operators, it is sufficient to assume that $a(y,\eta) = \psi(y)A(\eta)$ with $\psi \in C^\infty(U)$ and $A \in \mathcal{B}^{\mu,d}(X;\mathbf{R})$. The assertion is certainly true for regularizing A: In this case, op a already has the form (3); hence the pushforward is of the same type and (2) holds with $b = 0$, for ∂_r is not affected. We can therefore localize with respect to a coordinate neigborhood Ω_j for Ω and assume that A is given locally by a quintuple of parameter-dependent symbols in Boutet de Monvel's calculus, $(p_j, g_j, k_j, t_j, s_j)$, where $p_j = p_j(x,\xi,\eta) \in S^\mu_{tr}(X_j, \mathbf{R}^n_\xi \times \mathbf{R}_\eta)$, $X_j = \Omega_j \cap \overline{X}$, is a pseudodifferential symbol with the transmission property, g is a parameter–dependent singular Green symbol, etc., cf. 1.12. We then have to show that their pushforward is preserved.

In order to see this, let us focus on p_j; the arguments for the other symbols are similar. We have

$$S^\mu_{tr}(U \times X_j, \mathbf{R}^n \times \mathbf{R}) = C^\infty(U)\hat{\otimes}_\pi S^\mu_{tr}(X_j, \mathbf{R}^n \times \mathbf{R}); \tag{4}$$

thus $\psi(y)p(x,\xi,\eta) \in S^\mu_{tr}(U \times X_j, \mathbf{R}^n \times \mathbf{R})$. We know that $S^\mu_{tr}(U \times X_j, \mathbf{R}^n \times \mathbf{R})$ is invariant under coordinate transforms, therefore the pushforward $\chi_*[\psi(y)p(x,\xi,\eta)]$ belongs to $S^\mu_{tr}(V \times X_j, \mathbf{R}^n \times \mathbf{R})$ modulo $S^{-\infty}(V \times X_j, \mathbf{R}^n \times \mathbf{R})$. Employing now (5) with with U replaced by V plus the fact that $C^\infty(V, \mathcal{F}) = C^\infty(V)\hat{\otimes}_\pi \mathcal{F}$ for every Fréchet space $\mathcal{F}$, we see that $\chi_*[\psi(y)p(x,\xi,\eta)] \in C^\infty(V, S^\mu_{tr}(X_j, \mathbf{R}^n \times \mathbf{R}))$ may be considered the pseudodifferential part (with transmission property) of a parameter-dependent symbol tuple for an operator in $C^\infty(V, \mathcal{B}^{\mu,d}(X_j;\mathbf{R}))$. Applying the same argument for the four other components g_j, k_j, t_j, and s_j we obtain the symbol $b \in C^\infty(V, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$. $\lhd$

2.9 Pseudodifferential and Mellin Symbols. Given $f \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ let

$$b(y,\eta) = f(e^y, -i\eta) \quad , \quad y, \eta \in \mathbf{R} \ . \tag{1}$$

Denoting by exp the diffeomorphism $y \mapsto e^y$ from $\mathbf{R} \to \mathbf{R}_+$ we have

$$\mathrm{op}_M^{\frac{1}{2}} f = \exp_* \mathrm{op}\, b \ . \tag{2}$$

In more detail: For $u \in C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X}, V) \oplus C^\infty(Y, V_2))$ let $u^*(y) = u(e^y)$; then $[\mathrm{op}_M^{\frac{1}{2}} f(u)](e^y) = [\mathrm{op}\, b(u^*)](y)$. This is a simple consequence of the identity

$$\frac{1}{2\pi i} \int_{\Gamma_0} \int_0^\infty \left(\frac{e^y}{t'}\right)^{-z} f(e^y, z) u(t') \frac{dt'}{t'} dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(y-y')\eta} f(e^y, -i\eta) u^*(y') dy' d\eta \ .$$

Equation (1) implies that $b \in C^\infty(\mathbf{R}, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$. According to Corollary 2.8, we will have $\exp_* \mathrm{op}\, b \equiv \mathrm{op}\, a$ modulo $\mathcal{B}^{-\infty,d}(X^\wedge)$. Hence,

$$\mathrm{op}_M^{\frac{1}{2}} f \equiv \mathrm{op}\, a \quad \text{modulo} \quad \mathcal{B}^{-\infty,d}(X^\wedge) \quad .$$

We shall now analyze the relationship between f and a.

2.10 Definition and Remark. For $\mu \in \mathbf{Z}$ and $d \in \mathbf{N}$ let

$$\mathring{M}\mathcal{B}^{\mu,d}(X^\wedge) = \{\mathrm{op}_M^{\frac{1}{2}} f + G \ : \ f \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0)),\ G \in \mathcal{B}^{-\infty,d}(X^\wedge)\} \ .$$

For $f \in C^\infty(\mathbf{R}_+, \mathcal{B}^{-\infty,0}(X;\Gamma_0))$, $\mathrm{op}_M^{\frac{1}{2}} f$ is an integral operator with smooth kernel on $\overline{X}^\wedge$. Hence $\mathring{M}\mathcal{B}^{-\infty,d}(X^\wedge) := \bigcap_\mu \mathring{M}\mathcal{B}^{\mu,d}(X^\wedge) = \mathcal{B}^{-\infty,d}(X^\wedge)$.

The following lemma may be considered a 'coarse' quantization result. It shows that pseudodifferential and Mellin symbols induce the same operators modulo $\mathcal{B}^{-\infty,d}(X^\wedge)$ as long as we consider symbol classes with arbitrary behavior near $t = 0$.

2.11 Lemma.

$$\mathring{M}\mathcal{B}^{\mu,d}(X^\wedge)/\mathring{M}\mathcal{B}^{-\infty,d}(X^\wedge) \cong \mathcal{B}^{\mu,d}(X^\wedge)/\mathcal{B}^{-\infty,d}(X^\wedge) \ .$$

The isomorphism is given by $f \mapsto \exp_* b$ with $b(y,\eta) = f(e^y, -i\eta)$; the inverse by $a \mapsto f$ with $f(s,z) = [\ln_* a](\ln s, iz)$.

Proof. By 2.8 and 2.9 the mapping $f \mapsto \exp_* b$, where $b = f(e^y, -i\eta)$, maps the left hand side to the right hand side injectively. A direct computation then yields the above inverse. ◁

For what follows it will be interesting to know more precisely what the pushforward by exp looks like. We start with a formal calculation.

2.12 Lemma. *Let $p \in S^\mu(\mathbf{R}_+, \mathbf{R})$. Then $\exp_* \mathrm{op}\, p$ is the pseudodifferential operator with the 'double' symbol*

$$(\exp_* p)(t, t', \tau) = p(\ln t, M(t,t')^{-1}\tau)\frac{1}{t'}M(t,t')^{-1} \quad . \tag{1}$$

Here $M(t,t') = \frac{\ln t - \ln t'}{t - t'}$ is C^∞ and strictly positive on $\mathbf{R}_+ \times \mathbf{R}_+$.

Proof. For $u \in C_0^\infty(\mathbf{R}_+), t' = e^{y'}$, we have

$$\begin{aligned}
[\mathrm{op}\, p(u \circ \exp)](\ln t) &= \frac{1}{2\pi}\int\int_0^\infty e^{i(\ln t - y')\eta} p(\ln t, \eta) u(e^{y'}) dy' d\eta \\
&= \frac{1}{2\pi}\int\int e^{i(\ln t - \ln t')\eta} p(\ln t, \eta) u(t') \frac{dt'}{t'} d\eta \\
&= \frac{1}{2\pi}\int\int e^{i(t-t')M(t,t')\eta} p(\ln t, \eta) u(t') \frac{dt'}{t'} d\eta \\
&= \frac{1}{2\pi}\int\int e^{i(t-t')\tau} p(\ln t, M(t,t')^{-1}\tau) u(t') \frac{1}{t'} M(t,t')^{-1} dt' d\tau.
\end{aligned}$$

This gives (1). The function $M(t,t')$ is smooth and ≥ 0, for ln is monotonely increasing. Moreover, M has no zero since, for $t = t'$, we have $M(t,t) = \frac{1}{t} > 0$. ◁

2.13 Lemma. (a) $\partial_{t'}^k M(t,t')\mid_{t'=t}= c_k t^{-k-1}$ *for suitable* $c_k \in \mathbf{R}$, $k = 0,1,\ldots$. *In particular,* $(t'\partial_{t'})^k[t'M(t,t')]|_{t'=t}$ *is smooth up to* $t = 0$.
(b) $t^{k-1}\partial_{t'}^k[M(t,t')^{-1}]|_{t'=t}$ *is smooth up to* $t = 0, k = 0,1,\ldots$.

Proof. (a) Let $u, v \in \mathbf{R}_+$. We have for $1 + x = \frac{u}{v}$, $|x| < 1$

$$\ln u - \ln v = \ln(\frac{u}{v}) = \ln(1+x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} x^j = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \frac{(u-v)^j}{v^j} \quad , \tag{1}$$

hence

$$M(u,v) = \frac{\ln u - \ln v}{u - v} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \frac{(u-v)^k}{v^{k+1}}. \tag{2}$$

Therefore

$$\partial_u^k M(u,v)\mid_{u=v}= k!\frac{(-1)^k}{k+1} v^{-k-1} \quad . \tag{3}$$

This proves the first statement. Now it is easily checked that, for $k \geq 1$, $(t\partial_t)^k$ is a linear combination of terms $t^j\partial_t^j$, $j = 1,\ldots,k$, so we obtain the second statement, too.

(b) By induction, $\partial_{t'}^k[M(t,t')^{-1}]$ is a linear combination of terms of the form

$$M(t,t')^{-r-1} \prod_{l=1}^{r} \partial_{t'}^{j_l} M(t,t'),$$

where $r \leq k$ and $\sum_{l=1}^{r} j_l = k$. This implies that $\partial_{t'}^k[M(t,t')^{-1}]\mid_{t'=t}$ is a linear combination of terms $t^{r+1}t^{-r-k}$, $0 \leq r \leq k$. $\triangleleft$

2.14 Definition. Let $\mu \in \mathbf{Z}$, $d \in \mathbf{N}$. By $C^\infty(\overline{\mathbf{R}}_+, \widetilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$ we denote the set of all $a \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$ for which there is a $b \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$ such that

$$a(t,\tau) = b(t, t\tau) \quad .$$

2.15 Theorem. *For* $f \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ *there is an* $a \in C^\infty(\overline{\mathbf{R}}_+, \widetilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$ *with*

$$\operatorname{op} a \equiv \operatorname{op}_M^{\frac{1}{2}} f \quad \textit{modulo} \;\; \mathcal{B}^{-\infty,d}(X^\wedge) \quad . \tag{1}$$

Proof. We know from 2.9 that $\operatorname{op}_M^{\frac{1}{2}} f \equiv \operatorname{op}(\exp_* p)$ where $p(y,\eta) = f(e^y, -i\eta)$, and, according to Lemma 2.12, $c_1(t,t',\tau) = [\exp_* p](t,t',\tau) = p(\ln t, M(t,t')^{-1}\tau)\frac{1}{t'}M(t,t')^{-1} = f(t, -iM(t,t')^{-1}\tau)\frac{1}{t'}M(t,t')^{-1}$ with the notation of 2.7. Let us convert the 'double' symbol c_1 to a symbol $c \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$ independent of t':

$$c(t,\tau) \sim \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{t'}^k D_\tau^k c_1(t,t',\tau)\mid_{t'=t} \quad . \tag{2}$$

Now

$$\partial_{t'}^k D_\tau^k c_1(t,t',\tau) = \partial_{t'}^k\{(-i)^k(\partial_z^k f)(t, -iM(t,t')^{-1}\tau)\frac{1}{t'}M(t,t')^{-k-1}\} \quad . \tag{3}$$

By induction this is a linear combination of terms of the form

$$(\partial_z^{k+j} f)(t, -iM(t,t')^{-1}\tau)\, \tau^j\, g_{kj}(t,t') \quad , \quad j = 0,\ldots,k, \tag{4}$$

where $g_{kj}(t,t')$ is a linear combination of terms of the form

$$(t')^{-1-l_0}\prod_{i=1}^{r}\partial_{t'}^{l_i}\{M(t,t')^{-1}\}.$$

Here $r=k+1+j$, and $l_0+\sum_{i=1}^r l_i=k$. Using Lemma 2.13 we conclude that $t^{-j}g_{kj}(t,t)$ is smooth up to $t=0$.

Combining (3) and (4) we see that $\partial_{t'}^k D_\tau^k c_1(t,t',\tau)\mid_{t'=t}$ is a linear combination of terms of the form $(\partial_z^{k+j}f)(t,-it\tau)\,(t\tau)^j s_{kj}(t)$, where s_{kj} is a smooth function on $\overline{\mathbf{R}}_+$. Since $(\partial_z^{k+j}f)(t,-itz)\in C^\infty(\overline{\mathbf{R}}_+,\widetilde{\mathcal{B}}^{\mu-k-j,d}(X;\mathbf{R}))$, we obtain the symbol a by asymptotic summation in $C^\infty(\overline{\mathbf{R}}_+,\widetilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$. Note that there is asymptotic summation in this class: Given a sequence $\{a_j\}$ with $a_j\in C^\infty(\overline{\mathbf{R}}_+,\widetilde{\mathcal{B}}^{\mu-j,d}(X;\mathbf{R}))$ and $a_j(t,\tau)=b_j(t,t\tau)$ for $b_j\in C^\infty(\overline{\mathbf{R}}_+,\mathcal{B}^{\mu-j,d}(X;\mathbf{R}))$ choose $b\sim\sum b_j$ and let $a(t,\tau)=b(t,t\tau)$. We will then have $a-\sum_{j=0}^N a_j\in C^\infty(\overline{\mathbf{R}}_+,\widetilde{\mathcal{B}}^{\mu-N,d}(X;\mathbf{R}))\subseteq C^\infty(\mathbf{R}_+,\mathcal{B}^{\mu-N,d}(X;\mathbf{R}))$; hence $\operatorname{op}c-\operatorname{op}a\in\mathcal{B}^{-\infty,d}(X^\wedge)$. ◁

2.16 Theorem. (Mellin Quantization) *Let $a\in C^\infty(\overline{\mathbf{R}}_+,\widetilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$. Then there is an $f\in C^\infty(\overline{\mathbf{R}}_+,\mathcal{B}^{\mu,d}(X;\Gamma_0))$ such that*

$$\operatorname{op}_M^{\frac12}f\equiv\operatorname{op}a\quad \textit{modulo}\ \ \mathcal{B}^{-\infty,d}(X^\wedge)\ . \tag{1}$$

Proof. We know from Lemma 2.11 that $\operatorname{op}a\equiv\operatorname{op}_M^{\frac12}g$ with

$$g(t,t',z)=[\ln_* a](\ln t,\ln t',iz)\ ; \tag{2}$$

here, we use the 'double' symbol of $[\ln_* a]$ one obtains by straightforward substitution in the oscillatory integral. Given a symbol $p\in S^\mu(\mathbf{R}\times\mathbf{R},\mathbf{R})$ a computation similar to that in 2.12 shows that

$$(\ln_* p)(y,y',\eta)=p(e^y,M(e^y,e^{y'})\eta)e^{y'}M(e^y,e^{y'})$$

with the function $M(t,t')=\frac{\ln t-\ln t'}{t-t'}$ introduced in 2.12. Hence, in our case,

$$g(t,t',i\tau)=a(t,-M(t,t')\tau)\,t'M(t,t')\ . \tag{3}$$

Now we apply Theorem 2.6. We have $\operatorname{op}_M^{\frac12}g\equiv\operatorname{op}_M^{\frac12}f$ modulo $\mathcal{B}^{-\infty,d}(X^\wedge)$ whenever $f\in C^\infty(\mathbf{R}_+,\mathcal{B}^{\mu,d}(X;\Gamma_0))$ has the asymptotic expansion

$$f(t,z)\sim\sum_{k=0}^\infty\frac{1}{k!}(-t'\partial_{t'})^k\partial_z^k g(t,t',z)\mid_{t'=t}\ . \tag{4}$$

By assumption, the symbol $b(t,\tau)=a(t,t^{-1}\tau)$ is an element of $C^\infty(\overline{\mathbf{R}}_+,\mathcal{B}^{\mu,d}(X;\mathbf{R}))$. Thus $t^{-k}(\partial_\tau^k a)(t,t^{-1}\tau)=\partial_\tau^k b(t,\tau)\in C^\infty(\overline{\mathbf{R}}_+,\mathcal{B}^{\mu,d}(X;\mathbf{R}))$. The function $(t'\partial_{t'})^j(t'M(t,t'))\mid_{t'=t}$ is smooth up to $t=0$ for $j=0,1,\ldots$, by Lemma 2.13. So all the terms on the right hand side of (4) are smooth up to $t=0$ and the asymptotic summation can be carried out in $C^\infty(\overline{\mathbf{R}}_+,\mathcal{B}^{\mu,d}(X;\Gamma_0))$. ◁

2.3 Mellin Quantization for Arbitrary Weights

In the previous section we studied the question how to associate to a totally characteristic pseudodifferential symbol $a \in C^\infty(\overline{\mathbf{R}}_+, \widetilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$ a Mellin symbol $f_{1/2} \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ with $\operatorname{op} a = \operatorname{op}_M^{\frac{1}{2}} f_{1/2}$ modulo $\mathcal{B}^{-\infty,d}(X^\wedge)$. Given an arbitrary weight $\gamma \in \mathbf{R}$ this result allows us to easily find a Mellin symbol $f_\gamma \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{\frac{1}{2}-\gamma}))$ such that $\operatorname{op} a = \operatorname{op}_M^\gamma f_\gamma$ modulo $\mathcal{B}^{-\infty,d}(X^\wedge)$:

2.17 Theorem. *For every $a \in C^\infty(\overline{\mathbf{R}}_+, \widetilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$ and every $\gamma \in \mathbf{R}$ there is an $f_\gamma \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{\frac{1}{2}-\gamma}))$ such that*

$$\operatorname{op}_M^\gamma f_\gamma \equiv \operatorname{op} a \quad \textit{modulo } \mathcal{B}^{-\infty,d}(X^\wedge) \ . \tag{1}$$

Proof. The Mellin symbol f_γ can be computed in terms of the function $f = f_{1/2}$ in Theorem 2.16. The definition of $\operatorname{op}_M^\gamma$ shows that

$$\operatorname{op} a \equiv \operatorname{op}_M^{\frac{1}{2}} f_{1/2} = \operatorname{op}_M^\gamma g_\gamma,$$

where $g_\gamma(t,t',z) = (t/t')^{1/2-\gamma} f_{1/2}(t, z - \frac{1}{2} + \gamma)$. We convert g_γ to a t'-independent symbol f_γ with

$$\begin{aligned} f_\gamma(t,z) &\sim \sum_{k=0}^\infty \frac{1}{k!}(-t'\partial_{t'})^k \partial_z^k g_\gamma(t,t',z)|_{t'=t} \\ &\sim \sum_{k=0}^\infty \frac{1}{k!}(-t'\partial_{t'})^k (\frac{t}{t'})^{\frac{1}{2}-\gamma}|_{t'=t}\ \partial_z^k f_{1/2}(t, z-\frac{1}{2}+\gamma) \\ &\sim \sum_{k=0}^\infty \frac{1}{k!}(\frac{1}{2}-\gamma)^k \partial_z^k f_{1/2}(t, z - \frac{1}{2}+\gamma), \end{aligned} \tag{2}$$

where we used that $(-t'\partial_{t'})^k (t/t')^{1/2-\gamma}|_{t'=t} = (x\partial_x)^k x^{1/2-\gamma}|_{x=1} = (1/2-\gamma)^k$. Since $f_{1/2}$ is smooth up to $t=0$, the asymptotic summation can be carried out in $C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{\frac{1}{2}-\gamma}))$, and we obtain the assertion. ◁

2.18 Remark. In [14] we defined spaces $M_P^{\mu,d}(X)$ of (t,t')-independent Mellin symbols of order μ and type d. They are meromorphic functions on $\mathbf{C}$, their only singularities are poles described in terms of the 'asymptotic type' P. We can then consider the classes $C^\infty(\overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$ and the associated Mellin operators according to 2.3, cf. [15]. If the singularity set P is empty we shall write $h \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$. Then $h(t,\cdot)$ is an entire function, and Cauchy's theorem implies that $\operatorname{op}_M^\gamma h = \operatorname{op}_M^{\gamma'} h$ for all $\gamma, \gamma' \in \mathbf{R}$. We now let $f_{1/2} = h|_{\Gamma_0}$. According to Theorem 2.17 $\operatorname{op}_M^\gamma f_\gamma \equiv \operatorname{op}_M^\gamma (h|_{\Gamma_{\frac{1}{2}-\gamma}})$ modulo $\mathcal{B}^{-\infty,d}(X^\wedge)$. Therefore, $f_\gamma - h|_{\Gamma_{\frac{1}{2}-\gamma}} \in C^\infty(\mathbf{R}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_{\frac{1}{2}-\gamma})$. This can be viewed as a slightly different convergence result for the Taylor series on the right hand side of 2.17(2).

2.19 Remark. The present Mellin quantization which ensures a control of the operators in Boutet de Monvel's algebra up to the conical singularity will play a crucial role in [15] as part of a cone algebra without asymptotics. This will then be an important step towards the construction of the corresponding cone algebra *with* asymptotics that will also be established in [15].

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