SINGULARITIES WITH CRITICAL LOCUS AN COMPLETE INTERSECTION AND TRANSVERSAL TYPE A_1

MAMUKA SHUBLADZE

Abstract. In this paper we study germs of holomorphic functions $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ with the following two properties:

(i) the critical locus Σ of f is an isolated complete intersection singularity (icis);

(ii) the transversal singularity of f in points of $\Sigma\backslash\{0\}$ is of type A_1 we first compute the homology of the Milnor fibre and then show that the homotopy type of the Milnor fibre F of f is a bouquet of spheres.

AMS Subjclass: 32S05, 14B05, 32S25, 32S50, 32S55, 57R45.
Keywords: Nonisolated singularity, Milnor fibre.

1. Introduction

Let O be the ring of holomorphic germs $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$. Let $I \subset O$ be a reduced ideal defining an icis Σ of arbitrary dimension k. As usual $J(f)$ denote the jacobian ideal of f, namely:

$$J(f) = \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}\Big).$$

We consider, as in [Pe-1, Pe-2], the group D_I of local analytic isomorphisms $\varphi : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ such that $\varphi^*(I) = I$.

Let $f \in O$ be a germ whose critical set contains Σ. Then by [Pe-1, Pe-2], $f \in I^2$. The group D_I acts an I^2, and the extended codimension of the orbit of f with respect to this action is

$$c_e(f) = \dim \frac{I^2}{I^2 \cap J(f)}$$

we shall focus our attention on germs $f \in I^2$ with $c_e(f) < \infty$. We are interested in the topology of Milnor fibre of f. We known if k dimension of singular locus Σ is 1 then Milnor fibre F is homotopy equivalent of bouquet of some dimensional sphere [Si-1, Si-2].

If $k = m-1$ i.e. codim $\Sigma = 1$, then again F is homotopy equivalent of bouquet of some dimensional sphere [Sh-1, Sh-2, Ne-1]. If $k = 2$ bouquet theorem also are valid the Milnor fibre F is homotopy equivalent bouquet of sphere [Za, Ne-2].

We consider case when $k \geq 3$ and give the properties in which case we can prove the

Theorem. *The Milnor fibre F of $f = (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ is homotopy equivalent of bouquet of spheres $F \simeq S^n \vee S^{m-1} \vee S^{m-1} \vee \cdots \vee S^{m-1}$, where $n = m - k$.*

Acknowledgements. This work was done while the author was a guest at the Max-Planck-Institut für Mathematik. He thanks the institute and its staff members for the support and the good mathematical atmosphere. The author would like to express his gratitude to Prof. D. Siersma for helpful discussions and suggestions.

2. Non-Isolated Singularities with Transversal Type A_1

Let as above $I \subset O$ be a reduced ideal defining an icis (isolated complete intersection singularity) Σ of dimension k and suppose that $I = (g_1, \ldots, g_n)$ with $n = m-k$. We shall assume that $n \geq 2$ and $k \geq 3$; the cases $k = 1$, $k = 2$ and $n = 1$ are situated in [Si-2], [Ne-2, Za] and respectively [Sh-1] and [Ne-1]. Let $f \in O$ be a germ whose critical set contains Σ. It follows that $f \in I^2$ and we have decomposition

$$f = \sum_{i,j=1}^{n} h_{ij}\, g_i\, g_j$$

with $h_{ij} = h_{ji}$ [Pe-1, Pe-2]. Moreover, the class of h_{ij} in O/I is uniquely determined by f [Za].

In [Pe-1] and [Pe-2] were introduced $D(k,p)$ singularity. Their local equations, in a suitable coordinate system x_{ij} $(1 \leq i \leq j \leq p)$, $z_1, \ldots, z_q, y_1, \ldots, y_n$, is

$$f(x,y,z) = \sum_{1 \leq i \leq j \leq p} x_{ij}\, y_i\, y_j + \sum_{l=p+1}^{n} y_l^2.$$

Note also the singular locus of a $D(k,p)$ singularity is smooth and of dimension $k = \frac{1}{2}p(p+1) + q$, while $m = k + n$. $D(k,0)$ singularity in [Pe-1, Pe-2] is also called $A(k)$

$A(k) := D(k,0) : \sum\limits_{l=1}^{n} y_l^2.$

We note also:

$D(k,1) : xy_1^2 + \sum\limits_{l=2}^{n} y_l^2.$

Remark 2.1. As in [Sh-3], see also [Za], it is easy to see that following are valid

(1) A singular point $z \in \Sigma$ is a singular point of type $D(k, 0)$ if the matrix $(h_{ij}(z))$ has rank n.

(2) A singular point $z \in \Sigma$ is a singular point of type $D(k, 1)$ if the matrix rank $(h_{ij}(z)) = n - 1$ and $\mathrm{grad}_z(\det(h_{ij}(z)))|_\Sigma \neq 0$.

Let D be defined as in [Za] by $D(z) = \det(h_{ij}(z))_{ij}$ then if $D(0) = 0$ then the ideal $I + D = (g_1, \ldots, g_n, D)$ defines a complete intersection in $(\mathbb{C}^m, 0)$, which depends only of f [Za]. Let us denote by Δ the zero set of $I + (D)$.

The following result is similar to [Si-1, Sh-1] criterion of finite codimen.

Theorem 2.2. *Let* $f \in I^2$, $f = \sum\limits_{ij=1}^{n} h_{ij}\, g_i\, g_j$ *and* I, *and* $I + (D)$ *is isolated complete intersection and* Δ *is an isolated singularity. Then*

(a) *the critical locus of* f *is* Σ *and the germ of* f *in every points of* $\Sigma\backslash\{0\}$ *outside* Δ *is equivalent to a* $D(k, 0)$ *singularity and point an* Δ *is equivalent to a* $D(k, 1)$.

(b) $c_e(f) < +\infty$.

Proof. (a) If $z \in \Delta$ and $z \neq 0$ then $\mathrm{rank}((h_{ij})_{ij}) = n - 1$ since Σ is icis of dimension $k = m - n$. Since $\Delta = \det((h_{ij})_{ij})$ is isolated singularity on Σ so grad $\Delta|_\Sigma \neq 0$ at the point of $\Delta\backslash\{0\}$, which means that f at z is of type $D(k, 1)$ by the remark of 2.2. Let us now $z\overline{\in}\Delta$ and $z \neq 0$. Then we have $\det(h_{ij})_{ij}) \neq 0$ at this point z, so $\mathrm{rank}((h_{ij})_{ij}) = n$ and using Remark 2.2 f at this points z has $D(k, 0)$ singularity.

(b) Let f be some representative of the germ of given singularity. In the domain where it is given we define a sheaf of O modules as follows

$$\mathcal{F}(u) = I^2 / \tau_e(f),$$

where I^2 and $\tau_e(f)$ are considered as modules over the holomorphic functions on u. It is clear that $\mathcal{F}$ is coherent. We will use the fact: $\mathcal{F}$ is concentrated in a point $\Leftrightarrow \dim \Gamma(\mathcal{F}) < \infty$. For $z \in \mathbb{C}^m\backslash\Sigma$, f is regular at z and we have $\dim \mathcal{F}_z = 0$ since $I_z^2 \cong O_z$ and $(\tau_e(f))_z \cong O_z$. If $z \in \Sigma\backslash\{0\}$ then as we proved above f is of type $D(k, p)$, $p \leq 1$ at z and we have $\dim \mathcal{F}_z = 0$ since $c_e(D(k, p)) = 0$. So $\mathcal{F}$ is concentrated at 0, hence $c_e(f) < \infty$. □

3. The Deformation of Nonisolated Singularities

First consider the case when singular locus Σ of $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ is smooth k-dimensional submanifolds. Consider coordinates

$(x_1, \ldots, x_{m-n}, y_1, \ldots, y_n)$ in $\mathbb{C}^m$. Then $f = \sum\limits_{i,j=1}^{n} h_{ij}\, y_i\, y_j$. Let us $\det(h_{ij})_{ij} = D(z)$ and $D(z)|_{\Sigma}$ is isolated singularity at $0 \in \Sigma$.

In case of an ordinary isolated singularity it is useful to consider a generic approximation g of with only ordinary Morse point [Br]. At every Morse point one can study its local Milnor fibration, with Milnor fibre homotopy equivalent to one n-sphere S^n ("the vanishing cycle"). The Milnor fibre of the original f then has the homotopy type of the wedge of those spheres.

We like to mimic the constructions in our case.

Let us Σ is k-dimensional complete intersection defining by the ideal $I = (g_1, \ldots, g_n)$. Then $f = \sum\limits_{i,j=1}^{n} h_{ij}\, g_i\, g_j$. Assume that $D(z) = \det((h_{ij})_{i,j})$ is an isolated singularity and $I + (D)$ is complete intersection.

Let $G : (\mathbb{C}^m \times \mathbb{C}^r, 0) \to (\mathbb{C}^m \times \mathbb{C}^r, 0)$ be a versal deformation of $(\Sigma, 0)$ with $G(z, v) = (G_1(z, v), \ldots, G_n(z, v), v)$ and $G_i(z, 0) = g_i(z)$ [Loo]. Consider the deformation

$$f_s : (\mathbb{C}^m \times S, 0) = \mathbb{C}^m \times \mathbb{C}^r \times \mathbb{C}^n \times \mathbb{C}^{n(n+1)/2} \times \mathbb{C}^{m-n}, 0) \to \mathbb{C}$$

given by

$$\begin{aligned} f_s(z) = f(z, v, u, a, b) = \sum_{i,j=1}^{n} \Big(h_{ij}(z) + a_{ij} + \sum_{t=1}^{m} b_{tj}\, x_t\, \delta_{ij} \Big) \cdot \\ \cdot (G_i(z, v) - u_i)(G_j(z, v) - u_j), \end{aligned}$$

where $a_{ij} = a_{ji}$, and S is the space of parameters (a, b, u, v). In case $k = \dim \Sigma = 1, 2$ or $m - 1$, there exists a dense subset U in S such that for every $s \in U$, the germ of f_s at the points of Σ_s is of type $D(k, 0)$ or $D(k, 1)$. Moreover, the set of points of Σ_s where f_s is of type $D(k, 1)$ is exactly Δ_s and this set is a Milnor fibre of the icis Δ. [Si-2, Sh-1, Za] For an arbitrary k, we know at least two cases when such deformation exists: i) the germ f at any point $z \in \Sigma \backslash \{0\}$ is of type $D(k, 0)$, ii) the matrix $(h_{ij}(0))_{ij}$ has rank $n-1$. From this page assume the existence of such deformation for the arbitrary k. The following are valid [Za-Bo-Ne-2].

Lemma 3.1. *There exist an ε-ball B_ε with center $D \in \mathbb{C}^m$, a proper analytic set $(A, 0) \subset (S, 0)$, and a neighborhood U of $0 \in S$, such that for any $s \in U \backslash A$ has the following:*

(a) $\Sigma_s = \{z \in B_z : G_i(x, v) = u_i,\ i = 1, \ldots, n\}$ *is the Milnor fibre of* Σ.

(b) *The zero set* $D_s(z) = \det(h_{ij}(z) + a_{ij} + \Sigma_t\, b_{tj}\, z_t\, \delta_{ij})$ *intersets* Σ_s *transversally; hence* $\Delta_s = D_s^{-1}(0) \cap \Sigma_s$ *is smooth. In particular* Δ_s *is* (*diffeomorphic to*) *the Milnor fibre of* Δ.

(c) *The singularities of* f_s *in* $B_\varepsilon \backslash \Sigma_s$ *are of type* A_1.

(d) *The germ of* f_s *at any point of* $\Sigma_s \backslash \Delta_s$ *is of type* $D(k, 0)$ *and at any point of* Δ_s *is of type* $D(k, 1)$.

(e) *Fix* ε *sufficiently small and* δ *sufficiently small with respect to* ε. *If* U *is sufficiently small with respect to* ε *and* δ, *then* $f_s^{-1}(t)$ (*as a stratified set*) *intersects* ∂B_ε *transversally for any* $s \in U$ *and* $t \in \Lambda = \{|t| \le \delta\}$. *In particular, the topological type of the smooth fibres of the maps*

$$f_s : X_s = f_s^{-1}(\Lambda) \cap B_\varepsilon \to \Lambda \quad (s \in U)$$

is independent of the parameter $s \in U$. (*In fact, even the vibrations* $f_s : f_s^{-1}(\partial\Lambda) \cap B_\varepsilon \to \partial\Lambda$ *are equivalent to the fibration* $f : f^{-1}(\partial\Lambda) \cap B_\varepsilon \to \partial\Lambda$. *In particular, the corresponding fibres are homotopically equivalent*).

(f) *The spaces* X_s $(s \in U)$ *are contractible.*

4. The Topology of Milnor Fibre

Let f_s be a deformation of f obtained by Lemma 3.1 and let us suppose that the number of A_1 (Morse) points is σ. The critical set of f consists of

(a) A manifold Σ_0 with is the Milnor fibre Σ_s of k-dimensional isolated complete intersection singularity Σ. The local singularities of f on Σ_0 are $D(k, 0)$ and $D(k, 1)$.

(b) $\Sigma_1 = \{b_1\}, \ldots, \Sigma_\sigma = \{b_\sigma\}$, where the local singularity of f is isolated of type A_1.

Define $B_1, B_2, \ldots, B_\sigma$ as disjoint $2m$ dimensional balls in the space $\mathbb{C}^m$ with centered of the points $b_1, \ldots, b_\sigma$ and $D_1, D_2, \ldots, D_\sigma$ a disjoint two dimensional disks at the points $f_s(b_1), \ldots, f_s(b_\sigma)$. Choose them such that $\widetilde{f} : B_i \cap \widetilde{f}^{-1}(D_i) \to D_i$ define a locally trivial Milnor fibration, the following transversality condition holds: $f_s^{-1}(t) \cap \partial B_i$, $\forall t \in D_i$, $i = 1, \ldots, \sigma$.

The situation at the points of $b_1, \ldots, b_\sigma$ is well known we consider the situation along Σ_s.

Firstly we define B^0 a tabular neighborhood

$$B^0 = \Big\{ z \in B_\varepsilon : \sum_{i=1}^{n} \big| G_i(z, v) - u \big|^2 \le \rho \Big\} \quad \text{of} \quad \Sigma_s \cap B_\varepsilon,$$

which is diffeomorphic for sufficiently small ρ to the product $(\Sigma_s \cap B_\varepsilon) \times Q^n$, where Q^n is a closed n-ball in $\mathbb{C}^n$ with center at the origin [Si-1].

Let us denote $X_{t,s} = f_s^{-1}(t) \cap B_\varepsilon$ and $F^0 = B^0 \cap X_{t,s}$ then for the sufficiently small t we have

$$H_{*-1}(X_{s,t}) = H_*(X_s, X_{s,t}) = \begin{cases} H_*(B^0, F^0) & \text{if } \quad * \neq m, \\ H_m(B^0, F^0) \oplus \mathbb{Z}^\sigma & \text{if } \quad * = m, \end{cases}$$

[Si-2].

First compute the homology of the point (B^0, F^0). Following [Si-2, Za] we shall consider in B^0 coordinates $(w_1, \dots, w_n, w_{m-k+1}, \dots, w_m)$ such that $(w_1, \dots, w_n) \in Q^n$ are the functions defined by $w_i(z) = G_i(x, v) - u_i$ and $w_{m-k+1}, \dots, w_m \in \Sigma_s$ (recall that $\dim \Sigma = k$ and $m = n + k$). Then $(w_1, \dots, w_n)$ are holomorphic functions on B^0 and $(w_{m-k+1}, \dots, w_m)$ are real differentiable [Si-2]. Now consider the projection $\pi : (w_{m-k+1}, \dots, w_m) : (B^0, F^0) \to \Sigma_s$. Then similarly [Si-2, Za, Sh-3] we can prove

Lemma 4.1. *If ρ and tubular neighborhoods $U_1 \subset U_2 \subset \Sigma_s$ of $\Delta_s \subseteq \Sigma_s$ are sufficiently small then*

(a) $\pi : (B^0 \backslash \pi^{-1}(U_1), F^0 \backslash \pi^{-1}(U_1)) \to \Sigma_s \backslash U_1$ *is locally trivial fibration with fibre equal to the pair* ($\mathbb{C}^{m-k}$, *Milnor fibre of* $x_1^2 + \dots + x_n^2$),

(b) *the map given by the superposition* $\pi^{-1}(U_2) \to U_2 \to \Delta_s$ *is a fibration of the pair* $(\pi^{-1}(U_2), F^0 \cap \pi^{-1}(U_2))$ *with fibre equal to the pair* ($\mathbb{C}^{n+k}$, *Milnor fibre of* $x_{n+1}x_1^2 + x_2^2 + \dots + x_n^2$).

For a subset $W \subseteq \Sigma_s$ we shall denote by F_W the following set: $F_W = \pi^1(W) \cap F^0$.

The following statements holds

Lemma 4.2. *$H_q(F_{\Sigma_s \backslash U_1}) = 0$ for $q = n-2$ and $q = n$. Moreover $H_{n-1}(F_{\Sigma_s \backslash U_1}) = \mathbb{Z}_2$, $H_{m-1}(F_{\Sigma_s \backslash U_1}) = \mathbb{Z}^{\mu_\Delta + \mu_\Sigma}$, $H_1(F_{\Sigma_s \backslash U_1}) = 0$, $q \geq n-2$.*

Proof. If $n = 1$ this case was studied in [Sh-1, Ne-1]. $n = 2$ in [Ne-2], so $n \geq 3$. We may assume that $n > k + 2$ because of if w is a new variable, then the Milnor fiber F_w of $f(z) + w^2$ is the suspension of the Milnor fibre F of f, in particular $H_*(F) = H_{*+1}(F_w)$.

Consider the fibration $\pi : F_{\Sigma_s \backslash U_1} \to \Sigma_s \backslash U_1$.

The base space $\Sigma_s \backslash U_1$ is homotopy equivalent $\Sigma_s \backslash U_1 \simeq \Sigma_s \underset{\Delta_s}{\sqcup} U_1 \times S^1 \simeq S^1 \vee S^k \vee \dots \vee S^k$ bouquet of circle S^1 and k-dimensional spheres. The number of k-dimensional spheres μ is equal same of $\mu_\Sigma + \mu_\Delta$ [Za, Sh-1]. The homotopy type of fibre of π is S^{n-1} but unfortunately we cold'nt use Gysin exact sequence for this fibration π because π is not orientable. But the total space $F_s^0\backslash F_{U_1}$ is homotopy equivalent to $E' \underset{S^{n-1}}{\cup} E''$, where $E' \to S^1$ and $E'' \to \overset{\mu}{\underset{k=1}{\vee}} S_i^k$ are fibre bundles with fibre S^{n-1} and in $E' \underset{S^{n-1}}{\cup} E''$ a fibre of E' is identified with a fiber of E''. For the fibration $E' \to S^1$ which is nonorientable and its monodromy is equal -1 [Ne-2] we may use Wang exact sequence. We obtain

$$\to H_q(S^{n-1}) \to H_q(E') \to H_{q-1}(S^{n-1}) \to H_{q-1}(S^{n-1}) \to \cdots$$

Finally, we receive short exact sequence

$$0 \longrightarrow H_n(E') \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 0,$$

α is multiplication by 2. Therefore $H_{n-1}(E') = \mathbb{Z}_2$, $H_q(E') = 0$, $q \neq 0$, $q \neq n-1$. □

On the other hands, we have orientable fibration $E'' \to \overset{\mu}{\underset{i=1}{\vee}} S_i^k$ because of $k \geq 3$. Hence we may use Gysin exact sequence we obtain

$$\to H_q(E'') \to H_q(\overset{\mu}{\underset{i=1}{\vee}} S_i^k) \to H_{q-n}(\overset{\mu}{\underset{i=1}{\vee}} S_i^k) \to H_{q-1}(E'') \to \cdots$$

Since $n \geq k+2$ and $k \geq 3$ we receive $H_{n-2}(E'') = H_n(E'') = 0$ and $H_{n-1}(E'') \simeq \mathbb{Z}$, $H_{m-1}(E'') \simeq \mathbb{Z}^{\mu_\Delta + \mu_\Sigma}$.

The total space $F_{\Sigma_s\backslash U_1} = E' \cup E''$, where $E' \cap E'' \simeq S^{n-1}$. Using Mayer-Vietoris theorem we obtain

$$\to H_q(E'\cap E'') \to H_q(E')\oplus H_q(E'') \to H_q(E'\cup E'') \to H_{q-1}(E'\cup E'') \to \cdots$$

After short computations we receive short exact sequence

$$0 \to H_n(E) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H_{n-1}(E) \to 0.$$

Therefore $H_q(E) = 0$, $H_{n-1}(E) = \mathbb{Z}_2$ and $H_{n-2}(E) = 0$.

Similarly we receive $H_{m-1}(E) = \mathbb{Z}^{\mu_\Delta + \mu_\Sigma}$ and $H_q(E) = 0$, $q \geq n-2$.

Lemma 4.3. *$H_{n-2}(F_{U_2\backslash U_1}) = 0$, $H_{n-1}(F_{U_2\backslash U_1}) = \mathbb{Z}_2$, $H_{m-2}(F_{U_2\backslash U_1}) = \mathbb{Z}_2^{\mu_\Delta}$ and $H_q(F_{U_2\backslash U_1}) = 0$ if $q \geq n-2$ and $q \neq n-1, m-2$.*

Proof. We have fibration $\pi : F_{U_2\backslash U_1} \to U_2\backslash U_1$ with fibre S^{n-1}. Since $U_2\backslash U_1$ is homotopy equivalent to $S^1 \times \Delta_s$ using homotopy exact sequence of fibration π we receive $H_{n-2}(F_{U_2\backslash U_1}) = 0$. Because of π is not orientable $H_{n-1}(F_{U_2\backslash U_1}) = \mathbb{Z}_2$. As in [Ne-2], since the base space has a product structure, one can write $F_{U_2\backslash U_1}$ as a fibre bundle over Δ_s with fibre $\mathbb{Z}$ is the total space of a fibre bundle with base S^1 and fibre S^{n-1}. Using Wang exact sequence we obtain $H_{n-1}(Z) = \mathbb{Z}_2$, $H_q(Z) = 0$, $q \neq 0, n-1$. Because Δ_s is simply connected, it follows from the Serre spectral sequence [Me] $H_*(\Delta_s; H_*(Z)) \Rightarrow H_*(F_{U_2\backslash U_1})$

that $H_{m-2}(F_{U_2\setminus U_1}) = \mathbb{Z}_2^{\mu_\Delta}$ and $H_q(F_{U_2\setminus U_1}) = 0$ if $q \geq n-2$, $q \neq n-1, m-2$. $\square$

Lemma 4.4. *$H_{n-1}(F_{U_2}) = 0$, $H_n(F_{U_2}) = \mathbb{Z}$ and $H_{m-1}(F_{U_2}) = \mathbb{Z}^{\mu_\Delta}$.*

Proof. This follows from the fibration $F_{U_2} \to \Delta_s$ (cf. Lemma 3.1 (b)), whose fibre has the homotopy type of S^n. $\square$

Corollary 4.5.

$$H_q(F^0, F_{U_2}) = \begin{cases} \mathbb{Z}, & \text{if} \quad q = 0, \\ \mathbb{Z}^{\mu_\Delta + \mu_\Sigma}, & \text{if} \quad q = m-1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Using the long exact sequence for the pair $(F_{\Sigma_s\setminus U_1}, F_{U_2\setminus U_1})$ we receive

$$\to H_q(F_{U_2\setminus U_1}) \to H_q(F_{\Sigma_s\setminus U_1}) \to H_q(F_{\Sigma_s\setminus U_1}, F_{U_2\setminus U_1}) \to \\ \to H_{q-1}(F_{U_2\setminus U_1}) \to \cdots$$

Since $F_{U_2\setminus U_1} \hookrightarrow F_{\Sigma_s\setminus U_1}$ is inclusion using excision $H_q(F_{\Sigma_s\setminus U_1}, F_{U_2\setminus U_1}) = H_q(F^0, F_{U_2})$, and Lemma 4.2, 4.3 we obtain $H_q(F_s^0, F_{U_2}) = 0$ if $q \neq 0, n, m-1$. For n-dimensional homology group we have exact sequence

$$0 \to H_n(F^0, F_{U_2}) \to \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \to H_{n-1}(F^0, F_{U_2}) \to 0.$$

So $H_n(F^0, F_{U_2}) = 0$ and $H_{n-1}(F^0, F_{U_2}) = 0$. For $m-1$ dimensional homology group we have following exact sequence

$$0 \to \mathbb{Z}^{\mu_\Delta + \mu_\Sigma} \to H_{m-1}(F^0, F_{U_2}) \to \mathbb{Z}_2^{\mu_\Delta} \to 0.$$

As we known we have fibrations

$$\begin{array}{ccc} F_{\Sigma_s\setminus U_1} & \xleftarrow[i_1]{} & F_{U_2\setminus U_1} \\ \downarrow & & \downarrow \\ \Sigma_s \bigsqcup_{\Delta_s} S^1 \times \Delta_s & \xleftarrow[i_2]{} & S^1 \times \Delta_s \end{array}$$

Let $b_1, \ldots, b_{\mu_\Sigma}$ generators of $H_{m-1}(F_{\Sigma_s\setminus U_1})$. Take into account $\Delta_s \simeq \bigvee_{i=1}^{\mu_\Delta} S_i^{k-1}$. Let $f_{i,\pm}$ be the map

$$D_{i,\pm}^k = \left[0, \frac{1}{2}\right] \times S_i^{k-1}/\{1\} \times S_i^{k-1} \to S^1 \times \Delta_s/\{1\} \times \Delta_s \to \\ \to \Sigma_s \bigsqcup_{\Delta_s} (S^1 \times \Delta_s), \quad i = 1, \ldots, \mu_0.$$

The pullback of the fibration $F_{\Sigma_s\setminus U_1} \to \Sigma_s \underset{\Delta_s}{\sqcup} (S^1 \times \Delta_s)$ along $f_{i,+}$ is trivial. Therefore we have following diagram

$$\begin{array}{ccc} (D^k_{i,+} \times S^{n-1}, S^{k-1}_i \times S^{n-1}) & \xrightarrow{\widetilde{f}_{i,+}} & (F_{\Sigma_s\setminus U_1}, F_{U_2\setminus U_1}) \\ \downarrow & & \downarrow \\ (D^k_{i,+}, S^{k-1}_i) & \xrightarrow{f_{i,+}} & (\Sigma_s \underset{\Delta_s}{\sqcup} (S^1 \times \Delta_s), S^1 \times \Delta_s) \end{array} .$$

Let $a_i \in H_{n-1}(F_{\Sigma_s\setminus U_1}, F_{U_2\setminus U_1})$ be image of a generator of $H_{m-1}(D^k_{i,+} \times S^{n-1}, S^{k-1}_i \times S^{n-1}) \cong \mathbb{Z}$ under $(\widetilde{f}_{i,+})$. There is a homotopy between $f_{i,+}$ and $f_{i,-}$ (as a map of pairs), namely

$$D^k_i \times [0,1] = \big([0,1] \times S^{k-1}_i/\{0\} \times S^{k-1}_i\big) \times [0,1] \to \Sigma_s \underset{\Delta_s}{\sqcup} (S^1 \times \Delta_s).$$

$$([t,x],S) \mapsto \begin{cases} f_{i,+}\big([1-2s)t,x\big), & 0 \le s \le \dfrac{1}{2}, \\ f_{i,-}\big([2s-1)t,x\big), & \dfrac{1}{2} \le s \le 1. \end{cases}$$

Therefore $(\widetilde{f}_{i,+})_*$ and $(\widetilde{f}_{i,-})_*$ define the same element a_i. Hence $2a_i$ as an element $H_{m-1}(F_{\Sigma_s\setminus U_1})$ is represented by $\widetilde{f}_{i,+} \cup \widetilde{f}_{i,-}$, which means that

$$H_{m-1}(F_{\Sigma_s\setminus U_1}, F_{U_2\setminus U_1}) = H_{m-1}(F^0, F_{U_2}) = \mathbb{Z}^{\mu_\Delta + \mu_\Sigma}. \qquad \square$$

Corollary 4.6.

$$H_q(F^0) = \begin{cases} \mathbb{Z}, & \textit{if} \quad q = q = n, \quad q = 0, \\ \mathbb{Z}^{2\mu_\Delta + \mu_\Sigma}, & \textit{if} \quad q = m-1, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Use the long exact sequence of the pair (F^0, F_{U_2}) and above lemmas.

Now we consider the pair (B^0, F^0) and the corresponding exact sequence in homology we obtain $H_*(B^0, F^0) = H_{*-1}(F^0)$. As we mentioned in the beginning of this section for the Milnor fibre $F = X_{t,s}$ the homology group is equal

$$H_{*-1}(F) = \begin{cases} H_{*-1}(F^0) & \text{if} \quad * \neq m, \\ H_{m-1}(F^0) \oplus \mathbb{Z}^\sigma & \text{if} \quad * = m. \end{cases}$$

Therefore finally we receive

$$H_*(F) = \begin{cases} \mathbb{Z} & \text{if} \quad * = 0, n, \\ \mathbb{Z}^{2\mu_\Delta + \mu_\Sigma + \sigma} & \text{if} \quad * = m-1, \\ 0 & \text{otherwise.} \end{cases}$$

□

Now we will show that our Milnor fibre is homotopy equivalent to a wedge of spheres $S^n \vee S^{m-1} \vee \cdots \vee S^{m-1}$ following are valid.

Lemma 4.7. *Let X be a $(n-2)$-connected CW complex of dimension $n \geq 3$ with given homology $H_n(X,\mathbb{Z}) = \mathbb{Z}$, $H_{m-1}(X,\mathbb{Z}) = \mathbb{Z}^\mu$, $\widetilde{H}_k(X,\mathbb{Z}) = 0$ if $k \neq n, m-1$. Then we have a homotopy equivalence*

$$X \simeq S^n \vee S^{m-1} \vee \cdots \vee S^{m-1}.$$

Proof. For $n \geq 3$ we have that X is simple connected. According to Herewicz theorem $\pi_n(X) \simeq H_n(X) = \mathbb{Z}$. We may attach an n-cell e_n corresponding to a generator φ of $\pi_{n-1}(X)$. Let $\widetilde{X} = X \underset{\Phi}{\cup} e_n$. So we have $\pi_{n-1}(\widetilde{X}) = 0$ and $\pi_k(\widetilde{X}) = \pi_k(X) = 0$, $k \leq n-2$.

Moreover we can prove that $\widetilde{X}$ is homotopy equivalent bouquet of $n-1$ dimensional μ copies of sphere (see [Si-2], Prop. 6.1).

Consider the following Hurewicz diagram

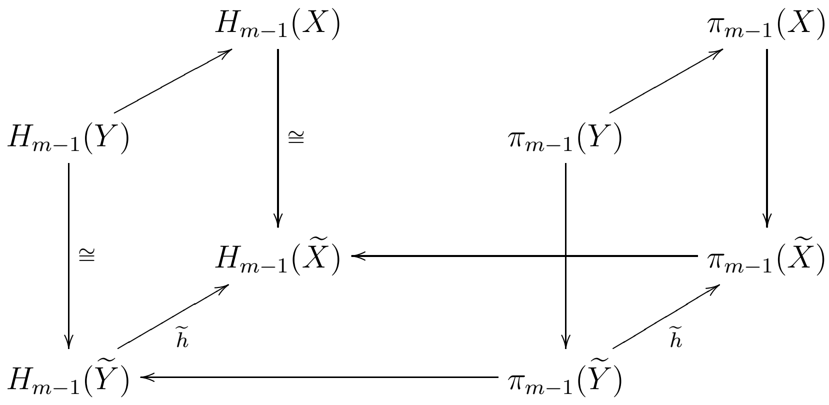
$$\begin{array}{ccc}
0 & & \\
\downarrow & & \downarrow \\
\mathbb{Z}^\mu = H_{m-1}(X) & \longleftarrow & \pi_{m-1}(X) \\
\downarrow \alpha_1 & & \downarrow \alpha_2 \\
H_{m-1}(\widetilde{X}) & \overset{\cong}{\longleftarrow} & \pi_{m-1}(\widetilde{X}) \\
\downarrow \beta_1 & & \downarrow \beta_2 \\
H_{m-1}(\widetilde{X}, X) & \longleftarrow & \pi_{m-1}(\widetilde{X}, X) \\
\downarrow \delta_1 & & \downarrow \gamma_2 \\
H_{m-2}(X) & \longleftarrow & \pi_{m-2}(X) \\
\downarrow & & \downarrow
\end{array}$$

This implies $\beta_2 = 0$ so α_2 is surjective. □

Let now $Y = S^n \vee S^{m-1} \vee \cdots \vee S^{m-1}$, and $\widetilde{Y} = D^{n+1} \vee S^{m-1} \vee \cdots \vee S^{m-1}$, where $\partial D^{n+1} = S^n$. Define $h : Y \to X$ and $\widetilde{h} : \widetilde{Y} \to \widetilde{X}$ as follows

$$h \mid S^n = \text{generator of} \quad \pi_n(X),$$
$$h \mid S^{m-1} = \text{lifted generator of} \quad \pi_{m-1}(\widetilde{X}),$$
$$h \mid D = e_n.$$

It is obvious that $H_q(X) = H_q(Y)$, if $q \neq m-1$. For $q = m-1$ consider



The following maps are isomorphisms

$$\begin{aligned}
\widetilde{h} : \pi_{m-1}(\widetilde{Y}) \to \pi_{m-1}(\widetilde{X}) &\quad \text{by construction,} \\
\pi_{m-1}(\widetilde{X}) \to H_{m-1}(\widetilde{X}) &\quad \text{by Hyrevicz-theorem,} \\
\pi_{m-1}(\widetilde{Y}) \to H_{m-1}(\widetilde{Y}) &\quad \text{by Hyrevicz-theorem,} \\
H_{m-1}(Y) \to H_{m-1}(\widetilde{Y}) &\quad \text{by exactness,} \\
H_{m-1}(X) \to H_{m-1}(\widetilde{X}) &\quad \text{by exactness.}
\end{aligned}$$

It follows that h is homotopy equivalence, because of $H_*(Y) \cong H_*(X)$, X and Y are simple connected, as a consequence of whiteheads theorem.

Main Theorem 4.8. *Let $\Sigma = \{g_1 = \cdots = g_n = 0\}$ be a isolated complete intersection and $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ a holomorphic function with singular locus $\Sigma(f) = \Sigma$ i.e. $f = \sum_{i,j=1}^{n} h_{ij} g_i g_j$, with $D = \det((h_{ij})_{ij})$ isolated singularity at the origin and $(g_1, \ldots, g_n, D)$ icis and deformation f_s described above exists. Then the Milnor fibre of f is homotopy equivalent of to a bouquet of $\mu_{m-1}(f) = 2\mu_\Delta + \mu_\Sigma + \sigma$ copies of S^{m-1} and one copy of S^n, where μ_Σ (respectively μ_Δ) is the Milnor number of Σ (respectively of Δ), and σ is the number of Morse points which occur in a special deformation of f.*

Proof. We know that Milnor fibre F is $n-2$ connected (see [Ka-Ma]). As we mansion above $n \geq 3$ so F is simple connected and we can apply Lemma 4.6 and find $F \simeq S^n \vee S^{m-1} \vee \cdots \vee S^{m-1}$. This finishes the proof of the main theorem. □

References

[Bo] J. Bobadilla, Approximations of non-isolated singularities of finite codimension with respect to an isolated complete intersection singularity. *Bull. London Math. Soc.* **35** (2003), no. 6, 812–816.

[Br] E. Brieskorn, Die Monodromie der isolierten Singularitaten von Hyperflachen. (German) *Manuscripta Math.* **2** 1970 103–161.

[Go-Gu] M . Golubitsky, V. Guillemin, Stable mappings and their singularities. *Graduate Texts in Mathematics*, Vol. 14. *Springer-Verlag, New York-Heidelberg*, 1973.

[Ka-Na] M. Kato, Y. Matsumoto, On the connectivity of the Milnor fiber of a holomorphic function at a critical point. *Manifolds—Tokyo* 1973 (*Proc. Internat. Conf., Tokyo*, 1973), 131–136. *Univ. Tokyo Press, Tokyo*, 1975.

[Loo] E. J. N. Looijenga, Isolated singular points on complete intersections. *London Mathematical Society Lecture Note Series,* 77*. Cambridge University Press, Cambridge*, 1984.

[Me] J. McCleary, A user's guide to spectral sequences. Second edition. *Cambridge Studies in Advanced Mathematics*, 58. *Cambridge University Press, Cambridge*, 2001.

[Ne-1] A. Nemethi, The Milnor fiber and the zeta function of the singularities of type $f = P(h, g)$. *Compositio Math.* **79** (1991), no. 1, 63–97.

[Ne-2] A. Nemethi, Hypersurface singularities with 2-dimensional critical locus. *J. London Math. Soc.* **(2) 59** (1999), no. 3, 922–938.

[Pe-1] G. R. Pellikaan, Hypersurface singularities and resolutions of Jacobi modules. *Dissertation, Rijksuniversiteit te Utrecht, Utrecht,* 1985*. With a Dutch summary. Drukkerij Elinkwijk B. V., Utrecht*, 1985.

[Pe-1] G. R. Pellikaan, Finite determinacy of functions with nonisolated singularities. *Proc. London Math. Soc.* **(3) 57** (1988), no. 2, 357–382.

[Sh-1] M. Shubladze, Flat isolated singularities of analytic functions of three complex variables. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **128** (1987), no. 2, 241–244.

[Sh-2] M. Shubladze, Isolated hypersurface singularities of the transversal type A_1. *Bull. Georgian Acad. Sci.* **153** (1996), no. 1, 7–10.

[Sh-3] M. Shubladze, Nonisolated singularities of codimension 1. Complex analysis. *J. Math. Sci.* (N. Y.) **118** (2003), no. 5, 5467–5505.

[Sh-4] M. Shubladze, On the homotopy type of the complement to the Milnor fiber of an isolated singularity. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **126** (1987), no. 3, 489–492.

[Si-1] D. Siersma, Isolated line singularities. *Singularities, Part* 2 (*Arcata, Calif.*, 1981), 485–496, *Proc. Sympos. Pure Math.,* 40*, Amer. Math. Soc., Providence*, RI, 1983.

[Si-2] D. Siersma, Singularities with critical locus a 1-dimensional complete intersection and transversal type A_1. *Topology Appl.* **27** (1987), no. 1, 51–73.

[Za] A. Zaharia, Topological properties of certain singularities with critical locus a 2-dimensional complete intersection. *Topology Appl.* **60** (1994), no. 2, 153–171.

Tbilisi State University, 0128 Chavchavadze av. 1, Tbilisi, Georgia