

**HINGES AND GEOMETRIC CONSTRUCTIONS OF
BOUNDARIES OF RIEMANN SYMMETRIC
SPACES**

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To A.A.Kirilov in his 60 birthday

ABSTRACT. We give elementary constructions for Satake-Furstenberg, Martin and Karpelevich boundaries of symmetric spaces. We also construct some new" boundaries

It is well-known that symmetric spaces have nontrivial and nice boundaries. There are two (disjoint) scientific traditions of investigation of such boundaries. The first tradition is related to enumerative algebraic geometry of quadrics. It was begun by the paper of Study (1886) on the geometry of the space $PGL(3, \mathbb{C})/SO(3, \mathbb{C})$ (this is the space of all nondegenerate conics in $\mathbb{C}P^2$). This construction was extended by Semple (1948-1951) to the spaces $PGL(n, \mathbb{C})/SO(n, \mathbb{C})$ and to the groups $PGL(n, \mathbb{C})$ itself. Later (1983) De Concini and Procesi constructed analogical compactification for arbitrary symmetric space G/K where G is a semisimple group without center and K is a *complex* symmetric subgroup.

Another scientific tradition is related to harmonic analysis on symmetric spaces. In 1960 Satake constructed nice compactifications of Riemannian symmetric spaces (these compactifications are real forms of compactifications of complex symmetric spaces mentioned above). In 1961-1969 in Karpelevich, Dynkin and Olshanetsky constructed more complicated boundaries (their works were devoted to analysis of harmonic functions on the symmetric spaces).

The purpose of these notes (it is a part of the paper [33]) is to give elementary description for Satake-Furstenberg boundary, Karpelevich boundary, Martin (Dynkin-Olshanetsky) boundary for Riemann noncompact symmetric spaces, we also construct some "new" boundaries (velocity boundaries in section 3 and sea urchins in section 6). We discuss only the boundaries of symmetric spaces $PGL(n, \mathbb{R})/SO(n, \mathbb{R})$ (boundaries of other classical symmetric spaces can be described by the same way).

Key words and phrases. Hausdorff distance, symmetric space, compactification, complete symmetric varieties, linear relation, Satake-Furstenberg boundary, Martin boundary.

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$

1. PRELIMINARIES. HINGES

1.1. Linear relations. Let V, W be linear spaces. A *linear relation* $V \rightrightarrows W$ is arbitrary linear subspace in $V \oplus W$.

Example 1.1. Let $A : V \rightarrow W$ be a linear operator. Then its graph $graph(A)$ is a linear relation.

Let $P : V \rightrightarrows W$ be a linear relation. Then we define

1. the *kernel* $Ker(P) = P \cap (V \oplus 0)$
2. the *image* $Im(P)$ is the projection of P to $0 \oplus W$
3. the *domain* $Dom(P)$ is the projection of P to $V \oplus 0$
4. the *indefinitness* $Indef(P) = P \cap (0 \oplus W)$

Remark 1.2. Let $P = graph(A)$. Then $Im(P)$ is the usual image of the linear operator A and $Ker(P)$ is the usual kernel of the linear operator A .

We also define the *rank* of a linear relation P :

$$\begin{aligned} rk(P) &= \dim Dom(P) - \dim Ker(P) = \dim Im(P) - \dim Indef(P) = \\ &= \dim P - \dim Ker(P) - \dim Indef(P) \end{aligned}$$

Remark 1.3. Let us consider a linear relation $P : V \rightrightarrows W$. Then it defines by the obvious way the invertible linear operator

$$[P] : Dom(P)/Ker(P) \rightarrow Im(P)/Indef(P)$$

1.2. Nonseparated quotient of grassmanian. We denote by \mathbb{R}^* the multiplicative group of \mathbb{R} . We denote by Gr_n the grassmanian of all n -dimensional subspaces in $\mathbb{R}^n \oplus \mathbb{R}^n$. Let $P : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a element of Gr_n . Let $\lambda \in \mathbb{R}^*$. We define $\lambda \cdot P \in Gr_n$ by the condition

$$(v, w) \in P \Leftrightarrow (v, \lambda w) \in \lambda P$$

Remark 1.4. If P has the form $graph(A)$ then $\lambda \cdot P = graph(\lambda A)$.

Let us consider the quotient space Gr_n/\mathbb{R}^* equipped with the usual quotient topology (see [29]). Let us consider a sequence $x_j \in Gr_n/\mathbb{R}^*$ and a point $y \in Gr_n/\mathbb{R}^*$. Let P_j, Q be representatives of x_j and y in Gr_n . Then the sequence x_j converges to y if there exist $\lambda_j, \lambda \in \mathbb{R}^*$ such that $\lambda_j \cdot P_j$ converges to $\lambda \cdot Q$ in the topology of Gr_n .

We will use the same notations for points $P \in Gr_n$ and their \mathbb{R}^* -orbits, i.e. we denote the orbit $\mathbb{R}^* \cdot P$ by P .

There are two types of orbits of \mathbb{R}^* on Gr_n . If $rk(P) = 0$ (i.e $P = Ker(P) \oplus Indef(P)$) then P is a fixed point of the group \mathbb{R}^* . If $rk(P) \neq 0$ then the stabilizer of P in \mathbb{R}^* is trivial and hence the orbit is isomorphic to the group \mathbb{R}^* itself. The orbit of the first type are closed. The orbits of the second type are not closed. Hence the topology in the space Gr_n/\mathbb{R}^* is not separated in the Hausdorff sence. A point $P \in Gr_n$ is closed set only in the case $rk(P) = 0$.

Example 1.5. Let us consider a sequence

$$A_j = \begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix}$$

of linear operators in \mathbb{R}^2 . Let $P^j \in Gr_2$ be their graphs. Let us consider the sequences

$$j \cdot P_j ; P_j ; j^{-1/2} \cdot P_j ; j^{-1} \cdot P_j ; j^{-2} \cdot P_j$$

in Gr_2 . Their limits in Gr_2 are the subspaces R_1, \dots, R_5 having the form

$$R_1 : (0, 0; x, y)$$

$$R_2 : (0, y; x, y)$$

$$R_3 : (0, y; x, 0)$$

$$R_4 : (x, y; x, 0)$$

$$R_5 : (x, y; 0, 0)$$

Hence the sequence P_j has 5 limits in the quotient space Gr_2/\mathbb{R}^* .

Remark 1.6. Let us consider a sequence of invertible operators $A_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Let P_j be their graphs. Evidently subspaces $\mathbb{R}^n \oplus 0$ and $0 \oplus \mathbb{R}^n$ are limits of the sequence P_j in the quotient space Gr_n/\mathbb{R}^* . By the official topological definition this sequence is convergent (and moreover it has at least 2 limits). It is quite clear that official definition of convergence (the sequence converges if it has limit) is bad.

Let A_j be a sequence of invertible operators. Let P_j be their graphs. We say that the sequence P_j is *seriously convergent* if each limit point of P_j in the quotient space Gr_n/\mathbb{R}^* is the limit of P_j in the quotient space.

Remark 1.7. We define serious convergence only for sequences of *invertible* operators!

We say that the subset $S \in Gr_n/\mathbb{R}^*$ is *admissible* if there exists seriously convergent sequence P_j such that the set of limits of P_j coincides with S .

Example 1.8. The sequence P_j described in example 1.5 is seriously convergent. Hence the set R_1, \dots, R_5 is admissible.

1.3. Hinges.

Definition 1.9. A *hinge*

$$\mathfrak{P} = (P_1, \dots, P_k)$$

is a family of elements of Gr_n/\mathbb{R}^* such that

0°. For all j $rk(P_j) > 0$

1°. For all j

$$Ker(P_j) = Dom(P_{j+1})$$

$$Im(P_j) = Indef(P_{j+1})$$

2°.

$$\text{Indef}(P_1) = 0$$

$$\text{Ker}(P_k) = 0$$

i.e. P_1 is the graph of a operator $(\mathbb{R}^n \oplus 0) \rightarrow (0 \oplus \mathbb{R}^n)$ and P_k is the graph of a operator $(\mathbb{R}^n \oplus 0) \leftarrow (0 \oplus \mathbb{R}^n)$

We denote space of all hinges in \mathbb{R}^n by $\text{Hinge}(n)$

Remark 1.10. The condition 2° is interpretation of the condition 1° if $j = 0$ and $j = k$.

Example 1.11. The graph of a invertible operator is a hinge ($k=1$). The graph of a noninvertible operator is not hinge (see the condition 2°)

Example 1.12. In Example 1.4 the set

$$(R_4, R_2)$$

is a hinge. Note that the rank of R_1, R_3, R_5 is 0.

By the definition of hinge we have

$$\mathbb{R}^n = \text{Dom}(P_1) \supset \text{Ker}(P_1) = \text{Dom}(P_2) \supset \text{Ker}(P_2) = \dots$$

Hence (by the condition 0°) we have $k \leq n - 1$

Theorem 1.13. Let us consider a hinge

$$\mathfrak{P} = (P_1, \dots, P_k)$$

Let

$$Q_j = \text{Ker}(P_j) \oplus \text{Im}(P_j) = \text{Dom}(P_{j+1}) \oplus \text{Indef}(P_{j+1}) \in \text{Gr}_n/\mathbb{R}^*$$

$$Q_0 = \mathbb{R}^n \oplus 0$$

Then the set

$$\{Q_0, P_1, Q_1, P_2, \dots, P_k, Q_k\}$$

is a admissible subset in Gr_n/\mathbb{R}^* . Moreover each admissible subset has such form.

Remark 1.14. Uniformally speaking hinges are limits in Gr_n/\mathbb{R}^* of sequences of invertible operators. For instance sequence A_j described in the Example 1. converges to the hinge (R_4, R_2) . Hinges are slightly different from admissible sets. Nevertheless it is better for us to forget about fixed points Q_0, Q_1, \dots (since they can be reconstructed by P_1, P_2, \dots)

1.4. The topology on the space of hinges. Let M be a compact metric space with a metric $\rho(\cdot, \cdot)$. Let $\mathcal{S}(M)$ be the space of all closed subsets in M . Let $X \in \mathcal{S}(M)$. We denote by $\mathcal{O}_\epsilon(X)$ the set of points $m \in M$ such that exists $x \in X$ satisfying the condition $\rho(m, x), \epsilon$.

Let X_1, X_2 be closed subsets. *Hausdorff distance* (see [31]) between X_1 and X_2 is infimum of ϵ such that

$$X_1 \subset \mathcal{O}_\epsilon(X_2) \ ; \ X_2 \subset \mathcal{O}_\epsilon(X_1)$$

It is well known that the space $\mathcal{S}(M)$ equipped with the Hausdorff distance is a compact metric space.

Let us consider a invertible operator A and its graph P . Let us consider the curve $\mathbb{R}^* \cdot P$ in grassmanian. Let us consider its closure $\sigma(A)$. It contains the curve $\mathbb{R}^* \cdot P$ itself and two points $\mathbb{R}^n \oplus 0$, $0 \oplus \mathbb{R}^n$. We denote family of curves $\sigma(A) \in \mathcal{S}(Gr_n)$ by $\widetilde{PGL}(n, \mathbb{R})$. We have the obvious bijection

$$PGL(n, \mathbb{R}) \leftrightarrow \widetilde{PGL}(n, \mathbb{R})$$

We denote by $\overline{PGL}(n, \mathbb{R})$ the closure of $\widetilde{PGL}(n, \mathbb{R})$ in the Hausdorff metric.

Theorem 1.15. *Let*

$$\mathfrak{P} = (P_1, \dots, P_k)$$

be a hinge. Let Q_j be the same as in the theorem 1.13. Let us denote by $\gamma(\mathfrak{P})$ the curve

$$Q_0 \cup (\mathbb{R}^* \cdot P_1) \cup Q_1 \cup (\mathbb{R}^* \cdot P_2) \cup Q_2 \cdots \cup Q_k$$

Then the map

$$\mathfrak{P} \mapsto \gamma(\mathfrak{P})$$

is the bijection

$$Hinge(n) \rightarrow \overline{PGL}(n, \mathbb{R})$$

We see that $Hinge(n)$ has the natural structure of a compact metric (metrizable) space containing $PGL(n, \mathbb{R})$ as open dense set(if $A \in PGL(n, \mathbb{R})$ then its graph is a one-element hinge $\mathfrak{P} = (P)$).

The space $Hinge(n)$ has the natural structure of $(n^2 - 1)$ -dimensional real analytic manifold (it is not obvious). The set $Hinge(n) \setminus PGL(n, \mathbb{R})$ is the union of $(n - 1)$ submanifolds of codimension 1 (see below bibliographical remarks).

2. SATAKE-FURSTENBERG BOUNDARY

2.1.Symmetric space $SL(n, \mathbb{R})/SO(n)$. Let us consider the space Q of real symmetric positive definite matrices defined up to multiplier. The action of the group $SL(n, \mathbb{R})$ on this space is defined by the formula

$$g : A \mapsto gAg^t$$

where A is symmetric matrix, $g \in SL(n, \mathbb{R})$ and g^t is the transposed matrix. Obviously the stabilizer of the point E is the group $SO(n)$. Hence we obtain

$$Q \simeq SL(n, \mathbb{R})/SO(n)$$

2.2. *Positive linear relations.* We want to describe the closure of the space Q in $Hinge(n)$. For this purpose we need in some preliminaries. Let us consider in the space \mathbb{R}^n the standard scalar product

$$\langle v, w \rangle = \sum_k v_k w_k$$

We define in the space $\mathbb{R}^n \oplus \mathbb{R}^n$ the skew-symmetric bilinear form by the formula

$$\{(v, v'); (w, w')\} = \langle v, w' \rangle - \langle w, v' \rangle$$

We define also indefinite symmetric bilinear form on $\mathbb{R}^n \oplus \mathbb{R}^n$ by the formula

$$[(v, w); (v', w')] := \langle v, w' \rangle + \langle v', w \rangle$$

We say that a n -dimensional linear relation $P : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *symmetric* if P is a maximal isotropic subspace with respect to the skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$.

Remark 2.1. Let A be a symmetric linear operator (i.e $A = A^t$). Then its graph is a symmetric linear relation.

Let us consider a symmetric linear relation $P : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Then $Im(P)$ is the orthogonal complement in \mathbb{R}^n to $Ker(P)$ (with respect to the scalar product $\langle \cdot, \cdot \rangle$) and $Indef(P)$ is the orthogonal complement to $Dom(P)$ (with respect to the standard scalar product in \mathbb{R}^n). Hence the linear relation P defines the nondegenerate pairing

$$(2.1) \quad Dom(P)/Ker(P) \times Im(P)/Indef(P) \rightarrow \mathbb{R}$$

The linear relation P also defines the operator

$$(2.2) \quad Dom(P)/Ker(P) \rightarrow Im(P)/Indef(P)$$

Hence each symmetric linear relation P defines nondegenerated symmetric bilinear form q_P on the space $Dom(P)/Ker(P)$.

We say that a symmetric linear relation P is *nonnegative definite* if the form $[\cdot, \cdot]$ is nonnegative definite on the subspace P . It is equivalent to the positivity of quadratic form q_P .

Remark 2.2. Let a linear relation P be the graph of a operator A . Then P is nonnegative definite if and only if A is nonnegative definite.

2.3. *Satake-Furstenberg boundary.* Let us consider the closure \overline{Q} of the space Q in the space $Hinge(n)$. It is easy to show that a hinge \mathfrak{P} belongs to \overline{Q} if and only if all linear relations P are nonnegative definite. It appears that this closure coincides with *Satake-Furstenberg compactification* of the symmetric space $SL(n, \mathbb{R})/SO(n)$.

Hence a point of Satake-Furstenberg compactification is given by the following data:

$$1^* . s = 1, 2, \dots, n - 1$$

2*. A hinge

$$\mathfrak{P} = (P_1, \dots, P_s)$$

such that all linear relations P_j are nonnegative definite.

Let us consider a point of Satake-Furstenberg compactification (i.e data 1* – 2*)

. Let us consider the subspaces

$$V_j = Ker(P_j) = Dom(P_{j+1})$$

Then the form q_{P_j} is positive definite on $Dom(P_j)/Ker(P_j)$. Now we can say that a point of Satake-Furstenberg boundary is defined by the following data

1*. $s = 1, 2, \dots, n - 1$

2*. A flag

$$0 \subset V_1 \subset V_2 \subset \dots \subset V_s \subset \mathbb{R}^n$$

where all subspaces $0, V_1, \dots, V_s, \mathbb{R}^n$ are different .

3*. A positive definite quadratic form R_j in each quotient space $Dom(P_j)/Ker(P_j)$.

3. VELOCITY COMPACTIFICATIONS OF SYMMETRIC SPACES.

3.1. Simplest velocity compactification.

Let $A \in Q = SL(n, \mathbb{R})/SO(n)$ be a positive definite matrix. Let

$$a_1 \geq a_2 \geq \dots \geq a_n$$

be eigenvalues of A . Let

$$\lambda_j = \ln a_j$$

We denote by $\Lambda(A)$ the collection

$$(3.1) \quad \Lambda(A) = (\lambda_1, \lambda_2, \dots, \lambda_n) ; \quad \lambda_1 \geq \lambda_2 \geq \dots$$

The matrix A is defined up to multiplier and hence $\Lambda(A)$ is defined up to additive constant:

$$(3.2) \quad (\lambda_1, \lambda_2, \dots, \lambda_n) \sim (\lambda_1 + \sigma, \lambda_2 + \sigma, \dots, \lambda_n + \sigma)$$

We denote by Σ_n the space of all collections $\Lambda(A)$ (see (3.2)). It is easy to see that $\Lambda(A)$ is a $(n - 1)$ -dimensional simplicial cone. We can assume $\lambda_n = 0$ and hence the cone Σ_n can be considered as the space of collections

$$\{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq 0\}$$

We denote by $\Delta_n = \partial\Sigma_n$ the $(n - 2)$ -dimensional simplex

$$1 \geq \mu_2 \geq \mu_3 \geq \dots \geq \mu_{n-1} \geq 0$$

It is natural to think that $\mu_1 = 1, \mu_n = 0$. We say that Δ_n is the *velocity simplex*. Let us consider the natural projection

$$\pi : (\Sigma_n \setminus 0) \rightarrow \Delta_n$$

defined by the rule

$$\pi(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0) = \left(\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}, \dots, \frac{\lambda_{n-1}}{\lambda_1} \right)$$

Now we define the compactification

$$\bar{\Sigma}_n = \Sigma_n \cup \Delta_n$$

of Σ_n . A sequence $L_j = (\lambda_1^{(j)}, \dots, \lambda_n^{(j)}) \in \Sigma_n$ converges to $M \in \Delta_n$ if

1. $\lambda_1^{(j)} - \lambda_n^{(j)} \rightarrow \infty$ if $j \rightarrow \infty$
2. The sequence $\pi(L_j) \in \Delta_n$ converges to M .

We also define the *velocity compactification*

$$\bar{Q}^{vel} = SL(n, \mathbb{R})/SO(n) \cup \Delta_n$$

of the symmetric space $SL(n, \mathbb{R})/SO(n)$. A sequence A_j in Q converges to $M \in \Delta_n$ if $\Lambda(A_j)$ converges to M in the topology of $\bar{\Sigma}_n$.

3.2. Polyhedron of Karpelevich velocities.

Now we want to describe more delicate compactification of the simplicial cone Σ_n (compactification by Karpelevich velocities). Let us consider a sequence

$$\lambda^{(j)} = \{\lambda_1^{(j)} \geq \dots \geq \lambda_n^{(j)}\} \in \Sigma_n$$

Let $1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq 0$ be its limit in Δ_n . It can happens that some of numbers μ_i are equals:

$$\mu_k = \mu_{k+1} = \dots = \mu_l$$

In this case we will separate velocities of

$$\{\lambda_k^{(j)} \geq \dots \geq \lambda_l^{(j)}\} \in \Sigma_{l-k+1}$$

by the same rule as above.

Definition of the polyhedron. We denote by $I_{\alpha, \beta}$ set $\{\alpha, \alpha + 1, \dots, \beta\} \subset \mathbb{N}$

Let us consider a interval $I_{\alpha, \beta} = \{\alpha, \alpha + 1, \dots, \beta\}$. We denote by $\Sigma(I_{\alpha, \beta})$ the simplicial cone

$$\lambda_\alpha \geq \lambda_{\alpha+1} \geq \dots \geq \lambda_\beta$$

the elements of the cone $\Sigma(I_{\alpha, \beta})$ are defined up to additive constant (see (3.2)). We also define the simplex $\Delta(I_{\alpha, \beta})$ given by the inequalities

$$1 = \mu_\alpha \geq \mu_{\alpha+1} \geq \dots \geq \mu_{\beta-1} \geq \mu_\beta = 0$$

Let us consider the compactification

$$\bar{\Sigma}(I_{\alpha, \beta}) = \Sigma(I_{\alpha, \beta}) \cup \Delta(I_{\alpha, \beta})$$

Remark 3.1. Let us consider the case $\alpha = \beta$. The set $\Sigma(I_{\alpha,\alpha}) = \overline{\Sigma}(I_{\alpha,\alpha})$ consist of the unique point (it is one real number defined up to additive constant).

Let $k \leq \alpha \leq \beta \leq l$. We define the map

$$\Pi_{\alpha,\beta}^{k,l} : \Sigma(I_{k,l}) \rightarrow \Sigma(I_{\alpha,\beta})$$

given by the formula

$$\Pi_{\alpha,\beta}^{k,l}(\lambda_k, \dots, \lambda_l) = (\lambda_\alpha, \dots, \lambda_\beta)$$

We define two polyhedra

$$\Xi(k, l) := \prod_{\alpha,\beta:k \leq \alpha \leq l \leq \beta} \Sigma(I_{\alpha,\beta})$$

$$\overline{\Xi}(k, l) := \prod_{\alpha,\beta:k \leq \alpha \leq l \leq \beta} \overline{\Sigma}(I_{\alpha,\beta})$$

Obviously $\Xi(k, l) \subset \overline{\Xi}(k, l)$. Let us consider the natural (diagonal) embedding

$$i : \Xi(I_{k,l}) \rightarrow \Xi(k, l)$$

(it is the product of the maps $\Pi_{\alpha,\beta}^{k,l}$)

The *polyhedron of Karpelevich velocities* (k, l) is the closure of the set $i(\Sigma(I_{k,l}))$ in $\overline{\Xi}(k, l)$.

Criterion of convergence of a sequence of interior points to a point of the boundary.

Let us consider a sequence

$$\Lambda^{(j)} = \{\lambda_k^{(j)}, \lambda_{k+1}^{(j)}, \dots, \lambda_l^{(j)}\}$$

Then the necessary and sufficient condition of convergence of the sequence $\Lambda^{(j)}$ in $\mathcal{K}(k, l)$ is the convergence of all sequences

$$\Pi_{\alpha,\beta}^{k,l}(\Lambda^{(j)}) = (\lambda_\alpha^{(j)}, \dots, \lambda_\beta^{(j)})$$

in $\overline{\Sigma}(I_{\alpha,\beta})$.

The Karpelevich velocity polyhedron is defined. Now we want to give explicit description of its combinatorial structure.

Tree-partitions. Let us consider the set $I_{k,l} := \{k, k+1, \dots, l\}$. We say that a *partition* of $I_{k,l}$ is a representation of $I_{k,l}$ as

$$I_{k,m_1} \cup I_{m_1+1,m_2} \cup \dots \cup I_{m_{s-1}+1,l}$$

where $s > 1$.

We say that a system \mathfrak{a} of subsets of $I_{k,l}$ is a *tree-partition* if

- a) $I_{k,l} \in \mathfrak{a}$
- b) Each element $J \in \mathfrak{a}$ has the form $I_{\alpha,\beta} = \{\alpha, \alpha+1, \dots, \beta\}$

c) If $J_1, J_2 \in \mathfrak{a}$ then

$$J_1 \cap J_2 = \emptyset \text{ or } J_1 \supset J_2 \text{ or } J_2 \subset J_1$$

d) Let $J = I_{\alpha, \beta} \in \mathfrak{a}$. Then there are only two possibilities

1*. There is no $K \in \mathfrak{a}$ such that $K \subset J$ (in this case we say that $I_{\alpha, \beta}$ is *irreducible*).

2*. $J = I_{\alpha, \beta}$ can be decomposed as the union

$$(3.3) \quad I_{\alpha, \beta} = I_{\alpha, \gamma_1} \cup I_{\gamma_1+1, \gamma_2} \cup I_{\gamma_2+1, \gamma_3} \cup \dots \cup I_{\gamma_{s-1}+1, \beta}$$

where $I_{\alpha, \gamma_1}, I_{\gamma_1+1, \gamma_2}, \dots, I_{\gamma_{s-1}+1, \beta} \in \mathfrak{a}$. In this case we say that J is *reducible* and (3.3) is the *canonical decomposition* of J .

Remark 3.2. Let $I_{\alpha, \beta} \in \mathfrak{a}$. Let \mathfrak{b} be the set of all $J \subset I_{\alpha, \beta}$ such that $J \in \mathfrak{a}$. Then \mathfrak{b} is the tree-partition of $I_{\alpha, \beta}$.

Remark 3.3. In the other words tree-partition is given by the following data. We consider a partition of the segment $I_{k, l} \subset \mathbb{N}$ to subsegments, then we consider partitions of some subsegments, etc.

We denote by $TP(k, l)$ the set of all tree-partitions of $I_{k, l}$. Let us define the partial canonical ordering on $TP(k, l)$. Let $\mathfrak{a}, \mathfrak{b} \in TP(k, l)$. We say that $\mathfrak{a} > \mathfrak{b}$ if $J \in \mathfrak{a}$ implies $J \in \mathfrak{b}$ (i.e $\mathfrak{b} \supset \mathfrak{a}$).

The partially ordered set $TP(k, l)$ contains the unique maximal element \mathfrak{a}_0 . This is the tree-partition which contains the unique element $I_{k, l}$.

A element $\mathfrak{b} \in TP(k, l)$ is minimal if

a) Each irreducible element of \mathfrak{b} contains only one point.

b) If $J \in \mathfrak{b}$ is reducible then the canonical decomposition of J contains exactly two elements ($s = 2$ in (3.3)).

Description of the polyhedron.

Let us consider a partition \mathfrak{t} of $I_{\alpha, \beta}$:

$$(3.4) \quad I_{\alpha, \beta} = I_{\alpha, \gamma_1} \cup I_{\gamma_1+1, \gamma_2} \cup I_{\gamma_2+1, \gamma_3} \cup \dots \cup I_{\gamma_{s-1}+1, \beta}$$

We denote by $\tilde{\Delta}(I_{\alpha, \beta} | \mathfrak{t})$ the open simplex

$$(3.5) \quad 1 = \mu_{\alpha} = \dots = \mu_{\gamma_1} > \mu_{\gamma_1+1} = \mu_{\gamma_1+2} = \dots = \mu_{\gamma_2} > \dots > \mu_{\gamma_{s-1}+1} = \dots = \mu_{\beta} = 0$$

We denote by $\Delta(I_{\alpha, \beta} | \mathfrak{t})$ the compact simplex

$$(3.6) \quad 1 = \mu_{\alpha} = \dots = \mu_{\gamma_1} \geq \mu_{\gamma_1+1} = \mu_{\gamma_1+2} = \dots = \mu_{\gamma_2} \geq \dots \geq \mu_{\gamma_{s-1}+1} = \dots = \mu_{\beta} = 0$$

It is natural to consider in $\Delta(I_{\alpha, \beta} | \mathfrak{t})$ and $\tilde{\Delta}(I_{\alpha, \beta} | \mathfrak{t})$ the coordinates

$$\tau_2 := \mu_{\gamma_1+2} = \dots = \mu_{\gamma_2} =$$

.....

$$\tau_{s-1} := \mu_{\gamma_{s-2}+1} = \dots = \mu_{\gamma_{s-1}}$$

Remark 3.4. If $s=2$ then $\Delta(J | t) = \tilde{\Delta}(J | t)$ consist of the unique point $\{1 > 0\}$.

Remark 3.5.

$$\Delta(I_{\alpha,\beta}) = \bigcup_t \tilde{\Delta}(I_{\alpha,\beta} | t)$$

where the union is given by the all partitions of $I_{\alpha,\beta}$

Fix a tree-partition $\mathfrak{a} \in TP(k, l)$. For each element $J \in \mathfrak{a}$ consider its canonical decomposition t . We denote the simplex $\tilde{\Delta}(J | t)$ by $\tilde{\Delta}(\mathfrak{a}, J)$.

For each $\mathfrak{a} \in TP(k, l)$ we define the *face*

$$(3.7) \quad F(\mathfrak{a}) = \left(\prod_{J=I_{\alpha,\beta} \in \mathfrak{a} \text{ is irreducible}} \Sigma(I_{\alpha,\beta}) \right) \times \prod_{J \in \mathfrak{a} \text{ is redicible}} \tilde{\Delta}(\mathfrak{a}, J)$$

Remark 3.6. For the trivial tree-partition \mathfrak{a}_0 we have $F(\mathfrak{a}_0) = \Sigma(I_{k,l})$. If \mathfrak{b} is a minimal tree -partition then $F(\mathfrak{b})$ is a one-point-set.

We define *Karpelevich velocity polyhedron* $\mathcal{K}(k, l)$ by

$$\mathcal{K}(k, l) = \bigcup_{\mathfrak{a} \in TP(k,l)} F(\mathfrak{a})$$

We want to define a topology of a compact metric space on $\mathcal{K}(k, l)$. The face $F(\mathfrak{a}_0) = \Sigma(I_{k,l})$ will be a open dense subset in $\mathcal{K}(k, l)$.

Remark 3.7. Let $l = k$. Then $\mathcal{K}(k, k)$ consist of one point. Let $l = k + 1$. Then we have two tree-partitions of the set $\{k, k + 1\}$: The trivial tree-partition \mathfrak{a}_0 and maximal tree-partition \mathfrak{a}_1 (its elements are $(k, k + 1), (k), (k + 1)$). The face $F(\mathfrak{a}_0)$ is closed half-line $\lambda_1 > 0$. The face $F(\mathfrak{a}_1)$ is one-point-set. Hence $\mathcal{K}(k, k + 1)$ is the segment $[0, \infty]$.

Convergence of interior points to the boundary.

The definition of convergence is inductive. We assume the convergence is defined for all Karpelevich polyhedra $\mathcal{K}(\alpha, \beta)$ such that $\beta - \alpha < l - k$.

We define the convergence of a sequence

$$x^{(j)} = \{x_k^{(j)} \geq \dots \geq x_l^{(j)}\} \in \Sigma(I_{k,l}) = F(\mathfrak{a}_0)$$

in two steps.

The first step. The convergence of $x^{(j)}$ in $\bar{\Sigma}(I_{k,l})$ is a nessesary condition for the convergence in $\mathcal{K}(k, l)$.

If $y \in \Sigma(k, l)$ then the limit of $x^{(j)}$ in $\mathcal{K}(k, l)$ is defined to be y .

The second step. Let $y \notin \Sigma(I_{k,l})$. Then y is a element of some open simplex $\tilde{\Delta}(I_{k,l} | t)$, i.e y has the form

$$\{1 = y_k = \cdots = y_{\gamma_1} > y_{\gamma_1+1} = \cdots = y_{\gamma_2} > \cdots > y_{\gamma_{s-1}+1} = \cdots = \gamma_l = 0\}$$

In this case the sufficient and necessary condition of convergence of the sequence $x^{(j)}$ in $\mathcal{K}(k, l)$ is the convergence of all sequences

$$x_{[\psi]}^{(j)} := (x_{\gamma_\psi+1}^{(j)}, \dots, x_{\gamma_{\psi+1}}^{(j)}) \in \Sigma(I_{\gamma_\psi+1, \gamma_{\psi+1}})$$

in the Karpelevich velocity polyhedra $\mathcal{K}(\gamma_\psi + 1, \gamma_{\psi+1})$ (this convergence is defined by the inductive assumption)

This concludes the definition.

Example 3.8. Let $k = 1, l = 8$.

$$\begin{aligned} x_1^{(j)} &= 2j^3 & x_2^{(j)} &= j^3 \\ x_3^{(j)} &= j^2 + j + 2 & x_4^{(j)} &= j^2 + j + 1 & x_5^{(j)} &= j^2 + j \\ x_6^{(j)} &= 2j & x_7^{(j)} &= j & x_8^{(j)} &= 0 \end{aligned}$$

Then the associated tree-partition has the form

$$\begin{aligned} &(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8) \\ &(1) \ (2) \ (3 \ 4 \ 5 \ 6 \ 7 \ 8) \\ &\quad (3 \ 4 \ 5) \ (6 \ 7 \ 8) \\ &\quad\quad (6) \ (7) \ (8) \end{aligned}$$

The limit of $x^{(j)}$ in $\bar{\Sigma}(I_{1,8})$ is the collection

$$(3.8) \quad \{1 > 1/2 > 0 = 0 = 0 = 0 = 0 = 0\} \in \Delta(I_{1,8})$$

The sequence $x^{(j)}$ induces the sequence

$$y^{(j)} = (x_3^{(j)}, \dots, x_8^{(j)}) \in \Sigma(I_{3,8})$$

The limit of $y^{(j)}$ in $\bar{\Sigma}(I_{3,8})$ is the collection

$$(3.9) \quad \{1 \geq 1 \geq 1 \geq 0 \geq 0 \geq 0\} \in \Delta(I_{3,8})$$

Now we obtain the sequences

$$z^{(j)} = (x_3^{(j)}, x_4^{(j)}, x_5^{(j)}) \in \Sigma(I_{3,5})$$

$$u^{(j)} = (x_6^{(j)}, x_7^{(j)}, x_8^{(j)}) \in \Sigma(I_{6,8})$$

We have

$$z^{(j)} = (j^2 + j + 2, j^2 + j + 1, j^2 + j) = (2, 1, 0)$$

(recall that the collection $z^{(j)}$ is defined up to additive constant) and $\lim z^{(j)}$ is the collection

$$(3.10) \quad \{2 > 1 > 0\} \in \Sigma(I_{3,5})$$

At last

$$u^{(j)} = (2j, j, 0)$$

and the limit of $u^{(j)}$ in $\bar{\Sigma}(I_{6,8})$ is the point

$$(3.11) \quad \{1 > 1/2 > 0\} \in \tilde{\Delta}(I_{6,8})$$

The limit of the sequence $x^{(j)}$ is the collection of collections (3.8)-(3.11).

Topology on the boundary of $I_{k,l}$. This topology satisfies the following property: the closure of $F(\mathbf{a})$ consists of all faces $F(\mathbf{b})$ such that $\mathbf{b} < \mathbf{a}$.

We assume the topology is defined for all polyhedra $\mathcal{K}(\alpha, \beta)$ such that $\beta - \alpha < l - k$.

We define the convergence of a sequence

$$Z^{(j)} \in F(\mathbf{a})$$

in two steps.

The first step Let

$$h^{(j)} = \{1 = h_k^{(j)} \geq h_{k+1}^{(j)} \geq \dots \geq h_l^{(j)} = 0\}$$

be the component of $Z^{(j)}$ associated to multiplier $\tilde{\Delta}(\mathbf{a}, I_{k,l})$ in the product (3.7). Then the convergence of $h^{(j)}$ in $\Delta(\mathbf{a}, I_{k,l})$ is a necessary condition for its convergence in $\mathcal{K}(k, l)$. We denote the limit of $h^{(j)}$ in $\Delta(\mathbf{a}, I_{k,l})$ by u .

Second step. Let us consider the partition of $I_{k,l}$ associated to \mathbf{a} :

$$I_{k,l} = I_{k,\gamma_1} \cup I_{\gamma_1+1,\gamma_2} \cup \dots \cup I_{\gamma_{s-1}+1,l}$$

Then the collection u has the form

$$u = \{1 = u_k = \dots = u_{\gamma_1} \geq u_{\gamma_1+1} = \dots = u_{\gamma_2} \geq \dots\}$$

Let us consider τ_1, τ_2, \dots such that

$$\{1 = u_k = \dots = u_{\tau_1} > u_{\tau_1+1} = \dots = u_{\tau_2} > \dots\}$$

The set $\{\tau_1, \tau_2, \dots\}$ is a subset in the set $\{\gamma_1, \gamma_2, \dots\}$ and hence each segment $I_{\tau_\alpha+1, \tau_{\alpha+1}}$ is the union of the segments $I_{\gamma_m+1, \gamma_{m+1}}$.

Let us consider on each set

$$\{\tau_\alpha + 1, \tau_\alpha + 2, \dots, \tau_{\alpha+1}\}$$

the tree-partition \mathfrak{b}_α induced by the tree-partition \mathfrak{a} . The sequence $Z^{(j)}$ induces the sequence $Z_{[\alpha]}^{(j)}$ in each face $F(\mathfrak{b}_\alpha) \subset \mathcal{K}(\tau_\alpha + 1, \tau_{\alpha+1})$.

The necessary and sufficient condition of the convergence of $Z^{(j)}$ is the convergence of each sequence $Z_{[\alpha]}^{(j)}$ in Karpelevich polyhedron $\mathcal{K}(\tau_\alpha + 1, \tau_{\alpha+1})$.

3.3. The compactification of symmetric space by Karpelevich velocities. Let us consider the boundary

$$\partial\mathcal{K}(1, n) := \mathcal{K}(1, n) \setminus \Sigma(I_{1, n})$$

of the polyhedron $\mathcal{K}(1, n)$.

We define the compactification

$$(SL(n, \mathbb{R})/SO(n)) \cup (\partial\mathcal{K}(1, n))$$

of the symmetric space $SL(n, \mathbb{R})/SO(n)$. Let $x^{(j)} \in SL(n, \mathbb{R})/SO(n)$ be a sequence and $y \in \partial\mathcal{K}(I_{1, n})$. The $x^{(j)} \rightarrow y$ if

1. distance $d(x^{(j)}, 0) \rightarrow \infty$
2. $\Lambda(x^{(j)}) \rightarrow y$ in the topology of $\mathcal{K}(I_{1, n})$ (where $\Lambda(\cdot)$ is defined by the formula (3.1))

4. TITS BUILDING ON MATRIX SKY

We recall that geodesics in the space $SL(n, \mathbb{R})/SO(n)$ have the form

$$(4.1) \quad \gamma(s) = A \begin{pmatrix} \exp(\lambda_1 s) & & & \\ & \exp(\lambda_2 s) & & \\ & & \ddots & \\ & & & \exp(\lambda_n s) \end{pmatrix} A^t$$

where

$$(4.2) \quad A \in SO(n), \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

The term *geodesic* below means the oriented geodesics without fixed parametrization.

4.1. Matrix sky (visibility boundary). Let us consider a Riemann noncompact symmetric space G/K . Fix a point $x_0 \in G/K$ (in our case $G/K = SL(n, \mathbb{R})/SO(n)$ it is natural to assume $x_0 = E$). Let T_{x_0} be the tangent space in the point x_0 (in our case $G/K = SL(n, \mathbb{R})/SO(n)$ the tangent space is the space of symmetric matrices defined up to addition of a scalar matrix, i.e. $A \simeq A + \lambda E$). Let S be the space of rays in T_{x_0} with origins in zero (i.e. $S = (T_{x_0} \setminus 0)/\mathbb{R}_+^*$ where \mathbb{R}_+^* is the multiplicative group of positive real numbers). Let $v \in S$, let $\tilde{v} \in T_{x_0}$ be a tangent vector on the ray v . Let

$$\gamma_v = \gamma_v(t)$$

be the geodesic such that

$$\gamma_v(0) = x_0 \quad \gamma'_v(0) = \tilde{v}$$

We don't interested by the parametrization of the geodesic $\gamma(s)$ but its direction is essential for us.

Let Sk be another copy of the sphere S . Points of the sphere Sk we consider as infinitely far points of G/K . We will call the sphere Sk by *the matrix sky* or by *the visibility boundary*. Let us describe the topology on the space

$$\overline{(G/K)}^{vis} := G/K \cup Sk$$

We equip the spaces G/K and Sk with the usual topology. Let y_j be a sequence in G/K . Let $v \in Sk$. Let $\gamma^{(j)}$ be the geodesic joining points x_0 and y_j . Let us consider the vectors $v_j \in S$ such that

$$\gamma^{(j)} = \gamma_{v_j}$$

The convergence of the sequence $y_j \in G/K$ to a point $v \in Sk$ is defined by the conditions

1. $\rho(x_0, y) \rightarrow \infty$
2. $v_j \rightarrow v$ in the natural topology of the sphere Sk

4.2. *The projection of the matrix sky to the velocity simplex.* Let $G/K = SL(n, \mathbb{R})/SO(n)$. Let us consider a geodesic γ with the origin in $x_0 = E$. Then γ has the form

$$(4.3) \quad \gamma(s) = A \begin{pmatrix} \exp(\lambda_1 s) & & & \\ & \exp(\lambda_2 s) & & \\ & & \ddots & \\ & & & \exp(\lambda_n s) \end{pmatrix} A^t$$

where $A \in SO(n)$ and

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = 0$$

Let $\Delta = \Delta_n$ be the simplex

$$1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq 0$$

(see 3.1) . We associate to each geodesic $\gamma(s)$ the point

$$D(\gamma) : 1 \geq \frac{\lambda_2}{\lambda_1} \geq \frac{\lambda_3}{\lambda_1} \geq \dots \geq \frac{\lambda_{n-1}}{\lambda_1} \geq 0$$

of the simplex Δ .

Obviously $D(\gamma)$ is the limit of the geodesics γ in the simplest velocity compactification of $SL(n, \mathbb{R})/SO(n)$. We say that $D(\gamma) \in \Delta$ is the *velocity of geodesic* γ

4.3. *The projection of the matrix sky to the space of flags.* Let \mathcal{F} be the set of all flags

$$0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_s = \mathbb{R}^n$$

in \mathbb{R}^n ($s = 0, 1, 2, \dots, n$), see section 7. Denote by $\mathcal{F}_{complete}$ the space of complete flags (i.e $i = n$)

Let us consider the geodesic $\gamma(s)$ given by the expression (4.1). Let the collection $\lambda_1, \lambda_2, \dots, \lambda_n$ has the form

$$(4.4) \quad \lambda_1 = \lambda_2 = \dots = \lambda_{s_1} > \lambda_{s_1+1} = \lambda_{s_1+2} = \dots = \lambda_{s_2} > \dots$$

Let T_α be the subspace in \mathbb{R}^n which consists of vectors

$$(x_1, \dots, x_{s_\alpha}, 0, 0, \dots)$$

Let $V_\alpha = AT_\alpha$ (see (4.3)). We denote by $F(\gamma)$ the flag

$$(4.5) \quad V_1 \subset V_2 \subset V_3 \subset \dots$$

We obtain the map $F : Sky \rightarrow \mathcal{F}$. It is easy to see that the geodesic γ is determined by the pair

$$(D(\gamma); F(\gamma)) \in \Delta \times \mathcal{F}$$

A pair (velocity (4.4), flag (4.5)) is not arbitrary. It has to satisfy the condition $\dim V_j = s_j$.

4.4. Limits of geodesics on the matrix sky. Let us consider arbitrary geodesic $\gamma(s)$ given by the formula (4.1)-(4.2). Let us consider the geodesics $\kappa_s(t)$ joining the points 0 and $\gamma(s)$. We want to calculate $\lim_{s \rightarrow \infty} \kappa(s)$.

For this purpose let us represent the matrix $A \in GL(n, \mathbb{R})$ in the form $A = UB$ where $U \in O(n)$ and B is uppertriangle matrix. It is easy to prove that the limit of the family of geodesics γ_s is the geodesics $\sigma(t)$ given by the formula

$$\sigma(t) = U \begin{pmatrix} \exp(\lambda_1 t) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \exp(\lambda_n t) \end{pmatrix} U^{-1}$$

This remark has several simple corollaries .

Construction of the matrix sky doesn't depend on the point x_0 .

Indeed let us consider two points x_0 and x_1 and denote the associated matrix skies by $Sk(x_0), Sk(x_1)$. Let us consider a geodesic $\gamma(s)$ with the origin in x_1 . Then $\gamma(s)$ has limit on $Sk(x_0)$. Hence we obtain the canonical map $\psi_{10} : Sk(x_1) \rightarrow Sk(x_0)$. We also have canonical map $\psi_{01} : Sk(x_0) \rightarrow Sk(x_1)$. It is easy to show that $\psi_{01} \circ \psi_{10} = id, \psi_{10} \circ \psi_{01} = id$ and we obtain the canonical bijection $Sk(x_0) \leftrightarrow Sk(x_1)$.

In particular for each point $x \in G/K$ and each point $y \in Sk$ there exists the unique geodesic joining x and y .

The group G/K act by the natural way on the space $(G/K)^{vis}$.

Indeed the group G acts on the space of geodesics.

For each $g \in G$ and each $\gamma \in Sk$

$$D(g \cdot \gamma) = D(\gamma) \quad F(g \cdot \gamma) = g \cdot F(\gamma)$$

4.5. *Simplicial structure on the matrix sky.* Let us consider a complete flag $L \in \mathcal{F}_{complete}$

$$L : 0 \subset W_1 \subset W_2 \subset \dots \subset W_{n-1} \subset \mathbb{R}^n; \quad \dim W_j = j$$

Let us consider the embedding

$$\sigma_L : \Delta \rightarrow Sk$$

defined by the conditions

1. $G \circ \sigma_L$ is the identity map $\Delta \rightarrow \Delta$
2. The image of the map $F \circ \sigma_L : \Delta \rightarrow \mathcal{F}$ consists of subflags of the flag L .

Now we will give a explicit construction of the map σ_L . Without loss of generality we can consider the flag

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \dots \subset \mathbb{R}^{n-1}$$

in \mathbb{R}^n , the subspace \mathbb{R}^j consists of vectors $(x_1, \dots, x_j, 0, \dots, 0)$. Let

$$1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq 0$$

be a point of Δ . Then the associated geodesic (we remind that geodesic is identified with the point of Sk) has the form

$$\gamma(s) = \begin{pmatrix} \exp(s) & & & & \\ & \exp(\mu_2 s) & & & \\ & & \ddots & & \\ & & & \exp(\mu_{n-1} s) & \\ & & & & 1 \end{pmatrix}$$

Hence we obtain the tiling of the sphere Sk by the simplices $\sigma_L(\Delta)$. These simplices are enumerated by the points L of the space of *complete* flags. It is easy to show that this tiling satisfies the conditions

- a) Let $g \in SL(n, \mathbb{R})$. Then

$$\sigma_{gL}(\Delta) = g \cdot \sigma_L(\Delta)$$

- b) If $L \neq L'$ then the interiors of simplices $\sigma_L(\Delta)$ and $\sigma_{L'}(\Delta)$ doesn't intersect
- c) Let

$$L : V_1 \subset V_2 \subset \dots \subset V_{n-1}$$

$$L' : V'_1 \subset V'_2 \subset \dots \subset V'_{n-1}$$

be complete flags. If $V_j \neq V'_j$ for all j then $\sigma_L(\Delta) \cap \sigma_{L'}(\Delta) = \emptyset$. In the opposite case the intersection

$$\Lambda = \sigma_L(\Delta) \cap \sigma_{L'}(\Delta)$$

is a joint face of simplices $\sigma_L(\Delta)$ and $\sigma_{L'}(\Delta)$. Let us describe Λ . Let $\alpha_1, \dots, \alpha_s$ be all indices j such that $V_j = V'_j$ (i.e $V_{\alpha_i} = V'_{\alpha_i}$ and $V_j \neq V'_j$ for all $j \neq \alpha_i$). Let us consider the face

$$1 = \lambda_1 = \dots = \lambda_{\alpha_1} > \lambda_{\alpha_1+1} = \lambda_{\alpha_1+2} = \dots = \lambda_{\alpha_2} > \dots$$

of the simplex Δ . Then

$$\Lambda = \Sigma_L(N) = \Sigma_{L'}(N)$$

Now we obtain on the sphere Sk the structure of a Tits building (see [30])

4.6. Tits metric on the matrix sky. Let us consider points $y_1, y_2 \in \sigma_L(\Delta)$. We define the distance $d(y_1, y_2)$ as the angle between geodesics x_0y_1 and x_0y_2 . Let $z, u \in Sk$. Let us consider a chain

$$z = z_1, z_2, \dots, z_\beta = u \quad (z_j \in Sk)$$

such that for all j points z_j, z_{j+1} belongs to one element of our tiling.

Let us define the *Tits metric* $D(\cdot, \cdot)$ on Sk by the formula

$$D(z, u) = \inf\left(\sum_j d(z_j), d(z_{j+1})\right)$$

(we consider the infimum by the all chains z_1, \dots, z_β).

Remark 4.1. The topology on the Sk defined by the Tits metric is not equivalent to the usual topology of the sphere.

Example 4.2. Let $n = 3, G/K = SL(3, \mathbb{R})/SO(3)$. Then Sk is the 4-dimensional sphere S^4 , $\dim \Delta = 1$, i.e the simplices $\sigma_L(\Delta)$ are segments. We will describe the simplicial structure on $Sk = S^4$. Let P be the space of all 1-dimensional linear subspaces in \mathbb{R}^3 and Q be the space of 2-dimensional subspaces in \mathbb{R}^3 (evidently $P \simeq Q$ are the projective planes). We want to construct some graph Γ . The set of vertices of Γ is $P \cup Q$. Let $p \in P, q \in Q, p \subset q$. Then p and q are adjacent to the same edge and all edges have such form. Assume that the length of each edge is $\pi/3$. Then graph Γ is isometric to the sphere $Sk = S^4$ endowed with the Tits metric.

4.7. Abel subspaces. Let A be a orthogonal matrix. Let us consider the submanifold $R[A] \subset SL(n, \mathbb{R})/SO(n)$ consisting of matrices of the form

$$\psi_A(s_1, \dots, s_{n-1}) = A \begin{pmatrix} \exp(s_1) & & & & \\ & \exp(s_2) & & & \\ & & \ddots & & \\ & & & \exp(s_{n-1}) & \\ & & & & 1 \end{pmatrix} A^{-1}$$

where $s_1, \dots, s_{n-1} \in \mathbb{R}$.

The map

$$(s_1, \dots, s_{n-1}) \mapsto \psi(s_1, \dots, s_{n-1})$$

is the isometric embedding $\mathbb{R}^{n-1} \rightarrow SL(n, \mathbb{R})/SO(n)$ (with respect to the standard metrics in \mathbb{R}^{n-1} and in $SL(n, \mathbb{R})/SO(n)$).

Let us consider the trace $S[A]$ of the space $R[A]$ on the surface Sk . It is easy to see that $S[A]$ is the union of $(n-1)!$ simplexes $\sigma_L(\Delta)$. These simplices are separated by the hyperplanes $s_i = s_j$.

5. HYBRIDIZATION: DYNKIN-OLSHANETSKY AND KARPELEVICH BOUNDARIES

5.1 *Hybridization.* Let

$$i_1 : G/K \rightarrow X$$

$$i_2 : G/K \rightarrow Y$$

be embeddings of symmetric space G/K to compact metric spaces X and Y . Let the images of G/K in X and Y be dense.

Let us consider the embedding

$$i_1 \times i_2 : G/K \rightarrow X \times Y$$

defined by the formula

$$h \mapsto (i_1(h), i_2(h))$$

where $h \in G/K$. Let Z be the closure of the image of G/K in $X \times Y$. Then Z is the new compactification of G/K . We say that Z is the *hybrid* of X and Y .

We want to apply this construction in the case then X is a velocity compactification and Y is Satake-Furstenberg compactification.

5.2. *Dynkin-Olshanetsky boundary.* Let us consider the hybrid Z of the simplest velocity compactification (see 3.1) and Satake-Furstenberg compactification of Riemann noncompact symmetric space. Again let us consider only the case $G/K = SL(n, \mathbb{R})/O(n)$.

A point of the space Z is given by the following data

0*. $s = 1, 2, \dots, n - 1$

1*. A hinge

$$\mathfrak{P} = (P_1, \dots, P_s)$$

such that P_j are nonnegative definite (see section 9)

2*. A point of the simplex Δ_s :

$$1 \geq \mu_2 \geq \dots \geq \mu_{s-1} \geq 0$$

Let $x^{(j)} \in SL(n, \mathbb{R})/O(n)$ be a unbounded sequence. Let $a_1^{(j)} \geq \dots \geq a_n^{(j)}$ be the eigenvalues of $x^{(j)}$. Let $\lambda_\alpha^{(j)} = \ln a_\alpha^{(j)}$. Then the point $\Lambda(x^{(j)}) := (\lambda_1^{(j)}, \lambda_2^{(j)}, \dots)$ be a point of the the simplicial cone Σ_n (see 4.1). The sequence $x^{(j)} \in SL(n, \mathbb{R})/O(n)$ converges in Z if $x^{(j)}$ converges in Furstenberg-Satake compactification and $\Lambda(x^{(j)})$ converges in the velocity simplex $\bar{\Sigma}_n = \Sigma_n \cup \Delta$.

Now we want to explain how to calculate $\lim x^{(j)}$. Let $\mathfrak{P} = (P_1, \dots, P_s)$ be the limit of $x^{(j)}$ in Satake-Furstenberg compactification. Let $\gamma_j = \dim \text{Im}(P_j)$. Let $(\tau_2, \dots, \tau_{s-1})$ be the limit of $\Lambda(x^{(j)})$ in the simplex Δ . Then the collection $\tau_2 \geq \tau_3 \geq \dots$ has the form

$$(5.1) \quad 1 = \tau_1 = \dots = \tau_{\gamma_1} > \tau_{\gamma_1+1} = \dots = \tau_{\gamma_2} > \dots$$

We assume

$$(5.2) \quad \mu_j := \tau_{\gamma_{j-1}+1} = \dots = \tau_{\gamma_j}$$

and we obtain the data $0^* - 2^*$.

5.3. The projection of the Dynkin-Olshanetsky boundary to the matrix sky. Let we have data $0^* - 2^*$ Let us consider the new data

1⁺. The flag

$$\text{Ker}(P_1) \supset \text{Ker}(P_2) \supset \text{Ker}(P_3) \supset \dots$$

2⁺. The collection of numbers $\tau_2, \dots, \tau_{s-1}$ defined by the formula (5.2)

These data define the point of the matrix sky (see 4.2-4.3)

5.4. Limits of geodesics. Let us consider a geodesics

$$\gamma(s) = A \begin{pmatrix} \exp(\lambda_1 s) & & & \\ & \exp(\lambda_2 s) & & \\ & & \dots & \\ & & & 1 \end{pmatrix} A^t$$

where $A \in SL(n, \mathbb{R})$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = 0$. Let $\mathfrak{P} = (P_1, \dots, P_s)$ be the limit of $\gamma(s)$ in the space of hinges. The limit of $\gamma(s)$ in the velocity simplex Δ_n is $\tau_2, \dots, \tau_{n-1}$ where $\tau_j = \lambda_j / \lambda_1$.

Let $\gamma_\alpha = \dim \text{Im}(P_\alpha)$. We define numbers

$$\mu_\alpha := \tau_{\gamma_{\alpha-1}+1} = \dots = \tau_{\gamma_\alpha}$$

Now we obtain the data $0^* - 2^*$.

Remark 5.1. Not all points of Dynkin-Olshanetsky boundary are limits of geodesics. A point defined by the data $0^* - 2^*$ is the limit of a geodesics if and only if $1 > \mu_2 > \dots > \mu_{n-1} > 0$

5.5. Karpelevich compactification. The Karpelevich compactification is the hybrid of the compactification by Karpelevich velocities and Satake-Furstenberg compactification.

A point of the Karpelevich compactification is given by the following data

0^{*}. $s = 1, \dots, n - 1$

1^{*}. A hinge

$$\mathfrak{P} = (P_1, \dots, P_s)$$

such that are positive definite (see section 10)

2^{*}. A point of the boundary of the Karpelevich velocity polyhedron $\mathcal{K}(1, s)$ (see 3.2)

The topology on Karpelevich compactification is defined by the obvious way .

The natural projection $\partial\mathcal{K}(1, s) \rightarrow \Delta(I_{1,s})$ defines the projection of Karpelevich boundary to Dynkin-Olshanetsky boundary

6. SPACE OF GEODESICS AND SEA URCHINS

6.1. Spase of geodesics. Let us consider a Riemann noncompact symmetric space $G/K = SL(n, \mathbb{R})/SO(n)$. Denote by \mathfrak{G} the spase of all oriented geodesics in G/K .

The question about topologies on \mathfrak{G} is delicate. I'll describe the topology which seems to me the most natural.

Let us consider a collection of integers $A = (\alpha_0, \dots, \alpha_\sigma)$ such that

$$1 = \alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_\sigma = n$$

Let us denote by $\Delta(A)$ the open simplex

$$1 = \lambda_1 = \dots = \lambda_{\alpha_1} > \lambda_{\alpha_1+1} = \dots = \lambda_{\alpha_2} > \dots > \lambda_{\alpha_{\sigma-1}+1} = \dots = \lambda_n = 0$$

Simplices $\Delta(A)$ don't intersect and $\cup_A \Delta(A)$ coincides with the simplex Δ_n .

Let us consider a geodesic $\gamma \in \mathfrak{G}$. Its velocity is a point of one of the simplices $\Delta(A)$. The space of all geodesics with a given velocity $\Lambda \in \Delta(A)$ is a $SL(n, \mathbb{R})$ -homogeneous space. The stabilizer $G(A)$ of the geodesic γ (up to conjugacy) depends only of the the collection A (it doesn't depend of Λ and the geodesic itself):

$$G(A) = \mathbb{R}_+^* \times \prod O(\alpha_{j+1} - \alpha_j).$$

We denote by $\mathfrak{G}(A)$ the space of all geodesics which velocities are elements of $\Delta(A)$. Then

$$\mathfrak{G}(A) \simeq \Delta(A) \times (SL(n, \mathbb{R})/G(A))$$

We equip this space with the usual topology of the direct product. We equip the space

$$\mathfrak{G} = \bigcup_A \mathfrak{G}(A) \simeq \bigcup_A \Delta(A) \times (SL(n, \mathbb{R})/G(A))$$

with the topology of disjoint union.

Remark 6.1. Hence the space of geodesics is disconnected set. It is not strange. Let $A_0 = \{0, 1, 2, \dots, n\}$ Let us consider the set of limits of the geodesics $\gamma \in \mathfrak{G}(A_0)$ on matrix sky. Then this set is open and dence. The set of limits of $\gamma \in \mathfrak{G}(A)_0$ in Satake boundary is compact. Hence it is natural to think that \mathfrak{G} is disconnected.

6.2. Spase of geodesics as boundary of symmetric space. We define the natural topology on the space

$$\mathfrak{R} = G/K \cup \mathfrak{G}$$

We equip the space G/K with the natural topology. The space \mathfrak{G} is equipped with the topology mentioned above and the spase \mathfrak{G} is closed in \mathfrak{R} . Fix a point $b_0 \in G/K$. Let $x_j \in G/K$ be a unbounded sequence. The sequence x_j converges in \mathfrak{R} if it satisfies the following conditions

1. Sequence of geodesics $b_0 x_j$ converges. Denote by y its limit on the matrix sky.

2. There exists a limit z of the sequence of geodesics yx_j .

The limit of the sequence x_j is defined to be the geodesic z .

Remark 6.2. In our case the dimension of the boundary

$$\dim \mathfrak{G} = 2 \dim G/K - 2$$

is greater than $\dim G/K$ (even in the case then $G/K = SL(2, \mathbb{R})/SO(2)$ is Lobachevskii plane)

Remark 6.3. The space \mathfrak{R} is not compact (since \mathfrak{G} is not compact)

6.3. Sea urchins. Recall that each geodesic $\gamma \in \mathfrak{G}$ has a velocity $\{\mu_2, \mu_3, \dots\}$ which is a point of the simplex Δ (see 3.1). We denote by \mathfrak{G}^{rat} the space of geodesics having rational velocities (i.e. μ_j are rational). Let us consider the set (*sea urchin*)

$$\mathfrak{R}^{rat} := G/K \cup \mathfrak{G}^{rat} \subset \mathfrak{R}$$

We don't interested by the topology on sea urchin (it is seems natural to consider the discrete topology on the set of velocities, the usual topology on the space of geodesics with a given velocity and the natural (see 6.2) convergence of sequences in G/K to geodesics)

6.4. Projective universality. Let ρ_j be a finite family of linear irreducible representations of the group G in the spaces V_j . We assume that for each j there exists a K -fixed nonzero vector $v_j \in V_j$. Let us consider the direct sum $\rho = \oplus \rho_j$ of representations ρ_j and the vector $w = \oplus v_j \in \oplus V_j$. Let us consider the projective space $\mathbb{P}(\oplus V_j)$. Let $\mathcal{O} \simeq G/K$ be the G -orbit of the vector $w \in \mathbb{P}(\mathcal{O})$. Let $\overline{\mathcal{O}}$ be the closure of \mathcal{O} in $\mathbb{P}(\oplus V_j)$.

The G -spaces $\overline{\mathcal{O}}$ are called *projective compactifications* of G/K

We will construct the map

$$\pi : \mathfrak{R} = G/K \cup \mathfrak{G} \rightarrow \overline{\mathcal{O}}$$

The map $G/K \rightarrow \mathcal{O}$ is obvious. Let us consider a geodesic $\gamma(s) \in \mathfrak{G}$. It is easy to prove that there exists $\lim_{s \rightarrow \infty} \rho(\gamma(s))$ in $\mathbb{P}(\oplus V_j)$. By definition $\pi(\gamma)$ is this limit.

Proposition 6.4. *a) The map $\pi : \mathfrak{R} \rightarrow \overline{\mathcal{O}}$ is surjective.*

b) Moreover the π -image of sea urchin \mathfrak{R}^{rat} is the whole $\overline{\mathcal{O}}$.

7. BIBLIOGRAPHICAL REMARKS

Remarks to section 1-2. The Satake-Furstenberg boundary is a version of Study-Semple-Satake- Furstenberg-De Concini-Procesi-Oshima boundary (see [1-7]) of symmetric space G/H where G is a semisimple group and H is a symmetric subgroup (i.e. subgroup H is set of fixed points for some involution on the group G). The usual definition is the following. Let us consider a finite-dimensional irreducible representation of G having a H -fixed vector v (the representation ρ have to satisfy some nondegeneracy conditions). Then our compactification is the closure of the orbit $G \cdot v$ in the projective space. The coincidence of our construction with classical is not obvious, for construction of projective embedding of space $Hinge(n)$ see [8,10].

Hinges were defined in [8], see also [10]. For construction of separated quotient space through Hausdorff metric see [9]. For construction of separated quotient space it is also possible to use closure in Chow scheme, see [11-12].

Our space $Hinge(n)$ is one of the real form of Semple complete collineation variety. The Satake-Furstenberg compactification of $SL(n, \mathbb{R})$ is one of the real forms of Study-Semple complete quadrics.

Data $1^* - 2^*$ were introduced in [3].

Remarks to section 3. I haven't seen this construction in literature. The analogy of the collection $\{\ln \lambda_j\}$ for arbitrary symmetric space is so-called complex (or compound) distance (see for instance [10])

Kaprelevisch velocity polyhedron is the closure of a Weyl chamber in Kaprelevisch compactification.

Remarks to section 4. See [15,32].

The most of constructions described in this paper are very exotic from the point of view of the official differential geometry. The visibility boundary is exception. It is more or less general differential-geometric object, see [13-15].

Tits metric on the infinitely distant sphere (see also [16] for boundaries of Bruhat-Tits buildings) also is more or less general construction (see [15]). Nevertheless nice tiling of the sphere also seems exceptional phenomena.

Remarks to section 5. . Kaprelevisch boundary was constructed in [17] in terms of geometry of geodesics. Dynkin-Olshanetsky boundary (see [18-20]) is Martin boundary (see [22-25]) for the diffusion on symmetric spaces Discussion of these boundaries see also [21].

Remarks to section 6. I havn' seen sea urchin construction in literature. See [26-27] for universal projective compactification of symmetric space (see urchin is not compact).

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