Symmetric spaces, adapted complex structures and hypercomplex structures

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Abstract. We study two complex structures, I_* and J, defined on domains in the tangent bundle of a hermitian manifold. We define I_* using the complex structure on M, while J is constructed using the Riemannian metric on M. We show that if M is a hermitian symmetric space associated to a classical group then the pullback of J by a suitable diffeomorphism of domains in TM anticommutes with I_*. A corollary is the existence of a hypercomplex structure on a domain in TM.

0. Introduction.

Let (M,g) be a Riemannian manifold and $0 < S \leq \infty$. We shall denote by $T^S M$ the subset of the tangent bundle TM consisting of those vectors whose norm is less than S. We similarly define the subset $T^{*S}M$ of the cotangent bundle.

The second author [5] (see also [2], [4]) has shown that for each compact real-analytic Riemannian manifold (M,g) there is a positive real number S, such that we may define a canonical complex structure J on $T^S M$ (see section 1 for details). We can use the metric to identify TM and T^*M, and so also define a complex structure J^* on $T^{*S}M$. The zero section M is totally real with respect to J and J^*. We call J the *adapted complex structure*. It was shown in [5] that if (M,g) is a compact symmetric space then J can be defined on the entire tangent bundle.

On the other hand, if M is itself a complex manifold, then the complex structure on M induces complex structures I, I^* on the tangent and cotangent bundles respectively. The zero section is a complex submanifold with respect to these complex structures, so they are distinct from J and J^*.

In this paper we make a first step towards exploring the relation between the adapted complex structure and that induced by a complex structure on M. For technical reasons we prefer to work on the tangent bundle instead of the cotangent bundle. Again using the metric to identify TM and T^*M we can pull back I^* to obtain a complex structure I_* on then tangent bundle. We shall prove the following theorem.

Theorem 0.1

Let M be a compact irreducible hermitian symmetric space U/K where U is one of the classical groups. Let I_*, J denote the complex structures on TM discussed above.

Then there exists a real-analytic diffeomorphism ϕ of TM such that the pullback J^ϕ of J by ϕ anticommutes with I_*. □

When M is a noncompact irreducible hermitian symmetric space of classical type we shall introduce in §4 a continuous non-negative function $G : TM \to \mathbb{R}$, invariant with respect to the isometry group of M, such that

$$G(tX) = |t| G(X)$$

whenever $t \in \mathbb{R}$ and $X \in TM$. The open unit disc bundle T^1M is contained in $G^{-1}((0,1])$.

Theorem 0.2

Let M be a noncompact irreducible hermitian symmetric space U^*/K associated to a classical group. Then J is defined on $G^{-1}([0,\frac{\pi}{4}))$ and there exists a real-analytic diffeomorphism ϕ of $G^{-1}([0,1))$ onto $G^{-1}([0,\frac{\pi}{4}))$ such that J^ϕ anticommutes with I_* on $G^{-1}([0,1))$. □

It follows that the endomorphism $aI_* + bJ^\phi + cI_*J^\phi$ is a complex structure on TM (respectively $G^{-1}([0,1))$) whenever $a^2+b^2+c^2 = 1$, so I_* and J^ϕ generate a hypercomplex structure on TM (respectively $G^{-1}([0,1))$).

In fact, we shall see that the diffeomorphisms of Theorems 0.1, 0.2 can be chosen to be equivariant with respect to the isometry group of M. As I_* and J are invariant with respect to this group, we can deduce the existence of a hypercomplex structure on T^1M whenever M is a locally symmetric quotient of a classical irreducible hermitian symmetric space.

The adapted complex structure for a product of manifolds is just obtained by taking the product of the individual adapted complex structures. It is straightforward therefore to make the appropriate generalisations of Theorems 0.1 and 0.2 to arbitrary symmetric spaces.

Hypercomplex (indeed hyperkähler) structures have been shown to exist on the cotangent bundles of compact hermitian symmetric spaces by Burns [1] using twistor methods. We conjecture that the hypercomplex structure generated by I_* and J^ϕ coincides with that of Burns.

It easily follows from the results of §1 that if M is a hermitian symmetric space then I_* and J anticommute only if M is flat. This explains why we need to conjugate J by a suitable diffeomorphism in the statements of Theorems 0.1 and 0.2.

1. Complex structures.

We shall briefly discuss the theory of adapted complex structures developed in [4], [5]. Consider a complete Riemannian manifold (M,g). If $\gamma : \mathbb{R} \mapsto M$ is a geodesic, we can define a map $\psi_\gamma : \mathbb{C} \mapsto TM$ by

$$\psi_\gamma(\sigma + i\tau) = \tau\dot\gamma(\sigma).$$

For each γ the image of $\mathbb{C} \setminus \mathbb{R}$ under ψ_γ is a leaf of a foliation of $TM \setminus M$, called the Riemann foliation.

Definition 1.1

Let D be a domain in TM containing the zero section. Assume moreover (to avoid problems with analytic continuation) that for every geodesic γ in M the open set $\psi_\gamma^{-1}(D)$ is a simply connected domain in $\mathbb{C}$.

A complex structure J on D is called *adapted* if for every geodesic γ in M, the map ψ_γ is holomorphic on $\psi_\gamma^{-1}(D)$.

Theorem 1.2 [4], [5]

If an adapted complex structure exists it is unique. Moreover if (M,g) is a compact or homogeneous real-analytic Riemannian manifold then there exists $S > 0$ such that T^SM admits an adapted complex structure. □

If M is locally symmetric we can be more precise.

Theorem 1.3 [4]

Let M be a complete locally symmetric space.

(i) If M has nonnegative sectional curvature (in particular if M is compact and symmetric) then an adapted complex structure exists on the whole tangent bundle.

(ii) If the sectional curvatures are bounded below by $\theta < 0$, then an adapted complex structure exists on T^SM where $S = \pi/(2\sqrt{-\theta})$. □

The adapted complex structure can be described more explicitly as follows.

Let z be a point of $TM \setminus M$; here we regard z as a tangent vector to M at a point m. Using the Levi-Civita connexion defined by the Riemannian metric on M we can express $T_z(TM)$ as the direct sum of horizontal and vertical spaces T_z^H and T_z^V. The latter is just the tangent space of the fibre T_mM through z, and can be canonically identified with T_mM. Moreover the derivative of the projection map $\pi : TM \to M$ identifies T_z^H with T_mM.

Therefore any vector v in $T_m M$ defines tangent vectors $\bar{\xi}_v$ in T_z^H and $\bar{\eta}_v$ in T_z^V; these are the horizontal and vertical lifts of v respectively.

Now let γ be the geodesic in M with $\gamma(0) = m$ and $\dot\gamma(0) = z/\parallel z \parallel$. Let $v_2, \dots v_n$ be tangent vectors to M at m such that $\dot\gamma(0), v_2, \dots, v_n$ is an orthonormal basis of $T_m M$. We shall let $v_1 = \dot\gamma(0)$.

We can associate to v_i Jacobi fields along γ. We let ξ_i, η_i be the Jacobi fields along γ with initial conditions

$$\xi_i(0) = v_i, \quad \nabla_{\dot\gamma}\xi_i(0) = 0,$$

$$\eta_i(0) = 0, \quad \nabla_{\dot\gamma}\eta_i(0) = v_i.$$

In particular, $\xi_1(t) = \dot\gamma(t)$ and $\eta_1(t) = t\dot\gamma(t)$.

If $i > 1$ the Jacobi fields ξ_i, η_i are *normal*, that is, orthogonal to the velocity vector field of the geodesic.

The ξ_i are pointwise linearly independent (except possibly on a discrete subset of $\mathbb{R}$) so there exist smooth functions Φ_{jk} such that

$$\eta_k = \sum_{j=2}^{n} \Phi_{jk}\xi_j, \quad (k = 2, \dots, n).$$

Now suppose that $T^S M$ admits an adapted complex structure for some $S \leq \infty$. Then the functions Φ_{jk} have meromorphic extensions F_{jk} to the strip $D_S = \{\sigma + i\tau \in \mathbb{C} : |\tau| < S\}$ such that the poles of F_{jk} lie on $\mathbb{R}$ and the matrix $(\text{Im } F_{jk})$ is invertible on $D_S \backslash \mathbb{R}$. Now let (e_{jk}) be the matrix whose inverse is $(\text{Im } F_{jk})$.

The adapted complex structure at z is now given by

$$J_z\bar{\xi}_{v_h} = \sum_{k=2}^{n}\left(e_{kh}(i \parallel z \parallel) \times \left(\parallel z \parallel \bar{\eta}_{v_k} - \sum_{j=2}^{n} \text{Re}F_{jk}(i \parallel z \parallel)\bar{\xi}_{v_j}\right)\right), \quad (h = 2, \dots, n), \tag{1}$$

$$J_z\bar{\xi}_{v_1} = \bar{\eta}_{v_1}.$$

This formula is slightly different from that given in [5], because the vectors $\bar{\eta}$ as defined in [5] are obtained by multiplying the $\bar{\eta}$ of our definition by a factor of $\parallel z \parallel$.

If M is locally symmetric our formula simplifies dramatically. The Jacobi operator $R_{\dot\gamma(t)}$: $T_{\gamma(t)}M \to T_{\gamma(t)}M$ defined by

$$v \mapsto \mathcal{R}(v, \dot\gamma(t))\dot\gamma(t),$$

(where $\mathcal{R}$ is the curvature tensor) is symmetric at each t, so can be diagonalised at $t = 0$ by an orthonormal basis of eigenvectors v_j with eigenvalues Λ_j. We can take $v_1 = \dot\gamma(0)$. Let V_j be the vector field obtained by parallelly transporting v_j along γ. If M is locally symmetric then the curvature tensor is parallel, so $R_\gamma(V_j)$ and $\Lambda_j V_j$ are parallel vector fields along γ agreeing at $t = 0$, and hence agreeing everywhere. That is, $V_j(t)$ is an orthonormal basis of eigenvectors of the Jacobi operator for each t. Note that $V_1(t) = \dot\gamma(t)$.

The Jacobi equation

$$\nabla_{\dot\gamma}^2 X + R_\gamma(X) = 0$$

is thus diagonalised, and splits into a set of first order ODEs. For rV_j is a Jacobi field precisely when

$$\ddot{r} + \Lambda_j r = 0. \tag{2}$$

Therefore the Jacobi fields ξ_j, η_j are defined by $\xi_j = g_j V_j, \eta_j = h_j V_j$, where g_j, h_j satisfy (2) and

$$g_j(0) = 1 \ , \ \dot{g}_j(0) = 0,$$

$$h_j(0) = 0 \ , \ \dot{h}_j(0) = 1.$$

We find that $\Phi_{jk} = 0$ for $j \neq k$, and

$$\Phi_{jj}(t) = \begin{cases} t & \text{if } \Lambda_j = 0, \\ \frac{\tan(\sqrt{\Lambda_j}t)}{\sqrt{\Lambda_j}} & \text{if } \Lambda_j > 0, \\ \frac{\tanh(\sqrt{-\Lambda_j}t)}{\sqrt{-\Lambda_j}} & \text{if } \Lambda_j < 0. \end{cases}$$

These equations, together with formula (1), yield (for $j = 2, \ldots, n$)

$$J_z \bar{\xi}_{v_j} = \bar{\eta}_{v_j} \ \text{ if } \ \Lambda_j = 0, \tag{3}$$

$$J_z \bar{\xi}_{v_j} = \sqrt{\Lambda_j} \parallel z \parallel \coth(\sqrt{\Lambda_j} \parallel z \parallel) \bar{\eta}_{v_j} \ \text{ if } \ \Lambda_j > 0, \tag{4}$$

$$J \bar{\xi}_{v_j} = \sqrt{-\Lambda_j} \parallel z \parallel \cot(\sqrt{-\Lambda_j} \parallel z \parallel) \bar{\eta}_{v_j} \ \text{ if } \ \Lambda_j < 0. \tag{5}$$

We also have

$$J_z \bar{\xi}_{v_1} = \bar{\eta}_{v_1}. \tag{6}$$

If (M, g) is locally symmetric then its universal cover splits isometrically as a product of a Euclidean space $\mathbb{R}^n$, a compact symmetric space $(M^{(1)}, g^{(1)})$ and a noncompact symmetric space $(M^{(2)}, g^{(2)})$. The adapted complex structure is defined everywhere on $T(\mathbb{R}^n \times M^{(1)})$ and the formulae (3)-(6) are valid away from the zero section. If X is a tangent vector to $M^{(2)}$ at m, let $\theta(X)$ be the most negative value of the sectional curvatures of planes in $T_m M^{(2)}$ containing X. Then the maximal domain in $TM^{(2)}$ where the adapted complex structure exists consists of those vectors X whose norm in the metric $g^{(2)}$ is less than $\pi/2\sqrt{-\theta(X)}$. Our formulae (3)-(6) are valid for all such nonzero X.

We shall conclude this section by discussing the complex structure on TM arising from a complex structure on M. We shall restrict ourselves to the case of Kähler manifolds.

Consider a Kähler manifold M with metric g and complex structure I_0. At each point z in TM we split the tangent space at z as

$$T_z(TM) = T_z^H \oplus T_z^V, \tag{7}$$

as discussed earlier. The Kähler condition on M implies that the complex structure I on TM induced by I_0 preserves T_z^H as well as T_z^V. With respect to the decomposition (7) it is just given by

$$I = I_0 \oplus I_0.$$

Now, I_0 also induces a complex structure I^* on the cotangent bundle T^*M. Identifying TM with T^*M using the metric we can pull back I^* to obtain a new complex structure I_* on TM. With respect to (7), I_* is defined by

$$I_* = I_0 \oplus (-I_0). \tag{8}$$

2. Symmetric spaces.

We now restrict ourselves to the case when M is a Hermitian (hence Kähler) irreducible symmetric space. Our aim is to find a diffeomorphism ϕ of the tangent bundle of M such that the pullback of J by ϕ commutes with I_*.

Our strategy is to first consider a diffeomorphism of the tangent space at one point, equivariant with respect to the isotropy action, and then extend it to the whole tangent bundle by homogeneity.

We first review the Cartan theory for symmetric spaces. Our reference for this material is Helgason [3] and we follow his notation.

Let $M = U/K$ be a compact irreducible symmetric space, with Cartan decomposition

$$\mathfrak{u} = \mathfrak{k}_0 + \mathfrak{p}_*, \tag{9}$$

where $\mathfrak{u}, \mathfrak{k}_0$ are the Lie algebras of U and K respectively.

If we fix a basepoint o in U/K, then we can identify $\mathfrak{p}_*$ with the tangent space at this point. Denote by $\mathcal{R}$ the curvature tensor on U/K, so **[3, Ch. IV, §4]** if $X, Y, Z \in \mathfrak{p}_*$ we have

$$\mathcal{R}(X,Y)Z = -[[X,Y],Z].$$

It follows that the Jacobi operator $R_X = \mathcal{R}(.,X)X$ is equal to $-(\mathrm{ad}_X)^2$.

Letting $\mathfrak{p}_0 = i\mathfrak{p}_*$ we have the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \tag{10}$$

for the noncompact dual symmetric space U^*/K.

We denote by $\mathfrak{g}$ the complexification of $\mathfrak{u}$, so $\mathfrak{u}$ and $\mathfrak{g}_0$ are real forms of the complex Lie algebra $\mathfrak{g}$.

Let $\mathfrak{h}_{\mathfrak{p}_*}$ be a maximal abelian subspace of $\mathfrak{p}_*$; then $\mathfrak{h}_{\mathfrak{p}_0} = i\mathfrak{h}_{\mathfrak{p}_*}$ is a maximal abelian subspace of $\mathfrak{p}_0$. The dimension r of $\mathfrak{h}_{\mathfrak{p}_*}$ is the *rank* of the symmetric space.

Let $\mathfrak{h}_0$ be a maximal abelian subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{h}_{\mathfrak{p}_0}$. Then the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ generated by $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}$. Finally, we denote by $\mathfrak{h}_{\mathfrak{p}}$ the subspace of $\mathfrak{g}$ generated by $\mathfrak{h}_{\mathfrak{p}_0}$.

Let $\Delta_{\mathfrak{p}}$ be the set of nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ whose restriction to $\mathfrak{h}_{\mathfrak{p}}$ is not identically zero. For each $\alpha \in \Delta_{\mathfrak{p}}$ the kernel of the restriction of α to the abelian subspace $\mathfrak{h}_{\mathfrak{p}_*}$ is a hyperplane $L(\alpha)$ in $\mathfrak{h}_{\mathfrak{p}_*}$. The elements of Δ_p take real values on the subspace $\mathfrak{h}_{\mathfrak{p}_0}$. We let Σ be the set of elements of the dual of $\mathfrak{h}_{\mathfrak{p}_0}$ obtained by restriction of elements of Δ_p. We call Σ the set of restricted roots.

Definition 2.1

(i) The *Weyl group* $\mathcal{W}(U,K)$ is the quotient of the group of elements of K preserving $\mathfrak{h}_{\mathfrak{p}_*}$ by the subgroup of elements of K acting trivially on $\mathfrak{h}_{\mathfrak{p}_*}$.

(ii) The *open Weyl chambers* are the connected components of the complement in $\mathfrak{h}_{\mathfrak{p}_*}$ of the union of hyperplanes $\bigcup\{L(\alpha) : \alpha \in \Delta_{\mathfrak{p}}\}$.

Theorem 2.2 ([3] Ch. V §6, Ch. VII §2)

Each orbit of K on $\mathfrak{p}_*$ intersects the maximal abelian subspace $\mathfrak{h}_{\mathfrak{p}_*}$. Moreover if two points of $\mathfrak{h}_{\mathfrak{p}_*}$ lie in the same orbit of K, they lie in the same orbit of the Weyl group. □

Theorem 2.3 ([3] VII §2)

The Weyl group is generated by the reflexions in the hyperplanes $L(\alpha)$, and acts simply transitively on the set of Weyl chambers. Moreover, the closure of any Weyl chamber contains exactly one point from each orbit of the Weyl group on $\mathfrak{h}_{\mathfrak{p}_*}$. □

Corollary 2.4

The closure of any Weyl chamber is a transversal for the action of K on $\mathfrak{p}_*$. □

We shall construct K-equivariant diffeomorphisms of $\mathfrak{p}_* = T_o(U/K)$ by extending maps of a closed Weyl chamber onto itself.

Lemma 2.5

Let C be a Weyl chamber with closure $\bar{C}$. Let f be a bijection of $\bar{C}$ onto itself, such that for any $x \in \bar{C}$ the stabiliser of x for the K action equals the stabiliser of $f(x)$. Then we can extend f to a K-equivariant bijection of $\mathfrak{p}_*$.

Proof

Let $y \in \mathfrak{p}_*$; then from Corollary 2.4 there exists $k \in K$ and $x \in \bar{C}$ with $y = k.x$. Define $\phi(y) = k.f(x)$. If $k_1.x_1 = k_2.x_2$ for $k_j \in K, x_j \in \bar{C}$ then $k_2^{-1}k_1x_1 = x_2$ so by Corollary 2.4 we must have $x_1 = x_2$, and $k_2^{-1}k_1$ stabilises x_1. By our hypothesis $k_2^{-1}k_1$ stabilises $f(x_1)$ also, so $\phi(y)$ is well defined. Clearly ϕ is K-equivariant. If we define ϕ^{-1} in the same way using f^{-1}, then ϕ^{-1} is an inverse for ϕ. □

The closure $\bar{C}$ of a Weyl chamber C is a convex subset of $\mathfrak{h}_{\mathfrak{p}_*}$ bounded by hyperplanes $L(\alpha_1), \ldots, L(\alpha_m)$ where $L(\alpha_j) = \operatorname{Ker} \alpha_j$. If $\mathcal{S}$ is a subset of $\{\alpha_1, \ldots, \alpha_m\}$ we let $L_{\mathcal{S}} = \cap\{L(\alpha) : \alpha \in \mathcal{S}\}$. If $\mathcal{S}$ is empty we take $L_{\mathcal{S}} = \mathfrak{h}_{\mathfrak{p}_*}$.

Let Stab_x^U denote the stabiliser of $x \in \mathfrak{p}_*$ with respect to the action of U, and let Stab_x^K be the stabiliser of x with respect to the action of K.

The argument of Lemma 2.14 of Chapter VII of [3] shows that Stab_x^U depends only on the set of roots vanishing at x. Taking the intersection of Stab_x^U with K we see that this conclusion also holds for Stab_x^K. Equivalently, the stabiliser Stab_x^K depends only on the set of hyperplanes $L(\alpha_j)$ which contain x. We have thus established the following Lemma.

Lemma 2.6

Let f be a bijection of $\bar{C}$ onto itself which maps $\bar{C} \cap L_{\mathcal{S}}$ bijectively onto itself for each subset $\mathcal{S}$ of $\{\alpha_1, \ldots, \alpha_m\}$. Then we can extend f to a K-equivariant bijection of $\mathfrak{p}_*$ onto itself. □

Identifying $\mathfrak{p}_*$ with the tangent space to U/K at our basepoint, we have a bijection of $T_o(U/K)$ which is equivariant with respect to the isotropy action of K. We can now extend this map using the action of U to a U-equivariant bijection ϕ of $T(U/K)$.

We can argue similarly for the noncompact dual symmetric space U^*/K, whose Cartan decomposition is given by (10). We can take $\mathfrak{h}_{\mathfrak{p}_0} = i\mathfrak{h}_{\mathfrak{p}_*}$ as a maximal abelian subspace in $\mathfrak{p}_0$, and if $\bar{C}$ is the closure of a Weyl chamber in $\mathfrak{h}_{\mathfrak{p}_*}$, then $i\bar{C}$ is a transversal for the action of K on $\mathfrak{p}_0$. Then any bijection satisfying the hypotheses of Lemma 2.6 will extend to a K-equivariant bijection of $\mathfrak{p}_0$, and hence define a U^*-equivariant bijection of $T(U^*/K)$.

More generally, let $\mathcal{D}_1, \mathcal{D}_2$ be domains in $\mathfrak{h}_{\mathfrak{p}_0}$ and let f be a bijection of $\mathcal{D}_1 \cap i\bar{C}$ onto $\mathcal{D}_2 \cap i\bar{C}$ mapping $\mathcal{D}_1 \cap i\bar{C} \cap L_{\mathcal{S}}$ bijectively onto $\mathcal{D}_2 \cap i\bar{C} \cap L_{\mathcal{S}}$ for each $\mathcal{S}$. Let $\mathcal{E}_1, \mathcal{E}_2$ be the subsets of $T(U^*/K)$ associated to $\mathcal{D}_1 \cap \bar{C}, \mathcal{D}_2 \cap \bar{C}$ by the U^*-action. Then f extends to a U^*-equivariant bijection of $\mathcal{E}_1$ onto $\mathcal{E}_2$.

In sections 4 and 5 we shall consider special choices of f whose equivariant extensions are real-analytic diffeomorphisms of appropriate domains in the tangent bundle.

Finally, we prove a lemma which will be neded for the calculations of the next section.

Given a real-valued linear functional λ on $\mathfrak{h}_{\mathfrak{p}_*}$ we define a linear functional $\tilde{\lambda}$ on $\mathfrak{h}_{\mathfrak{p}_0}$ by seting $\tilde{\lambda}(w) = \lambda(-iw)$.

Lemma 2.7

Let λ be a real-valued linear functional on $\mathfrak{h}_{\mathfrak{p}_*}$, and suppose there exists a nonzero vector $X \in \mathfrak{p}_*$ with the property that

$$[[H, X], H] = \lambda(H)^2 X \quad \text{for all } H \in \mathfrak{h}_{\mathfrak{p}_*}. \tag{11}$$

Then $\tilde{\lambda}$ is a restricted root, and

$$[[H_1, X], H_2] = \lambda(H_1)\lambda(H_2)X \quad \text{for all } H_1, H_2 \in \mathfrak{h}_{\mathfrak{p}_*}. \tag{12}$$

Proof

Let $\tilde{H} \in \mathfrak{h}_{\mathfrak{p}_0}$. Then $-i\tilde{H} \in \mathfrak{h}_{\mathfrak{p}_*}$ and

$$[\tilde{H}, [\tilde{H}, iX]] = -i[-i\tilde{H}, [-i\tilde{H}, X]] = i\lambda(-i\tilde{H})^2 X = \tilde{\lambda}(\tilde{H})^2 iX.$$

As $\tilde{H}$ was arbitrary, it follows from Corollary 2.10 of Chapter VII of [3] that $\tilde{\lambda}$ is a restricted root.

We obtain (12) by putting $H = H_1 + H_2$ in (11) and using the Jacobi identity and the relation $[H_1, H_2] = 0$. □

3. Equivariant diffeomorphisms

In this section we shall establish two lemmas which will enable us to calculate the derivative of a U-equivariant diffeomorphism of the tangent bundle of a symmetric space U/K. We shall use these results in §4, when we study the pullback of the adapted complex structure by a diffeomorphism of $T(U/K)$.

Lemma 3.1
Let $M = U/K$ be a symmetric space, and ϕ a U-equivariant diffeomorphism of TM. Let o be the basepoint $[K]$ of M, and identify $\mathfrak{p}_*$ with T_oM. Suppose that ϕ restricts to a diffeomorphism of $\mathfrak{h}_{\mathfrak{p}_*}$ onto itself.

Let $v \in \mathfrak{p}_*$ and let z be a nonzero element of $\mathfrak{h}_{\mathfrak{p}_*}$. Let $\bar{\xi}_v(z), \bar{\xi}_v(\phi(z))$ denote the horizontal lifts of v to z and $\phi(z)$ respectively. Then

$$\phi_* \bar{\xi}_v(z) = \bar{\xi}_v(\phi(z)).$$

Proof

Let γ be the geodesic with $\gamma(0) = o$ and $\gamma'(0) = v$. As M is a symmetric space, γ is given [3] by

$$\gamma(t) = \exp(tv)o. \tag{13}$$

Moreover, on a symmetric space, parallel transport along the geodesic with equation (13) is given by $Y \mapsto \exp(tv)_* Y$. (We are regarding $\exp(tv) \in U$ as defining a transformation of M).

Therefore the vector field χ defined by

$$\chi(t) = \exp(tv)_* z$$

is parallel along γ and satisfies $\chi(0) = z$. We deduce that $\xi_v(z) = \dot{\chi}(0)$.

Now, using the equivariance of ϕ we have

$$\phi(\chi(t)) = \phi(\exp(tv)_*z) = \exp(tv)_*\phi(z), \tag{14}$$

so the vector field $t \mapsto \phi(\chi(t))$ is also parallel along γ. Differentiating (14) proves our claim. □.

Lemma 3.2

Let M, ϕ, o, v, z be as in Lemma 3.1. Let R_H be the Jacobi operator defined by $R_H v = \mathcal{R}(v, H)H$, where $\mathcal{R}$ is the curvature tensor.

Suppose that λ is a linear functional on $\mathfrak{h}_{\mathfrak{p}_*}$ and that

$$R_H v = \lambda(H)^2 v$$

for all $H \in \mathfrak{h}_{\mathfrak{p}_*}$. Assume moreover that $\lambda(z) \neq 0$. Denote by $\bar{\eta}_v(z)$ and $\bar{\eta}_v(\phi(z))$ the vertical lifts of v to z and $\phi(z)$ respectively.

Then

$$\phi_*\bar{\eta}_v(z) = \frac{\lambda(\phi(z))}{\lambda(z)}\bar{\eta}_v(\phi(z)).$$

Proof
Consider the curve in $\mathfrak{p}_* = T_oM$ defined by

$$\kappa : t \mapsto Ad(t\Theta)z,$$

where $\Theta = \lambda(z)^{-2}[z, v]$. Now $[\Theta, z] = v$, so $\kappa'(0)$ is the vertical lift $\bar{\eta}_v(z)$ of v to z.

Using the equivariance of ϕ again, we have

$$\phi(\kappa(t)) = \phi(Ad(t\Theta)z) = Ad(t\Theta)\phi(z).$$

Differentiating at $t = 0$ shows that $\phi_*\bar{\eta}_v(z)$ is the vertical lift to $\phi(z)$ of $[\Theta, \phi(z)]$, but by Lemma 2.7

$$[\Theta, \phi(z)] = \frac{1}{\lambda(z)^2}[[z, v], \phi(z)] = \frac{\lambda(z)\lambda(\phi(z))}{\lambda(z)^2}v,$$

giving the required result. □

4. Anticommuting complex structures.

For any compact hermitian symmetric space $M = U/K$ we have an adapted complex structure J on TM. If ϕ is a diffeomorphism of TM we can pull back J to obtain a new complex structure J^ϕ, defined by

$$J^\phi_z\zeta = \phi_*^{-1}J_{\phi(z)}\phi_*\zeta,$$

where $z \in TM$ and $\zeta \in T_z(TM)$. Our aim is to show the existence of a U-equivariant diffeomorphism of TM such that the complex structure I_* anticommutes with J^ϕ. We shall simplify the calculations by making a suitable choice of bases for the tangent space to TM at the points z and $\phi(z)$, and by exploiting the equivariance of ϕ.

Definition 4.1

Let M be a compact irreducible hermitian symmetric space of rank r, with its complex structure defined by an endomorphism I_0 of $\mathfrak{p}_*$, We shall say that M satisfies condition (*) if there exists a maximal abelian subspace $\mathfrak{h}_{\mathfrak{p}_*}$ in $\mathfrak{p}_*$, an orthonormal basis $e_1, \ldots, e_r$ for $\mathfrak{h}_{\mathfrak{p}_*}$, and an orthogonal direct sum decomposition

$$\mathfrak{p}_* = \mathfrak{h}_{\mathfrak{p}_*} \oplus I_0\mathfrak{h}_{\mathfrak{p}_*} \oplus_{1\leq j<k\leq r} (\mathcal{V}_{jk} \oplus I_0\mathcal{V}_{jk}) \oplus_{k=1}^{s} \mathcal{Q}_k \tag{15}$$

satisfying the following conditions.

(i) Let x be an arbitary element of $\mathfrak{h}_{\mathfrak{p}_*}$, with coordinates $\lambda_k(x)$ with respect to the basis elements e_k, so $x = \sum \lambda_k(x)e_k$. Denote by R_x the Jacobi operator associated to x. Let v and q be arbitrary elements of $\mathcal{V}_{jk}$ and $\mathcal{Q}_k$ respectively. Then

$$\begin{aligned}
R_x I_0 e_j &= 4\lambda_j^2(x) I_0 e_j, && (16)\\
R_x v &= (\lambda_j(x) - \lambda_k(x))^2 v, && (17)\\
R_x I_0 v &= (\lambda_j(x) + \lambda_k(x))^2 I_0 v, && (18)\\
R_x q &= \lambda_k(x)^2 q. && (19)
\end{aligned}$$

(ii) Each $\mathcal{Q}_k$ is I_0-invariant (and possibly zero).

Theorem 4.2

Every compact irreducible hermitian symmetric space associated to one of the classical groups satisfies condition (*).

Proof

This is established by a case-by-case check. For future reference we record the appropriate choices of $\mathfrak{h}_{\mathfrak{p}_*}, \mathcal{V}_{jk}$ and $\mathcal{Q}_k$, as well as the complex structure I_0. We let E_{jk} denote the matrix of appropriate size with 1 in the jk position and zeroes in all other positions.

(i) Complex Grassmannians $SU(p+q)/S(U(p) \times U(q))$ with $p \leq q$.

The rank of the symmetric space is p.

Let

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} 0 & -\bar{Z}^T \\ Z & 0 \end{pmatrix} : Z \in M_{p\times q}(\mathbb{C}) \right\}.$$

The complex structure is given by multiplication of Z by i.

We let $\mathfrak{h}_{p_*}$ be the subset of $\mathfrak{p}_*$ obtained by taking

$$Z = \begin{pmatrix} \Delta & 0 \end{pmatrix},$$

where $\Delta \in M_{p\times p}(\mathbb{R})$ is diagonal. Letting $Z = E_{jj}$ $(j = 1, \ldots, p)$ defines an orthonormal basis $e_1, \ldots, e_p$.

For $1 \leq j < k \leq p$, $\mathcal{V}_{jk}$ is spanned (over $\mathbb{R}$) by the two elements of $\mathfrak{p}_*$ defined by taking $Z = E_{jk} + E_{kj}$ and $Z = i(E_{jk} - E_{kj})$. For $1 \leq k \leq p$ we obtain a basis for $\mathcal{Q}_k$ over $\mathbb{C}$ by taking $Z = E_{kl}; (l = p+1, \ldots, q)$.

(ii) $SO(2n)/U(n)$.
Here the rank is $[\frac{n}{2}]$.

Let

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} Z & W \\ W & -Z \end{pmatrix} : Z, W \in M_{n\times n}(\mathbb{R}) : Z^T = -Z, W^T = -W \right\}.$$

The complex structure sends $(Z\ W)$ to $(-W\ Z)$.

We choose $\mathfrak{h}_{\mathfrak{p}_*}$ to be the subspace of $\mathfrak{p}_*$ where $W = 0$ and Z belongs to the standard Cartan algebra of $\mathfrak{so}(n)$. The matrices where one of the 2×2 blocks on the diagonal of Z is

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

and the other blocks are zero, form an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_*}$.

In order to define the other spaces in the decomposition (*) we must introduce some more notation.

Let $\mathcal{A}$ be an $n\times n$ real skew-symmetric matrix. If n is even, we write $\mathcal{A}$ as

$$\begin{pmatrix} \mathcal{A}_{11} & \dots & \mathcal{A}_{1r} \\ -\mathcal{A}_{12}^T & \dots & \mathcal{A}_{2r} \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ -\mathcal{A}_{1r}^T & \dots & \mathcal{A}_{rr} \end{pmatrix},$$

where each $\mathcal{A}_{jk}$ is a 2×2 matrix.

If n is odd, we write $\mathcal{A}$ as

$$\begin{pmatrix} \mathcal{A}_{11} & \dots & \mathcal{A}_{1r} & \mathcal{B}_1 \\ -\mathcal{A}_{12}^T & \dots & \mathcal{A}_{2r} & \mathcal{B}_2 \\ \cdot & \dots & \cdot & \cdot \\ \cdot & \dots & \cdot & \cdot \\ -\mathcal{A}_{r1}^T & \dots & \mathcal{A}_{rr} & \mathcal{B}_r \\ -\mathcal{B}_1^T & \dots & -\mathcal{B}_r^T & 0 \end{pmatrix},$$

where the $\mathcal{A}_{jk}$ are as above and the $\mathcal{B}_j$ are 2×1 matrices.

For every 2×2 matrix Ψ, let E_{jk}^{Ψ} be the $n\times n$ matrix with $\mathcal{A}_{jk} = \Psi$ and all the other $\mathcal{A}_{mq}$ $(m \le q)$, as well as all the matrices $\mathcal{B}_m$, equal to zero.

If Ω is a 2×1 matrix, let E_j^{Ω} be the $n\times n$ matrix with $\mathcal{B}_j = \Omega$ and all the other $\mathcal{B}_k$, as well as all the $\mathcal{A}_{mq}$, equal to zero.

Then for $1 \le j < k \le r$,

$$\mathcal{V}_{jk} = \left\{ \begin{pmatrix} E_{jk}^{\Psi} & E_{jk}^{\Upsilon} \\ E_{jk}^{\Upsilon} & -E_{jk}^{\Psi} \end{pmatrix} : \Psi = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \Upsilon = \begin{pmatrix} c & d \\ d & -c \end{pmatrix}, a,b,c,d \in \mathbb{R} \right\}$$

Also, for $k = 1,\dots,r$ we have

$$\mathcal{Q}_k = \left\{ \begin{pmatrix} E_k^{\Omega} & E_k^{\Xi} \\ E_k^{\Xi} & -E_k^{\Omega} \end{pmatrix} : \Omega, \Xi \in \mathbb{R}^2 \right\}.$$

The $\mathcal{Q}_k$ terms only occur if n is odd.

(iii) $Sp(n)/U(n)$.

The rank is n.

Let

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & -Z_1 \end{pmatrix} Z_1, Z_2 \in iM_{n\times n}(\mathbb{R}) : Z_1, Z_2 \in \mathfrak{u}(n) \right\}.$$

The complex structure sends $(Z_1\ Z_2)$ to $(-Z_2\ Z_1)$.

We choose $\mathfrak{h}_{\mathfrak{p}_*}$ to be the subspace defined by taking Z_1 to be diagonal and Z_2 to be zero. Letting $Z_1 = iE_{jj}$ $(j = 1, \ldots, n)$ defines an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_*}$.

We define a basis for $\mathcal{V}_{jk}$ over $\mathbb{R}$ by letting $Z_1 = i(E_{jk} + E_{kj}), Z_2 = 0$. There are no Q_k terms.

(iv) Quadrics $SO(n+2)/SO(n) \times SO(2)$, $\quad (n \geq 2)$.

The rank is 2.

Let

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} 0 & -Z^T \\ Z & 0 \end{pmatrix} : Z \in M_{n\times 2}(\mathbb{R}) \right\},$$

and let $\mathfrak{h}_{\mathfrak{p}_*}$ be defined by taking Z to be of the form

$$\begin{pmatrix} a & b \\ b & a \\ 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 \end{pmatrix} \qquad (a, b \in \mathbb{R}).$$

We define an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_*}$ by taking $a = 1/\sqrt{2}, b = 0$ and $a = 0, b = 1/\sqrt{2}$.

If we write Z as $(Z_1\ Z_2)$ where Z_1, Z_2 are column vectors, then the complex structure is defined by

$$I_0 : (Z_1\ Z_2) \mapsto (-Z_2, Z_1).$$

As the rank of the symmetric space is two, the only $\mathcal{V}_{jk}$ term which occurs is $\mathcal{V}_{12}$. A basis of $\mathcal{V}_{12}$ over $\mathbb{R}$ is defined by taking $Z = E_{k1} - E_{k2}$ $(k = 3, \ldots, n)$. There are no Q_k terms. □

Condition (*) will enable us to choose a good basis in which to do the calculations of the next theorem.

Theorem 4.3

Let $M = U/K$ be a compact hermitian symmetric space satisfying condition (*). Let $\mathfrak{h}_{\mathfrak{p}_*}$ be a maximal abelian subspace of $\mathfrak{p}_*$ and $e_1, \ldots, e_r$ an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_*}$ as in (15).

Let ϕ be a U-equivariant diffeomorphism of TM, restricting to a diffeomorphism of $\mathfrak{h}_{\mathfrak{p}_*}$ onto itself which preserves each open Weyl chamber.

Then J^ϕ anticommutes with I_* if and only if there exists a positive constant p such that

$$\phi(z) = \frac{1}{2} \sum_{j=1}^{r} \sinh^{-1}(p\lambda_j) e_j, \tag{20}$$

when

$$z = \sum_{j=1}^{r} \lambda_j e_j \in \mathfrak{h}_{\mathfrak{p}_*}.$$

Proof

We regard the coordinates λ_j on $\mathfrak{h}_{\mathfrak{p}_*}$ as defining real-valued linear functionals on this space. As discussed in section 2, to each λ_j we can associate a linear functional $\tilde{\lambda}_j$ on $\mathfrak{h}_{\mathfrak{p}_0}$ by setting $\tilde{\lambda}_j(w) = \lambda_j(-iw)$.

We see from Corollary 2.10 of Chapter VII of [3] that the set of restricted roots is

$$\Sigma = \{\pm 2\tilde{\lambda}_m, \pm(\tilde{\lambda}_j - \tilde{\lambda}_k), \pm(\tilde{\lambda}_j + \tilde{\lambda}_k), \pm\tilde{\lambda}_m \ : \ 1 \leq m \leq r, 1 \leq j < k \leq r\},$$

so

$$C = \{x \in \mathfrak{h}_{\mathfrak{p}_*} : \lambda_1(x) > \ldots > \lambda_r(x) > 0\}$$

is an open Weyl chamber in $\mathfrak{h}_{\mathfrak{p}_*}$. The set of points conjugate to points in C by the action of U is an open dense subset of TM.

The action of U on TM is holomorphic with respect to both I_* and J, so it is sufficient to see when anticommutation holds on C. Let z be a point of C. We shall choose special bases of $T_z(TM)$ and $T_{\phi(z)}(TM)$, with respect to which we shall calculate ϕ_*, I_{*z} and J_z^ϕ.

For each pair (j,k) with $1 \leq j < k \leq r$ choose an orthonormal basis for $\mathcal{V}_{jk}$. Applying I_0 to these vectors gives an orthonormal basis for $I_0\mathcal{V}_{jk}$. Finally, for each k pick an orthonormal basis for $\mathcal{Q}_k$. Then the union of these bases, together with the elements

$$e_j, I_0 e_j \quad (j = 1, \ldots, r)$$

forms an orthonormal basis for $\mathfrak{p}_*$. The horizontal and vertical lifts of this basis to z and $\phi(z)$ give bases for $T_z(TM)$ and $T_{\phi(z)}(TM)$ respectively.

As discussed earlier, we can split tangent spaces to TM into vertical and horizontal spaces. From (3)-(6), (8) and Lemmas 3.1 and 3.2, we see that I_{*z} and ϕ_* preserve horizontal and vertical spaces while J interchanges them. Using the above bases we have that $\phi_* : T_z(TM) \to T_{\phi(z)}(TM)$ is represented by a matrix of the form

$$\begin{pmatrix} Id & 0 \\ 0 & B \end{pmatrix}.$$

The maps $J_{\phi(z)} : T_{\phi(z)}(TM) \to T_{\phi(z)}(TM)$ and $I_{*z} : T_z(TM) \to T_z(TM)$ are represented by

$$\begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix},$$

$$\begin{pmatrix} I_0 & 0 \\ 0 & -I_0 \end{pmatrix},$$

respectively, for some A.

It readily follows that the pulled back complex structure J_z^ϕ commutes with I_{*z} if and only if

$$I_0 A^{-1} B = A^{-1} B I_0. \tag{21}$$

We shall use the decomposition of Definition 4.1 to calculate A, B, I_0 explicitly and see when the anticommutation relation (21) holds. We shall see that each of the spaces $\mathcal{V}_{jk} \oplus I_0\mathcal{V}_{jk}, \mathcal{Q}_k$ and $\mathfrak{h}_{\mathfrak{p}_*} + I_0\mathfrak{h}_{\mathfrak{p}_*}$ is invariant under I_0, A, B so it is sufficient to work on each of these spaces separately.

We shall denote by $\phi_k(z)$ the k th. component of $\phi(z)$ with respect to the basis $e_1, \ldots, e_r$.

Let us first study the space $\mathcal{Q}_k$. From condition (*) we know that for any $q \in \mathcal{Q}_k$ and any $x \in \mathfrak{h}_{\mathfrak{p}_*}$ we have

$$R_x q = \lambda_k(x)^2 q.$$

In particular $R_{\phi(z)/\|\phi(z)\|}$ is a scalar operator on Q_k with eigenvalue $\phi_k(z)^2/\parallel \phi(z) \parallel^2$. The equations (3)-(6) defining the adapted complex structure now show that

$$A\mid_{Q_k} = \phi_k(z)\coth(\phi_k(z))Id.$$

Moreover, by Lemma 3.2 we have

$$B\mid_{Q_k} = \frac{\phi_k(z)}{\lambda_k(z)} Id,$$

so the relation (21) holds automatically on Q_k.

On the subspace $\hat{\mathfrak{h}} = \mathfrak{h}_{\mathfrak{p}_*} + I_0\mathfrak{h}_{\mathfrak{p}_*}$ with respect to the basis $e_1, \ldots, e_r, Ie_1, \ldots, Ie_r$ the complex structure I_0 has matrix

$$\begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}.$$

Now, for any $x \in \mathfrak{h}_{\mathfrak{p}_*}$ we have

$$\begin{aligned} R_x e_j &= 0, \\ R_x I_0 e_j &= 4\lambda_j^2(x) I_0 e_j, \end{aligned}$$

so, applying equations (3)-(6), we find that

$$A^{-1}\mid_{\hat{\mathfrak{h}}} = \begin{pmatrix} Id & 0 \\ 0 & \nu \end{pmatrix},$$

where $\nu = \operatorname{diag}(\nu_1, \ldots, \nu_r)$ and

$$\nu_j = \frac{\tanh(2\phi_j(z))}{2\phi_j(z)}.$$

It follows from Lemma 3.2 that

$$B\mid_{\hat{\mathfrak{h}}} = \begin{pmatrix} d\phi & 0 \\ 0 & \mu \end{pmatrix},$$

where $d\phi$ is the derivative of $\phi : \mathfrak{h}_{\mathfrak{p}_*} \to \mathfrak{h}_{\mathfrak{p}_*}$ in the coordinates given by the basis $e_1, \ldots, e_r$. Here $\mu = \operatorname{diag}(\mu_1, \ldots, \mu_r)$, and

$$\mu_j = \frac{\phi_j(z)}{\lambda_j}.$$

We find that $\hat{\mathfrak{h}}$ is indeed invariant under A, B and I_0, and on $\hat{\mathfrak{h}}$ (21) is equivalent to

$$d\phi = \nu\mu,$$

which in turn is equivalent to

$$2\lambda_i \frac{\partial \phi_i(z)}{\partial \lambda_i} = \tanh(2\phi_i(z)), \tag{22}$$

$$\frac{\partial \phi_i(z)}{\partial \lambda_j} = 0, \quad \text{if } i \neq j, \tag{23}$$

where $z = \sum \lambda_j e_j$, and $\lambda_1 > \ldots > \lambda_r > 0$.

The solution to (22-23) is

$$\phi_i = \frac{1}{2}\sinh^{-1}(p_i\lambda_i), \tag{24}$$

where p_i are constants. In fact, as ϕ is equivariant with respect to the action of the Weyl group, the p_i must all be equal to some constant p. The requirement that ϕ preserves each open Weyl chamber means that p is positive.

Equation (24) shows that the restriction of ϕ to $\mathfrak{h}_{\mathfrak{p}_*}$ must be of the form (20). As ϕ is U-equivariant, we see that ϕ is determined up to the choice of constant p.

(ii) We shall now demonstrate the converse implication. In order to show that the anticommutation relation (21) holds we must look at the spaces $P_{jk} = \mathcal{V}_{jk} \oplus I_0\mathcal{V}_{jk}$. We see that

$$I_0 \mid_{P_{jk}} = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}.$$

Using condition (*) and equations (3)-(6) as before we find that

$$A^{-1} \mid_{P_{jk}} = \begin{pmatrix} \rho_1 Id & 0 \\ 0 & \rho_2 Id \end{pmatrix},$$

where

$$\rho_1 = \frac{\tanh(\phi_j(z) - \phi_k(z))}{\phi_j(z) - \phi_k(z)} \quad , \quad \rho_2 = \frac{\tanh(\phi_j(z) + \phi_k(z))}{\phi_j(z) + \phi_k(z)}.$$

Lemma 3.2 tells us that

$$B \mid_{P_{jk}} = \begin{pmatrix} \sigma_1 Id & 0 \\ 0 & \sigma_2 Id \end{pmatrix},$$

where

$$\sigma_1 = \frac{\phi_j(z) - \phi_k(z)}{\lambda_j - \lambda_k} \quad , \quad \sigma_2 = \frac{\phi_j(z) + \phi_k(z)}{\lambda_j + \lambda_k}.$$

It follows that I_* and J^ϕ anticommute on P_{jk} precisely when

$$\frac{\tanh(\phi_j(z) + \phi_k(z))}{\lambda_j + \lambda_k} = \frac{\tanh(\phi_j(z) - \phi_k(z))}{\lambda_j - \lambda_k}. \tag{25}$$

We have already seen that if $\phi_i = \frac{1}{2}\sinh^{-1}(p\lambda_i)$ for each i then the anticommutation relation holds on each Q_k and on $\hat{\mathfrak{h}}$. It is easy to check that for this choice of ϕ the equation (26) holds, so anticommutation holds on each P_{jk} also. It follows that I_* and J^ϕ anticommute at all points of the open Weyl chamber C, and hence everywhere on TM. □

The final ingredient we need for the proof of Theorem 0.1 is to show that the map

$$\sum \lambda_j e_j \mapsto \sum \frac{1}{2}\sinh^{-1}(p\lambda_j)e_j \tag{26}$$

extends to a real-analytic U-equivariant diffeomorphism of TM.

Proposition 4.4

Let $M = U/K$ be a compact irreducible hermitian symmetric space of classical type. Define a real-analytic diffeomorphism of $\mathfrak{h}_{\mathfrak{p}_*}$ by

$$\sum \lambda_j e_j \mapsto \sum \frac{1}{2} \sinh^{-1}(p\lambda_j) e_j .$$

Then this map extends uniquely to a K-equivariant real-analytic diffeomorphism of $\mathfrak{p}_*$, and hence to a U-equivariant real-analytic diffeomorphism of TM.

Proof

The existence and uniqueness of a U-equivariant bijective extension ϕ follows from Lemma 2.6. We must show that ϕ, ϕ^{-1} are real-analytic. We proceed case-by-case. Without loss of generality we take the constant p to be 1.

Case (i) $Sp(n)/U(n)$, $SO(2n)/U(n)$, $SU(p+q)/S(U(p) \times U(q))$.

As discussed in the proof of Theorem 4.2, we regard $\mathfrak{p}_*$ as a subspace of the vector space $\mathfrak{u}(N)$ for suitable N. Under this identification, each element X of $\mathfrak{p}_*$ has pure imaginary spectrum.

It is easy to verify that the restriction of ϕ to $\mathfrak{h}_{\mathfrak{p}_*}$ is given in some neighbourhood of the origin by a power series

$$\phi(X) = \sum a_j X^j,$$

with scalar coefficients. The a_j are the coefficients of the Taylor expansion about $x = 0$ of the function $F(x) = -\frac{1}{2} i \sinh^{-1}(ix)$. Now F is holomorphic on some open set D containing the imaginary axis, so we can define a real-analytic function $\tilde{F} : \mathfrak{u}(N) \to M_{N\times N}(\mathbb{C})$ by

$$\tilde{F} : X \mapsto \frac{1}{2\pi i} \int_\Gamma F(\lambda)(\lambda Id - X)^{-1} d\lambda,$$

where Γ is a contour in D enclosing the spectrum of X. Then $\tilde{F}$ agrees with ϕ on a neighbourhood of the origin in $\mathfrak{h}_{\mathfrak{p}_*}$. As ϕ is also real-analytic on $\mathfrak{h}_{\mathfrak{p}_*}$, it follows that ϕ and $\tilde{F}$ are equal on $\mathfrak{h}_{\mathfrak{p}_*}$. Since $\tilde{F}$ is Ad K-equivariant, we deduce that $\tilde{F}$ and ϕ agree on $\mathfrak{p}_*$, so ϕ is real-analytic on $\mathfrak{p}_*$, and hence on TM.

Similarly the restriction of ϕ^{-1} to $\mathfrak{h}_{\mathfrak{p}_*}$ is given by

$$\phi^{-1}(X) = \sum b_j X^j,$$

where $-\frac{1}{2} i \sinh(ix) = \sum b_j x^j$. It follows that the equivariant extension of ϕ^{-1} to $\mathfrak{p}_*$ is also given by this formula, so ϕ^{-1} is real-analytic on $\mathfrak{p}_*$, and hence on TM. (As the power series of $-i\sinh(ix)$ converges everywhere we do not need to use the symbolic calculus in this case).

(ii) $SO(n+2)/SO(n) \times SO(2)$.

As in 4.2, we identify points of $\mathfrak{p}_*$ with pairs (Z_1, Z_2) of $n \times 1$ column vectors.

With the choice of $\mathfrak{h}_{\mathfrak{p}_*}$ made in 4.2, it is straightforward to calculate that the restriction of ϕ to $\mathfrak{p}_*$ is given by

$$\phi : (Z_1\ Z_2) \mapsto (Z_1\ Z_2) \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{pmatrix},$$

where

$$\begin{aligned}
\beta_1 &= \frac{\delta_1 + \delta_2}{2} + \frac{(\delta_1 - \delta_2) < Z_2, Z_2 >}{2\sqrt{\omega}}, \\
\beta_2 &= \frac{(\delta_2 - \delta_1) < Z_1, Z_2 >}{2\sqrt{\omega}}, \\
\beta_3 &= \frac{\delta_1 + \delta_2}{2} + \frac{(\delta_1 - \delta_2) < Z_1, Z_1 >}{2\sqrt{\omega}},
\end{aligned}$$

$$\delta_1 = \frac{\sinh^{-1}(\epsilon_1)}{2\epsilon_1},$$
$$\delta_2 = \frac{\sinh^{-1}(\epsilon_2)}{2\epsilon_2},$$

$$\epsilon_1 = \frac{1}{2}\sqrt{< Z_1, Z_1 > + < Z_2, Z_2 > +2\sqrt{\omega}},$$
$$\epsilon_2 = \frac{1}{2}\sqrt{< Z_1, Z_1 > + < Z_2, Z_2 > -2\sqrt{\omega}},$$

and

$$\omega =< Z_1, Z_1 >< Z_2, Z_2 > - < Z_1, Z_2 >^2 .$$

Here ϵ_1, ϵ_2 are the coordinates of the point in the closed Weyl chamber $\bar{C}$ conjugate to (Z_1, Z_2) under the adjoint action of $SO(n) \times SO(2)$. We have $\epsilon_1 \geq \epsilon_2 \geq 0$.

It is sufficient to check that $\delta_1 + \delta_2$ and $(\delta_1 - \delta_2)/\sqrt{\omega}$ are real-analytic.

Now, the function $t \mapsto \sinh^{-1}(t)/2t$ is real-analytic and even, so it may be written as $l(t^2)$ where l is real-analytic on some open interval of $\mathbb{R}$ including the closed half-line $\{t : t \geq 0\}$.

It can be readily checked that, for such a function l, the functions l_1 and l_2 defined by

$$l_1(x,y) = l(x + \sqrt{y}) + l(x - \sqrt{y}),$$
$$l_2(x,y) = \frac{l(x + \sqrt{y}) - l(x - \sqrt{y})}{\sqrt{y}},$$

extend to real-analytic functions on some open neighbourhoods of the region $\{(x,y) : x, y \geq 0 \,; x \geq \sqrt{y}\}$.

Taking $x = \frac{1}{4}(< Z_1, Z_1 > + < Z_2, Z_2 >)$ and $y = \frac{1}{4}\omega$ (so $x \geq \sqrt{y}$) concludes the proof.

The only properties of ϕ that were needed for this argument were that $\sinh^{-1}$ is odd and real-analytic. Hence the same argument also applies to ϕ^{-1}. □.

We can now finish the proof of Theorem 0.1.

Proof of Theorem 0.1

The map

$$\sum \lambda_j e_j \mapsto \sum \frac{1}{2} \sinh^{-1} \lambda_j e_j$$

is a real-analytic diffeomorphism of $\mathfrak{h}_{\mathfrak{p}_*}$ onto itself, preserving each open Weyl chamber. Proposition 4.4 shows that it extends to a U-equivariant real-analytic diffeomorphism of TM, and Theorem 4.3 now implies that the pullback of J by ϕ anticommutes with I_*. □

We have shown the existence of two anti-commuting complex structures I_* and J^ϕ on TM. It follows that $aI_* + bJ^\phi + cI_*J^\phi$ is a complex structure whenever $a^2 + b^2 + c^2 = 1$. In other words I_* and J^ϕ generate a *hypercomplex structure*.

5. The noncompact case.

The arguments of the preceding section can be adapted with only minor changes to the case when M is a noncompact irreducible hermitian symmetric space asociated to a classical group. Such spaces are precisely the duals of the compact examples we have already considered.

In each case we have a decomposition of $\mathfrak{p}_0$ analogous to that of (15), but now the eigenvalues of the operator R_x have the opposite sign to those of (16)-(19).

The orthonormal basis $e_1, \ldots, e_r$ for $\mathfrak{h}_{\mathfrak{p}_*}$ determines an orthonormal basis $ie_1, \ldots, ie_r$ of $\mathfrak{h}_{\mathfrak{p}_0}$. If $w \in \mathfrak{h}_{\mathfrak{p}_0}$ has coordinates $\lambda_1, \ldots, \lambda_r$ with respect to this basis, let

$$G(w) = \max \mid \lambda_j \mid .$$

We can extend G to a continuous U^*-equivariant function (which we shall also denote by G) defined on TM and taking values in $[0, \infty)$. We have the equation

$$G(tX) = \mid t \mid G(X) \qquad (t \in \mathbb{R},\ X \in TM).$$

Let

$$\mathcal{D}_1 = \{w \in \mathfrak{h}_{\mathfrak{p}_0} : G(w) < 1\},$$

$$\mathcal{D}_2 = \{w \in \mathfrak{h}_{\mathfrak{p}_0} : G(w) < \frac{\pi}{4}\},$$

$$\mathcal{E}_1 = \{X \in TM : G(X) < 1\},$$

and

$$\mathcal{E}_2 = \{X \in TM : G(X) < \frac{\pi}{4}\}.$$

The map

$$\lambda_j \mapsto \frac{1}{2}\sin^{-1}(\lambda_j) \quad (j = 1, \ldots, r) \tag{27}$$

is then a real-analytic diffeomorphism of $\mathcal{D}_1$ onto $\mathcal{D}_2$.

The discussion after Lemma 2.6 shows that the map defined by (27) extends to a U^*-equivariant bijection ϕ between the regions $\mathcal{E}_1$ and $\mathcal{E}_2$. Moreover, the arguments of Proposition 4.4 show that ϕ and its inverse are in fact real-analytic. From §1 and the decomposition in 4.1 it follows that the maximal domain on which the adapted complex structure J is defined is $\mathcal{E}_2$. Therefore the pulled back complex structure J^ϕ is defined and is smooth on $\mathcal{E}_1$.

Proceeding as in the proof of Theorem 4.3, we find that the relations that ϕ must satisfy for J^ϕ to anticommute with I_* on $\mathcal{E}_1$ are

$$\begin{aligned}
2\lambda_i \frac{\partial\phi_i(z)}{\partial\lambda_i} &= \tan(2\phi_i(z)), \\
\frac{\partial\phi_i(z))}{\partial\lambda_j} &= 0 \quad \text{if } i \neq j, \\
\frac{\tan(\phi_i(z) - \phi_j(z))}{(\lambda_i - \lambda_j)} &= \frac{\tan(\phi_i(z) + \phi_j(z))}{(\lambda_i + \lambda_j)},
\end{aligned}$$

where $z = \sum \lambda_i e_i \in \mathcal{D}_1$ and $\phi(z) = \sum \phi_i(z) e_i$. It is easy to verify that these equations are satisfied if the restriction of ϕ to $\mathcal{D}_1$ is given by (27), so we have established Theorem 0.2.

It is clear that $G^{-1}([0,1))$ contains the open unit disc bundle $T^1 M$, so Theorem 0.2 shows the existence of a hypercomplex structure on $T^1 M$.

In Theorems 0.1 and 0.2 the diffeomorphism ϕ is equivariant with respect to the action of the isometry group of M . Moreover this action preserves the complex structures $I_{\bullet}, J$. We see therefore that a hypercomplex structure exists on $T^1 M$ whenever M is a locally symmetric quotient of a classical hermitian symmetric space.

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