NEW SYMPLECTIC V-MANIFOLDS OF DIMENSION FOUR VIA THE RELATIVE COMPACTIFIED PRYMIAN

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Abstract. Three new examples of 4-dimensional irreducible symplectic V-manifolds are constructed. Two of them are relative compactified Prymians of a family of genus-3 curves with involution, and the third one is obtained from a Prymian by Mukai's flop. They have the same singularities as two of Fujiki's examples, namely, 28 isolated singular points analytically equivalent to the Veronese cone of degree 8, but a different Euler number. The family of curves used in this construction forms a linear system on a K3 surface with involution. The structure morphism of both Prymians to the base of the family is a Lagrangian fibration in abelian surfaces with polarization of type (1,2). No example of such fibration is known on *nonsingular* irreducible symplectic varieties.

0. Introduction

Historically, the first constructions of nontrivial compact Kähler holomorphically symplectic varieties Y of dimension > 2 belong to Beauville [Beau-1] and Fujiki [F]. Fujiki's notion of nontriviality means that Y is not obtained as a finite quotient from a product of a complex torus with symplectic surfaces. Fujiki constructed one nonsingular example in dimension 4, the blowup $S^{[2]}$ of the diagonal in the symmetric square $S^{(2)}$ of a K3 surface S, and his other examples are 4-dimensional V-manifolds, that is varieties having finite quotient singularities.

Beauville [Beau-1], [Beau-2] constructed two deformation classes of nonsingular irreducible symplectic manifolds in all even dimensions $2n$. Here a manifold is called *irreducible symplectic* if it is simply connected and has a unique symplectic structure up to proportionality; this is equivalent to Fujiki's condition of nontriviality at least in the category of nonsingular symplectic varieties. The Beauville's examples are: 1) $S^{[n]} = \mathrm{Hilb}^n(S)$, the Hilbert scheme of 0-dimensional subschemes of length n in a K3 surface S, and 2) $K_n(A)$, the generalized Kummer

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variety associated to an abelian surface A. The latter is defined as the fiber of the summation map $A^{[n+1]} \to A$.

Mukai [Mu-1] showed that the moduli spaces of semistable sheaves on a K3 or abelian surface are symplectic. According to [Hu-1], [Hu-2], [O'G-1] and [Y], whenever such a moduli space is nonsingular, it is deformation equivalent to $S^{[n]}$ or $K_n(A) \times T$ with $T = A$ or $A \times A$. Thus, for years, two Beauville's examples provided the only known moduli components of irreducible symplectic manifolds, until O'Grady [O'G-2], [O'G-3] constructed two essentially new such manifolds. They are obtained as symplectic desingularizations of singular moduli spaces of semistable sheaves. The first one is associated to a K3 surface and is of dimension 10, the second one is associated to an abelian surface and is of dimension 6. It is still unknown whether there exist irreducible symplectic 4-folds that are not deformation equivalent to $S^{[2]}$ or $K_2(A)$. O'Grady studies in [O'G-4], [O'G-5] the irreducible symplectic 4-folds whose intersection 4-linear form on H^2 is isomorphic to that of $S^{[2]}$ and conjectures that they are deformation equivalent to $S^{[2]}$.

The results of [KL], [KLS], [LS], [CK-1], [CK-2] show that, informally speaking, no new examples of *nonsingular* irreducible symplectic manifolds can be obtained by the method of [O'G-2], [O'G-3]. More precisely, for any singular moduli space $\mathcal{M}$ of semistable sheaves on a K3 surface, either $\mathcal{M}$ has no symplectic resolution, or such a resolution exists and up to deformations coincides with one of the known examples: Beauvilles's or O'Grady's. A weaker result, concerning only rank-2 sheaves, is obtained for moduli of sheaves on abelian surfaces.

Thus the problem of extending the very short list of known deformation classes of irreducible symplectic *manifolds* is very hard. Leaving aside this hard problem, we turn back to the original setting of Fujiki, who considered symplectic *V-manifolds*. All of his examples, up to deformation of a complex structure, are partial resolutions of finite quotients of the products of two symplectic surfaces.

In the present article, we provide a new construction of irreducible symplectic V-manifold of dimension 4, the relative compactified Prym variety of some family of curves with involution. The fibration in Prym surfaces is Lagrangian.

Many features of the theory of irreducible symplectic manifolds are very similar to those of K3 surfaces, and in view of this similarity, the manifolds with a Lagrangian fibration constitute an important class of irreducible symplectic manifolds which is an analog of the class of K3 surfaces with an elliptic pencil. Earlier examples of Lagrangian fibrations on irreducible symplectic manifolds were constructed in [Beau-3],

[D], [HasTsch-1], [HasTsch-2], [IR], [S-2]. There are more examples if we relax the hypothesis that the fibration map is a regular morphism, but admit *rational* Lagrangian fibrations [Mar-2], [Gu].

By Liouville's Theorem, the general fiber of these Lagrangian fibrations is an abelian variety. It turns out, that in all of the known examples it is either the Jacobian of a curve, or a $(1,\ldots,1,k)$-polarized abelian variety with $k \geq 3$. The first possibility occurs for Lagrangian fibrations $f : Y \to B$ with Y deformation equivalent to $S^{[n]}$, and the second one for Y birational to $K_n(A)$, where A is an abelian surface of polarization $(1,k)$ (see [S-1], Remark 3.9).

Thus there are no examples of Lagrangian fibrations on irreducible symplectic 4-folds with $(1,2)$-polarized abelian surfaces as fibers. On the other hand, there are classical integrable systems integrated on Prym surfaces of such polarization, for example, the complexified Kowalevski top [HvM]. However, the corresponding symplectic manifolds, which are the (complexified) phase spaces of these systems, are always rational, or at least unirational, and hence they are very far from having a symplectic compactification, neither nonsingular, nor in the class of V-manifolds. The phase space of the Kowalevski top is identified with the relative Prym variety $\mathrm{Prym}^k(\mathcal{C},\tau)$ of a family $\mathcal{C}/\mathbb{P}^2$ of genus-3 curves endowed with an involution τ such that the quotients by τ form a family of elliptic curves.

In the present paper, we use this idea in taking for $\mathcal{C}/\mathbb{P}^2$ the family of τ-invariant members of a linear system $|H|$ of genus-3 curves on a K3 surface S with an involution τ. In order that the construction might work, τ should leave the symplectic form $\omega \in H^0(S,\Omega_S^2)$ anti-invariant.

We denote by $\overline{\mathrm{Prym}}^{\,k,\kappa}(\mathcal{C},\tau)$ $(k \in \mathbb{Z})$ the relative compactified Prym variety defined as a connected component of the fixed locus of some involution κ in the relative compactified Picard variety $\overline{\mathrm{Pic}}^{\,k}(|H|)$. The latter is a compactification of the relative Picard variety $\mathrm{Pic}^k(|H|)$ parametrizing divisor classes of degree k on the curves from the complete linear system $|H|$ of an ample genus-3 curve H. The compactification depends on the choice of a polarization on S, which we fix once and forever to be H. There are at most 4 non-isomorphic compactified Picard varieties, corresponding to $k = 0,1,2,3$, and $\overline{\mathrm{Pic}}^{\,k}(|H|)$ is birational to $\overline{\mathrm{Pic}}^{\,k+2}(|H|)$, so there are at most two nonbirational ones. The definiton of κ depends on some arbitrary choices if $k = 1$ or 3, but is canonical for even k, so for even k, we suppress the superscript κ from the notation. We work out in full detail the case of even $k = 2m$, proving that $\boldsymbol{\mathcal{P}}^{2m} = \overline{\mathrm{Prym}}^{\,2m}(\mathcal{C},\tau)$ is an irreducible symplectic V-manifold.

As $\boldsymbol{\mathcal{P}}^{2m} \simeq \boldsymbol{\mathcal{P}}^{2m+4}$, there are at most two non-isomorphic compactified Prymians, $\boldsymbol{\mathcal{P}}^0$ and $\boldsymbol{\mathcal{P}}^2$. We do not know whether they are really non-isomorphic.

The Prymian $\boldsymbol{\mathcal{P}}^0$ of degree 0 contains a family of groups, hence has the zero section whose image Π is isomorphic to the base of the family, that is to $\mathbb{P}^2$. Performing Mukai's flop with center Π, we obtain a third 4-dimensional symplecti V-manifold M' (see Corollary 5.7). It has the same topological invariants and Hodge numbers as $\boldsymbol{\mathcal{P}}^0$, but is conjectured to be non-isomorphic neither to $\boldsymbol{\mathcal{P}}^0$, nor to $\boldsymbol{\mathcal{P}}^2$. We also identify M' as a partial desingularization of the quotient of $S^{[2]}$ by a symplectic involution.

In the case of odd degree k, it is unlikely that $\overline{\mathrm{Prym}}^{k,\kappa}(\mathcal{C},\tau)$ is symplectic. We show that it is not symplectic for one of the possible choices of κ, for which it contains a 3-dimensional rational variety, see Remark 5.8.

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1. Definition and basic properties of varieties $\overline{\mathrm{Pic}}^k(|H|)$

Let X be a Del Pezzo surface of degree 2 obtained as a double cover of $\mathbb{P}^2$ branched in a generic quartic curve B_0, $\mu : X \longrightarrow \mathbb{P}^2$ the double cover map, $B = \mu^{-1}(B_0)$ the ramification curve. Let Δ_0 be a generic curve from the linear system $|-2K_X|$, $\rho : S \longrightarrow X$ the double cover branched in Δ_0 and $\Delta = \rho^{-1}(\Delta_0)$. Then S is a K3 surface, and $H = \rho^*(-K_X)$ is a degree-4 ample divisor class on S, which we fix once and forever as a polarization of S. We will denote by ι (resp. τ) the Galois involution of the double cover μ (resp. ρ).

The plane quartic B_0 has 28 bitangent lines $m_1, \ldots, m_{28}$, and $\mu^{-1}(m_i)$ is the union of two rational curves $\ell_i \cup \ell_i'$ meeting in 2 points. The 56 curves ℓ_i, ℓ_i' are all the *lines* on X, that is, curves of degree 1 with respect to $-K_X$. Further, the curves $C_i = \rho^{-1}(\ell_i)$, $C_i' = \rho^{-1}(\ell_i')$ are *conics* on S, that is, curves of degree 2 with respect to H. Each pair C_i, C_i' meets in 4 points, thus forming a reducible curve of arithmetic genus 3 belonging to the linear system $|H|$. Throughout the paper we assume that B_0, Δ_0 are sufficiently generic. This implies, in particular, that each line ℓ_i meets only one of the two lines ℓ_j, ℓ_j' for $j \neq i$.

Lemma 1.1. *The linear system $|H|$ is very ample and embeds S into $\mathbb{P}^3$ as a quartic surface. Every curve in $|H|$ is reduced, and the only reducible members of $|H|$ are the 28 curves $\Gamma_i = C_i + C_i'$, $i = 1, \ldots, 28$.*

Proof. A generic curve in $|H|$ is isomorphic to a plane quartic, for $\Delta \in |H|$ and $\Delta \simeq \Delta_0 \simeq \mu(\Delta_0) \in |\mathcal{O}_{\mathbb{P}^2}(4)|$. According to Saint-Donat's description of ample linear systems on K3 surfaces [SD] (see also [Mor], Section 6), $|H|$ is very ample and embeds S into $\mathbb{P}^3$.

Let $C \in |H|$ be reducible or non-reduced. Then the same is true for $\underline{C} = \rho_*(C) \in |-2K_X|$. If $\underline{C} = D_1 + D_2$ is reducible and reduced, then we have the following 3 possibilities: (a) $\mu(D_1)$ is a line and $\mu(D_2)$ is a cubic; (b) $\mu(D_1), \mu(D_2)$ are conics; (c) μ is of degree 2 over one or both components D_i.

In the cases (a) and (b), $\mu^{-1}\mu(D_i)$ decomposes into two components D_i and $\iota(D_i)$ for both $i = 1, 2$. Hence $\mu(D_i)$ are totally tangent to B_0. Hence the family of such curves $D_1 + D_2$ is 3-dimensional in case (a) and 2-dimensional in case (b). Similarly, $\rho^{-1}(D_i)$ is the union of two components permuted by τ, and D_i are totally tangent to Δ_0 for both $i = 1, 2$. This is impossible for generic B_0, Δ_0 by dimension reasons.

In the case (c), let $E = \rho^{-1}(D_i)$, where D_i is the component mapped onto the line $\mu(D_i)$ with degree 2. Then $E^2 = 0$, $H \cdot E = 2$, which is impossible by loc. cit., for then every smooth member of $|H|$ should be hyperelliptic.

By a similar argument, one can eliminate the case when $\rho(C)$ is reducible and $\rho_*(C)$ is non-reduced. Thus, the only remaining case is when $\deg \rho|_C = 2$ and $\deg \mu|_{\rho(C)} = 2$, in which $\mu\rho(C)$ is a bitangent to B_0. $\square$

Mukai [Mu-2] has endowed the integer cohomology $H^*(Y)$ of a K3 surface Y with the following bilinear form:

$$\langle (v_0, v_1, v_2), (w_0, w_1, w_2) \rangle = v_1 \cup w_1 - v_0 \cup w_2 - v_2 \cup w_0,$$

where $v_i, w_i \in H^{2i}(X)$. For a sheaf $\mathcal{F}$ on Y, the *Mukai vector* is $v(\mathcal{F}) = (\operatorname{rk} \mathcal{F}, c_1(\mathcal{F}), \chi(\mathcal{F}) - \operatorname{rk} \mathcal{F}) \in H^*(Y)$, where $H^4(Y)$ is naturally identified with $\mathbb{Z}$. We refer to [Sim] or to [HL] for the definition and the basic properties of the Simpson (semi-)stable sheaves. Let $M_Y^{H,s}(v)$ (resp. $M_Y^{H,ss}(v)$) denote the moduli space of Simpson stable (resp. semistable) [Sim] sheaves $\mathcal{F}$ on Y with respect to an ample class H with Mukai vector $v(\mathcal{F}) = v$. According to Mukai ([Mu-1], [Mu-2], see also [HL]), $M_Y^{H,s}(v)$, if nonempty, is smooth of dimension $\langle v, v \rangle + 2$ and carries a holomorphic symplectic structure. We will study the moduli space $\mathcal{M}^k = M_S^{H,ss}(v)$ on the above special K3 surface S with Mukai vector $v = (0, H, k-2)$.

Proposition 1.2. *(i) $\mathcal{M} = \mathcal{M}^k$ is an irreducible projective variety of dimension 6. The open part $\mathcal{M}^* = M_S^{H,s}(0, H, k-2)$ corresponding to the stable sheaves is contained in the smooth locus of $\mathcal{M}$ and is a holomorphically symplectic manifold with symplectic form $\alpha \in H^0(\mathcal{M}^*, \Omega^2)$ induced by the Yoneda pairing*

$$\alpha_{[\mathcal{L}]} : \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) \times \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) \longrightarrow \mathrm{Ext}^2(\mathcal{L}, \mathcal{L}) \xrightarrow{\mathrm{Tr}} H^2(S, \mathcal{O}_S) \simeq \mathbb{C},$$

where $[\mathcal{L}] \in \mathcal{M}^*$ is the class of a stable sheaf $\mathcal{L}$ and the tangent space $T_{[\mathcal{L}]}\mathcal{M}^*$ is identified with $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L})$.

(ii) $\mathcal{M}^k$ parametrizes the S-equivalence classes of pure 1-dimensional sheaves $\mathcal{L}$ whose supports are curves from the linear system $|H|$ and such that $\mathcal{L}|_C$ is a torsion free $\mathcal{O}_C$-module of rank 1 with $\chi(\mathcal{L}) = k-2$, where $C = \mathrm{Supp}\,\mathcal{L}$. In the case when $\mathcal{L}$ is invertible as a sheaf on its support, the condition $\chi(\mathcal{L}) = k-2$ is equivalent to saying that $\deg \mathcal{L} = k$.

(iii) For any $k \in \mathbb{Z}$, $\mathcal{M}^k \simeq \mathcal{M}^{k+4}$. For odd k, any semistable sheaf from $\mathcal{M}^k$ is stable, so $\mathcal{M}^k$ is nonsingular. For even k, $\mathcal{M}^k$ contains exactly 28 S-equivalence classes of strictly semistable sheaves. Each of them is the class of the sheaf $\mathcal{O}_{C_i}(\frac{k-4}{2}pt) \oplus \mathcal{O}_{C'_i}(\frac{k-4}{2}pt)$ $(i = 1, \ldots, 28)$, where pt stands for the class of a point on either one of the conics C_i, C'_i.

Proof. (i) The projectivity of $\mathcal{M}^k$ follows by Theorem 1.21 of [Sim]. The stable sheaves being simple, the remaining assertions follow by Theorem 0.1 of [Mu-1].

(ii) If $[\mathcal{L}] \in \mathcal{M}^k$, then $v(\mathcal{L}) = (0, H, k-2)$, so $\mathcal{L}$ is, by definition, an equidimensional torsion sheaf with $c_1(\mathcal{L}) = H$ and $\chi(\mathcal{L}) = k-2$. Hence the support of $\mathcal{L}$ is a curve from $|H|$ and the rank of $\mathcal{L}$ is 1 at the generic points of all the components of H. It is torsion-free when considered as a sheaf on C since it is equidimensional.

(iii) The isomorphism $\mathcal{M}^k \to \mathcal{M}^{k+4}$ is given by $[\mathcal{L}] \mapsto [\mathcal{L}(1)]$ for all $[\mathcal{L}] \in \mathcal{M}^k$. Further, if $[\mathcal{L}] \in \mathcal{M}^k$ and $C = \mathrm{Supp}\,\mathcal{L}$ is an integral curve, then any rank-1 torsion-free sheaf on C is stable according to Simpson's definition, whether it is considered as a sheaf on C or on S, for it has no proper 1-dimensional subsheaves. By (i), this implies that $[\mathcal{L}]$ is a smooth point of $\mathcal{M}^k$. Now suppose that C is not integral. By Lemma 1.1, C is one of the curves $C_i + C'_i$. Hence the only possibility for a strictly semistable sheaf is to be the central term of an extension

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L} \longrightarrow \mathcal{F}' \longrightarrow 0, \tag{1}$$

where $\mathcal{F}, \mathcal{F}'$ are pure 1-dimensional, $\mathrm{Supp}\,\mathcal{F} = C_i$, $\mathrm{Supp}\,\mathcal{F}' = C'_i$, or vice versa, and $\chi(\mathcal{F}(n)) = \chi(\mathcal{F}'(n))$ for $n \gg 0$. Hence $\mathcal{F}, \mathcal{F}'$ are invertible on their supports, and $\chi(\mathcal{F}) = \chi(\mathcal{F}') = \frac{k-2}{2}$, which implies that k is even and $\mathcal{F} \simeq \mathcal{O}_{C_i}(\frac{k-4}{2}pt)$, $\mathcal{F}' \simeq \mathcal{O}_{C_i'}(\frac{k-4}{2}pt)$. $\square$

Definition 1.3. (i) A V-manifold is an algebraic variety having at worst finite quotient singularities. We reserve the term "manifold" for nonsingular algebraic varieties.

(ii) A symplectic variety is a normal algebraic variety Y such that its nonsingular locus Y_{ns} has a symplecitc structure, that is a 2-form $\omega \in H^0(Y_{\mathrm{ns}}, \Omega^2_{Y_{\mathrm{ns}}})$ which is closed and everywhere nondegenerate on Y_{ns}. The nondegeneracy means that $\omega^{\wedge \frac{1}{2}\dim Y}$ has no zeros on Y_{ns}. If Y is nonsingular, we also call it a symplectic manifold.

(iii) A closed irreducible subvariety $W \subset Y$ of a symplectic variety Y endowed with a symplectic structure ω is called Lagrangian if $\dim W = \frac{1}{2}\dim Y$, $W_0 := Y_{\mathrm{ns}} \cap W_{\mathrm{ns}} \neq \varnothing$ and $\omega|_{W_0} \equiv 0$.

(iv) A symplectic manifold (or V-manifold) Y is said to be irreducible symplectic if Y is complete, simply connected, and $h^0(Y, \Omega^2_Y) = 1$.

(v) A morphism $f : Y \to B$ from a symplectic variety of dimension $2n$ to another variety B of dimension n is called a Lagrangian fibration if it is surjective and if the generic fiber of f is a connected Lagrangian subvariety of Y.

Proposition 1.4. *In the above notation, the map $f : \mathcal{M}^k \to |H| \simeq \mathbb{P}^3$ sending $[\mathcal{L}] \in \mathcal{M}^k$ to the curve $C_{\mathcal{L}} = \operatorname{Supp} \mathcal{L} \in |H|$ is a Lagrangian fibration. The following properties are verified:*

(i) If $C \in |H|$ is smooth, then the fiber $f^{-1}(\{C\})$ is canonically isomorphic to $\operatorname{Pic}^k(C)$. Here $\{C\}$ denotes the point of the projective 3-space $|H|$ representing the curve C. Further, if $U \subset |H|$ is the open set parametrizing integral curves, $U = |H| \setminus \{\{\Gamma_1\}, \dots, \{\Gamma_{28}\}\}$, then the restriction $f_U : f^{-1}(U) \to U$ of f over U is identified with the relative compactified Picard variety of Altman–Kleiman.

(ii) Let $\Gamma = C + C'$ be one of the reducible curves $\Gamma_i = (i = 1, \dots, 28)$.

If k is even, then $f^{-1}(\{C\})$ is the union of three 3-dimensional rational components $\overline{J}^{\frac{k-2}{2},\frac{k+2}{2}}$, $\overline{J}^{\frac{k}{2},\frac{k}{2}}$, $\overline{J}^{\frac{k+2}{2},\frac{k-2}{2}}$.

If k is odd, then $f^{-1}(\{C\})$ is the union of four 3-dimensional rational components $\overline{J}^{\frac{k-3}{2},\frac{k+3}{2}}, \dots, \overline{J}^{\frac{k+3}{2},\frac{k-3}{2}}$.

Each $\overline{J}^{d,d'} = \overline{J}^{d,d'}(\Gamma)$ contains an open subset $J^{d,d'} = J^{d,d'}(\Gamma) \simeq (\mathbb{C}^*)^3$ parametrizing the invertible $\mathcal{O}_\Gamma$-modules $\mathcal{L}$ such that $\deg \mathcal{L}|_C = d$, $\deg \mathcal{L}|_{C'} = d'$.

(iii) Let q be one of the 56 conics C_i, C_i' $(i = 1, \dots, 28)$. Then $\mathcal{M}^k$ is birational to $\mathcal{M}^{k+2}$ via the map $\psi : [\mathcal{L}] \mapsto [\mathcal{L}(q)]$. Let us set $q = C_i$, and fix the notation for the 56 conics in such a way that $C_i \cap C_j' = \varnothing$ and $C_i \cap C_j = 2$ points for all $j \neq i$.

If k is even, then the indeterminacy locus of ψ is given by the formula

$$\mathrm{Indet}(\psi) = f^{-1}(\{\Gamma_i\}) \cup \bigcup_{j\neq i} \overline{J}^{\frac{k+2}{2},\frac{k-2}{2}}(\Gamma_j).$$

If k is odd, then the indeterminacy locus of ψ is

$$\mathrm{Indet}(\psi) = \overline{J}^{\frac{k-3}{2},\frac{k+3}{2}}(\Gamma_i) \cup \overline{J}^{\frac{k-1}{2},\frac{k+1}{2}}(\Gamma_i) \cup \overline{J}^{\frac{k+1}{2},\frac{k-1}{2}}(\Gamma_i) \cup \bigcup_{j\neq i} \overline{J}^{\frac{k+3}{2},\frac{k-3}{2}}(\Gamma_j). \quad (2)$$

The formulas for $\mathrm{Indet}(\psi)$ *in the case when $q = C'_i$ are obtained by replacing all the $\overline{J}^{m,n}$ by $\overline{J}^{n,m}$.*

Proof. (i) The map $f : \mathcal{M}^k \longrightarrow |H|$ can be defined as a map from the moduli functor of sheaves on S to the Hilbert functor of curves on S, using the 0-th Fitting ideal of a torsion sheaf, and it obviously commutes with base change and descends to the schemes $\mathcal{M}^k, \mathrm{Hilb}_S$ representing these functors.

Let $U \subset |H|$ be the complement of the 28 reducible curves, and $\varphi : \mathcal{C}_U \longrightarrow U$ the universal curve of the linear system $|H|$, restricted over U. Every fiber $C_t = \varphi^{-1}(t)$ for $t \in U$ is an integral curve, so the Altman-Kleiman relative compactified Jacobian $\overline{J}^k\varphi : \overline{J}^k(\mathcal{C}_U/U) \longrightarrow U$ is defined [AK], which is the relative moduli space parametrizing the isomorphism classes of degree-k torsion-free rank-1 sheaves on the fibers of φ ($d \in \mathbb{Z}$). It commutes with base change, so $(\overline{J}^k\varphi)^{-1}(t) = \overline{J}^k(C_t)$. Since the curves C_t lie on a smooth surface, they may have only planar singularities. Then by [AIK], $\overline{J}^k(\mathcal{C}_U/U), \overline{J}^k(C_t)$ are reduced and irreducible and are compactifications of the Picard schemes $\mathrm{Pic}^k(\mathcal{C}_U/U)$, resp. $\mathrm{Pic}^k(C_t)$. By the universal property of moduli spaces, there is a natural morphism $\overline{J}^k(\mathcal{C}_U/U) \longrightarrow \mathcal{M}^k$ which is bijective onto its image, equal to $f^{-1}(U)$. As $f^{-1}(U)$ is nonsingular, $\overline{J}^k(\mathcal{C}_U/U)$ is nonsingular (= smooth over $\mathbb{C}$) as well and the last map is an isomorphism identifying $f_U = f|_{f^{-1}(U)}$ with $\overline{J}^k\varphi$. By [Beau-3], f is Lagrangian for $k = 3$, the genus of the curves from $|H|$. As $\mathrm{Pic}^k(\mathcal{C}_U/U)$ is a torsor under $\mathrm{Pic}^0(\mathcal{C}_U/U)$ in the étale topology, f is Lagrangian for any k by [MarT], Lemma 5.7.

(ii) Let k be even; the case of odd k is completely similar. We are to show that the special fibers $f^{-1}(t_i)$, where $\{t_1, \dots, t_{28}\} = |H| \setminus U$, are unions of 3 components. Let Γ_i be represented by t_i, and look again at the exact triple (1), but now $\mathcal{F}, \mathcal{F}'$ are invertible on their supports with $\chi(\mathcal{F}) \leq 0 \leq \chi(\mathcal{F}')$. Let $C_i \cap C'_i = \{z_1, \dots, z_4\}$. In each point

z_k, the stalk of the sheaf $\mathcal{E}xt^{\,1}_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}')$ is a 1-dimensional vector space $\mathbb{C}_{z_k}$, so, locally at z_k, there are only two non-isomorphic extensions: $\mathcal{L}_{z_k}\simeq\mathcal{O}_{\Gamma_i,z_k}$ (the non-trivial extension) and $\mathcal{L}_{z_k}\simeq\mathcal{O}_{C_i,z_k}\oplus\mathcal{O}_{C_i',z_k}$ (the trivial one). We have

$$\mathrm{Ext}^1(\mathcal{F},\mathcal{F}')=H^0(\mathcal{E}xt^{\,1}_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}'))\simeq\bigoplus_{k=1}^{4}\mathbb{C}_{z_k},\tag{3}$$

so that every $\xi\in\mathrm{Ext}^1(\mathcal{F},\mathcal{F}')$ can be viewed as a vector in $\mathbb{C}^4$ with components ξ_{z_k}, and the extension with class ξ provides a sheaf $\mathcal{L}$ locally free at z_k as an $\mathcal{O}_{\Gamma_i}$-module if and only if $\xi_{z_k}\neq 0$.

Let s be the number of points z_k in which $\mathcal{L}$ is locally free as an $\mathcal{O}_{\Gamma_i}$-module. Then $\mathcal{F}\simeq\mathcal{L}_i(-s\cdot pt)$ and $\mathcal{F}'\simeq\mathcal{L}_i'$, where $\mathcal{L}_i=\gamma_i^{-1}\mathcal{L}\simeq\mathcal{L}|_{C_i}/(\text{torsion})$, $\mathcal{L}_i'=\gamma_i'^{-1}\mathcal{L}\simeq\mathcal{L}|_{C_i'}/(\text{torsion})$, and γ_i (resp. γ_i') is the natural inclusion of C_i (resp. C_i') into Γ_i. Thus (1) acquires the form

$$0\longrightarrow\mathcal{L}_i(-s\cdot pt)\longrightarrow\mathcal{L}\longrightarrow\mathcal{L}_i'\longrightarrow 0\tag{4}$$

Let $d=\deg\mathcal{L}_i$, $d'=\deg\mathcal{L}_i'$. We will call (d,d') the bidegree of $\mathcal{L}$. Then the semistability of $\mathcal{L}$ implies $d-s\leq d'$. Reversing the roles of C_i,C_i', we can represent the same sheaf as an extension

$$0\longrightarrow\mathcal{L}_i'(-s\cdot pt)\longrightarrow\mathcal{L}\longrightarrow\mathcal{L}_i\longrightarrow 0,\tag{5}$$

which implies $d'-s\leq d$. We have $\chi(\mathcal{L})=k-2=d+d'-s+2$ and $|d-d'|\leq s\leq 4$. Taking $s=4$, we obtain all the locally free extensions; the only possible bidegrees are given by $(d-\frac{k}{2},d'-\frac{k}{2})\in\{(-1,3),(0,2),(1,1),(2,0),(3,-1)\}$. The extremal cases $(-1,3),(3,-1)$ correspond to $\deg\mathcal{F}=\deg\mathcal{F}'=\frac{k-4}{2}$, so all such extensions represent one and the same S-equivalence class of the *trivial* extension, that is the direct sum $\mathcal{O}_{C_i}(\frac{k-4}{2}pt)\oplus\mathcal{O}_{C_i'}(\frac{k-4}{2}pt)$. For the remaining three bidegrees, the non-isomorphic *locally-free* extensions provide non-isomorphic stable sheaves. The locally free extensions are parametrized by the complements $J^{d,d'}\simeq(\mathbb{C}^*)^3$ to the coordinate hyperplanes in $\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{C_i'}(d'pt),\mathcal{O}_{C_i}((d-4)pt)))\simeq\mathbb{P}^3$, so $J^{d,d'}$ are mapped injectively into $\mathcal{M}^k$. The non-locally-free extensions deform in the corresponding Ext-groups to the locally free ones, so they lie in the closures $\overline{J}^{d,d'}$ of the images of $J^{d,d'}$.

(iii) Tensoring by $\mathcal{O}_S(q)$ for $q=C_i$ preserves the support and the property of being torsion-free rank-1 sheaf considered as a sheaf on its support. Thus it preserves the stability of all the sheaves from $\mathcal{M}^k$ supported on the integral curves. But it changes the distribution of degrees on the components of reducible ones. If we denote by $(\tilde{d},\tilde{d}')$

the bidegree of $\mathcal{L}(q)$ for $\mathcal{L}$ supported on Γ_j we have:

$$(\tilde{d},\tilde{d}') = \begin{cases} (d-2, d'+4) & \text{if } \ j = i \\ (d+2, d') & \text{if } \ j \neq i. \end{cases}$$

This immediately implies the formulas for the indeterminacy locus of ψ. $\square$

Remark 1.5. For odd $k = 2m+1$, $\mathcal{M}^{2m+1}$ is smooth and is birational to $\mathcal{M}^3$. In its turn, $\mathcal{M}^3$ is birational to the punctual Hilbert scheme $S^{[3]}$ (see [Beau-3]). Then, by [Hu-0], $\mathcal{M}^{2m+1}$ is deformation equivalent to $S^{[3]}$.

Definition 1.6. We will call $\mathcal{M}^k$ the degree-k relative compactified Picard variety of the linear system $|H|$ and denote it $\overline{\mathrm{Pic}}^{\,k}(|H|)$.

2. Local structure of $\overline{\mathrm{Pic}}^{\,2m}(|H|)$

We will use the approach of [O'G-2] to describe the local structure of the moduli space at a point representing a strictly semistable sheaf $\mathcal{F}$ as a quotient of the versal deformation of $\mathcal{F}$ by $\mathrm{Aut}(\mathcal{F})$.

Let us fix an integer m and consider the relative compactified Picard variety $\mathcal{M} = \overline{\mathrm{Pic}}^{\,2m}(|H|)$. First we will describe Simpson's construction for $\mathcal{M}$. Let $\mathcal{L} \in \mathcal{M}$ and $k \gg 0$ a sufficiently big integer. Then $\mathcal{L}(k)$ is generated by global sections, and denoting by V the vector space $H^0(\mathcal{L}(k))$, we will consider the Grothendieck Quot-scheme $\mathcal{Q}$ parametrizing all the quotients $V \otimes \mathcal{O}_X(-k) \twoheadrightarrow \mathcal{L}'$ such that $\chi(\mathcal{L}(n)) = \chi(\mathcal{L}'(n))$ for all $n \in \mathbb{Z}$. Let $\mathcal{Q}_c^{ss} \subset \mathcal{Q}$ be the open subscheme parametrizing the semistable pure 1-dimensional sheaves and $\mathcal{Q}_c$ the closure of $\mathcal{Q}_c^{ss}$ in $\mathcal{Q}$. There is a natural action of $G = GL(V)$ on $\mathcal{Q}$, $\mathcal{Q}_c$ and a G-linearized ample invertible sheaf L on $\mathcal{Q}$, such that $\mathcal{Q}_c^{ss}$ coincides with the set of L-semistable points of the action of G on $\mathcal{Q}_c$, and $\mathcal{M}$ is obtained as the Mumford quotient $\mathcal{Q}_c//G$.

Let $z \in \mathcal{Q}_c^{ss}$ be a point with closed orbit $G \cdot z$, $[z]$ the corresponding point in $\mathcal{M}$, $\mathcal{L}_z$ the quotient sheaf represented by z, and H the stabilizer of z; we have $H \simeq \mathrm{Aut}(\mathcal{L}_z)$. Luna's Slice Theorem ([Lu], [Sim]) affirms that there exists a H-invariant affine subscheme $W \subset \mathcal{Q}_c^{ss}$ passing through z such that the map $W//H \longrightarrow \mathcal{Q}_c//G$ of GIT quotients is étale. Such a W is called Luna's slice of the action of G. Let (W, z) be the germ of W at z and $\boldsymbol{\mathcal{L}}$ the restriction of the universal quotient sheaf on $\mathcal{Q} \times S$ to $(W, z) \times S$. By [O'G-2], Proposition (1.2.3), $((W, z), \boldsymbol{\mathcal{L}})$ is a versal deformation of $\mathcal{L}_z$.

There is a standard method for constructing a versal deformation of a sheaf which provides the following proposition:

Proposition 2.1. *Let X be a smooth projective variety, $\mathcal{F}_0$ a coherent sheaf on X. Then there exists a germ of a nonsingular algebraic variety $(M,0)$ together with a morphism $\Upsilon : (M,0) \longrightarrow (\mathrm{Ext}^2(\mathcal{F}_0,\mathcal{F}_0),0)$, called the obstruction map, such that the following properties are verified:*

(i) $(\Upsilon^{-1}(0),0)$ is the base of a versal deformation of $\mathcal{F}_0$, that is, there exists a coherent sheaf $\mathcal{F}$ on $(\Upsilon^{-1}(0),0)\times X$ such that $((\Upsilon^{-1}(0),0),\mathcal{F})$ is a versal deformation of $\mathcal{F}_0$. The Kodaira-Spencer map of this deformation provides a natural isomorphism KS : $T_0M \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{F}_0,\mathcal{F}_0)$.

(ii) Let

$$\Upsilon = \sum_{i=1}^{\infty} \Upsilon_i, \quad \Upsilon_i \in \mathrm{Hom}_{\mathbb{C}-\mathrm{lin}}(S_i(T_0M), \mathrm{Ext}^2(\mathcal{F}_0,\mathcal{F}_0))$$

be a Taylor expansion of Υ. Then $\Upsilon_1 = 0$ and Υ_2 is the composition

$$T_0M \xrightarrow{KS\times KS} \mathrm{Ext}^1(\mathcal{F}_0,\mathcal{F}_0) \times \mathrm{Ext}^1(\mathcal{F}_0,\mathcal{F}_0) \xrightarrow{(\xi,\xi)\mapsto\xi\cup\xi} \mathrm{Ext}^2(\mathcal{F}_0,\mathcal{F}_0)$$

where $\xi\cup\eta$ denotes the Yoneda product of two elements of $\mathrm{Ext}^1(\mathcal{F}_0,\mathcal{F}_0)$.

Proof. The Appendix of Bingener to [BH] provides a scheme of the proof of this statement. The existence of a formal versal deformation was proven in [Rim]. By [Art], the formal versal deformation is the formal completion of a genuine versal deformation. The identification of the obstruction Υ_2 on the formal level with the Yoneda pairing was done in [Ar], [Mu-2]. See also [HL], I.2.A.6 and historical comments, for the case when $\mathcal{F}_0$ is simple. For the construction of Υ_i for all i, see Proposition A.1 of [LS]. See also the paper [Lau], which provides a similar construction in the deformation theory of modules over a k-algebra and uncovers its relation to the Steenrod squares. $\square$

Lemma 2.2. *In the situation of Proposition 2.1, let us assume in addition that X is a K3 or abelian surface. Then the image of Υ_2 lies in the codimension-1 subspace $\mathrm{Ext}^2(\mathcal{F}_0,\mathcal{F}_0)_0$, defined as the kernel of the trace map $\mathrm{Ext}^2(\mathcal{F}_0,\mathcal{F}_0) \xrightarrow{\mathrm{Tr}} H^2(\mathcal{O}_X)$.*

Proof. The surjectivity of the trace map on a K3 or abelian surface is proved in [Ar], [Mu-2]. The fact that $\mathrm{Tr}\circ\Upsilon_2 = 0$ follows from [HL], Lemma 10.1.3. $\square$

Let now z be a point of $\mathcal{Q}_c^{ss}$ representing one of the 28 polystable sheaves $\mathcal{L}_z = \mathcal{O}_{C_i}((m-2)pt) \oplus \mathcal{O}_{C_i'}((m-2)pt)$. To shorten the formulas, we will denote $\mathcal{L}_z = \mathcal{L}$, $\mathcal{L}_1 = \mathcal{O}_{C_i}((m-2)pt)$, $\mathcal{L}_2 = \mathcal{O}_{C_i'}((m-2)pt)$, so that $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$. As in the previous section, denote by $z_1,\dots,z_4$ the intersection points of C_i and C_i'. The orbit of a polystable sheaf is closed in $\mathcal{Q}_c^{ss}$ (see [LP], 2.9), so the above local description of $\mathcal{M}$ at $[z]$ applies.

We have for $i = 1, 2$:

$$\mathcal{E}xt\,^1_{\mathcal{O}_S}(\mathcal{L}_i, \mathcal{L}_{2-i}) \simeq \bigoplus_{q=1}^{4} \mathbb{C}_{z_q},\ \mathcal{E}xt\,^k_{\mathcal{O}_S}(\mathcal{L}_i, \mathcal{L}_{2-i}) = 0 \text{ if } k \neq 1,$$

$$\mathcal{E}xt\,^0_{\mathcal{O}_S}(\mathcal{L}_i, \mathcal{L}_i) = \mathcal{O}_C,\ \mathcal{E}xt\,^1_{\mathcal{O}_S}(\mathcal{L}_i, \mathcal{L}_i) \simeq \mathcal{O}_C(-2pt),$$
$$\mathcal{E}xt\,^k_{\mathcal{O}_S}(\mathcal{L}_i, \mathcal{L}_i) = 0 \text{ if } k \notin \{0, 1\},$$

where $C = C_i$ for $i = 1$ and $C = C_i'$ for $i = 2$. Thus

$$T_zW \simeq \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) = \mathrm{Ext}^1(\mathcal{L}_1, \mathcal{L}_2) \oplus \mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1),\ \mathrm{Ext}^1(\mathcal{L}_i, \mathcal{L}_i) = 0,$$

$$\mathrm{Ext}^1(\mathcal{L}_i, \mathcal{L}_{2-i}) = H^0(\mathcal{E}xt\,^1_{\mathcal{O}_S}(\mathcal{L}_i, \mathcal{L}_{2-i})) \simeq \bigoplus_{q=1}^{4} \mathbb{C}_{z_q},$$

$$\mathrm{Ext}^2(\mathcal{L}, \mathcal{L}) = \bigoplus_{i=1,2} \mathrm{Ext}^2(\mathcal{L}_i, \mathcal{L}_i), \quad \mathrm{Ext}^2(\mathcal{L}_i, \mathcal{L}_{2-i}) = 0, \quad i = 1, 2.$$

By Serre duality ([Mu-2], Proposition 2.3), $\mathrm{Ext}^2(\mathcal{L}_i, \mathcal{L}_i) \xrightarrow{\mathrm{Tr}} H^2(\mathcal{O}_S)$ is an isomorphism, and

$$\mathrm{Ext}^1(\mathcal{L}_i, \mathcal{L}_j) \times \mathrm{Ext}^1(\mathcal{L}_j, \mathcal{L}_i) \xrightarrow{\text{Yoneda}} \mathrm{Ext}^2(\mathcal{L}_i, \mathcal{L}_i) \xrightarrow{\mathrm{Tr}} H^2(\mathcal{O}_S),$$

where $j = 2 - i$, is a nondegenerate pairing. Let us fix once and forever a generator of $H^2(\mathcal{O}_S)$, then denote by e_i its preimage in $\mathrm{Ext}^2(\mathcal{L}_i, \mathcal{L}_i)$. Denote $E_i = \mathrm{Ext}^1(\mathcal{L}_i, \mathcal{L}_{2-i})$, $E = \mathrm{Ext}^1(\mathcal{L}, \mathcal{L})$. Our choice of the e_i allows us to identify E_{2-i} with the dual of E_i in such a way that $E = E_1 \oplus E_2$ and Υ_2 is given by

$$\Upsilon_2 : E_1 \oplus E_2 \longrightarrow \mathbb{C}e_1 \oplus \mathbb{C}e_2, \quad (\xi_1, \xi_2) \mapsto \langle \xi_1, \xi_2 \rangle (e_1 - e_2) \qquad (6)$$

Thus we have proved:

Lemma 2.3. *The first obstruction map Υ_2 for the sheaf $\mathcal{L}$ is a nondegenerate quadratic form on the 8-dimensional vector space $E = \mathrm{Ext}^1(\mathcal{L}, \mathcal{L})$ with values in the 1-dimensional vector space $\mathrm{Ext}^2(\mathcal{L}, \mathcal{L})_0 = \mathbb{C}(e_1 - e_2)$, given by formula (6).*

This implies, in particular, that the base of the versal deformation $\Upsilon^{-1}(0)$ is at most 7-dimensional. Further, the stabilizer H of z is just the automorphism group $\mathrm{Aut}(\mathcal{L}) = \mathbb{C}^* \,\mathrm{id}_{\mathcal{L}_1} \times \mathbb{C}^* \,\mathrm{id}_{\mathcal{L}_2}$ acting on $\Upsilon^{-1}(0)$ via its quotient by the center, hence with 1-dimensional orbits. As $\dim \mathcal{M} = 6$, we conclude:

Corollary 2.4. $(\Upsilon^{-1}(0), 0)$ *is a nondegenerate* 7*-dimensional quadratic singularity with tangent cone* $\Upsilon_2^{-1}(0)$*. In particular,* $(\Upsilon^{-1}(0), 0)$ *and* $(\Upsilon_2^{-1}(0), 0)$ *are analytically equivalent.*

The linearized action of H is given by the following lemma.

Lemma 2.5. *In the above notation, let* $g \in H = \mathrm{Aut}(\mathcal{L})$ *and* $g : W \longrightarrow W$ *the map given by the group action of* H *on* W*. Let* $\xi \in T_z W \simeq \mathrm{Ext}^1(\mathcal{L}, \mathcal{L})$*. Then* $g_*(\xi) = g \cup \xi \cup g^{-1}$

Proof. See [O'G-2], (1.4.16), or [Dr], (7.4.1). □

Thus the linearized action on E of an element $(\lambda_1, \lambda_2) \in \mathrm{Aut}(\mathcal{L})$ is given by:

$$(\lambda_1, \lambda_2)_* : E_1 \oplus E_2 \longrightarrow E_1 \oplus E_2, \quad (\xi_1, \xi_2) \mapsto (\lambda_1^{-1}\lambda_2\xi_1, \lambda_1\lambda_2^{-1}\xi_2).$$

Passing to the quotient $\mathrm{P}H = \mathrm{PAut}(\mathcal{L})$ by the center, we have: $\mathrm{P}H \simeq \mathbb{C}^*$ via the map $(\lambda_1, \lambda_2) \mapsto \lambda = \lambda_1^{-1}\lambda_2$, and for $\lambda \in \mathrm{P}H$, λ_* acts with weight λ on E_1 and λ^{-1} on E_2. Let us introduce coordinates $x_1, \ldots, x_4$ on E_1 in such a way that the i-th coordinate axis is $H^0(\mathbb{C}_{z_i}) \subset E_1$. Let $y_1, \ldots, y_4$ be the dual coordinates on E_2. We obtain:

Corollary 2.6. *The linearized action of* $\mathrm{P}H$ *on* $\Upsilon_2^{-1}(0)$ *is identified with the action of* $\mathbb{C}^*$ *on the nondegenerate quadraric cone* $Q = \{x_1y_1 + \ldots + x_4y_4 = 0\}$ *in* $E \simeq \mathbb{C}^8$ *given by*

$$\lambda_* : (x_1, \ldots, x_4, y_1, \ldots, y_4) \mapsto (\lambda x_1, \ldots, \lambda x_4, \lambda^{-1}y_1, \ldots, \lambda^{-1}y_4), \quad \lambda \in \mathbb{C}^*. \quad (7)$$

Now we will use the birational modification $\pi : \tilde{\mathcal{M}} \longrightarrow \mathcal{M}$ of $\mathcal{M}$ constructed by the method of Kirwan [Kir]. Given a GIT quotient $Z//G$, Kirwan constructs its partial desingularization in blowing up successively closed semistable orbits of G until the stability of a point under the action of G on the blown up variety $\tilde{Z}$ becomes equivalent to the semistability: $\tilde{Z}^{ss} = \tilde{Z}^s$. Equivalently, one may require that, the projectivized stabilizer of any semistable point of $\tilde{Z}$ is finite. Then Kirwan's modification of $Z//G$ is the induced birational morphism $\pi : \tilde{Z}//G \to Z//G$.

In our case, we consider just one blowup $\sigma : \tilde{\mathcal{Q}}_c \longrightarrow \mathcal{Q}_c$ with center at the union of all the closed orbits in the strictly semistable locus of $\mathcal{Q}_c$. The induced map of GIT quotients $\pi : \tilde{\mathcal{M}} = \tilde{\mathcal{Q}}_c//G \longrightarrow \mathcal{M} = \mathcal{Q}_c//G$ is a morphism by [Kir], 3.1, 3.2.

Proposition 2.7. *In the above notation, the following assertions hold:*

(i) $\pi : \tilde{\mathcal{M}} \longrightarrow \mathcal{M}$ *is Kirwan's modification of* $\mathcal{M}$*.*

(ii) $\tilde{\mathcal{M}}$ *is nonsingular and projective, thus* π *is a resolution of singularities of* $\mathcal{M}$*. The construction of* $\tilde{\mathcal{M}}$ *consists in blowing up the* 28 *singular points* $\zeta_1, \dots, \zeta_{28}$ *of* $\mathcal{M}$ *taken with their reduced structure.*

(iii) The exceptional divisors $I_i = \pi^{-1}(\zeta_i)$ $(i = 1, \dots, 28)$ *can be identified with the flag variety* $\mathrm{Fl}(0, 2; \mathbb{P}^3)$*.*

(iv) The normal bundle $\mathcal{N}_{I_i/\tilde{\mathcal{M}}}$ *is isomorphic to* $\mathcal{O}_{I_i}(-1)$*, the restriction of* $\mathcal{O}_{\mathbb{P}^{14}}(-1)$ *to the flag variety* $\mathrm{Fl}(0, 2; \mathbb{P}^3)$ *in its standard embedding into* $\mathbb{P}^{14}$*.*

Proof. (i) We have to show that one blowup suffices to get a complete Kirwan's modification. For a strictly semistable point $z \in \mathcal{Q}_c$, σ induces on the étale slice W at z the blowup $\sigma_z : \tilde{W} \longrightarrow W$ with center z, and $\tilde{W}//H$ is an étale neighborhood of the exceptional fiber $\pi^{-1}(\zeta)$ in $\tilde{\mathcal{Q}}_c//G$, where H is the stabilizer of z and $\zeta = [z]$ is the image of z in $\mathcal{M}$. By Corollary 2.6, $F = \sigma_z^{-1}(z)$ is isomorphic to the projectivized quadratic cone $\mathbb{P}Q$ with $\mathrm{P}H$ acting by formula (7). The two projective 3-spaces $\mathbb{P}E_1$, $\mathbb{P}E_2$ contained in $\mathbb{P}Q$ consist of unstable points, and the stabilizer of any point of $\mathbb{P}Q \smallsetminus (\mathbb{P}E_1 \cup \mathbb{P}E_2)$ in $\mathrm{P}H$ is $\{\pm 1\}$, so $\mathbb{P}Q^{ss} = \mathbb{P}Q^s = \mathbb{P}Q \smallsetminus (\mathbb{P}E_1 \cup \mathbb{P}E_2)$. As all the semistable points of $\mathbb{P}Q$ are stable, π is Kirwan's modification at ζ. Remark also that the strictly semistable points of $\mathcal{Q}_c$ (or W) with *non-closed* orbits become unstable when lifted to $\tilde{\mathcal{Q}}_c$ (resp.$\tilde{W}$).

(ii) The blowup $\tilde{W}$ at z is nonsingular over z since, by Corollary 2.4, (W, z) is a nondegenerate quadratic singularity. As the stabilizer in $\mathrm{P}H$ of all the semistable points of $\sigma^{-1}(z)$ is constant, equal to $\{\pm 1\}$, the quotient $\tilde{W}//H$ is nonsingular at every point of $F = \pi^{-1}(z)$ by Luna's slice theorem. By Lemma 3.11 of [Kir], π is the blowup of the reduced point $\zeta = [z]$.

(iii) The exceptional fiber $I = \pi^{-1}(\zeta)$ is isomorphic to the quotient $\mathbb{P}Q//\mathbb{C}^*$ by the action (7). The algebra of invariants of this action is generated by the quadratic monomials $u_{ij} = x_i y_j$, and the generating relations are of two types: one linear, $u_{11} + \ldots + u_{44} = 0$, and the quadratic ones $u_{ij}u_{kl} = u_{kj}u_{il}$. The quadratic relations define the standard Segre embedding of $\mathbb{P}^3 \times \mathbb{P}^3$ in $\mathbb{P}^{15}$, and the linear one cuts out the incidence variety: if we identify the second factor $\mathbb{P}^3$ with $\mathbb{P}^{3\vee}$, parametrizing the hyperplanes h in the first factor $\mathbb{P}^3$, then $I = \{(p, h) \in \mathbb{P}^3 \times \mathbb{P}^{3\vee} \mid p \in h\}$. This is just the flag variety $\mathrm{Fl}(0, 2; \mathbb{P}^3)$.

(iv) Let A denote the algebra of regular functions on W, so that $W = \operatorname{Spec} A$. Let $\mathfrak{M} = \mathfrak{M}_z \subset A$ be the maximal ideal of $z \in W$. As any representation of $\mathrm{P}H$ is completely reducible, $\mathfrak{M}$ contains a subrepresentation V of $\mathrm{P}H$ that projects down isomorphically and equivariantly onto $\mathfrak{M}/\mathfrak{M}^2 = T_z^*W$ via the differential $d : \mathfrak{M} \to \mathfrak{M}/\mathfrak{M}^2$. The map $d|_V$ extends to a $\mathrm{P}H$-equivariant epimorphism of $\mathbb{C}$-algebras $S_\bullet V \to A$ giving rise to a $\mathrm{P}H$-equivariant morphism $W \to T_zW$. Its image is a hypersurface W_1, and it is étale at z if considered as a morphism $W \to W_1$. In shrinking W, we can assume that this morphism is everywhere étale.

We can choose functions $x_i, y_i \in V \simeq T_z^*W$ on which $\mathrm{P}H$ acts according to formula (7). Then W_1 is defined by the equation $F = 0$, where $F \in \mathbb{C}[x_i, y_i]$ is the sum of homogeneous components of even degree, $F = F_2 + \ldots + F_{2r}$, which are $\mathrm{P}H$-invariant and such that $F_2 = x_1y_1 + \ldots + x_4y_4$. We can write $W_1 = \operatorname{Spec} A_1$, where $A_1 = \mathbb{C}[x_i, y_i]/(F)$. Let $U = W//\mathrm{P}H$, $U_1 = W_1//\mathrm{P}H$. We have $U = \operatorname{Spec} B$, $U_1 = \operatorname{Spec} B_1$, where $B = A^{\mathrm{P}H}$, $B_1 = A_1^{\mathrm{P}H}$, and the étale morphism $W \to W_1$ descends to the quotients as an étale morphism $U \to U_1$. Let $\zeta = [z]$ denote the image of z in U or in U_1, and $\mathfrak{m} = \mathfrak{m}_\zeta$ the maximal ideal of ζ in either one of the rings B, B_1. The blowup $\tilde{U}$ of U at ζ can be given by $\tilde{U} = \operatorname{Proj}_B(\bigoplus_{k\geq 0} \mathfrak{m}^k)$, and the exceptional divisor $I = \operatorname{Proj}_{\mathbb{C}}(\bigoplus_{k\geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1})$, its normal bundle being $\mathcal{O}_I(-1)$, the dual of the Grothendieck tautological sheaf $\mathcal{O}_I(1)$ on the latter Proj. As U and U_1 are locally isomorphic at z in the étale topology, I and its normal bundle do not depend on whether $\mathfrak{m}$ is considered in B or in B_1. So the wanted normal bundle $\mathcal{N}_{I_i/\tilde{\mathcal{M}}}$ can be computed as the normal bundle to the blowup of ζ in U_1.

Choosing $u_{ij} = x_iy_j$ as the generators of $B_1 = A_1^{\mathrm{P}H}$, we represent U_1 as a hypersurface in the cone $\mathfrak{C} = \{(u_{ij}) \in \mathbb{C}^{16} \mid u_{ij}u_{kl} = u_{kj}u_{il},\ \ 1 \leq i,j,k,l \leq 4\}$, defined by the equation $f = 0$, where f has a decomposition into homogeneous components of the form $f = f_1 + f_2 + \ldots + f_r$, $f_1 = u_{11} + \ldots + u_{44}$. The proper transform $\tilde{U}_1$ of the hypersurface $f = 0$ in the blowup $\tilde{\mathfrak{C}}$ of $\mathfrak{C}$ at ζ meets the exceptional divisor $E \simeq \mathbb{P}^3 \times \mathbb{P}^3$ transversely along the flag variety $I \subset E$. Hence $\mathcal{N}_{I/\tilde{U}_1} \simeq \left(\mathcal{N}_{E/\tilde{\mathfrak{C}}}\right)|_I$. But the latter normal bundle is just the restriction of $\mathcal{O}_{\mathbb{P}^{15}}(-1)$, and we are done. □

Remark 2.8. Our argument in part (iv) is a kind of "equivariant deformation to the normal cone", compare to Sect. 5 of [LS].

The exceptional divisor I_j over any of the points ζ_j has two distinct projections to $\mathbb{P}^3$ which are $\mathbb{P}^2$-bundles, and which we will refer to as *rulings* of I_j. By Proposition 2.7 (iv), the normal bundle to I_j restricts as $\mathcal{O}(-1)$ to the fibers $\mathbb{P}^2$ of each ruling. By Moishezon's contractibility criterion [Mo], both projections of I_j to $\mathbb{P}^3$ can be extended to a morphism $f : \tilde{\mathcal{M}} \longrightarrow Y$ such that Y is a smooth compact complex 6-dimensional manifold, not necessarily projective. Applying this argument successively to different $j = 1, \dots, 28$, we obtain:

Corollary 2.9. *There are 2^{28} distinct bimeromorphic morphisms $f : \tilde{\mathcal{M}} \longrightarrow Y$ onto smooth, compact, complex, not necessarily projective 6-dimensional manifolds Y which contract each one of the divisors I_j onto a projective 3-space $f(I_j) \simeq \mathbb{P}^3$. For any of these morphisms f, Kirwan's desingularization $\pi : \tilde{\mathcal{M}} \longrightarrow \mathcal{M}$ factors through f, $\pi = g \circ f$, so that g is a small contraction, that is, a contraction without exceptional divisors. Moreover, the symplectic form α on the nonsingular locus $\mathcal{M}^*$ of $\mathcal{M}$ lifts to a global symplectic form α_Y on Y, and hence Y is a holomorphically symplectic manifold.*

Proof. The small contraction map g induces an isomorphism $g : g^{-1}(\mathcal{M}^*) \xrightarrow{\sim} \mathcal{M}^*$, so $g^*(\alpha)$ is a symplectic form on $g^{-1}(\mathcal{M}^*)$. It extends to a regular 2-form α_Y on all of Y by Riemann–Hartogs extension theorem since the complement $Y \setminus g^{-1}(\mathcal{M}^*)$ is a union of $\mathbb{P}^3$'s and thus is of codimension > 1. Finally, α_Y is nondegenerate, and hence is a symplectic form. Indeed, the degeneracy locus of α_Y is nothing else but the zero locus of $\alpha_Y^{\wedge 3} \in H^0(Y, \Omega_Y^6)$. The zero locus of a section of an invertible sheaf, if nonempty, is either Y itself, or a divisor in Y, but we know that α_Y is nondegenerate on an open set whose complement contains no divisors, so α_Y is everywhere nondegenerate. □

In fact, there are projective varieties among the complex manifolds Y from Corollary 2.9. One of them is given by the next proposition.

Proposition 2.10. *Let $H_\epsilon = H + \epsilon \sum_{i=1}^{28} (C_i - C_i')$. Then there exists a sufficiently small $\epsilon_0 > 0$ such that for any $\epsilon \in \mathbb{Q} \cap]0, \epsilon_0[$, the following assertions hold:*

(i) H_ϵ is an ample $\mathbb{Q}$-divisor on S.

(ii) The (semi-) stability of a sheaf with Mukai vector $v = (0, H, 2m-2)$ with respect to H_ϵ does not depend on ϵ, and every H_ϵ-semistable sheaf with Mukai vector v is stable.

(iii) The moduli space $Y = M_S^{H_\epsilon, ss}(v) = M_S^{H_\epsilon, s}(v)$ is an irreducible symplectic manifold which does not depend on ϵ.

(iv) The natural map $g : Y \to \mathcal{M}$ is a small resolution of singularities such that $g^{-1}(\zeta_j) \simeq \mathbb{P}^3$ $(j = 1, \dots, 28)$.

Proof. (i) follows from the openness of the cone of ample classes in $\operatorname{Pic} S \otimes \mathbb{R}$. For (ii), remind that the (semi)-stability of a sheaf supported on an integral curve does not depend on polarization. So we have only to examine the sheaves supported on the reducible curves Γ_i. This is similar to the proof of Proposition 1.4, (iii): any H_ϵ-semistable sheaf which is rank-1 and torsion-free as a sheaf on its support $\Gamma_i = C + C'$ is given by extensions (4) and (5) such that

$$(1-3\epsilon)(d-s) \leq (1+3\epsilon)d', \quad (1+3\epsilon)(d'-s) \leq (1-3\epsilon)d, \\ s = 0, \ldots, 4, \quad d + d' = 2m.$$

For all sufficiently small $\epsilon > 0$, the solutions of these inequalities are the same triples of integers d, d', s as in the proof of Proposition 1.4, (iii), except for $d = d' = m$, $s = 0$ which does not satisfy the second inequality. For all the solutions, the inequalities are strict, hence there are no strictly semistable sheaves. This ends the proof of (ii). The assertion (iii) follows by [HL], 6.2.5. To prove (iv), remark, that by the above argument, any H_ϵ-semistable sheaf is also H-semistable, so there is a natural morphism $g : Y \to \mathcal{M}$. Further, all the nontrivial extensions

$$0 \longrightarrow \mathcal{O}_C((m-2)pt) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{C'}((m-2)pt) \longrightarrow 0$$

provide H_ϵ-stable sheaves with the same image $[\mathcal{O}_C((m-2)pt) \oplus \mathcal{O}_{C'}((m-2)pt)] = \zeta_i$ in $\mathcal{M}$, and two such sheaves are isomorphic if and only if they correspond to proportional extension classes. Thus $g^{-1}(\zeta_i) = \mathbb{P}\operatorname{Ext}^1(\mathcal{O}_{C'}((m-2)pt), \mathcal{O}_C((m-2)pt)) \simeq \mathbb{P}^3$. $\square$

3. The relative compactified Prymian $\overline{\mathrm{Prym}}^k(\mathcal{C}, \tau)$

The Galois involution τ of the double covering $\rho : S \to X$ is H-linear and induces an involution on $|H| \simeq \mathbb{P}^3$, whose fixed locus consists of two components: a point and a plane. The plane parametrizes the curves of the form $\rho^{-1}\mu^{-1}(\ell)$, where ℓ runs over the lines in $\mathbb{P}^2$, thus this plane is identified with the dual of the $\mathbb{P}^2$ which is the image of μ. We will denote it $\mathbb{P}^{2\vee}$. The other component of the fixed locus, a point, corresponds to the ramification curve $\Delta \in |H|$. Let $\varphi : \mathcal{C} \to \mathbb{P}^{2\vee}$ be the linear subsystem of τ-invariant curves parametrized by $\mathbb{P}^{2\vee}$. A generic $t \in \mathbb{P}^{2\vee}$ represents a line $\ell = \ell_t$ which is not tangent to B_0, neither to $\overline{\Delta}_0 := \mu(\Delta_0)$. The corresponding curve $C_t = \varphi^{-1}(t) = \rho^{-1}\mu^{-1}(\ell_t)$ is a smooth genus-3 curve, and $E_t = C_t/\tau$ is elliptic; the double cover $\rho_t = \rho|_{C_t} : C_t \to E_t$ is branched at 4 points of the intersection $\Delta_0 \cap E_t$.

Definition 3.1. Let $\eta : C \to D$ be a double covering map of integral projective curves and τ the Galois involution of η. Then τ acts as a linear involution on the generalized Jacobian $J(C)$, and the Prym variety $\mathrm{Prym}(C, \tau)$ is defined as $\mathrm{im}(\mathrm{id} - \tau) = [\ker(\mathrm{id} + \tau)]^\circ$, where G°

denotes the connected component of the neutral element in a subgroup G of $J(C)$.

If C is smooth, then $J(C)$ and $\mathrm{Prym}(C,\tau)$ are abelian varieties, but for singular curves, they can be extensions of abelian varieties by a number of copies of $\mathbb{C}^*$ or $\mathbb{C}$.

Lemma 3.2. *Let C be a smooth genus-3 curve with an involution τ such that $D = C/\tau$ is an elliptic curve. Then $\ker(\mathrm{id}+\tau)$ has only one connected component in $J(C)$, and the restriction of the principal polarization from $J(C)$ to $\mathrm{Prym}(C,\tau) = \ker(\mathrm{id}+\tau)$ is a polarization of type $(1,2)$.*

Proof. It is well-known that under the hypotheses of the lemma, $P = \mathrm{Prym}(C,\tau)$ has a polarization of type $(1,2)$, see [B]. A very explicit proof of the fact that this polarization is the restriction of the standard principal polarization of the Jacobian $J = J(C)$ is given in the paper [P], in which the author identifies the intersection $\Theta_a \cap P$, where $\Theta_a = a + \Theta$ is an appropriate translate of the theta-divisor $\Theta \subset J$, as a genus-3 curve $C^\vee$ obtained by a bigonal construction from C.

Let $\eta : C \to D$ be the natural double covering map. As C is not hyperelliptic, $\eta^* : J(D) \to J$ is injective. Let $E = \eta^*(JD) \subset J$. Then $E + P = J$, and $K = E \cap P \subset J_{(2)}$, where $J_{(n)}$ denotes the n-torsion subgroup of J. It is obvious that $\ker(\mathrm{id}+\tau) = \bigcup_{z\in E_{(2)}}(z + P)$, so $\ker(\mathrm{id}+\tau)$ is connected if and only if $K = E_{(2)}$. By [BM], 7.6 and 7.10, $K = \ker\lambda_{\Xi^1} = \ker\lambda_{\Xi^2}$, where $\Xi^1 = \Theta|_E$, $\Xi^2 = \Theta|_P$, and λ_Ξ denotes the polarization map associated to an ample divisor Ξ on an abelian variety A. It is defined by the formula

$$\lambda_\Xi : A \longrightarrow \hat{A} = \mathrm{Pic}^0(A), \quad a \mapsto \mathrm{Cl}[\Xi_a - \Xi].$$

Since we already know that Ξ^2 is a polarization of type $(1,2)$, we have $\#(\ker\lambda_{\Xi^2}) = 4$, hence $K = E_{(2)}$ and we are done. $\square$

The lemma allows us to define $\mathrm{Prym}(C,\tau)$ as the fixed locus of the involution $\kappa = \tau \circ \iota$, where $\iota : J(C) \to J(C)$ is defined by $[\mathcal{L}] \mapsto [\mathcal{L}^{-1}]$. Now we will relativize the construction of κ in the linear system $|H|$.

Let $\mathcal{M}^k = \overline{\mathrm{Pic}}^{\,k}(|H|)$ be as in the previous sections. First, let $k = 2m$ be even. The naive extension of ι to the sheaves that are not invertible on $C = \mathrm{Supp}\,\mathcal{L}$ is $[\mathcal{L}] \mapsto [\mathcal{H}om_{\mathcal{O}_C}(\mathcal{L}, \mathcal{O}_C(mH))]$. But this does not commute with base change. The proper definition is

$$\iota : \mathcal{M}^{2m} \longrightarrow \mathcal{M}^{2m}, \quad [\mathcal{L}] \mapsto [\mathcal{E}xt^1_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_S((m-1)H))].$$

This duality functor for pure 1-dimensional sheaves was used by Maruyama in [Maru], Proposition 2.9, over $\mathbb{P}^2$, but it can be applied on any smooth surface. The fixed locus $\mathrm{Fix}(\kappa)$ of $\kappa = \tau \circ \iota$ has one connected component of dimension 4, parametrizing sheaves with supports C_t, $t \in \mathbb{P}^{2\vee}$, and $2^6 = 64$ isolated points representing the invertible sheaves $\mathcal{L}$ on Δ such that $\mathcal{L}^2 \simeq \mathcal{O}_S(mH)|_\Delta$.

To define ι for $k = 2m+1$, we need to fix a class $c \in \mathrm{Pic}(S)$ of degree 2, that is such that $(c \cdot H) = 2$. Then we define

$$\iota = \iota_c : \mathcal{M}^{2m+1} \dashrightarrow \mathcal{M}^{2m+1}, \quad [\mathcal{L}] \mapsto [\mathcal{E}xt^{\,1}_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_S((m-1)H - c))].$$

This is a rational involution whose indeterminacy locus consists of H-stable sheaves $\mathcal{L} \in \mathcal{M}^{2m+1}$ such that $\mathcal{L} \otimes \mathcal{O}_S(-c)$ is unstable. One can choose for c one of the 56 conics C_i, C_i'. For example, if $c = C_i'$, then the indeteminacy locus of ι coincides with $\mathrm{Indet}\,\psi$ as given by formula (2). Thus $\kappa = \tau \circ \iota$ is a rational involution in this case and we define $\mathrm{Fix}(\kappa)$ as the closure of the fixed point set of the restriction of κ to its regular locus. It also has one 4-dimensional and 64 zero-dimensional components.

Definition 3.3. The relative compactified Prymian $\overline{\mathrm{Prym}}^{\,k,\kappa}(\mathcal{C}, \tau)$, or simply $\overline{\mathrm{Prym}}^{\,k}(\mathcal{C}, \tau)$, is the 4-dimensional component of $\mathrm{Fix}(\kappa)$ in $\mathcal{M}^k$.

We will study in more detail the variety $\overline{\mathrm{Prym}}^{\,k}(\mathcal{C}, \tau)$ for even $k = 2m$, which we will denote by $\boldsymbol{\mathcal{P}}^{2m}$, or simply $\boldsymbol{\mathcal{P}}$ when there is no risk of ambiguity. Remark that $\boldsymbol{\mathcal{P}}^{2m} \simeq \boldsymbol{\mathcal{P}}^{2m+4}$ via the map $\mathcal{F} \mapsto \mathcal{F}(H)$, so that there are at most two different varieties $\boldsymbol{\mathcal{P}}^{2m}$: $\boldsymbol{\mathcal{P}}^0$ and $\boldsymbol{\mathcal{P}}^2$. We ignore if they are really non-isomorphic, or even non-birational.

Theorem 3.4. *Let $\boldsymbol{\mathcal{P}} = \overline{\mathrm{Prym}}^{\,2m}(\mathcal{C}, \tau)$ with $m \in \mathbb{Z}$. Identifying, as above, the 2-dimensional linear subsystem of τ-invariant curves in $|H|$ with $\mathbb{P}^{2\vee}$, let $f_{\boldsymbol{\mathcal{P}}} = f^{2m}_{\boldsymbol{\mathcal{P}}}$ denote the map $\boldsymbol{\mathcal{P}} \to \mathbb{P}^{2\vee}$ sending each sheaf to its support. Let $C_t = \varphi^{-1}(t)$, $E_t = C_t/\tau$, and $\rho_t = \rho|_{C_t} : C_t \to E_t$, where $\varphi : \mathcal{C} \to \mathbb{P}^{2\vee}$ is the natural map and $t \in \mathbb{P}^{2\vee}$.*

Then the following assertions hold:

(i) $\boldsymbol{\mathcal{P}}$ is nonsingular out of the 28 points $\zeta_i = [\mathcal{L}_i]$ representing the S-equivalence classes of the sheaves $\mathcal{L}_i = \mathcal{O}_{C_i}((m-2)pt) \oplus \mathcal{O}_{C_i'}((m-2)pt)$, $i = 1, \ldots, 28$. The singularities $(\boldsymbol{\mathcal{P}}, \zeta_i)$ are analytically equivalent to $(\mathbb{C}^4/\{\pm 1\}, 0)$.

(ii) $\boldsymbol{\mathcal{P}}$ is a symplectic V-manifold, and $f_{\boldsymbol{\mathcal{P}}}$ is a Lagrangian fibration on it. The generic fiber $f^{-1}_{\boldsymbol{\mathcal{P}}}(t)$ is the $(1,2)$-polarized Prym surface $\mathrm{Prym}(C_t, \tau)$ of the double covering $\rho_t : C_t \to E_t$.

Proof. (i) It is obvious that the fixed point set of any biregular involution on a smooth variety is also smooth. The sheaves $\mathcal{L}_i$ are invariant under τ and ι, hence also under κ. So $\zeta_i \in \boldsymbol{\mathcal{P}}$, and we only have to determine the analytic type of the singularity at ζ_i. To this end, we will write out the action of κ on the tangent cone of $\mathcal{M}^{2m}$ at ζ_i.

Let us change, for convenience, the notation, so that $C_+ = C_i$, $C_- = C_i'$, $C = \Gamma_i = C_+ \cup C_-$, $\mathcal{L}_\pm = \mathcal{O}_{C_\pm}((m-2)pt)$, $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$, $\zeta = \zeta_i = [\mathcal{L}]$. As τ leaves invariant both curves $C_\pm$, it has two fixed points on each of them, which we will denote by $\lambda_{1\pm}, \lambda_{2\pm}$. We can choose homogeneous coordinates $(x_{0\pm}, x_{1\pm})$ on $C_\pm \simeq \mathbb{P}^1$ in such a way that $\lambda_{1\pm} = (0:1)$, $\lambda_{2\pm} = (1:0)$, and τ is given by $\tau : (x_{0\pm}, x_{1\pm}) \mapsto (x_{0\pm}, -x_{1\pm})$. As the cross-ratio of 4 points of intersection of two conics is the same on both of them, we can adjust the choice of the above coordinates in such a way that the 4 points $z_1, \dots, z_4$ of $C_+ \cap C_-$ have equal coordinates on both curves, and we will number them in such a way that they are permuted by τ in pairs $z_1 \leftrightarrow z_2$, $z_3 \leftrightarrow z_4$.

The 4-dimensional vector space $F = \mathrm{Ext}^1(\mathcal{L}_+, \mathcal{L}_-)$ parametrizes the extensions $0 \to \mathcal{L}_- \to \mathcal{F} \to \mathcal{L}_+ \to 0$. Let x_i, resp. y_i be the coordinates on F, resp. $F^\vee = \mathrm{Ext}^1(\mathcal{L}_-, \mathcal{L}_+)$ obtained in the same way as those used in Corollary 2.6. The choice of x_i made in Section 2 is not unique, it depends on the choice of a basis in each of the 1-dimensional stalks $\mathbb{C}_{z_i}$ of the sheaf $\mathcal{E}xt^{\,1}(\mathcal{L}_+, \mathcal{L}_-)$. Now we will make this choice more precise. Let s be the number of the points z_i in which $\mathcal{F}$ is locally free. Then $\mathcal{F}$ is the result of gluing of the sheaves $\mathcal{L}_-(s \cdot pt)$ and $\mathcal{L}_+$. The gluing consists in the identification of the fibers at z_i via isomorphisms $\varphi_i : \mathcal{L}_{+,z_i} = \mathcal{L}_+ \otimes \mathbb{C}_{z_i} \xrightarrow{\sim} \mathcal{L}_-(s \cdot pt)_{,z_i} = \mathcal{L}_-(s \cdot pt) \otimes \mathbb{C}_{z_i}$ for those i, for which the $\mathbb{C}_{z_i}$-component of the extension class is non-zero. Let us denote the resulting sheaf $\mathcal{F}$ by $\mathcal{L}_-(s \cdot pt) \#_{(\varphi_i)} \mathcal{L}_+$.

Consider the case $s = 4$. Let us fix some isomorphisms $\mathcal{L}_-(4 \cdot pt) \simeq \mathcal{O}_{\mathbb{P}^1}(m+2)$ and $\mathcal{L}_+ \simeq \mathcal{O}_{\mathbb{P}^1}(m-2)$. Now, fix $e_+ = x_{0+}^{m-2}$, resp. $e_- = x_{0-}^{m+2}$ as a trivializing section of $\mathcal{L}_+$, resp. $\mathcal{L}_-(4 \cdot pt)$ over an open set containing all the points z_i. Define the four isomorphisms φ_i as above by $e_{+,z_i} \mapsto e_{-,z_i}$. Finally, we fix the choice of (x_i) by the condition that (x_i) are the coordinates of the extension class of the sheaf

$$\mathcal{F}_{(x_1,x_2,x_3,x_4)} = \mathcal{L}_-(s \cdot pt) \#_{(x_i \varphi_i)} \mathcal{L}_+$$

whenever $x_i \neq 0$ for all $i = 1, \dots, 4$. This determines also the coordinates y_i, dual to x_i.

The action of τ lifts to $\mathcal{L}_\pm$ in such a way that it preserves e_-, e_+. Further, τ interchanges z_1 with z_2, z_3 with z_4, hence $\tau^*(\mathcal{F}_{(x_1,x_2,x_3,x_4)}) \simeq \mathcal{F}_{(x_2,x_1,x_4,x_3)}$. From here we deduce the action of τ on $E = \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) = F \oplus F^\vee$:

$$\tau : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \mapsto (x_2, x_1, x_4, x_3, y_2, y_1, y_4, y_3).$$

As ι interchanges x_i with y_i, we obtain

$$\kappa : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \mapsto (y_2, y_1, y_4, y_3, x_2, x_1, x_4, x_3).$$

The tangent cone to $\mathcal{M}^{2m}$ is obtained by taking the quotient of the quadric $\sum x_i y_i = 0$ by $\mathbb{C}^*$ (see the proof of Proposition 2.7, (iii)). The quotient is identified with the cone over the hyperplane section $\sum u_{ii} = 0$ of the Segre variety, given by the parametrization $u_{ij} = x_i y_j$ in $\mathbb{P}^{15}$. As we have already noticed, this hyperplane section is the flag variety $\mathrm{Fl}(0, 2; \mathbb{P}^3)$ embedded in $\mathbb{P}^{14}$. Restricting further to the fixed locus of κ is equivalent to intersecting with 6 hyperplanes

$$u_{11} = u_{22},\ u_{33} = u_{44},\ u_{13} = u_{42},\ u_{14} = u_{32},\ u_{23} = u_{41},\ u_{24} = u_{31}.$$

These equations cut out the Veronese image of $\mathbb{P}^3$ in $\mathbb{P}^9$. Thus the tangent cone to $\mathcal{M}^{2m}$ is the cone over the the Veronese image of $\mathbb{P}^3$, or in other words, the quotient $\mathbb{C}^4/\pm 1$. This ends the proof of (i).

The assertions of (ii) follow from (i), Lemma 3.2 and [Mar-1], Section 6. $\square$

We can use some of the settings of the above proof to determine the fiber of $f_{\boldsymbol{\mathcal{P}}}$ over a point $t \in \mathbb{P}^{2\vee}$ representing a reducible quartic C_t.

Lemma 3.5. *Let, in the notation of Theorem 3.4, $t \in \mathbb{P}^{2\vee}$ be a point representing a reducible quartic $C = C_+ \cup C_-$, and $P^{2m} = (f^{2m}_{\boldsymbol{\mathcal{P}}})^{-1}(t)$. Then $P = P^{2m}$ is an irreducible projective surface having a stratification $P = P_0 \sqcup P_1 \sqcup P_2$ such that $P_0 \simeq \mathbb{C}^* \times \mathbb{C}^*$, $P_1 \simeq \mathbb{C}^* \sqcup \mathbb{C}^*$ and P_2 is a single point. The isomorphism class of P does not depend on m.*

Proof. We choose the coordinates $(x_{0\pm}, x_{1\pm})$ on $C_\pm$ as in the proof of Theorem 3.4. Let$z_\pm = x_{1\pm}/x_{0\pm}$ denote the associated affine coordinate on $C_\pm \setminus \{\lambda_{1\pm}\}$. Then the 4 points of $C_+ \cap C_-$ have the same values of the coordinates $z_\pm$: $z_{i+} = z_{i-} = z_i$ for $i = 1, \ldots, 4$. Moreover, $z_2 = -z_1$, $z_4 = -z_3$, for the involution τ acts by $z_+ \mapsto -z_+$, $z_- \mapsto -z_-$.

Consider first the case $m = 0$. Any invertible sheaf of degree 0 on C can be represented as the result of gluing $\mathcal{O}_{C_+}(a\,pt)$ with $\mathcal{O}_{C_-}(-a\,pt)$ at the 4 points of $C_+ \cap C_-$. Let us fix the convention that the sheaf $\mathcal{O}_{C_\pm}(n\,pt)$ is trivialized by the rational section $e_\pm = x^n_{0\pm}$. Denote the result of the above gluing via the maps $e_{+,z_i} \mapsto \lambda_i e_{-,z_i}$ as $\mathcal{F}(a; \lambda_1, \ldots, \lambda_4)$ or $\mathcal{O}_{C_-}(-a\,pt) \underset{(\lambda_1,\ldots,\lambda_4)}{\#} \mathcal{O}_{C_+}(a\,pt)$. We have $\mathcal{F}(a; (\lambda_i)) \simeq \mathcal{F}(a'; (\lambda'_i))$ if and only if $a' = a$ and $\lambda'_i = c\lambda_i$ $(i = 1, \ldots, 4)$ for some $c \in \mathbb{C}^*$. Since τ^* preserves the value of a and $\iota : \mathcal{F} \mapsto \mathcal{F}^\vee$ changes a to $-a$, we have: $\mathcal{F}(a; (\lambda_i)) \in P \;\Rightarrow\; a = 0$. Let P_0 be the open subset in P that parametrizes the locally free sheaves. To determine P_0, we compute the action of $\kappa = \iota\circ\tau^*$ on $\mathcal{F} = \mathcal{O}_{C_-} \underset{(\lambda_1,\ldots,\lambda_4)}{\#} \mathcal{O}_{C_+}$:

$$\mathcal{F} \xmapsto{\tau^*} \mathcal{O}_{C_-} \underset{(\lambda_2,\lambda_1,\lambda_4,\lambda_3)}{\#} \mathcal{O}_{C_+} \xmapsto{\iota} \mathcal{O}_{C_-} \underset{(\frac{1}{\lambda_2},\frac{1}{\lambda_1},\frac{1}{\lambda_4},\frac{1}{\lambda_3})}{\#} \mathcal{O}_{C_+},$$

thus $\mathcal{F} \in P_0 \iff \mathrm{rk}\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_2^{-1} & \lambda_1^{-1} & \lambda_4^{-1} & \lambda_3^{-1}\end{pmatrix} = 1$, and P_0 is the quotient of the subtorus of $(\mathbb{C}^*)^4$ with equation $\lambda_1\lambda_2 = \lambda_3\lambda_4$ by the diagonal action of $\mathbb{C}^*$. Hence $P_0 \simeq \mathbb{C}^*\times\mathbb{C}^*$.

We define the next stratum, P_1, as the locus of the sheaves in P that are invertible in at least one of the points z_i, but are not invertible in all of them. If $\mathcal{F}\in P_1$, then either $\mathcal{F} \simeq \mathcal{F}'(a;\lambda_3,\lambda_4) = \mathcal{O}_{C_-}((-a-1)pt) \underset{(\cdot,\cdot,\lambda_3,\lambda_4)}{\#} \mathcal{O}_{C_+}((a-1)pt)$, or $\mathcal{F} \simeq \mathcal{F}''(a;\lambda_1,\lambda_2) = \mathcal{O}_{C_-}((-a-1)pt) \underset{(\lambda_1,\lambda_2,\cdot,\cdot)}{\#} \mathcal{O}_{C_+}((a-1)pt)$, where we put a dot on the i-th place to indicate that the gluing in z_i is not effectuated, that is, $\mathcal{F}_{z_i} = \mathcal{O}_{C_-}((-a-1)pt)_{z_i} \oplus \mathcal{O}_{C_+}((a-1)pt)_{z_i}$.

To determine the dual of such a sheaf, represent it as the direct image $\sigma_*(\mathcal{L})$, where $\sigma : \tilde{S}\to S$ is the blowup of S at the two points in which $\mathcal{F}$ is not locally free, and $\mathcal{L}$ is supported on the proper transform C' of C and is invertible as a $\mathcal{O}_{C'}$-module. Then, by the relative duality for σ (see [Ha], p. 210),

$$\mathcal{F}^\vee \simeq \sigma_*(\mathcal{E}xt^{\,1}_{\mathcal{O}_{\tilde{S}}}(\mathcal{L}, \mathcal{O}_{\tilde{S}}(-\sigma^*(C))\otimes\omega_{\tilde{S}/S})) \simeq \sigma_*(\mathcal{L}^\vee(-E\cdot C')),$$

where E is the union of two (-1)-curves which form the exceptional locus of σ. Let, for example, $\mathcal{F} = \mathcal{F}'(a;\lambda_3,\lambda_4)$. Then $\mathcal{L}$ is the result of gluing $\mathcal{O}_{C'_-}((-a-1)pt) \underset{(\lambda_3,\lambda_4)}{\#} \mathcal{O}_{C'_+}((a-1)pt)$ at the two points of $C'_+\cap C'_-$, where $C'_\pm$ is the proper transform of $C_\pm$, and $\mathcal{L}^\vee = \mathcal{L}^{-1} \simeq \mathcal{O}_{C'_-}((a+1)pt) \underset{(\lambda_3^{-1},\lambda_4^{-1})}{\#} \mathcal{O}_{C'_+}((1-a)pt)$. Thus we obtain:

$$\mathcal{F}'(a;\lambda_3,\lambda_4)^\vee = \sigma_*\Big(\mathcal{O}_{C'_-}((a-1)pt) \underset{(\lambda_3^{-1},\lambda_4^{-1})}{\#} \mathcal{O}_{C'_+}((-a-1)pt)\Big) \simeq \mathcal{F}'(-a;\lambda_3^{-1},\lambda_4^{-1}).$$

It is much easier to determine the action of τ: obviously, $\tau^*\big(\mathcal{F}'(a;\lambda_3,\lambda_4)\big) \simeq \mathcal{F}'(a;\lambda_4,\lambda_3)$. We conclude that $\mathcal{F}'(a;\lambda_3,\lambda_4)\in P_1$ if and only if $a=0$, and the sheaves $\mathcal{F}'(0;\lambda_3,\lambda_4)$ fill a component of P_1, isomorphic to $\mathbb{C}^*$. The other copy of $\mathbb{C}^*$ contained in P_1 is formed by the sheaves $\mathcal{F}''(0;\lambda_1,\lambda_2)$.

Finally, define P_2 as the locus of sheaves which are noninvertible in all the 4 points z_i. By Theorem 3.4, $\boldsymbol{\mathcal{P}}$ contains only one such sheaf with support C: $\mathcal{F} = \mathcal{O}_{C_-}(-2pt) \oplus \mathcal{O}_{C_+}(-2pt)$. This ends the proof for $m = 0$. As $\boldsymbol{\mathcal{P}}^{2m} \simeq \boldsymbol{\mathcal{P}}^{2m+4}$, it remains to consider the case $m = 1$. An isomorphism $P^0 \xrightarrow{\sim} P^2$ can be given by $\mathcal{F} \mapsto \mathcal{F} \otimes \theta$, where θ is a τ-invariant theta-characteristic on C. One easily verifies that there are two such theta-characteristics: $\theta = \mathcal{O}_{C_-}(pt) \underset{(1,1,\epsilon,\epsilon)}{\#} \mathcal{O}_{C_+}(pt)$, where $\epsilon = \pm 1$.

□

4. Compactified Prymians of integral curves

We will use the notation of the previous section. Thus $\boldsymbol{\mathcal{P}}$ will denote $\overline{\mathrm{Prym}}^{2m}(\mathcal{C}, \tau)$, and $f_{\boldsymbol{\mathcal{P}}}$ or $f_{\boldsymbol{\mathcal{P}}}^{2m}$ the map $\boldsymbol{\mathcal{P}} \to \mathbb{P}^{2\vee}$ sending each sheaf to its support. For $t \in \mathbb{P}^{2\vee}$, let $C_t = \varphi^{-1}(t)$, $E_t = C_t/\tau$, and $\rho_t = \rho|_{C_t} : C_t \to E_t$, where $\varphi : \mathcal{C} \to \mathbb{P}^{2\vee}$ is the natural map. We call the fiber $P_t = (f_{\boldsymbol{\mathcal{P}}}^{2m})^{-1}(t)$ of the support map the compactified Prymian of the pair (C_t, τ). In this section, we will describe the structure of P_t for all irreducible singular members C_t of the linear system $\mathcal{C}/\mathbb{P}^{2\vee}$.

Lemma 4.1. *Let us assume that S is generic, that is, the curves $B_0 \in |\mathcal{O}_{\mathbb{P}^2}(4)|$ and $\Delta_0 \in |-2K_X|$ are generic. Let $\overline{\Delta}_0 = \mu(\Delta_0) \subset \mathbb{P}^2$, and let $B_0^\vee$, $\overline{\Delta}_0^\vee$ denote the dual curves in $\mathbb{P}^{2\vee}$. Let T be the finite set of points which are singularities of the curve $B_0^\vee \cup \overline{\Delta}_0^\vee$. Then the linear system $\mathcal{C}/\mathbb{P}^{2\vee}$ contains the following singular members C_t:*

(i) C_t has a unique node p if $t \in \overline{\Delta}_0^\vee \setminus T$; p is τ-invariant, and τ permutes the branches of C_t at p.

(ii) C_t has a unique cusp if t is one of the 24 cusps of $\overline{\Delta}_0^\vee$.

(iii) C_t has two nodes permuted by τ if $t \in B_0^\vee \setminus T$.

(iv) C_t has two τ-invariant nodes if t is one of the 28 nodes of $\overline{\Delta}_0^\vee$.

(v) C_t has two cusps permuted by τ if t is one of the 24 cusps of $B_0^\vee$.

(vi) C_t has 3 nodes, only one of which is invariant under τ, if t is one of the 128 points of transversal intersection of $B_0^\vee$ and $\overline{\Delta}_0^\vee$.

(vii) C_t has one tacnode if t is one of the 8 points of tangency of $B_0^\vee$, $\overline{\Delta}_0^\vee$.

(viii) C_t is a union of two smooth conics meeting transversely in 4 points if t is one of the 28 nodes of $B_0^\vee$.

Proof. The proof is obvious. Remark that B_0, $\overline{\Delta}_0$ are totally tangent to each other; if $f_4(u_0, u_1, u_2) = 0$, $g_4(u_0, u_1, u_2) = 0$ are their equations, then the pencil $\langle f_4, g_4 \rangle$ contains the square q^2 of some quadratic form q in u_0, u_1, u_2. This follows from the fact that the inverse image of $\overline{\Delta}_0$ in

X decomposes into two components, one of which is Δ_0. The number 28, resp. 24 is the number of bitangents, resp. flexes of a smooth plane quartic. The eight tangency points of B_0, $\overline{\Delta}_0$ are sent by the Gauss map to 8 tangency points of $B_0^\vee$, $\overline{\Delta}_0^\vee$. As the degree of each of the two dual curves is 12, there remains $12^2 - 8 \cdot 2 = 128$ points of transversal intersection, corresponding to the non-tacnodal common tangents of B_0, $\overline{\Delta}_0$. $\square$

Lemma 4.2. *Let $t \in \mathbb{P}^{2\vee}$ and $P_t = (f^{2m}_{\boldsymbol{\mathcal{P}}})^{-1}(t)$. Assume that C_t is irreducible. Then the following assertions hold:*

(i) The varieties P_t constructed for different values of m are isomorphic to each other.

(ii) P_t has an action of the 2-dimensional algebraic group $G_t = \mathrm{Prym}(C_t, \tau)$, and the locus P^*_t of invertible sheaves in P_t is a finite union of orbits of G°_t on which the action is free.

Proof. (i) There is a canonical isomorphism $\overline{\mathrm{Prym}}^{2m}(\mathcal{C}, \tau) \simeq \overline{\mathrm{Prym}}^{2m+4}(\mathcal{C}, \tau)$ given by $[\mathcal{F}] \mapsto [\mathcal{F} \otimes \mathcal{O}_S(H)]$. Thus it suffices to consider only the values $m = 0$ and 1. In this case there is no isomorphism of the relative Prymians, but there are noncanonical isomorphisms of the individual fibers $P^0_t = (f^0_{\boldsymbol{\mathcal{P}}})^{-1}(t) \simeq P^2_t = (f^2_{\boldsymbol{\mathcal{P}}})^{-1}(t)$. Such an isomorphism can be associated to any of the τ-invariant theta-characteristics θ of C_t by $[\mathcal{F}] \mapsto [\mathcal{F} \otimes \theta]$. One can choose $\theta = \rho_t^{-1}(\xi)$, where ξ is a ramification point of the double cover $\mu_t : E_t \to \ell_t \simeq \mathbb{P}^1$, and $\mu_t = \mu|_{E_t}$.

(ii) In the case when $\boldsymbol{\mathcal{L}}$ is an invertible sheaf on C_t and $\mathcal{F}$ is a rank-1 torsion-free sheaf, we have $\tau(\mathcal{F} \otimes \mathcal{L}) = \tau(\mathcal{F}) \otimes \tau(\mathcal{L})$, and similarly for ι. This implies that G_t acts on P_t by tensor multiplication of the corresponding sheaves. The action is obviously free on P^*_t.

By (i), we can assume that $m = 0$. By [AIK], $\overline{J}(C)$ is irreducible. Further, $\boldsymbol{\mathcal{P}}$ is the 4-dimensional fixed locus of the involution κ on $\mathcal{M}$ whose differential has exactly 2 eigenvalues -1 at any point of $P^*_t = \boldsymbol{\mathcal{P}} \cap J(C_t)$, and one of these eigenvalues corresponds to the reflection with respect to a plane in the base $\mathbb{P}^3$, while the other to a reflection in the fiber $J(C_t)$. Thus every connected component of P^*_t is 2-dimensional, hence isomorphic to G°_t. $\square$

Remark that we have not proved that P_t has no 2-dimensional components contained entirely in the non-locally-free locus. We will get this as a consequence of a case-by-case description of a natural stratification of P_t for the degenerate curves C_t listed in Lemma 4.1. Let us fix t and omit the subscript t from the symbols C_t, P_t, etc. In this section, we consider only the case when C is irreducible. By Lemma 4.2, (i), P does not depend on m, so we can assume $m = 0$. According to [Cook-1], $\overline{J}(C)$ has a stratification in smooth strata whose codimension is equal to the index $i(\mathcal{F})$ of the sheaves $\mathcal{F}$ represented by the points of these strata. The index is defined as follows. Let $\nu : \tilde{C} \to C$ be the normalization map. Then there exists a factorization $\tilde{C} \xrightarrow{\nu''} C' \xrightarrow{\nu'} C$ of ν such that $\nu'^*(\mathcal{F})/(\text{torsion})$ is invertible, and $i(\mathcal{F})$ is the minimum of $\text{length}(\nu'_*\mathcal{O}_{C'}/\mathcal{O}_C)$ over such factorizations. The index takes values between 0 and $\delta(C) = \text{length}(\nu_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C) = p_a(C) - g(C)$, and $\mathcal{F}$ is invertible if and only if $i(\mathcal{F}) = 0$.

Let $J_i(C)$ be the union of strata of codimension i in $\overline{J}(C)$ ($0 \leq i \leq 3$); obviously, $J_0(C) = J(C)$. We will denote by P_i the intersection $J_i(C) \cap P$. Then $P_0 = \text{Prym}(C, \tau)$ is an algebraic group of dimension 2, which we denoted by G_t in Lemma 4.2. As we will see, for $i > 0$, the value of i may differ from the codimension of P_i in P. We will determine the strata P_i for all the singular members of the linear system $\mathcal{C}/\mathbb{P}^{2\vee}$.

Proposition 4.3. *Assume that $m = 0$. In the notation of Lemma 4.1, let $t \in B_0^\vee \cup \overline{\Delta}_0^\vee$. Denote by $\nu : \tilde{C} \to C$ the normalization of $C = C_t$. The map $[\mathcal{F}] \mapsto [\nu^*(\mathcal{F})/(\text{torsion})]$, when restricted to $J_i(C)$, is a morphism $J_i(C) \to \text{Pic}^{-i}(\tilde{C})$, which will be denoted by υ_i. The involution τ lifts to $\tilde{C}$ or to any partial normalization of C, and we will use the same symbol τ to denote such a lift.*

In the first seven cases of Lemma 4.1, all the nonempty strata P_i are described as follows:

(i) $P = P_0 \sqcup P_1$, $P_0 = \upsilon_0^{-1}(\text{Prym}(\tilde{C}, \tau))$, $P_1 \simeq \text{Prym}(\tilde{C}, \tau)$. Here $\text{Prym}(\tilde{C}, \tau)$ is an elliptic curve lying in the abelian surface $J(\tilde{C})$, and $\upsilon_0 : J(C) \to J(\tilde{C})$ is a group morphism with kernel $\mathbb{C}^*$. Thus P_1 is an elliptic curve, and P_0 is an extension of an elliptic curve by $\mathbb{C}^*$.

(ii) $P = P_0 \sqcup P_1$, $P_0 = \upsilon_0^{-1}(\text{Prym}(\tilde{C}, \tau))$, $P_1 \simeq \text{Prym}(\tilde{C}, \tau)$ as in (i), but now $\ker \upsilon_0 \simeq \mathbb{C}$ and P_0 is an extension of an elliptic curve by the additive group $\mathbb{C}$.

(iii) $P = P_0 \sqcup P_2$, P_0 is a $\mathbb{C}^*$-extension of the elliptic curve $J(\tilde{C}) \simeq \tilde{C}$, and $P_2 \simeq J(\tilde{C})$ (thus $\text{codim}_P P_2 = 1$).

(iv) $P = P_0 \sqcup P_1 \sqcup P_2$, $P_0 \simeq \bigsqcup_{k=1}^{4} \mathbb{C}^* \times \mathbb{C}^*$, $P_1 \simeq \bigsqcup_{k=1}^{8} \mathbb{C}^*$, and P_2 is a finite set, consisting of 4 points.

(v) $P = P_0 \sqcup P_2$, P_0 is a $\mathbb{C}$-extension of the elliptic curve $J(\tilde{C}) \simeq \tilde{C}$, and $P_2 \simeq J(\tilde{C})$ (thus $\text{codim}_P P_2 = 1$).

(vi) $P = P_0 \sqcup \ldots \sqcup P_3$, $P_0 \simeq \mathbb{C}^* \times \mathbb{C}^*$, $P_1 \simeq P_2 \simeq \mathbb{C}^*$, and P_3 is one point, corresponding to the sheaf $\nu_*(\mathcal{O}_{\mathbb{P}^1}(-3))$, where we have identified $\tilde{C}$ with $\mathbb{P}^1$.

(vii) $P = P_0 \sqcup P_2$, P_0 is an irreducible extension of the elliptic curve $J(\tilde{C}) \simeq \tilde{C}$ by $\mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})$, and $P_2 \simeq \tilde{C}$.

Proof. (i) Here $\delta(C) = 1$, so $\overline{J}(C) = J_0(C) \sqcup J_1(C)$, hence $P = P_0 \sqcup P_1$. Let $\{p', p''\} = \nu^{-1}(p)$, $\{p\} = \operatorname{Sing} C$. If $\mathcal{F} \in J_0(C)$, then $\mathcal{L} = \nu^*(\mathcal{F})$ is invertible and $\mathcal{F}$ is a subsheaf of $\nu_*(\mathcal{L})$ with quotient $\mathbb{C}_p$. For fixed $\mathcal{L}$, the invertible subsheaves of $\nu_*(\mathcal{L})$ with quotient $\mathbb{C}_p$ are parametrized by $\mathbb{C}^*$. One can easily describe them in terms of the corresponding line bundles. We will not distinguish in the notation the invertible sheaves and the corresponding line bundles. So, we represent the line bundle $\mathcal{F}$ as the result of gluing together the fibers $\mathcal{L}_{,p'}, \mathcal{L}_{,p''}$ of the line bundle $\mathcal{L}$. To this end, we choose some rational section e of $\mathcal{L}$ which trivializes $\mathcal{L}$ on an open set containing both p' and p'', and then the gluing is determined by a factor $\lambda \in \mathbb{C}^*$ by the following rule: $\mathcal{F} = \mathcal{L}/(e_{,p'} \sim \lambda e_{,p''})$. Let us denote it by $\mathcal{L}[{}^{\lambda}_{e}]$. In the language of sheaves, $\mathcal{F} = \mathcal{L}[{}^{\lambda}_{e}]$ is described as the subsheaf of $\nu_*(\mathcal{L})$ with the following stalks:

$$\mathcal{F}_z = (\nu_*(\mathcal{L}))_z \;\; \forall \;\; z \in C \setminus \{p\}, \;\; \text{and} \;\; \mathcal{F}_p = \mathcal{O}_p \cdot (e_{p'} + \lambda e_{p''}) \; .$$

Now we will seak the triples $(\mathcal{L}, e, \lambda)$ for which $\mathcal{L}[{}^{\lambda}_{e}] \in P$, that is, $\tau^*(\mathcal{L}[{}^{\lambda}_{e}]) \otimes \mathcal{L}[{}^{\lambda}_{e}] \simeq \mathcal{O}_C$. The involution $\tau : \tilde{C} \to \tilde{C}$ lifts in a natural way to a map $\tau_\# : \tau^*(\mathcal{L}) \to \mathcal{L}$, understood as a map of the total spaces either of sheaves, or of line bundles, and we choose $e^\tau := \tau_\#^{-1}(e)$ as a trivialization of $\tau^*(\mathcal{L})$ in the neighborhood of p', p''. As τ permutes p', p'', we have $\tau^*(\mathcal{L}[{}^{\lambda}_{e}]) = \tau^*(\mathcal{L})[{}^{\lambda^{-1}}_{e^\tau}]$ and $\tau^*(\mathcal{L}[{}^{\lambda}_{e}]) \otimes \mathcal{L}[{}^{\lambda}_{e}] = (\tau^*(\mathcal{L}) \otimes \mathcal{L})\left[{}^{\;1}_{e^\tau \otimes e}\right]$. Thus the necessary condition for $\mathcal{L}[{}^{\lambda}_{e}] \in P$ is $\tau^*(\mathcal{L}) \otimes \mathcal{L} \simeq \mathcal{O}_{\tilde{C}}$. Assume it is satisfied, then fix an isomorphism and denote by g the image of $e^\tau \otimes e$ in $\mathcal{O}_{\tilde{C}}$. Then $(\tau^*(\mathcal{L}) \otimes \mathcal{L})\left[{}^{\;1}_{e^\tau \otimes e}\right] \simeq \mathcal{O}_{\tilde{C}}\left[{}^{g(p'')/g(p')}_{\;\;1}\right]$. Using the canonical isomorphisms $\tau^*(\mathcal{L}) \otimes \mathcal{L} = \mathcal{L} \otimes \tau^*(\mathcal{L})$ and $\tau^*(\tau^*(\mathcal{L})) = \mathcal{L}$, we may claim that $\tau_\# \circ \tau_\# = \mathrm{id}_{\mathcal{L}}$ and that $e^\tau \otimes e$ is τ-invariant. Hence g is also τ-invariant and $g(p'')/g(p') = 1$. Thus $(\tau^*(\mathcal{L}) \otimes \mathcal{L})\left[{}^{\;1}_{e^\tau \otimes e}\right] \simeq \mathcal{O}_{\tilde{C}}[{}^{1}_{1}] = \mathcal{O}_C$, and we conclude that $\mathcal{L}[{}^{\lambda}_{e}] \in P$ as soon as $\mathcal{L} \in \operatorname{Prym}(\tilde{C}, \tau)$, and this does not depend on the choice of e, λ. We have proved the part of (i) concerning P_0.

Let now $\mathcal{F} \in J_1(C)$. Then $\mathcal{F} \simeq \nu_*(\mathcal{L})$ for $\mathcal{L} \in \operatorname{Pic}^{-1}(\tilde{C})$. To express the dual $\mathcal{F}^\vee = \mathcal{E}xt^{\,1}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S(-C))$ in terms of $\mathcal{L}^{-1}$, we consider ν as an embedded resolution: let $\sigma : \tilde{S} \to S$ be the blowup at p, $\tilde{C}$ the proper transform of C in $\tilde{S}$, and $\nu = \sigma|_{\tilde{C}}$. Then, by the relative duality for σ (see [Ha], p. 210),

$$\mathcal{F}^\vee \simeq \sigma_*(\mathcal{E}xt^{\,1}_{\mathcal{O}_{\tilde{S}}}(\mathcal{L}, \mathcal{O}_{\tilde{S}}(-\sigma^*(C)) \otimes \omega_{\tilde{S}/S})) \simeq \nu_*(\mathcal{L}^{-1}(-p' - p'')).$$

We have:

$$\begin{array}{rl} & [\mathcal{F}] \in P_1 = \mathrm{Fix}(\kappa) \cap J_1(C) \\ \iff & (\tau^*\mathcal{F})^\vee \simeq \mathcal{F} \\ \iff & (\tau^*\mathcal{L})^{-1}(-p'-p'') \simeq \mathcal{L} \\ \iff & [\mathcal{L}(p')] \in \mathrm{Prym}(\tilde{C},\tau). \end{array}$$

Thus $P_1 \simeq \mathrm{Prym}(\tilde{C},\tau)$ via the mutually inverse maps

$$\begin{array}{ccc} P_1 & \longrightarrow & \mathrm{Prym}(\tilde{C},\tau) \\ [\mathcal{F}] & \longmapsto & \upsilon_1(\mathcal{F})\cdot[\mathcal{O}_{\tilde{C}}(p')] \end{array}, \quad \begin{array}{ccc} \mathrm{Prym}(\tilde{C},\tau) & \longrightarrow & P_1 \\ [\mathcal{L}] & \longmapsto & [\nu_*(\mathcal{L}(-p'))] \end{array}.$$

(ii) As in (i), $\delta(C)=1$, so $P=P_0\sqcup P_1$. Denote by p the singular point of C and set $p'=\nu^{-1}(p)$. Fix a local parameter t of $\mathcal{O}_{\tilde{C},p'}$ in such a way that $\tau^*(t)=-t$, and an invertible sheaf $\mathcal{L}$ on $\tilde{C}$ together with a local trivialization e at p'. Then any invertible sheaf $\mathcal{F}$ on C such that $\nu^*(\mathcal{F})\simeq\mathcal{L}$ can be represented as the subsheaf of $\nu_*(\mathcal{L})$ given by its stalks:

$$\mathcal{F}_z = (\nu_*(\mathcal{L}))_z \ \ \forall \ \ z\in C\setminus\{p\}, \ \text{ and } \ \mathcal{F}_p = \mathcal{O}_p\cdot(1+bt)e_{p'}$$

for some constant $b\in\mathbb{C}$. Denote this sheaf by $\mathcal{L}\left[{b \atop e;t}\right]$. Similarly to the proof of part (i), $\tau^*(\mathcal{L}\left[{b \atop e;t}\right]) = \mathcal{L}\left[{-b \atop e^\tau;t}\right]$, and one easily verifies that $\mathcal{L}\left[{b \atop e;t}\right]\in P_0$ if and only if $\mathcal{L}\in\mathrm{Prym}(\tilde{C},\tau)$ independently of the choice of e,t,b. This implies the assertion about P_0. The proof for P_1 is based on the formula for the dual $(\nu_*(\mathcal{L}))^\vee\simeq\nu_*(\mathcal{L}^{-1}(-2p'))$, which implies that $\nu_*(\mathcal{L})\in P_1 \Leftrightarrow \mathcal{L}(p')\in\mathrm{Prym}(\tilde{C},\tau)$.

(iii) Here $\delta(C)=2$ and C has two singular points p_1,p_2 permuted by τ. Let $\nu^{-1}(p_i)=\{p_i',p_i''\}$. We can choose the notation in such a way that $\tau(p_1')=p_2'$. $\overline{J}(C)$ has three strata, but $P_1=J_1\cap P=\varnothing$, because if a sheaf belonging to P is not locally free at p_1, it is not locally free at p_2 either. Hence $P=P_0\sqcup P_2$. Let $\mathcal{L}$ be an invertible sheaf on $\tilde{C}$, e its rational section, regular at p_i',p_i'', and $(\lambda_1,\lambda_2)\in\mathbb{C}^*\times\mathbb{C}^*$. Then we define $\mathcal{L}\left[{\lambda_1,\lambda_2 \atop e}\right]$ as the subsheaf $\mathcal{F}$ of $\nu_*(\mathcal{L})$ which coincides with $\nu_*(\mathcal{L})$ over $C\setminus\{p_1,p_2\}$ and such that $\mathcal{F}_{p_i}=\mathcal{O}_{p_i}\cdot(e_{p_i'}+\lambda_i e_{p_i''})$, $i=1,2$. Similarly to the part (i), one easily verifies that $\tau^*(\mathcal{L}\left[{\lambda_1,\lambda_2 \atop e}\right]) = (\tau^*(\mathcal{L}))\left[{\lambda_2,\lambda_1 \atop e^\tau}\right]$. This implies that $\mathcal{L}\left[{\lambda_1,\lambda_2 \atop e}\right]\in P_0$ if and only if $\mathcal{L}\in\mathrm{Prym}(\tilde{C},\tau)$ and $\lambda_1\lambda_2=1$. Here $\tilde{C}$ is elliptic and $\tilde{C}/\tau\simeq\mathbb{P}^1$, hence $\mathrm{Prym}(\tilde{C},\tau)\simeq J(\tilde{C})\simeq\tilde{C}$, and thus P_0 is an extension of $\tilde{C}$ by $\mathbb{C}^*$.

Further, any sheaf from $J_2(C)$ is of the form $\mathcal{F}=\nu_*(\mathcal{L})$ for $\mathcal{L}\in\mathrm{Pic}^{-2}(\tilde{C})$, and its dual is given by $\mathcal{F}^\vee\simeq\nu_*(\mathcal{L}^{-1}(-p_1'-p_1''-p_2'-p_2''))$. This implies that P_2 consists of the sheaves $\nu_*(\mathcal{J}(-p_1'-p_1''))$, where $\mathcal{J}$ runs over $J(\tilde{C})$.

(iv) Here $\delta(C)=2$ and C has two τ-invariant nodes p_1,p_2, in which τ permutes the branches. Let $\nu^{-1}(p_i)=\{p_i',p_i''\}$. When lifted to $\tilde{C}$, τ is fixed point free. As $\tau(p_i')=p_i''$, τ is a translation on $\tilde{C}$ by a point of order two $[p_1''-p_1']=[p_2''-p_2']$, so that $C/\tau=\tilde{C}/\tau=E$ is an elliptic curve. $\overline{J}(C)$ has three strata, and we first consider a sheaf $\mathcal{F}\in J_0(C)$. As in *(iii)*, we represent it in the form $\mathcal{F}=\mathcal{L}\left[\begin{smallmatrix}\lambda_1,\lambda_2\\ e\end{smallmatrix}\right]$ with $(\lambda_1,\lambda_2)\in\mathbb{C}^*\times\mathbb{C}^*$. Then $\tau^*(\mathcal{F})=(\tau^*(\mathcal{L}))\left[\begin{smallmatrix}\lambda_1^{-1},\lambda_2^{-1}\\ e^\tau\end{smallmatrix}\right]$ and $\tau^*(\mathcal{F})\otimes\mathcal{F}=(\tau^*(\mathcal{L})\otimes\mathcal{L})\left[\begin{smallmatrix}1\ ,\ 1\\ e^\tau\otimes e\end{smallmatrix}\right]$. On $\tilde{C}$, $\tau^*(\mathcal{L})\simeq\mathcal{L}$ for any degree-0 invertible $\mathcal{L}$, so $\mathcal{L}\in P_0$ if and only if $\mathcal{L}$ is of order 2 in $J(\tilde{C})$. Thus $P_0=\upsilon_0^{-1}(J(\tilde{C})_{(2)})$ is the disjoint union of 4 copies of $\mathbb{C}^*\times\mathbb{C}^*$.

$J_1(C)$ consists of the subsheaves $\mathcal{F}\subset\nu_*(\mathcal{L})$ with $\mathcal{L}\in\mathrm{Pic}^{-1}(\tilde{C})$, which coincide with $\nu_*(\mathcal{L})$ over $C\setminus\{p_i\}$ and such that $\mathcal{F}_{p_i}=\mathcal{O}_{p_i}\cdot(e_{p_i'}+\lambda e_{p_i''})$ for one of the values of $i=1$ or 2 ($\lambda\in\mathbb{C}^*$). Let us denote such a sheaf by $\mathcal{L}\left[\begin{smallmatrix}\lambda,\cdot\\ e\end{smallmatrix}\right]$ if $i=1$ and $\mathcal{L}\left[\begin{smallmatrix}\cdot,\lambda\\ e\end{smallmatrix}\right]$ if $i=2$. Let, for example, $\mathcal{F}=\mathcal{L}\left[\begin{smallmatrix}\lambda,\cdot\\ e\end{smallmatrix}\right]$. We have $\mathcal{F}^\vee\simeq(\mathcal{L}^{-1}(-p_2'-p_2''))\left[\begin{smallmatrix}\lambda^{-1},\cdot\\ e^\vee\end{smallmatrix}\right]$, so that $\kappa(\mathcal{F})=(\tau^*(\mathcal{F}))^\vee\simeq\left(\tau^*(\mathcal{L})^{-1}(-p_2'-p_2'')\right)\left[\begin{smallmatrix}\lambda,\cdot\\ (e^\tau)^\vee\end{smallmatrix}\right]$. A necessary condition for $\mathcal{F}\in P_1$ is $\tau^*(\mathcal{L})^{-1}(-p_2'-p_2'')\simeq\mathcal{L}$, or equivalently, $(\tau^*(\mathcal{L}(p_2')))^{-1}\simeq\mathcal{L}(p_2')$. Let it be satisfied, and let us fix such an isomorphism. Via a natural embedding $\mathcal{L}\hookrightarrow\mathcal{L}(p_2')$, we can consider e as a rational section of $\mathcal{L}(p_2')$, regular and nonvanishing at p_1',p_1'', and the latter isomorphism sends $(e^\tau)^\vee$ to ge for some $g\in\mathbb{C}(C)$. Then

$$\begin{aligned}
&\mathcal{F}\in P_1\\
\Leftrightarrow\quad &(\tau^*(\mathcal{L}(p_2')))^{-1}\left[\begin{smallmatrix}\lambda,\cdot\\ (e^\tau)^\vee\end{smallmatrix}\right]\simeq\mathcal{L}(p_2')\left[\begin{smallmatrix}\lambda,\cdot\\ e\end{smallmatrix}\right]\\
\Leftrightarrow\quad &\mathcal{L}(p_2')\begin{bmatrix}\frac{g(p_1'')}{g(p_1')}\lambda,\ \cdot\\ e\end{bmatrix}\simeq\mathcal{L}(p_2')\left[\begin{smallmatrix}\lambda,\cdot\\ e\end{smallmatrix}\right]\\
\Leftrightarrow\quad &g(p_1')=g(p_1'')
\end{aligned}$$

It is easily seen that g is τ-invariant, so the last condition is satisfied. Thus $\mathcal{L}\left[\begin{smallmatrix}\lambda,\cdot\\ e\end{smallmatrix}\right]\in P_1$ if and only if $\mathcal{L}(p_2')$ is one of the 4 points of second order in $J(\tilde{C})$, independently of e,λ, and this gives 4 components of P_1, each isomorphic to $\mathbb{C}^*$. The other four are given by the sheaves $\mathcal{L}\left[\begin{smallmatrix}\cdot,\lambda\\ e\end{smallmatrix}\right]$ for which $\mathcal{L}(p_1')$ is a point of second order in $J(\tilde{C})$.

Finally, P_2 consists of 4 sheaves $\nu_*\mathcal{L}$, for which $\mathcal{L}(p_1'+p_2')\in J(\tilde{C})_{(2)}$.

(v) As in (iii), $P_1=\varnothing$. Denote $\mathrm{Sing}\,C=\{p_1,p_2\}$, $p_i'=\nu^{-1}(p_i)$. We represent the sheaves from $J_0(C)$ in the form $\mathcal{L}\left[\begin{smallmatrix}b_1,b_2\\ e;t_1,t_2\end{smallmatrix}\right]$, where $\mathcal{L}$ runs over $J(\tilde{C})$, e is a rational section of $\mathcal{L}$ trivializing it at p_1',p_2', and t_i are local parameters at p_i' such that $\tau^*(t_i)=t_{3-i}$ ($i=1,2$). The sheaf $\mathcal{F}=\mathcal{L}\left[\begin{smallmatrix}b_1,b_2\\ e;t_1,t_2\end{smallmatrix}\right]$ is defined as the subsheaf of $\nu_*\mathcal{L}$ which coincides with $\nu_*\mathcal{L}$ over $C\setminus\{p_1,p_2\}$ and such that $\mathcal{F}_{p_i}=\mathcal{O}_{p_i}\cdot(1+b_it_i)e_{p_i'}$ for

$i=1,2$. Then $\tau^*(\mathcal{F})=\tau^*(\mathcal{L})\left[\begin{smallmatrix} b_2,b_1\\ e^\tau;t_1,t_2\end{smallmatrix}\right]$, and the stalk of $\tau^*(\mathcal{F})\otimes\mathcal{F}$ at p_i, as an $\mathcal{O}_{p_i}$-submodule of the stalk of $\nu_*(\tau^*(\mathcal{L})\otimes\mathcal{L})$, is generated by $(1+b_1t_i)(1+b_2t_i)e^\tau_{p'_i}\otimes e_{p'_i}$. As $t_i^2\in\mathfrak{m}_{p_i}$, we conclude that $\tau^*(\mathcal{F})\otimes\mathcal{F}=(\tau^*(\mathcal{L})\otimes\mathcal{L})\left[\begin{smallmatrix} b_1+b_2,b_1+b_2\\ e^\tau\otimes e;t_1,t_2\end{smallmatrix}\right]$ and that $\mathcal{F}\in P_0\ \Leftrightarrow\ b_1+b_2=0$. Thus P_0 is a $\mathbb{C}$-extension of $J(\tilde{C})$.

The stratum $J_2(C)$ consists of the sheaves $\nu_*(\mathcal{L})$, where $\mathcal{L}$ runs over $\mathrm{Pic}^{-2}(\tilde{C})$, and $(\nu_*(\mathcal{L}))^\vee\simeq\nu_*(\mathcal{L}^{-1}(-2p'_1-2p'_2))$. This implies that P_2 consists of the sheaves $\nu_*(\mathcal{J}(-p'_1-p'_2))$, where $\mathcal{J}$ runs over $J(\tilde{C})$, hence $P_2\simeq\tilde{C}$.

(vi) Here $\delta(C)=3$ and $\overline{J}(C)$ has 4 strata. We set $\mathrm{Sing}\,C=\{p_0,p_1,p_2\}$, $\tau(p_0)=p_0$, $\tau(p_1)=p_2$, $\nu^{-1}(p_i)=\{p'_i,p''_i\}$, $\tau(p'_0)=p''_0$, $\tau(p'_1)=p'_2$. As $\tilde{C}\simeq\mathbb{P}^1$, the open stratum $J_0(C)$ consists of the subsheaves $\mathcal{O}_{\tilde{C}}\left[\begin{smallmatrix}\lambda_0,\lambda_1,\lambda_2\\ 1\end{smallmatrix}\right]$ of $\nu_*\mathcal{O}_{\tilde{C}}$ which coincide with $\nu_*\mathcal{O}_{\tilde{C}}$ over C_{ns} and whose stalk at p_i is generated by $1_{p'_i}+\lambda_i1_{p''_i}$. We have $\tau^*\mathcal{F}\simeq\mathcal{O}_{\tilde{C}}\left[\begin{smallmatrix}\lambda_0^{-1},\lambda_2,\lambda_1\\ 1\end{smallmatrix}\right]$ and $\tau^*\mathcal{F}\otimes\mathcal{F}\simeq\mathcal{O}_{\tilde{C}}\left[\begin{smallmatrix}1,\lambda_1\lambda_2,\lambda_1\lambda_2\\ 1\end{smallmatrix}\right]$. Thus $P_0\simeq(\mathbb{C}^*)^2$ is the subtorus of $J(C)\simeq(\mathbb{C}^*)^3$ singled out by the equation $\lambda_1\lambda_2=1$. Similarly, we can describe the other strata:

$$\begin{aligned}
P_1&=\left\{\mathcal{O}_{\tilde{C}}(-pt)\left[\begin{smallmatrix}\cdot,\lambda,\lambda^{-1}\\ 1\end{smallmatrix}\right]\right\}_{\lambda\in\mathbb{C}^*}\simeq\mathbb{C}^*;\\
P_2&=\left\{\mathcal{O}_{\tilde{C}}(-2pt)\left[\begin{smallmatrix}\lambda,\cdot,\cdot\\ 1\end{smallmatrix}\right]\right\}_{\lambda\in\mathbb{C}^*}\simeq\mathbb{C}^*;\\
P_3&=\{\nu_*(\mathcal{O}_{\tilde{C}}(-3pt))\}\ =\ 1\text{ point}.
\end{aligned}$$

To clarify the notation, we remind that $\tilde{C}\simeq\mathbb{P}^1$, so that τ has two fixed points on $\tilde{C}$. We use one of them, denoted pt, to embed the sheaves $\nu^*\mathcal{F}/(\text{torsion})$ with $\mathcal{F}\in P_i$ into $\mathcal{O}_{\tilde{C}}$ as the τ-invariant subsheaves $\mathcal{O}_{\tilde{C}}(-ipt)$, and the section 1 of $\mathcal{O}_C$ is considered as a rational trivialization of $\mathcal{O}_{\tilde{C}}(-i\,pt)$.

(vii) Here C has one tacnode p, $\delta(C)=2$, so that $\overline{J}(C)$ has three strata. Denote by p_1,p_2 the preimages of p in $\tilde{C}$ and fix some local parameters t_i at p_i. The points p_i are τ-invariant, and we can choose the t_i in such a way that $\tau^*(t_i)=-t_i$. We will identify the formal completion $B=(\nu_*\mathcal{O}_{\tilde{C}})\hat{\ }_p$ of $\nu_*\mathcal{O}_{\tilde{C}}$ at p with $\mathbb{C}[[t_1]]\times\mathbb{C}[[t_2]]$. We can further restrict the choice of the t_i so that the formal completion $A=\mathcal{O}_p\hat{\ }$ is given by

$$A=\{(a_0+a_1t_1+a_2t_1^2+\dots,b_0+b_1t_2+b_2t_2^2+\dots)\in B\mid a_0=b_0,\ a_1=b_1\}.$$

Denote by $\mathfrak{c}$ the conductor of $\mathcal{O}_p$ in $(\nu_*\mathcal{O}_{\tilde{C}})_p$:

$$\mathfrak{c}=\{u\in\mathcal{O}_p\mid u(\nu_*\mathcal{O}_{\tilde{C}})_p\subset\mathcal{O}_p\}.$$

For its completion, we have $\hat{\mathfrak{c}}=A(t_1^2,0)+A(0,t_2^2)=B(t_1^2,t_2^2)$.

The description of $\overline{J}(C)$ that we will expose here is similar to that given in [Cook-1]. To each $\mathcal{F} \in \overline{J}(C)$, we have assigned the invertible sheaf $\mathcal{L} = \nu^*\mathcal{F}/(\text{torsion})$ on $\tilde{C}$. Let us fix a local trivialization of $\mathcal{L}$ by a rational section e, regular and nonvanishing at p_1, p_2. Then $\mathcal{F}$ can be described as a subsheaf of $\nu_*\mathcal{L}$ which coincides with $\nu_*\mathcal{L}$ out of p and such that $\mathcal{F}_p \subset (\nu_*\mathcal{L})_p$ is an $\mathcal{O}_p$-submodule of colength $2 - i(\mathcal{F})$.

Consider the case when $\mathcal{F} \in J_0(C)$. Then $\mathcal{L}$ is of degree 0 and $\mathcal{F}_p$ is of colength 2. Quotienting by $\mathfrak{c}$, we obtain the vector plane $\mathcal{F}_p/\mathfrak{c}\mathcal{F}_p$ in the 4-dimensional vector space $V = (\nu_*\mathcal{L})_p/\mathfrak{c}(\nu_*\mathcal{L})_p$. This gives a one-to-one correspondence between the sheaves $\mathcal{F} \in J_0(C)$ with the same assigned $\mathcal{L}$ and the vector planes in V which are principal $\mathcal{O}_p/\mathfrak{c}$-modules. Such planes form a locally closed subset $U_{\mathcal{L}}$ of the Grassmannian $G(2, V)$, and to describe it, we can go over to the formal completions. Let $\overline{A} = A/\hat{\mathfrak{c}}$, $\overline{B} = B/\hat{\mathfrak{c}}$. Using e as a generator of $(\nu_*\mathcal{L})_p$, we will identify V with $\overline{B}$.

Thus we can choose $(1,0), (\bar{t}_1, 0), (0,1), (0, \bar{t}_2)$ as a basis of V, where the bar over t_i means taking the coset modulo $\hat{\mathfrak{c}}$. Then the 2-planes in $\overline{B}$, invariant under the multiplication by the elements of $\overline{A} = \langle (1,1), (\bar{t}_1, \bar{t}_2)\rangle$, form a 2-dimensional quadratic cone $Q_{\mathcal{L}}$. If we introduce the Plücker coordinates p_{ij} associated to the above basis of $\overline{B}$, then $G(2, \overline{B}) = G(2,4)$ is the Plücker quadric in $\mathbb{P}^5$ with equation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$, and $Q_{\mathcal{L}}$ is the linear section of $G(2,4)$ defined by $p_{13} = p_{14} + p_{23} = 0$. The 2-planes that are principal $\overline{A}$-modules are parametrized by the open subset of $Q_{\mathcal{L}}$, the complement of two generators of the cone: $U_{\mathcal{L}} = Q_{\mathcal{L}} \setminus (\ell_1 \cup \ell_2)$. If we denote by e_{ij} the point of $G(2,4)$ for which $p_{ij} = 1$ and all the other p_{kl} are zero, then $\ell_1 = \langle e_{12}, e_{24}\rangle$ and $\ell_2 = \langle e_{24}, e_{34}\rangle$. The following map is an isomorphic parametrization of $U_{\mathcal{L}}$:

$$\Pi : \mathbb{C}^* \times \mathbb{C} \longrightarrow U_{\mathcal{L}}, \qquad (\lambda, b) \mapsto [\overline{A} \cdot (1, \lambda + b\bar{t}_2)],$$

or in Plücker coordinates,

$$(p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}) = (1 : 0 : \lambda : -\lambda : -b : \lambda^2).$$

Remark, that $U_{\mathcal{L}}$ is an orbit of the group of units $\overline{B}^{\times}$ and does not depend on the choice of e, for different e's differ by a unit of $(\nu_*\mathcal{O}_{\tilde{C}})_p$. Thus $J_0(C)$ is a $\mathbb{C}^* \times \mathbb{C}$-bundle over $J(\tilde{C})$. We will denote the invertible sheaf on C corresponding to the plane $\Pi(\lambda, b)$ by $\mathcal{L}\left[\begin{smallmatrix}\lambda, b\\ e; t_2\end{smallmatrix}\right]$.

Now we compute the acton of $\kappa = \iota \circ \tau^*$ on $U_{\mathcal{L}}$:

$$\mathcal{L}\left[\begin{smallmatrix}\lambda, b\\ e; t_2\end{smallmatrix}\right] \xmapsto{\tau^*} (\tau^*\mathcal{L})\left[\begin{smallmatrix}\lambda, -b\\ e^{\tau}; t_2\end{smallmatrix}\right] \xmapsto{\iota} (\tau^*\mathcal{L})^{-1}\left[\begin{smallmatrix}\lambda^{-1}, b\lambda^{-2}\\ (e^{\tau})^{\check{}};\ t_2\end{smallmatrix}\right].$$

As $\tilde{C}/\tau \simeq \mathbb{P}^1$, we have $(\tau^*\mathcal{L})^{-1} \simeq \mathcal{L}$ for any $\mathcal{L} \in J(\tilde{C})$. Fix such an isomorphism; then it sends $(e^\tau)\check{}$ to ge for some $g \in \mathbb{C}(C)$. The τ-invariance of g implies that it has no linear terms in t_i, and we deduce that $\kappa\left(\mathcal{L}\left[\begin{smallmatrix}\lambda,b\\e;t_2\end{smallmatrix}\right]\right) = \mathcal{L}\left[\begin{smallmatrix}a\lambda^{-1}, & ab\lambda^{-2}\\ e \;\; ; & t_2\end{smallmatrix}\right]$, where $a = \frac{g(p_1)}{g(p_2)} \neq 0$. Hence the fixed locus of κ in $U_{\mathcal{L}}$ is given by $\lambda = \pm\sqrt{a}$, which singles out two generators of the cone (with deleted vertex). Remark, that κ is a restriction of a linear involution on $\mathbb{P}^5$:

$$(p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}) \xmapsto{\kappa} \\ (p_{34} : p_{13} : ap_{14} : ap_{23} : ap_{24} : a^2p_{12}) \quad (8)$$

Thus P_0 is fibered over $J(\tilde{C})$ with each fiber the disjoint union of two copies of $\mathbb{C}$. To see that the family of components of the fibers is an irreducible double cover of $J(\tilde{C})$, one can argue as follows. Write down the double cover $\tilde{C} \to \tilde{C}/\tau \simeq \mathbb{P}^1$ in coordinates:

$$\tilde{C} = \{y^2 = (x - x_1)(x - x_2)(x - x_3)\} \;\to \mathbb{P}^1, \quad (x, y) \mapsto\; x.$$

It is ramified at the 4 points $p_i = (x_i, 0)$ $(i = 1, 2, 3)$ and $p_4 = \infty$. Parametrize $J(\tilde{C})$ by the map $\tilde{C} \to J(\tilde{C})$, $q \mapsto [\mathcal{O}(q - p_4)]$. Embed $\mathcal{O}(q - p_4)$ into the constant sheaf $\mathbb{C}(C)$ in the natural way and use $1 \in \mathbb{C}(C)$ as the rational trivialization e of $\mathcal{L}$. Then the function g introduced in the previous paragraph is given by $g = x - x(q)$, where $q = (x(q), y(q))$, and the equation $\lambda^2 = \frac{g(p_1)}{g(p_2)}$, whose two solutions provide two components of the fiber of the fibration $f : P_0 \to J(\tilde{C})$ over $[\mathcal{O}(q - p_4)]$, becomes $\lambda^2 = \frac{x_1 - x(q)}{x_2 - x(q)}$. Varying $x = x(q)$, we obtain the curve Γ with equation $\lambda^2 = \frac{x_1 - x}{x_2 - x}$, and the connected components of fibers of f are parametrized by the normalization of $\Gamma \times_{\mathbb{P}^1} \tilde{C}$. The latter is a nonramified double cover of $\tilde{C}$.

Now we will determine the lower-dimensional strata of P. Instead of looking for the non-invertible sheaves $\mathcal{F}$ in $\overline{J}(C)$ as $\mathcal{O}_C$-submodules of colength $2 - i$ in $\nu_*\mathcal{L}$ with $\deg \mathcal{L} = -i$, we can get all of them as $\mathcal{O}_C$-submodules of colength 2 with $\mathcal{L}$ of degree 0, parametrized by the points of $Q_{\mathcal{L}} \setminus U_{\mathcal{L}}$. In fact, it is easy to see that the cones $Q_{\mathcal{L}}$ fit into an algebraic family over $J(\tilde{C})$ and that this family is the normalization of $\overline{J}(C)$ (see [Cook-2]), thus any non-invertible sheaf in $\overline{J}(C)$ is in the closure of $U_{\mathcal{L}}$ for some $\mathcal{L} \in J(\tilde{C})$.

As follows from (8), κ permutes ℓ_1 and ℓ_2, thus the only fixed point of κ in $Q_{\mathcal{L}} \setminus U_{\mathcal{L}}$ is the vertex e_{24} of the cone. It corresponds to the 2-plane $\langle(\bar{t}_1, 0), (0, \bar{t}_2)\rangle$ in $\overline{B}$. Thus the associated sheaf $\mathcal{F}$ has for its stalk at p

$$\mathcal{F}_p = \mathcal{O}_p \cdot t_1 e_{p_1} + \mathcal{O}_p \cdot t_2 e_{p_2} = (\nu_*(\mathcal{L}(-p_1 - p_2)))_p,$$

and as $\mathcal{F}$, $\nu_*\mathcal{L}$, $\nu_*(\mathcal{L}(-p_1 - p_2))$ coincide on $C \setminus \{p\}$, we conclude that $\mathcal{F} \simeq \nu_*(\mathcal{L}(-p_1 - p_2))$. It is of index 2, and we see that $P_2 \simeq J(\tilde{C})$, $P_1 = \varnothing$. This ends the proof of the proposition. $\square$

5. Further properties of $\boldsymbol{\mathcal{P}}^{2m}$

Fujiki has constructed a number of irreducible symplectic V-manifolds of dimension 4 with at worst isolated singularities as partial desingularizations of a finite quotient of the product of two symplectic surfaces. Among his examples, there are two with 28 singular points of the same type that the singular points of $\boldsymbol{\mathcal{P}}^{2m}$, see Table 1 on p. 225 and Remark 13.2.4 on p. 227 of [F].

These two examples are obtained by the following construction. Let H be a finite group of symplectic automorphisms of a K3 surface S, and $\theta \in \operatorname{Aut} H$ such that $\theta^2 = \mathrm{id}$. Then H acts on $S \times S$ by the rule $h : (s,t) \mapsto (hs, \theta(h)t)$. Define $G \subset \operatorname{Aut}(S \times S)$ as the subgroup generated by H and the involution $(s,t) \mapsto (t,s)$. Then $K = S \times S/G$ is a symplectic V-manifold, in general, with non-isolated singularities. The two examples under consideration correspond to $H = \mathbb{Z}/2\mathbb{Z}$ or $H = (\mathbb{Z}/2\mathbb{Z})^3$ and $\theta : h \mapsto h^{-1}$. For these H, θ, the blowup of the 2-dimensional components of the singular locus of K provides two irreducible symplectic V-manifolds Y_1, Y_2 with 28 singular points of analytic type of $(\mathbb{C}^4/\{\pm 1\}, 0)$. They have the same Euler characteristic and the Hodge numbers. The symmetries for the Hodge diamond of a symplectic V-manifold imply that the whole Hodge diamond of Y_i is determined by the three of them, $h^{1,1} = 14$, $h^{1,2} = 0$, $h^{2,2} = 162$, and the Euler number is $\chi(Y_i) = 8 + 4h^{1,1} + h^{2,2} - 4h^{1,2} = 226$.

The easiest way to prove that Fujiki's examples are different from $\boldsymbol{\mathcal{P}}^{2m}$ is to compute the Euler number. Recall that there are at most two non-isomorphic varieties among the $\boldsymbol{\mathcal{P}}^{2m}$: $\boldsymbol{\mathcal{P}}^{0}$ and $\boldsymbol{\mathcal{P}}^{2}$.

Proposition 5.1. *The varieties $\boldsymbol{\mathcal{P}}^{0}$ and $\boldsymbol{\mathcal{P}}^{2}$ have the same topological Euler number, equal to* 268.

Proof. Let $\boldsymbol{\mathcal{P}}$ denote either one of the varieties $\boldsymbol{\mathcal{P}}^{2m}$, $f : \boldsymbol{\mathcal{P}} \to \mathbb{P}^{2\vee}$ the natural map. We introduce a stratification $(\Pi_i)_{i=0,\dots,8}$ of $\mathbb{P}^{2\vee}$ as follows: $\Pi_0 = \mathbb{P}^{2\vee} \setminus (B_0 \cap \overline{\Delta}_0)$, the complement of the discriminant divisor of f, and Π_k for $k = 1, \dots, 8$ is the locus of points $t \in B_0 \cap \overline{\Delta}_0$ for which the k-th case of Lemma 4.1 is realized. Then we can compute the topological Euler number of $\boldsymbol{\mathcal{P}}$ by the formula $\chi(\boldsymbol{\mathcal{P}}) = \sum_{k=0}^{8} \chi(\Pi_k)\chi(P_{t_k})$. ¿From Lemma 3.5 and Proposition 4.3, we see that $\chi(P_{t_k})$ is the number of 0-dimensional strata in P_{t_k} and that it is different from zero only for $k = 4, 6, 8$. For these values of k, Π_k is finite and $\chi(\Pi_k) = \#\Pi_k$. Thus

$$\chi(\boldsymbol{\mathcal{P}}) = 28 \cdot 4 + 128 \cdot 1 + 28 \cdot 1 = 268.$$

$\square$

To show that $\boldsymbol{\mathcal{P}}^{2m}$ are irreducible symplectic V-manifolds in the sense of Definition 1.3, it remains to prove their simple connectedness. We will start by the case $m = 0$, in which we will use a certain rational map $\Phi : S^{[2]} \dashrightarrow \boldsymbol{\mathcal{P}}^0$, an analog of the Abel-Jacobi map for Prym varieties. Recall some notation from Section 1: $\tau : S \to S$ is the Galois involution of the double cover $\rho : S \to X$, $\mu : X \to \mathbb{P}^2$ is the double cover map, $B \subset X$ (resp. $\Delta \subset S$) the ramification curve of μ (resp. ρ), $B_0 = \mu(B)$, $\Delta_0 = \rho(\Delta)$. Let $\xi \in S^{[2]}$ be generic. Then ξ is a pair of distinct points, $\xi = \{p_1, p_2\}$, and the line $\ell_\xi = \langle \mu\rho(\xi) \rangle$ spanned by $\mu\rho(p_1), \mu\rho(p_2)$ in $\mathbb{P}^2$ is well-defined. Let $C_\xi = (\mu\rho)^{-1}(\ell_\xi)$. Then $\mathrm{Prym}(C_\xi, \tau|_{C_\xi})$ is a subvariety of $\boldsymbol{\mathcal{P}}^0$, the fiber $f^{-1}(\{\ell_\xi\})$, where $f : \boldsymbol{\mathcal{P}}^0 \to \mathbb{P}^{2\vee}$ is the natural map and $\{\ell\}$ denotes the point of $\mathbb{P}^{2\vee}$ representing a line $\ell \subset \mathbb{P}^2$. Define

$$\Phi : S^{[2]} \dashrightarrow \boldsymbol{\mathcal{P}}^0, \quad \xi = \{p_1, p_2\} \mapsto \sum_{i=1}^{2} [p_i - \tau(p_i)] \in \mathrm{Prym}(C_\xi, \tau|_{C_\xi}).$$

Obviously, Φ is dominant. To describe the fibers of Φ, we will introduce the involution

$$\iota_0 : S^{[2]} \longrightarrow S^{[2]}, \quad \xi \mapsto \xi' = (\langle \xi \rangle \cap S) - \xi.$$

Here S is considered in its embedding as a quartic surface in $\mathbb{P}^3$, given by the linear system $|H|$, $\langle \xi \rangle$ stands for the line in $\mathbb{P}^3$ spanned by ξ, and ξ' is the residual intersection of $\langle \xi \rangle$ with S. This involution is regular whenever S contains no lines, which is the case for sufficiently generic S (see Lemma 1.1). Further, τ induces on $S^{[2]}$ an involution which we will denote by the same symbol. As τ on S is the restriction of a linear involution on $\mathbb{P}^3$, ι_0 commutes with τ, and the composition $\iota_1 = \iota_0 \circ \tau$ is also an involution.

Lemma 5.2. *Φ is a rational double covering with Galois involution ι_1, so that the quotient $M = S^{[2]}/\iota_1$ is birational to $\boldsymbol{\mathcal{P}}^0$.*

Proof. Let $\xi = \{p_1, p_2\}$ be generic. We have to determine all the divisors $p_1' + p_2'$ on C_ξ such that $p_1 - \tau(p_1) + p_2 - \tau(p_2) \sim p_1' - \tau(p_1') + p_2' - \tau(p_2')$. Assume this relation satisfied, and set $\delta = p_1 + p_2 + \tau(p_1') + \tau(p_2')$, $\delta' = p_1' + p_2' + \tau(p_1) + \tau(p_2)$. Then either $\delta' \neq \delta$ and $\dim |\delta| > 0$, or $\delta = \delta'$.

Let us consider the first case. There are three subcases:

(1) $\dim|\delta| = 2$. Then $\delta, \delta' \sim K = K_{C_\xi}$ are intersections of C_ξ, considered as a plane quartic, with two different lines L_1, L_2, and $\tau(p_1' + p_2')$ is uniquely determined as the residual intersection $(L_1 \cap C_\xi) - p_1 - p_2$. Thus there is a unique solution $p_1' + p_2'$, different from $p_1 + p_2$: $p_1' + p_2' = \tau((L_1 \cap C_\xi) - p_1 - p_2) = \iota_1(p_1 + p_2)$.

(2) $\dim|\delta| = 1$ and $|\delta|$ is base point free. Then there exist 4 points $\tilde{\delta}$ on C_ξ, such that no three of them are aligned, and $|\delta|$ consists of the residual intersections $(q \cap C_\xi) - \tilde{\delta}$, where q runs over the pencil of conics $|2H - \tilde{\delta}|$ in the plane $\langle C_\xi \rangle$ spanned by C_ξ. Remark that τ acts as a linear involution on this plane with fixed line L_τ, and when q runs over $|2H - \tilde{\delta}|$, the symmetric conic $\tau(q)$ runs over another pencil of the same type, $|2H - \tau(\tilde{\delta})|$. As $\delta' = \tau(\delta)$, δ' belongs to both pencils, hence the two pencils coincide. We conclude that $\tilde{\delta}$ is τ-invariant, and hence every conic in $|2H - \tilde{\delta}|$ is τ-invariant. Hence $p_1' + p_2' = p_1 + p_2$, which is absurd.

(3) $\dim|\delta| = 1$ and $|\delta|$ has a base point. There are two points $u, v \in C_\xi$ such that $|\delta| = \{u - v + L \cap C_\xi\}$, where the line L runs over the pencil $|H - v|$. As $\delta' = \tau(\delta)$, $u, v \in L_\tau \cap C_\xi$, hence either $\{p_1, p_2\} \cap L_\tau \cap C_\xi \neq \varnothing$, or p_1, p_2 are aligned with one of the 4 points of $L_\tau \cap C_\xi$. In both cases ξ is non-generic, which contradicts our assumption.

It remains to consider the second case $\delta = \delta'$. Then δ is τ-invariant, and modulo the transpositions $p_1 \leftrightarrow p_2$, $p_1' \leftrightarrow p_2'$, there are only two possibilities for which $p_1' + p_2' \neq p_1 + p_2$:

(a) $p_i' = \tau(p_i)$, $i = 1, 2$. Then $2(p_1 + p_2) \sim 2(\tau(p_1) + \tau(p_2))$, hence $p_1 + p_2$ is nongeneric.

(b) $p_1' = p_1$, $p_2' = \tau(p_2)$. Then $2p_2 \sim 2\tau(p_2)$, hence $p_1 + p_2$ is nongeneric.

We conclude that the generic fiber of Φ consists of two elements: $\{\xi, \iota_1(\xi)\}$. □

Lemma 5.3. *The fixed locus* $\mathrm{Fix}(\iota_1)$ *of* ι_1 *is the union of a nonsingular irreducible surface* $\Sigma \subset S^{[2]}$ *and of* 28 *isolated points.*

Proof. It is obvious that the fixed point set of any biregular involution on a smooth variety is also smooth. Consider S as a quartic in $\mathbb{P}^3$. As τ has invariant curves in the linear system of hyperplane sections H, it acts linearly on $\mathbb{P}^3$, and its fixed locus is the union of a plane H_τ and a point $\infty_\tau \notin S$. If $\iota_1(\xi) = \xi$, then the line $\langle \xi \rangle$ is τ-invariant. Hence either $\langle \xi \rangle \subset H_\tau$, or $\langle \xi \rangle$ passes through ∞_τ. The first case provides the 28 isolated points of $\mathrm{Fix}(\iota_1)$, each of them being the pair of tangency points of a bitangent to the plane quartic $\Delta_0 = H_\tau \cap S$. The second case provides the remaining part of $\mathrm{Fix}(\iota_1)$:

$$\Sigma = \{\ \xi \in S^{[2]} \mid \infty_\tau \in \langle\xi\rangle,\ \tau(\xi) \neq \xi\}.$$

Let us call the lines through ∞_τ vertical. A generic vertical line L meets S in 4 points which represent one fiber of $\mu\rho$. These 4 points form 6 pairs. When we vary L, the 6 pairs sweep a surface $\tilde{\Sigma} \subset S^{[2]}$, a 6-sheeted covering of $\mathbb{P}^2$. Two of the 6 pairs are τ-invariant, so $\tilde{\Sigma}$ contains an irreducible component Σ_0 which is a double covering of $\mathbb{P}^2$ and is identified with $X = S/\tau$. The other 4 pairs sweep Σ, a 4-sheeted covering of $\mathbb{P}^2$, and we have $\tilde{\Sigma} = \Sigma \cup \Sigma_0$. If we assume that Σ is reducible, then the two components of Σ would meet along the curve (identified with $\rho^{-1}(B)$) of pairs of tangency points of the vertical bitangents to S. This would contradict the smoothness of Σ. Hence Σ is irreducible. $\square$

Proposition 5.4. *The varieties $\boldsymbol{\mathcal{P}}^0$ and $M = S^{[2]}/\iota_1$ are simply connected.*

Proof. We will first prove that M and its resolution of singularities $\tilde{M}$ are simply connected. Denote by Ψ the quotient map $S^{[2]} \to M$. Let $F = \mathrm{Fix}(\iota_1)$ and $\overline{F}$ the image of F in M. Choose any point $z_0 \in F$ and denote by $\overline{z}_0$ its image under Ψ. Then any loop based at $\overline{z}_0$ lifts to a loop based at z_0, just because z_0 is the unique preimage of $\overline{z}_0$. Hence the map $\Psi_* : \pi_1(S^{[2]}, z_0) \to \pi_1(M, \overline{z}_0)$ is surjective. But $\pi_1(S^{[2]}, z_0) = 1$, so M is simply connected.

The singularities of M are analytically equivalent to $(\mathbb{C}^4/\{\pm 1\}, 0)$ at the 28 isolated points of $\overline{F}$ and to $((\mathbb{C}^2/\{\pm 1\}) \times \mathbb{C}^2, 0)$ along $\overline{\Sigma} = \Psi(\Sigma)$. Thus a resolution of singularities can be obtained by a single blowup $\sigma : \tilde{M} \to M$ with center $\overline{F}$, and the fibers of σ over the points of $\overline{F}$ are the projective spaces $\mathbb{P}^3$ and $\mathbb{P}^1$. Hence σ does not change the fundamental group and $\tilde{M}$ is simply connected. Similarly, the blowup $\tilde{\boldsymbol{\mathcal{P}}}^0 \to \boldsymbol{\mathcal{P}}^0$ of the 28 singular points of $\boldsymbol{\mathcal{P}}^0$ pastes in 28 copies of $\mathbb{P}^3$ and hence does not change the fundamental group. We have obtained two complete smooth varieites $\tilde{\boldsymbol{\mathcal{P}}}^0$, $\tilde{M}$ which are birational. It follows that their fundamental groups are isomorphic. This can be deduced from the Weak Factorization Theorem [AKMW], saying that a birational map between complete smooth varieites over an algebraically closed field of characteristic 0 decomposes into blowups with smoth centers or their inverses, and from an obvious observation that a blowup of a smooth variety with smooth center does not change the fundamental group. We have thus proved the simple connectedness of $\boldsymbol{\mathcal{P}}^0$. $\square$

Lemma 5.5. *Let* $\mathbf{G} \subset \boldsymbol{\mathcal{P}}^0$ *be the open subscheme parametrizing invertible sheaves on the curves* C_t, $t \in T$, *where* $T = \mathbb{P}^{2\vee}$; *it is a group scheme over* T *with a regular action on* $\boldsymbol{\mathcal{P}}^0$. *Let* $\boldsymbol{\mathcal{G}}$ *denote the sheaf of cross-sections of* $\mathbf{G}$ *in the étale topology over* T, *and* $\boldsymbol{\mathcal{G}}_2$ *the constructible subsheaf of* 2*-torsion points. Then there exists a* 1*-cocycle* β *representing an element of* $H^1_{\text{ét}}(T, \boldsymbol{\mathcal{G}}_2)$ *such that* $\boldsymbol{\mathcal{P}}^2 \simeq \boldsymbol{\mathcal{P}}^0 \times_{\mathbf{G}} \mathbf{G}^\beta$, *where* $\mathbf{G}^\beta$ *is the* $\mathbf{G}$*-torsor defined by* β.

Proof. The theta-characteristics of the curves C_t, that is, invertible sheaves θ on C_t such that $\theta^{\otimes 2} \simeq \omega_{C_t}$, form a constructible sheaf Θ with finite stalks over T. Let Θ^τ denote the subsheaf of τ-invariant theta-characteristics. As we saw in the proofs of Lemmas 3.5 and 4.2 (i), Θ^τ has nonempty stalks at all the points $t \in T$. Thus there exists an étale covering $(i_j : U_j \to T)_{j \in J}$ together with local sections $\theta_j \in \Gamma(U_j, i_j^* \Theta^\tau)$. The translation by θ_j defines an isomorphism $T(\theta_j) : \boldsymbol{\mathcal{P}}^0_{U_j} \xrightarrow{\sim} \boldsymbol{\mathcal{P}}^2_{U_j}$. We can define the cocycle $\beta = (\beta_{jk})$ over $U_{jk} = U_j \times_T U_k$ by $\beta_{jk} = \mathrm{pr}_j^* \theta_j \otimes (\mathrm{pr}_k^* \theta_k)^{-1}$, where $U_{jk} \overset{\mathrm{pr}_j}{\underset{\mathrm{pr}_k}{\rightrightarrows}} {U_j \atop U_k}$ are natural projections. $\square$

Proposition 5.6. $\boldsymbol{\mathcal{P}}^2$ *is simply connected.*

Proof. Let $\boldsymbol{\mathcal{P}}$ denote either one of the varieties $\boldsymbol{\mathcal{P}}^0$ or $\boldsymbol{\mathcal{P}}^2$, $f : \boldsymbol{\mathcal{P}} \to \mathbb{P}^{2\vee}$ the natural map, $D = B_0 \cap \overline{\Delta}_0$ the discriminant divisor of f, $U = \mathbb{P}^{2\vee} \setminus D$, $E = f^{-1}(D)$, $V = \boldsymbol{\mathcal{P}} \setminus E$, so that $f_V = f|_V : V \to U$ is a smooth projective morphism. Then f_U is a locally trivial fiber bundle in the C^∞-category with general fiber P_t, and there is an exact sequence of homotopy groups:

$$\ldots \to \pi_2(U) \xrightarrow{\partial} \pi_1(P_t) \xrightarrow{\epsilon} \pi_1(V) \to \pi_1(U) \to 1.$$

It allows us to identify $\pi_1(P_t)/\operatorname{im}\partial$ with a subgroup of $\pi_1(V)$. Let $(c_j)_{j\in J}$ be any generating system for $\pi_1(P_t)/\operatorname{im}\partial$. Let us also fix one lift $\tilde{\gamma}$ in $\pi_1(V)$ for each element γ of $\pi_1(U)$ different from 1. Then, by Proposition 0.2 of [Lei] and taking into account the fact that $\pi_1(\mathbb{P}^{2\vee}) = 1$, we obtain a surjection $\pi_1(P_t)/\operatorname{im}\partial \twoheadrightarrow \pi_1(\boldsymbol{\mathcal{P}})$ whose kernel is generated, as a normal subgroup, by the following set of commutators:

$$R = \big\{ \, [\tilde{\gamma}, c_j] \;\; | \;\; j \in J, \; \gamma \in \pi_1(U) \setminus \{1\} \big\}. \tag{9}$$

The description of R in our case simplifies drastically due to the fact that $\pi_1(P_t) \simeq \mathbb{Z}^4$ is abelian. As $\pi_1(P_t) = H_1(P_t, \mathbb{Z})$, the monodromy action $M : \pi_1(U) \to \operatorname{Aut}\pi_1(P_t)$ is well-defined, and for any $c \in \pi_1(P_t)$, we have $\tilde{\gamma}\epsilon(c)\tilde{\gamma}^{-1} = M_\gamma(c)$ (as above, $f_{V*}(\tilde{\gamma}) = \gamma \in \pi_1(U) \setminus \{1\}$). Thus we can write $\pi_1(\boldsymbol{\mathcal{P}}) \simeq \pi_1(P_t)/N$, where $N = <R_1, R_2>_{\mathrm{norm}}$ is the normal subgroup of $\pi_1(P_t)$ generated by the two sets of elements:

R_1 : the elements of the form $M_\gamma(c_j)c_j^{-1}$, where γ runs over $\pi_1(U)\setminus\{1\}$, and (c_j) is a basis of $\pi_1(P_t)$ $(j=1,\dots,4)$;

R_2 : the image of any generating subset of $\pi_2(U)$.

We will show that if $\boldsymbol{\mathcal{P}}=\boldsymbol{\mathcal{P}}^2$, then R_1 generates the whole of $\pi_1(P_t)$, and thus $\pi_1(\boldsymbol{\mathcal{P}}^2)=1$. By Lemma 5.5, the smooth locus $V=\boldsymbol{\mathcal{P}}^2_U$ of $\boldsymbol{\mathcal{P}}^2/\mathbb{P}^{2\vee}$ can be obtained by gluing together pieces $\boldsymbol{\mathcal{P}}^0_{U_j}$ of $\boldsymbol{\mathcal{P}}^0/\mathbb{P}^{2\vee}$ over $U_i\cap U_j$ via transition maps which are translations in the fibers. A translation in a fiber induces a canonical isomorphism of the homology groups, hence the local systems of the groups $H_1(P_t,\mathbb{Z})$ for $\boldsymbol{\mathcal{P}}^2_U$ and $\boldsymbol{\mathcal{P}}^0_U$ are isomorphic. Thus it suffices to see that R_1 generates the whole of $H_1(P_t,\mathbb{Z})=\pi_1(P_t)$ in the case when $\boldsymbol{\mathcal{P}}=\boldsymbol{\mathcal{P}}^0$. Here we can use Propositioon 0.3 of op. cit. The latter applies to the situation when f has a global cross-section meeting all the components of E, which is the case for the cross-section of neutral elements of the group scheme $\mathbf{G}$ inside $\boldsymbol{\mathcal{P}}^0$. It permits to replace the description of the relations in the fundamental group given in (9) by the following one: $\pi_1(\boldsymbol{\mathcal{P}}^0)=\pi_1(P_t)/<\tilde{R}>_{\mathrm{norm}}$, where $\tilde{R}$ is the set of all the commutators $[c_j,h_k]$, in which c_j (resp. h_k) runs over any set of generators of $\pi_1(P_t)$ (resp. of $\ker(\pi_1(\boldsymbol{\mathcal{P}}^0_U)\to\pi_1(\mathbb{P}^{2\vee}))$. Using the commutativity of $\pi_1(P_t)$, as above, we obtain that $[c_j,h_k]=M_\gamma(c_j)c_j^{-1}$, where $\gamma=f(h_k)\in\pi_1(U)$. Thus for $\boldsymbol{\mathcal{P}}=\boldsymbol{\mathcal{P}}^0$, $<R_1>_{\mathrm{norm}}=<\tilde{R}>_{\mathrm{norm}}=\pi_1(P_t)$, and we are done. □

Corollary 5.7. *The partial resolution of singularities M' of M obtained by blowing up the image of Σ is an irreducible symplectic V-manifold whose singularities are 28 points of analytic type $(\mathbb{C}^4/\{\pm1\},0)$. The natural birational map $\boldsymbol{\mathcal{P}}^0 \dashrightarrow M'$ is the Mukai flop with center at the image $\Pi\simeq\mathbb{P}^2$ of the zero section of $\boldsymbol{\mathcal{P}}^0$, that is, it blows up Π and then blows down the obtained exceptional divisor $\tilde{\Pi}\simeq\mathbb{P}(\Omega^1_\Pi)$ along the second ruling. The image $\Pi'\simeq\mathbb{P}^2$ of $\tilde{\Pi}$ in M' coincides with the proper transform of Σ_0/ι_1.*

Proof. To construct M', we may first blow up Σ and then quotient by ι_1. Let $N=S^{[2]}$, N_1 the blowup of N at Σ, and N_2 the blowup of N_1 at the proper transform of Σ_0. Denote the proper transforms of Σ, Σ_0 in N_2 by Σ', Σ_0' respectively. The curve of intersection $\tilde{B}=\Sigma\cap\Sigma_0$ is a common fixed curve of the pair of regular commuting involutions ι_0, τ on N, hence it is smooth and Σ', Σ_0' intersect transversely along a smooth surface which is a $\mathbb{P}^1$-bundle over $\tilde{B}$. As the two blowups are done at ι_1-invariant centers, ι_1 lifts to a regular involution, denoted by the same symbol, on N_2. The 3-fold Σ_0' and the natural $\mathbb{P}^1$-bundle $\Sigma_0'\to\Sigma_0$ are ι_1-invariant, and $\mathrm{Fix}(\iota_1|_{\Sigma_0'})=\Sigma'\cap\Sigma_0'$. We deduce that the image $\overline{\Sigma}_0'$ of Σ_0' in N_2/ι_1 is smooth and is a $\mathbb{P}^1$-bundle over $\overline{\Sigma}_0 = \Sigma_0/\iota_1$. As we noticed earlier, Σ_0 is identified with X; under this identification, $\iota_1|_{\Sigma_0'} = \iota$, the Galois involution of $\mu : X \to \mathbb{P}^2$. Thus $\overline{\Sigma}_0 \simeq \mathbb{P}^2$ and $\overline{\Sigma}_0' \to \overline{\Sigma}_0$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$.

We have $M' = N_1/\iota_1$, and as the proper transform of Σ_0 in N_1 is isomorphic to Σ_0, $\overline{\Sigma}_0 \simeq \mathbb{P}^2$ embeds naturally into M'. Denote its image in M' by Π'. Then $M'' = N_2/\iota_1$ is nothing but the blowup of M' at Π', and we denote by $\tilde{\Pi}$ the exceptional divisor of this blowup. The fibers of the blowdown map $\tilde{\Pi} \to \Pi'$ represent one ruling of Π', and we are to verify that the map to $\boldsymbol{\mathcal{P}}^0$ contracts another ruling of Π'.

Let $\Phi_2 : N_2 \to \boldsymbol{\mathcal{P}}^0$ be the composition of $N_2 \to N$ with Φ. The indeterminacy locus of Φ consists of those $\xi \in S^{[2]}$ for which ξ is vertical (that is, contained in a fiber of $\mu\rho$). We omit a fastidious calculation in local coordinates on N_2 which shows that the indeterminacy is resolved on N_2, so that Φ_2 is regular. We can represent a point $\hat{\xi} \in N_2$ as a pair (ξ, ℓ_ξ), where ℓ_ξ is a line in $\mathbb{P}^2$ containing $\mu\rho(\xi)$, and the curve $C_{\hat{\xi}} = (\mu\rho)^{-1}(\ell_\xi)$ is well defined. Then Φ_2 contracts to the neutral element $0_{\hat{\xi}}$ of $\mathrm{Prym}(C_{\hat{\xi}}, \tau)$ all the vertical divisors of $C_{\hat{\xi}}$. The latter form a curve, isomorphic to $C_{\hat{\xi}}/\tau \simeq E_{\hat{\xi}} := \mu^{-1}(\ell_\xi)$. Quotienting further by ι_1, we see that the fiber of the induced map $\Phi'' : M'' \to \boldsymbol{\mathcal{P}}^0$ over $0_{\hat{\xi}}$ is $E_{\hat{\xi}}/\iota \simeq \mathbb{P}^1$. Thus Φ'' contracts another ruling of $\tilde{\Pi}$ to the locus Π of neutral elements of the Prymians P_t.

As remarked Mukai [Mu-1], the normal bundle of a plane $\mathbb{P}^2$ in a symplectic 4-fold is isomorphic to $\Omega^1_{\mathbb{P}^2}$, so that the exceptional divisor $\mathbb{P}(\Omega^1_{\mathbb{P}^2})$ of the blowup centered at this $\mathbb{P}^2$ has exactly two different rulings that can be blown down. The map $\Phi' : M' \to \boldsymbol{\mathcal{P}}^0$ induced by Φ blows up Π' and contracts the exceptional divisor along another ruling. It is easily seen, along the lines of the proof of Lemma 5.2, that Φ' is bijective on the complements to Π, Π', and this ends the proof. $\square$

We conclude this section by several miscellaneous remarks.

Remark 5.8. *Odd-degree Prymians.* It is plausible that all the odd-degree Prymians $\overline{\mathrm{Prym}}^{2k+1,\kappa}(\mathcal{C}, \tau)$ contain 3-dimensional rational subvarieties and thus cannot be symplectic. We will produce such a subvariety in degree $2k + 1 = 3$ for κ defined by $c = C_i'$ as in the paragraph preceding Definition 3.3. The case of degree 3 is particularly handy, because $\overline{\mathrm{Pic}}^3(|H|)$ is fiberwise birational to the relative symmetric cube of the linear system $|H|$. For the fiber $\overline{\mathrm{Pic}}^3(C)$ corresponding to the reducible curve $C = C_i' \cup C_i''$, this means that the

Abel-Jacobi map

$$AJ : C^{(3)} \dashrightarrow \overline{\mathrm{Pic}}^{\,3}(C), \quad p_1+p_2+p_3 \mapsto [\mathcal{O}_C(p_1+p_2+p_3)]$$

maps birationally all the 4 components of the symmetric cube $C^{(3)}$ onto the 4 respective components of $\overline{\mathrm{Pic}}^{\,3}(C)$. We will see that $AJ(C_i'^{(2)} \times C_i'') = \overline{J}^{2,1}(C)$ is contained entirely in $\overline{\mathrm{Prym}}^{\,3,\kappa}(\mathcal{C},\tau)$.

Let us suppress the subscript i from the notation, so that $c = C', C = C' \cup C''$. On a typical fiber C_t, the involution ι acts by

$$\begin{aligned} \iota : [x_1+x_2+x_3] &\mapsto [(H+C')\cdot C_t - x_1 - x_2 - x_3] \\ &= [(2H-C'')\cdot C_t - x_1 - x_2 - x_3] \\ &= [q\cdot C_t - z_1'' - z_2'' - x_1 - x_2 - x_3] \\ &= [y_1+y_2+y_3], \end{aligned}$$

where $z_1''+z_2'' = C''\cdot C_t,\ \ q \in |2H - z_1'' - z_2'' - x_1 - x_2 - x_3|$ is a (generically unique) conic passing through the 5 points, and $y_1+y_2+y_3$ is the residual intersection of this conic with C_t. Now assume C_t τ-invariant, that is C_t is of the form $(\mu\rho)^{-1}(\ell_t)$ for a sufficiently general line ℓ_t. For generic $x_1, x_2, x_3 \in C_t$ ("generic" here means: which do not vary in a pencil g^1_3), we have:

$$\begin{aligned} [x_1+x_2+x_3] \in \overline{\mathrm{Prym}}^{\,3,\kappa}(\mathcal{C},\tau) &\iff y_1+y_2+y_3 = \tau(x_1+x_2+x_3) \\ &\iff q \text{ is } \tau\text{-invariant.} \end{aligned}$$

We obtain that the birational transform $\tilde{P}_t$ of $P_t := \overline{\mathrm{Prym}}^{\,3,\kappa}(\mathcal{C},\tau) \cap \mathrm{Pic}^3(C_t)$ in $C^{(3)}$ can be described as follows:

$$\tilde{P}_t = \{x_1+x_2+x_3 \in C^{(3)} \mid \exists\ q \in \mathbb{P}^2_{\tau,t} :\ x_1+x_2+x_3 \in q\},$$

where $\mathbb{P}^2_{\tau,t}$ denotes the 2-dimensional linear system of τ-invariant conics through the two points $z_1''+z_2'' = C''\cdot C_t$ in the plane spanned by C_t. Now let ℓ_t tend to $\ell_0 := \mu\rho(C)$ in the pencil with a fixed intersection point $p = \ell_0 \cap \ell_t$. Then $z_1''+z_2''$ remains fixed, and the limits of $\tilde{P}_t$ contain all the triples $x_1+x_2+x_3$ extracted from the 6 points $q\cdot C - z_1'' - z_2''$, where q runs over the linear system $\mathbb{P}^2_\tau(z_1'', z_2'')$ of τ-invariant conics through z_1'', z_2'' in the plane $\langle C\rangle$. Varying p, and hence the pair $z_1'', z_2'' = \tau(z_1')$, we allow all the triples $x_1+x_2+x_3$ extracted from the 8-tuples $q\cdot C$, where q runs over the linear system $\mathbb{P}^3_\tau$ of all the τ-invariant conics in $\langle C\rangle$, with the only restriction that at least one of the two τ-invariant pairs of points of $q\cap C''$ has empty intersection with $\{x_1, x_2, x_3\}$. Taking generic points $x_1, x_2 \in C'$, $x_3 \in C''$, we find a unique τ-invariant conic through x_1, x_2, x_3, which satisfies the above restriction, and thus $C'^{(2)} \times C''$ lies in the closure of the family of $\tilde{P}_t$. Hence the 3-dimensional rational variety $\overline{J}^{2,1}(C)$ is contained in $\overline{\mathrm{Prym}}^{3,\kappa}(\mathcal{C},\tau)$.

Remark 5.9. *More on the structure of P_t.* In Lemma 3.5 and Proposition 4.3, we only enumerated the strata of the fibers P_t; to determine the topological structure of P_t, one should also describe the adjacencies of these strata. We are going to produce several examples of such calculation.

In the situation of Lemma 3.5, the open piece P_0 of P_t consists of the sheaves

$$\mathcal{F} = \mathcal{F}(0;\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \mathcal{O}_{C_-} \underset{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{\#} \mathcal{O}_{C_+}$$

with $\lambda_1\lambda_2 = \lambda_3\lambda_4$. Let us fix λ_3,λ_4 and make $\lambda_1 \to 0$; then automatically $\lambda_2 \to \infty$. The sheaf $\mathcal{F}$ can be defined as the subsheaf of $\mathcal{O}_{C_-} \oplus \mathcal{O}_{C_+}$, whose stalks at all the points coincide with the stalks of the ambient sheaf, except at z_i, where

$$\mathcal{F}_{z_i} = \{(f_-,f_+) \in \mathcal{O}_{C_-,z_i} \oplus \mathcal{O}_{C_+,z_i} \mid \quad f_-(z_i) = \lambda_i f_+(z_i)\}.$$

Thus the stalks of the limiting sheaf $\mathcal{F}(0;0,\infty,\lambda_3,\lambda_4)$ coincide with the stalks of $\mathcal{F}$ everywhere, except for the stalks $\mathfrak{m}_{C_-,z_1} \oplus \mathcal{O}_{C_+,z_1}$ at z_1 and $\mathcal{O}_{C_-,z_2} \oplus \mathfrak{m}_{C_+,z_2}$ at z_2. Hence

$$\mathcal{F}(0;0,\infty,\lambda_3,\lambda_4) = \mathcal{O}_{C_-}(-z_1) \underset{(\cdot,\cdot,\lambda_3,\lambda_4)}{\#} \mathcal{O}_{C_+}(-z_2),$$

where the bases of the two sheaves used to define the gluings at z_3, z_4 are the functions $1 \in \Gamma(C_\pm, \mathcal{O}_{C_\pm})$ considered as rational sections of $\mathcal{O}_{C_-}(-z_1)$, $\mathcal{O}_{C_+}(-z_2)$. In the same way, we determine the limit when λ_3,λ_4 are fixed and $\lambda_1 \to \infty$:

$$\mathcal{F}(0;\infty,0,\lambda_3,\lambda_4) = \mathcal{O}_{C_-}(-z_2) \underset{(\cdot,\cdot,\lambda_3,\lambda_4)}{\#} \mathcal{O}_{C_+}(-z_1).$$

Changing to the standard bases $e_\pm$ for the sheaves $\mathcal{O}_{C_\pm}(-pt)$, we see that

$$\mathcal{F}(0;0,\infty,\lambda_3,\lambda_4) \simeq \mathcal{F}'(0;\frac{z_3-z_2}{z_3-z_1}\lambda_3,\frac{z_4-z_2}{z_4-z_1}\lambda_4),$$

$$\mathcal{F}(0;\infty,0,\lambda_3,\lambda_4) \simeq \mathcal{F}'(0;\frac{z_3-z_1}{z_3-z_2}\lambda_3,\frac{z_4-z_1}{z_4-z_2}\lambda_4).$$

Similarly, we find the limits when λ_1,λ_2 are fixed and $\lambda_3 \to 0$ or ∞. And when λ_1,λ_3 tend simultaneously to elements of $\{0,\infty\}$, then the limit is the unique sheaf in P_t which is non-invertible at all the 4 points z_i: $\mathcal{O}_{C_-}(-2\,pt) \oplus \mathcal{O}_{C_+}(-2\,pt)$. Finally, we conclude:

In the situation of Lemma 3.5, P_t is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by the following gluings:

– *the horizontal sections* $0 \times \mathbb{P}^1$ *and* $\infty \times \mathbb{P}^1$ *are glued together according to the rule* $(0, \lambda) \sim (\infty, [z_1, z_2; z_3, z_4]^2\lambda)$*;*
– *the vertical sections* $\mathbb{P}^1 \times 0$ *and* $\mathbb{P}^1 \times \infty$ *are glued together according to the rule* $(\lambda, 0) \sim ([z_3, z_4; z_1, z_2]^2\lambda, \infty)$*;*
– *the* 4 *"vertices"* $(0, 0), (0, \infty), (\infty, 0), (\infty, \infty)$ *are glued together.*

Here $[z_1, z_2; z_3, z_4]$ *stands for the cross ratio of* 4 *complex numbers.*

We will also provide the answers for two cases of Proposition 4.3, using the notation used in the proof of this proposition.

Case (i). *The normalization* $\tilde{P}_t$ *of* P_t *is a* $\mathbb{P}^1$*-bundle over the elliptic curve* $E = \mathrm{Prym}(\tilde{C}, \tau)$ *having two distinguished cross-sections* $0, \infty$*. Let* 0_x*,* ∞_x *denote the point of the respective cross-section lying in the fiber over* $x \in E$*. Then* P_t *is obtained from* $\tilde{P}_t$ *by gluing* 0 *to* ∞ *with a translation according to the rule* $0_x \sim \infty_{x+[p''-p']}$*.*

Case (iii). *The normalization* $\tilde{P}_t$ *of* P_t *is a* $\mathbb{P}^1$*-bundle over the elliptic curve* $\tilde{C}$ *having two distinguished cross-sections* $0, \infty$*, and* P_t *is obtained from* $\tilde{P}_t$ *by gluing* 0 *to* ∞ *with a translation according to the rule* $0_x \sim \infty_{x+[p_1'-p_2'-p_1''+p_2'']}$*.*

Case (vii). P_t *is a locally trivial bundle over the elliptic curve* $\tilde{C}$ *with fiber* $\mathbb{P}^1 \bigvee \mathbb{P}^1$*, the bouquet of two copies of* $\mathbb{P}^1$*.*

As concerns the compactified Jacobians of the curves C_t, one can find examples of their calculation in [Cook-2].

Remark 5.10. *Moduli spaces with involution.* One can pursue our approach to constructing new symplectic varieties in a generalized setting: search for pairs $(\mathcal{M}, \kappa)$ formed by a moduli space of sheaves on a K3 surface and a symplectic birational involution. Then one may expect to get new symplectic manifolds either as a (partial) desingularization of the quotient $\mathcal{M}/\kappa$, or as the fixed locus $\mathcal{M}^\kappa$. We can obtain an example of this kind with $\mathcal{M}$ parametrizing non-torsion sheaves by a birational transformation from the compactified Jacobian $\overline{\mathrm{Pic}}^{\,2}(|H|)$ of Section 2. Let C_i' be one of the 56 conics in S, $\mathcal{L} \in \overline{\mathrm{Pic}}^{\,2}(|H|)$ invertible on its support, and $V = \mathrm{Ext}^1_S(\mathcal{L} \otimes \mathcal{O}(-C_i'), \mathcal{O}_S) \simeq \mathbb{C}^2$. Then the ext-group classifying the extensions

$$0 \longrightarrow V^\vee \otimes \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \otimes \mathcal{O}(-C_i') \longrightarrow 0$$

is canonically isomorphic to $\mathrm{Hom}(V, V)$, and we can define a vector bundle $\mathcal{E}$ as the middle term of this extension with extension class $\mathrm{id}_V \in \mathrm{Hom}(V, V)$. This provides a birational isomorphism $\overline{\mathrm{Pic}}^{\,2}(|H|) \dashrightarrow M_S^{H,ss}(2, H, 0)$ in the notation using the Mukai vector $(2, H, 0) =$

$(\operatorname{rk}\mathcal{E}, c_1(\mathcal{E}), \chi(\mathcal{E}) - \operatorname{rk}\mathcal{E})$, and the (regular) symplectic involution κ on $\overline{\operatorname{Pic}}^{\,2}(|H|)$ induces a birational symplectic involution on $M_S^{H,ss}(2,H,0)$.

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