ON SOME SYSTEMS OF DIFFERENCE EQUATIONS. Part 7.

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To 100th birthday of
Professor A.O.Gelfond.

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§7.0. Foreword.

Let

$$|z| \geq 1, -3\pi/2 < \arg(z) \leq \pi/2, \log(z) = \ln(|z|) + i\arg(z).$$

Then $\log(-z) = \log(z) - i\pi$, if $\Re(z) > 0$ and $\log(z) = \log(-z) - i\pi$, if $\Re(z) < 0$.

Let

$$f_{l,1}^{\vee}(z,\nu) = f_{l,1}(z,\nu) = \sum_{k=0}^{\nu}(-1)^{(\nu+k)l}(z)^k \binom{\nu}{k}^{2+l}\binom{\nu+k}{\nu}^{2+l}, \tag{1}$$

where $l = 0,\ 1,\ 2,\ \nu \in [0, +\infty) \cap \mathbb{Z}$. Let

$$R(t,\nu) = \frac{\prod\limits_{j=1}^{\nu}(t-j)}{\prod\limits_{j=0}^{\nu}(t+j)}, \tag{2}$$

where $\nu \in [0, +\infty) \cap \mathbb{Z}$,

$$f_{l,2}^{\vee}(z,\nu) = f_{l,2}(z,\nu) = \sum_{t=1+\nu}^{+\infty} z^{-t}(R(t,\nu))^{2+l}, \tag{3}$$

where $l = 0,\ 1,\ 2,\ |z| \geq 1$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$. Since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1, \ldots, \nu$, the zeros of the order $2 + l$, it follows that

$$f_{l,2}(z,\nu) = \sum_{t=1}^{+\infty} z^{-t}(R(t,\nu))^{2+l}, \tag{4}$$

for $l = 0,\ 1,\ 2,\ |z| \geq 1$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$. Let

$$f_{l,3}^{\vee}(z,\nu) = f_{l,3}(z,\nu) = (\log(z)) f_{l,2}(z,\nu) + f_{l,4}(z,\nu), \tag{5}$$

where

$$f_{l,4}(z,\nu) = -\sum_{t=1+\nu}^{+\infty} z^{-t} \left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu), \tag{6}$$

where $l = 0,\ 1,\ 2,\ |z| \geq 1$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$. Since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1, \ldots, \nu$, the zeros of the order $2 + l$, it follows that

$$f_{l,4}(z,\nu) = -\sum_{t=1}^{+\infty} z^{-t} \left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu) \tag{7}$$

for $l = 0,\ 1,\ 2,\ |z| \geq 1$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$. Let

$$f_{l,5}^{\vee}(z,\nu) = -i\pi f_{l,3}(z,\nu) + f_{l,5}(z,\nu), \tag{8}$$

with $l = 1,\ 2,\ \nu \in [0, +\infty) \cap \mathbb{Z},\ |z| \geq 1$ and

$$f_{l,5}(z,\nu) = \tag{9}$$

$$2^{-1}(\log(z))^2 f_{l,2}(z,\nu) + (\log(z)) f_{l,4}(z,\nu) + f_{l,6}(z,\nu) =$$
$$= -2^{-1}(\log(z))^2 f_{l,2}(z,\nu) + (\log(z)) f_{l,3}(z,\nu) + f_{l,6}(z,\nu),$$

where

$$f_{l,6}(z,\nu) = 2^{-1} \sum_{t=1+\nu}^{\infty} z^{-t} \left(\left(\frac{\partial}{\partial t}\right)^2 (R^{2+l})\right)(t,\nu). \tag{10}$$

Since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1, \ldots, \nu$, the zeros of the order $2 + l$, and $l = 1,\ 2$ now, it follows that

$$f_{l,6}(z,\nu) = 2^{-1} \sum_{t=1+\nu}^{\infty} z^{-t} \left(\left(\frac{\partial}{\partial t}\right)^2 (R^{2+l})\right)(t,\nu) \tag{11}$$

for $l = 1,\ 2,\ |z| \geq 1$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$. Let

$$f^{\vee}_{l,7}(z,\nu) = f_{l,7}(z,\nu) + (2\pi^2/3) f_{l,3}(z,\nu). \tag{12}$$

with $l = 2$, $\nu \in [0,+\infty) \cap \mathbb{Z}$, $|z| \geq 1$ and

$$\begin{gathered} f_{l,7}(z,\nu) = \\ -3^{-1}(\log(z))^3 f_{l,2}(z,\nu) + 2^{-1}(\log(z))^2 f_{l,3}(z,\nu) + f_{l,8}(z,\nu) + \\ (\log(z))(f_{l,5}(z,\nu) + 2^{-1}(\log(z))^2 f_{l,2}(z,\nu) - (\log(z)) f_{l,3}(z,\nu)) = \\ 6^{-1}(\log(z))^3 f_{l,2}(z,\nu) - 2^{-1}(\log(z))^2 f_{l,3}(z,\nu) + (\log(z)) f_{l,5}(z,\nu) + f_{l,8}(z,\nu) = \\ (1/6)(\log(z))^3 f_{l,2}(z,\nu) + (1/2)(\log(z))^2 f_{l,4}(z,\nu) + \\ (\log(z)) f_{l,6}(z,\nu) + f_{l,8}(z,\nu), \end{gathered} \tag{13}$$

where

$$f_{l,8}(z,\nu) = -6^{-1} \sum_{t=\nu+1}^{\infty} z^{-t} \left(\left(\frac{\partial}{\partial t} \right)^3 (R^{2+l}) \right)(t,\nu). \tag{14}$$

Since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ have in the points $t = 1, \ldots, \nu$, the zeros of the order $2+l$, and $l = 2$ now, it follows that

$$f_{l,8}(z,\nu) = -6^{-1} \sum_{t=1}^{\infty} z^{-t} \left(\left(\frac{\partial}{\partial t} \right)^3 (R^{2+l}) \right)(t,\nu) \tag{15}$$

for $l = 1, 2$, $|z| \geq 1$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$. Let

$$\mathfrak{K}_0 = \{1, 2, 3\},\ \mathfrak{K}_1 = \{1, 2, 3, 5\},\ \mathfrak{K}_2 = \{1, 2, 3, 5, 7\}.$$

Let λ be a variable. Let $T_{n,\lambda}$ be diagonal $n \times n$-matrix, i-th diagonal element of which is equal to λ^{i-1} for $i = 1, \ldots, n$. We denote by δ the operator $z\frac{d}{dz}$. Let further $l = 0, 1, 2$, $k \in \mathfrak{K}_l$, $|z| > 1, \nu \in \mathbb{N}$, and let $Y_{l,k}(z;\nu)$ be the columnn with $4+2l$ elements, i-th of which is equal to $(\nu^{-1}\delta)^{i-1} f^{\vee}_{l,k}(z,\nu)$, where $i = 1, \ldots, 4+2l$.

Theorem 1. *The following equalities hold*

$$A^{\sim}_l(z;\nu) Y_{l,k}(z;\nu) = T_{4+2l,1-\nu^{-1}} Y_{l,k}(z;\nu-1), \tag{16}$$

$$Y_{l,k}(z;\nu) = T_{4+2l,-1} A^{\sim}_l(z;-\nu) T_{4+2l,-1+\nu^{-1}} Y_{l,k}(z;\nu-1), \tag{17}$$

where $l = 0, 1, 2$, $k \in \mathfrak{K}_l$, $|z| > 1, \nu \in \mathbb{N}, \nu \geq 2$,

$$A^{\sim}_l(z;\nu) = S^{\sim}_l + z \sum_{i=0}^{1+l} \nu^{-i} V^{\sim *}_l(i) \tag{18}$$

with

$$S^{\sim}_0 = \begin{pmatrix} 1 & -4 & 8 & -12 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

$$S_1^{\sim} = \begin{pmatrix} -1 & 6 & -18 & 38 & -66 & 102 \\ 0 & -1 & 6 & -18 & 38 & -66 \\ 0 & 0 & -1 & 6 & -18 & 38 \\ 0 & 0 & 0 & -1 & 6 & -18 \\ 0 & 0 & 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{20}$$

$$S_2^{\sim} = \begin{pmatrix} 1 & -8 & 32 & -88 & 192 & -360 & 608 & -952 \\ 0 & 1 & -8 & 32 & -88 & 192 & -360 & 608 \\ 0 & 0 & 1 & -8 & 32 & -88 & 192 & -360 \\ 0 & 0 & 0 & 1 & -8 & 32 & -88 & 192 \\ 0 & 0 & 0 & 0 & 1 & -8 & 32 & -88 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{21}$$

$$V_0^{\sim *}(0) = 4 \begin{pmatrix} 4 & -5 & -2 & 3 \\ -3 & 4 & 1 & -2 \\ 2 & -3 & 0 & 1 \\ -1 & 2 & -1 & 0 \end{pmatrix},$$

$$V_0^{\sim *}(1) = 4 \begin{pmatrix} 3 & -6 & 3 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_1^{\sim *}(0) = \begin{pmatrix} 146 & -198 & -180 & 268 & 66 & -102 \\ -102 & 146 & 108 & -180 & -38 & 66 \\ 66 & -102 & -52 & 108 & 18 & -38 \\ -38 & 66 & 12 & -52 & -6 & 18 \\ 18 & -38 & 12 & 12 & 2 & -6 \\ -6 & 18 & -20 & 12 & -6 & 2 \end{pmatrix},$$

$$V_1^{\sim *}(1) = \begin{pmatrix} 240 & -516 & 108 & 372 & -204 & 0 \\ -160 & 348 & -84 & -236 & 132 & 0 \\ 96 & -212 & 60 & 132 & -76 & 0 \\ -48 & 108 & -36 & -60 & 36 & 0 \\ 16 & -36 & 12 & 20 & -12 & 0 \\ 0 & -4 & 12 & -12 & 4 & 0 \end{pmatrix},$$

$$V_1^{\sim *}(2) = \begin{pmatrix} 102 & -306 & 306 & -102 & 0 & 0 \\ -66 & 198 & -198 & 66 & 0 & 0 \\ 38 & -114 & 114 & -38 & 0 & 0 \\ -18 & 54 & -54 & 18 & 0 & 0 \\ 6 & -18 & 18 & -6 & 0 & 0 \\ -2 & 6 & -6 & 2 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(0) = 8 \begin{pmatrix} 176 & -249 & -364 & 545 & 280 & -431 & -76 & 119 \\ -119 & 176 & 227 & -364 & -169 & 280 & 45 & -76 \\ 76 & -119 & -128 & 227 & 92 & -169 & -24 & 45 \\ -45 & 76 & 61 & -128 & -43 & 92 & 11 & -24 \\ 24 & -45 & -20 & 61 & 16 & -43 & -4 & 11 \\ -11 & 24 & -1 & -20 & -5 & 16 & 1 & -4 \\ 4 & -11 & 8 & -1 & 4 & -5 & 0 & 1 \\ -1 & 4 & -7 & 8 & -7 & 4 & -1 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(1) = 8 \begin{pmatrix} 455 & -1020 & -113 & 1552 & -603 & -628 & 357 & 0 \\ -300 & 682 & 44 & -996 & 404 & 394 & -228 & 0 \\ 185 & -428 & -3 & 592 & -253 & -228 & 135 & 0 \\ -104 & 246 & -16 & -316 & 144 & 118 & -72 & 0 \\ 51 & -124 & 19 & 144 & -71 & -52 & 33 & 0 \\ -20 & 50 & -12 & -52 & 28 & 18 & -12 & 0 \\ 5 & -12 & 1 & 16 & -9 & -4 & 3 & 0 \\ 0 & -2 & 8 & -12 & 8 & -2 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(2) = 8 \begin{pmatrix} 400 & -1243 & 972 & 542 & -1028 & 357 & 0 & 0 \\ -259 & 808 & -642 & -332 & 653 & -228 & 0 & 0 \\ 156 & -489 & 396 & 186 & -384 & 135 & 0 & 0 \\ -85 & 268 & -222 & -92 & 203 & -72 & 0 & 0 \\ 40 & -127 & 108 & 38 & -92 & 33 & 0 & 0 \\ -15 & 48 & -42 & -12 & 33 & -12 & 0 & 0 \\ 4 & -13 & 12 & 2 & -8 & 3 & 0 & 0 \\ -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(3) = 8 \begin{pmatrix} 119 & -476 & 714 & -476 & 119 & 0 & 0 & 0 \\ -76 & 304 & -456 & 304 & -76 & 0 & 0 & 0 \\ 45 & -180 & 270 & -180 & 45 & 0 & 0 & 0 \\ -24 & 96 & -144 & 96 & -24 & 0 & 0 & 0 \\ 11 & -44 & 66 & -44 & 11 & 0 & 0 & 0 \\ -4 & 16 & -24 & 16 & -4 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The above matices $A_l^{\sim}(z;\nu), S_l^{\sim}$ and $V_l^{\sim *}(i)$ have the following properties:

(22) $$A_l^{\sim}(z;-\nu)T_{4+2l,-1}A_l^{\sim}(z;\nu) = T_{4+2l,-1},$$

(23) $$S_l^{\sim}T_{4+2l,-1} = (S_l^{\sim}T_{4+2l,-1})^{-1}$$

(24) $$S_l^{\sim}T_{4+2l,-1}V_l^{\sim *}(i) = -(-1)^i V_l^{\sim *}(i)T_{4+2l,-1}S_l^{\sim},$$

(25) $$V_l^{\sim *}(i)T_{4+2l,-1}V_l^{\sim *}(k) = 0T_{4+2l,-1},$$

where

$$l = 0,\, 1,\, 2,\ i \in [0, 1+l] \cap \mathbb{Z},\ k \in [0, 1+l] \cap \mathbb{Z}.$$

Proof. Full proof can be found in [52] – [56].

In [56] I promised to give arithmetical applications of the Theorem 1. I had given in [58] short deduction of the famous Apéry's equation from the Theorem 1 (see correction of misprint there in §7.5 below) . Here I continue to fulfill this promise and begin the proof of the Theorem 2, which joins the Apéry's Theorem and my result in [23], [43] in one Theorem. Let

$$z \in \mathbb{Q},\ |z| \geq 1, x = 1/z,\ b \in \mathbb{N}, bz \in \mathbb{Z},$$

$$\tilde{\eta}_i(z) = \left(\sum_{k=0}^{1} \sqrt{\sqrt{|z|} + k(-1)^i)}\right)^2 =$$

$$2\sqrt{|z|} + (-1)^i + 2\sqrt{|z| + (-1)^i\sqrt{|z|}}$$

for $i = 0, 1$,

$$\tilde{\eta}_2(z) = \sqrt{|z|} + \sqrt{|z| + 1} + \sum_{k=0}^{1} \sqrt{|z| + (-1)^k\sqrt{|z|}} =$$

$$\sqrt{|z|} + \sqrt{|z| + 1} + \sqrt{2(\sqrt{|z|^2 + |z|} + |z|)},$$

$$L_{i,s}(x) = (i/x + (-1)^i) \sum_{n=1}^{+\infty} x^n/n^s$$

for $i = 0, 1$,

$$\beta_k(z) = \frac{\ln((\tilde{\eta}_{2[k/2]}(z))^2 e^3 b)}{\ln((\tilde{\eta}_k(z))^2/e^3 b)},$$

where $k = 0,\ 1,\ 2$

$$\alpha_k(z) = \beta_k + \frac{(1 - (-1)^k)(\ln(\tilde{\eta}_1(z)/\tilde{\eta}_0(z)}{\ln((\tilde{\eta}_1(z))^2/e^3 b)},$$

$$D_k(b) = \{y \in \mathbb{R}\colon (-1)^{[k/2]}y > ((\sqrt{e^3 b} + 1)^4/(e^3 b + 1)^2)^{[k/2]}/16e^3 b\},$$

where $k = 0,\ 1,\ 2$,

$$L_{i,s}(x) = (i/x + (-1)^i) \sum_{n=1}^{+\infty} x^n/n^s,$$

where $i = 0,\ 1,\ s \in \mathbb{N},\ |x - 1| + s > 1$,

$$L_{1,1}(1) = 0,$$

$$x_1 \in \mathbb{R},\ x_2 \in \mathbb{Q},\ |x_1| + |x_2| > 0,$$

$$\phi_i = \phi_i(x_1, x_2, x) = \tilde{\phi}_i(z, x_1, x_2) = x_1 L_{2-i,i}(x) + i x_2 L_{2-i,i+1}(x),$$

where $i = 1,\ 2$. Let further, $\phi_3 = x_1,\ \hat{\alpha}_0(x) = \alpha_0(z),\ \tilde{\alpha}_0(x) = \alpha_2(z)$,

$$\tilde{\alpha}_i(x) = \alpha_1(z),\ \tilde{\beta}_i(x) = \beta_1(z) \text{ for } i = 1,\ 2,\ \varepsilon > 0,$$

and $\|\psi\|$ denotes the distance from ψ to $\mathbb{Z}$.

Theorem 2. *There exsist effective positive*

$$\hat{\gamma}_i(x_1, x_2, x, \varepsilon) = \gamma_i^*(z, x_1, x_2\varepsilon),$$

where $i = 1, 2,$

$$\hat{\gamma}_0(x, \varepsilon) = \gamma_0(z, \varepsilon),$$

$$\tilde{\gamma}_1(x, \varepsilon) = \gamma_1(z, \varepsilon),\ \tilde{\gamma}_0(x, \varepsilon) = \gamma_2(z, \varepsilon),$$

such that, if

$$z \in D_0(b),\ x_1 = \ln(z),\ x_2 = 1,$$

then

$$\max_{i=1,2,3} \|q\phi_i\| q^{\alpha_0(z)+\varepsilon} \geq \gamma_0(z, \varepsilon) \tag{26}$$

for any $q \in \mathbb{N}$*, if*

$$z \in D_k(b),\ x_1 \in \mathbb{Z},\ x_2 \in \mathbb{Z},\ x_2 \neq 0,$$

then

$$\max_{i=1,2} \|q\phi_i\| q^{\alpha_0(z)+\varepsilon} \geq \gamma_k^*(z, x_1, x_2\varepsilon) \tag{27}$$

for any $q \in \mathbb{N}$*,*

$$\max_{i=1,2} \|\tilde{\phi}_i(z, x_1, x_2)\| (|x_1| + |x_2|)^{\alpha_k(z)+\varepsilon} \geq \gamma_0(z, \varepsilon) \tag{28}$$

for any $x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z}$*, for which*

$$|x_1| + |x_2| > 0.$$

If $z = x = b = 1$, $k = 0$, then $4ln(e^{3/2} - 1) - 4ln2 - 3 = -0,7825... < 0$, $1 > (e^{3/2} - 1)^4/16e^3$ and therfore $1 \in D_0(1)$. Moreover,

$$\phi_1 = \phi_3 = 0,\ \phi_2 = 2\zeta(3),$$

$$\tilde{\eta}_0(1) = (1 + \sqrt{2})^2,\ \tilde{\eta}_1(1) = 1,$$

$$\beta_0 = \alpha_0 = 1 + 6/(4\ln(1 + \sqrt{2}) - 3) = 12,417820...$$

The inequality (39) takes in this case the following form

$$\|2q\zeta(3)\| q^{\alpha_0(z)+\varepsilon} \geq \gamma_0(z, \varepsilon)$$

and includes Ape̒ry theorem; moreover it give for the mesure $\mu(\zeta(3))$ of irrationality of the number $\zeta(3)$ the estimate

$$\mu(\zeta(3)) < 12,417821 + 1 = 13,417821.$$

This estimate was decreased by several authors; best result, which I know, belongs to G.Rhinn and C.Viola; they obtain the inequality

$$\mu(\zeta(3)) < 5,513891.$$

If $-z = -x = b = 1$, $k = 2$, then $4ln(e^3 - 1) - 2ln(e^3 + 1) - 4ln2 - 3 = -0,07404... < 0$, $1 > (e^3 - 1)^4/(16e^3(e^3 + 1)^2$ and therfore $-1 \in D_2(1)$. Moreover,

$$\phi_1 = 2x_1 \ln(2) + x_2\zeta(2),\ 2\phi_2 = x_1\zeta(2) + x_2 3\zeta(3),$$

$$\beta_2 = \alpha_2 = 1+$$

$$6/(2\ln(1 + \sqrt{2} + \sqrt{\sqrt{2}+1}\sqrt{\sqrt{2}-1}) - 3) = 106,00187...,$$

and it follows from (28) that The $\mathbb{Z}$-module with generators

$$f_1 = \begin{pmatrix} \log 4 \\ \zeta(2) \end{pmatrix},\ f_2 = \begin{pmatrix} \zeta(2) \\ 3\zeta(3) \end{pmatrix},\ f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ f_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

has these four generators as free generators, it is not a Liouville module and measure of nondiscreteness of this $\mathbb{Z}$- module is not bigger than

$$\beta_2 = 106,00187....$$

The constant 106, 00187... was decreased to 22, 42693 in [27] (in Russian) and in [51] (in English). Theorem 2 was announced in [23], the proof was published in [26] in Russian. It was not easy job to restore the calculations, which was made 20 years ago. As it is mentioned above, I split the Proof of the Theorem 2 in two parts; here is presented the first part.

Let $\mu = \nu(\nu + 1)$,

(29) $$\omega_{-2}(z; \mu) := 9 - 45z + (30 - 126z)\mu + (8 - 32z)\mu^2,$$

(30) $$\omega_2(z; \mu) := -3 + 15z + (-16 + 72z)\mu + (-16 + 64z)\mu^2.$$

(31) $$\omega_1(z; \mu) = (-9 + 9z + 180z^2) + (3 - 39z - 96z^2)\mu +$$

$$(72 + 168z - 1248z^2)\mu^2+$$

$$(-128 - 368z + 2944z^2)\mu^3 + (64 + 256z - 2048z^2)\mu^4,$$

(32) $$\omega_{-1}(z; \mu) = (9 - 9z - 180z^2) + (-21 + 57z + 456z^2)\mu +$$

$$(12 - 60z - 288z^2)\mu^2,$$

(33) $$\omega_0(z; \mu) = (8 - 224z + 696z^2 - 480z^3)\mu^2 +$$

$$(22 - 238z + 320z^2 + 256z^3)\mu^3 - (76 - 1996z + 7808z^2 - 6656z^3)\mu^4+$$

$$(48 - 1728z + 8192z^2 - 8192z^3)\mu^5,$$

(34) $$a_{0,-2}(z; \nu) = (\nu(\nu - 1))^3 \times$$

$$((2\nu + 1)\omega_{-2}(z; \nu(\nu + 1)) - 3\omega_2(z; \nu(\nu + 1))),$$

(35) $$a_{0,2}(z; \nu) = -a_{0,-2}(z; -\nu - 1) = ((\nu + 1)(\nu + 2))^3 \times$$

$$((2\nu+1)\omega_{-2}(z;\nu(\nu+1))+3\omega_2(z;\nu(\nu+1))),$$

(36) $$a_{0,-1}(z;\nu)=\nu^3\times$$

$$((2\nu+1)\omega_{-1}(z;\nu(\nu+1))-\omega_1(z;\nu(\nu+1)),$$

(37) $$a_{0,1}(z;\nu)=-a_{0,-1}(z;-\nu-1)=-(\nu+1)^3\times$$

$$((2\nu+1)\omega_{-1}(z;\nu(\nu+1))+\omega_1(z;\nu(\nu+1)),$$

(38) $$a_{0,0}(z;\nu)=-a_{0,0}(z;-\nu-1)=(2\nu+1)\omega_0(z;\nu(\nu+1)).$$

Previously I prove the following

Lemma 7.0. *The set of all the solutions of difference equation*

(39) $$\sum_{\kappa=-2}^{2} a_{0,\kappa}(z;\nu)y(z;\nu+\kappa)=0$$

contains the solutions

$$y(z;\nu)=f_{0,2}(z;\nu),\ y(\nu)=f_{0,4}(z;\nu) \text{ for } |z|>1,\ \nu\in\mathbb{Z},$$

$$y(z;\nu)=\alpha^*_{0,i}(z;\nu) \text{ for } z\in\mathbb{C},\ i=1,\ 2,\ \nu\in\mathbb{Z}$$

and

$$y(z;\nu)=\beta^*_{0,j}(z;\nu) \text{ for } z\in\mathbb{C},\ j=0,\ 1,\ \nu\in\mathbb{Z},$$

where $\alpha^*_{0,i}(z;\nu)$, $\beta^*_{0,i}(z;\nu)$ *are specified in [58] for* $\nu\in[0,+\infty)\cap\mathbb{Z}$, *and*

$$\alpha^*_{0,i}(z;-\nu-1):=\alpha^*_{0,i}(z;\nu)$$

$$\beta^*_{0,j}(z;-\nu-1):=\beta^*_{0,j}(z;\nu),$$

$$y(z;-\nu-1)=y(z;\nu)$$

for $\nu\in[0,+\infty)\cap\mathbb{Z}$.

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§7.1. Difference equations for my auxiliary functios.

The field $\mathbb{Q}(z,\nu)$ has authomorphism ω, which turns ν into $\nu^\omega=-\nu-1$. Clearly, $\mu^\omega=\mu$; if $p\in\mathbb{Q}(z,\nu)$, the we denote by p^ω the result of applying of ω to p. If A is the matrix (in particular row or column), all elements of which belong to $\mathbb{Q}(z,\nu)$, then we denote by A^ω the result of applying of ω to all elements qf A. If A_k, where $k=1$, ..., n is a sequence of $s\times s$-matrices, all elements of of which belong to $\mathbb{Q}(z,\nu)$, then

$$\prod_{k=1}^{m} A_k, \text{ where } m=0,\ ...,\ n,$$

is defined sequentially by means the equalities

$$\prod_{k=1}^{0} A_k = E_s,$$

where E_s is unit $s \times s$-matrix,

$$\prod_{k=1}^{m} A_k = \left(\prod_{k=1}^{m-1} A_k\right) A_m$$

for $m = 1, ..., n$. Let $a^*_{l,i,j}(z;\nu)$ stands in the matrix,

$$A^*_l(z;\nu) = \nu^{3+2l} T_{4+2l,\nu} A^\sim_l(z;\nu)(T_{4+2l,\nu}))^{-1} \tag{40}$$

on the intetrsection of its i-th row and j-th column. Clearly,

$$a^*_{l,i,j}(z;\nu) \in \mathbb{Q}[z;\nu], \tag{41}$$

where $l = 0, 1, 2, i = 1, ..., 4+2l, j = 1, ..., 4+2l$. Let

$$X_{l,k}(z;\nu) = T_{4+2l,\nu} Y_{l,k}(z;\nu), \tag{42}$$

where $l = 0, 1, 2, k \in \mathfrak{K}_l, i = 1, ..., 4+2l, j = 1, ..., 4+2l$ and $\nu \in \mathbb{N}$. Then according to the Theorem 1,

$$\nu^{3+2l} X_{l,k}(z;\nu-1) = A^*_l(z;\nu) X_{l,k}(z;\nu), \tag{43}$$

where $|z| > 1, l = 0, 1, 2, k \in \mathfrak{K}_l$ and $\nu \in \mathbb{N}$. In [52] – [58] it is proved that the equality

$$X_{l,k}(z;-\nu-1) = X_{l,k}(z;\nu) \tag{44}$$

extends the set of ν, for which is defined $X_{l,k}(z;\nu)$, from $[0,+\infty) \cap \mathbb{Z}$ to $\mathbb{Z}$, and (43) holds for all the $\nu \in \mathbb{Z}$ also. Therefore

$$-(\nu+1^{3+2l} X_{l,k}(z;\nu+1) = A^*_l(z;-\nu-1) X_{l,k}(z;\nu), \tag{45}$$

where $|z| > 1, l = 0, 1, 2, k \in \mathfrak{K}_l$ and $\nu \in \mathbb{Z}$. Let further

$$Q_l(z;\nu;\nu-m) := \tag{46}$$

$$\prod_{k=1}^{m} A^*_0(z;\nu-m+k),$$

$$Q_l(z;\nu;\nu+m) := \tag{47}$$

$$\prod_{k=1}^{m} A^*_0(z;-\nu-1+m-k) = (Q_l(z;\nu;\nu-m))^\omega,$$

where $m = 0, ..., 2+l$, and let $\bar{q}_{l,i}(z;\nu;\nu+\kappa)$ denotes the first row of the matrix $Q_l(z;\nu;\nu+\kappa)$ for $\kappa = -2-l, ..., 2+l$. In view of (43) – (47),

$$\left(\prod_{k=1}^{m}(\nu-m+k)\right)^{3+2l} X_{l,k}(z;\nu-m) = \tag{48}$$

$$Q_l(z;\nu;\nu-m)X_{l,k}(z;\nu),$$

(49) $$(-1)^m\left(\prod_{k=1}^{m}(\nu+1+m-k)\right)^{3+2l} X_{l,k}(z;\nu+m)=$$

$$Q_l(z;\nu;\nu+m)X_{l,k}(z;\nu).$$

Hence

(50) $$\left(\prod_{k=1}^{m}(\nu-m+k)\right)^{3+2l} f^{\vee}_{l,k}(z;\nu-m)=$$

$$q_l(z;\nu;\nu-m)X_{l,k}(z;\nu)$$

(51) $$(-1)^m\left(\prod_{k=1}^{m}(\nu+1+m-k)\right)^{3+2l} X_{l,k}(z;\nu+m)=$$

$$q_l(z;\nu;\nu+m)f_{l,k}(z;\nu).$$

where $|z|>1$, $l=0,\,1,\,2$, $k\in\mathfrak{K}_l$, $\nu\in\mathbb{Z}$ and $m=0,\,...,\,2+l$.

Lemma 7.1. *Let L be a field and K be its subfield, such that $[L:K]=2$. Let ω denotes the unique non-identity automorhism of the extension L/K. Let further $n\in\mathbb{N}$, $\bar{a}_1$, ..., $\bar{a}_{2n+1}$ are rows with $2n$ elements lying in L and such that $\bar{a}_{2n+2-k}=(\bar{a}_k)^\omega$ for $k=1$, ..., $2n+1$. Let A be a $(2n+1)\times 2n$-matrix having $\bar{a}_k$ as its k-th row for $k=1$, ..., $2n+1$. Let A_k be a result of removing from A its k-th row, and $M_k=\det(A_k)$, where $k=1$, ..., $2n+1$. Then*

$$M_{2n+2-k}=(-1)^n(M_k)^\omega.$$

Proof. Let B be a $(2n+1)\times 2n$-matrix having inverse order of rows relatively to this order in A and let B_k for $k=1$, ..., $2n+1$ be a result of removing from B its k-th row. Then cleraly

$$(det(A_k))^\omega=det(B_k)=(-1)^{2n(2n-1)/2}det(A_{2n+2-k})$$

for $k=1$, ..., $2n+1$. ■

Let $w_{l,\kappa}$ be independent variables, where $\kappa=-2-l$, ..., $2+l$. Let $H_l(z;\nu)$ be the $(5+2l)\times(4+2l)$-matrix having $\bar{q}^*_l(z;\nu+i-l-3)$ as its i-th row. Let W_l be the column, consisting of $5+2l$ elements, i-th of which is equal to the variable w_{i-l-3},where $i=1$, ..., $5+2l$.

We consider the $(5+2l)\times(5+2l)$-matrix $V_l(z,\nu)=(H_l(z;\nu),W_l)$ now. Let $H_l^{(\kappa)}(z;\nu)$ be the result of remowing from $H_l(z;\nu)$ its $(\kappa+3+l)$-th row, and let

(52) $$M_l^{(\kappa)}(z;\nu)=\det(H_l^{(\kappa)}(z;\nu))$$

for $\kappa=-2-l$, ..., $2+l$. Applying the Lemma 7.1 with $n=2+l$, $k=\kappa+3+l$, we see that

(53) $$(M_l^{(\kappa)}(z;\nu))^\omega=(-1)^l M_l^{(-\kappa)}(z;\nu)).$$

Clearly, $(-1)^{\kappa+l}M_l^{(\kappa)}(z;\nu)$ is cofactor to w_κ in the matrix $V_l(z,\nu)$.

Let $\bar{m}_l(z;\nu)$ be the row consisting of $5+2l$ elements, $(\kappa+3+l)$-th of which is equal to $(-1)^{\kappa+l}M_l^{(\kappa)}(z;\nu)$, where $l=0$, 1, 2, $\kappa=-2-l$, ..., $2+l$.

Clearly, $\bar{m}_l(z;\nu)$ is the last row of the adjoint matrix to the matrix $V_l(z,\nu)$. Hence,

$$\bar{m}_l(z;\nu)H_l(z;\nu) = 0\bar{m}_l(z;\nu). \tag{54}$$

Let $\mathfrak{X}_{l,k}(z;\nu)$ be a column, consisting from $5+2l$ elements, $(3+l-m)$-th of which is equal to

$$\left(\prod_{k=1}^{m}(\nu-m+k)\right)^{3+2l} f_{l,k}(z;\nu-m),$$

and $(3+l+m)$-th of which is equal to

$$(-1)^m\left(\prod_{k=1}^{m}(\nu+1+m-k)\right)^{3+2l} f_{l,k}(z;\nu+m),$$

where $m=0$, ..., $2+l$. It follows from (50), (51) that

$$\mathfrak{X}_{l,k}(z;\nu) = H_l(z;\nu)X_{l,k}(z;\nu). \tag{55}$$

It follows from (54), (55), that

$$\bar{m}_l(z;\nu)\mathfrak{X}_{l,k}(z;\nu) = \bar{m}_l(z;\nu)H_l(z;\nu)X_{l,k}(z;\nu) = 0. \tag{56}$$

Let

$$A_{l,-m}(z;\nu) = \left(\prod_{k=1}^{m}(\nu-m+k)\right)^{3+2l} (-1)^{m+l}M_l^{-m}(z;\nu), \tag{57}$$

$$A_{l,m}(z;\nu) = \left(\prod_{k=1}^{m}(\nu+m+1-k)\right)^{3+2l} (-1)^{l}M_l^{m)}(z;\nu), \tag{58}$$

where $m=0$, ..., $2+l$. Since,

$$\left(\left(\prod_{k=1}^{m}(\nu-m+k)\right)^{3+2l} (-1)^m\right)^{\omega} = \left(\prod_{k=1}^{m}(\nu+m+1-k)\right)^{3+2l},$$

it follows from (53) that

$$(A_l^{\kappa)}(z;\nu))^{\omega} = (-1)^l A_l^{(-\kappa)}(z;\nu) \tag{59}$$

for $l=0,1,2$, $\kappa=-2-l$, ..., $2+l$. Clearly,

$$A_{l,\kappa}(z;\nu)\in\mathbb{Q}[z;\nu] \tag{60}$$

for $l = 0, 1, 2,\ \kappa = -2 - l,\ ...,\ 2 + l$, and in view of (56),

$$\sum_{\kappa=-2-l}^{2+l} A_{l,\kappa}(z;\nu) f^{\vee}_{l,k}(z;\nu+\kappa) = 0 \tag{61}$$

for $|z| > 1, l = 0, 1, 2,\ k \in \mathfrak{K}_l,\ \nu \in \mathbb{Z}$. In view of (5), (8), (9), (12) and (13), that

$$\sum_{\kappa=-2-l}^{2+l} A_{l,\kappa}(z;\nu) f_{l,k}(z;\nu+\kappa) = 0 \tag{62}$$

for $|z| > 1, l = 0, 1, 2,\ k \in \mathfrak{K}^{\wedge}_l := [1, 4 + 2l] \cap \mathbb{Z}, \nu \in \mathbb{Z}$. In view of (99) in [58],

$$f_{l,2+2j}(z;\nu) = \left(\sum_{i=1}^{2+l} \alpha^*_{l,i}(z;\nu) L_{i+j}(1/z)\right) - \beta^*_{l,j}(z;\nu), \tag{63}$$

where $|z| > 1,\ l = 0, 1, 2,\ j \in [0, 1 + l] \cap \mathbb{Z},\ \nu \in \mathbb{Z}$,

$$L_{i+j}(1/z) = \sum_{n=1}^{+\infty} \frac{z^n}{n^{i+j}},$$

and $\alpha_{l,i}(z;\nu)$, $\beta_{l,j}(z;\nu)$ are specified in [58] for $\nu \in [0, +\infty) \cap \mathbb{Z}$ and

$$\alpha^*_{0,i}(z;-\nu-1) := \alpha^*_{0,i}(z;\nu) \tag{64}$$

$$\beta^*_{0,j}(z;-\nu-1) := \beta^*_{0,j}(z;\nu) \tag{65}$$

for $\nu \in [0, +\infty) \cap \mathbb{Z}$. Since the functions

$$1, L_1(1/z),\ ...,\ L_n(1/z),\ ...$$

compose a linear independent system over $\mathbb{C}(z, \nu)$ (see for example [37],§2) it follows from (62) that

$$\sum_{\kappa=-2-l}^{2+l} A_{l,\kappa}(z;\nu) \alpha^*_{l,i}(z;\nu+\kappa) = 0 \tag{66}$$

for $z \in \mathbb{C}, l = 0, 1, 2,\ i \in [1, 2 + l] \cap \mathbb{Z},\ \nu \in \mathbb{Z}$,

$$\sum_{\kappa=-2-l}^{2+l} A_{l,\kappa}(z;\nu) \beta^*_{l,j}(z;\nu+\kappa) = 0 \tag{67}$$

for $z \in \mathbb{C}, l = 0, 1, 2,\ j \in [0, 1 + l] \cap \mathbb{Z},\ \nu \in \mathbb{Z}$. So,

$$y(\nu) = f_{l,2+2j}(z;\nu) \text{ for } |z| > 1,\ l = 0, 1, 2,\ j \in [0, 1 + l] \cap \mathbb{Z},\ \nu \in \mathbb{Z},$$

$$y(\nu) = \alpha^*_{l,i}(z;\nu) \text{ for } z \in \mathbb{C}, l = 0, 1, 2,\ i \in [1, 2 + l] \cap \mathbb{Z},\ \nu \in \mathbb{Z}$$

and

$$y(\nu) = \beta^*_{l,j}(z;\nu) \text{ for } z \in \mathbb{C}, l = 0, 1, 2,\ j \in [0, 1 + l] \cap \mathbb{Z},\ \nu \in \mathbb{Z}$$

satisfy to difference equation

$$\sum_{\kappa=-2-l}^{2+l} A_{l,\kappa}(z;\nu) y(z;\nu+\kappa) = 0. \tag{68}$$

§7.2. Proof of the Lemma 7.0.

It is sufficient to prove that

(69) $$A_0^{(0,\kappa)}(z;\nu) = 128(z-1)^3(2\nu+1)a_{0,\kappa}(z;\nu)$$

for $\kappa = -2,\ -1,\ 0,\ 1,\ 2$. Let $q_{l,i,k}(z;\nu;\nu+\kappa)$ stands in $Q_l(z;\nu;\nu+\kappa)$ on the intetrsection of its i-th row and k-th column, where

$l = 0, 1, 2,\ \kappa = -2-l,\ -1,\ 0,\ 1,\ 2+l,\ i = 1,\ ...,\ 4+2l$ and $k = 1,\ ...,\ 4+2l$.

Let, as before $\mu = \nu(\nu+1), \tau = \nu + \frac{1}{2}$. In view of (140) – (145) in [58],

(70) $$q_{0,1,1}(z;\nu,\nu-1) = a^*_{0,1,1}(z;\nu) = \nu^3 + z(16\nu^3 + 12\nu^2) =$$
$$-\frac{1}{2} - \frac{3}{2}\mu - 2z - 12z\mu + \tau(1+\mu+4z+16z\mu),$$

(71)
$$q_{0,1,1}(z;\nu,\nu+1) = a^*_{0,1,1}(z;-\nu-1) = -\nu^3 - 3\nu^2 - 3\nu - 1 +$$
$$z(-16\nu^3 - 36\nu^2 - 24\nu - 4) =$$
$$-\frac{1}{2} - \frac{3}{2}\mu - 2z - 12z\mu - \tau(1+\mu+4z+16z\mu) =$$
$$(a^*_{0,1,1}(z;\nu))^\omega = (q_{0,1,1}(z;\nu,\nu-1)^\omega = (q_{0,1,1}(z;\nu,\nu-1))^\omega,$$

(72) $$q_{0,1,2}(z;\nu,\nu-1) = a^*_{0,1,2}(z;\nu) = -4\nu^2 + z(-20\nu^2 - 24\nu) =$$
$$-2 - 4\mu + 2z - 20z\mu + t(4-4z)$$

(73) $$q_{0,1,2}(z;\nu,\nu+1) = a^*_{0,1,2}(z;-\nu-1) =$$
$$-4\nu^2 - 8\nu - 4 + z(-20\nu^2 - 16\nu + 4) = -2 - 4\mu + 2z - 20z\mu - t(4-4z) =$$
$$(a^*_{0,1,2}(z;\nu))^\omega = (q_{0,1,2}(z;\nu,\nu-1))^\omega,$$

(74) $$q_{0,1,3}(z;\nu,\nu-1) = a^*_{0,1,3}(z;\nu) = 8\nu + z(-8\nu+12) =$$
$$8\tau - 4 + z(-8\tau+16) = -4 + 16z + \tau(8-8z),$$

(75) $$q_{0,1,3}(z;\nu,\nu+1) = a^*_{0,1,3}(z;-\nu-1) =$$
$$-8\nu - 8 + z(8\nu+20) = -8\tau - 4 + z(8\tau+16) =$$
$$-4 + 16z - \tau(8-8z) = (a^*_{0,1,3}(z;\nu))^\omega =$$
$$(q_{0,1,3}(z;\nu,\nu-1))^\omega,$$

(76) $$q_{0,1,4}(z;\nu,\nu-1) = a^*_{0,1,4}(z;\nu) = -12 + 12z =$$
$$a^*_{0,1,4}(z;-\nu-1) = q_{0,1,4}(z;\nu,\nu+1) =$$
$$(a^*_{0,1,4}(z;\nu))^\omega = (q_{0,1,4}(z;\nu,\nu-1))^\omega,$$

$$a^*_{0,2,1}(z;\nu) = z(-12\nu^4 - 8\nu^3), \tag{77}$$

$$a^*_{0,2,2}(z;\nu) = \nu^3 + z(16\nu^3 + 16\nu^2), \tag{78}$$

$$a^*_{0,2,3}(z;\nu) = -4\nu^2 + z(4\nu^2 - 8\nu), \tag{79}$$

$$a^*_{0,2,4}(z;\nu) = 8\nu - 8z\nu, \tag{80}$$

$$a^*_{0,3,1}(z;\nu) = z(8\nu^5 + 4\nu^4), \tag{81}$$

$$a^*_{0,3,2}(z;\nu) = z(-12\nu^4 - 8\nu^3), \tag{82}$$

$$a^*_{0,3,3}(z;\nu) = \nu^3 + 4z\nu^2, \tag{83}$$

$$a^*_{0,3,4}(z;\nu) = -4\nu^2 + 4z\nu^2, \tag{84}$$

$$a^*_{0,4,1}(z;\nu) = -4z\nu^6, \tag{85}$$

$$a^*_{0,4,2}(z;\nu) = 8z\nu^5, \tag{86}$$

$$a^*_{0,4,3}(z;\nu) = -4z\nu^4, \tag{87}$$

$$a^*_{0,4,4}(z;\nu) = \nu^3, \tag{88}$$

$$\begin{aligned} a^*_{0,1,1}(z;\nu-1) = \nu^3 - 3\nu^2 + 3\nu - 1 + \\ z(16\nu^3 - 36\nu^2 + 24\nu - 4), \end{aligned} \tag{89}$$

$$a^*_{0,1,2}(z;\nu-1) = -4\nu^2 + 8\nu - 4 + z(-20\nu^2 + 16\nu + 4), \tag{90}$$

$$a^*_{0,1,3}(z;\nu-1) = 8\nu - 8 + z(-8\nu + 20), \tag{91}$$

$$a^*_{0,1,4}(z;\nu-1) = -12 + 12z, \tag{92}$$

$$a^*_{0,2,1}(z;\nu-1) = z(-12\nu^4 + 40\nu^3 - 48\nu^2 + 24\nu - 4), \tag{93}$$

$$\begin{aligned} a^*_{0,2,2}(z;\nu-1) = \nu^3 - 3\nu^2 + 3\nu - 1 + \\ z(16\nu^3 - 32\nu^2 + 16\nu), \end{aligned} \tag{94}$$

(95) $$a^*_{0,2,3}(z;\nu-1) = -4\nu^2+8\nu-4+z(4\nu^2-16\nu+12),$$

(96) $$a^*_{0,2,4}(z;\nu) = 8\nu-8-8z(\nu-1),$$

(97) $$a^*_{0,3,1}(z;\nu-1) = z(8\nu^5-36\nu^4+64\nu^3-56\nu^2+24\nu-4),$$

(98) $$a^*_{0,3,2}(z;\nu-1) = z(-12\nu^4+40\nu^3-48\nu^2+24\nu-4),$$

(99) $$a^*_{0,3,3}(z;\nu-1) = \nu^3-3\nu^2+3\nu-1+z(4\nu^2-8\nu+4),$$

(100) $$a^*_{0,3,4}(z;\nu-1) = -4\nu^2+8\nu-4+z(4\nu^2-8\nu+4),$$

(101) $$a^*_{0,4,1}(z;\nu-1) =$$
$$z(-4\nu^6+24\nu^5-60\nu^4+80\nu^3-60\nu^2+24\nu-4),$$

(102)
$$a^*_{0,4,2}(z;\nu-1) = z(8\nu^5-40\nu^4+80\nu^3-80\nu^2+40\nu-8),$$

(103) $$a^*_{0,4,3}(z;\nu-1) = z(-4\nu^4+16\nu^3-24\nu^2+16\nu-4),$$

(104) $$a^*_{0,4,4}(z;\nu-1) = \nu^3-3\nu^2+3\nu-1.$$

(105) $$q_{0,1,1}(z;\nu;\nu-2) = a^*_{0,1,1}(z;\nu-1)a^*_{0,1,1}(z;\nu)+$$
$$a^*_{0,1,2}(z;\nu-1)a^*_{0,2,1}(z;\nu)+a^*_{0,1,3}(z;\nu-1)a^*_{0,3,1}(z;\nu)+a^*_{0,1,4}(z;\nu-1)a^*_{0,4,1}(z;\nu) =$$
$$-\nu^3+3\nu^4-3\nu^5+\nu^6+z(-12\nu^2+48\nu^3-12\nu^4-168\nu^5+192\nu^6)+$$
$$z^2(-48\nu^2+192\nu^3-144\nu^4-288\nu^5+384\nu^6 = 4+144z+144z^2+$$
$$((18+888z+1248z^2)\mu+(15+1272z+2304z^2)\mu^2+(1+192z+384z^2)\mu^3+$$
$$(-8-288z-288z^2)t+(-20-1200z-1920z^2)t\mu+(-6-744z-1440z^2)t\mu^2,$$

(106) $$q_{0,1,1}(z;\nu;\nu+2) = a^*_{0,1,1}(z;-\nu-2)a^*_{0,1,1}(z;-\nu-1)+$$
$$a^*_{0,1,2}(z;-\nu-2)a^*_{0,2,1}(z;-\nu-1)+a^*_{0,1,3}(z;-\nu-2)a^*_{0,3,1}(z;-\nu-1)+$$
$$a^*_{0,1,4}(z;-\nu-2)a^*_{0,4,1}(z;-\nu-1) = 4+144z+144z^2+$$
$$(18+888z+1248z^2)\mu+(15+1272z+2304z^2)\mu^2+(1+192z+384z^2)\mu^3-$$
$$(-8-288z-288z^2)\tau-(-20-1200z-1920z^2)\tau\mu-(-6-744z-1440z^2)\tau\mu^2 =$$
$$(q_{0,1,1}(z;\nu,\nu-2))^\omega$$

(107) $$q_{0,1,2}(z;\nu;\nu-2) = a^*_{0,1,1}(z;\nu-1)a^*_{0,1,2}(z;\nu)+$$

$$a^*_{0,1,2}(z;\nu-1)a^*_{0,2,2}(z;\nu)+a^*_{0,1,3}(z;\nu-1)a^*_{0,3,2}(z;\nu)+$$
$$a^*_{0,1,4}(z;\nu-1)a^*_{0,4,2}(z;\nu)=$$
$$4\nu^2-16\nu^3+20\nu^4-8\nu^5+24z\nu-100z\nu^2+48z\nu^3+292z\nu^4-360\nu^5+$$
$$96z^2\nu-432z^2\nu^2+544z^2\nu^3+96z^2\nu^4-448z^2\nu^5=$$
$$-\nu^3+3\nu^4-3\nu^5+\nu^6-12z\nu^2+48z\nu^3-12z\nu^4-168z\nu^5+192z\nu^6-$$
$$48z^2\nu^2+192z^2\nu^3-144z^2\nu^4-288z^2x^5+384z^2\nu^6=$$
$$24+240z-264z^2+(88+1312z+64z^2)\mu+(40+1192z+1216z^2)\mu^2+$$
$$(-48-480z+528z^2)\tau+(-80-1616z-992z^2)\tau\mu+(-8-360z-448z^2)\tau\mu^2,$$

(108) $$q_{0,1,2}(z;\nu;\nu+2)=a^*_{0,1,1}(z;-\nu-2)a^*_{0,1,2}(z;-\nu-1)+$$
$$a^*_{0,1,2}(z;-\nu-2)a^*_{0,2,2}(z;-\nu-1)+a^*_{0,1,3}(z;-\nu-2)a^*_{0,3,2}(z;-\nu-1)+$$
$$a^*_{0,1,4}(z;-\nu-2)a^*_{0,4,2}(z;-\nu-1)=$$
$$24+240z-264z^2+(88+1312z+64z^2)\mu+(40+1192z+1216z^2)\mu^2-$$
$$(-48-480z-528z^2)\tau-(-80-1616z-992z^2)\tau\mu-(-8-360z-448z^2)\tau\mu^2=$$
$$(q_{0,1,2}(z;\nu,\nu-2))^\omega$$

(109) $$q_{0,1,3}(z;\nu;\nu-2)=a^*_{0,1,1}(z;\nu-1)a^*_{0,1,3}(z;\nu)+$$
$$a^*_{0,1,2}(z;\nu-1)a^*_{0,2,3}(z;\nu)+a^*_{0,1,3}(z;\nu-1)a^*_{0,3,3}(z;\nu)+a^*_{0,1,4}(z;\nu-1)a^*_{0,4,3}(z;\nu)=$$
$$-8\nu+40\nu^2-64\nu^3+32\nu^4-12z+44z\nu+4z\nu^2-200z\nu^3+224z\nu^4-$$
$$48z^2+288z^2\nu-656z^2\nu^2+672z^2\nu^3-256z^2\nu^4=$$
$$72+180z-984z^2+(200+752z-2176z^2)\mu+(32+224z-256z^2)\mu^2+$$
$$(-144-384z+1872z^2)\tau+(-128-648z+1184z^2)\tau\mu,$$

(110) $$q_{0,1,3}(z;\nu;\nu+2)=a^*_{0,1,1}(z;-\nu-2)a^*_{0,1,3}(z;-\nu-1)+$$
$$a^*_{0,1,2}(z;-\nu-2)a^*_{0,2,3}(z;-\nu-1)+a^*_{0,1,3}(z;-\nu-2)a^*_{0,3,3}(z;-\nu-1)+$$
$$a^*_{0,1,4}(z;-\nu-2)a^*_{0,4,3}(z;-\nu-1)=$$
$$144+408\nu+424\nu^2+192\nu^3+32\nu^4+372z+1460z\nu+1948z\nu^2+1096z\nu^3+$$
$$224z\nu^4-1920z^2-4640z^2\nu-4208z^2\nu^2-1696z^2\nu^3-256z^2\nu^4=$$
$$72+180z-984z^2+(200+752z-2176z^2)\mu+(32+224z-256z^2)\mu^2-$$
$$(-144-384z+1872z^2)\tau-(-128-648z+1184z^2)\tau\mu=$$
$$(q_{0,1,3}(z;\nu,\nu-2))^\omega$$

(111) $$q_{0,1,4}(z;\nu;\nu-2)=a^*_{0,1,1}(z;\nu-1)a^*_{0,1,4}(z;\nu)+$$
$$a^*_{0,1,2}(z;\nu-1)a^*_{0,2,4}(z;\nu)+a^*_{0,1,3}(z;\nu-1)a^*_{0,3,4}(z;\nu)+a^*_{0,1,4}(z;\nu-1)a^*_{0,4,4}(z;\nu)=$$
$$12-68\nu+132\nu^2-88\nu^3+36z-188z\nu+348z\nu^2-232z\nu^3-$$

$$48z^2+256z^2\nu-480z^2\nu^2+320z^2\nu^3=$$
$$156+420z-576z^2+(264+696z-960z^2)\mu+(-288-768z+1056z^2)\tau+$$
$$(-88-232z+320z^2)\tau\mu,$$

(112) $$q_{0,1,4}(z;\nu;\nu+2)=a^*_{0,1,1}(z;-\nu-2)a^*_{0,1,4}(z;-\nu-1)+$$
$$a^*_{0,1,2}(z;-\nu-2)a^*_{0,2,3}(z;-\nu-1)+a^*_{0,1,3}(z;-\nu-2)a^*_{0,3,3}(z;-\nu-1)+$$
$$a^*_{0,1,4}(z;-\nu-2)a^*_{0,4,3}(z;-\nu-1)=$$
$$300+804z-1104z^2+(596+1580z-2176z^2)\nu+(396+1044z-1440z^2)\nu^2+$$
$$(88+232z-320z^2)\nu^3=156+420z-576z^2+(264+696z-960z^2)\mu-$$
$$(-288-768z+1056z^2)\tau-(-88-232z+320z^2)\tau\mu=$$
$$(q_{0,1,4}(z;\nu,\nu-2))^\omega.$$

We put $H^{(0)}_0(z;\nu)=H_0(z;\nu)$, where $H_l(z;\nu)$ for $l=0,1,2$ is defined in the section 1, i.e. $H_{0,0}(z;\nu)$ is the 5×4-matrix such that its i-th row coincides with the row $\vec{q}^*_0(z;\nu+i-3)$ for $i=1,\dots,5$. We will transform below the matrix $H^{(0)}_0(z;\nu)$ in other 5×4-matrices $H^{(k)}_0(z;\nu)$, where $k=1,\dots,14$; we denote by $\bar{h}^{(k)}_{0,i-3}(z;\nu)$ and $\vec{h}^{(k)}_{0,j}(z;\nu)$ respectively the i-th row and j-th column of the matrix $H^{(k)}_0(z;\nu)$; we denote by $H^{(k,i-3)}_0(z;\nu)$ the matrix, which one obtains by remowing from $H^{(k)}_0(z;\nu)$ its i-th row. Let $h^{(k)}_{0,i-3,j}(z;\nu)$ denotes the polynomial, which stands in of the matrix $H^{(k)}_0(z;\nu)$ on the intersection of its i-th row and j-th column. We put, finally,

$$M^{(k,\kappa)}_0(z;\nu)=\det(H^{(k,\kappa)}_0(z;\nu)),$$
$$M^{(k,2\kappa)}_0(z;\nu)=\det(H^{(k,2\kappa)}_0(z;\nu))$$

(113) $$A^{(k)}_{0,\kappa}(z;\nu)=\kappa((\nu+(\kappa+1)/2))^3M^{(k,\kappa)}_0(z;\nu))$$

(114) $$A^{(k)}_{0,2\kappa}(z;\nu)=(\nu+\kappa)^3(\nu+\kappa+1)^3M^{(k,2\kappa)}_0(z;\nu)),$$

for $\kappa=-1,\,1$,

(115) $$A^{(k)}_{0,0}(z;\nu)=M^{(k,0)}_0(z;\nu)=\det(H^{(k,0)}_0(z;\nu)),$$

where $k=1,\dots 14$. Clearly,

(116) $$A^{(0)}_{0,\kappa}(z;\nu)=A_{0,\kappa}(z;\nu),$$

where $A_{l,\kappa}(z;\nu)$ are defined (57), (58) for $\kappa=-2-l,\dots,2+l$.

In view of (116) and (59),

(117) $$A^{(0,-\kappa)}_0(z;\nu)=(A^{(0,\kappa)}_0(z;\nu))^\omega$$

for $\kappa=-2,\,-1,\,0,\,1,\,2$.

It follows from (76), (111) , (112) that $4(z-1)$ is a common divisor of all the elements of the last column of the matrix $H^{(0)}_0(z;\nu)$. Let $H^{(1)}_0(z;\nu)$ be the matrix, which one obtains after division of all the elements of last column in the matrix $H_0^{(0)}(z;\nu)$ by $4(z-1)$. Then

$$h^{(1)}_{0,-1,4}(z;\nu) = 3 = h^{(1)}_{0,1,4}(z;\nu) = (h^{(1)}_{0,-1,4}(z;\nu))^{\omega}, \tag{118}$$

$$h^{(1)}_{0,-2,4}(z;\nu) = \tag{119}$$
$$-39 - 144z + (-66 - 240z)\mu + (72 + 264z)\tau + (22 + 80z)\tau\mu =$$
$$-3 + 17\nu - 33\nu^2 + 22\nu^3 + (-12 + 64\nu - 120\nu^2 + 80\nu^3)z,$$

$$h^{(1)}_{0,2,4}(z;\nu) = \tag{120}$$
$$-39 - 144z + (-66 - 240z)\mu - (72 + 264z)\tau - (22 + 80z)\tau\mu =$$
$$(h^{(1)}_{0,-2,4}(z;\nu))^{\omega},$$

$$M_0^{(0,i-3)}(z;\nu) = 4(z-1)M_0^{(1,i-3)}(z;\nu), \tag{121}$$

where $i = 1, ..., 5$, $\kappa = -2, -1, 0, 1, 2$. We denote by $H_0^{(2)}(z;\nu)$ the matrix, which apears after replacement of second column in the matrix $H_0^{(1)}(z;\nu)$ by the sum $\vec{h}^{(1)}_{0,2}(z;\nu) + 2\mu\vec{h}^{(1)}_{0,3}(z;\nu)$. Then

$$h^{(2)}_{0,-1,2}(z;\nu) = 2(z-1) + \tag{122}$$
$$12\mu(z-1) - 4\tau(z-1) - 16\mu\tau(z-1),$$

$$h^{(2)}_{0,1,2}(z;\nu) = 2(z-1) + 12\mu(z-1) + 4\tau(z-1) + \tag{123}$$
$$16\mu\tau(z-1) = (h^{(2)}_{0,-1,2}(z;\nu))^{\omega},$$

$$h^{(2)}_{0,-2,2}(z;\nu) = \tag{124}$$
$$24 + 240z - 264z^2 + (232 + 1672z - 1904z^2)\mu +$$
$$(440 + 2696z - 3136z^2)\mu^2 + (64 + 448z - 512z^2)\mu^3 +$$
$$(-48 - 480z + 528z^2)\tau + (-368 - 2384z + 2752z^2)\mu\tau +$$
$$(-264 - 1656z + 1920z^2)\mu^2\tau,$$

$$h^{(2)}_{0,2,2}(z;\nu) = \tag{125}$$
$$24 + 240z - 264z^2 + (232 + 1672z - 1904z^2)\mu +$$
$$(440 + 2696z - 3136z^2)\mu^2 + (64 + 448z - 512z^2)\mu^3 - (-48 - 480z + 528z^2)\tau -$$
$$(-368 - 2384z + 2752z^2)\mu\tau - (-264 - 1656z + 1920z^2)\mu^2\tau =$$
$$(h^{(2)}_{0,-2,2}(z;\nu))^{\omega},$$

(126) $$M_0^{(0,i-3)}(z;\nu) = 4(z-1)M_0^{(2,i-3)}(z;\nu),$$

where $i = 1, ..., 5$, $\kappa = -2, -1, 0, 1, 2$. In view of (122) – (125), $2(z-1)$ is a common divisor of all the elements of the second column of the matrix $H_0^{(2)}(z;\nu)$. We denote by $H_0^{(3)}(z;\nu)$ the matrix, which one obtains after division of all the elements of second column in $H_0^{(2)}(z;\nu)$ by $2(z-1)$. Then

(127) $$h_{0,-1,2}^{(3)}(z;\nu) = 1 + 6\mu(z-1) - 2\tau - 8\mu\tau,$$

(128) $$h_{0,1,2}^{(3)}(z;\nu) = 1 + 6\mu + 2\tau(z-1) + 8\mu\tau(z-1) = (h_{0,-1,2}^{(3)}(z;\nu))^{\omega},$$

(129) $$h_{0,-2,2}^{(3)}(z;\nu) = -12 - 132z + (-116 - 952z)\mu + (-220 - 1568z)\mu^2 + (-32 - 256z)\mu^3 + (24 + 264z)\tau + (184 + 1376z)\mu\tau + (132 + 960z)\mu^2\tau,$$

(130) $$h_{0,2,2}^{(3)}(z;\nu) = -12 - 132z^2 + (-116 - 952z)\mu + (-220 - 1568z)\mu^2 + (-32 - 256z)\mu^3 - (24 + 264z)\tau - (184 + 1376z)\mu\tau - (132 + 960z)\mu^2\tau = (h_{0,-2,2}^{(2)}(z;\nu))^{\omega},$$

(131) $$M_0^{(0,i-3)}(z;\nu) = 8(z-1)^2 M_0^{(3,i-3)}(z;\nu),$$

where $i = 1, ..., 5$, $\kappa = -2, -1, 0, 1, 2$. Let $H_0^{(4)}(z;\nu)$ be a result of replacement of third column in the matrix $H_0^{(3)}(z;\nu)$ by $\vec{h}_{0,3}^{(2)}(z;\nu) - 4z\mu\vec{h}_{0,3}^{(2)}(z;\nu)$. Then

(132) $$h_{0,-1,3}^{(4)}(z;\nu) = 4z - 8z\tau - 4 + 8\tau,$$

(133) $$h_{0,1,3}^{(4)}(z;\nu) = 4z + 8z\tau - 4 - 8\tau = (h_{0,-1,3}^{(4)}(z;\nu))^{\omega}$$

(134) $$h_{0,-2,3}^{(4)}(z;\nu) = 72 + 336z - 408z^2 + (200 + 1016z - 1216z^2)\mu + (32 + 224z - 256z^2)\mu^2 + (-144 - 672z + 816z^2)\tau + (-128 - 736z + 864z^2)\mu\tau,$$

(135) $$h_{0,2,3}^{(4)}(z;\nu) = 72 + 336z - 408z^2 + (200 + 1016z - 1216z^2)\mu + (32 + 224z - 256z^2)\mu^2 - (-144 - 672z + 816z^2)\tau - (-128 - 736z + 864z^2)\mu\tau = (h_{0,-2,3}^{(4)}(z;\nu))^{\omega},$$

$$M_0^{(0,i-3)}(z;\nu) = 8(z-1)^2 M_0^{(4,i-3)}(z;\nu), \tag{136}$$

where $i = 1, ..., 5$, $\kappa = -2, -1, 0, 1, 2$. In view of (132) – (135), $4(z-1)$ is a common divisor of all the elements of the third column in $H_0^{(4)}(z;\nu)$. We denote $H_0^{(5)}(z;\nu)$ be the matrix, which one obtains after division of all the elements of third column in $H_0^{(4)}(z;\nu)$ by $4(z-1)$. Then

$$h_{0,-1,3}^{(5)}(z;\nu) = 1 - 2\tau = -2\nu, \tag{137}$$

$$h_{0,1,3}^{(5)}(z;\nu) = 1 + 2\tau = (h_{0,-1,3}^{(5)}(z;\nu))^{\omega} = 2\nu + 2 \tag{138}$$

$$h_{0,-2,3}^{(5)}(z;\nu) = -18 - 102z + (-50 - 304z)\mu + \tag{139}$$

$$(-8z - 64z)\mu^2 + (36 + 204z)\tau + (32 + 216z)\mu\tau =$$

$$2\nu - 10nu^2 + 16\nu^3 - 8\nu^4 + (8\nu - 44\nu^2 + 88\nu^3 - 64\nu^4)z,$$

$$h_{0,2,3}^{(5)}(z;\nu) = -18 - 102z + (-50 - 304z)\mu + \tag{140}$$

$$(-8z - 64z)\mu^2 - (36 + 204z)\tau - (32 + 216z)\mu\tau = (h_{0,-2,3}^{(5)}(z;\nu))^{\omega},$$

$$M_0^{(0,i-3)}(z;\nu) = 32(z-1)^3 M_0^{(5,i-3)}(z;\nu), \tag{141}$$

where $i = 1, ..., 5$, $\kappa = -2, -1, 0, 1, 2$. In view of (118), (127) and (137)

$$h_{0,-1,2}^{(5)}(z;\nu) - h_{0,-1,3}^{(5)}(z;\nu) = 6\mu - 8\mu\tau,$$

$$h_{0,-1,2}^{(5)}(z;\nu) - h_{0,-1,3}^{(5)}(z;\nu) -$$

$$2\mu h_{0,-1,4}^{(5)}(z;\nu) = -8\mu\tau,$$

$$h_{0,-1,2}^{(5)}(z;\nu) - (1 + 6\mu) h_{0,-1,3}^{(5)}(z;\nu) = 4\mu\tau$$

and, finally,

$$3h_{0,-1,2}^{(5)}(z;\nu) - (3 + 12\mu) h_{0,-1,3}^{(5)}(z;\nu) - \tag{142}$$

$$2\mu h_{0,-1,4}^{(5)}(z;\nu) = 0.$$

We denote by $H_0^{(6)}(z;\nu)$ the matrix, which apears after replacement of the second column in the matrix $H_0^{(5)}(z;\nu)$ by

$$3\vec{h}_{0,2}^{(5)}(z;\nu) - (3 + 12\mu)\vec{h}_{0,3}^{(5)}(z;\nu) - 2\mu\vec{h}_{0,4}^{(5)}(z;\nu).$$

Then, in view of (142), (29), (30),

$$h_{0,-1,2}^{(6)}(z;\nu) = 0 = h_{0,1,2}^{(6)}(z;\nu) = (h_{0,-1,2}^{(6)}(z;\nu))^{\omega} \tag{143}$$

$$h_{0,-2,2}^{(6)}(z;\nu) = \tag{144}$$

$$18-90z+(96-432z)\mu+(96-384z)\mu^2+$$
$$(-36+180z)\tau+(-120+504z)\tau\mu+(-32+128z)\tau\mu^2=$$
$$-2(3\omega_2(z;\nu(\nu+1))+(2\nu+1)\omega_{-2}(z;\nu(\nu+1))=$$
$$-72-336\mu-288\mu^2+(144+384\mu+96\mu^2)\tau+$$
$$(z-1)(-90-432\mu-384\mu^2+\tau(180+504\mu+128\mu^2)=$$
$$-4\nu^2+8\nu^3+16\nu^4-32\nu^5+(4\nu^2-8\nu^3-64\nu^4+128\nu^5)z=$$
$$96nu^5-48\nu^4+(128nu^5-64\nu^4-8\nu^3+4\nu^2)(z-1)=$$
$$4(2\nu-1)\nu^2(12\nu^2+(16\nu^2-1)(z-1))=$$
$$2\nu^2(2\nu-1)h^{\vee 1}_{1,1}(z;\nu),$$

where

(145) $$h^{\vee 1}_{1,1}(z;\nu)=2-8\nu^2-2z+32\nu^2z,$$

(146) $$h^{(6)}_{0,2,2}(z;\nu)=$$
$$18-90z+(90-432z)\mu+(96-384z)\mu^2-$$
$$(-36+180z)\tau-(-120+504z)\tau\mu-(-32+128z)\tau\mu^2=$$
$$(h^{(6)}_{0,-2,2}(z;\nu))^{\omega}=$$
$$-2(3\omega_2(z;\nu(\nu+1))-(2\nu+1)\omega_{-2}(z;\nu(\nu+1));$$

(147) $$M_0^{(0,3-i)}(z;\nu)=\frac{32}{3}(z-1)^3M_0^{(6,i-3)}(z;\nu),$$

where $i=1,\ ...,\ 5$, $\kappa=-2,\ -1,\ 0,\ 1,\ 2$. Clearly,

$$M_0^{(6,2)}(z;\nu)=\det\begin{pmatrix} h^{(6)}_{0,-2,2}(z;\nu) & h^{(6)}_{0,-2,3}(z;\nu) & h^{(6)}_{0,-2,4}(z;\nu)\\ 0 & 1-2\tau & 3\\ 0 & 1+2\tau & 3\end{pmatrix}=$$
$$-12h^{(6)}_{0,-2,2}\tau,$$

and, in view of (147),

(148) $$M_0^{(0,2)}(z;\nu)=-128(z-1)^3h^{(6)}_{0,-2,2}\tau.$$

It follows from (114), (148), (35) and (144) that

(149) $$A_0^{(0,2)}(z;\nu)=(\nu+1)^3(\nu+2)^3M_0^{(0,2)}=$$
$$128(z-1)^3(2\nu+1)(\nu+1)^3(\nu+2)^3\times$$
$$((2\nu+1)\omega_{-2}(z;\nu(\nu+1))+3\omega_2(z;\nu(\nu+1)))=$$
$$128(z-1)^3(2\nu+1)a_{0,2}(z;\nu)=$$
$$-256(z-1)^3(\nu+1)^3(\nu+2)^3\nu^2(12\nu^2+(16\nu^2-1)(z-1))(4\nu^2-1)=$$
$$-3072(z-1)^3(2\nu+1)(\nu+1)^3(\nu+2)^3\nu^4(2\nu-1)-$$

$$-256(z-1)^4(2\nu+1)(\nu+1)^3(\nu+2)^3\nu^2)(4\nu+1)(4\nu-1)(2\nu-1).$$

Therefore, according to the Lemma 7.1 and (34) – (35),

$$A_0^{(0,-2)}(z;\nu)=(A_0^{(0,2)}(z;\nu))^{\omega}= \tag{150}$$

$$128(z-1)^3(2\nu+1)(\nu+1)^3(\nu+2)^3\times$$

$$((2\nu+1)\omega_{-2}(z;\nu(\nu+1))-3\omega_2(z;\nu(\nu+1)))=$$

$$128(z-1)^3(2\nu+1)a_{0,-2}(z;\nu)=$$

$$-3072(z-1)^3(2\nu+1)\nu^3(\nu-1)^3(\nu+1)^4(2\nu+3)-$$

$$256(z-1)^4(2\nu+1)\nu^3(\nu-1)^3(\nu+1)^2(4\nu+3)(4\nu+5)(2\nu+3).$$

In view of (149) and (150), the equality (69) is proved for $\kappa=\pm 2$.

We denote by $H_0^{(7)}(z;\nu)$ the matrix, which apears after replacement of the last row in the matrix $H_0^{(6)}(z;\nu)$ by

$$\frac{1}{4\tau}(\bar{h}_{0,2}^{(6)}(z;\nu)-\bar{h}_{0,-2}^{(6)}(z;\nu)).$$

Then, in view of (144), (146), (139), (140), (119), (120),

$$h_{0,2,2}^{(7)}(z;\nu)= \tag{151}$$

$$18-90z+60\mu-252z\mu+16\mu^2-64\mu^2z=$$

$$18+60\nu+76\nu^2+32\nu^3+16\nu^4+(-90-252\nu-316\nu^2-128\nu^3-64\nu^4)z=$$

$$2h_{2,1}^{\vee}(z;\nu),$$

where

$$h_{2,1}^{\vee}(z;\nu)=9+30\nu+38\nu^2+16\nu^3+8\nu^4+ \tag{152}$$

$$(-45-126\nu-158\nu^2-64\nu^3-32\nu^4)z,$$

$$h_{0,2,3}^{(7)}(z;\nu)=-18-102z-16\mu-108z\mu= \tag{153}$$

$$-18-16\nu-16\nu^2+(-102-108\nu-108\nu^2)z,$$

$$h_{0,2,4}^{(7)}(z;\nu)=-36-11\mu-132z-40\mu z= \tag{154}$$

$$-36-11\nu-11\nu^2-(132+40\nu+40\nu^2)z,$$

$$M_0^{(6,1)}(z;\nu)=4\tau M_0^{(7,1)}(z;\nu), \tag{155}$$

We denote by $H_0^{(8)}(z;\nu)$ the matrix, which apears after devision of the second row in the matrix $H_0^{(7)}(z;\nu)$ by -2ν and multiplication of the last column of the obtained matrix by -2ν. Then, in view of (119) and (154),

$$h_{0,-2,4}^{(8)}(z;\nu)= \tag{156}$$

$$6\nu-34\nu^2+66\nu^3-44\nu^4+(24\nu-128\nu^2+240\nu^3-160\nu^4)z,$$

(157) $$h^{(8)}_{0,2,4}(z;\nu) = -36 - 11\mu + (-132z + 140\mu z =$$

$$72\nu + 22\nu^2 + 22\nu^3 + (264\nu + 80\nu^2 + 80\nu^3)z,$$

(158) $$M_0^{(7,1)}(z;\nu) = M_0^{(8,1)}(z;\nu).$$

We denote by $H_0^{(9)}(z;\nu)$ the matrix, which apears after replacement of the last column in the matrix $H_0^{(8)}(z;\nu)$ by $\vec{h}^{(8)}_{0,4}(z;\nu) - 3\vec{h}^{(8)}_{0,3}$.

Then, in view of (156), (139), (157), (153),

(159) $$h^{(9)}_{0,-2,4}(z;\nu) = -4\nu^2 + 18\nu^3 - 20\nu^4 +$$

$$(4\nu^2 - 24\nu^3 + 32\nu^4)z = 2\nu^2(2nu - 1)h^{\vee 1}_{1,2}(z;\nu),$$

where

(160) $$h^{\vee 1}_{1,2}(z;\nu) = -5\nu + 2 + (8\nu - 2)z,$$

(161) $$h^{(9)}_{0,2,4}(z;\nu) = 54 + 120\nu + 70\nu^2 + 22\nu^3 +$$

$$(306 + 588\nu + 404\nu^2 + 80\nu^3)z = 2h^{\vee 1}_{2,2}(z;\nu),$$

where

(162) $$h^{\vee 1}_{2,2}(z;\nu) =$$

$$27 + 60\nu + 35\nu^2 + 11\nu^3 + (153 + 294\nu + 202\nu^2 + 40\nu^3)z,$$

(163) $$M_0^{(8,1)}(z;\nu) = M_0^{(9,1)}(z;\nu).$$

Since

$$h^{(9)}_{0,0,2}(z;\nu) = h^{(9)}_{0,0,3}(z;\nu) = h^{(9)}_{0,0,4}(z;\nu) = 0,$$

$$h^{(9)}_{0,-1,2}(z;\nu) = h^{(9)}_{0,-1,4}(z;\nu) = 0,$$

$$h^{(9)}_{0,0,1}(z;\nu) = h^{(9)}_{0,-1,3}(z;\nu) = 1,$$

it follows from (144), (145), (159), (160), (151), (152), (161), (162) that

(164) $$M_0^{(8,1)}(z;\nu) = 4\nu^2(2\nu - 1)\det(H^{\vee 1}),$$

where

$$H^{\vee 1} = \begin{pmatrix} h^{\vee 1}_{1,1}(z;\nu) & h^{\vee 1}_{1,2}(z;\nu) \\ h^{\vee 1}_{2,1}(z;\nu) & h^{\vee 1}_{2,2}(z;\nu) \end{pmatrix}.$$

Further we have

(165) $$h^{\vee 1}_{1,1}(z;\nu) - (4\nu + 1)h^{\vee 1}_{1,2}(z;\nu) =$$

$$12\nu^2 - 3\nu = 3h^{\vee 2}_{1,1}(z;\nu),$$

(166) $$h^{\vee 1}_{2,1}(z;\nu) - (4\nu + 1)h^{\vee 1}_{2,2}(z;\nu) =$$

$$-18-138\nu-237\nu^2-135\nu^3-36\nu^4-(198+1032\nu+1536\nu^2+912nu^3+192\nu^4)z = 3h^{\vee 2}_{2,1}(z;\nu),$$

where

$$h^{\vee 2}_{1,1}(z;\nu) = \nu h^{\vee 3}_{1,1}(z;\nu), \text{ with } h^{\vee 3}_{1,1}(z;\nu) = 4\nu - 1, \tag{167}$$

$$h^{\vee 2}_{2,1}(z;\nu) = -6 - 46\nu - 79\nu^2 - 45\nu^3 - 12\nu^4 - (66 + 344\nu + 512\nu^2 + 304\nu^3 + 64\nu^4)z. \tag{168}$$

In view of (164)

$$M_0^{(8,1)}(z;\nu) = 12\nu(2\nu - 1)\det(H^{\vee 2}), \tag{169}$$

where

$$H^{\vee 2} = \begin{pmatrix} h^{\vee 2}_{1,1}(z;\nu) & h^{\vee 1}_{1,2}(z;\nu)\nu \\ h^{\vee 2}_{2,1}(z;\nu) & h^{\vee 1}_{2,2}(z;\nu)\nu \end{pmatrix}.$$

Further we have

$$h^{\vee 1}_{1,2}(z;\nu)\nu - 2zh^{\vee 2}_{1,1}(z;\nu) = h^{\vee 3}_{1,2}(z;\nu)\nu, \tag{170}$$

$$h^{\vee 1}_{2,2}(z;\nu)\nu - 2zh^{\vee 2}_{2,1}(z;\nu) = h^{\vee 3}_{2,2}(z;\nu)\nu, \tag{171}$$

where

$$h^{\vee 3}_{1,2}(z;\nu) = 2 - 5\nu,\ h^{\vee 3}_{2,2}(z;\nu) = 27\nu + 60\nu^2 + 35\nu^3 + 11\nu^4 + (12 + 245\nu + 452\nu^2 + 292\nu^3 + 64\nu^4)z + (132 + 688\nu + 1024\nu^2 + 608\nu^3 + 128\nu^4)z^2. \tag{172}$$

Therefore, in view of (167), (171), (172), (169),

$$M_0^{(8,1)}(z;\nu) = 12\nu^2(2\nu - 1)\det(H^{\vee 3}), \tag{173}$$

where

$$H^{\vee 3} = \begin{pmatrix} h^{\vee 3}_{1,1}(z;\nu) & h^{\vee 3}_{1,2}(z;\nu) \\ h^{\vee 2}_{2,1}(z;\nu) & h^{\vee 3}_{2,2}(z;\nu) \end{pmatrix}.$$

In view of (167), (172), (168), (173),

$$h^{\vee 4}_{1,2}(z;\nu) := h^{\vee 3}_{1,2}(z;\nu) + h^{\vee 3}_{1,1}(z;\nu) = 1 - \nu, \tag{174}$$

$$h^{\vee 4}_{2,2}(z;\nu) := h^{\vee 3}_{2,2}(z;\nu) + h^{\vee 2}_{2,1}(z;\nu) = -6 - 19nu - 19\nu^2 - 10\nu^3 - \nu^4 - (54 + 99\nu + 60\nu^2 - 12\nu^3)z + (132 + 688\nu + 1024\nu^2 + 608\nu^3 + 128\nu^4)z^2, \tag{175}$$

$$M_0^{(8,1)}(z;\nu) = 12\nu^2(2\nu - 1)\det(H^{\vee 4}), \tag{176}$$

where

$$H^{\vee 4} = \begin{pmatrix} h^{\vee 3}_{1,1}(z;\nu) & h^{\vee 4}_{1,2}(z;\nu) \\ h^{\vee 2}_{2,1}(z;\nu) & h^{\vee 4}_{2,2}(z;\nu) \end{pmatrix}.$$

In view of (167), (174), (168),

$$h^{\vee 5}_{1,1}(z;\nu) := h^{\vee 3}_{1,1}(z;\nu) + 4h^{\vee 4}_{1,2}(z;\nu) = 3, \tag{177}$$

$$h^{\vee 5}_{2,1}(z;\nu) := h^{\vee 2}_{1,1}(z;\nu) + 4h^{\vee 4}_{2,2}(z;\nu) = \tag{178}$$

$$-30 - 122\nu - 155\nu^2 - 85\nu^3 - 16\nu^4 - (282 + 740\nu + 752\nu^2 + 352\nu^3 + 64\nu^4)z +$$
$$(528 + 2752\nu + 4096\nu^2 + 2432\nu^3 + 512\nu^4)z^2,$$

$$M_0^{(8,1)}(z;\nu) = 12\nu^2(2\nu - 1)\det(H^{\vee 5}), \tag{179}$$

where

$$H^{\vee 3} = \begin{pmatrix} 3 & 1-\nu \\ h^{\vee 5}_{1,1}(z;\nu) & h^{\vee 4}_{2,2}(z;\nu) \end{pmatrix}. \tag{180}$$

Therefore, in view of (180), (178), (175),

$$\det(H^{\vee 5}) = 3h^{\vee 4}_{2,2}(z;\nu) + (\nu - 1)h^{\vee 5}_{1,1}(z;\nu) = \tag{181}$$

$$(12 + 35\nu - 24\nu^2 - 100\nu^3 - 72\nu^4 - 16\nu^5 +$$
$$(120 + 161\nu - 168\nu^2 - 436nu^3 - 288nu^4 - 64\nu^5)z +$$
$$+(-132 - 160\nu + 1728\nu^2 + 3488\nu^3 + 2304\nu^4 + 512\nu^5)z^2) =$$
$$(2\nu + 3)(4 + 9\nu - 14\nu^2 - 24\nu^3 - 8\nu^4) +$$
$$(2\nu + 3)(40 + 27\nu - 74\nu^2 - 96\nu^3 - 32\nu^4)z +$$
$$(2\nu + 3)(-44 - 24\nu + 592\nu^2 + 768\nu^3 + 256\nu^4)z^2 =$$
$$12\nu(2\nu + 3)(1 + 42\nu + 54\nu^2 + 18\nu^3) + (z - 1)(2\nu + 3)\times$$
$$(-4 + 3\nu + 518\nu^2 + 672\nu^3 + 224\nu^4 + (-44 - 24\nu + 592\nu^2 + 768\nu^3 + 256\nu^4)z).$$

In view of (179), (180), (31), (32), (37),

$$M_0^{(8,1)}(z;\nu) = 6(4\nu^3 - 2\nu^2)\det(H^{\vee 5}) = \tag{182}$$

$$-6(\omega_1(z;\nu(\nu+1)) + (2\nu + 1)\omega_{-1}(z;\nu(\nu+1))).$$

In view of (179), (182) (158), (155), (147),

$$M_0^{(0,1)}(z;\nu) = \frac{32}{3}(z-1)^3 4\tau M_0^{(8,1)}(z;\nu) = \tag{183}$$

$$-128(z-1)^3(2\nu+1)((\omega_1(z;\nu(\nu+1)) + (2\nu+1)\omega_{-1}(z;\nu(\nu+1))) =$$
$$256(z-1)^3(2\nu+1)\nu^2(2\nu-1)\det(H^{\vee 5}) =$$
$$3072(z-1)^3(2\nu+1)\nu^3(2\nu-1)(2\nu+3)(1 + 42\nu + 54\nu^2 + 18\nu^3) +$$
$$256(z-1)^4(2\nu+1)(2\nu+3)\times$$

$(-4+3\nu+518\nu^2+672\nu^3+224\nu^4+(-44-24\nu+592\nu^2+768\nu^3+256\nu^4)z).$

In wiew of (113), (183), (182),(183),

$$A_0^{(0,1)}(z;\nu) = (\nu+1)^3 M_0^{(0,1)})(z;\nu)) = \tag{184}$$

$$-128(z-1)^3(2\nu+1)(\nu+1)^3((\omega_1(z;\nu(\nu+1)) + (2\nu+1)\omega_{-1}(z;\nu(\nu+1))) =$$
$$128(z-1)^3(2\nu+1)a_{0,1}(z;\nu) =$$
$$3072(z-1)^3(2\nu+1)(\nu+1)^3\nu^3(2\nu-1)(2\nu+3)(1+42\nu+54\nu^2+18\nu^3)+$$
$$256(z-1)^4(2\nu+1)(2\nu+3)(\nu+1)^3\times$$
$$(-4+3\nu+518\nu^2+672\nu^3+224\nu^4+(-44-24\nu+592\nu^2+768\nu^3+256\nu^4)z) =$$
$$3072(z-1)^3(2\nu+1)\mu^3(4\mu-3)(27\mu-2+(9\mu+3)(2\nu+1))+$$
$$128(z-1)^4(2\nu+1)(2\nu+3)(\nu+1)^3\times$$
$$((448+512z)\mu^2-(84+96z)\mu+59+16z+((224+256z)\mu-67-104z)(2\nu+1)).$$

Therefore, according to the Lemma 7.1, (36) – (37),

$$A_0^{(0,-1)}(z;\nu) = (A_0^{(0,1)}(z;\nu))^{\omega} = \tag{185}$$

$$-128(z-1)^3(2\nu+1)\nu^3((\omega_1(z;\nu(\nu+1)) - (2\nu+1)\omega_{-1}(z;\nu(\nu+1))) =$$
$$128(z-1)^3(2\nu+1)a_{0,-1}(z;\nu) =$$
$$-3072(z-1)^3(2\nu+1)\mu^3(4\mu-3)(27\mu-2-(9\mu+3)(2\nu+1))-$$
$$128(z-1)^4(2\nu+1)(2\nu-1)\nu^3\times$$
$$((448+512z)\mu^2-(84+96z)\mu+59+16z-((224+256z)\mu-67-104z)(2\nu+1)) =$$
$$+3072(z-1)^3(2\nu+1)(\nu+1)^3\nu^3(2\nu-1)(2\nu+3)(5-12\nu+18\nu^3)-$$
$$128(z-1)^4(2\nu+1)(2\nu-1)\nu^3\times$$
$$((448+512z)\mu^2-(84+96z)\mu+59+16z-((224+256z)\mu-67-104z)(2\nu+1))$$

with $\mu = \nu(\nu+1)$.

In view of (184) and (185), the equality (69) is proved for $\kappa = \pm 1$.

We denote by $H_0^{(10)}(z;\nu)$ the matrix, which apears after sequential replacement of the first, second, fourth and fifth rows in the matrix $H_0^{(6)}(z;\nu)$ respetively by

$$\bar{h}_{0,-2}^{(10)}(z;\nu) = \frac{1}{4\tau}(\bar{h}_{0,2}^{(6)}(z;\nu) - \bar{h}_{0,-2}^{(6)}(z;\nu)),$$

$$\bar{h}_{0,-1}^{(10)}(z;\nu) = \frac{1}{4\tau}(\bar{h}_{0,1}^{(6)}(z;\nu) - \bar{h}_{0,-1}^{(6)}(z;\nu)),$$

$$\bar{h}_{0,1}^{(10)}(z;\nu) = \frac{1}{2}(\bar{h}_{0,1}^{(6)}(z;\nu) + \bar{h}_{0,-1}^{(6)}(z;\nu)) =$$
$$\bar{h}_{0,1}^{(6)}(z;\nu) + 2\tau\bar{h}_{0,-1}^{(10)}(z;\nu))$$

$$\bar{h}_{0,2}^{(10)}(z;\nu) = \frac{1}{2}(\bar{h}_{0,2}^{(6)}(z;\nu) + \bar{h}_{0,-2}^{(6)}(z;\nu)) =$$
$$\bar{h}_{0,2}^{(6)}(z;\nu) + 2\tau\bar{h}_{0,-2}^{(10)}(z;\nu)).$$

Then, in view of (70), (71),

$$h^{(10)}_{0,-1,1}(z;\nu) = -(1+\mu+4z+16z\mu)/2, \tag{186}$$

$$h^{(10)}_{0,1,1}(z;\nu) = -\frac{1}{2} - \frac{3}{2}\mu - 2z - 12z\mu, \tag{187}$$

in view of (143), (70), (71),

$$h^{(10)}_{0,-1,2}(z;\nu) = h^{(10)}_{0,1,2}(z;\nu) = 0, \tag{188}$$

in view of (137), (138),

$$h^{(10)}_{0,-1,3}(z;\nu) = h^{(10)}_{0,1,3}(z;\nu) = 1, \tag{189}$$

in view of (118),

$$h^{(10)}_{0,-1,4}(z;\nu) = 0,\ h^{(10)}_{0,1,4}(z;\nu) = 3, \tag{190}$$

in view of (105), (106),

$$h^{(10)}_{0,-2,1}(z;\nu) = h^{(7)}_{0,2,1}(z;\nu) = 4+144z+144z^2 + (10+600z+960z^2)\mu + (3+372z+720z^2)\mu^2, \tag{191}$$

$$h^{(10)}_{0,2,1}(z;\nu) = 4+144z+144z^2+(18+888z+1248z^2)\mu + (15+1272z+2304z^2)\mu^2 + (1+192z+384z^2)\mu^3, \tag{192}$$

in view of (144), (146), (151),

$$h^{\wedge 1}_{1,2}(z;\nu) := h^{(10)}_{0,-2,2}(z;\nu) = h^{(7)}_{0,2,2}(z;\nu) = 18-90z+60\mu-252z\mu+16\mu^2-64\mu^2 z, \tag{193}$$

$$h^{(10)}_{0,2,2}(z;\nu) = 18-90z+96\mu-432z\mu+96\mu^2-384\mu^2 z, \tag{194}$$

in view of (139), (140), (153),

$$h^{\wedge}_{1,2}(z;\nu) := h^{(10)}_{0,-2,3}(z;\nu) = h^{(7)}_{0,2,3}(z;\nu) = -18-102z-16\mu-108z\mu, \tag{195}$$

$$h^{(10)}_{0,2,3}(z;\nu) = -18-102z-50\mu-304z\mu-8\mu^2-64z\mu^2, \tag{196}$$

in view of (119), (120), (154),

$$h^{(10)}_{0,-2,4}(z;\nu) = h^{(7)}_{0,2,4}(z;\nu) = -36-11\mu-132z-40\mu z, \tag{197}$$

(198) $$h^{(10)}_{0,2,4}(z;\nu) = h^{(7)}_{0,2,4}(z;\nu) = -39 - 144z - 66\mu - 240z\mu,$$

and, cleraly,

(199) $$M^{(6,0)}(z;\nu) = 16\tau^2 M^{(10,0)}(z;\nu) =$$

$$(16\mu + 4)M^{(10,0)}(z;\nu).$$

We denote by $H^{(11)}_0(z;\nu)$ the matrix, which apears after sequential replacement of fourth and fifth rows in the matrix $H^{(10)}_0(z;\nu)$ respetively by

$$\bar{h}^{(11)}_{0,1}(z;\nu) = \bar{h}^{(10)}_{0,1}(z;\nu) - \bar{h}^{(10)}_{0,-1}(z;\nu)),$$

$$\bar{h}^{(11)}_{0,2}(z;\nu) = \frac{1}{\mu}\times$$

$$(\bar{h}^{(10)}_{0,2}(z;\nu) - \bar{h}^{(10)}_{0,-2}(z;\nu) + (4z+1)\bar{h}^{(11)}_{0,1}(z;\nu)).$$

Then, in view of (186), (187), (188), (189), (190),

(200) $$h^{(11)}_{0,1,1}(z;\nu) = -\mu - 4z\mu,$$

(201) $$h^{(11)}_{0,1,2}(z;\nu) = h^{(11)}_{0,1,3}(z;\nu) = 0,\ h^{(11)}_{0,1,4}(z;\nu) = 3,$$

in view of (191), (192), (200),

(202) $$h^{(11)}_{0,2,1}(z;\nu) = 7 + 280z + 272z^2 +$$

$$(12 + 900z + 1584z^2)\mu + (1 + 192z + 384z^2)\mu,$$

in view of (193), (194), (201),

(203) $$h^{\wedge 1}_{2,2}(z;\nu) := h^{(11)}_{0,2,2}(z;\nu) = 36 - 180\mu + 80\mu - 320z\mu,$$

in view of (195), (196), (201),

(204) $$h^{\wedge}_{2,2}(z;\nu) := h^{(11)}_{0,2,3}(z;\nu) = -34 - 196z - 8\mu - 64z\mu,$$

in view of (197), (198), (201),

(205) $$h^{(11)}_{0,2,4}(z;\nu) = -55 + 200z,$$

and, clearly,

(206) $$M^{(10,0)}(z;\nu) = \mu M^{(11,0)}(z;\nu),\ M^{(6,0)}(z;\nu) =$$

$$\mu(16\mu + 4)M^{(11,0)}(z;\nu).$$

We denote by $H^{(12)}_0(z;\nu)$ the matrix, which apears after replacement of the first column in the matrix $H^{(11)}_0(z;\nu)$ by

$$\vec{h}^{(12)}_{0,1}(z;\nu) = \vec{h}^{(11)}_{0,1}(z;\nu) - h^{(11)}_{0,-1,1}(z;\nu)\vec{h}^{(11)}_{0,3}(z;\nu)).$$

Then, in view of (189), (191), (195),

(207) $$h^{(12)}_{0,-2,1}(z;\nu) = h^{(10)}_{0,-2,1}(z;\nu) - h^{(11)}_{0,-1,1}(z;\nu)h^{(10)}_{0,-2,3}(z;\nu) =$$

$$-5 + 57z - 60z^2 - (7 - 319z + 72z^2)\mu - (5 - 190z + 144z^2)\mu^2,$$

in view of (189),

(208) $$h^{(12)}_{0,-1,1}(z;\nu) = h^{(10)}_{0,-1,1}(z;\nu) -$$

$$h^{(11)}_{0,-1,1}(z;\nu)h^{(10)}_{0,-2,3}(z;\nu) = 0,$$

in view of (189), (201), (200),

(209) $$h^{(12)}_{0,1,1}(z;\nu) = h^{(11)}_{0,1,1}(z;\nu) - h^{(11)}_{0,-1,1}(z;\nu)h^{(11)}_{0,1,3}(z;\nu) =$$

$$h^{(11)}_{0,1,1}(z;\nu) = -\mu - 4z\mu,$$

in view of (189), (202), (204),

(210) $$h^{(12)}_{0,2,1}(z;\nu) = h^{(11)}_{0,2,1}(z;\nu) - h^{(11)}_{0,-1,1}(z;\nu)h^{(11)}_{0,2,3}(z;\nu) =$$

$$-10 + 114z - 120z^2 - (9 - 482z + 112z^2)\mu - (3 - 96z + 128z^2)\mu^2,$$

and

(211) $$M^(6,0)(z;\nu) = \mu(16\mu + 4)M^(12,0)(z;\nu).$$

We denote by $H^{(13)}_0(z;\nu)$ the matrix, which apears after replacement of the first column in the matrix $H^{(12)}_0(z;\nu)$ by

$$\vec{h}^{(13)}_{0,1}(z;\nu) = 3\vec{h}^{(12)}_{0,1}(z;\nu) - h^{(12)}_{0,1,1}(z;\nu)\vec{h}^{(12)}_{0,4}(z;\nu)).$$

Then, in view of (209), (207), (197),

(212) $$h^{\wedge 1}_{1,1}(z;\nu) := h^{(13)}_{0,-2,1}(z;\nu) =$$

$$3h^{(12)}_{0,-2,1}(z;\nu) - h^{(12)}_{0,1,1}(z;\nu)h^{(12)}_{0,-2,4}(z;\nu) =$$

$$-15 + 171z - 180z^2 - (57 - 681z + 744z^2)\mu - (26 - 486z + 592z^2)\mu^2,$$

in view of (209), (208), (190),

(213)
$$h^{(13)}_{0,-1,1}(z;\nu) = 3h^{(12)}_{0,-1,1}(z;\nu) - h^{(12)}_{0,1,1}(z;\nu)h^{(10)}_{0,-1,4}(z;\nu) = 0,$$

in view of (209), (208), (190),

(214) $$h^{(13)}_{0,1,1}(z;\nu) = 3h^{(12)}_{0,1,1}(z;\nu) - h^{(12)}_{0,1,1}(z;\nu)h^{(11)}_{0,1,4}(z;\nu) = 0,$$

in view of (209), (210), (205),

(215)
$$h^{\wedge 1}_{2,1}(z;\nu) := h^{(13)}_{0,2,1}(z;\nu) = 3h^{(12)}_{0,2,1}(z;\nu) - h^{(12)}_{0,1,1}(z;\nu)h^{(11)}_{0,2,4}(z;\nu) =$$

$$-30 + 342z - 360z^2 - (82 - 1026z + 1136z^2)\mu - (9 - 288z + 384z^2)\mu^2,$$

and

$$M_0^{(6,0)}(z;\nu) = \frac{1}{3}\mu(16\mu + 4)M_0^{(13,0)}(z;\nu). \tag{216}$$

In view of (213), (214), (188), (189), (190), (201) ,

$$h^{(13)}_{0,-1,1}(z;\nu) = h^{(13)}_{0,1,1}(z;\nu) = 0,$$

$$h^{(13)}_{0,-1,2}(z;\nu) = h^{(10)}_{0,-1,2}(z;\nu) = 0,$$
$$h^{(13)}_{0,1,2}(z;\nu) = h^{(11)}_{0,1,2}(z;\nu) = 0,$$
$$h^{(13)}_{0,-1,3}(z;\nu) = h^{(10)}_{0,-1,3}(z;\nu) = 1,$$
$$h^{(13)}_{0,1,3}(z;\nu) = h^{(11)}_{0,1,3}(z;\nu) = 0$$
$$h^{(13)}_{0,-1,4}(z;\nu) = h^{(10)}_{0,-1,4}(z;\nu) = 0,$$
$$h^{(13)}_{0,1,4}(z;\nu) = h^{(11)}_{0,1,4}(z;\nu) = 3;$$

therefore, in view of (212), (215), (193), (203),

$$M_0^{(13,0)}(z;\nu) = 3\det(H^{\wedge 1}), \tag{217}$$

where

$$H^{\wedge 1} = \begin{pmatrix} h^{\wedge 1}_{1,1}(z;\nu) & h^{\wedge 1}_{1,2}(z;\nu) \\ h^{\wedge 1}_{2,1}(z;\nu) & h^{\wedge 1}_{2,2}(z;\nu) \end{pmatrix}.$$

In view of (212), (215),

$$h^{\wedge 2}_{2,1}(z;\nu) := \tag{218}$$

$$(h^{\wedge 1}_{2,1}(z;\nu) - 2h^{\wedge 1}_{1,1}(z;\nu))/\mu =$$

$$(h^{(13)}_{0,2,1}(z;\nu) - 2h^{(13)}_{0,-2,1}(z;\nu))/\mu =$$

$$32 - 336z + 352z^2 + (43 - 684z + 800z^2)\mu;$$

in view of (193), (203),

$$h^{\wedge 2}_{2,2}(z;\nu) := \tag{219}$$

$$(h^{\wedge 1}_{2,2}(z;\nu) - 2h^{\wedge 1}_{1,2}(z;\nu))/\mu =$$

$$(h^{(11)}_{0,2,2}(z;\nu) - h^{(10)}_{0,-2,2}(z;\nu))/\mu =$$

$$-40 + 184z - (32 - 128z)\mu.$$

In view of (216), (217),

$$M_0^{(6,0)}(z;\nu) = \mu^2(16\mu + 4)\det(H^{\wedge 2}), \tag{220}$$

where

$$H^{\wedge 2} = \begin{pmatrix} h^{\wedge 1}_{1,1}(z;\nu) & h^{\wedge 1}_{1,2}(z;\nu) \\ h^{\wedge 2}_{2,1}(z;\nu) & h^{\wedge 2}_{2,2}(z;\nu) \end{pmatrix}.$$

In view of (212), (218),

(221) $$h^{\wedge 3}_{1,1}(z;\nu) := 4h^{\wedge 1}_{1,1}(z;\nu)+$$
$$4h^{\wedge 1}_{1,1}(z;\nu)+(2\mu+5)h^{\wedge 2}_{2,1}(z;\nu)=$$
$$4h^{(13)}_{0,-2,1}(z;\nu)+(2\mu+5)h^{\wedge 2}_{2,1}(z;\nu)=100-996z-1040z^2+$$
$$(51-1368z+1728z^2)\mu-(18-576z+768z^2)\mu^2;$$

In view of (193), (219),

(222) $$h^{\wedge 3}_{1,2}(z;\nu) :=$$
$$(4h^{\wedge 1}_{1,2}(z;\nu)+(2\mu+5)h^{\wedge 2}_{2,2}(z;\nu))/8=$$
$$(4h^{(10)}_{0,-2,2}(z;\nu)+(2\mu+5)h^{\wedge 2}_{2,2}(z;\nu))/8=-16+70z,$$

(223) $$h^{\wedge 3}_{2,2}(z;\nu) := h^{\wedge 3}_{2,2}(z;\nu))/8=$$
$$(-40+184z-(32-128z)\mu)/5=-5+23z-(4-16z)\mu),$$

In view of (221) – (223), (220),

(224) $$M_0^{(6,0)}(z;\nu)=8\mu^2(4\mu+1)\det(H^{\wedge 3}),$$

where

$$H^{\wedge 3}=\begin{pmatrix} h^{\wedge 3}_{1,1}(z;\nu) & h^{\wedge 3}_{1,2}(z;\nu)\\ h^{\wedge 2}_{2,1}(z;\nu) & h^{\wedge 3}_{2,2}(z;\nu)\end{pmatrix}.$$

In view of (218), (221) – (223),

(225) $$h^{\wedge 4}_{1,1}(z;\nu) :=$$
$$(h^{\wedge 3}_{1,1}(z;\nu)+(1+z)h^{\wedge 2}_{1,2}(z;\nu))/3=$$
$$28-314z+370z^2+(17-456z+576z^2)\mu-(6-192z+256z^2)\mu^2,$$

(226) $$h^{\wedge 4}_{2,1}(z;\nu) :=$$
$$(h^{\wedge 2}_{2,1}(z;\nu)+(1+z)h^{\wedge 3}_{2,2}(z;\nu))/3=$$
$$9-106z+125z^2+(13-224z+272z^2)\mu.$$

In view of (221) – (226), (224),

(227) $$M_0^{(6,0)}(z;\nu)=24\mu^2(4\mu+1)\det(H^{\wedge 4}),$$

where

$$H^{\wedge 4}=\begin{pmatrix} h^{\wedge 3}_{1,1}(z;\nu) & h^{\wedge 3}_{1,2}(z;\nu)\\ h^{\wedge 4}_{2,1}(z;\nu) & h^{\wedge 3}_{2,2}(z;\nu)\end{pmatrix}.$$

In view of (222) – (226), (227),

(228) $$\det(H^{\wedge 4})=h^{\wedge 4}_{1,1}(z;\nu)h^{\wedge 3}_{2,2}(z;\nu)-h^{\wedge 4}_{2,1}(z;\nu)h^{\wedge 3}_{1,2}(z;\nu)=$$
$$4-112z+348z^2-240z^3+(11-119z+160z^2+128z^3)\mu-$$

$$(38-998z+3904z^2-3328z^3)\mu^2+(24-864z+4096z^2-4096z^3)\mu^3.$$

Therefore, in view of (115), (147), (227), (33),(38),

(229) $$A^{(0)}_{0,0}(z;\nu)=M^{(0,0)}_0(z;\nu)=\frac{32}{3}(z-1)^3M^{(6,0)}_0(z;\nu)=$$

$$256\mu^2(z-1)^3(4\mu+1)\det(H^{\wedge 3})=128(z-1)^3(2\nu+1)^2\omega_0(z,\mu)=$$

$$128(z-1)^3(2\nu+1)a_{0,0}(z;\nu)=$$

$$-3072(z-1)^3\mu^3(2\nu+1)^2(70\mu^2-32\mu-15)-$$

$$256(z-1)^4\mu^2(2\nu+1)(240z^2-108z+4)-$$

$$256(z-1)^4\mu^2(2\nu+1)(169+288z+128z^2)\mu-$$

$$256(z-1)^4\mu^2(2\nu+1)(422-576z+3328z^2)\mu^2-$$

$$256(z-1)^4\mu^2(2\nu+1)(864-4096z^2)\mu^3=$$

$$-3072(z-1)^3\nu^3(\nu+1)^3(2\nu+1)^2(70\nu^4+140\nu^3+38\nu^2-32\nu-15)-$$

$$256(z-1)^4\mu^2(2\nu+1)\times$$

$$256(z-1)^4\mu^2(2\nu+1)(240z^2-108z+4)-$$

$$256(z-1)^4\mu^2(2\nu+1)(169+288z+128z^2)\mu-$$

$$256(z-1)^4\mu^2(2\nu+1)(422-576z+3328z^2)\mu^2-$$

$$256(z-1)^4\mu^2(2\nu+1)(864-4096z^2)\mu^3.$$

In view of (229), the equality (69) is proved for $\kappa=0$. The Lemma 7.0 is proved.

§7.3. Properties of the obtained difference equation in the case $l=0$. Comparison with the previous results.

It follows from (29) – (38) that the equivalent to equation (39) difference equation

(230)

$$\sum_{\kappa=-2}^{2}-1/(16(4z-1)(\nu+1/2)^{11})a_{0,\kappa}(z;\nu)y(z;\nu+\kappa)=0,$$

where $|z|\geq 1$, $\nu\in\mathbb{N}_0$, is a difference equution of Poincaré type, and characteristic polynomial of this equation is equal to

(231) $$T_0(z;\lambda)=$$

$$1-4(8z+1)\lambda+(256z^2-192z+6)\lambda^2-4(8z+1)\lambda^3+\lambda^4.$$

In (231) is represented the exact form of difference equation of Poincaré type, existence of which I had proved in [43] (see [43], Lemma 3.1.9), and I proved there (see [43], §2) that η^2 runs trough the roots of of characteristic polynomial of this equation, when η runs trough the roots of the polynomial

(232) $$D^{\wedge}(z;\eta)=(\eta+1)^4-2^4z\eta^2=$$

$$\eta^4 + 4\eta^3 + (6-16z)\eta^2 + 4\eta + 1$$

I want to check this assertion. Let σ_k and $\sigma_k^\wedge$ are k-th elementary symmetric sums of the roots respectively of the polynomials $T_0(z;\lambda)$ and $D^\wedge(z;\eta)$. Then

$$\sigma_1^\wedge = \sigma_3^\wedge = -4, \sigma_2^\wedge = 6 - 16z,$$

$$\sigma_1 = (\sigma_1^\wedge)^2 - 2\sigma_2^\wedge = 16 - 12 + 32z = 4(8z+1),$$

$$\sigma_2 = (\sigma_2^\wedge)^2 - 2\sigma_1^\wedge\sigma_3^\wedge +$$

$$2\sigma_4^\wedge = 36 - 192z + 256z^2 - 32 + 2 = 6 - 192z + 256z^2,$$

$$\sigma_3 = (\sigma_3^\wedge)^2 - 2\sigma_2^\wedge\sigma_4^\wedge =$$

$$16 - 2(6-16z) = 4(8z+1)$$

So, our assertion is checked.

§7.4. Where is hidden one of fundamental solutions in the case $l = 0$, $z = 1$.

According to the Lemma 3.2.1 in [43], for any $\kappa \in \mathbb{N}$ the four sequences

$$\{\alpha^*_{0,i}(-1;\kappa+k)\}_{k=1}^{+\infty}), \{\beta^*_{0,j}(-1;\kappa+k)\}_{k=1}^{+\infty}), \tag{233}$$

where $i = 1, 2$, $j = 0, 1$, compose a linearly independent system over $\mathbb{C}$. But

$$\alpha(0, 1^*(1;\nu) = -Res_{t=\infty}((R(t;\nu))^2) = 0.$$

Where is hidden one of fundamental solutions of the equation (39), if $z = 1$? Let we study this situation more detailed. According to the Lemma 7.0,

$$\sum_{\kappa=-2}^{2} a_{0,\kappa}(z;\nu)\frac{\alpha_{0,1}(z;\nu+\kappa)}{z-1} = 0, \tag{234}$$

where $z \neq 1$. We turn z to 1 now. Then we obtain the equality

$$\sum_{\kappa=-2}^{2} a_{0,\kappa}(1;\nu)\frac{d\alpha^*_{0,1}}{dz}(1;\nu+\kappa) = 0. \tag{235}$$

Therefore $y(\nu) = \frac{d\alpha^*_{0,1}}{dz}(1;\nu)$ is a solution of the equation

$$\sum_{\kappa=-2}^{2} a_{0,\kappa}(1;\nu)y(\nu+\kappa) = 0. \tag{236}$$

In view of (92) and (101) in [58],

$$t(R(t;\nu))^2 = \sum_{i=1}^{2}\left(\sum_{k=0}^{\nu}\alpha_{0,i,k,\nu}\frac{t}{(t+k)^{-i}}\right) = \tag{237}$$

$$\alpha^*_{0,1}(1;\nu) +$$

$$\left(\sum_{k=0}^{\nu}(-k\alpha_{0,1,k,\nu}+\alpha_{0,2,k,\nu})\frac{1}{t+k}\right)-$$

$$\left(\sum_{k=0}^{\nu}(-k\alpha_{0,2,k,\nu})\frac{1}{(t+k)^2}\right);$$

since

$$-Res_{t=\infty}(t(R(t;\nu))^2)=1,$$

it follows that

(238) $$1=\sum_{k=0}^{\nu}(-k\alpha_{0,1,k,\nu}+\alpha_{0,2,k,\nu})=$$

$$\frac{d\alpha_{0,1}}{dz}(1;\nu)+\alpha_{0,2}(1;\nu).$$

Therefore $y(\nu)=1$ is a solution of the equation (236). This allow us to check previous calculations once more. We must check the equality

(239) $$\sum_{\kappa=-2-l}^{2+l}A_{0,\kappa}(1;\nu)=0.$$

According to the Lemma 2.0, (149), (150), (184), (185),

$$a_{0,2}(1;\nu)=-24(\nu+1)^3(\nu+2)^3\nu^4(2\nu-1),$$

$$a_{0,-2}(1;\nu)=-24\nu^3(\nu-1)^3(\nu+1)^4(2\nu+3),$$

$$a_{0,1}(1;\nu)=24(\nu+1)^3\nu^3(2\nu-1)(2\nu+3)(1+42\nu+54\nu^2+18\nu^3),$$

$$a_{0,-1}(1;\nu)=24(\nu+1)^3\nu^3(2\nu-1)(2\nu+3)(5-12\nu+18\nu^3),$$

$$a_{0,0}(1;\nu)=-24\nu^3(\nu+1)^3(2\nu+1)(70\nu^4+140\nu^3+38\nu^2-32\nu-15)$$

We must check the equality

$$P(\nu):=(\nu+2)^3\nu(2\nu-1)+(\nu-1)^3(\nu+1)(2\nu+3)-$$

$$(2\nu-1)(2\nu+3)(1+42\nu+54\nu^2+18\nu^3)-$$

$$(2\nu-1)(2\nu+3)(5-12\nu+18\nu^3)+$$

$$(2\nu+1)(70\nu^4+140\nu^3+38\mu^2-32\mu-15)=$$

$$(\nu+2)^3\nu(2\nu-1)+(\nu-1)^3(\nu+1)(2\nu+3)-$$

$$(2\nu-1)(2\nu+3)(6+30\nu+54\nu^2+36\nu^3)+$$

$$(2\nu+1)(70\nu^4+140\nu^3+38\nu^2-32\nu-15)=0$$

If $P(\nu)\neq 0$, then, clearly, $\deg(P(\nu))\leq 4$, moreover, $P(\nu)\in\mathbb{R}[\nu]$, therefore it is sufficient to check equality $P(\nu)=0$ in the points $\nu=0,1,-1,i$. We have

$$P(0)=-3+3+15-15=0,\ P(1)=27-630+3\times 201=0,$$

$$P(-1)=3+3\times(-6)+15=0,$$

$$P(i) = -(2+i)^4 - 2(1-i)^2(3+2i) - (-7+4i)(-48-6i)+$$
$$(2i+1)(17-172i) = -(-7+24i) + 4i(3+2i) - (360-150i) + 361 - 138i = 0.$$
So, $P(\nu) = 0$. Let

$$P_0(\nu) := \frac{d\alpha^*_{0,2}}{dz}(1;\nu),$$

$$P_1(\nu) := 1,\ P_2(\nu) := \alpha^*_{0,2}(1;\nu), \tag{240}$$

$$P_{3+j}(\nu) := \beta^*_{0,j}(1;\nu) \text{ for } j = 0, 1.$$

Lemma 7.4.1. *For any $\kappa \in \mathbb{N}$ four sequences*

$$\{P_i(1;\kappa + k)\}_{k=1}^{+\infty}, \tag{241}$$

where $i = 1, 2, 3, 4$ compose a linearly independent system over $\mathbb{R}$.

Proof. First we prove that for each $\kappa \in \mathbb{N}$ four sequences (241) compose a linearly independent system over $\mathbb{Q}$. Suppose the contrary. Then there exist $\kappa \in \mathbb{N}$ $a_i \in \mathbb{Z}$, where $i = 1, ..., 4$,such that

$$\sum_{i=1}^{4} |a_i| > 0$$

and

$$\sigma := \sum_{i=1}^{4} a_i P_i(\nu = 0, \tag{242}$$

where $\nu \in \mathbb{N}$, $\nu > \kappa$. We let in (92) in [58]

$$\alpha^{\sim}_{k,i}(\nu) = \alpha^{\sim}_{0,i,k,\nu}$$

Let p is prime number, $p > 5$. According to the Lemma 9 in [39], and, in view of (3.2.39) in [43],

$$\alpha^{\sim}_{0,2}(p) = 1, \alpha^{\sim}_{p,2}(p) \in 4 + p\mathbb{Z}, \tag{243}$$

$$\alpha^{\sim}_{k,2}(p) \in p^2\mathbb{Z},\ k = 1, \dots, p-1. \tag{244}$$

Therefore, according to (101) in [58], and, in view of (252),

$$P_2(p) \in 5 + p\mathbb{Z}. \tag{245}$$

We denote by $ord_p(x)$ additive $p-adic$ valuation on $\mathbb{Q}$ such that $ord_p(p) = 1$. In view of (3.2.35) in [43],

$$ord_p(\alpha^{\sim}_{0,1}(p) + 2/p) \geq 0,\ ord_p(\alpha^{\sim}_{p,1}(p) - 2/p) \geq 0, \tag{246}$$

$$ord_p(\alpha^{\sim}_{k,1}(p)) \geq 1,\ k = 1, \dots, p-1. \tag{247}$$

In view of (102) in (58), (3.2.38) in [43], and in view of (243), (244), (246), (247),

$$ord_p\left(P_{3+\mu}(p)-\sum_{l=0}^{p-1}\sum_{\lambda=1}^{2}\binom{\mu+\lambda-1}{\mu}\frac{\alpha^{\sim}_{p,\lambda}(p)}{(p-l)^{\lambda+\mu}}\right)\geq 1, \tag{248}$$

(249)
$$ord_p\left(\sum_{l=0}^{p-1}\sum_{\lambda=1}^{2}\binom{\mu+\lambda-1}{\mu}\frac{\alpha^{\sim}_{p,\lambda}(p)}{(p-l)^{\lambda+\mu}}-\sum_{\lambda=1}^{2}\binom{\mu+\lambda-1}{\mu}\frac{\alpha^{\sim}_{p,\lambda}(p)}{p^{\lambda+\mu}}\right)\geq -1,$$

(250)
$$ord_p\left(\left(\sum_{\lambda=1}^{2}\binom{\mu+\lambda-1}{\mu}\frac{\alpha^{\sim}_{p,\lambda}(p)}{(p)^{\lambda+\mu}}\right)-\binom{\mu+1}{\mu}\frac{\alpha^{\sim}_{p,2}(p)}{(p)^{2+\mu}}\right)\geq -1-\mu,$$

where $\mu=0,\ 1$. It follows from (240), (245), (248) – (250), that

(251)
$$ord_p(P_1(p))=ord_p(P_2(p))=0, ord_p(P_{3+\mu}(p))=-2-\mu,$$

where $\mu=0,1$. According to (251), if in (242) $a_4\neq 0$, then $ord_p(\sigma)=-3$, if in (242) $a_3\neq 0, a_4=0$ then $ord_p(\sigma)=-2$, if in (242) $a_2\neq 0, a_3=a_4=0$ then, clearly $\lim\limits_{\nu\to\infty}(s)=\infty$. So, for each $\kappa\in\mathbb{N}$ four sequences (241) compose a linearly independent system over $\mathbb{Q}$. Consequently the composed by these sequences $4\times\mathbb{N}$ infinite matrix contain an invertible submatrix. ■

Corollary. *For any $\kappa\in\mathbb{N}$ four sequences*

$$\{P_i(1;\kappa+k)\}_{k=1}^{+\infty}, \tag{252}$$

where $i=0,\ 2,\ 3,\ 4$ compose a linearly independent system over $\mathbb{R}$.

§7.5. Corrections in the previous my papers.

Of course, the (184) in §6.5 of Part 6 must have the form

$$(\nu+1)^3y(1;\nu+1)+\nu^3y(\nu-1)=(17\nu^3+51\nu^2+27\nu+5)y(\nu).$$

instead of

$$(\nu+1)^3y(1;\nu+1)+\nu^3y\nu-1)=(17\nu^3+51\nu^2+27\nu+5)y(\nu)(34\nu^3+85\nu^2.$$

In (3.2.48) in [43] must stand

$$ord_p(P_{1+\mu}(-1,p))=\mu-1,\ ord_p(P_{3+\mu}(-1,p))=-2-\mu,$$

instead of

$$ord_p(P_{1+\mu}(-1,p))=1-\mu,\ ord_p(P_{3+\mu}(-1,p))=2+\mu.$$

References.

[1] R.Apéry, Interpolation des fractions continues
et irrationalite de certaines constantes,
Bulletin de la section des sciences du C.T.H., 1981, No 3, 37 – 53;
[2] F.Beukers, A note on the irrationality of $\zeta(2)$ and $\zeta(3)$,
Bull. London Math. Soc., 1979, 11, 268 – 272;
[3] A.van der Porten, A proof that Euler missed...Apéry's proof of the irrationality of $\zeta(3)$,
Math Intellegencer, 1979, 1, 195 – 203;
[4] W. Maier, Potenzreihen irrationalen Grenzwertes,
J.reine angew. Math., 156, 1927, 93 – 148;
[5] E.M. Nikišin, On irrationality of the values of the functions F(x,s) (in Russian),
Mat.Sb. 109 (1979), 410 – 417;
English transl. in Math. USSR Sb. 37 (1980), 381 – 388;
[6] G.V. Chudnovsky, Pade approximations to the generalized hyper-geometric functions
I,J.Math.Pures Appl., 58, 1979, 445 – 476;
[7] ________________, Transcendental numbers, Number Theory,Carbondale,
Lecture Notes in Math, Springer-Verlag, 1979, 751, 45 – 69;
[8] ________________, Approximations rationelles des logarithmes de nombres rationelles
C.R.Acad.Sc. Paris, Série A, 1979, 288, 607 – 609;
[9] ________________, Formules d'Hermite pour les approximants de Padé de logarithmes
et de fonctions binômes, et mesures d'irrationalité,
C.R.Acad.Sc. Paris, Série A, 1979, t.288, 965 – 967;
[10] _______________,Un systme explicite d'approximants de Padé
pour les fonctions hypérgéometriques généraliesées,
avec applications a l'arithmétique,
C.R.Acad.Sc. Paris, Série A, 1979, t.288, 1001 – 1004;
[11] _______________, Recurrenses defining Rational Approximations
to the irrational numbers, Proceedings
of the Japan Academie, Ser. A, 1982, 58, 129 – 133;
[12] _______________, On the method of Thue-Siegel,
Annals of Mathematics, 117 (1983), 325 – 382;
[13] K.Alladi and M. Robinson, Legendre polinomials and irrationality,
J. Reine Angew.Math., 1980, 318, 137 – 155;
[14] S.Eckmann, Über die lineare Unaqbhangigkeit der Werte gewisser Reihen,
Results in Mathematics, 11, 1987, 7 – 43;
[15] M.Hata, Legendre type polinomials and irrationality mesures,
J. Reine Angew. Math., 1990, 407, 99 – 125;
[16] M.Hata, A new irrationality mesure for $\zeta(3)$,
Acta Arithmetica, XCII, 1 (2001), 47 – 57;
[17] George Rhinn and Carlo Viola, The group structure for $\zeta(3)$.
Acta Arithmetica, XCVII, 3 (2001), 269 – 293;
[18] A.O. Gelfond, Transcendental and algebraic numbers (in Russian),
GIT-TL, Moscow, 1952;
[19] O.Perron, Über die Poincaresche Differenzengleichumg,
Journal für die reine und angewandte mathematik,
1910, 137, 6 – 64;
[20] A.O.Gelfond, Differenzenrechnung (in Russian), 1967, Nauka, Moscow.
[21] A.O.Gelfond and I.M.Kubenskaya, On the theorem of Perron
in the theory of differrence equations (in Russian),
IAN USSR, math. ser., 1953, 17, 2, 83 – 86.
[21] L.A.Gutnik, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$,
Max-Plank-Institut für Mathematik,
Bonn, Preprint Series, 2000, 3, 1 – 13;
[22] ___________, On the Irrationality of Some Quantyties Containing $\zeta(3)$ (in Russian),
Uspekhi Mat. Nauk, 1979, 34, 3(207), 190;
[23] ___________, On the Irrationality of Some Quantities Containing $\zeta(3)$,
Eleven papers translated from the Russian,
American Mathematical Society, 1988, 140, 45 - 56;
[24] ___________, Linear independence over $\mathbb{Q}$ of dilogarithms at rational points
(in Russian), UMN, 37 (1982), 179-180;

english transl. in Russ. Math. surveys 37 (1982), 176-177;
[25] ____________, On a measure of the irrationality of dilogarithms at rational points (in Russian), VINITI, 1984, 4345-84, 1 − 74;
[26] ____________, On the smallness of some linear forms (in Russian), VINITI, 1987, 6279-B87, 1 − 23;
[27] ____________, To the question of the smallness of some linear forms (in Russian), VINITI, 1993, 2413-B93, 1 − 94;
[28] ____________, On linear forms, whose coefficients are logarithms of algebraic numbers (in Russian), VINITI, 1995, 135-B95, 1 − 149;
[29] ____________, On systems of vectors, whose coordinates are linear combinations of logarithms of algebraic numbers with algebraic coefficients (in Russian), VINITI, 1994, 3122-B94, 1 − 158;

[30] ____________, On the linear forms, whose coefficients are $\mathbb{A}$ - linear combinations of logarithms of $\mathbb{A}$ - numbers, VINITI, 1996, 1617-B96, pp. 1 − 23.
[31] ____________, On systems of linear forms, whose coefficients are $\mathbb{A}$ - linear combinations of logarithms of $\mathbb{A}$ - numbers, VINITI, 1996, 2663-B96, pp. 1 − 18.

[32] ____________, On linear forms, whose coefficients are $\mathbb{Q}$-proportional to the number $\log 2$, and the values of $\zeta(s)$ for integer s (in Russian), VINITI, 1996, 3258-B96, 1 − 70;
[33] ____________, The lower estimate for some linear forms, coefficients of which are proportional to the values of $\zeta(s)$ for integer s (in Russian), VINITI, 1997, 3072-B97, 1 − 77;
[34] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2000, 3, 1 − 13;
[35] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version,part 1), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2001, 15, 1 − 20;
[36] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version,part 2), Max-Plank-Institut für Mathematik, Bonn, Preprint Series,2001, 104, 1 − 36;
[37] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version,part 3), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 57, 1 − 33;
[38] ____________, On the rank over $\mathbb{Q}$ of some real matrices (in Russian), VINITI, 1984, 5736-84; 1 − 29;
[39] ____________, On the rank over $\mathbb{Q}$ of some real matrices, Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 27, 1 − 32;
[40] ____________, On the linear forms with coefficients proportional to zeta values(in Russian), VINITI, 1999, 3207-B99; 1 − 71;
[41] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version, part 4), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 142, 1 − 27;
[42] ____________, On the dimension of some linear spaces over finite extension of $\mathbb{Q}$ (part 2), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 107, 1 − 37;

[43] ____________, On the dimension of some linear spaces over $\mathbb{Q}$ (part 3), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 16, 1 − 45.

[44] ____________, On the dimension of some linear spaces over $\mathbb{Q}$, (part 4) Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 73, 1 − 38.

[45] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version, part 5), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 83, 1 − 13.

[46] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version, part 6), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 99, 1 − 33.

[47] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$, Bonner Mathematishe Schriften Nr. 360, Bonn, 2003, 360.

[48] ____________, On the dimension of some linear spaces over $\mathbb{Q}$, (part 5) Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2004, 46, 1 − 42.

[49] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version, part 7), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2004, 88, 1 − 27.

[50] ____________,On the difference equation of Poincaré type (Part 4). Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2004, 129, 1 − 28.

[51] ____________, On the measure of nondiscreteness of some modules, Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2005, 32, 1 − 51.

[52] [52] ____________, On some systems of difference equations. Part 1. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 23, 1 − 37.

[53] ____________, On some systems of difference equations. Part 2. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 49, 1 − 31.

[54] ____________, On some systems of difference equations. Part 3. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 91, 1 − 52.

[55] ____________, On some systems of difference equations. Part 4. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 101, 1 − 49.

[56] ____________, On some systems of difference equations. Part 5. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 115, 1 − 9.

[57] ____________, On the Diophantine approximations of logarithmes in cyclotomic fields. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 147, 1 − 36.

[58] ____________, On some systems of difference equations. Part 6. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2007, 16, 1 − 30.

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