Automorphisms of the Weyl algebra

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I report on a joint work with Alexei Belov-Kanel, which was started about 1,5 years ago, and is not finished yet.

1 Main Conjecture

For integer $n \geq 1$ denote by $A_{n,\mathbb{C}}$ the Weyl algebra of rank n over $\mathbb{C}$

$$\mathbb{C}\langle \hat{x}_1, \ldots, \hat{x}_{2n} \rangle / (\text{ relations } [\hat{x}_i, \hat{x}_j] = \omega_{ij},\ \forall i, j, \quad 1 \leq i, j \leq 2n\ )\ ,$$

where $(\omega_{ij})_{1 \leq i,j \leq 2n}$ is the standard skew-symmetric matrix,

$$\omega_{ij} = \delta_{i,n+j} - \delta_{i+n,j}\ .$$

Algebra $A_{n,\mathbb{C}}$ is isomorphic to the algebra $D(\mathbb{A}^n_{\mathbb{C}})$ of polynomial differential operators in n variables $x_1, \ldots, x_n$:

$$\hat{x}_i \mapsto x_i,\ x_{n+i} \mapsto \partial / \partial x_i,\ \ i = 1, \ldots, n\ .$$

Denote by $P_{n,\mathbb{C}}$ the Poisson algebra over $\mathbb{C}$ which is the usual polynomial algebra $\mathbb{C}[x_1, \ldots, x_{2n}] \simeq \mathcal{O}(\mathbb{A}^{2n}_{\mathbb{C}})$ endowed with the Poisson bracket

$$\{x_i, x_j\} = \omega_{ij},\ \ 1 \leq i, j \leq 2n\ .$$

Conjecture 1 *There exists a canonical isomorphism* $\Phi = \Phi_n$

$$Aut(A_{n,\mathbb{C}}) \simeq Aut(P_{n,\mathbb{C}})$$

between the automorphism group of the Weyl algebra and the group of polynomial symplectomorphisms of $\mathbb{A}^{2n}_{\mathbb{C}}$. *Isomorphism* Φ *is covariant with respect to the natural action of the Galois group* $Aut(\mathbb{C})$.

1.1 First positive evidence: case $n = 1$

The structure of group $Aut(P_{1,\mathbb{C}})$ is known after H.W.E.Young (1942) and W. van der Kulk (1953). This group contains the group $G_1 = SL(2,\mathbb{C}) \ltimes \mathbb{C}^2$ of special affine transformations, and solvable group G_2 of polynomial transformations of the form

$$(x_1, x_2) \mapsto (\lambda x_1 + F(x_2), \lambda^{-1} x_2), \ \ \lambda \in \mathbb{C}^\times, \ \ F \in \mathbb{C}[x] \ .$$

Group $Aut(P_{1,\mathbb{C}})$ is equal to the amalgamated product of G_1 and G_2 over their intersection. L.Makar-Limanov (1984) proved that if one replaces commuting variables x_1, x_2 by noncommuting variables $\hat{x}_1, \hat{x}_2$ in above formulas, one obtains the description of group $Aut(A_{1,\mathbb{C}})$. Hence, in the case $n = 1$ two automorphism groups are isomorphic.

1.2 Negative evidence: Lie algebras are not isomorphic

Groups $Aut(A_{n,\mathbb{C}})$ and $Aut(P_{n,\mathbb{C}})$ are in fact groups of complex points of certain group-like ind-affine schemes $\underline{Aut}(A_{n,\mathbb{C}})$ and $\underline{Aut}(P_{n,\mathbb{C}})$. Namely, an automorphism ϕ of $A_{n,\mathbb{C}}$ such that both ϕ, ϕ^{-1} map all generators $\hat{x}_i$ to expressions of degree bounded by some integer $N \geq 1$, is a solution of certain system of polynomial equations in variables $c_{i,I}, c'_{i,I}$ (coefficients of ϕ and ϕ^{-1}):

$$\phi(\hat{x}_i) = \sum_{I:\,|I|\leq N} c_{i,I} \hat{x}^I, \ \ \phi^{-1}(\hat{x}_i) = \sum_{I:\,|I|\leq N} c'_{i,I} \hat{x}^I \ ,$$

where for multi-index $I = (i_1, \ldots, i_{2n}) \in \mathbb{Z}^{2n}_{\geq 0}$ we define

$$\hat{x}^I := \hat{x}_1^{i_1} \ldots \hat{x}_{2n}^{i_{2n}}, \ \ |I| := i_1 + \cdots + i_{2n} \ .$$

Similarly, one has an ind-scheme $\underline{Aut}(P_{n,\mathbb{C}})$, an inductive limit of a chain of affine schemes of finite type over $\mathbb{C}$. Using dual numbers $\mathbb{C}[t]/(t^2)$ one defines as usual Lie algebras of ind-schemes $\underline{Aut}(A_{n,\mathbb{C}})$ and $\underline{Aut}(P_{n,\mathbb{C}})$. These algebras are derivations $Der(A_{n,\mathbb{C}})$ and $Der(P_{n,\mathbb{C}})$ respectively. It is well-known that all derivations of $A_{n,\mathbb{C}}$ are inner,

$$Der(A_{n,\mathbb{C}}) \simeq A_{n,\mathbb{C}} \,/\, \mathbb{C} \cdot 1_{A_{n,\mathbb{C}}} \ .$$

Similarly, derivations of $P_{n,\mathbb{C}}$ are hamiltonian vector fields,

$$Der(P_{n,\mathbb{C}}) \simeq P_{n,\mathbb{C}} \,/\, \mathbb{C} \cdot 1_{P_{n,\mathbb{C}}} \ .$$

These two Lie algebras are *not* isomorphic: $Der(P_{n,\mathbb{C}})$ contains many non-trivial Lie subalgebras of finite codimension (e.g. vector fields vanishing at some point of $\mathbb{A}^{2n}_{\mathbb{C}}$), while $Der(A_{n,\mathbb{C}})$ does not contain such subalgebras (in a sense it is similar to sl_∞).

Thus, we conclude that the conjectural isomorphism Φ can not be an isomorphism of ind-schemes. In fact, we expect that Φ preserves the filtration of automorphism groups by degree, and it is a constructible continuous map, both in Zariski and usual topology.

1.3 Another positive evidence: tame automorphisms

Symplectic group $Sp(2n,\mathbb{C})$ acts by automorphisms of $A_{n,\mathbb{C}}$ and of $P_{n,\mathbb{C}}$ by linear transformations of generators. Also, for any polynomial $F\in\mathbb{C}[x_1,\dots,x_n]$ we define non-linear transvections

$$T_F^P\in Aut(A_{n,\mathbb{C}}),\ T_F^A\in Aut(P_{n,\mathbb{C}})$$

by formulas

$$T_F^P(x_i)=x_i,\ T_F^P(x_{n+i})=x_{n+i}+\partial_i F(x_1,\dots,x_n),\ 1\le i\le n\,,$$

$$T_F^A(\hat{x}_i)=\hat{x}_i,\ T_F^P(\hat{x}_{n+i})=\hat{x}_{n+i}+\partial_i F(\hat{x}_1,\dots,\hat{x}_n),\ 1\le i\le n\,.$$

The last formual makes sense, as variables $\hat{x}_1,\dots,\hat{x}_n$ commute with each other. Correspondence $F\mapsto T_F^P$ (resp. $F\mapsto T_F^A$) gives a group homomorphism $\mathbb{C}[x_1,\dots,x_n]/\mathbb{C}\cdot 1\to Aut(P_{n,\mathbb{C}})$ (resp. to $Aut(A_{n,\mathbb{C}})$). Automorphisms of $A_{n,\mathbb{C}}$ and of $P_{n,\mathbb{C}}$ generated by $Sp(2n,\mathbb{C})$ and transvections are called *tame*.

We prove in the next section that groups of tame automorphisms of $A_{n,\mathbb{C}}$ and of $P_{n,\mathbb{C}}$ are canonically isomorphic. Symplectic group $Sp(2n,\mathbb{C})$ is generated by Fourier transform

$$x_i\mapsto x_{n+i},\, x_{n+i}\mapsto -x_i;\ \ \hat{x}_i\mapsto\hat{x}_{n+i},\,\hat{x}_{n+i}\mapsto-\hat{x}_i\,,\ \ i=1,\dots,n$$

and by transvections corresponding to quadratic polynomials F. Our assertion follows from the following theorem:

Theorem 1 *For any sequence of polynomials $F_1,\dots,F_k\in\mathbb{C}[x_1,\dots,x_n]/\mathbb{C}\cdot 1$ the composition*

$$Fourier\circ T_{F_1}^A\circ Fourier\circ\dots\circ T_{F_k}^A$$

is identity in $Aut(A_{n,\mathbb{C}})$ if and only if the composition

$$Fourier\circ T_{F_1}^P\circ Fourier\circ\dots\circ T_{F_k}^P$$

is identity in $Aut(P_{n,\mathbb{C}})$.

2 Proof of theorem 1

Notice that in the definition of algebras $A_{n,\mathbb{C}}$ and $P_{n,\mathbb{C}}$ one can replace $\mathbb{C}$ by arbitrary commutative ring.

The only known proof of Theorem 1 is based on considerations in finite characteristic. It is a challenge to find a purely complex proof. In fact, in the case $n=1$, J.Dixmier in 1968 described automorphisms of A_1 in finite characteristic, and L.Makar-Limanov used the result of Dixmier in zero characteristic. Here we follow the same line.

Let us assume that the composition in $Aut(A_{n,\mathbb{C}})$ is equal to identity. We want to prove that the corresponding composition in $Aut(P_{n,\mathbb{C}})$ is also equal to identity.

Let us denote by $R\subset\mathbb{C}$ the subring of $\mathbb{C}$ generated by all coefficients of polynomials $F_1,\dots,F_k$. It is a finitely generated integral domain. Hence for all sufficiently large primes $p \gg 1$ the reduction $R/p := R \otimes \mathbb{Z}/p\mathbb{Z}$ is a non-zero ring. For any prime p Weyl algebra $A_{n,R/p}$ has a large center,

$$Center(A_{n,R/p}) = R/p\,[\hat{x}_1^p, \dots, \hat{x}_{2n}^p] \ .$$

Any automorphism ϕ_p of $A_{n,R/p}$ induces an automorphism of its center. If we replace generators $\hat{x}_i^p$, $i = 1, \dots, 2n$ of $Center(A_{n,R/p})$ by letters x_i, we obtain a polynomial automorphism ϕ_p^{centr} in characteristic p. One sees immediately that the Fourier transfrom maps to Fourier transform. Almost the same happens for transvections:

$$(T_F^A \pmod{p})^{centr} = T^P_{Fr_p(F)} \ ,$$

where $Fr_p(F) := \sum_I c_I^p x^I$ for $F = \sum_I c_I x^I$. It follows immediately from a simple identity in zero characteristic for differential operators in one variable:

$$(\partial/\partial x + g'(x))^p = (\partial/\partial x)^p + (g'(x))^p \pmod{p} \ .$$

Thus, we see that the composition $Fourier \circ T^P_{F_1} \circ Fourier \circ \dots \circ T^P_{F_k}$ (after application of the Frobenius map Fr_p to its coefficients) coinsides with the identity morphism modulo p for all sufficiently large primes p. Hence it is equal to identity in $Aut(A_{n,R}) \subset Aut(A_{n,\mathbb{C}})$. Implication in one direction is proven.

Conversely, let us assume that the composition ϕ^A is *not* equal to identity in $Aut(A_{n,R})$. Then it is either a non-trivial affine map on generators, or it maps some generator $\hat{x}_i$ to an expression of degree $d > 1$. In the first case one can show that the composition ϕ^P is not equal to identity, as after application of Frobenius one obtains affine map $\phi_p^{centr} \neq id$. In the second case the image of $\hat{x}_i^p$ has degree strictly equal to pd as its symbol is p-th power of the degree d symbol of $\hat{x}_i$. Hence, we have again $\phi_p^{centr} \neq id$. This finishes the proof of the inverse implication.

3 Conjectural description of Φ

The above prooof give an indication how the homomorphism $\Phi : Aut(A_{n,\mathbb{C}}) \to Aut(P_{n,\mathbb{C}})$ should be defined in general. Namely, as before, we can assume that $\phi \in Aut(A_{n,\mathbb{C}})$ is defined over a finitely generated integral domain $R \subset \mathbb{C}$. For any prime p we obtain a polynomial map $\phi_p^{centr} \in Aut(R/p\,[x_1, \dots, x_{2n}]$ by restriction of $\phi \mod p$ to the center of $A_{n,R/p}$.

The next step is to prove that ϕ_p^{centr} is a symplectomorphism. For sufficiently large p ring R is flat over $p \in Spec(\mathbb{Z})$, hence $A_{n,R}/p^2$ is flat over Z/p^2. We can consider $A_{n,R/p^2}$ as an infinitesimal one-step deformation of associative algebra $A_{n,R/p}$. In the usual way, one associates to this deformation a canonical Poisson bracket on the center of $A_{n,R/p}$:

$$\{a, b\} := \frac{[\tilde{a}, \tilde{b}]}{p} \pmod{p} \ ,$$

where $\tilde{a}, \tilde{b} \in A_{n,R}/p^2$ are arbitrary lifts of central elements $a, b \in Center(A_{n,R/p})$. A straightforward calculation:

$$[(\partial/\partial x)^p, x^p] = -p \pmod{p^2}$$

shows that one get the usual Poisson bracket on $R/p\,[x_1,\ldots,x_{2n}] \simeq Center(A_{n,R/p})$. Hence, we conclude that ϕ_p^{centr} is a symplectomorphism for $p \gg 1$. For any p the degree of ϕ_p^{centr} is bounded from above by the degree of ϕ.

Let us denote by R_∞ the following ring:

$$\prod_{primes\ p} (R/p) \;/ \bigoplus_{primes\ p} (R/p)$$

It contains $R \otimes \mathbb{Q}$, and it has a "universal" Frobenius endomorphism $Fr_R : R_\infty \to R_\infty,\ (a_p) \mapsto (a_p^p)$. Our construction of symplectomorphisms ϕ_p^{centr} for all $p \gg 1$ can be interpreted as a map

$$\Phi^{centr} : Aut(A_{n,R}) \to Aut(P_{n,R_\infty}),\ \ \phi \mapsto (\phi_p^{centr})\ .$$

The example of tame automorphisms shows that it is not exactly the conjectural map Φ which we want to construct, one should first untwist by Frobenius coefficents of symplectomorphisms ϕ_p^{centr}. This can be done in general, by the following result:

Theorem 2 *For any finitely generated ring R which is smooth and dominant over $\mathbb{Z}$, the image of the map Φ^{centr} belongs to $Aut(P_{n,Fr_R(R_\infty)}) \subset Aut(P_{n,R})$.*

This theorem is applicable to automorphisms over $\mathbb{C}$ because any finitely generated integral domain $R \subset \mathbb{C}$ can be made smooth over $\mathbb{Z}$ by adding inverses of finitely many non-zero elements.

3.1 Proof of Theorem 2

Let us choose (under the assumption that R is smooth and dominant over $\mathbb{Z}$) a finite collection of derivations $\delta_1,\ldots,\delta_k \in Der(R)$ which span the tangent space at every closed point of $Spec(R)$. An element $a \in R/p$ belongs to $(R/p)^p$ if and only if it is killed by all derivations δ_j. Therefore, we have to prove that for all $p \gg 1$ and any i,j, one has

$$\delta_j(\phi_p^{centr}(x_i)) = 0 \in R/p\,[x_1,\ldots,x_{2n}]\ .$$

Equivalently, we have to show that

$$\delta_j(\phi_p(\hat{x}_i^p)) = 0 \in A_{n,R/p}\ .$$

The l.h.s of the above expression is equal to

$$\delta_j((\phi_p\hat{x}_i)^p) = a^{p-1}b + a^{p-2}ba + \cdots + ba^{p-1} \pmod p$$

where

$$a := \phi(\hat{x}_i),\ b := \delta_j(a),\ \ a,b \in A_{n,R}\ .$$

Notice that for any i element $\hat{x}_i$ is locally ad-nilpotent in $A_{n,R}$, that is for any element $f \in A_{n,R}$ the iterated commutator $(ad(\hat{x}_i))^m(f)$ vanishes for sufficiently large m (in fact $m = deg(f)+1$ suffices). Hence, element $a = \phi(\hat{x}_i)$ is also locally ad-nilpotent, and there exists positive integer N such that $(ad(a))^N(b) = 0$. Finally, for any prime $p \geq N+1$ one has

$$0 = (ad(a))^{p-1}(b) = a^{p-1}b - \binom{p-1}{1}a^{p-2}ba + \cdots + ba^{p-1} =$$
$$= a^{p-1}b + a^{p-2}ba + \cdots + ba^{p-1} \pmod p\ .$$

3.2 Lifting to characteristic zero

By Theorem 2 we know that the $\Phi^{centr}(\phi)$ is a polynomial symplectomorphism with coefficients in $Fr_R(R_\infty) \subset R_\infty$. We define $\Phi^{untwisted}(\phi)$ to be an element of $Aut(P_{n,R_\infty})$ obtained from $\Phi^{centr}(\phi)$ by the inverse Frobenius map (it is well-defined because Fr_R is injective for any integral domain R).

Conjecture 2 *For any finitely generated integral domain R smooth and dominant over $\mathbb{Z}$ and any $\phi \in Aut(A_{n,R})$, element $\Phi^{untwisted}(\phi)$ belongs to subgroup $Aut(P_{n,R\otimes\mathbb{Q}}) \subset Aut(P_{n,R_\infty})$.*

If we assume Conjecture 2, then the map Φ from Conjecture 1 is defined by

$$\Phi(\phi) := \Phi^{untwisted}(\phi) \subset Aut(P_{n,R\otimes\mathbb{Q}}) \subset Aut(A_{n,\mathbb{C}})$$

We know that conjecture 2 holds for tame automorphisms. In general, I can prove a good approximation:

Theorem 3 *Under the same assumptions as in Conjecture 2, element $\Phi^{untwisted}(\phi)$ belongs to $Aut(P_{n,\tilde{R}})$, where $\tilde{R} \subset R_\infty$ is a finitely generated ring (not necessarily integral) containing R and finite over the generic point of $Spec(R)$.*

The above theorem implies that there exists a homomorphism

$$Aut(A_{n,\overline{\mathbb{Q}}}) \to Aut(P_{n,\overline{\mathbb{Q}}})$$

which may be not covariant with respect to the natural $Aut(\overline{\mathbb{Q}})$ action.

I will not give here the proof of theorem 3. It is based on some geometric considerations relating general holonomic modules over the Weyl algebra, and singular lagrangian subvarieties of $\mathbb{A}^{2n}$.

4 Conjectural description of Φ^{-1}

Here we propose a candidate for the inverse map

$$\Phi^{-1} : Aut(P_{n,\mathbb{C}}) \to Aut(A_{n,\mathbb{C}}) \ .$$

As before, we may assume that $\psi \in Aut(P_{n,\mathbb{C}})$ is defined over a finitely generated integral domain $R \subset \mathbb{C}$. We know already the automorphism $\Phi^{-1}(\psi) \pmod p$ of algebra $A_{n,R/p}$ should induce symplectomorphism $Fr_p(\psi_p)$ on its center for any prime $p \gg 1$. We will show here that it lifts to at most unique automorphism of $A_{n,R/p}$.

Algebra $A_{n,R/p}$ is an Azumaya algebra over its center, a twisted form of matrix algebra $Mat(p^n \times p^n, R/p\,[\hat{x}_1^p, \dots, \hat{x}_{2n}^p])$. It has an invariant

$$[A_{n,R/p}] \in Br(R/p\,[x_1, \dots, x_{2n}])$$

in the Brauer group of its center $Center(A_{n,R/p}) \simeq R/p\,[x_1, \dots, x_{2n}]$, the group of Morita equivalence classes of Azumaya algebras.

For any commutative algebra S in finite characteristic $p > 0$ there is a canonical map

$$\Omega^1_{abs}(S)/d\Omega^0(S) \to Br(S), \quad \Omega^1_{abs}(S) := \Omega^1(S/\mathbb{Z}) \ ,$$

given by the formula $fdg \mapsto [A_{f,g}]$. Here for any two elements $f, g \in S$ one defines Azumaya algebra $A_{f,g}$ over S as

$$A_{f,g} := S\langle \xi, \eta \rangle / \text{relations } \xi^p = f,\ \eta^p = g,\ [\xi, \eta] = 1 \ .$$

It follows from definitions that the class $[A_{n,R/p}]$ is given by the class of 1-form

$$\alpha := \sum_{i=1}^{n} x_i dx_{n+i} \in \Omega^1_{abs}(R[x_1, \dots, x_{2n}]) \ .$$

Let us denote by α_0 the same 1-form considered as a relative form over R. Symplectomorphism ψ preserves 2-form $d\alpha_0$, hence (by Poincaré lemma) there exists a function $W \in R \otimes \mathbb{Q}[x_1, \dots, x_{2n}]$ such that $\psi^*(\alpha_0) = \alpha_0 + dW$. We enlarge R by adding inverses to prime numbers which appear in denominators of coefficients of W. Symplectomorphism $Fr_p(\psi)$ for $p \gg 1$ preserves the class of the *absolute* form α modulo differentials, because all coefficients of $Fr_p(\psi)$ are p-th powers, and therefore can be treated as "constants" (they have vanishing derivatives).

The conclusion is that for all $p \gg 1$ there exists a Morita self-equivalence of algebra $A_{n,R/p}$ inducing automorphism $Fr_p(\psi)$ on its center. This Morita self-equivalence is given by a bimodule $M_{\psi,p}$ over $A_{n,R/p}$, which is finitely generated projective both as left and as right module. Moreover, it is easy to see that such bimodule is unique up to an isomorphism, and its automorphism group is the group of multiplications by scalars $(R/p)^\times$.

Conjecture 1 follows from Conjecture 2 and the following

Conjecture 3 *Bimodule $M_{\psi,p}$ is free rank one as left and as right $A_{n,R/p}$-module.*

Namely, if $M_{\psi,p}$ is free rank one module on both sides, then choosing the generator of it we obtain an automorphism of $A_{n,R/p}$. Conjecture 3 implies that ind-schemes $\underline{Aut}(A_n)$ and $\underline{Aut}(P_n)$ over $\mathbb{Z}$ have the same points over finite fields of sufficiently large characteristic (when we keep the degree of automorphisms bounded), therefore (assuming Conjefcture 2) there will be a constructible homeomorphism of schemes over $\mathbb{Q}$.

There is no good reasons to beleive in Conjecture 3, as an analog of Serre conjecture fails for Weyl algebras. Namely, there are non-trivial projective modules of "rank one" over $A_{1,\mathbb{C}}$ which are not free (J.T.Stafford (1987), Yu.Berest and J.Wilson (2000)). Maybe, one should replace in Conjecture 1 the group $Aut(A_{n,\mathbb{C}})$ by the group of Morita self-equivalences of $A_{n,\mathbb{C}}$.

5 Generalizations

There are many generalizations of Conjecture 1. Here I'll describe few of them. First of all, the idea of the use of large centers in finite characteristic can be applied to holonomic D-modules. For example, with any holonomic D_X-module M where X is a smooth algebraic variety over $\mathbb{C}$ one can associate the collections of its supports $\pmod p$ in $Fr_p(T^*X)$. One can prove that these supports are Lagrangian varieties. In general, these supports behave in a complicated way as functions of prime p, the dependence on p is (presumably) related to the motivic Galois group in the crystalline realization. Nevertheless, I expect that in some situations (including the one considered in Conjecture 1), there exists a clean correspondence:

Conjecture 4 *For any smooth variety $X/\mathbb{C}$ and a smooth closed Lagrangian subvariety $L \subset T^*X$ such that $H_1(L(\mathbb{C});\mathbb{Z}) = 0$ there exists a canonical holonomic D_X-module M_L characterized uniquely by the property that after reduction to finite characteristic $p \gg 1$ module M_L considered as a module over the center of algebra of differential operators, is supported over $Fr_p(L)$ and has rank p^{dimX}.*

For example, any polynomial symplectomorphim ψ of $\mathbb{A}^{2n}_{\mathbb{C}}$ gives a contractible Lagrangian subvariety in $\mathbb{A}^{4n}_{\mathbb{C}}$, its graph. The corresponding holonomic module can be interpreted as a bimodule over $A_{n,\mathbb{C}}$.

One of motivations for me to propose Conjecture 1 came from the deformation quantization. Namely, I proved several years ago the following result, which is in fact not a theorem but a construction:

Theorem 4 *Let $X/\mathbb{C}$ be a smooth affine variety endowed with an algebraic Poisson bracket $\alpha \in \Gamma(X, \wedge^2 T_X)$. Assume that X is rational, and there exists a smooth compactification $\overline{X}$ by divisor $D = \overline{X} \setminus X$ such that D is ample, α extends to $\overline{X}$, and the ideal of D is a Poisson ideal. Then there exists a canonical associative algebra $A/\mathbb{C}[[\hbar]]$ which is free as $\mathbb{C}[[\hbar]]$-module and quantizes $\mathcal{O}(X)$ in direction α.*

In fact, I expect that the quantized algebra does not depend on the choice of compactification if it exists. In the case when there exists a symmetry of X identified α and rescaled bivector field $\lambda\alpha$ where $\lambda \in \mathbb{C}^\times$ is arbitrary, one can remove parameter $\hbar$. In the case $X = \mathbb{A}^{2n}_{\mathbb{C}}$ with the flat bracket we get Conjecture 1. Also we get

Conjecture 5 *For any finite-dimensional Lie algebra $\mathbf{g}/\mathbb{C}$ the group of automorphisms of the universal enveloping algebra $U\mathbf{g}$ is canonically isomorphic to the group of polynomial automorphisms of $\mathbf{g}^*$ preserving the standard Kirillov bracket.*

Finally, using the same circle of ideas, together with A.Belov-Kanel we proved that the stable Jacobian Conjecture:

étale maps $\mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^n_{\mathbb{C}}$, $n \geq 1$ are invertible

is equivalent to the stable Dixmier Conjecture:

Endomorphisms of $A_{n,\mathbb{C}}$, $n \geq 1$ are invertible .