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ON KV, AT AND COMPATIBILITY BETWEEN CUP PRODUCTS

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Abstract. For a finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic 0 which contains $\mathbb{C}$, we deduce from the compatibility between cup products [8, Section 8] and the vanishing of the weights associated to wheel graphs with spokes pointing inwards [9] an alternative way of re-writing the Kontsevich $\star$-product on $\mathrm{S}(\mathfrak{g})$ using the Alekseev–Torossian flat connection [3]. Further, we prove that a similar way of re-writing the Kontsevich $\star$ on $\mathrm{S}(\mathfrak{g})$ can be deduced directly from the Kashiwara–Vergne conjecture.

1. Introduction

For a general finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic 0 which contains $\mathbb{C}$, we consider the symmetric algebra $\mathrm{S}(\mathfrak{g})$.

Deformation quantization *à la* Kontsevich [8] permits to endow $\mathrm{S}(\mathfrak{g})$ with an associative, non-commutative product $\star$: the universal property of the Universal Enveloping Algebra (shortly, from now on, UEA) $\mathrm{U}(\mathfrak{g})$ and a degree argument imply that there is an isomorphism of associative algebras $\mathcal{I}$ from $(\mathrm{S}(\mathfrak{g}),\star)$ to $(\mathrm{U}(\mathfrak{g}),\cdot)$. In fact, the algebra isomorphism $\mathcal{I}$ has been characterized explicitly in [4,5,11] as the composition of the Poincaré–Birkhoff–Witt (shortly, from now on, PBW) isomorphism (of vector spaces) with an invertible differential operator with constant coefficients and of infinite order associated to the well-known Duflo element $\sqrt{j(\bullet)}$ in the completed symmetric algebra $\widehat{\mathrm{S}}(\mathfrak{g}^*)$.

In this short note, we try and answer a question asked to us by M. Vergne about a possible way of re-writing the product $\star$ on $\mathrm{S}(\mathfrak{g})$ in terms of the Lie series F, G appearing in the Kashiwara–Vergne (shortly, from now on, KV) conjecture [6]. More precisely, M. Vergne kindly shared with us the thought that such a result should follow from the KV conjecture and asked if it were also possible to prove a similar claim starting from the famous compatibility between cup products for Kontsevich's formality quasi-isomorphism [8, Section 8]. We prove both claims in a constructive way (see later on Formulæ (5), (9) and (14)).

In Section 2, we quickly review the main notation and conventions.

In Subsection 3.1, we recall the main features of Kontsevich's deformation quantization, in particular, the graphical language and the construction of the product $\star$, which will be central in the forthcoming computations.

In Subsection 3.2, we re-prove in a slightly different way the famous result about compatibility between cup products in 0-th degree for the tangent cohomology in the Lie algebra case: this permits to simplify considerably computations and to prove Formulæ (5) and (9), by recalling certain results in [4,9].

In Subsection 3.3, we compute symbols of all (bi)differential operators in the aforementioned formulæ and find, as one could expect, the Alekseev–Torossian (shortly, from now on, AT) connection [3,10], which had been central in the proof of the KV conjecture presented in [1].

Remark 1.1. In our humble opinion, the results of Subsections 3.2 and 3.3 were already somehow present in the work [10] of C. Torossian: we just present them here in a slightly different fashion, pointing out more explicitly their relationship with compatibility between cup products.

Finally, in Subsection 3.4, we consider the (combinatorial) KV conjecture and from it we deduce Formula (14), which also yields compatibility between cup products in 0-th cohomology.

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2. Notation and conventions

We consider a field $\mathbb{K}$ of characteristic 0 which contains $\mathbb{C}$.

We denote by $\mathfrak{g}$ a finite-dimensional Lie algebra over $\mathbb{K}$ of dimension d; by $\{x_i\}$ we denote a $\mathbb{K}$-basis of $\mathfrak{g}$. To $\mathfrak{g}$ we associate the linear variety $X=\mathfrak{g}^*$ over $\mathbb{K}$: the basis $\{x_i\}$ defines a set of global coordinates over X, and the canonical Kirillov–Kostant Poisson bivector field π on X can be written as $\pi = f_{ij}^k x_k \partial_i \partial_j$, where we have omitted wedge product for the sake of simplicity, and f_{ij}^k denote the structure constants of $\mathfrak{g}$ w.r.t. the basis $\{x_i\}$.

3. Compatibility between cup products and the AT connection

In the present section, we consider a slightly different approach to the compatibility between cup products from [8, Subsection 8.2] on the 0-th cohomology. We then specialize to the case of the Poisson variety (X,π), where $X=\mathfrak{g}^*$ for $\mathfrak{g}$ as in Section 2.

3.1. **Explicit formulæ for Kontsevich's star product.** Let first $X=\mathbb{K}^d$ and let $\{x_i\}$ a system of global coordinates on X, for $\mathbb{K}$ as above.

For a pair (n,m) of non-negative integers, by $\mathcal{G}_{n,m}$ we denote the set of admissible graphs of type (n,m): an element Γ of $\mathcal{G}_{n,m}$ is a directed graph with n, resp. m, vertices of the first, resp. second type, such that $i)$ there is no directed edge departing from any vertex of the second type and $ii)$ Γ admits whether multiple edges nor short loops (*i.e.* given two distinct vertices v_i, $i=1,2$, of Γ there is at most one directed edge from v_1 to v_2 and there is no directed edge, whose endpoint coincides with the initial point). By $E(\Gamma)$ we denote the set of edges of Γ in $\mathcal{G}_{n,m}$.

We denote by $C^+_{n,m}$ the configuration space of n points in the complex upper half-plane $\mathbb{H}^+$ and m ordered points on the real axis $\mathbb{R}$ *modulo* the componentwise action of rescalings and real translations: provided $2n+m-2\geq 0$, $C^+_{n,m}$ is a smooth, oriented manifold of dimension $2n+m-2$. We denote by $\mathcal{C}^+_{n,m}$ a suitable compactification *à la* Fulton–MacPherson introduced in [8, Section 5]: $\mathcal{C}^+_{n,m}$ is a compact, oriented, smooth manifold with corners of dimension $2n+m-2$. We will be interested mostly in its boundary strata of codimension 1.

We denote by ω the closed, real-valued 1-form

$$\omega(z_1,z_2)=\frac{1}{2\pi}d\arg\left(\frac{z_1-z_2}{\overline{z}_1-z_2}\right),\ (z_1,z_2)\in(\mathbb{H}^+\sqcup\mathbb{R})^2,\ z_1\neq z_2,$$

where $\arg(\bullet)$ denotes the $[0,2\pi)$-valued argument function on $\mathbb{C}\smallsetminus\{0\}$ such that $\arg(i)=\pi/2$. The main feature of ω is that it extends to a smooth, closed 1-form on $\mathcal{C}^+_{2,0}$, such that $i)$ when the two arguments approach to each other in $\mathbb{H}^+$, ω equals the normalized volume form $d\varphi$ on S^1 and $ii)$ when the first argument approaches $\mathbb{R}$, ω vanishes.

We introduce $T_{\mathrm{poly}}(X)=A[\theta_1,\dots,\theta_d]$, $A=C^\infty(X)$, where $\{\theta_i\}$ denotes a set of graded variables of degree 1, which commute with A and anticommute with each other (one may think of θ_i as ∂_i with a shifted degree). We further consider the well-defined linear endomorphism τ of $T_{\mathrm{poly}}(X)^{\otimes 2}$ of degree -1 defined *via*

$$\tau=\partial_{\theta_i}\otimes\partial_{x_i},$$

where of course summation over repeated indices is understood. We set $\omega_\tau=\omega\otimes\tau$.

To Γ in $\mathcal{G}_{n,m}$ such that $|E(\Gamma)|=2n+m-2$, γ_i, $i=1,\dots,n$, elements of $T_{\mathrm{poly}}(X)$ and a_j, $j=1,\dots,m$, elements of A, we associate a map *via*

(1)

$$(\mathcal{U}_\Gamma(\gamma_1,\dots,\gamma_n))(a_1\otimes\cdots\otimes a_m)=\mu_{m+n}\left(\int_{\mathcal{C}^+_{n,m}}\omega_{\tau,\Gamma}(\gamma_1\otimes\cdots\otimes\gamma_n\otimes a_1\otimes\cdots\otimes a_m)\right),\ \omega_{\tau,\Gamma}=\prod_{e\in E(\Gamma)}\omega_{\tau,e},\ \omega_{\tau,e}=\pi_e^*(\omega)\otimes\tau_e,$$

τ_e being the graded endomorphism of $T_{\mathrm{poly}}(X)^{\otimes(m+n)}$ which acts as τ on the two factors of $T_{\mathrm{poly}}(X)$ corresponding to the initial and final point of the edge e, and μ_{m+n} denotes the multiplication map from $T_{\mathrm{poly}}(X)^{m+n}$ to $T_{\mathrm{poly}}(X)$, followed by the natural projection from $T_{\mathrm{poly}}(X)$ onto A by setting $\theta_i=0$, $i=1,\dots,d$. We may re-write (1) by splitting the form-part and the polydifferential operator part as

$$(\mathcal{U}_\Gamma(\gamma_1,\dots,\gamma_n))(a_1\otimes\cdots\otimes a_m)=\varpi_\Gamma(\mathcal{B}_\Gamma(\gamma_1,\dots,\gamma_n))(a_1,\dots,a_m),\ \varpi_\Gamma=\int_{\mathcal{C}^+_{n,m}}\omega_\Gamma.$$

In [8, Theorem 6.4], the following theorem has been proved.

Theorem 3.1. *For a Poisson bivector field π on X, and a formal parameter $\hbar$, the formula*

$$f_1\star_\hbar f_2=\sum_{n\geq 0}\frac{\hbar^n}{n!}\sum_{\Gamma\in\mathcal{G}_{n,2}}(\mathcal{U}_\Gamma(\underbrace{\pi,\dots,\pi}_{n}))(f_1,f_2),\ f_i\in A,\ i=1,2, \quad (2)$$

defines a $\mathbb{K}_\hbar=\mathbb{K}[\![\hbar]\!]$-linear, associative product on $A_\hbar=A[\![\hbar]\!]$.

3.2. **The 1-form governing the compatibility between cup products.** We now consider $\mathfrak{g}$ as in Section 2, to which we associate the Poisson variety $(X=\mathfrak{g}^*,\pi)$. Observe that the commutative algebra $\mathbb{K}[X]$ of regular functions on X identifies with $A=\mathrm{S}(\mathfrak{g})$.

Since π is linear, Formula (2) restricts to $A_\hbar$ and moreover the $\hbar$-dependence is polynomial: we may thus safely set $\hbar=1$ and consider the associative algebra $(A,\star)$.

For a non-negative integer n, let us consider the projection $\pi_{n,2}$ from $C^+_{n+2,0}$ onto $C^+_{2,0}$ which forgets all points in $\mathbb{H}^+$ except the last two: it extends smoothly to a projection from $\mathcal{C}^+_{n+2,0}$ onto $\mathcal{C}^+_{2,0}$, which we denote by the same symbol. It is clear that $\pi_{n,2}$ defines a fibration onto $\mathcal{C}^+_{2,0}$, whose typical fiber is a smooth, oriented manifold with corners of dimension $2n$.

To Γ in $\mathcal{G}_{n+2,0}$ such that $|E(\Gamma)| = 2n$, we associate a smooth 0-form on $C^+_{2,0}$ with values in the bidifferential operators on A defined as

$$\mathcal{T}^\pi_\Gamma(f_1,f_2) = \mu_{n+2}(\pi_{n,2,*}(\omega_{\mathcal{T},\Gamma}(\underbrace{\pi\otimes\cdots\otimes\pi}_{n}\otimes f_1\otimes f_2))) = \widehat{\varpi}_\Gamma(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n}))(f_1,f_2),\ \widehat{\varpi}_\Gamma = \pi_{n,2,*}(\omega_\Gamma), \tag{3}$$

where $\pi_{n,2,*}$ denotes the integration along the fiber of the operator-valued form $\omega_{\mathcal{T},\Gamma}$ w.r.t. the projection $\pi_{n,2}$ and where we have borrowed previous notation.

Remark 3.2. Observe that (3) is well-defined, as ω extends to $\mathcal{C}^+_{2,0}$. If Γ contains an edge connecting the last two vertices, (3) vanishes by dimensional reasons. Further, no arrow may depart from any of the last two vertices, and exactly two arrows depart from any vertex different from the last two: thus, we may identify graphs Γ in $\mathcal{G}_{n+2,0}$ as in (3) with graphs in $\mathcal{G}_{n,2}$ as in (2).

We finally set

$$\mathcal{T}^\pi(f_1,f_2) = \sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\|E(\Gamma)|=2n}}\mathcal{T}^\pi_\Gamma(f_1,f_2),\ f_i\in A,\ i=1,2. \tag{4}$$

Formula (4) yields a well-defined smooth function on $C^+_{2,0}$ with values in the bidifferential operators on A.

Theorem 3.3. *There exist smooth* 1*-forms* Ω^π_i, $i=1,2$, *with values in* $\mathfrak{g}\otimes\widehat{\mathrm{S}}(\mathfrak{g})^{\otimes 2}$, *such that the following identity holds true:*

$$d(\mathcal{T}^\pi(f_1,f_2)) = \mathcal{T}^\pi(\Omega^\pi_1([\pi,f_1],f_2)) + \mathcal{T}^\pi(\Omega^\pi_2(f_1,[\pi,f_2])),\ f_i\in A,\ i=1,2. \tag{5}$$

Proof. First of all, for Γ in $\mathcal{G}_{n+2,0}$ such that $|E(\Gamma)|=2n$, $n\geq 1$, let us compute

$$d(\mathcal{T}^\pi_\Gamma(f_1,f_2)) = d\widehat{\varpi}_\Gamma(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n}))(f_1,f_2) = \pi^\partial_{n,2,*}(\omega_\Gamma)(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n}))(f_1,f_2),$$

where the second equality follows by means of the generalized Stokes Theorem for integration along the fiber, and $\pi^\partial_{n,2,*}$ denotes integration along the boundary of the compactification of the typical fiber of the projection $\pi_{n,2}$.

The boundary strata of codimension 1 of the compactification of the typical fiber of $\pi_{n,2}$ can be deduced from the boundary strata of codimension 1 of $\mathcal{C}^+_{n+2,0}$:

i) there is a subset A of $[n]=\{1,\dots,n\}$, $1\leq|A|\leq n$ which contains either $n+1$ or $n+2$, such that points in $\mathbb{H}^+$ labeled by A collapse either to the $n+1$-st or $n+2$-nd point in $\mathbb{H}^+$;
ii) there is a subset A of $[n]$, $2\leq|A|\leq n$, $n+1,n+2\notin A$, such that points in $\mathbb{H}^+$ labeled by A collapse to a single point in $\mathbb{H}^+$, distinct from the last two points;
iii) there is a subset A of $[n]$, which either contains both $n+1$, $n+2$ or contains neither of them, such that the points in $\mathbb{H}^+$ labeled by A approach $\mathbb{R}$.

For Γ as above, we denote by Γ_A, resp. Γ^A, the subgraph of Γ, whose edges have both endpoints labeled by A, resp. the quotient graph obtained by collapsing to a single vertex the subgraph Γ_A of Γ.

The boundary strata of type *iii*) yield trivial contributions. Namely, let us consider first a subset A such that $n+1,n+2\notin A$: Fubini's Theorem implies that

$$\pi^{\partial,A}_{n,2,*}(\omega_\Gamma)\propto\int_{\mathcal{C}^+_{A,0}}\omega_{\Gamma_A},$$

and the aforementioned properties of ω imply that the form degree of ω_{Γ_A} equals $2|A|$, while the dimension of $C^+_{A,0}$ equals $2|A|-2$. If A contains both $n+1$, $n+2$, we may repeat the previous arguments *verbatim* by replacing A by A^c.

Let us consider a general boundary stratum of type *ii*): Fubini's Theorem and the properties of ω imply

$$\pi^{\partial,A}_{n,2,*}(\omega_\Gamma)\propto\int_{\mathcal{C}_A}\omega_{\Gamma_A},$$

where $\mathcal{C}_A$ is the compactified configuration space of $|A|$ points in $\mathbb{C}$ *modulo* rescalings and complex translations; by abuse of notations, we have denoted by ω_{Γ_A} a product of 1-forms $d\arg(z_i-z_j)$, i, j in A, on $\mathcal{C}_A$. If $|A|\geq 3$, the above integral on the right-hand side vanishes by [8, Lemma 6.6]. Thus, it remains to consider the case $|A|=2$. If no edge connects the two vertices labeled by A, there is nothing to integrate over $\mathcal{C}_2=S^1$, while, if there is a cycle between the two vertices, ω_{Γ_A} is the square of a 1-form, hence both contributions vanish. We thus assume that there is a single edge connecting the two vertices labeled by A, in which case Fubini's Theorem together with the properties of ω when its arguments collapse in $\mathbb{H}^+$ yields

$$\pi^{\partial,A}_{n,2,*}(\omega_\Gamma)=\pi_{n-1,2,*}(\omega_{\mathcal{T},\Gamma^A}).$$

The quotient graph Γ^A belongs to $\mathcal{G}_{n+1,0}$ and no edge departs from $n+1$, $n+2$, all other vertices are bivalent except one, which is trivalent (here, the valence of a vertex is the number of outgoing edges from the said vertex).

Finally, let us consider a boundary stratum of type $i)$, labeled by a subset $n+1\in A$, $n+2\notin A$. Assume first $|A|\geq 2$: then, in a way similar to the analysis of a boundary stratum of type $ii)$, we find

$$\pi^{\partial,A}_{n,2,*}(\omega_\Gamma)\propto\int_{\mathcal{C}_{A\sqcup\{n+1\}}}\omega_{\Gamma_A}=0$$

by [8, Lemma 6.6], as $|A|\geq 2$. It remains to consider the case $|A|=1$. As before, we may safely assume that Γ_A consists of a single edge connecting with endpoint $n+1$ and, provided $n\geq 1$, initial point different from $n+2$, in which case we find

$$\pi^{\partial,A}_{n,2,*}(\omega_\Gamma)=\pi_{n-1,2,*}(\omega_{\Gamma^A}).$$

Due modifications of the previous arguments yield a similar formula in the situation $n+1\notin A$, $n+2\in A$. The quotient graph Γ^A, if $n+1$ is in A, belongs to $\mathcal{G}_{n+1,0}$, exactly one edge departs from $n+1$, no edge departs from $n+2$, and all other vertices are bivalent; when $n+2$ belongs to A, Γ^A is described in a similar way by replacing $n+1$ by $n+2$.

The previous computations yield

(6)

$$\begin{aligned}
d(\mathcal{T}^\pi(f_1,f_2))&=\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\|E(\Gamma)|=2n}}\sum_{\substack{A\subseteq[n],\ |A|=2\\n+1\in A,\ n+2\notin A}}\widehat{\varpi}_{\Gamma^A}(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n}))(f_1,f_2)+\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\|E(\Gamma)|=2n}}\sum_{\substack{A\subseteq[n],\ |A|=2\\n+1\notin A,\ n+2\in A}}\widehat{\varpi}_{\Gamma^A}(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n}))(f_1,f_2)+\\
&+\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\|E(\Gamma)|=2n}}\sum_{\substack{A\subseteq[n],\ |A|=2\\n+1,n+2\notin A}}\widehat{\varpi}_{\Gamma^A}(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n}))(f_1,f_2)=\\
&=\sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\|E(\Gamma)|=2n+1}}\widehat{\varpi}_{\Gamma}(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n}))([\pi,f_1],f_2)+\sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\|E(\Gamma)|=2n+1}}\widehat{\varpi}_{\Gamma}(\mathcal{B}_\Gamma(\underbrace{\pi,\dots,\pi}_{n}))(f_1,[\pi,f_2])+\\
&+\sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{\Gamma\in\mathcal{G}_{n+2,0}\\|E(\Gamma)|=2n+1}}\widehat{\varpi}_{\Gamma}(\mathcal{B}_\Gamma([\pi,\pi],\underbrace{\pi,\dots,\pi}_{n-1}))(f_1,f_2),
\end{aligned}$$

recalling the explicit shape of the quotient subgraph Γ^A in the three previous cases and using Leibniz rule to re-write the sums over A in the bidifferential operators; $[\pi,\pi]$ denotes the trivector field on x, whose components are given by the sum over the cyclic permutations of $\{j,k,l\}$ in $\pi_{ij}\partial_i\pi_{kl}$. We recall the previous discussion about the shape of the quotient graph Γ^A in order to understand the shape of the admissible graphs in the interior sums in all three terms in the last expression of (6). In particular, $\widehat{\varpi}_\gamma$ in the last expression in (6) is the integral along the fiber of $\pi_{n,2}$ of a form of degree $2n+1$: such an integral exists, because ω extends to $^+_{2,0}$, and ϖ_γ yields a smooth 1-form on $c^+_{2,0}$.

Observe now that the third term in the final expression of (6) vanishes because of the Jacobi identity, hence only the first and second term matter in our discussion. The linearity of π on $X=\mathfrak{g}^*$ permits to re-write both 1-forms in a more elegant way.

If we consider a general graph Γ in $\mathcal{G}_{n+2,0}$ as in the first term on the right-hand side of (6), any bivalent vertex different from $n+1$, $n+2$ may be the endpoint of at most one arrow. Thus, by slightly adapting [4, Subsections 3.1.2-3.1.4], Γ factorizes uniquely into the union of its simple components [1]: the main novelty is that in the present situation, there are three types of simple components, namely

i) rooted, bivalent trees with 2 leaves,
ii) wheel-like graphs with 2 leaves, whose legs may be attached to rooted, bivalent trees,
iii) rooted, bivalent trees with 2 leaves and an edge connecting either one of the two leaves to the root.

Pictorially, we have

[1] An element Γ of $\mathcal{G}_{n,2}$ is simple, if the graph obtained from Γ by removing all arrows connecting to the vertices of the second type is connected. In the present situation, we may regard Γ in $\mathcal{G}_{n+2,0}$ as an element of $\mathcal{G}_{n,2}$ by interpreting the last two vertices of the first type as vertices of the second type.

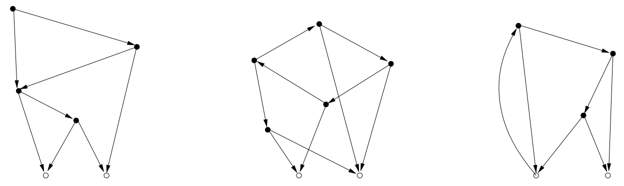

Figure 1 - Simple graphs of type $i)$, $ii)$ and $iii)$

By definition, Γ has exactly one simple component of type $iii)$.

Let us consider a simple graph ${}_{\curvearrowright}\Gamma$, resp. $\Gamma_{\curvearrowleft}$, of type $iii)$ with exactly one edge connecting $n+1$, resp. $n+2$, to the root: then, borrowing previous notation, we may define

$$\Omega^{\pi}_{1,{}_{\curvearrowright}\Gamma}(f_1 \otimes \xi, f_2) = \varpi_{{}_{\curvearrowright}\Gamma}\left((\xi \otimes 1 \otimes 1) \circ (\mu_n \otimes 1 \otimes 1) \circ \tau_\Gamma\right)(\underbrace{\pi \otimes \cdots \otimes \pi}_{n} \otimes f_1 \otimes f_2), \tag{7}$$

$$\Omega^{\pi}_{2,\Gamma_{\curvearrowleft}}(f_1, f_2 \otimes \xi) = \varpi_{\Gamma_{\curvearrowleft}}\left((\xi \otimes 1 \otimes 1) \circ (\mu_n \otimes 1 \otimes 1) \circ \tau_\Gamma\right)(\underbrace{\pi \otimes \cdots \otimes \pi}_{n} \otimes f_1 \otimes f_2), \tag{8}$$

where f_i in A, $i = 1,2$, ξ in $\mathfrak{g}^*$, and Γ is the rooted, bivalent tree obtained from ${}_{\curvearrowright}\Gamma$ or $\Gamma_{\curvearrowleft}$ by removing the edge from $n+1$ or $n+2$ to the root.

As already observed, $\varpi_{{}_{\curvearrowright}\Gamma}$ and $\varpi_{\Gamma_{\curvearrowleft}}$ are well-defined, smooth 1-forms on $C^+_{2,0}$. Further, since Γ is a rooted, bivalent tree, $(\mu_n \otimes 1 \otimes 1) \circ \tau_\Gamma$ is a linear map from $A^{\otimes 2}$ to $\mathfrak{g} \otimes A^{\otimes 2}$: hence, contraction of $\mathfrak{g}$ with $\mathfrak{g}^*$ yields an endomorphism of $A^{\otimes 2}$ consisting of differential operators with constant coefficients. Summing up over all simple graphs of type $iii)$ (7) and (8) we obtain well-defined, smooth 1-forms Ω^π_i, $i = 1,2$, on $C^+_{2,0}$ with values in $\mathfrak{g} \otimes \widehat{\mathrm{S}}(\mathfrak{g}^*)^{\otimes 2}$, where we identify $\widehat{\mathrm{S}}(\mathfrak{g}^*)$ with the algebra of differential operators on A with constant coefficients.

On the other hand, the sum over all simple graphs of type $i)$ and $ii)$ yield the bidifferential operator $\mathcal{T}_\pi(\bullet,\bullet)$ by the arguments of [4, Subsubsections 3.1.2-3.1.4]. (We will come back to the simple graphs of type $i)$ and $ii)$ in Subsection 3.4 about the KV conjecture, where their relevance will be clearer.)

Therefore, Fubini's Theorem and the decomposition of admissible graphs into simple components of type $i)$, $ii)$ and $iii)$ yield (5). □

The 0-form $\mathcal{T}^\pi$ and the 1-forms Ω^π_i, $i = 1,2$, are smooth on $C^+_{2,0}$ and extend to the class L^1 when restricted on piecewise differentiable curves on $\mathcal{C}^+_{2,0}$.

Now, let us evaluate $\mathcal{T}^\pi(f_1,f_2)$ at a point in the boundary stratum $\mathcal{C}_2 = S^1$ of $\mathcal{C}^+_{2,0}$, corresponding to the situation, where the two distinct points in $\mathbb{H}^+$ collapse together along a prescribed direction: the skew-symmetry of π eliminates all contributions coming from simple graphs of type $i)$ and of type $ii)$, where at least one rooted, bivalent tree is attached to a wheel-like graph. The only possibly non-trivial contributions come from wheel-like graphs with the spokes pointing inwards (the two leaves have collapsed to a single point in $\mathbb{H}^+$, which we may fix to i): the corresponding integral weights vanish by the famous result of [9]. The only non-trivial contribution comes from the unique graph in $\mathcal{G}_{2,0}$ with no edges.

Let us evaluate $\mathcal{T}^\pi(f_1,f_2)$ at the boundary stratum $\mathcal{C}^+_{0,2} = \{0,1\}$ of codimension 2 of $\mathcal{C}^+_{2,0}$, which corresponds to the approach of the two distinct points in $\mathbb{H}^+$ to 0 and 1 on $\mathbb{R}$: resorting to local coordinates on $\mathcal{C}^+_{2,0}$ near the said boundary stratum and recalling the projection $\pi_{n,2}$, the corresponding integral weights factorize as

$$\widehat{\varpi}_\Gamma = \varpi_{\Gamma_1}\varpi_{\Gamma_2}\varpi_{\Gamma_3},$$

where Γ_1 is in $\mathcal{G}_{n_1,2}$, Γ_2, Γ_3 are in $\mathcal{G}_{n_2,0}$ and $\mathcal{G}_{n_3,0}$. Dimensional reasons and the linearity of π force Γ_2 and Γ_3 to be wheel-like graphs with spokes points inwards, thus again in virtue of [9], the corresponding weights are non-trivial only if $n_2 = n_3 = 0$.

If we consider a piecewise differentiable curve γ on $\mathcal{C}^+_{2,0}$ connecting the said point in $\mathcal{C}_2 = S^1$ with $\mathcal{C}^+_{0,2} = \{0,1\}$ and whose interior is in $C^+_{2,0}$, we may integrate (5) along γ: the previous arguments yield

$$f_1 \star f_2 - f_1 f_2 = \int_\gamma \left(\mathcal{T}^\pi(\Omega^\pi_1([\pi,f_1],f_2)) + \mathcal{T}^\pi(\Omega^\pi_2(f_1,[\pi,f_2]))\right), \tag{9}$$

which is precisely a special case of the famous compatibility between cup products [8, Theorem 8.2].

3.3. **Relationship with the AT connection.** By their very construction, $\mathcal{T}^\pi$ and Ω^π_i, $i = 1,2$, extend to the completed symmetric algebra $\widehat{A} = \widehat{\mathrm{S}}(\mathfrak{g}) = \mathbb{K}[\![x_1,\dots,x_d]\!]$. For y_i, $i = 1,2$, in $\mathfrak{g}$, we consider e^{y_i} in $\widehat{A}$: e^{y_i} may be also regarded as a smooth function on X *via* $e^{y_i}(\xi) = e^{\langle \xi, y_i\rangle}$, ξ in X, and $\langle\bullet,\bullet\rangle$ denotes the canonical duality pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

First of all, borrowing previous notation, let us compute the symbol of Ω_i^π, $i = 1,2$, *i.e.*

$$\Omega_1^\pi(e^{y_1}\otimes\xi, e^{y_2}),\ \Omega_1^\pi(e^{y_1}, e^{y_2}\otimes\xi),\ \xi\in\mathfrak{g}^*.$$

Recalling Formulæ (7), (8), a direct computation yields

$$\Omega_1^\pi(e^{y_1}\otimes\xi, e^{y_2}) = \langle\xi,\omega_1(y_1,y_2)\rangle e^{y_1}\otimes e^{y_2},\ \Omega_2^\pi(e^{y_1}\otimes\xi, e^{y_2}) = \langle\xi,\omega_2(y_1,y_2)\rangle e^{y_1}\otimes e^{y_2},$$

where ω_i denotes here the AT connection [3,10]. In fact, $\omega_i(y_1,y_2)$, $i=1,2$, denotes a 1-form on $C_{2,0}^+$ with values in the formal Lie series w.r.t. y_i in $\mathfrak{g}$. In a more precise way, the AT connection ω_i, $i=1,2$, is a connection 1-form on $C_{2,0}^+$ with values in the Lie algebra $\mathfrak{tder}_2$ of tangential derivations of the degree completion of the free Lie algebra $\mathfrak{lie}_2$ with two generators [2].

Following the same patterns, it is not difficult to prove by direct computations the following identities:

$$\Omega_1^\pi([\pi,e^{y_1}],e^{y_2}) = ([y_1,\omega_1(y_1,y_2)]e^{y_1})\otimes e^{y_2} + \mathrm{tr}_\mathfrak{g}(\mathrm{ad}(y_1)\partial_{y_1}\omega_1(y_1,y_2))\, e^{y_1}\otimes e^{y_2},$$
$$\Omega_2^\pi([\pi,e^{y_1}],e^{y_2}) = e^{y_1}\otimes([y_2,\omega_2(y_1,y_2)]e^{y_2}) + \mathrm{tr}_\mathfrak{g}(\mathrm{ad}(y_2)\partial_{y_2}\omega_2(y_1,y_2))\, e^{y_1}\otimes e^{y_2},$$

where $\mathrm{tr}_\mathfrak{g}(\bullet)$ denotes the trace of endomorphisms of $\mathfrak{g}$, $\mathrm{ad}(\bullet)$ denotes the adjoint representation of $\mathfrak{g}$ and $\partial_{y_1}\omega_1(y_1,y_2)$ denotes the endomorphism of $\mathfrak{g}$ defined *via*

$$\left(\partial_{y_1}\omega_1(y_1,y_2)\right)(x) = \frac{d}{dt}\omega_1(y_1+tx,y_2)\Big|_{t=0},\ x\in\mathfrak{g}.$$

It is possible to re-write (9) in the following form:

$$\begin{aligned} e^{y_1}\star e^{y_2} - e^{y_1}e^{y_2} &= \int_\gamma \left(\mathcal{T}^\pi(\langle[y_1,\omega_1(y_1,y_2)],\partial_{y_1}\rangle(e^{y_1}),e_{y_2}) + \mathcal{T}^\pi(e^{y_1},\langle[y_1,\omega_1(y_1,y_2)],\partial_{y_1}\rangle(e_{y_2}))\right) + \\ &+ \left(\mathrm{tr}_\mathfrak{g}(\mathrm{ad}(y_1)\partial_{y_1}\omega_1(y_1,y_2)) + \mathrm{tr}_\mathfrak{g}(\mathrm{ad}(y_2)\partial_{y_2}\omega_1(y_1,y_2))\right)\int_\gamma \mathcal{T}^\pi(e^{y_1},e^{y_2}) = \\ &= \int_\gamma \left(\langle[y_1,\omega_1(y_1,y_2)],\partial_{y_1}\rangle + \langle[y_2,\omega_2(y_1,y_2)],\partial_{y_1}\rangle + \mathrm{div}(\omega(y_1,y_2))\right) D_\mathrm{T}(y_1,y_2)e^{Z_T(y_1,y_2)}, \end{aligned}$$

where $\langle[y_1,\omega_1(y_1,y_2)],\partial_{y_1}\rangle(e^{y_1})$ denotes the tangent vector $[y_1,\omega_1(y_1,y_2)]$ of the adjoint type acting on e^{y_1}, and similarly for $\langle[y_1,\omega_1(y_1,y_2)],\partial_{y_1}\rangle(e_{y_2})$, and, following notation from [3],

$$\mathrm{div}(\omega(y_1,y_2)) = \mathrm{tr}_\mathfrak{g}(\mathrm{ad}(y_1)\partial_{y_1}\omega_1(y_1,y_2)) + \mathrm{tr}_\mathfrak{g}(\mathrm{ad}(y_2)\partial_{y_2}\omega_1(y_1,y_2)).$$

Finally, by $D_\mathrm{T}(\bullet,\bullet)$ and $Z_\mathrm{T}(\bullet,\bullet)$ we denote the functions over $C_{2,0}^+$, providing deformations of the Duflo density function $D(\bullet,\bullet)$ and the Baker–Campbell–Hausdorff (shortly, BCH) formula $Z(\bullet,\bullet)$ respectively, introduced in [10]. We will discuss the Duflo density function in the next Subsection, as well as its relationship with the product $\star$.

3.4. **Relationship with the KV conjecture.** The AT connection had been introduced in [10] in a (successful, albeit not complete until the appearance of the paper [1]; see also [2] for an alternative approach which does not use deformation quantization techniques) attempt to solve the combinatorial KV conjecture [6].

Given $\mathfrak{g}$ as in Section 2, the KV conjecture states the existence of two Lie series F, G, which are convergent in a neighborhood U of $(0,0)$ in $\mathfrak{g}\times\mathfrak{g}$, which satisfy the two identities

$$y_1+y_2-\log(e^{y_2}e^{y_1}) = \left(1-e^{-\mathrm{ad}(y_1)}\right)F(y_1,y_2) + \left(e^{\mathrm{ad}(y_2)}-1\right)G(y_1,y_2), \tag{10}$$

$$\mathrm{tr}_\mathfrak{g}(\mathrm{ad}(y_1)\partial_{y_1}F(y_1,y_2)) + \mathrm{tr}_\mathfrak{g}(\mathrm{ad}(y_2)\partial_{y_2}G(y_1,y_2)) = \frac{1}{2}\mathrm{tr}_\mathfrak{g}\left(\frac{\mathrm{ad}(y_1)}{e^{\mathrm{ad}(y_1)}-1} + \frac{\mathrm{ad}(y_2)}{e^{\mathrm{ad}(y_2)}-1} - \frac{\mathrm{ad}(Z(y_1,y_2))}{e^{\mathrm{ad}(Z(y_1,y_2))}-1} - 1\right), \tag{11}$$

for (y_1,y_2) in U, such that the BCH Lie series $Z(y_1,y_2) = \log(e^{y_1}e^{y_2})$ converges.

We now recall *e.g.* from [5,11] the following general formula relating the product $\star$ with the product in the UEA $\mathrm{U}(\mathfrak{g})$ of $\mathfrak{g}$ as in Section 2:

$$\mathcal{I}(f_1\star f_2) = \mathcal{I}(f_1)\cdot\mathcal{I}(f_2),\ f_i\in A,\ i=1,2, \tag{12}$$

where $\cdot$ denotes the product in $\mathrm{U}(\mathfrak{g})$, and $\mathcal{I}$ is the isomorphism (of vector spaces) from A to $\mathrm{U}(\mathfrak{g})$ given by post-composing the PBW isomorphism from A to $\mathrm{U}(\mathfrak{g})$ with the automorphism of A given by the differential operator with constant coefficients and infinite order associated to the Duflo function

$$\sqrt{j(x)} = \sqrt{\det{}_\mathfrak{g}\left(\frac{1-e^{-\mathrm{ad}(x)}}{\mathrm{ad}(x)}\right)},\ x\in\mathfrak{g}.$$

(In fact, $j(\bullet)$ defines an invertible element of $\widehat{\mathrm{S}}(\mathfrak{g}^*)$.)

[2]A derivation of $\mathfrak{lie}_2$ is uniquely defined on the generators y_1, y_2: thus, a derivation u of $\mathfrak{lie}_2$ is called tangential, if it obeys $u(y_i) = [y_i,u_i]$, for u_i in $\mathfrak{lie}_2$, $i=1,2$.

As a corollary of (12), we have the identity

$$e^{y_1} \star e^{y_2} = D(y_1, y_2) e^{Z(y_1,y_2)},\ D(y_1,y_2) = \frac{\sqrt{j(y_1)}\sqrt{j(y_2)}}{\sqrt{j(Z(y_1,y_2))}},\ y_i \in \mathfrak{g},\ i = 1,2. \tag{13}$$

We observe that (13) has been proved by different methods in [7] and [4, Subsubsections 3.1.2-3.1.4].

Remark 3.4. More precisely, in [7], it had been proved that the simple graphs of type $i)$ contribute to the BCH Lie series $Z(\bullet,\bullet)$, while in [4], recalling also [9], it had been proved that the simple graphs of type $ii)$ contribute to the density function $D(\bullet,\bullet)$.

Let us replace in (13) π by $t\pi$, for t in the unit interval: we write $\star_t$ for the corresponding product, and a direct computation yields

$$e^{y_1} \star_t e^{y_2} = D(ty_1, ty_2) e^{Z_t(y_1,y_2)},\ Z_t(y_1,y_2) = \frac{Z(ty_1,ty_2)}{t}.$$

It follows directly from (2) that $e^{y_1} \star_1 e^{y_2} = e^{y_1} \star e^{y_2}$ and $e^{y_1} \star_0 e^{y_2} = e^{y_1} e^{y_2}$, y_i in $\mathfrak{g}$.

Let us compute the derivative w.r.t. t of both sides of (13): in particular, we are interested into the derivative of the right-hand side.

Identity (10) implies that (see e.g. [6, Lemma 3.2])

$$\frac{d}{dt} Z_t(y_1,y_2) = \left(\langle [y_1, F_t(y_1,y_2)], \partial_{y_1}\rangle + \langle [y_2, G_t(y_1,y_2)], \partial_{y_2}\rangle\right) Z_t(y_1,y_2),$$

where $F_t(y_1,y_2) = F(ty_1,ty_2)/t$ and similarly for $G_t(y_1,y_2)$.

On the other hand, combining [6, Lemma 3.2] with [6, Lemma 3.3] and observing that $\sqrt{j(\bullet)}$ is $\mathfrak{g}$-invariant, we get

$$\begin{aligned}\frac{d}{dt} D(ty_1,ty_2) &= \frac{1}{2t}\mathrm{tr}_{\mathfrak{g}}\left(\frac{\mathrm{ad}(ty_1)}{e^{\mathrm{ad}(ty_1)}-1} + \frac{\mathrm{ad}(ty_2)}{e^{\mathrm{ad}(ty_2)}-1} - \frac{\mathrm{ad}(tZ_t(y_1,y_2))}{e^{\mathrm{ad}(tZ_t(y_1,y_2))}-1} - 1\right) D(ty_1,ty_2) + \\ &\quad + \left(\langle [y_1, F_t(y_1,y_2)], \partial_{y_1}\rangle + \langle [y_2, G_t(y_1,y_2)], \partial_{y_2}\rangle\right) D(ty_1,ty_2) = \\ &= \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(y_1)\partial_{y_1} F_t(y_1,y_2)) + \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(y_2)\partial_{y_2} G_t(y_1,y_2)) D(ty_1,ty_2) + \\ &\quad + \left(\langle [y_1, F_t(y_1,y_2)], \partial_{y_1}\rangle + \langle [y_2, G_t(y_1,y_2)], \partial_{y_2}\rangle\right) D(ty_1,ty_2),\end{aligned}$$

where the second equality is a consequence of (11).

Combining both previous results, we get

$$\begin{aligned}\frac{d}{dt}(e^{y_1} \star_t e^{y_2}) &= \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(y_1)\partial_{y_1} F_t(y_1,y_2)) + \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(y_2)\partial_{y_2} G_t(y_1,y_2)) D(ty_1,ty_2)(e^{y_1} \star_t e^{y_2}) + \\ &\quad + \left(\langle [y_1, F_t(y_1,y_2)], \partial_{y_1}\rangle + \langle [y_2, G_t(y_1,y_2)], \partial_{y_2}\rangle\right)(e^{y_1} \star_t e^{y_2}) = \\ &= \left(\langle [y_1, F_t(y_1,y_2)], \partial_{y_1}\rangle + \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(y_1)\partial_{y_1} F_t(y_1,y_2))\right)(e^{y_1}) \star_t e^{y_2} + \\ &\quad + e^{y_1} \star_t \left(\langle [y_2, G_t(y_1,y_2)], \partial_{y_1}\rangle + \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(y_1)\partial_{y_1} G_t(y_1,y_2))\right)(e^{y_2}).\end{aligned}$$

Recalling the computations at the beginning of Subsection 3.3, it is not difficult to verify that to the Lie series F_t, G_t, one may associate two smooth 1-forms Ω_i^{KV}, $i = 1,2$, on the unit interval with values in $\mathfrak{g} \otimes \widehat{\mathrm{S}}(\mathfrak{g}^*)$, such that the following identities hold true:

$$\begin{aligned}\Omega_1^{\mathrm{KV}}([\pi, e^{y_1}], e^{y_2}) &= ([y_1, F_t(y_1,y_2)dt] e^{y_1}) \otimes e^{y_2} + \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(y_1)\partial_{y_1}(F_t(y_1,y_2)dt))\, e^{y_1} \otimes e^{y_2}, \\ \Omega_2^{\mathrm{KV}}([\pi, e^{y_1}], e^{y_2}) &= e^{y_1} \otimes ([y_2, G_t(y_1,y_2)dt] e^{y_2}) + \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(y_2)\partial_{y_2}(G_t(y_1,y_2)dt))\, e^{y_1} \otimes e^{y_2},\end{aligned}$$

whence, denoting by $\mathcal{T}_t(\bullet,\bullet)$ the t-dependent bidifferential operator of infinite order $\mathcal{T}_t(f_1,f_2) = f_1 \star_t f_2$, f_i in A, we find the homotopy formula

$$f_1 \star f_2 - f_1 f_2 = \int_0^1 \left(\mathcal{T}_t(\Omega_1^{\mathrm{KV}}([\pi, f_1], f_2)) + \mathcal{T}_t(\Omega_2^{\mathrm{KV}}(f_2, [\pi, f_1]))\right), \tag{14}$$

which is similar in its structure to the homotopy formula (9) obtained by deforming the product $\star$ on $C_{2,0}^+$.

We finally observe that both (9) and (14) imply that

$$f_1 \star f_2 = f_1 f_2,\ f_i \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}},\ i = 1,2.$$

References

[1] A. Alekseev and E. Meinrenken, *On the Kashiwara-Vergne conjecture*, Invent. Math. **164** (2006), no. 3, 615–634, DOI 10.1007/s00222-005-0486-4. MR2221133 (2007g:17017)

[2] Anton Alekseev and Charles Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, accepted for publication in Ann. Math. (2008), available at `arXiv:0802.4300`.

[3] ———, *Kontsevich deformation quantization and flat connections*, Comm. Math. Phys. **300** (2010), no. 1, 47–64, DOI 10.1007/s00220-010-1106-8. MR2725182 (2011j:53181)

[4] Martin Andler, Siddhartha Sahi, and Charles Torossian, *Convolution of invariant distributions: proof of the Kashiwara-Vergne conjecture*, Lett. Math. Phys. **69** (2004), 177–203, DOI 10.1007/s11005-004-0979-x. MR2104443 (2005j:53108)

[5] Damien Calaque, Giovanni Felder, and Carlo A. Rossi, *Deformation quantization with generators and relations*, J. Algebra **337** (2011), 1–12, DOI 10.1016/j.jalgebra.2011.03.037. MR2796061

[6] Masaki Kashiwara and Michèle Vergne, *The Campbell-Hausdorff formula and invariant hyperfunctions*, Invent. Math. **47** (1978), no. 3, 249–272. MR0492078 (58 #11232)

[7] Vinay Kathotia, *Kontsevich's universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula*, Internat. J. Math. **11** (2000), no. 4, 523–551, DOI 10.1142/S0129167X0000026X. MR1768172 (2002h:53154)

[8] Maxim Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216. MR2062626 (2005i:53122)

[9] Boris Shoikhet, *Vanishing of the Kontsevich integrals of the wheels*, Lett. Math. Phys. **56** (2001), no. 2, 141–149, DOI 10.1023/A:1010842705836. EuroConférence Moshé Flato 2000, Part II (Dijon). MR1854132 (2002j:53119)

[10] Charles Torossian, *Sur la conjecture combinatoire de Kashiwara-Vergne*, J. Lie Theory **12** (2002), no. 2, 597–616 (French, with English summary). MR1923789 (2004c:17023)

[11] Thomas Willwacher, *A counterexample to the quantizability of modules*, Lett. Math. Phys. **81** (2007), no. 3, 265–280. MR2355492 (2008j:53160)

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