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# ŁOJASIEWICZ-SIMON GRADIENT INEQUALITIES FOR COUPLED YANG-MILLS ENERGY FUNCTIONALS

PAUL M. N. FEEHAN AND MANOUSOS MARIDAKIS

Abstract. In this sequel to [19], we apply our abstract Łojasiewicz-Simon gradient inequality [19, Theorem 1] to prove Łojasiewicz-Simon gradient inequalities for coupled Yang-Mills energy functionals using Sobolev spaces which impose minimal regularity requirements on pairs of connections and sections. The Łojasiewicz-Simon gradient inequalities for coupled Yang-Mills energy functionals generalize that of the pure Yang-Mills energy functional due to the first author [13, Theorem 22.8] for base manifolds of arbitrary dimension and due to Råde [48, Proposition 7.2] for dimensions two and three.

## Contents



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## 1. Introduction

In this sequel to [19], we prove Łojasiewicz-Simon gradient inequalities for coupled Yang-Mills $L^2$ energy functionals using our abstract Łojasiewicz-Simon gradient inequality [19, Theorem 1].

We begin by recalling that a key feature of our version of the Łojasiewicz-Simon gradient inequality for the pure Yang-Mills energy functional [13, Theorem 22.8] is that it holds for $W^{1,p}$ Sobolev norms — considerably weaker than the $C^{2,\alpha}$ Hölder norms originally employed by Simon in [50, Theorem 3] and this affords considerably greater flexibility in applications. For example, when $(X, g)$ is a closed, four-dimensional, Riemannian manifold, the $W^{1,2}$ Sobolev norm on (bundle-valued) one-forms is (in a suitable sense) *quasi-conformally invariant* with respect to conformal changes in the Riemannian metric $g$. In particular, that observation is exploited in our proof of [11, Theorem 1], which asserts discreteness of $L^2$ energies of Yang-Mills connections on arbitrary $G$-principal bundles over $X$, for any compact Lie structure group $G$.

There are essentially three approaches to establishing a Łojasiewicz-Simon gradient inequality for a particular energy functional arising in geometric analysis or mathematical physics: *1*) establish the inequality from first principles, *2*) adapt the argument employed by Simon in the proof of his [50, Theorem 3], or *3*) apply an abstract version of the Łojasiewicz-Simon gradient inequality for an analytic or Morse-Bott functional on a Banach space. The first approach is exactly that employed by Simon in [50] and by Råde for the Yang-Mills energy functional. For this strategy to work well, one desires an abstract Łojasiewicz-Simon gradient inequality with the weakest possible hypotheses and a proof of such a gradient inequality (Theorem 1) is provided in our companion article [19].

In this article, we establish versions of the Łojasiewicz-Simon gradient inequality for coupled Yang-Mills energy functionals (for example, Theorem 5), using systems of Sobolev norms in these applications that are (as best we can tell) as *weak as possible*. In particular, our gradient inequalities use $W^{1,p}$ norms for coupled Yang-Mills pairs over manifolds of arbitrary dimension $d \geq 2$ (including the quasi-conformally invariant case $d = 4$ and $p = 2$).

In the remainder of our Introduction, we review our abstract Łojasiewicz-Simon gradient inequalitiy for an analytic functional on a Banach space in Section 1.1, state our results on existence of global transformation to Coulomb gauge valid for borderline Sobolev exponents in Section 1.2, and state our results on Łojasiewicz-Simon gradient inequalities for coupled Yang-Mills $L^2$ energy functionals in Section 1.3.

We refer to the Introduction of our companion article [19] for a discussion of the history of the Łojasiewicz-Simon gradient inequality and a survey of its many applications in geometric analysis, mathematical physics, and applied mathematics.

### 1.1. Łojasiewicz-Simon gradient inequalities for analytic functionals on Banach spaces.

We begin by recalling from [19] the following generalization of Simon's infinite-dimensional version [50, Theorem 3] of the Łojasiewicz gradient inequality [40]. As we noted in [19], Theorem 1 is stated by Huang as [32, Theorem 2.4.5] but no proof is given and it does not follow from his less general [32, Theorem 2.4.2]. Huang cites [33, Proposition 3.3] for the proof of Theorem 1 but the hypotheses of [33, Proposition 3.3] assume that $X$ is a Hilbert space. That distinction is important because we shall need Theorem 1 when $X$ is a Banach space, as in our application to the coupled Yang-Mills $L^2$ energy functionals in Section 1.3.

**Theorem 1** (Łojasiewicz-Simon gradient inequality for analytic functionals on Banach spaces)**.** [19, Theorem 1] *Let $\mathscr{X}$ be a Banach space that is continuously embedded in a Hilbert space $\mathscr{H}$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be an analytic function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, that is, $\mathscr{E}'(x_\infty) = 0$. Assume that $\mathscr{E}''(x_\infty) : \mathscr{X} \to \mathscr{X}^*$ is a Fredholm operator with index zero. Then there are positive constants, $Z$, $\sigma$, and $\theta \in [1/2, 1)$, with the following significance. If $x \in \mathscr{U}$ obeys*

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{1.1}$$

*then*

$$\|\mathscr{E}'(x)\|_{\mathscr{X}^*} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta. \tag{1.2}$$

A survey of the history of the Łojasiewicz-Simon gradient inequality is provided in [19].

### 1.2. Automorphisms and transformation to Coulomb gauge.

For some energy functionals, the associated Hessian is already an elliptic second-order partial differential operator on a Sobolev space, but for others the Hessian is only elliptic when combined with a type of Coulomb gauge condition [10, 22] and it is only then that one can apply Theorem 1. For example, in the first category, one has the harmonic map energy and Yamabe functionals, while in the second category one has the Yang-Mills and coupled Yang-Mills energy functionals.

The Yang-Mills energy functional is invariant under the action of gauge transformations (or bundle automorphisms) and so, in principle, one can always find a gauge transformation to produce the required Coulomb gauge condition with the aid of a *slice theorem*. However, in order to prove the most useful version of the Łojasiewicz-Simon gradient inequality, it is convenient to have a stronger version of the slice theorem for the action of the group of gauge transformations, going beyond the usual statements found in standard references such as Donaldson and Kronheimer [10] or Freed and Uhlenbeck [22] and proved by applying the Implicit Function Theorem. One stronger version of a slice theorem, valid in dimension four, was proved by the first author as [14, Theorem 1.1]. A purpose of this article is to prove a stronger version of [14, Theorem 1.1] for both connections and pairs rather than just connections as in [14], but using standard Sobolev norms rather than the critical-exponent norms employed in [14] and valid in all dimensions.

1.2.1. *Transformation to Coulomb gauge.* We first state the desired result for connections and then its analogue for pairs.

**Theorem 2** (Existence of $W^{2,q}$ Coulomb gauge transformations for $W^{1,q}$ connections that are $W^{1,\frac{d}{2}}$ close to a reference connection)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$. If $A_1$ is a $C^\infty$ connection on $P$, and $A_0$ is a Sobolev connection on $P$ of class $W^{1,q}$ with $d/2 < q < \infty$, and $p \in (1,\infty)$ obeys $d/2 \leq p \leq q$, then there exists a constant $\zeta = \zeta(A_0, A_1, g, G, p, q) \in (0,1]$ with the following significance. If $A$ is a $W^{1,q}$ connection on $P$ that obeys*

$$\|A - A_0\|_{W^{1,p}_{A_1}(X)} < \zeta, \tag{1.3}$$

*then there exists a gauge transformation $u \in \operatorname{Aut}(P)$ of class $W^{2,q}$ such that*

$$d^*_{A_0}(u(A) - A_0) = 0,$$

*and*

$$\|u(A) - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\|A - A_0\|_{W^{1,p}_{A_1}(X)},$$

*where $N = N(A_0, A_1, g, G, p, q) \in [1,\infty)$ is the constant in the forthcoming Proposition 2.11.*

For a description of the action of the group of gauge transformations in Theorem 2 and the definition of the Coulomb gauge condition for connections, we refer the reader to Section 2.6, and for an explanation of the remainder of the notation in Theorem 2, we refer the reader to Section 1.3.1.

The essential point in Theorem 2 is that the result holds for the critical exponent, $p = d/2$ with $d \geq 3$, when the Sobolev space $W^{2,p}(X)$ fails to embed in $C(X)$ (see [3, Theorem 4.12]) and a proof of Theorem 2 by the Implicit Function Theorem in the case $p > d/2$ fails when $p = d/2$. In this situation, a $W^{2,\frac{d}{2}}$ gauge transformation $u$ of $P$ is not continuous, the set $\operatorname{Aut}^{2,\frac{d}{2}}(P)$ of $W^{2,\frac{d}{2}}$ gauge transformations is not a manifold, and $\operatorname{Aut}^{2,\frac{d}{2}}(P)$ cannot act smoothly on the affine space $\mathscr{A}^{1,\frac{d}{2}}(P)$ of $W^{1,\frac{d}{2}}$ connections on $P$. When $d = 4$ and $p \geq 2$, this phenomenon is discussed by Freed and Uhlenbeck in [22, Appendix A].

The proof of Theorem 2 adapts *mutatis mutandis* to establish the following refinement of the [15, Proposition 2.8] and [45, Theorem 4.1].

**Theorem 3** (Existence of $W^{2,q}$ Coulomb gauge transformations for $W^{1,q}$ pairs that are $W^{1,\frac{d}{2}}$ close to a reference pair)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho : G \to \operatorname{Aut}_{\mathbb{C}}(\mathbb{E})$. If $A_1$ is a $C^\infty$ connection on $P$, and $(A_0, \Phi_0)$ is a Sobolev pair on $(P,E)$ of class $W^{1,q}$ with $d/2 < q < \infty$, and $p \in (1,\infty)$ obeys $d/2 \leq p \leq q$, then there exists a constant $\zeta = \zeta(A_1, A_0, \Phi_0, g, G, p, q) \in (0,1]$ with the following significance. If $(A,\Phi)$ is a $W^{1,q}$ pair on $(P,E)$ that obeys*

$$\|(A,\Phi) - (A_0,\Phi_0)\|_{W^{1,p}_{A_1}(X)} < \zeta, \tag{1.4}$$

*then there exists a gauge transformation $u \in \operatorname{Aut}(P)$ of class $W^{2,q}$ such that*

$$d^*_{A_0,\Phi_0}(u(A,\Phi) - (A_0,\Phi_0)) = 0,$$

*and*

$$\|u(A,\Phi) - (A_0,\Phi_0)\|_{W^{1,p}_{A_1}(X)} < 2N\|(A,\Phi) - (A_0,\Phi_0)\|_{W^{1,p}_{A_1}(X)},$$

*where $N = N(A_1, A_0, \Phi_0, g, G, p, q) \in [1,\infty)$ is the constant in the forthcoming Proposition 2.19.*

For a description of the action of the group of gauge transformations in Theorem 3 and the definition of the Coulomb gauge condition for pairs, we refer the reader to Section 2.8.

1.2.2. *Real analytic Banach manifold structures on quotient spaces.* In order to establish the analyticity of the pure or coupled Yang-Mills $L^2$ energy functionals on affine spaces of $W^{1,q}$ connections $\mathscr{A}^{1,q}(P)$ or pairs $\mathscr{P}^{1,q}(P,E)$, respectively, it is not necessary to know that their quotient spaces with respect to the action of the group $\mathrm{Aut}^{2,q}(P)$ of gauge transformations are analytic Banach manifolds. Nevertheless, because this readily follows from the proofs of Theorem 2 and Theorem 3, respectively, we include the relevant statements here for the case of connections, noting that the analogous statements for pairs are similar.

Theorem 2 provides the essential ingredient one needs to show not only that the quotient space $\mathscr{B}(P) := \mathscr{A}^{1,q}(P)/\mathrm{Aut}^{2,q}(P)$ is a $C^\infty$ but also a *real analytic* Banach manifold away from orbits $[A] = \{u(A) : u \in \mathrm{Aut}^{2,q}(P)\}$ corresponding to $W^{1,q}$ connections $A$ on $P$ whose stabilizers (or isotropy groups), $\mathrm{Stab}(A) := \{u \in \mathrm{Aut}^{2,q}(P) : u(A) = A\}$, are non-minimal, that is, contain the $\mathrm{Center}(G)$ as a proper subgroup. To show that

$$\mathscr{B}^*(P) = \left\{A \in \mathscr{A}^{1,q}(P) : \mathrm{Stab}(A) = \mathrm{Center}(G)\right\} / \mathrm{Aut}^{2,q}(P),$$

is a $C^\infty$ Banach manifold, one only needs the 'easy case' of Theorem 2 where $p = q$, as the condition $q > d/2$ ensures that the proofs using $H^{k+1}(X)$ Sobolev spaces (with $d = 4$ and $k \geq 2$) due to Donaldson and Kronheimer [10, Sections 4.2.1 and 4.2.2] or Freed and Uhlenbeck [22, pp. 48-51] apply *mutatis mutandis*. We have the following analogue of [10, Proposition 4.2.9], [22, Corollary, p. 50], for real analytic Banach manifolds and $X$ of dimension $d \geq 2$ rather than $C^\infty$ Hilbert manifolds and $X$ of dimension four.

**Corollary 4** (Real analytic Banach manifold structure on the quotient space of $W^{1,q}$ connections)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$, and $q$ obey $d/2 < q < \infty$. If $A_1$ is a $C^\infty$ reference connection on $P$ and $[A] \in \mathscr{B}(P)$, then there is a constant $\varepsilon = \varepsilon(A_1, [A], g, G, q) \in (0, 1]$ with the following significance. If*

$$\mathbf{B}_A(\varepsilon) := \left\{ a \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P) : d_A^* a = 0 \text{ and } \|a\|_{W^{1,q}_{A_1}(X)} < \varepsilon \right\},$$

*then the map,*

$$\pi_A : \mathbf{B}_A(\varepsilon)/\mathrm{Stab}_A \ni [a] \mapsto [A + a] \in \mathscr{B}(P),$$

*is a homeomorphism onto an open neighborhood of $[A] \in \mathscr{B}(P)$. For $a \in \mathbf{B}_A(\varepsilon)$, the stabilizer of $a$ in $\mathrm{Stab}_A$ is naturally isomorphic to that of $\pi_A(a)$ in $\mathrm{Aut}^{2,q}(P)$. In particular, the inverse coordinate charts, $\pi_A$, determine real analytic transition functions for $\mathscr{B}^*(P)$, giving it the structure of a real analytic Banach manifold, and each map $\pi_A$ is a real analytic diffeomorphism from the open subset of points $[a] \in \mathbf{B}_A(\varepsilon)/\mathrm{Stab}_A$ where $\pi_A(a)$ has stabilizer isomorphic to $\mathrm{Center}(G)$.*

As in [53, p. 328], one may consider the quotient space of framed connections modulo gauge transformations, $\mathscr{B}'(P) := (\mathscr{A}^{1,q}(P) \times P|_{x_0})/\mathrm{Aut}^{2,q}(P)$, for some fixed base point $x_0 \in X$, and now the obvious analogue of Theorem 2 shows that $\mathscr{B}'(P)$ is a real analytic Banach manifold.

Corollary 4 may be easily extended to the setting of pairs by applying Theorem 3 in place of Theorem 2. We leave such extensions to the reader, but refer to [45, Theorem 4.2] and [15, Proposition 2.8] for statements and proofs of $C^\infty$ Banach manifold structures for quotient spaces of pairs.

1.3. **Łojasiewicz-Simon gradient inequality for coupled Yang-Mills $L^2$ energy functionals.** In this subsection, we summarize consequences of Theorem 1 for coupled Yang-Mills $L^2$ energy functionals.

1.3.1. *Łojasiewicz-Simon gradient inequality for boson and fermion coupled Yang-Mills $L^2$ energy functionals.* We begin with a definition (due to Parker [45]) of two coupled Yang-Mills energy functionals.

**Definition 1.1** (Boson and fermion coupled Yang-Mills energy functionals)**.** [45, Section 2] Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ a smooth principal $G$-bundle over $X$, and $\mathbb{E}$ be a complex finite-dimensional $G$-module equipped with a $G$-invariant Hermitian inner product, $\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ be a unitary representation [8, Definitions 2.1.1 and 2.16], and $E = P \times_{\varrho} \mathbb{E}$ be a smooth Hermitian vector bundle over $X$, and $m$ and $s$ be smooth real-valued functions on $X$.

We define the *boson coupled Yang-Mills $L^2$-energy functional* by

$$\mathscr{E}_g(A, \Phi) := \frac{1}{2} \int_X \left( |F_A|^2 + |\nabla_A \Phi|^2 - m|\Phi|^2 - s|\Phi|^4 \right) \, d\,\mathrm{vol}_g, \tag{1.5}$$

for all smooth connections, $A$ on $P$, and smooth sections, $\Phi$ of $E$, where

$$\nabla_A : C^\infty(X; E) \to C^\infty(T^*X \otimes E),$$

is the covariant derivative induced on $E$ by the connection $A$ on $P$ and $F_A \in \Omega^2(X; \mathrm{ad}P)$ is the curvature of $A$ and $\mathrm{ad}P := P \times_{\mathrm{ad}} \mathfrak{g}$ denotes the real vector bundle associated to $P$ by the adjoint representation of $G$ on its Lie algebra, $\mathrm{Ad} : G \ni u \to \mathrm{Ad}_u \in \mathrm{Aut}(\mathfrak{g})$, with fiber metric defined through the Killing form on $\mathfrak{g}$.

Suppose that $X$ admits a spin$^c$ structure comprising a Hermitian vector bundle $W$ over $X$ and a *Clifford multiplication map*, $c : T^*X \to \mathrm{End}_{\mathbb{C}}(W)$, thus

$$c(\alpha)^2 = -g(\alpha, \alpha) \,\mathrm{id}_W, \quad \forall \, \alpha \in \Omega^1(X), \tag{1.6}$$

and

$$D_A := c \circ \nabla_A : C^\infty(X; W \otimes E) \to C^\infty(X; W \otimes E),$$

is the corresponding *Dirac operator* [38, Appendix D], [36, Sections 1.1 and 1.2], where $\nabla_A$ denotes the covariant derivative induced on $\otimes^n(T^*X) \otimes E$ (for $n \geq 0$) and $W \otimes E$ by the connection $A$ on $P$ and Levi-Civita connection for the metric $g$ on $TX$.

We define the *fermion coupled Yang-Mills $L^2$-energy functional* by

$$\mathscr{F}_g(A, \Psi) := \frac{1}{2} \int_X \left( |F_A|^2 + \langle \Psi, D_A \Psi \rangle - m|\Psi|^2 \right) \, d\,\mathrm{vol}_g, \tag{1.7}$$

for all smooth connections, $A$ on $P$, and smooth sections, $\Psi$ of $W \otimes E$.

We recall from [38, Corollary D.4] that a closed orientable smooth manifold $X$ admits a spin$^c$ structure if and only if the second Stiefel-Whitney class $w_2(X) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ is the mod 2 reduction of an integral class. One calls $W$ the *fundamental spinor bundle* and it carries irreducible representations of $\mathrm{Spin}^c(d)$; when $X$ is even-dimensional, there is a splitting $W = W^+ \oplus W^-$ and Clifford multiplication restricts to give $\rho : T^*X \to \mathrm{Hom}_{\mathbb{C}}(W^\pm, W^\mp)$ [38, Definition D.9].

Although initially defined for smooth connections and sections, the energy functionals $\mathscr{E}_g$ and $\mathscr{F}_g$ in Definition 1.1, extend to the case of Sobolev connections and sections of class $W^{1,2}$.

A short calculation shows that the gradient of the boson coupled Yang-Mills energy functional $\mathscr{E}_g$ in (1.5) with respect to the $L^2$ metric on $\Omega^1(X;\mathrm{ad}P)\oplus C^\infty(X;E)$,

$$(1.8)\qquad \left(\mathscr{E}_g'(A,\Phi),(a,\phi)\right)_{L^2(X,g)} := \left.\frac{d}{dt}\mathscr{E}_g(A+ta,\Phi+t\phi)\right|_{t=0},$$

for all $(a,\phi)\in\Omega^1(X;\mathrm{ad}P)\oplus C^\infty(X;E)$, is given by

$$(1.9)\qquad \begin{aligned}\left(\mathscr{E}_g'(A,\Phi),(a,\phi)\right)_{L^2(X,g)} &= (d_A^*F_A,a)_{L^2(X)} + \mathrm{Re}(\nabla_A^*\nabla_A\Phi,\phi)_{L^2(X)} + \mathrm{Re}(\nabla_A\Phi,\rho(a)\Phi)_{L^2(X)}\\ &\quad - \mathrm{Re}(m\Phi,\phi)_{L^2(X)} - 2\,\mathrm{Re}\int_X s|\Phi|^2\langle\Phi,\phi\rangle\,d\,\mathrm{vol}_g,\end{aligned}$$

where $d_A^* = d_A^{*,g}:\Omega^l(X;\mathrm{ad}P)\to\Omega^{l-1}(X;\mathrm{ad}P)$ is the $L^2$ adjoint of the exterior covariant derivative $d_A:\Omega^l(X;\mathrm{ad}P)\to\Omega^{l+1}(X;\mathrm{ad}P)$, for integers $l\geq l$. We call $(A,\Phi)$ a *boson Yang-Mills pair* (with respect to the Riemannian metric $g$ on $X$) if it is a critical point for $\mathscr{E}_g$, that is, $\mathscr{E}_g'(A,\Phi)=0$.

Similarly, one finds that the gradient of the fermion coupled Yang-Mills energy functional $\mathscr{F}_g$ in (1.7) with respect to the $L^2$ metric on $\Omega^1(X;\mathrm{ad}P)\oplus C^\infty(X;W\otimes E)$,

$$(1.10)\qquad \left(\mathscr{F}_g'(A,\Psi),(a,\psi)\right)_{L^2(X,g)} := \left.\frac{d}{dt}\mathscr{F}_g(A+ta,\Psi+t\psi)\right|_{t=0},$$

for all $(a,\psi)\in\Omega^1(X;\mathrm{ad}P)\oplus C^\infty(X;W\otimes E)$, is given by

$$(1.11)\qquad \left(\mathscr{F}_g'(A,\Psi),(a,\psi)\right)_{L^2(X,g)} = (d_A^*F_A,a)_{L^2(X)} + \mathrm{Re}(D_A\Psi - m\Psi,\psi)_{L^2(X)} + \frac{1}{2}(\Psi,\rho(a)\Psi)_{L^2(X)},$$

where the action of $a\in\Omega^1(X;\mathrm{ad}P)\equiv C^\infty(T^*X\otimes\mathrm{ad}P)$ on $\Psi\in C^\infty(X;W\otimes E)$ is defined by

$$\begin{aligned}&\rho(\alpha\otimes\xi)(\phi\otimes\eta) := c(\alpha)\phi\otimes\varrho_*(\xi)\eta,\\ &\qquad\qquad \forall\,\alpha\in\Omega^1(X),\quad \xi\in C^\infty(X;\mathrm{ad}P),\quad \phi\in C^\infty(X;W),\quad \eta\in C^\infty(X;E),\end{aligned}$$

where $\varrho_*:\mathfrak{g}\to\mathrm{End}_{\mathbb{C}}(\mathbb{E})$ is the representation of the Lie algebra induced by the representation $\varrho:G\to\mathrm{End}_{\mathbb{C}}(\mathbb{E})$ of the Lie group.

We call $(A,\Psi)$ a *fermion Yang-Mills pair* (with respect to the Riemannian metric $g$ on $X$) if it is a critical point for $\mathscr{F}_g$, that is, $\mathscr{F}_g'(A,\Psi)=0$.

Note that both the boson and fermion coupled Yang-Mills $L^2$-energy functionals reduce to the pure *Yang-Mills $L^2$-energy functional* when $\Phi\equiv 0$ or $\Psi\equiv 0$, respectively,

$$(1.12)\qquad \mathscr{E}_g(A) := \frac{1}{2}\int_X |F_A|^2\,d\,\mathrm{vol}_g,$$

and $A$ is a *Yang-Mills connection* (with respect to the Riemannian metric $g$ on $X$) if it is a critical point for $\mathscr{E}_g$, that is,

$$\mathscr{E}_g'(A) = d_A^{*,g}F_A = 0.$$

Given a Hermitian or Riemannian vector bundle, $V$, over $X$ and covariant derivative, $\nabla_A$, which is compatible with the fiber metric on $V$, we denote the Banach space of sections of $V$ of Sobolev class $W^{k,p}$, for any $k\in\mathbb{N}$ and $p\in[1,\infty]$, by $W_A^{k,p}(X;V)$, with norm,

$$(1.13)\qquad \|v\|_{W_A^{k,p}(X)} := \left(\sum_{j=0}^k\int_X|\nabla_A^j v|^p\,d\,\mathrm{vol}_g\right)^{1/p},$$

when $1 \leq p < \infty$ and

$$\|v\|_{W_A^{k,\infty}(X)} := \sum_{j=0}^{k} \operatorname*{ess\,sup}_X |\nabla_A^j v|, \tag{1.14}$$

otherwise, where $v \in W_A^{k,p}(X;V)$. If $k = 0$, then we denote $\|v\|_{W^{0,p}(X)} = \|v\|_{L^p(X)}$. For $p \in [1,\infty)$ and nonnegative integers $k$, we use [3, Theorem 3.12] (applied to $W_A^{k,p}(X;V)$ and noting that $X$ is a closed manifold) and Banach space duality to define

$$W_A^{-k,p'}(X;V) := \left(W_A^{k,p}(X;V)\right)^*,$$

where $p' \in (1,\infty]$ is the dual exponent defined by $1/p + 1/p' = 1$. Elements of the Banach space dual $(W_A^{k,p}(X;V))^*$ may be characterized via [3, Section 3.10] as distributions in the Schwartz space $\mathscr{D}'(X;V)$ [3, Section 1.57].

As our first application of Theorem 1, we have the following generalization of [13, Theorem 22.8] from the case of the pure Yang-Mills energy functional (1.12), when $X$ has arbitrary dimension $d \geq 2$, and Råde's [48, Proposition 7.2], when $X$ has dimension $d = 2$ or 3. Because gauge transformations of class $W^{2,2}$ are continuous when $d = 2$ or 3 and standard versions of the slice theorem [10, Proposition 2.3.4], [22, Theorem 3.2], [37, Theorem 10.4] for the action of gauge transformations are applicable, the proof of the analogue of Theorem 5 for the pure Yang-Mills $L^2$ energy functional due to Råde is simpler for $d = 2, 3$ and $p = 2$ [48, Proposition 7.2].

**Theorem 5** (Łojasiewicz-Simon gradient inequality for the boson coupled Yang-Mills energy functional)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho : G \to \operatorname{Aut}_{\mathbb{C}}(\mathbb{E})$. Let $A_1$ be a $C^\infty$ reference connection on $P$, and $(A_\infty, \Phi_\infty)$ a boson coupled Yang-Mills pair on $(P,E)$ for $g$ of class $W^{1,q}$, with $q \in [2,\infty)$ obeying $q > d/2$. If $p \in [2,\infty)$ obeys $d/2 \leq p \leq q$ and, in addition $p \geq 4d/(d+4)$ for $d = 2,3$, and $p' \in (1,\infty)$ is the dual exponent defined by $1/p + 1/p' = 1$, then the gradient map,*

$$\mathscr{E}_g' : W_{A_1}^{1,p}(X;\Lambda^1 \otimes \operatorname{ad}P) \oplus W_{A_1}^{1,p}(X;E) \to W_{A_1}^{-1,p'}(X;\Lambda^1 \otimes \operatorname{ad}P) \oplus W_{A_1}^{-1,p'}(X;E),$$

*is* real analytic *and there are positive constants $Z \in [1,\infty)$, $\sigma \in (0,1]$, and $\theta \in [1/2,1)$, depending on $A_1$, $(A_\infty, \Phi_\infty)$, $g$, $G$, $p$, and $q$ with the following significance. If $(A,\Phi)$ is a $W^{1,q}$ Sobolev pair on $(P,E)$ obeying the* Łojasiewicz-Simon neighborhood *condition,*

$$\|(A,\Phi) - (A_\infty,\Phi_\infty)\|_{W_{A_1}^{1,p}(X)} < \sigma, \tag{1.15}$$

*then the boson coupled Yang-Mills energy functional* (1.5) *obeys the* Łojasiewicz-Simon gradient inequality

$$\|\mathscr{E}_g'(A,\Phi)\|_{W_{A_1}^{-1,p'}(X)} \geq Z|\mathscr{E}_g(A,\Phi) - \mathscr{E}_g(A_\infty,\Phi_\infty)|^\theta. \tag{1.16}$$

The statement of Theorem 5 simplifies with the addition of the rather mild assumption that $A_1 = A_\infty$ and that $(A_\infty, \Phi_\infty)$ is $C^\infty$ (which can be assumed, modulo a $W^{2,q}$ gauge transformation, provided by the regularity Theorem 2.23).

**Corollary 6** (Łojasiewicz-Simon gradient inequality for the boson coupled Yang-Mills energy functional)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation,*

$\varrho : G \to \operatorname{Aut}_{\mathbb{C}}(\mathbb{E})$. *Let* $(A_\infty, \Phi_\infty)$ *a smooth boson coupled Yang-Mills pair for* $g$ *on* $(P, E)$. *If* $p \in [2, \infty)$ *obeys* $p \geq d/2$ *and, in addition* $p \geq 4d/(d+4)$ *for* $d = 2, 3$, *and* $p' \in (1, \infty)$ *is the dual exponent defined by* $1/p + 1/p' = 1$, *then the gradient map,*

$$\mathscr{E}_g' : W^{1,p}_{A_\infty}(X; \Lambda^1 \otimes \operatorname{ad}P) \oplus W^{1,p}_{A_\infty}(X; E) \to W^{-1,p'}_{A_\infty}(X; \Lambda^1 \otimes \operatorname{ad}P) \oplus W^{-1,p'}_{A_\infty}(X; E),$$

*is* real analytic *and, for* $d/2 < q < \infty$ *obeying* $q \geq p$, *there are positive constants* $Z \in [1, \infty)$, $\sigma \in (0, 1]$, *and* $\theta \in [1/2, 1)$, *depending on* $(A_\infty, \Phi_\infty)$, $g$, $G$, $p$, *and* $q$ *with the following significance. If* $(A, \Phi)$ *is a* $W^{1,q}$ *Sobolev pair on* $(P, E)$ *that obeys the* Łojasiewicz-Simon neighborhood *condition,*

$$\|(A, \Phi) - (A_\infty, \Phi_\infty)\|_{W^{1,p}_{A_\infty}(X)} < \sigma, \tag{1.17}$$

*then the boson coupled Yang-Mills energy functional* (1.5) *obeys the* Łojasiewicz-Simon gradient inequality*,*

$$\|\mathscr{E}_g'(A, \Phi)\|_{W^{-1,p'}_{A_\infty}(X)} \geq Z |\mathscr{E}_g(A, \Phi) - \mathscr{E}_g(A_\infty, \Phi_\infty)|^\theta. \tag{1.18}$$

Similarly, for the fermion coupled Yang-Mills energy functional, we have the

**Theorem 7** (Łojasiewicz-Simon gradient inequality for the fermion coupled Yang-Mills energy functional)**.** *Assume the hypotheses of Theorem 5, except that we require that* $X$ *admit a spin*$^c$ *structure* $(\rho, W)$, *replace the role of* $\mathscr{E}_g$ *in* (1.5) *by* $\mathscr{F}_g$ *in* (1.7), *and replace the role of the pair* $(A, \Phi)$ *and critical point* $(A_\infty, \Phi_\infty)$ *of* $\mathscr{E}_g$ *by the pair* $(A, \Psi)$ *and critical point* $(A_\infty, \Psi_\infty)$ *of* $\mathscr{F}_g$, *where* $\Psi$ *and* $\Psi_\infty$ *are sections of* $W \otimes E$. *Then the conclusions of Theorem 5 hold* mutatis mutandis.

*Remark* 1.2 (Łojasiewicz-Simon gradient inequality for coupled Yang-Mills energy functionals on quotient spaces). We recall that the space of all smooth connections on $P$ is an affine space, $\mathscr{A}(p) = A_1 + \Omega^1(X; \operatorname{ad}P)$. While the energy functionals $\mathscr{E}_g$ and $\mathscr{F}_g$ in Definition 1.1 were initially defined on affine spaces modeled on $\Omega^1(X; \operatorname{ad}P) \oplus C^\infty(X; E)$ or $\Omega^1(X; \operatorname{ad}P) \oplus C^\infty(X; W \otimes E)$, the functionals are invariant under the action of the group of gauge transformations, $\operatorname{Aut}(P)$, and thus descend to the corresponding quotient spaces. The resulting configuration spaces may be given the structure of smooth Banach manifolds in a standard way [10, Sections 4.2.1], [22, Chapter 3], [26, Section 1] and, with minor modifications of standard proofs, the structure of real analytic Banach manifolds as discussed in Section 1.2.2.

We have chosen to derive the Łojasiewicz-Simon gradient inequalities (in Theorems 5 and 7) for two specific coupled Yang-Mills energy functionals, motivated by physical considerations, namely the properties of *regularity*, *naturality*, and *conformal invariance* (in dimension four) described by Parker in [45, Section 2].

However, it is clear from the proofs of Theorems 5 and 7 that one can expect the same conclusions for any $L^2$ energy functional on pairs of connections and sections with the same nonlinearity structure. Indeed, proofs of such results can be obtained by simple modifications of our proof of the Łojasiewicz-Simon gradient inequality for the boson coupled Yang-Mills energy functional, just as we do in this article for the case of the fermion coupled Yang-Mills energy functional.

1.3.2. *Łojasiewicz-Simon gradient inequality for the Yang-Mills-Higgs* $L^2$*-energy functional.* A well-known example in complex differential geometry of a coupled Yang-Mills $L^2$ energy functional is the *Yang-Mills-Higgs* functional, which we now describe. See Bradlow [4, 5], Bradlow and García-Prada [6], Hitchin [28], Hong [29], Li and Zhang [39], and Simpson [51] for additional details and further references.

We shall follow the description by Bradlow and García-Prada [6, Section 3], but refer the reader to Hong [29], Li and Zhang [39], and the cited references for variants of the Yang-Mills-Higgs functional described here. Let $E$ be a complex vector bundle with Hermitian metric $H$ over a compact Kähler manifold $(X, \omega)$. Let $\mathscr{A}_H$ denote the affine space of smooth connections on $E$ that are *unitary* (that is, compatible with the metric $H$), and $\Omega^0(X; E)$ denote the vector space of smooth sections of $E$, and $\tau \in \mathbb{R}$.

One defines the *Yang-Mills-Higgs $L^2$ energy functional* on $(A, \Phi) \in \mathscr{A}_H \times \Omega^0(X; E)$ by

$$\mathscr{E}_{H,\omega,\tau}(A, \Phi) := \frac{1}{2} \int_X \left( |F_A|^2 + 2|\nabla_A \Phi|^2 + |\Phi \otimes \Phi^* - \tau \, \mathrm{id}_E|^2 \right) d\,\mathrm{vol}_\omega, \tag{1.19}$$

where $\Phi^* := \langle \cdot, \Phi \rangle_H$, the dual of $\Phi$ with respect to the metric $H$.

By definition, a *Yang-Mills-Higgs pair* $(A, \Phi)$ is a critical point of the Yang-Mills-Higgs functional $\mathscr{E}_{H,\omega,\tau}$, so $\mathscr{E}'_{H,\omega,\tau}(A, \Phi) = 0$, or equivalently $(A, \Phi)$ satisfies the second-order *Yang-Mills-Higgs equations* (the Euler-Lagrange equations defined by the functional (1.19)). A calculation reveals that a pair is an *absolute minimum* of $\mathscr{E}_{H,\omega,\tau}$ if and only if it obeys the first-order *vortex equations*,

$$\begin{aligned} F_A^{0,2} &= 0, \\ \bar\partial \Phi &= 0, \\ \Lambda F_A &= \sqrt{1} \left( \Phi \otimes \Phi^* - \tau \, \mathrm{id}_E \right), \end{aligned} \tag{1.20}$$

where $\Lambda F_A$ denotes contraction of $F_A$ with $\omega$. Let $\mathfrak{u}(E) \subset \mathrm{End}_\mathbb{C}(E)$ denote the subbundle of skew-Hermitian endomorphisms of $E$.

The proof of Theorem 5 carries over *mutatis mutandis* to give

**Theorem 8** (Łojasiewicz-Simon gradient inequality for the Yang-Mills-Higgs $L^2$ energy functional)**.** *Let $X$ be a compact, Kähler manifold of complex dimension $n \geq 1$ and $E$ be a complex vector bundle with Hermitian metric $H$ over $X$. Let $A_1$ be a smooth reference connection on the principal frame bundle for $E$. Assume that $d = 2n \geq 2$ and $p \in (1, \infty)$ obey one of the conditions in Theorem 5 and let $p' \in (1, \infty)$ be the dual exponent defined by $1/p + 1/p' = 1$. Then the gradient map,*

$$\mathscr{E}'_{H,\omega,\tau} : W^{1,p}_{A_1}(X; \Lambda^1 \otimes \mathfrak{u}(E)) \oplus W^{1,p}_{A_1}(X; E) \to W^{-1,p'}_{A_1}(X; \Lambda^1 \otimes \mathfrak{u}(E)) \oplus W^{-1,p'}_{A_1}(X; E),$$

*is* real analytic *and the remaining conclusions of Theorem 5 hold* mutatis mutandis *for the Yang-Mills-Higgs functional* (1.19).

1.3.3. *Łojasiewicz-Simon gradient inequality for the Seiberg-Witten $L^2$-energy functionals.* For another example of a coupled Yang-Mills energy functional whose absolute minima can be readily identified, we consider the Seiberg-Witten equations.

Expositions of the Seiberg-Witten equations are now provided by many authors but, for the sake of consistency, we shall follow our development in [16]. Let $(\rho, W)$ denote a spin$^c$ structure on a four-dimensional manifold, $X$, with Riemannian metric, $g$. We recall from [16, Equation (2.55)] that a pair $(B, \Psi)$, comprising a spin$^c$ connection, $B$, on $W = W^+ \oplus W^-$ and a section, $\Psi$, of $W^+$ is a *Seiberg-Witten monopole* if

$$\begin{aligned} \mathrm{tr}(F_B^+) - \rho^{-1}(\Psi \otimes \Psi^*)_0 &= 0, \\ D_B \Psi &= 0, \end{aligned} \tag{1.21}$$

recalling that $\rho : \Lambda^+ \cong \mathfrak{su}(W^+)$ is the isomorphism of Riemannian vector bundles induced by Clifford multiplication, $D_B : C^\infty(X; W^+) \to C^\infty(X; W^-)$ is the Dirac operator, and $(\cdot)_0$

denotes the trace-free part of $\Psi \otimes \Psi^* \in \operatorname{End}_{\mathbb{C}}(W^+)$. We have restricted $B$ to $W^+$, so $F_B \in C^\infty(X; \mathfrak{u}(W^+) \otimes \Lambda^+) = \Omega^+(X; \mathfrak{u}(W^+))$ and $\operatorname{tr}(F_B^+) \in C^\infty(X; i\Lambda^+) = \Omega^+(X; i\mathbb{R})$, using the fiberwise trace homomorphism, $\operatorname{tr} : \mathfrak{u}(W^+) \to i\underline{\mathbb{R}}$. The Seiberg-Witten equations (1.21) are a system of first-order partial differential equations in $(B, \Psi)$ and thus cannot be the Euler-Lagrange equations of any action functional. However, as we recall from [30, 34, 43], Seiberg-Witten monopoles have a variational interpretation by an argument which is the reverse of those provided by Bradlow and García-Prada [6, Section 3] or Hong [29, Section 1] in their derivations of the vortex equations or Li and Zhang [39, Section 1] for the Hermitian-Einstein equations.

Thus, from [30, Equation (1.6)] or [43, Propsition 2.1.4], the Seiberg-Witten $L^2$ energy functional is

$$\mathscr{E}_g(B, \Psi) = \int_X \left( |\nabla_B \Psi|^2 + \frac{1}{2} |\operatorname{tr}(F_B)|^2 + \frac{R}{4} |\Psi|^2 + \frac{1}{8} |\Psi|^2 \right) d\operatorname{vol}_g + 2\pi^2 c_1(W^+)^2, \tag{1.22}$$

where $c_1(W^+)^2 := \int_X c_1(W^+)^2$. The topological term, $2\pi^2 c_1(W^+)^2$, is independent of the pair $(B, \Psi)$ and does not affect the critical points. In particular,

$$\mathscr{E}(B, \Psi) \geq 2\pi^2 c_1(W^+)^2,$$

and a pair $(B, \Psi)$ is a Seiberg-Witten monopole if and only if equality is achieved.

Hong and Schabrun derive a version of the Łojasiewicz-Simon gradient inequality [30, Lemma 5.3] based in part on an earlier proof due to Wilkin for the Yang-Mills-Higgs functional over a Riemann surface [59, Proposition 3.5]. However, the proof of Theorem 5 carries over *mutatis mutandis* to give

**Theorem 9** (Łojasiewicz-Simon gradient inequality for the Seiberg-Witten $L^2$ energy functional)**.** *Let $(X, g)$ be a closed, four-dimensional, oriented, Riemannian smooth manifold with spin$^c$ structure $(\rho, W)$. Let $B_1$ be a smooth reference spin$^c$ connection on $W$. Assume that $p \in (1, \infty)$ obeys the hypotheses of Theorem 5 with $d = 4$ and let $p' \in (1, \infty)$ be the dual exponent defined by $1/p + 1/p' = 1$. Then the gradient map,*

$$\mathscr{E}_g' : W_{B_1}^{1,p}(X; i\Lambda^1) \oplus W_{B_1}^{1,p}(X; W^+) \to W_{B_1}^{-1,p'}(X; i\Lambda^1) \oplus W_{B_1}^{-1,p'}(X; W^+),$$

*is* real analytic *and the remaining conclusions of Theorem 5 hold* mutatis mutandis *for the Seiberg-Witten functional* (1.22)*.*

1.3.4. *Łojasiewicz-Simon gradient inequality for the non-Abelian monopole $L^2$-energy functionals.* For our final example of a coupled Yang-Mills energy functional whose absolute minima can be readily identified, we have the non-Abelian monopoles arising in the work of the first author and Leness [16], Okonek and Teleman [44], and Pidstrigatch and Tyurin [46].

Following [16], we consider pairs $(A, \Phi)$ obeying

$$\begin{aligned} (F_A^+)_0 - \rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\ D_A \Phi &= 0, \end{aligned} \tag{1.23}$$

where $A$ is a unitary connection on a Hermitian vector bundle, $E$, with curvature $F_A \in C^\infty(X; \Lambda^2 \otimes \mathfrak{u}(E)) = \Omega^2(X; \mathfrak{u}(E))$ and $(F_A^+)_0 \in C^\infty(X; \Lambda^+ \otimes \mathfrak{su}(E)) = \Omega^+(X; \mathfrak{su}(E))$, while $\rho : \Lambda^+ \otimes \mathfrak{su}(E) \cong \mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$ is the isomorphism of Riemannian vector bundles induced by Clifford multiplication, $D_A : C^\infty(X; W^+ \otimes E) \to C^\infty(X; W^- \otimes E)$ is the Dirac operator, and $(\cdot)_{00}$ denotes the trace-free part of $\Phi \otimes \Phi^* \in \operatorname{End}_{\mathbb{C}}(W^+ \otimes E)$. Let $\mathfrak{su}(E) \subset \operatorname{End}_{\mathbb{C}}(E)$ denote the subbundle of skew-Hermitian, trace-zero endomorphisms of $E$.

By extending the derivations of the Seiberg-Witten $L^2$ energy functional in [30] or [43], we find that the *non-Abelian monopole $L^2$ energy functional* is

$$\begin{aligned}\mathscr{E}_g(A,\Phi) = \int_X \left(|\nabla_A\Phi|^2 + \frac{1}{2}|F_A|^2 + \frac{R}{4}|\Phi|^2 + \frac{1}{8}|\Phi|^2\right) d\,\mathrm{vol}_g \\ - 4\pi^2 c^2(E) + \frac{1}{2}\|F_{A_w}^+\|_{L^2(X)}^2 - 2\|F_{A_e}^+\|_{L^2(X)}^2.\end{aligned} \tag{1.24}$$

The connections $A_e$ on $\det E$ and $A_w$ on $\det W^+$ are fixed, with no dynamical role, so the true variables in the SO(3)-monopole equations are the SO(3) connection $\hat{A}$ induced by $A$ on the bundle $\mathfrak{su}(E)$ and the (spinor) section $\Phi$ of $W^+ \otimes E$. The action functional, $\mathscr{E}_g(A,\Phi)$, again has a universal lower bound and is achieved if and only if $(A,\Phi)$ is a *non-Abelian monopole*, namely a solution to (1.23). Again the proof of Theorem 5 carries over *mutatis mutandis* to give

**Theorem 10** (Łojasiewicz-Simon gradient inequality for the non-Abelian monopole $L^2$ energy functional)**.** *Let $(X,g)$ be a closed, four-dimensional, oriented, Riemannian, smooth manifold with spin$^c$ structure $(\rho,W)$. Let $E$ be a Hermitian vector bundle over $X$, and $A_e$ be a smooth connection on $\det E$, and $B$ be a smooth spin$^c$ connection on $W$, and $A_1$ be a smooth reference connection on $E$ inducing $A_e$ on $\det E$. Assume that $p \in (1,\infty)$ obeys the hypotheses of Theorem 5 with $d=4$ and let $p' \in (1,\infty)$ be the dual exponent defined by $1/p + 1/p' = 1$. Then the gradient map,*

$$\begin{aligned}\mathscr{E}_g' : W_{A_1}^{1,p}(X;\Lambda^1\otimes\mathfrak{su}(E)) \oplus W_{A_1}^{1,p}(X;W^+\otimes E) \\ \to W_{A_1}^{-1,p'}(X;\Lambda^1\otimes\mathfrak{su}(E)) \oplus W_{A_1}^{-1,p'}(X;W^+\otimes E),\end{aligned}$$

*is* real analytic *and the remaining conclusions of Theorem 5 hold* mutatis mutandis *for the non-Abelian monopole functional* (1.24)*.*

Our interest in Łojasiewicz-Simon gradient inequalities for coupled Yang-Mills and harmonic map energy functionals is motivated by the wealth of potential applications. We shall survey some of those applications below.

1.4. **Applications of the Łojasiewicz-Simon gradient inequality for the coupled Yang-Mills energy functionals.** In [13], we apply the Łojasiewicz-Simon gradient inequality for the pure Yang-Mills energy functional [13, Theorem 22.8] to prove global existence, convergence, convergence rate, and stability results for solutions $A(t)$ to the associated gradient flow,

$$\frac{\partial A}{\partial t} = -\mathscr{E}_g'(A(t)), \quad A(0) = A_0,$$

that is,

$$\frac{\partial A}{\partial t} = -d_{A(t)}^{*,g} F_{A(t)}, \quad A(0) = A_0.$$

Given our Łojasiewicz-Simon gradient inequalities for the boson and fermion coupled Yang-Mills energy functionals, Theorems 5 and 7, the main conclusions in [13] for pure Yang-Mills gradient flow carry over to the more general case of coupled Yang-Mills gradient flows. We describe those results in a sequel [18] to the present article.

In [12] and [11], we applied the Łojasiewicz-Simon gradient inequality to prove an energy gap result for Yang-Mills connections with small $L^{d/2}$ energy and, more generally, discreteness of $L^2$ energies of Yang-Mills connections over closed Riemannian smooth manifolds when $d=4$. We extend those results to the case of solutions to the coupled Yang-Mills equations in a sequel [17] to the present article.

1.5. **Outline of the article.** To apply Theorem 1 to pure or coupled Yang-Mills energy functionals and obtain the best possible results in those applications, one requires the global Coulomb gauge constructions provided by Theorems 2 or 3, and those results are proved in Section 2. In Section 3, we derive Łojasiewicz-Simon gradient inequalities for the coupled Yang-Mills energy functionals, proving Theorems 5, 7, 8, 9, 10, and Corollary 6. Appendix A discusses the equivalence of Sobolev norms defined by Sobolev and smooth connections while Appendix B establishes the Fredholm properties and computes the index of a Hodge Laplacian with Sobolev coefficients.

1.6. **Notation and conventions.** For the notation of function spaces, we follow Adams and Fournier [3], and for functional analysis, Brezis [7] and Rudin [49]. We let $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ denote the set of non-negative integers. We use $C = C(*, \dots, *)$ to denote a constant which depends at most on the quantities appearing on the parentheses. In a given context, a constant denoted by $C$ may have different values depending on the same set of arguments and may increase from one inequality to the next. If $\mathscr{X}, \mathscr{Y}$ is a pair of Banach spaces, then $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ denotes the Banach space of all continuous linear operators from $\mathscr{X}$ to $\mathscr{Y}$. We denote the continuous dual space of $\mathscr{X}$ by $\mathscr{X}^* = \mathscr{L}(\mathscr{X}, \mathbb{R})$. We write $\alpha(x) = \langle x, \alpha \rangle_{\mathscr{X} \times \mathscr{X}^*}$ for the pairing between $\mathscr{X}$ and its dual space, where $x \in \mathscr{X}$ and $\alpha \in \mathscr{X}^*$. If $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, then its adjoint is denoted by $T^* \in \mathscr{L}(\mathscr{Y}^*, \mathscr{X}^*)$, where $(T^*\beta)(x) := \beta(Tx)$ for all $x \in \mathscr{X}$ and $\beta \in \mathscr{Y}^*$.

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## 2. Existence of Coulomb gauge transformations for connections and pairs

In Section 2.6, we prove our refinement, Theorem 2, of the standard construction of a $W^{2,q}$ Coulomb gauge transformation $u$, with $q > d/2$, for a $W^{1,q}$ connection $A$ on a principal $G$-bundle $P$ over a closed Riemannian smooth manifold of dimension $d \geq 2$. We extend this result in Section 2.8 to the action of gauge transformations on affine spaces of $W^{1,q}$ pairs, obtaining our refinement, Theorem 3 of the standard constructions of Coulomb gauge transformations in that context due to Parker [45] and the first author and Leness [15]. Finally, in Section 2.9, we extend known regularity results for solutions to the Yang-Mills equations in dimensions greater than or equal to two and coupled Yang-Mills equations in dimension four to the case of solutions to the coupled Yang-Mills equations in dimensions greater than or equal to two.

2.1. **Action of Sobolev gauge transformations on Sobolev connections.** Suppose that $P$ is a smooth principal $G$-bundle over a smooth manifold $X = P/G$ of dimension $d$, where $P \times G \to P$ is a right action of $G$ on $P$. For $q > d/2$, let $\mathrm{Aut}^{2,q}(P)$ denote the Banach Lie group of Sobolev $W^{2,q}$ automorphisms (or gauge transformations) of $P$ [10, Section 2.3.1], [22, Appendix A and p. 32 and pp. 45–51], [23, Section 3.1.2]. We recall that there is a smooth left action,

$$\mathrm{Aut}^{2,q}(P) \times P \to P,$$

which commutes with the right action of $G$ on $P$. This induces a smooth right (affine) action on the affine space $\mathscr{A}^{1,q}(P)$ of Sobolev $W^{1,q}$ connections on $P$,

$$\mathscr{A}^{1,q}(P) \times \mathrm{Aut}^{2,q}(P) \ni (A,u) \to u(A) \in \mathscr{A}^{1,q}(P), \tag{2.1}$$

defined by pull-back,

$$u(A) := u^*A, \quad \forall\, u \in \mathrm{Aut}^{2,q}(P) \text{ and } A \in \mathscr{A}^{1,q}(P).$$

The constraint $q > d/2$ is required to ensure that $W^{2,q}(X) \subset C(X)$ by the Sobolev Embedding [3, Theorem 4.12] and thus $u \in \mathrm{Aut}^{2,q}(P)$ is a continuous gauge transformation of $P$ and that $W^{2,q}(X)$ is a Banach algebra by [3, Theorem 4.39].

Given a $W^{1,q}$ connection $A_0$ on $P$, the standard construction of a slice for the action of $\mathrm{Aut}^{2,q}(P)$ on $\mathscr{A}^{1,q}(P)$ provides constants $\varepsilon = \varepsilon(A_0, g, P) \in (0, 1]$ and $C = C(A_0, g, P) \in [1,\infty)$ such that if $A$ is close to $A_0$ in the sense that,

$$\|A - A_0\|_{W^{1,q}_{A_0}(X)} < \varepsilon,$$

then there exists $u \in \mathrm{Aut}^{2,q}(P)$ such that $u(A)$ is in *Coulomb gauge relative to* $A_0$, that is,

$$d^*_{A_0}(u(A) - A_0) = 0,$$

and $u(A)$ is close to $A_0$,

$$\|u(A) - A_0\|_{W^{1,q}_{A_0}(X)} < C\varepsilon.$$

For example, see [10, Proposition 2.3.4], [22, Theorem 3.2], [37, Theorem 10.4] or [14, Theorem 1.1] for statements of the Slice Theorem and their proofs using the Implicit Function Theorem for smooth maps of Banach spaces.

Our Theorem 2 relaxes the condition that $A$ be $W^{1,q}_{A_0}$ close to $A$ for $q > d/2$ to $W^{1,p}_{A_0}$ close for $p$ obeying $d/2 \leq p \leq q$ when $d \geq 3$ and provides $W^{1,p}$ bounds for $u(A) - A_0$ in terms of $A - A_0$. This is significant since $\mathrm{Aut}^{2,\frac{d}{2}}(P)$ is not a smooth manifold and the action (2.1) cannot be smooth when $q = d/2$, so the Implicit Function Theorem does not apply.

2.2. ***A priori* estimates for Laplace operators with Sobolev coefficients and existence and uniqueness of strong solutions.** Before proceeding to the proof of Theorem 2, we begin with some preparatory lemmata and remarks that have some interest in their own right. Standard theory for existence and uniqueness of strong solutions to (scalar) second-order elliptic partial differential equations, such as [24, Chapter 9], requires that the second-order coefficients be continuous and the lower-order coefficients be bounded. Here, we observe that one can relax those requirements on the lower-order coefficients and accommodate the setting we employ in this article.

For a smooth connection $A$ on $P$ and integers $l \geq 0$, we let

$$\Delta_A = d^*_A d_A + d_A d^*_A \quad \text{on } \Omega^l(X; \mathrm{ad}P) \tag{2.2}$$

denote the *Hodge Laplace operator*. Our proof of Theorem 2 will require *a priori* $L^p$ estimates, existence and uniqueness results, Fredholm properties, and Hodge decompositions involving the Hodge Laplacian (2.2) when $A$ is a $W^{1,q}$ *Sobolev* connection. When $A$ is a $C^\infty$ connection and we restrict our attention to $p = 2$, those properties are immediate consequences of more general results (for example, see Gilkey [25]) for elliptic operators on sections of vector bundles over closed manifolds.

**Proposition 2.1** (*A priori* $L^p$ estimate for a Laplace operator with Sobolev coefficients)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$, and $l \geq 0$ be an integer. If $A$ is a $W^{1,q}$ connection on $P$ with $q > d/2$, and $A_1$ is a $C^\infty$ connection on $P$, and $p$ obeys $d/2 \leq p \leq q$, then*

$$\Delta_A : W^{2,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P) \to L^p(X; \Lambda^l \otimes \mathrm{ad}P) \tag{2.3}$$

*is a bounded operator. If in addition $p \in (1, \infty)$, then there is a constant $C = C(A, A_1, g, G, l, p, q) \in [1, \infty)$ such that*

$$\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C \left(\|\Delta_A \xi\|_{L^p(X)} + \|\xi\|_{L^p(X)}\right), \quad \forall\, \xi \in W^{2,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P). \tag{2.4}$$

*Remark* 2.2 (Regularity of distributional solutions to elliptic partial differential equations). Suppose as in the hypotheses of Proposition 2.1 that $A_1$ is a smooth connection on $P$. By analogy with [41, Definition 2.56], we call $\xi \in L^1(X; \Lambda^l \otimes \mathrm{ad}P)$ a *distributional solution* to the equation $\Delta_{A_1}\xi = 0$ if

$$(\xi, \Delta_{A_1}\eta)_{L^2(X)} = 0, \quad \forall\, \xi \in C^\infty(X; \Lambda^l \otimes \mathrm{ad}P).$$

In the case of the scalar Laplace operator on functions, $C^\infty$-smoothness of distributional solutions is provided by *Weyl's Lemma* [60, Theorem 18.G]. More generally, the $C^\infty$-smoothness of a solution $u \in L^1_{\mathrm{loc}}(\Omega)$ to a scalar (second-order) elliptic equation on an open subset $\Omega \subset \mathbb{R}^d$ is a consequence of regularity theory for solutions in $H^s_{\mathrm{loc}}(\Omega)$, for $s \in \mathbb{R}$ [21, Theorem 6.33]. Such regularity results extend to the case of elliptic systems (see [13] and references therein) and so we conclude that if $\xi$ is a *distributional solution* to the equation $\Delta_{A_1}\xi = 0$, then $C^\infty(X; \Lambda^l \otimes \mathrm{ad}P)$.

Proposition 2.1 implies that the domain of the unbounded operator $\Delta_A$ on $L^p(X; \Lambda^l \otimes \mathrm{ad}P)$ is

$$\mathscr{D}_p(\Delta_A) = W^{2,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P).$$

(We omit the subscript $p$ when that is clear from the context.) In order to give criteria for when the term $\|\xi\|_{L^p(X)}$ can be eliminated from the right-hand side of the *a priori* estimate (2.4), we need to analyze the spectrum of the Hodge Laplacian with Sobolev coefficients. It is convenient to abbreviate $\mathfrak{H} := H^1_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P) = W^{1,2}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$ and recall that $(H^1_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P))^* = H^{-1}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$. Denote

$$I : \mathfrak{H} \to \mathfrak{H}^*, \quad \xi \mapsto (\cdot, \xi)_{L^2(X)},$$

by analogy with the embedding [24, Equation (8.12)]. The forthcoming Proposition 2.3 is an analogue of [24, Theorem 8.6], for a scalar, second-order, strictly elliptic equation in divergence form with homogeneous Dirichlet boundary condition over a bounded domain $\Omega \subset \mathbb{R}^d$. However, it is not a direct consequence since the first and zeroth-order coefficients of the Laplace operator $\Delta_A$ on $\Omega^l(X; \mathrm{ad}P)$ are not necessarily bounded unless $q > d$, which we do not wish to assume, for a $W^{1,q}$ connection $A$ on $P$.

**Proposition 2.3** (Spectral properties of a Laplace operator with Sobolev coefficients)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $l \geq 0$ be an integer. If $A$ is a $W^{1,q}$ connection on $P$ with $d/2 < q < \infty$, and $A_1$ is a $C^\infty$ reference connection on $P$, then the operator*

$$\Delta_A : \mathfrak{H} \to \mathfrak{H}^*$$

*is bounded and there is a countable subset $\Sigma \subset [0, \infty)$ without accumulation points and having the following significance. If $\lambda \in \mathbb{R} \setminus \Sigma$, then the equation*

$$(\Delta_A - \lambda I)\xi = \mathfrak{f}$$

*has a unique solution $\xi \in \mathfrak{H}$ for each $\mathfrak{f} \in \mathfrak{H}^*$. If $\lambda \in \Sigma$, then $\operatorname{Ker}(\Delta_A - \lambda I) \cap \mathfrak{H}$ has finite, positive dimension.*

*Remark* 2.4 (Spectral properties of a Laplace operator with Sobolev coefficients on $L^p$ spaces and compact perturbations)*.* We recall from Weyl's Theorem [35, Theorem IV.5.35] that if $T$ is a closed operator on a Banach space $\mathscr{X}$ and $K$ is an operator on $\mathscr{X}$ that is compact relative to $T$, then $T$ and $T+K$ have the same essential spectrum. In particular, under the hypotheses of Corollary B.4, the operator

$$\Delta_A - \Delta_{A_1} : W^{2,p}_{A_1}(X; \Lambda^l \otimes \operatorname{ad}P) \to L^p(X; \Lambda^l \otimes \operatorname{ad}P)$$

is compact by the proof of that corollary. Therefore, the essential spectrum of $\Delta_A$ as an unbounded operator on $L^p(X; \Lambda^l \otimes \operatorname{ad}P)$ is empty and consists purely of real eigenvalues with finite multiplicity, since the same is true of $\Delta_{A_1}$.

**Corollary 2.5** (*A priori* $L^p$ estimate for a Laplace operator with Sobolev coefficients)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$, and $l \geq 0$ be an integer. If $A$ is a $W^{1,q}$ connection on $P$ with $d/2 < q < \infty$, and $A_1$ is a $C^\infty$ connection on $P$, and $p \in (1, \infty)$ obeys $d/2 \leq p \leq q$, then the kernel $\operatorname{Ker}\Delta_A \cap W^{2,p}_{A_1}(X; \Lambda^l \otimes \operatorname{ad}P)$ of the operator* (2.3) *is finite-dimensional and*

$$\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C\|\Delta_A \xi\|_{L^p(X)}, \quad \forall\, \xi \in (\operatorname{Ker}\Delta_A)^\perp \cap W^{2,p}_{A_1}(X; \Lambda^l \otimes \operatorname{ad}P), \tag{2.5}$$

*where $\perp$ denotes $L^2$-orthogonal complement and $C = C(A, A_1, g, G, l, p, q) \in [1, \infty)$.*

Before proceeding to the proofs of these results proper, we begin with the

**Lemma 2.6** (*A priori* $L^p$ estimate for a Laplace operator with smooth coefficients)**.** *Assume the hypotheses on $A_1$, $d$, $G$, $l$, $P$, and $(X, g)$ in Proposition 2.1 and let $p \in (1, \infty)$. If $A$ is $C^\infty$, then there is a constant $C = C(A, A_1, g, l, p) \in [1, \infty)$ such that*

$$\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C\left(\|\Delta_A \xi\|_{L^p(X)} + \|\xi\|_{L^p(X)}\right), \quad \forall\, \xi \in W^{2,p}_{A_1}(X; \Lambda^l \otimes \operatorname{ad}P). \tag{2.6}$$

*Proof.* Suppose first that $\Delta_g$ is the Laplace-Beltrami operator on $C^\infty(X)$ defined by the Riemannian metric $g$. The *a priori* $L^p$ estimate for $\Delta_g$ analogous to (2.6) can be obtained from the *a priori* interior $L^p$ estimate provided by [24, Theorem 9.11] for a scalar, second-order, strictly elliptic operator with $C^\infty$ coefficients defined on a bounded domain $\Omega \subset \mathbb{R}^d$ with the aid of a $C^\infty$ partition of unity subordinate to a finite set of coordinate charts covering the closed manifold, $X$. For the general case, one first chooses in addition a set of local trivializations for $\Lambda^l \otimes \operatorname{ad}P$ corresponding to the coordinate neighborhoods, after shrinking those neighborhoods if needed. The Bochner-Weitzenböck formula [22, Equation (C.7)], [37, Equation (II.1)] for $\Delta_A$ implies that $\Delta_A - \nabla_A^*\nabla_A$ is a first-order differential operator with $C^\infty$ coefficients and that $\Delta_A$ has principal symbol given by the $C^\infty$ Riemannian metric $g$ times the identity endomorphism of $\Lambda^l \otimes \operatorname{ad}P$. (In fact, $\Delta_A = \nabla_A^*\nabla_A$ when $l = 0$.) The (first-order) covariant derivative of $\xi \in W^{2,p}_{A_1}(X; \Lambda^l \otimes \operatorname{ad}P)$ may be estimated with the following analogue of the interpolation inequality [24, Theorem 7.27], valid for $p \in [1, \infty)$,

$$\|\nabla_{A_1}\xi\|_{L^p(X)} \leq \varepsilon\|\xi\|_{W^{2,p}_{A_1}(X)} + C\varepsilon^{-1}\|\xi\|_{L^p(X)}, \tag{2.7}$$

where $C = C(A_1, g) \in [1, \infty)$ and $\varepsilon$ is any positive constant. The conclusion now follows by combining the preceding observations and using rearrangement with small $\varepsilon$ to remove the term $\|\nabla_{A_1}\xi\|_{L^p(X)}$ from the right-hand side. ∎

We can now proceed to the

*Proof of Proposition 2.1.* We choose a $C^\infty$ connection, $A_s$, on $P$ that we regard as a smooth approximation to $A$. We write $A = A_s + a$, with $a \in W^{1,q}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$ and a bound $\|a\|_{W^{1,q}_{A_1}(X)} \leq \varepsilon$ with small constant $\varepsilon \in (0,1]$ to be chosen during the proof, and write $A_s = A_1 + a_1$, where $a_1 \in C^\infty(X; \Lambda^l \otimes \mathrm{ad}P)$ may be 'large'. We expand $\Delta_A = \Delta_{A_s+a}$ to give

$$\Delta_A \xi = \Delta_{A_s}\xi + \nabla_{A_s} a \times \xi + a \times \nabla_{A_s}\xi + a \times a \times \xi,$$

and thus, for $\xi \in W^{2,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$,

$$\Delta_A \xi = \Delta_{A_s}\xi + \nabla_{A_1} a \times \xi + a_1 \times a \times \xi + a \times \nabla_{A_1}\xi + a \times a \times \xi, \tag{2.8}$$

We define $r \in [p, \infty]$ by $1/p = 1/q + 1/r$ and recall that by [3, Theorem 4.12] we have *i)* $W^{2,p}(X) \subset L^r(X)$ for any $r \in [1, \infty)$ when $p = d/2$, and *ii)* $W^{2,p}(X) \subset L^\infty(X)$ when $p > d/2$. The expansion (2.8) yields

$$\begin{aligned}\|(\Delta_A - \Delta_{A_s})\xi\|_{L^p(X)} &\leq z\|\nabla_{A_1} a\|_{L^q(X)}\|\xi\|_{L^r(X)} + \|a \times \nabla_{A_1}\xi\|_{L^p(X)} \\ &\quad + z\|a_1\|_{C(X)}\|a\|_{L^{2p}(X)}\|\xi\|_{L^{2p}(X)} + z\||a|^2\|_{L^q(X)}\|\xi\|_{L^r(X)},\end{aligned}$$

where $z = z(g, G, l) \in [1, \infty)$. To ensure a continuous Sobolev embedding $W^{1,p}(X) \subset L^d(X)$ by [3, Theorem 4.12], we need $p^* = dp/(d-p) \geq d$, that is, $p \geq d - p$ or $p \geq d/2$, which we assume in our hypotheses.

To ensure a continuous Sobolev embedding $W^{1,q}(X) \subset L^{2q}(X)$ when $q < d$, we need $q^* = dq/(d-q) \geq 2q$, that is, $d \geq 2d - 2q$ or $2q \geq d$ or $q \geq d/2$, which follows from our hypothesis that $q \geq p \geq d/2$; when $q \geq d$, the fact that $W^{1,q}(X) \subset L^{2q}(X)$ is a continuous Sobolev embedding is immediate from [3, Theorem 4.12].

Consequently, by the preceding continuous Sobolev embeddings and the Kato Inequality [22, Equation (6.20)],

$$\begin{aligned}\|(\Delta_A - \Delta_{A_s})\xi\|_{L^p(X)} &\leq z\left(\|\nabla_{A_1} a\|_{L^q(X)} + \|a\|^2_{W^{1,q}_{A_1}(X)}\right)\|\xi\|_{W^{2,p}_{A_1}(X)} \\ &\quad + z\|a_1\|_{C(X)}\|a\|_{W^{1,q}_{A_1}(X)}\|\xi\|_{W^{1,p}_{A_1}(X)} + \|a \times \nabla_{A_1}\xi\|_{L^p(X)},\end{aligned}$$

where $z = z(g, G, l, p, q) \in [1, \infty)$.

When $q < d$, we recall from [3, Theorem 4.12] that there is a continuous embedding $W^{1,q}(X) \subset L^{q^*}(X)$, where $q^* = dq/(d-q)$. Hence, $1/q^* = 1/q - 1/d$ or $1/q = 1/q^* + 1/d$ and so, using $p \leq q$,

$$\|a \times \nabla_{A_1}\xi\|_{L^p(X)} \leq z\|a \times \nabla_{A_1}\xi\|_{L^q(X)} \leq z\|a\|_{L^{q^*}(X)}\|\nabla_{A_1}\xi\|_{L^d(X)},$$

and therefore, by the preceding continuous Sobolev embeddings,

$$\|a \times \nabla_{A_1}\xi\|_{L^p(X)} \leq C\|a\|_{W^{1,q}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{W^{1,p}_{A_1}(X)}, \tag{2.9}$$

where $C = C(g, G, l, p, q) \in [1, \infty)$. When $q = d$ and $d/2 \leq p < d$, we can define $t \in [d, \infty)$ by $1/p = 1/t + 1/d$ and apply the continuous Sobolev embedding $W^{1,d}(X) \subset L^t(X)$ from [3, Theorem 4.12] to give

$$\|a \times \nabla_{A_1}\xi\|_{L^p(X)} \leq z\|a\|_{L^t(X)}\|\nabla_{A_1}\xi\|_{L^d(X)} \leq C\|a\|_{W^{1,q}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{W^{1,p}_{A_1}(X)},$$

and (2.9) again holds; when $q = d = p$, we can simply use the embedding $W^{1,d}(X) \subset L^t(X)$ for any $t \in [1, \infty)$ and observe that (2.9) holds from

$$\|a \times \nabla_{A_1}\xi\|_{L^d(X)} \leq z\|a\|_{L^{2d}(X)}\|\nabla_{A_1}\xi\|_{L^{2d}(X)} \leq C\|a\|_{W^{1,d}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{W^{1,d}_{A_1}(X)}.$$

Finally, when $q > d$ we have the continuous Sobolev embedding $W^{1,d}(X) \subset C(X)$ from [3, Theorem 4.12] and so

$$\|a \times \nabla_{A_1}\xi\|_{L^p(X)} \leq z\|a\|_{C(X)}\|\nabla_{A_1}\xi\|_{L^p(X)} \leq C\|a\|_{W^{1,q}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{L^p(X)},$$

which also yields (2.9).

Combining our previous $L^p$ bound for $(\Delta_A - \Delta_{A_s})\xi$ with the inequality (2.9) gives

$$\begin{aligned}\|(\Delta_A - \Delta_{A_s})\xi\|_{L^p(X)} \leq z\left(\|\nabla_{A_1}a\|_{L^q(X)} + \|a\|^2_{W^{1,q}_{A_1}(X)}\right)\|\xi\|_{W^{2,p}_{A_1}(X)} \\ + z\|a_1\|_{C(X)}\|a\|_{W^{1,q}_{A_1}(X)}\|\xi\|_{W^{1,p}_{A_1}(X)} \\ + z\|a\|_{W^{1,q}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{W^{1,p}_{A_1}(X)},\end{aligned} \tag{2.10}$$

where $z = z(g, G, l, p, q) \in [1, \infty)$. Combining the preceding bound with the *a priori* estimate (2.6) for $\Delta_{A_s}$ provided by Lemma 2.6,

$$\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C_0\left(\|\Delta_{A_s}\xi\|_{L^p(X)} + \|\xi\|_{L^p(X)}\right),$$

with constant denoted by $C_0 = C_0(A_1, A_s, g, p) \in [1, \infty)$ for clarity, yields

$$\begin{aligned}\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C_0\left(\|\Delta_A\xi\|_{L^p(X)} + \|\xi\|_{L^p(X)}\right) + z\left(\|a\|_{W^{1,q}_{A_1}(X)} + \|a\|^2_{W^{1,q}_{A_1}(X)}\right)\|\xi\|_{W^{2,p}_{A_1}(X)} \\ + z\|a_1\|_{C(X)}\|a\|_{W^{1,q}_{A_1}(X)}\|\xi\|_{W^{1,p}_{A_1}(X)}.\end{aligned}$$

We can choose $a = A - A_s$ so that $\|a\|_{W^{1,q}_{A_1}(X)} \leq \varepsilon$ for small constant $\varepsilon \in (0, 1]$, but we are not at liberty to choose $a_1 = A_s - A_1$ to be $W^{1,q}_{A_1}(X)$-small. Thus in our forthcoming rearrangement arguments we first apply the interpolation inequality (2.7),

$$\|\nabla_{A_1}\xi\|_{L^p(X)} \leq \delta\|\xi\|_{W^{2,p}_{A_1}(X)} + C_1\delta^{-1}\|\xi\|_{L^p(X)},$$

where $C_1 = C_1(A_1, g) \in [1, \infty)$ and $\delta = \delta(A_1, \|A_s - A_1\|_{C(X)}, g, G, l, p, q) \in (0, 1]$ is a positive constant chosen small enough that

$$\delta z\|a_1\|_{C(X)} \leq 1/2,$$

and thus, for a constant $C_2 = C_2(A_1, A_s, g, G, l, p, q) \in [1, \infty)$,

$$\begin{aligned}\|\xi\|_{W^{2,p}_{A_1}(X)} \leq 2C_0\left(\|\Delta_A\xi\|_{L^p(X)} + \|\xi\|_{L^p(X)}\right) \\ + 2z\left(\|a\|_{W^{1,q}_{A_1}(X)} + \|a\|^2_{W^{1,q}_{A_1}(X)}\right)\|\xi\|_{W^{2,p}_{A_1}(X)} + C_2\|\xi\|_{L^p(X)}.\end{aligned}$$

Provided $\|a\|_{W^{1,q}_{A_1}(X)} \leq \varepsilon$ and we choose $\varepsilon \equiv \varepsilon(g, G, l, p, q) = 1/(8z) \in (0, 1]$ in the preceding inequality, rearrangement yields the desired estimate (2.4).

Our proof of (2.4) also verifies that the operator $\Delta_A$ in (2.3) is bounded since $\Delta_{A_s}$ is bounded with the same domain and range spaces. □

Next, we have the

*Proof of Corollary 2.5.* The finite-dimensionality of $\operatorname{Ker}\Delta_A \cap W^{2,p}_{A_1}(X; \Lambda^l \otimes \operatorname{ad}P)$ follows from Proposition 2.3. We observe that, by increasing the constant $C$ as needed, the term $\|\xi\|_{L^p(X)}$ appearing on the right-hand side of the inequality (2.4) can be replaced by $\|\xi\|_{L^2(X)}$. This is clear when $p \leq 2$, while if $p > 2$, we can choose $s \in (p, \infty)$ and apply the interpolation inequality [24, Equation (7.10)],

$$\|\xi\|_{L^p(X)} \leq \delta\|\xi\|_{L^s(X)} + \delta^{-\nu}\|\xi\|_{L^2(X)},$$

for $\nu := (1/2 - 1/p)/(1/p - 1/s) > 0$ and arbitrary positive $\delta$. Because $p \geq d/2$, we have a continuous Sobolev embedding $W^{2,p}(X) \subset L^s(X)$ as already observed, so

$$\|\xi\|_{L^p(X)} \leq C_1\delta\|\xi\|_{W^{2,p}_{A_1}(X)} + \delta^{-\nu}\|\xi\|_{L^2(X)},$$

where $C_1 = C_1(A_1, g, l, p) \in [1, \infty)$. Hence, for $\delta(A_1, g, l, p) \in (0, 1]$ given by $\delta = 1/(2CC_1)$, we can use rearrangement in (2.4) to replace $\|\xi\|_{L^p(X)}$ by $\|\xi\|_{L^2(X)}$. Therefore, the estimate (2.4) implies

$$\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C\left(\|\Delta_A\xi\|_{L^p(X)} + \|\xi\|_{L^2(X)}\right), \quad \forall\, \xi \in W^{2,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P). \tag{2.11}$$

Proposition 2.3 and the forthcoming Remark 2.7 imply that the spectrum $\sigma(\Delta_A)$ of $\Delta_A$ on $L^2(X; \Lambda^l \otimes \mathrm{ad}P)$ consists purely of eigenvalues and is discrete with no accumulation points. By hypothesis, $\sigma(\Delta_A) \subset [0, \infty)$. Let $\mu[A]$ denote the least positive eigenvalue of $\Delta_A$ on $L^2(X; \Lambda^l \otimes \mathrm{ad}P)$ and recall from [9, Rayleigh's Theorem, p. 16] (or more generally [52, Theorem 6.5.1], applied to the Green's operator $G_A$ for $\Delta_A$) that

$$\mu[A] = \inf_{\xi \in (\mathrm{Ker}\,\Delta_A)^\perp} \frac{(\xi, \Delta_A\xi)_{L^2(X)}}{\|\xi\|^2_{L^2(X)}},$$

where $(\mathrm{Ker}\,\Delta_A)^\perp$ denotes the $L^2$-orthogonal complement of $\mathrm{Ker}\,\Delta_A \subset H^1_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$, with equality achieved in the infimum if and only if $\xi$ is an eigenvector with eigenvalue $\mu[A]$. Therefore, if $\xi \in (\mathrm{Ker}\,\Delta_A)^\perp \cap W^{2,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$, then

$$\|\xi\|_{L^2(X)} \leq \mu[A]^{-1}\|\Delta_A\xi\|_{L^2(X)}.$$

Hence, the inequality (2.5) follows from the preceding eigenvalue bound and (2.11) when $p \geq 2$. For the case $d/2 \leq p < 2$ (which forces $d = 2$ or 3), we observe that

$$\|\xi\|^2_{L^2(X)} \leq \frac{1}{\mu[A]}(\Delta_A\xi, \xi)|_{L^2(X)} \leq \frac{1}{\mu[A]}\|\Delta_A\xi\|_{L^p(X)}\|\xi\|_{L^{p'}(X)},$$

where $p'$ is defined by $1/p + 1/p' = 1$, and so

$$\|\xi\|_{L^2(X)} \leq \frac{1}{\sqrt{\mu[A]}}\|\Delta_A\xi\|^{1/2}_{L^p(X)}\|\xi\|^{1/2}_{L^{p'}(X)} \leq \frac{1}{2\sqrt{\mu[A]}}\left(\delta^{-1}\|\Delta_A\xi\|_{L^p(X)} + \delta\|\xi\|_{L^{p'}(X)}\right),$$

for arbitrary positive $\delta$. We have a continuous Sobolev embedding $W^{2,p}(X) \subset L^{p'}(X)$ by [3, Theorem 4.12] if $p' \leq p^* = dp/(d-2p)$ or $1/p' = 1 - 1/p \geq 1/p^* = 1/p - 2/d$ or $1 + 2/d \geq 2/p$ or $(d+2)/d \geq 2/p$, that is, if $p \geq 2d/(d+2)$. Moreover, $2d/(d+2) \leq d/2$ if and only if $4 \leq d+2$, that is, $d \geq 2$. Hence, for $d \geq 2$ and $p \geq d/2$, we have the bound

$$\|\xi\|_{L^{p'}(X)} \leq C_2\|\xi\|_{W^{2,p}_{A_1}(X)},$$

where $C_2 = C_2(A_1, g, l, p) \in [1, \infty)$. Combining the preceding inequalities for the case $d/2 \leq p < 2$ gives

$$\|\xi\|_{L^2(X)} \leq \frac{1}{2\sqrt{\mu[A]}}\left(\delta^{-1}\|\Delta_A\xi\|_{L^p(X)} + \delta C_2\|\xi\|_{W^{2,p}_{A_1}(X)}\right).$$

Combining the preceding inequality with (2.11) and applying rearrangement by choosing $\delta = \sqrt{\mu[A]}/(CC_2)$ yields the desired the inequality (2.5) for this case too. □

Finally, we complete the

*Proof of Proposition 2.3.* We adapt the proof of [24, Theorem 8.6] and by analogy with [24, Equation (8.2)] set

$$\mathscr{L}(\xi,\eta) := (d_A\xi, d_A\eta)_{L^2(X)} + (d_A^*\xi, d_A^*\eta)_{L^2(X)}, \quad \forall\, \xi,\eta \in \mathfrak{H} := H^1_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P),$$

and note that, for all $\xi,\eta\in\Omega^l(X;\mathrm{ad}P)$,

$$\mathscr{L}(\xi,\eta) = (\Delta_A\xi,\eta)_{L^2(X)} = ((d_A^*d_A + d_Ad_A^*)\xi,\eta)_{L^2(X)},$$

with $A = A_1 + a_1$ and $a_1 \in W^{1,q}(X;\Lambda^1\otimes\mathrm{ad}P)$. We combine the expansion (2.8) for $\Delta_A = \Delta_{A_1+a_1}$ with the Bochner-Weitzenböck formula [22, Equation (C.7)], [37, Equation (II.1)] for $\Delta_{A_1}$ to give

$$\Delta_A\xi = \nabla_{A_1}^*\nabla_{A_1}\xi + \mathrm{Riem}_g\times\xi + F_{A_1}\times\xi + \nabla_{A_1}a_1\times\xi + a_1\times\nabla_{A_1}\xi + a_1\times a_1\times\xi, \tag{2.12}$$

where $\mathrm{Riem}_g$ denotes the Riemann curvature tensor for $g$ on $X$. The linear map $I:\mathfrak{H}\to\mathfrak{H}^*$ is compact embedding by the Kondrachov-Rellich [3, Theorem 6.3] or the proof of [24, Lemma 8.5]. Define operators $L_\lambda := \Delta_A + \lambda I$ and $L = L_0 = \Delta_A$, for $\lambda\in\mathbb{R}$, and a bilinear form $\mathscr{L}_\lambda := \mathscr{L} + \lambda(\cdot,\cdot)_{L^2(X)}$ on $\mathfrak{H}$. Define $q'$ by $1/q + 1/q' = 1$ and note that $q'\in[1,d/(d-2))$ (if $d>2$) or $q'\in[1,\infty)$ (if $d=2$) since $q>1$ by hypothesis. Then (2.12) gives

$$\begin{aligned}\mathscr{L}_\lambda(\xi,\xi) &= \|\nabla_{A_1}\xi\|^2_{L^2(X)} + \lambda\|\xi\|^2_{L^2(X)} + (\mathrm{Riem}_g\times\xi,\xi)_{L^2(X)} + (F_{A_1}\times\xi,\xi)_{L^2(X)}\\ &\quad + (\nabla_{A_1}a_1\times\xi,\xi)_{L^2(X)} + (a_1\times\nabla_{A_1}\xi,\xi)_{L^2(X)} + (a_1\times a_1\times\xi,\xi)_{L^2(X)}\\ &\geq \|\nabla_{A_1}\xi\|^2_{L^2(X)} + \lambda\|\xi\|^2_{L^2(X)} - z\|\mathrm{Riem}_g\|_{C(X)}\|\xi\|^2_{L^2(X)} - z\|F_{A_1}\|_{C(X)}\|\xi\|^2_{L^2(X)}\\ &\quad - z\|\nabla_{A_1}a_1\|_{L^q(X)}\||\xi|^2\|_{L^{q'}(X)} - z\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)}\\ &\quad - z\||a_1|^2\|_{L^q(X)}\||\xi|^2\|_{L^{q'}(X)},\end{aligned}$$

for $z = z(g,G,l)\in[1,\infty)$, and thus

$$\begin{aligned}\mathscr{L}_\lambda(\xi,\xi) &\geq \|\nabla_{A_1}\xi\|^2_{L^2(X)} + \lambda\|\xi\|^2_{L^2(X)} - z\|\mathrm{Riem}_g\|_{C(X)}\|\xi\|^2_{L^2(X)} - z\|F_{A_1}\|_{C(X)}\|\xi\|^2_{L^2(X)}\\ &\quad - z\|a_1\|_{W^{1,q}_{A_1}(X)}\|\xi\|^2_{L^{2q'}(X)} - z\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)} - z\|a_1\|^2_{L^{2q}(X)}\|\xi\|^2_{L^{2q'}(X)}.\end{aligned} \tag{2.13}$$

When $d>2$, we have a continuous Sobolev embedding $H^1(X)\subset L^{2^*}(X)$ by [3, Theorem 4.12], where $2^* = 2d/(d-2)$. For any $t\in(2,2^*)$, the interpolation inequality [24, Equation (7.10)] gives, with $\nu := (1/2-1/t)/(1/t-1/2^*)$ and arbitrary $\zeta\in(0,1]$ to be determined,

$$\|\xi\|_{L^t(X)} \leq \zeta\|\xi\|_{L^{2^*}(X)} + \zeta^{-\nu}\|\xi\|_{L^2(X)}.$$

But when $d>2$ we have $2q' < 2^* = 2d/(d-2)$ if and only if

$$1/(2q') = 1/2 - 1/(2q) > 1/2 - 1/d,$$

or equivalently $1/d > 1/(2q)$, in turn assured by our hypothesis that $q>d/2$. On the other hand, $2q'>2$ if and only if $q'>1$ or equivalently $q<\infty$, again assured by our hypothesis on $q$. Hence, we can choose $t = 2q'$ to give

$$\|\xi\|_{L^{2q'}(X)} \leq \zeta\|\xi\|_{L^{2^*}(X)} + \zeta^{-\nu}\|\xi\|_{L^2(X)}.$$

The embedding $H^1(X)\subset L^{2q'}(X)$ is immediate by [3, Theorem 4.12] when $d=2$ since $q'<\infty$ by our hypothesis that $q>1$ when $d=2$. Combining the preceding inequality with (2.13) and applying the continuous Sobolev embeddings $H^1(X) \subset L^{2^*}(X)$ and $W^{1,q}(X) \subset L^{2q}(X)$ and the Kato Inequality [22, Equation (6.20)] yields

$$
\begin{aligned}
\mathscr{L}_\lambda(\xi,\xi) \geq \|\nabla_{A_1}\xi\|_{L^2(X)}^2 + \lambda\|\xi\|_{L^2(X)}^2 - z\|\operatorname{Riem}_g\|_{C(X)}\|\xi\|_{L^2(X)}^2 - z\|F_{A_1}\|_{C(X)}\|\xi\|_{L^2(X)}^2 \\
- z\left(\|a_1\|_{W_{A_1}^{1,q}(X)} + \|a_1\|_{W_{A_1}^{1,q}(X)}^2\right)\left(\zeta\|\xi\|_{H_{A_1}^1(X)} + \zeta^{-\nu}\|\xi\|_{L^2(X)}\right)^2 \\
- z\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)}.
\end{aligned}
\tag{2.14}
$$

Recall that if $q \geq d/2$ (we assume $q > d/2$), then there is a continuous Sobolev embedding $W^{1,q}(X) \subset L^{2q}(X)$ by [3, Theorem 4.12]. To bound the term,

$$
\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)},
$$

in (2.13) we shall separately consider the cases $q < d$, $q = d$, and $q > d$.

Suppose $q < d$ and denote $q^* = dq/(d-q)$, so we have a continuous Sobolev embedding $W^{1,q}(X) \subset L^{q^*}(X)$ by [3, Theorem 4.12]. We have $q^* = dq/(d-q) > 2$ if and only if $dq > 2d - 2q$ or $q > 2d/(d+2)$; as $d/2 \geq 2d/(d+2)$ for all $d \geq 2$, with strict inequality when $d > 2$, we are assured that $q^* > 2$ by our assumption that $q \geq d/2$ and $q > 1$. Therefore, we can define $r \in (2,\infty)$ by $1 = 1/q^* + 1/2 + 1/r$ and observe that

$$
\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)} \leq \|a_1\|_{L^{q^*}(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\|\xi\|_{L^r(X)}.
$$

Because $r \in (2,\infty)$, we can choose $s \in (r,\infty)$, to be determined, and define $\mu = (1/2 - 1/r)/(1/r - 1/s) > 0$ and apply the interpolation inequality [24, Equation (7.10)],

$$
\|\xi\|_{L^r(X)} \leq \delta\|\xi\|_{L^s(X)} + \delta^{-\mu}\|\xi\|_{L^2(X)},
$$

for arbitrary $\delta \in (0,1]$ to be determined. By combining the preceding two inequalities and then applying Young's inequality [24, Equation (7.6)] with arbitrary $\varepsilon \in (0,1]$ to be determined and the Sobolev embedding $W^{1,q}(X) \subset L^{q^*}(X)$, we see that

$$
\begin{aligned}
\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)} &\leq \delta\|a_1\|_{L^{q^*}(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\|\xi\|_{L^s(X)} + \delta^{-\mu}\|a_1\|_{L^{q^*}(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\|\xi\|_{L^2(X)} \\
&\leq \delta z\|a_1\|_{W_{A_1}^{1,q}(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\|\xi\|_{L^s(X)} \\
&\quad + \delta^{-\mu} z\|a_1\|_{W_{A_1}^{1,q}(X)}\left(\varepsilon\|\nabla_{A_1}\xi\|_{L^2(X)}^2 + \varepsilon^{-1}\|\xi\|_{L^2(X)}^2\right),
\end{aligned}
$$

where $z = z(g,q) \in [1,\infty)$. We have $H^1(X) \subset L^s(X)$ if $d = 2$ and any $s \in [1,\infty)$ or $d > 2$ and $s \leq 2^* = 2d/(d-2)$. But $1/2 = 1/q^* + 1/r$ and the condition $r < 2^* = 2d/(d-2)$ is equivalent to

$$
1/r = 1/2 - 1/q^* = 1/2 - 1/q + 1/d > (d-2)/(2d) = 1/2 - 1/d,
$$

that is, $2/d > 1/q$, as implied by our hypothesis that $q > d/2$; hence we can choose $s \in (r, 2^*]$ and obtain $H^1(X) \subset L^s(X)$ when $d > 2$ also. Therefore, when $q < d$,

$$
\begin{aligned}
\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)} \leq \delta z\|a_1\|_{W_{A_1}^{1,q}(X)}\|\xi\|_{H_{A_1}^1(X)}^2 \\
+ \delta^{-\mu} z\|a_1\|_{W_{A_1}^{1,q}(X)}\left(\varepsilon\|\xi\|_{H_{A_1}^1(X)}^2 + \varepsilon^{-1}\|\xi\|_{L^2(X)}^2\right),
\end{aligned}
$$

where $z = z(g,q) \in [1,\infty)$. Now choose $\varepsilon = \delta^{\mu+1}$, so that

$$
\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)} \leq 2\delta z\|a_1\|_{W_{A_1}^{1,q}(X)}\|\xi\|_{H_{A_1}^1(X)}^2 + \delta^{-1-2\mu} z\|a_1\|_{W_{A_1}^{1,q}(X)}\|\xi\|_{L^2(X)}^2.
\tag{2.15}
$$

by combining terms.

When $q = d$ and $d = 2$, we use the continuous Sobolev embeddings $W^{1,d}(X) \subset L^4(X)$ and $H^1(X) \subset L^4(X)$ and $H^1(X) \subset L^8(X)$ and the Kato Inequality [22, Equation (6.20)] and the interpolation inequality [24, Equation (7.10)] (with $\mu = (1/2 - 1/4)/(1/4 - 1/8) = 2$) to give

$$
\begin{aligned}
\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)} &\leq \|a_1\|_{L^4(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\|\xi\|_{L^4(X)} \\
&\leq \|a_1\|_{L^4(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\left(\delta\|\xi\|_{L^8(X)} + \delta^{-2}\|\xi\|_{L^2(X)}\right) \\
&\leq \delta z\|a_1\|_{W^{1,q}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\|\xi\|_{H^1_{A_1}(X)} \\
&\quad + \delta^{-2}z\|a_1\|_{W^{1,q}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\|\xi\|_{L^2(X)}.
\end{aligned}
$$

The preceding inequality recovers (2.15) when $q = d = 2$ by applying Young's inequality [24, Equation (7.6)] with $\varepsilon = \delta^3 \in (0, 1]$ just as before (the change in value of $\mu$ is immaterial). For $q = d$ and $d > 2$, choose a exponent finite $r \in (2, 2^*)$, where $2^* = 2d/(d - 2) > 2$, and define $s < \infty$ by $1 = 1/s + 1/2 + 1/r$, so $1/s = 1/2 - 1/r < 1/2 - 1/2^* = 1/2 - (1/2 - 1/d) = 1/d$ and $s > d$. We have

$$
\begin{aligned}
\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)} &\leq \|a_1\|_{L^s(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\|\xi\|_{L^r(X)} \\
&\leq z\|a_1\|_{W^{1,d}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\|\xi\|_{L^r(X)},
\end{aligned}
$$

where the second inequality follows from the continuous Sobolev embedding $W^{1,d}(X) \subset L^s(X)$ and the Kato Inequality [22, Equation (6.20)]. As before, we can appeal to the interpolation inequality [24, Equation (7.10)],

$$
\|\xi\|_{L^r(X)} \leq \delta\|\xi\|_{L^{2^*}(X)} + \delta^{-\mu}\|\xi\|_{L^2(X)},
$$

for arbitrary $\delta \in (0, 1]$ to be determined and $\mu = (1/2 - 1/r)/(1/r - 1/2^*) > 0$. We apply the continuous Sobolev embedding $H^1(X) \subset L^{2^*}(X)$ and the Kato Inequality [22, Equation (6.20)] and combine the preceding two inequalities to give

$$
\begin{aligned}
\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)} &\leq z\|a_1\|_{W^{1,d}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\left(\delta\|\xi\|_{L^{2^*}(X)} + \delta^{-\mu}\|\xi\|_{L^2(X)}\right) \\
&\leq z\|a_1\|_{W^{1,d}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\left(\delta\|\xi\|_{H^1_{A_1}(X)} + \delta^{-\mu}\|\xi\|_{L^2(X)}\right).
\end{aligned}
$$

By again applying Young's inequality [24, Equation (7.6)] with $\varepsilon = \delta^{\mu+1} \in (0, 1]$, we complete the proof of (2.15) for the case $q = d$.

Finally, when $q > d$, we have

$$
\begin{aligned}
\||a_1||\nabla_{A_1}\xi||\xi|\|_{L^1(X)} &\leq \|a_1\|_{C(X)}\|\nabla_{A_1}\xi\|_{L^2(X)}\|\xi\|_{L^2(X)} \\
&\leq z\|a_1\|_{W^{1,q}_{A_1}(X)}\left(\varepsilon\|\nabla_{A_1}\xi\|^2_{L^2(X)} + \varepsilon^{-1}\|\xi\|^2_{L^2(X)}\right),
\end{aligned}
$$

which again gives (2.15) by choosing $\varepsilon = 2\delta$.

Thus, combining (2.15) with our previous lower bound (2.14) for $\mathscr{L}_\lambda(\xi, \xi)$, the continuous Sobolev embeddings $W^{1,q}(X) \subset L^{2q}(X)$ and the Kato Inequality [22, Equation (6.20)],

$$
\begin{aligned}
\mathscr{L}_\lambda(\xi, \xi) \geq{} & \|\nabla_{A_1}\xi\|^2_{L^2(X)} + \lambda\|\xi\|^2_{L^2(X)} - z\|\operatorname{Riem}_g\|_{C(X)}\|\xi\|^2_{L^2(X)} - z\|F_{A_1}\|_{C(X)}\|\xi\|^2_{L^2(X)} \\
& - z\left(\|a_1\|_{W^{1,q}_{A_1}(X)} + \|a_1\|^2_{W^{1,q}_{A_1}(X)}\right)\left(\zeta\|\xi\|_{H^1_{A_1}(X)} + \zeta^{-\nu}\|\xi\|_{L^2(X)}\right)^2 \\
& - 2\delta z\|a_1\|_{W^{1,q}_{A_1}(X)}\|\xi\|^2_{H^1_{A_1}(X)} - \delta^{-1-2\mu}z\|a_1\|_{W^{1,q}_{A_1}(X)}\|\xi\|^2_{L^2(X)}.
\end{aligned}
$$

Consequently, for small enough constants $\zeta$ and $\delta$ in $(0,1]$ and possibly large constant $C \in [1,\infty)$, all depending on $g, G, l, q, \|a_1\|_{W^{1,q}_{A_1}(X)}$, we obtain

$$\begin{aligned}\mathscr{L}_\lambda(\xi,\xi) \geq \frac{1}{2}\|\nabla_{A_1}\xi\|^2_{L^2(X)} + \lambda\|\xi\|^2_{L^2(X)} - z\|\operatorname{Riem}_g\|_{C(X)}\|\xi\|^2_{L^2(X)} - z\|F_{A_1}\|_{C(X)}\|\xi\|^2_{L^2(X)} \\ - C\|\xi\|_{L^2(X)}.\end{aligned}$$

Hence, for large enough $\sigma_0 = \sigma_0(g, G, l, q, \|a_1\|_{W^{1,q}_{A_1}(X)}, \|\operatorname{Riem}_g\|_{C(X)}, \|F_{A_1}\|_{C(X)}) \in [1,\infty)$, we can ensure that

$$\mathscr{L}_{\sigma_0}(\xi,\xi) \geq \frac{1}{2}\left(\|\nabla_{A_1}\xi\|^2_{L^2(X)} + \|\xi\|^2_{L^2(X)}\right) = \frac{1}{2}\|\xi\|^2_{H^1_{A_1}(X)} = \frac{1}{2}\|\xi\|^2_{\mathfrak{H}},$$

and thus $\mathscr{L}_{\sigma_0} : \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ is a bounded, coercive bilinear form. For any $\lambda \in \mathbb{R}$ and $\mathfrak{f} \in \mathfrak{H}^*$, the equation

$$(L - \lambda I)\xi = \mathfrak{f}$$

for $\xi \in \mathfrak{H}$ is equivalent to

$$L_{\sigma_0}\xi - (\lambda + \sigma_0)I\xi = \mathfrak{f}.$$

By the Lax-Milgram Theorem [24, Theorem 5.8], the operator $L^{-1}_{\sigma_0} : \mathfrak{H}^* \to \mathfrak{H}$ is bijective and continuous, so the preceding equation for $\xi \in \mathfrak{H}$ is equivalent to

$$\xi - (\lambda + \sigma_0)L^{-1}_{\sigma_0}I\xi = L^{-1}_{\sigma_0}\mathfrak{f}.$$

The operator $I : \mathfrak{H} \to \mathfrak{H}^*$ is compact, so the operator $T_{\sigma_0} := L^{-1}_{\sigma_0}I : \mathfrak{H} \to \mathfrak{H}$ is also compact by [7, Proposition 6.3]. By [24, Theorem 5.11], there exists a countable subset $\Sigma \subset \mathbb{R}$ without limit points such that if $\lambda \notin \Sigma$ the equation

$$\xi - (\lambda + \sigma_0)T_{\sigma_0}\xi = L^{-1}_{\sigma_0}\mathfrak{f}$$

has a unique solution $\xi \in \mathfrak{H}$ for each $\mathfrak{f} \in \mathfrak{H}^*$. If $\lambda \in \Sigma$, then $\operatorname{Ker}(\mathrm{id} - (\lambda + \sigma_0)T_{\sigma_0})$ has finite, positive dimension. The conclusions of Proposition 2.3 regarding $\Sigma$ now follow immediately.

We also note that the arguments used to prove that $\mathscr{L}_{\sigma_0} : \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ is a coercive bilinear form can be applied, with trivial modifications, to show that $\mathscr{L} : \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ is a continuous bilinear form, that is,

$$|\mathscr{L}(\xi,\eta)| \leq C\|\xi\|_{\mathfrak{H}}\|\eta\|_{\mathfrak{H}}, \quad \forall\, \xi,\eta \in \mathfrak{H},$$

with $C = C(g, G, l, q) \in [1,\infty)$ and hence that the operator $\Delta_A : \mathfrak{H} \to \mathfrak{H}^*$ defined by

$$\Delta_A\xi = \mathscr{L}(\cdot,\xi), \quad \forall\, \xi \in \mathfrak{H},$$

is bounded. □

*Remark* 2.7 (Reality of the spectrum of the Laplace operator $\Delta_A$). Under the hypotheses of Proposition 2.3, one can assert more generally that the spectrum $\sigma(\Delta_A) \subset \mathbb{C}$ of $\Delta_A : \mathscr{D}(\Delta_A) \subset L^2(X; \Lambda^l \otimes \operatorname{ad}P) \to L^2(X; \Lambda^l \otimes \operatorname{ad}P)$ is real, where the domain $\mathscr{D}(\Delta_A) = H^2_{A_1}(X; \Lambda^l \otimes \operatorname{ad}P)$ is dense in $L^2(X; \Lambda^l \otimes \operatorname{ad}P)$. Indeed, because

$$(\Delta_A\xi,\eta)_{L^2(X)} = (\xi,\Delta_A\eta)_{L^2(X)}, \quad \forall\, \xi,\eta \in \mathscr{D}(\Delta_A),$$

then $\Delta_A$ is a symmetric unbounded operator. The unbounded operator $\Delta_A$ is self-adjoint since $\Delta_A$ is symmetric and $\mathscr{D}(\Delta_A^*) = \mathscr{D}(\Delta_A)$, where $\Delta_A^*$ is the $L^2$-adjoint of $\Delta_A$ [49, Section 13.1].

The $L^2$ self-adjointness of $\Delta_A$ implies that

$$\langle\eta,\Delta_A\xi\rangle_{\mathfrak{H}\times\mathfrak{H}^*} = (\eta,\Delta_A\xi)_{L^2(X)} = (\xi,\Delta_A\eta)_{L^2(X)} = \langle\xi,\Delta_A\eta\rangle_{\mathfrak{H}\times\mathfrak{H}^*}, \quad \forall\, \xi,\eta \in \mathfrak{H}.$$

Hence, the operator $L_{\sigma_0} : \mathfrak{H} \to \mathfrak{H}^*$ is self-adjoint (noting that $\mathfrak{H}^{**} \cong \mathfrak{H}$ via the canonical identification of a reflexive Banach space with its bidual) and thus $T_{\sigma_0} := L^{-1}_{\sigma_0}I : \mathfrak{H} \to \mathfrak{H}$ is self-adjoint.

The proof of Proposition 2.3 shows that $\lambda \in \sigma(\Delta_A)$ if and only if $\lambda \in \sigma(T_{\sigma_0})$, where $T_{\sigma_0}$ is a (compact) self-adjoint operator on a Hilbert space $\mathfrak{H}$ and because $\sigma(T_{\sigma_0}) \subset \mathbb{R}$ by [47, Theorem VI.8], then $\sigma(\Delta_A) \subset \mathbb{R}$.

2.3. **Regularity for distributional solutions to an elliptic equation with Sobolev coefficients.** Before proceeding further, we shall need to address a complication that arises when establishing regularity for distributional solutions to an elliptic equation with Sobolev coefficients. We shall confine our discussion to the Hodge Laplace operator, though one can clearly establish more general results of this kind.

**Lemma 2.8** (Regularity for distributional solutions to an equation defined by the Hodge Laplace operator for a Sobolev connection)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group and $P$ be a smooth principal $G$-bundle over $X$. Let $A_1$ be a $C^\infty$ connection on $P$, and $A$ be a $W^{1,q}$ connection on $P$ with $d/2 < q < \infty$, and $l \geq 0$ be an integer. If $q' \in (1, \infty)$ is the dual exponent defined by $1/q + 1/q' = 1$ and $\eta \in L^{q'}(X; \Lambda^l \otimes \mathrm{ad}P)$ is a distributional solution*[1] *to*

$$\Delta_A \eta = 0,$$

*then $\eta \in W^{2,q}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)$.*

*Proof.* We recall from (B.4) in the proof of Corollary B.4 (with $p = q$) that

$$\Delta_A - \Delta_{A_1} : W^{1,u}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P) \to L^q(X; \Lambda^1 \otimes \mathrm{ad}P)$$

is a bounded operator, where allowable values of $u \in (1, \infty)$ are given by

$$u = \begin{cases} d + \varepsilon & \text{if } d/2 < q < d, \\ 2d & \text{if } q = d, \\ q & \text{if } q > d, \end{cases} \tag{2.16}$$

and $\varepsilon \in (0, 1]$ is chosen small enough that $q^* = dq/(d-q) \geq d+\varepsilon$, which is possible when $q > d/2$. Consequently, the dual operator,

$$\Delta_A - \Delta_{A_1} : L^{q'}(X; \Lambda^1 \otimes \mathrm{ad}P) \to W^{-1,u'}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P),$$

is bounded, where the dual exponent $u' \in (1, \infty)$ is defined by $1/u + 1/u' = 1$.

We write $\Delta_A = \Delta_{A_1} + (\Delta_A - \Delta_{A_1})$, set $\alpha := (\Delta_{A_1} - \Delta_A)\eta \in W^{-1,u'}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)$, and observe that $\eta$ is a distributional solution to

$$\Delta_{A_1} \eta = \alpha. \tag{2.17}$$

We now appeal to regularity for distributional solutions to an equation (namely, (2.17)) defined by an elliptic operator $\Delta_{A_1}$ with $C^\infty$ coefficients (see Remark 2.2 and [13]) to conclude that $\eta \in W^{1,u}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)$. The range of exponents $u \in (d, \infty)$ given by (2.16) ensures that $\eta \in C(X; \Lambda^l \otimes \mathrm{ad}P)$, since $W^{1,u}(X) \subset C(X)$ by [3, Theorem 4.12] when $u > d$. Moreover, the estimate (B.6) for $(\Delta_{A_1} - \Delta_A)\eta$ in terms of $a = A - A_1$ and $\eta$ ensures that

$$(\Delta_{A_1} - \Delta_A)\eta \in L^q(X; \Lambda^l \otimes \mathrm{ad}P).$$

In particular, $\alpha \in L^q(X; \Lambda^l \otimes \mathrm{ad}P)$ and regularity for solutions to an elliptic equation (that is, (2.17)) with $C^\infty$ coefficients implies that $\eta \in W^{2,q}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)$, as desired. □

[1] In the sense of Remark 2.2.

2.4. **Surjectivity of a perturbed Laplace operator.** We now consider surjectivity properties of a perturbation of a Laplace operator, namely

**Lemma 2.9** (Surjectivity of a perturbed Laplace operator)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group and $P$ be a smooth principal $G$-bundle over $X$. Let $A_1$ be a $C^\infty$ connection on $P$ and $A$ be a $W^{1,q}$ connection on $P$ with $d/2 < q < \infty$. Then there is a constant $\delta = \delta(A,g) \in (0,1]$ with the following significance. If $a \in W^{1,q}_{A_1}(X;\Lambda^1\otimes \operatorname{ad}P)$ obeys*

$$\|a\|_{L^d(X)} < \delta \quad \text{when } d \geq 3 \quad \text{or} \quad \|a\|_{L^4(X)} < \delta \quad \text{when } d = 2, \tag{2.18}$$

*then the operator,*

$$d_A^* d_{A+a} : (\operatorname{Ker}\Delta_A)^\perp \cap W^{2,q}_{A_1}(X;\operatorname{ad}P) \to (\operatorname{Ker}\Delta_A)^\perp \cap L^q(X;\operatorname{ad}P), \tag{2.19}$$

*is surjective.*

*Remark* 2.10. Our proof of Lemma 2.9 is based on a similar argument arising in Donaldson and Kronheimer [10, p. 66], as part of their version of the proof of Uhlenbeck's local Coulomb gauge-fixing theorem [10, Theorem 2.3.7]. We make some adjustments to their argument for reasons which we briefly explain here. When $B$ is a Sobolev connection matrix [10, p. 66], one has to take into account the possibility that their operator $d^*d_B$, for a $\mathfrak{g}$-valued Sobolev one-form $B$ over $S^4$, could have dense range but still fail to be surjective, so one would first have to verify, for example, that $d^*d_B$ has closed range. Similarly, while elliptic regularity ensures that if $\eta \in L^2(S^4;\operatorname{ad}P)\cap \operatorname{Ker} d^*d_B$ then $\eta$ is $C^\infty(S^4;\operatorname{ad}P)$ when $B$ is $C^\infty$, those regularity issues become more subtle when $B$ is merely a Sobolev one-form.

*Proof of Lemma 2.9.* Note that when $a = 0$, the operator (2.19) is invertible by Corollary 2.5. It is convenient to abbreviate $\mathscr{X} := W^{2,q}_{A_1}(X;\operatorname{ad}P)$, and $K := \operatorname{Ker}\Delta_A \cap W^{2,q}_{A_1}(X;\operatorname{ad}P)$, and $\mathscr{Y} := L^q(X;\operatorname{ad}P)$, and $\mathscr{H} := L^2(X;\operatorname{ad}P)$, and denote $T = d_A^*d_{A+a}$ in (2.19). The kernel $K \subset \mathscr{X}$ is finite-dimensional by Proposition 2.3 and its $L^2$-orthogonal complements $K^\perp \cap \mathscr{X}$ and $K^\perp \cap \mathscr{Y}$ and $K^\perp$ provide closed complements of $K$ in $\mathscr{X}$ and $\mathscr{Y}$ and $\mathscr{H}$, respectively. Similarly, we let $K^\perp$ and $K^\perp \cap \mathscr{Y}^*$ and $K^\perp \cap \mathscr{X}^*$ denote the closed complements of $K^* \cong K$ in $\mathscr{H}^* \cong \mathscr{H}$ and $\mathscr{Y}^*$ and $\mathscr{X}^*$, respectively, where $\mathscr{Y}^* = L^{q'}(X;\operatorname{ad}P)$, with $q' \in (1,\infty)$ defined by $1/q + 1/q' = 1$, and $\mathscr{X}^* = W^{-2,q'}_{A_1}(X;\operatorname{ad}P)$.

The operator $T : \mathscr{X} \to \mathscr{Y}$ is bounded by Proposition 2.1 and according to [49, Theorem 4.12],

$$\operatorname{Ker}(T^* : \mathscr{Y}^* \to \mathscr{X}^*) = (\operatorname{Ran}(T : \mathscr{X} \to \mathscr{Y}))^\circ,$$

where $\circ$ denotes the annihilator in $\mathscr{Y}^*$. In particular, $T : K^\perp \cap \mathscr{X} \to K^\perp \cap \mathscr{Y}$ is bounded and well-defined by Corollary B.4 and surjective if and only if $T^* : K^\perp \cap \mathscr{Y}^* \to K^\perp \cap \mathscr{X}^*$ has zero kernel.

If $T^* : K^\perp \cap \mathscr{Y}^* \to K^\perp \cap \mathscr{X}^*$ were not injective, there would be a non-zero $\eta \in K^\perp \cap \mathscr{Y}^*$ such that $T^*\eta = 0$. In other words, because $T^* = d_{A+a}^*d_A$, there would be a non-zero $\eta \in K^\perp \cap L^{q'}(X;\operatorname{ad}P)$ such that

$$d_{A+a}^*d_A\eta = 0 \in W^{-2,q'}_{A_1}(X;\operatorname{ad}P),$$

that is, $(d_{A+a}^*d_A\eta)(\xi) = 0$ for all $\xi \in K^\perp \cap W^{2,q}_{A_1}(X;\operatorname{ad}P)$ or equivalently,

$$\langle \chi, d_{A+a}^*d_A\eta\rangle_{\mathscr{X}\times\mathscr{X}^*} = \langle \eta, d_A^*d_{A+a}\chi\rangle_{\mathscr{Y}\times\mathscr{Y}^*} = (\eta, d_A^*d_{A+a}\chi)_{L^2(X)} = 0, \quad \forall\, \chi \in K^\perp \cap W^{2,q}_{A_1}(X;\operatorname{ad}P).$$

Lemma 2.8 implies that $\eta \in W^{2,q}_{A_1}(X; \mathrm{ad}P)$. Observe that, by writing $d_{A+a}\chi = d_A\chi + [a,\chi]$,

$$\begin{aligned} 0 &= (d_A^* d_{A+a}\chi, \eta)_{L^2(X)} \\ &= (d_{A+a}\chi, d_A\eta)_{L^2(X)} \\ &= (d_A\chi, d_A\eta)_{L^2(X)} + ([a,\chi], d_A\eta)_{L^2(X)}. \end{aligned}$$

Since $\eta \perp \operatorname{Ker}\Delta_A \cap W^{2,q}_{A_1}(X; \mathrm{ad}P)$ and letting $\mu[A]$ denote the least positive eigenvalue of the Laplace operator $\Delta_A$ on $L^2(X; \mathrm{ad}P)$ provided by Proposition 2.3, we have

$$\mu[A] \leq \frac{(\eta, d_A^* d_A\eta)_{L^2(X)}}{\|\eta\|^2_{L^2(X)}} = \frac{\|d_A\eta\|^2_{L^2(X)}}{\|\eta\|^2_{L^2(X)}}$$

and thus,

$$\|\eta\|_{L^2(X)} \leq \mu[A]^{-1/2}\|d_A\eta\|_{L^2(X)}.$$

Hence, we obtain

$$\|\eta\|_{W^{1,2}_A(X)} \leq C_1\|d_A\eta\|_{L^2(X)}, \tag{2.20}$$

for a constant $C_1 = C_1(A,g) = 1+\mu[A]^{-1/2} \in [1,\infty)$. For $d \geq 3$ and using $1/2 = (d-2)/2d + 1/d$ and the continuous multiplication $L^{2d/(d-2)}(X) \times L^d(X) \to L^2(X)$, we see that

$$\begin{aligned} \left|([a,\chi], d_A\eta)_{L^2(X)}\right| &\leq \|[a,\chi]\|_{L^2(X)}\|d_A\eta\|_{L^2(X)} \\ &\leq \|a\|_{L^d(X)}\|\chi\|_{L^{2d/(d-2)}(X)}\|d_A\eta\|_{L^2(X)} \\ &\leq C_2\|a\|_{L^d(X)}\|\chi\|_{W^{1,2}_A(X)}\|d_A\eta\|_{L^2(X)}, \end{aligned}$$

where the constant $C_2 = C_2(g) \in [1,\infty)$ is the norm of the continuous Sobolev embedding $W^{1,2}(X) \subset L^{2d/(d-2)}(X)$ provided by [3, Theorem 4.12]. Hence, setting $\chi = \eta$ and applying the *a priori* estimate (2.20), the preceding identity and inequalities yield

$$\begin{aligned} \|d_A\eta\|^2_{L^2(X)} &= \left|([a,\chi], d_A\eta)_{L^2(X)}\right| \\ &\leq C_1C_2\|a\|_{L^d(X)}\|d_A\eta\|^2_{L^2(X)}, \end{aligned}$$

and so, if $\eta \not\equiv 0$, we have

$$\|a\|_{L^d(X)} \geq C_1C_2,$$

contradicting our hypothesis (2.18) that $\|a\|_{L^d(X)} < \delta$, with $\delta$ small.

For the case $d = 2$, we instead use the continuous multiplication $L^4(X) \times L^4(X) \to L^2(X)$ and continuous Sobolev embedding $W^{1,2}(X) \subset L^r(X)$ for $1 \leq r < \infty$ provided by [3, Theorem 4.12]. In particular, using the embedding $W^{1,2}(X) \subset L^4(X)$ with norm $C_3 = C_3(g) \in [1,\infty)$, we obtain

$$\begin{aligned} \|d_A\eta\|^2_{L^2(X)} &\leq \left|([a,\eta], d_A\eta)_{L^2(X)}\right| \\ &\leq \|[a,\eta]\|_{L^2(X)}\|d_A\eta\|_{L^2(X)} \\ &\leq \|a\|_{L^4(X)}\|\eta\|_{L^4(X)}\|d_A\eta\|_{L^2(X)} \\ &\leq C_3\|a\|_{L^4(X)}\|\eta\|_{W^{1,2}_A(X)}\|d_A\eta\|_{L^2(X)} \\ &\leq C_1C_3\|a\|_{L^4(X)}\|d_A\eta\|^2_{L^2(X)} \quad \text{(by (2.20))}, \end{aligned}$$

which again yields a contradiction to our hypothesis (2.18), just as in the case $d \geq 3$. □

2.5. ***A priori* estimates for Coulomb gauge transformations.** We now establish a generalization of [14, Lemma 6.6], which is in turn an analogue of [10, Lemma 2.3.10]. We allow $p \geq 2$ and any $d \geq 2$ rather than assume $d = 4$, as in [14, Lemma 6.6], but we use standard Sobolev norms rather than the 'critical exponent' Sobolev norms employed in the statement and proof of [14, Lemma 6.6], since we do not seek an explicit optimal dependence of constants on the reference connection, $A_0$.

Before stating our generalization of [14, Lemma 6.6], we digress to recall from [23, p. 231], that a gauge transformation, $u \in \mathrm{Aut}(P)$, may be viewed as a section of the fiber bundle $\mathrm{Ad}P := P \times_{\mathrm{Ad}} G \to X$, where we denote $\mathrm{Ad}(g) : G \ni h \to g^{-1}hg \in G$, for all $h \in G$. With the aid of a choice of a unitary representation, $\varrho : G \subset \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, we may therefore consider $\mathrm{Ad}P$ to be a subbundle of the Hermitian vector bundle $P \times_\varrho \mathrm{End}_{\mathbb{C}}(\mathbb{E})$. We can alternatively replace $\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ by $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ and $\mathrm{End}_{\mathbb{C}}(\mathbb{E})$ by $\mathrm{End}(\mathfrak{g})$. A choice of connection $A$ on $P$ induces covariant derivatives on all associated vector bundles, such as $E = P \times_\varrho \mathbb{E}$ and $P \times_\varrho \mathrm{End}_{\mathbb{C}}(\mathbb{E})$ or $\mathrm{ad}P = P \times_{\mathrm{ad}} \mathfrak{g}$ and $P \times_{\mathrm{Ad}} \mathrm{End}(\mathfrak{g})$. We can thus define Sobolev norms of sections of $\mathrm{Ad}P$, generalizing the construction of Freed and Uhlenbeck in [22, Appendix A]

A similar construction is described by Parker [45, Section 4], but we note that while the center of $\mathrm{Aut}(P)$ — which is given by $P \times_{\mathrm{Ad}} \mathrm{Center}(G)$ — acts trivially on $\mathscr{A}(P)$, it does not act trivially on $C^\infty(X; E)$.

**Proposition 2.11** (*A priori* $W^{1,p}$ estimate for $u(A) - A_0$ in terms of $A - A_0$)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$. Let $A_1$ be a $C^\infty$ connection on $P$, and $A_0$ be a $W^{1,q}$ connection on $P$ with $d/2 < q < \infty$, and $p \in (1, \infty)$ obey $d/2 \leq p \leq q$, and*[2] *$\delta \in (0, 1]$. Then there are constants $N \in [1, \infty)$ and $\varepsilon = \varepsilon(A_0, A_1, g, G, p, q) \in (0, 1]$ (with dependence on $p$ replaced by dependence on $\delta$ when $p = d$) with the following significance. If $A$ is a $W^{1,q}$ connection on $P$ and $u \in \mathrm{Aut}(P)$ is a gauge transformation of class $W^{2,q}$ such that*

$$d^*_{A_0}(u(A) - A_0) = 0, \tag{2.21}$$

*then the following hold. If*

$$\|A - A_0\|_{L^s(X)} \leq \varepsilon \quad \text{and} \quad \|u(A) - A_0\|_{L^s(X)} \leq \varepsilon, \tag{2.22}$$

*where*

$$s(p) := \begin{cases} d & \text{if } p < d, \\ d + \delta & \text{if } p = d, \\ p & \text{if } p > d, \end{cases} \tag{2.23}$$

*then*

$$\|u(A) - A_0\|_{L^p(X)} \leq N \|A - A_0\|_{W^{1,p}_{A_1}(X)}, \tag{2.24}$$

*where $N = N(A_0, A_1, g, G, p, q)$. If in addition $A$ obeys*[3]

$$\|A - A_0\|_{W^{1,p}_{A_1}(X)} \leq M, \tag{2.25}$$

*for some constant $M \in [1, \infty)$, then*

$$\|u(A) - A_0\|_{W^{1,p}_{A_1}(X)} \leq N \|A - A_0\|_{W^{1,p}_{A_1}(X)}, \tag{2.26}$$

[2] In applications of Proposition 2.11, we can choose $\delta = 1$ without loss of generality.

[3] The use of the constant $M$ could be avoided if we replaced the right-hand-side of Inequality (2.26) by $N(1 + \|A - A_0\|_{W^{1,p}_{A_1}(X)})\|A - A_0\|_{W^{1,p}_{A_1}(X)}$.

*where* $N = N(A_0, A_1, g, G, M, p, q)$.

*Proof.* Following the convention of [54, p. 32] for the action of $u \in \operatorname{Aut}(P)$ on connections $A$ on $P$ and setting $B := u(A)$ for convenience, we have

$$B - A_0 = u^{-1}(A - A_0)u + u^{-1}d_{A_0}u. \tag{2.27}$$

Our task is thus to estimate the term $d_{A_0}u$. Rewriting the preceding equality gives a first-order, linear elliptic equation in $u$ with $W^{1,q}$ coefficients,

$$d_{A_0}u = u(B - A_0) - (A - A_0)u. \tag{2.28}$$

Proposition 2.3 implies that the kernel,

$$K := \operatorname{Ker}\left(\Delta_{A_0} : W^{2,q}_{A_1}(X; \operatorname{ad}P) \to L^q(X; \operatorname{ad}P)\right),$$

is finite-dimensional. Let

$$\Pi : L^2(X; \operatorname{ad}P) \to K \subset L^2(X; \operatorname{ad}P)$$

denote the $L^2$-orthogonal projection and denote

$$\gamma := \Pi u \in K \subset W^{2,q}_{A_1}(X; \operatorname{ad}P),$$
$$u_0 := u - \gamma \in K^\perp \cap W^{2,q}_{A_1}(X; \operatorname{ad}P),$$

where $\perp$ is $L^2$-orthogonal complement. We may assume without loss of generality that we have a unitary representation, $G \subset \mathrm{U}(n)$. Recall that (due to [57, Equation (6.2)]),

$$d^*_{A_0} = (-1)^{-d(k+1)+1} * d_{A_0} * \quad \text{on } \Omega^k(X; \operatorname{ad}P), \tag{2.29}$$

where $* : \Omega^k(X) \to \Omega^{d-k}$ is the Hodge star operator on $k$-forms. Because $d^*_{A_0}(B - A_0) = 0$ and $d_{A_0}u = d_{A_0}u_0$, an application of $d^*_{A_0}$ to (2.28) yields

$$\begin{aligned} d^*_{A_0}d_{A_0}u_0 &= - * (d_{A_0}u \wedge *(B - A_0)) + ud^*_{A_0}(B - A_0) \\ &\quad - (d^*_{A_0}(A - A_0))u + *(*(A - A_0) \wedge d_{A_0}u) \\ &= - * (d_{A_0}u_0 \wedge *(B - A_0)) - (d^*_{A_0}(A - A_0))u + *(*(A - A_0) \wedge d_{A_0}u_0). \end{aligned}$$

We shall use the bound $\|u\|_{C(X)} \le 1$ for any $u \in \operatorname{Aut}(P)$ of class $W^{2,q}$, implied by the fact that the representation for $G$ is unitary. Recall that $\Delta_{A_0} = d^*_{A_0}d_{A_0}$. We first consider the case $p < d$ and setting $p^* = dp/(d-p)$, we have $1/p = 1/d + 1/p^*$ and the continuous multiplication $L^d(X) \times L^{p^*}(X) \to L^p(X)$, so

$$\begin{aligned} \|\Delta_{A_0}u_0\|_{L^p(X)} &\le \|d_{A_0}u_0\|_{L^{p^*}(X)}\|B - A_0\|_{L^d(X)} + \|d^*_{A_0}(A - A_0)\|_{L^p(X)}\|u\|_{C(X)} \\ &\quad + \|A - A_0\|_{L^d(X)}\|d_{A_0}u_0\|_{L^{p^*}(X)} \\ &\le \left(\|B - A_0\|_{L^d(X)} + \|A - A_0\|_{L^d(X)}\right)\|d_{A_0}u_0\|_{L^{p^*}(X)} \\ &\quad + \|d^*_{A_0}(A - A_0)\|_{L^p(X)}. \end{aligned}$$

Second, for the case $p > d$, we use the continuous multiplication $L^p(X) \times L^\infty(X) \to L^p(X)$, so

$$\begin{aligned} \|\Delta_{A_0}u_0\|_{L^p(X)} &\le \left(\|B - A_0\|_{L^p(X)} + \|A - A_0\|_{L^p(X)}\right)\|d_{A_0}u_0\|_{L^\infty(X)} \\ &\quad + \|d^*_{A_0}(A - A_0)\|_{L^p(X)}. \end{aligned}$$

Third, for the case $p = d$, we can instead use $r \in (d, \infty)$ defined by $1/d = 1/(d + \delta) + 1/r$ and the resulting continuous multiplication, $L^{d+\delta}(X) \times L^r(X) \to L^d(X)$, to give

$$\begin{aligned}\|\Delta_{A_0} u_0\|_{L^d(X)} &\leq \left(\|B - A_0\|_{L^{d+\delta}(X)} + \|A - A_0\|_{L^{d+\delta}(X)}\right) \|d_{A_0} u_0\|_{L^r(X)} \\ &\quad + \|d_{A_0}^*(A - A_0)\|_{L^d(X)}.\end{aligned}$$

By [3, Theorem 4.12], we have continuous Sobolev embeddings,

$$\tag{2.30} W^{1,p}(X) \subset \begin{cases} L^{dp/(d-p)}(X) & \text{if } 1 \leq p < d, \\ L^r(X) & \text{if } p = d \text{ and } 1 \leq r < \infty, \\ C(X) & \text{if } p > d. \end{cases}$$

Therefore, the Sobolev Embedding Theorem and Kato Inequality [22, Equation (6.20)] give, for $r \in (d, \delta)$ determined by $\delta$ as above,

$$\tag{2.31} \begin{aligned} \|d_{A_0} u_0\|_{L^{p^*}(X)} &\leq C_0 \|d_{A_0} u_0\|_{W^{1,p}_{A_1}(X)}, \quad p < d, \\ \|d_{A_0} u_0\|_{L^r(X)} &\leq C_0 \|d_{A_0} u_0\|_{W^{1,d}_{A_1}(X)}, \quad p = d, \\ \|d_{A_0} u_0\|_{L^\infty(X)} &\leq C_0 \|d_{A_0} u_0\|_{W^{1,p}_{A_1}(X)}, \quad p > d, \end{aligned}$$

where $C_0 = C_0(g, p)$ or $C_0(g, \delta) \in [1, \infty)$ is bounded below by the norm of the Sobolev embedding (2.30). Writing $A_0 = A_1 + a_0$, for $a_0 \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)$, so $d_{A_0} u_0 = d_{A_1} u_0 + [a_0, u_0]$, we see that

$$\begin{aligned}\|\nabla_{A_1} d_{A_0} u_0\|_{L^p(X)} &\leq \|\nabla_{A_1} d_{A_1} u_0\|_{L^p(X)} + \|\nabla_{A_1} [a_0, u_0]\|_{L^p(X)} \\ &\leq \|\nabla^2_{A_1} u_0\|_{L^p(X)} + \|\nabla_{A_1} a_0 \times u_0 + a_0 \times \nabla_{A_1} u_0\|_{L^p(X)}.\end{aligned}$$

By hypothesis, we have $p = d/2 < q$ or $d/2 < p \leq q$, and thus, defining $r \in [p, \infty]$ by $1/p = 1/q + 1/r$ and using the continuous Sobolev embeddings $W^{1,p} \subset L^{2p}(X)$ and $W^{2,p}(X) \subset L^r(X)$ (with norm bounded above by $C_0$) for $p \geq d/2$,

$$\begin{aligned}&\|\nabla_{A_1} d_{A_0} u_0\|_{L^p(X)} \\ &\quad \leq \|\nabla^2_{A_1} u_0\|_{L^p(X)} + z \|\nabla_{A_1} a_0\|_{L^q(X)} \|u_0\|_{L^r(X)} + z \|a_0\|_{L^{2p}(X)} \|\nabla_{A_1} u_0\|_{L^{2p}(X)} \\ &\quad \leq \|\nabla^2_{A_1} u_0\|_{L^p(X)} + z C_0 \|\nabla_{A_1} a_0\|_{L^q(X)} \|u_0\|_{W^{2,p}_{A_1}(X)} + z C_0^2 \|a_0\|_{W^{1,p}_{A_1}(X)} \|\nabla_{A_1} u_0\|_{W^{1,p}_{A_1}(X)} \\ &\quad \leq \|\nabla^2_{A_1} u_0\|_{L^p(X)} + z C_0 \|a_0\|_{W^{1,q}_{A_1}(X)} \|u_0\|_{W^{2,p}_{A_1}(X)} + z C_0^2 \|a_0\|_{W^{1,q}_{A_1}(X)} \|u_0\|_{W^{2,p}_{A_1}(X)},\end{aligned}$$

where $z = z(g) \in [1, \infty)$ and now $C_0 = C_0(A_1, g, p, q) \in [1, \infty)$. By substituting the preceding bound into (2.31), we find that

$$\begin{aligned} \|d_{A_0} u_0\|_{L^{p^*}(X)} &\leq C_1 \|u_0\|_{W^{2,p}_{A_1}(X)}, \quad p < d, \\ \|d_{A_0} u_0\|_{L^r(X)} &\leq C_1 \|u_0\|_{W^{2,d}_{A_1}(X)}, \quad p = d, \\ \|d_{A_0} u_0\|_{L^\infty(X)} &\leq C_1 \|u_0\|_{W^{2,p}_{A_1}(X)}, \quad p > d, \end{aligned}$$

for a constant $C_1 = C_1(A_1, g, p, q, \|a_0\|_{W^{1,q}_{A_1}(X)}) \in [1, \infty)$, with dependence on $p$ replaced by $\delta$ when $p = d$. By combining the preceding three cases ($p < d$, and $p = d$, and $p > d$), we obtain

$$\tag{2.32} \begin{aligned}\|\Delta_{A_0} u_0\|_{L^p(X)} &\leq C_1 \left(\|B - A_0\|_{L^s(X)} + \|A - A_0\|_{L^s(X)}\right) \|u_0\|_{W^{2,p}_{A_1}(X)} \\ &\quad + \|d^*_{A_0}(A - A_0)\|_{L^p(X)},\end{aligned}$$

where $s = s(p)$ is as in (2.23). From the *a priori* estimate (2.5) in Corollary 2.5 — and noting that this lemma also holds for $\mathrm{Ad}P$ in place of $\mathrm{ad}P$ via the definition of Sobolev norms of $u \in \mathrm{Aut}(P)$ described earlier — we have the *a priori* estimate,

$$\|u_0\|_{W^{2,p}_{A_1}(X)} \leq C_3 \|\Delta_{A_0} u_0\|_{L^p(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q, \tag{2.33}$$

where $C_3 = C_3(A_1, A_0, g, G, p, q) \in [1, \infty)$. Substituting the *a priori* estimate (2.33) into our $L^p$ bound (2.32) for $\Delta_{A_0} u_0$ gives, for $s$ as in (2.23),

$$\begin{aligned}\|\Delta_{A_0} u_0\|_{L^p(X)} \leq C_1 C_3 \left(\|B - A_0\|_{L^s(X)} + \|A - A_0\|_{L^s(X)}\right) \|\Delta_{A_0} u_0\|_{L^p(X)} \\ + \|d^*_{A_0}(A - A_0)\|_{L^p(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q.\end{aligned}$$

Provided

$$\|B - A_0\|_{L^s(X)} \leq 1/(4C_1C_3) \quad \text{and} \quad \|A - A_0\|_{L^s(X)} \leq 1/(4C_1C_3), \tag{2.34}$$

as assured by (2.22), then rearrangement in the preceding inequality yields

$$\|\Delta_{A_0} u_0\|_{L^p(X)} \leq 2\|d^*_{A_0}(A - A_0)\|_{L^p(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q. \tag{2.35}$$

Therefore, by combining the inequalities (2.35) and (2.33) we find that

$$\|u_0\|_{W^{2,p}_{A_1}(X)} \leq 2C_3\|d^*_{A_0}(A - A_0)\|_{L^p(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q. \tag{2.36}$$

Using $d_{A_0} u = d_{A_0} u_0$ and (2.27) and the facts that $|u| \leq 1$ and $|u^{-1}| \leq 1$ on $X$, we obtain

$$\|B - A_0\|_{L^p(X)} \leq \|A - A_0\|_{L^p(X)} + \|d_{A_0} u_0\|_{L^p(X)}. \tag{2.37}$$

From (2.37) and (2.36), we see that

$$\begin{aligned}\|B - A_0\|_{L^p(X)} &\leq \|A - A_0\|_{L^p(X)} + \|u_0\|_{W^{1,p}_{A_1}(X)} \\ &\leq \|A - A_0\|_{L^p(X)} + 2C_3\|d^*_{A_0}(A - A_0)\|_{L^p(X)}.\end{aligned}$$

Using $A_0 = A_1 + a_0$, we have $d^*_{A_0}(A - A_0) = d^*_{A_1}(A - A_0) + a_0 \times (A - A_0)$ and

$$\|d^*_{A_0}(A - A_0)\|_{L^p(X)} \leq z\|A - A_0\|_{W^{1,p}_{A_1}(X)} + z\|a_0\|_{L^{2p}(X)}\|A - A_0\|_{L^{2p}(X)}.$$

Applying the continuous Sobolev embedding, $W^{1,p}(X) \subset L^{2p}(X)$, with norm $C_0 = C_0(g, p) \in [1, \infty)$ and the Kato Inequality [22, Equation (6.20)],

$$\|d^*_{A_0}(A - A_0)\|_{L^p(X)} \leq z\|A - A_0\|_{W^{1,p}_{A_1}(X)} + zC_0^2\|a_0\|_{W^{1,p}_{A_1}(X)}\|A - A_0\|_{W^{1,p}_{A_1}(X)}. \tag{2.38}$$

Thus,

$$\|B - A_0\|_{L^p(X)} \leq C_4\|A - A_0\|_{W^{1,p}_{A_1}(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q, \tag{2.39}$$

where $C_4 = C_4(A_0, A_1, g, G, p, q) \in [1, \infty)$, giving the desired $L^p$ estimate (2.24) for $B - A_0$.

We now estimate the $L^p$ norms of the covariant derivatives of the right-hand side of the identity (2.27). Considering the term $u^{-1} d_{A_0} u$ in the right-hand side of the identity (2.27) and recalling that $\nabla_{A_0} u = d_{A_0} u = d_{A_0} u_0$, we have

$$\nabla_{A_0}(u^{-1} d_{A_0} u_0) = -u^{-1}(\nabla_{A_0} u_0)u^{-1} \otimes d_{A_0} u_0 + u^{-1}\nabla_{A_0} d_{A_0} u_0.$$

First, if $d/2 \le p < d$ and using the continuous multiplication, $L^{p^*}(X) \times L^d(X) \to L^p(X)$,

$$\begin{aligned}\|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^p(X)} &\le \|\nabla_{A_0}u_0\|_{L^{p^*}(X)}\|\nabla_{A_0}u_0\|_{L^d(X)} + \|\nabla^2_{A_0}u_0\|_{L^p(X)} \\ &\le C_0^2\|\nabla_{A_0}u_0\|^2_{W^{1,p}_{A_0}(X)} + \|\nabla^2_{A_0}u_0\|_{L^p(X)} \\ &\le C_0^2\|u_0\|^2_{W^{2,p}_{A_0}(X)} + \|u_0\|_{W^{2,p}_{A_0}(X)}.\end{aligned}$$

Second, if $p = d$ and using the continuous multiplication, $L^{2d}(X) \times L^{2d}(X) \to L^d(X)$,

$$\begin{aligned}\|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^d(X)} &\le \|\nabla_{A_0}u_0\|_{L^{2d}(X)}\|\nabla_{A_0}u_0\|_{L^{2d}(X)} + \|\nabla^2_{A_0}u_0\|_{L^d(X)} \\ &\le C_0^2\|\nabla_{A_0}u_0\|^2_{W^{1,d}_{A_0}(X)} + \|\nabla^2_{A_0}u_0\|_{L^d(X)} \\ &\le C_0^2\|u_0\|^2_{W^{2,d}_{A_0}(X)} + \|u_0\|_{W^{2,d}_{A_0}(X)}.\end{aligned}$$

Third, if $p > d$,

$$\begin{aligned}\|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^p(X)} &\le \|\nabla_{A_0}u_0\|_{L^\infty(X)}\|\nabla_{A_0}u_0\|_{L^p(X)} + \|\nabla^2_{A_0}u_0\|_{L^p(X)} \\ &\le C_0^2\|\nabla_{A_0}u_0\|^2_{W^{1,p}_{A_0}(X)} + \|\nabla^2_{A_0}u_0\|_{L^p(X)} \\ &\le C_0^2\|u_0\|^2_{W^{2,p}_{A_0}(X)} + \|u_0\|_{W^{2,p}_{A_0}(X)}.\end{aligned}$$

Thus, by combining the three preceding cases and applying Lemma A.2 (5),

$$\|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^p(X)} \le C_6^2C_0^2\|u_0\|^2_{W^{2,p}_{A_1}(X)} + C_6\|u_0\|_{W^{2,p}_{A_1}(X)}, \quad \text{for } d/2 \le p \le q, \tag{2.40}$$

where $C_6 = C_6(A_0, A_1, g, p, q) \in [1, \infty)$ is the constant in Lemma A.2 (5). By combining the bound (2.40) for $\|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^p(X)}$ with (2.36), we find that

$$\|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^p(X)} \le 2C_6^2C_0^2C_3\|d^*_{A_0}(A - A_0)\|^2_{L^p(X)} + 2C_6C_3\|d^*_{A_0}(A - A_0)\|_{L^p(X)},$$
$$\text{for } 1 < p < \infty \text{ and } d/2 \le p \le q.$$

But

$$\|d^*_{A_0}(A - A_0)\|_{L^p(X)} \le z\|A - A_0\|_{W^{1,p}_{A_0}(X)}, \tag{2.41}$$

for a generic constant $z = z(g) \in [1, \infty)$, and by Lemma A.2 (3),

$$\|A - A_0\|_{W^{1,p}_{A_0}(X)} \le C_7\|A - A_0\|_{W^{1,p}_{A_1}(X)}, \tag{2.42}$$

where $C_7 = C_7(A_0, A_1, g, p) \in [1, \infty)$ is the constant in Lemma A.2 (3), and because $A$ is now assumed to obey (2.25), that is,

$$\|A - A_0\|_{W^{1,p}_{A_1}(X)} \le M,$$

we obtain

$$\|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^p(X)} \le 2C_6C_3(zC_6C_7C_0^2M + 1)\|d^*_{A_0}(A - A_0)\|_{L^p(X)},$$
$$\text{for } 1 < p < \infty \text{ and } d/2 \le p \le q,$$

and thus by (2.41) and (2.42),

$$\|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^p(X)} \le 2z^2C_6C_7C_3(zC_6C_7C_0^2M + 1)\|A - A_0\|_{W^{1,p}_{A_1}(X)}, \tag{2.43}$$
$$\text{for } 1 < p < \infty \text{ and } d/2 \le p \le q.$$

Considering the term $u^{-1}(A-A_0)u$ in the right-hand side of (2.27), we discover that

$$\begin{aligned}\nabla_{A_0}(u^{-1}(A-A_0)u) &= -u^{-1}(\nabla_{A_0}u)u^{-1}\otimes(A-A_0)u + u^{-1}(\nabla_{A_0}(A-A_0))u \\ &\quad + u^{-1}(A-A_0)\otimes\nabla_{A_0}u.\end{aligned}$$

Noting that $\nabla_{A_0}u = \nabla_{A_0}u_0$ and $\|u\|_{C(X)} \leq 1$, the preceding identity gives, for $d/2 \leq p < d$,

$$\begin{aligned}\|\nabla_{A_0}(u^{-1}(A-A_0)u)\|_{L^p(X)} &\leq 2\|\nabla_{A_0}u_0\|_{L^{p^*}(X)}\|A-A_0\|_{L^d(X)} + \|\nabla_{A_0}(A-A_0)\|_{L^p(X)} \\ &\leq 2C_0^2\|\nabla_{A_0}u_0\|_{W^{1,p}_{A_0}(X)}\|A-A_0\|_{W^{1,p}_{A_0}(X)} + \|\nabla_{A_0}(A-A_0)\|_{L^p(X)} \\ &\leq 2C_0^2\|u_0\|_{W^{2,p}_{A_0}(X)}\|A-A_0\|_{W^{1,p}_{A_0}(X)} + \|\nabla_{A_0}(A-A_0)\|_{L^p(X)}.\end{aligned}$$

Second, for the case $p = d$,

$$\begin{aligned}\|\nabla_{A_0}(u^{-1}(A-A_0)u)\|_{L^d(X)} &\leq 2\|\nabla_{A_0}u_0\|_{L^{2d}(X)}\|A-A_0\|_{L^{2d}(X)} + \|\nabla_{A_0}(A-A_0)\|_{L^d(X)} \\ &\leq 2C_0^2\|\nabla_{A_0}u_0\|_{W^{1,d}_{A_0}(X)}\|A-A_0\|_{W^{1,d}_{A_0}(X)} + \|\nabla_{A_0}(A-A_0)\|_{L^d(X)} \\ &\leq 2C_0^2\|u_0\|_{W^{2,p}_{A_0}(X)}\|A-A_0\|_{W^{1,d}_{A_0}(X)} + \|\nabla_{A_0}(A-A_0)\|_{L^d(X)}.\end{aligned}$$

Third, for the case $d < p < \infty$,

$$\begin{aligned}\|\nabla_{A_0}(u^{-1}(A-A_0)u)\|_{L^p(X)} &\leq 2\|\nabla_{A_0}u_0\|_{L^\infty(X)}\|A-A_0\|_{L^p(X)} + \|\nabla_{A_0}(A-A_0)\|_{L^p(X)} \\ &\leq 2C_0\|\nabla_{A_0}u_0\|_{W^{1,d}_{A_0}(X)}\|A-A_0\|_{L^p(X)} + \|\nabla_{A_0}(A-A_0)\|_{L^p(X)} \\ &\leq 2C_0\|u_0\|_{W^{2,p}_{A_0}(X)}\|A-A_0\|_{L^p(X)} + \|\nabla_{A_0}(A-A_0)\|_{L^p(X)}.\end{aligned}$$

Hence, the combination of the preceding three cases gives

$$\|\nabla_{A_0}(u^{-1}(A-A_0)u)\|_{L^p(X)} \leq 2C_0^2\|u_0\|_{W^{2,p}_{A_0}(X)}\|A-A_0\|_{W^{1,p}_{A_0}(X)} + \|\nabla_{A_0}(A-A_0)\|_{L^p(X)}, \quad \text{for } p < \infty \text{ and } d/2 \leq p \leq q.$$

and applying Lemma A.2, Items (3) and (5),

$$\begin{aligned}\|\nabla_{A_0}(u^{-1}(A-A_0)u)\|_{L^p(X)} &\leq 2C_6C_7C_0^2\|u_0\|_{W^{2,p}_{A_1}(X)}\|A-A_0\|_{W^{1,p}_{A_1}(X)} \\ &\quad + C_7\|A-A_0\|_{W^{1,p}_{A_1}(X)}, \quad \text{for } p < \infty \text{ and } d/2 \leq p \leq q.\end{aligned} \tag{2.44}$$

Therefore, combining the inequalities (2.36) and (2.44) yields

$$\begin{aligned}&\|\nabla_{A_0}(u^{-1}(A-A_0)u)\|_{L^p(X)} \\ &\quad\leq 4C_6C_7C_0^2C_3\|d^*_{A_0}(A-A_0)\|_{L^p(X)}\|A-A_0\|_{W^{1,p}_{A_1}(X)} + C_7\|A-A_0\|_{W^{1,p}_{A_1}(X)}, \\ &\quad\quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q.\end{aligned} \tag{2.45}$$

From (2.25) and (2.41), we see that (2.45) simplifies to give

$$\|\nabla_{A_0}(u^{-1}(A-A_0)u)\|_{L^p(X)} \leq (4zC_6C_7C_0^2C_3M+1)C_7\|A-A_0\|_{W^{1,p}_{A_1}(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q. \tag{2.46}$$

From the identity (2.27) (noting again that $d_{A_0}u = d_{A_0}u_0$) we have

$$\|\nabla_{A_0}(B-A_0)\|_{L^p(X)} \leq \|\nabla_{A_0}(u^{-1}(A-A_0)u\|_{L^p(X)} + \|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^p(X)}.$$

Combining the preceding estimate with the inequalities (2.43) and (2.46) gives

$$\|\nabla_{A_0}(B-A_0)\|_{L^p(X)} \leq C_5\|A-A_0\|_{W^{1,p}_{A_1}(X)}, \quad \text{for } 1<p<\infty \text{ and } d/2\leq p\leq q, \tag{2.47}$$

with a constant $C_5 = C_5(A_0, A_1, g, G, M, p, q) \in [1,\infty)$. Finally, from (2.39) and (2.47) and Lemma A.2 (3) we obtain the desired $W^{1,p}_{A_1}$ bound (2.26) for $u(A)-A_0$ in terms of $A-A_0$, recalling that $B = u(A)$, with large enough constant $N = N(A_0, A_1, g, G, M, p, q) \in [1,\infty)$ and under the hypothesis (2.22) with small enough constant $\varepsilon = \varepsilon(A_0, A_1, g, G, p, q) \in (0,1]$. □

The proof of Proposition 2.11 also yields the following useful

**Lemma 2.12** (*A priori* $W^{2,p}$ estimate for a $W^{2,q}$ gauge transformation $u$ intertwining two $W^{1,q}$ connections)**.** *Assume the hypotheses of Proposition 2.11, excluding those on the connection $A$. Then there is a constant $C = C(A_0, A_1, g, G, p, q) \in [1,\infty)$ with the following significance. If $A$ obeys the hypotheses of Proposition 2.11 and $u \in \mathrm{Aut}^{2,q}(P)$ is the resulting gauge transformation, depending on $A$ and $A_0$, such that*

$$d^*_{A_0}(u(A)-A_0) = 0,$$

*then*

$$\|u\|_{W^{2,p}_{A_1}(X)} \leq C.$$

*Proof.* Write $u = u_0+\gamma$ as in the proof of Proposition 2.11, with $u_0 \in (\operatorname{Ker}\Delta_{A_0})^\perp$ and $\gamma \in \operatorname{Ker}\Delta_{A_0}$, and observe that

$$\begin{aligned}
\|u\|_{W^{2,p}_{A_1}(X)} &\leq C\left(\|\Delta_{A_0}u\|_{L^p(X)} + \|u\|_{L^p(X)}\right) \quad \text{(by Proposition 2.1)}\\
&= C\left(\|\Delta_{A_0}u_0\|_{L^p(X)} + \|u\|_{L^p(X)}\right)\\
&\leq C\left(\|d^*_{A_0}(A-A_0)\|_{L^p(X)} + \|u\|_{L^p(X)}\right) \quad \text{(by (2.35))}\\
&\leq C\left(1+\|A_0-A_1\|_{W^{1,p}_{A_1}(X)}\right)\|A-A_0\|_{W^{1,p}_{A_1}(X)} + C\|u\|_{L^p(X)} \quad \text{(by (2.38))}\\
&\leq C\left(1+\|A_0-A_1\|_{W^{1,p}_{A_1}(X)}\right)(\varepsilon + C\operatorname{Vol}_g(X)^{1/p},
\end{aligned}$$

where the last inequality follows from (2.22) with $\varepsilon \in (0,1]$, the Sobolev embedding $W^{1,p}(X) \subset L^s(X)$ given by [3, Theorem 4.12], the Kato Inequality [22, Equation (6.20)], and the fact that $|u| \leq 1$ pointwise. This completes the proof. □

2.6. **Existence of Coulomb gauge transformations for connections.** Finally, we can proceed to the

*Proof of Theorem 2.* We shall apply the method of continuity, modeled on the proofs of [54, Theorem 2.1] due to Uhlenbeck and [10, Proposition 2.3.13] due to Donaldson and Kronheimer. For a related application of the method of continuity, see the proof [14, Theorem 1.1] due to the first author.

We begin by defining a one-parameter family of $W^{1,q}$ connections by setting

$$A_t := A_0 + t(A-A_0), \quad \forall\, t \in [0,1],$$

and observe that their curvatures are given by

$$F(A_t) = F_{A_0} + t d_{A_0}(A-A_0) + \frac{t^2}{2}[A-A_0, A-A_0],$$

and they obey the bounds,

$$\begin{aligned}\|F(A_t)\|_{L^q(X)} &\leq \|F_{A_0}\|_{L^q(X)} + \|d_{A_0}(A-A_0)\|_{L^q(X)} + \frac{1}{2}\|A-A_0\|^2_{L^{2q}(X)} \\ &\leq \|F_{A_0}\|_{L^q(X)} + \|d_{A_0}(A-A_0)\|_{L^q(X)} + c\|A-A_0\|^2_{W^{1,q}_{A_1}(X)},\end{aligned}$$

with $c = c(g,q) \in [1,\infty)$ and where we use the Sobolev embedding, $W^{1,q}(X) \subset L^{2q}(X)$ [3, Theorem 4.12] and the Kato Inequality [22, Equation (6.20)] to obtain the last inequality. Note that we have a continuous embedding, $W^{1,q}(X) \subset L^{2q}(X)$, when $2q \leq q^* := dq/(d-q)$, that is $2d - 2q \leq d$ or $q \geq d/2$, as implied by our hypotheses. Therefore,

$$\|F(A_t)\|_{L^q(X)} \leq K, \quad \forall\, t \in [0,1], \tag{2.48}$$

for $K = K(A, A_0, G, g, q) \in [1,\infty)$, noting that $F_{A_0} \in L^q(X;\Lambda^2 \otimes \mathrm{ad}P)$ since $A_0$ is of class $W^{1,q}$.

Let $S$ denote the set of $t \in [0,1]$ such that there exists a $W^{2,q}$ gauge transformation $u_t \in \mathrm{Aut}(P)$ with the property that

$$d^*_{A_0}(u_t(A_t) - A_0) = 0 \quad \text{and} \quad \|u_t(A_t) - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\|A_t - A_0\|_{W^{1,p}_{A_1}(X)},$$

where $N$ is the constant in Proposition 2.11. Clearly, $0 \in S$ since the identity automorphism of $P$ is the required gauge transformation in that case, so $S$ is non-empty. As usual, we need to show that $S$ is an open and closed subset of $[0,1]$.

**Step 1** ($S$ is open)**.** To prove openness, we shall adapt the argument of Donaldson and Kronheimer in [10, Section 2.3.8]. We apply the Implicit Function Theorem to the gauge fixing equation,

$$d^*_{A_0}(u_t(A_t) - A_0) = d^*_{A_0}\left(u_t^{-1}(A_t - A_0)u_t + u_t^{-1}d_{A_0}u_t\right) = 0.$$

As usual, we denote $B_t = u_t(A_t)$ for convenience, for $t \in S$. Let $t_0 \in S$, so we have

$$d^*_{A_0}\left(u_{t_0}(A_{t_0}) - A_0\right) = 0 \quad \text{and} \quad \|u_{t_0}(A_{t_0}) - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\|A_{t_0} - A_0\|_{W^{1,p}_{A_1}(X)}.$$

Our task is to show that $t_0 + s \in S$ for $|s|$ sufficiently small, that is, there exists $u_{t_0+s} \in \mathrm{Aut}^{2,q}(P)$ such that the preceding two properties hold with $t_0$ replaced by $t_0 + s$. By the hypothesis (1.3) of Theorem 2, we have $\|A - A_0\|_{W^{1,p}_{A_1}(X)} < \zeta$ and thus, since $A_{t_0} - A_0 = A_0 + t_0(A - A_0) - A_0 = t_0(A - A_0)$ and $t_0 \in [0,1]$, we see that

$$\|A_{t_0} - A_0\|_{W^{1,p}_{A_1}(X)} < \zeta,$$

and so

$$\|u_{t_0}(A_{t_0}) - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\zeta.$$

It will be convenient to define $a \in W^{1,q}_{A_1}(X;\Lambda^1 \otimes \mathrm{ad}P)$ by

$$u_{t_0}(A_{t_0}) =: A_0 + a, \tag{2.49}$$

The preceding inequality ensures that

$$\|a\|_{W^{1,p}_{A_1}(X)} < 2N\zeta. \tag{2.50}$$

We shall seek a solution $u_{t_0+s} \in \mathrm{Aut}^{2,q}(P)$ to the gauge-fixing equation,

$$d^*_{A_0}(u_{t_0+s}(A_{t_0+s}) - A_0) = 0.$$

In particular, we shall seek a solution in the form

$$u_{t_0+s} = e^{\chi_s}u_{t_0}, \quad \text{for } \chi_s \in W^{2,q}_{A_1}(X;\mathrm{ad}P),$$

so the gauge-fixing equation becomes

$$d_{A_0}^*(e^{\chi_s} u_{t_0}(A_{t_0+s}) - A_0) = 0. \tag{2.51}$$

For $s \in \mathbb{R}$, it will be convenient to define $b_s \in W_{A_1}^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P)$ by

$$u_{t_0}(A_{t_0+s}) =: A_0 + a + b_s. \tag{2.52}$$

We can determine $b_s$ explicitly using $A_{t_0+s} = A_0 + (t_0+s)(A - A_0)$, so that

$$\begin{aligned} u_{t_0}(A_{t_0+s}) - A_0 &= u_{t_0}^{-1}((t_0+s)(A-A_0))u_{t_0} + u_{t_0}^{-1} d_{A_0} u_{t_0} \\ &= u_{t_0}^{-1}(t_0(A-A_0))u_{t_0} + u_{t_0}^{-1} d_{A_0} u_{t_0} + u_{t_0}^{-1}(s(A-A_0))u_{t_0} \\ &= u_{t_0}(A_{t_0}) - A_0 + s u_{t_0}^{-1}(A-A_0)u_{t_0} \\ &= a + s u_{t_0}^{-1}(A-A_0)u_{t_0} \quad \text{(by (2.49))}, \end{aligned}$$

and thus

$$b_s = s u_{t_0}^{-1}(A-A_0)u_{t_0}, \quad s \in \mathbb{R}. \tag{2.53}$$

Note that $u_{t_0} \in \mathrm{Aut}^{2,q}(P)$ and so we have the estimate

$$\|b_s\|_{W_{A_1}^{1,q}(X)} \leq |s| C_0 \|A - A_0\|_{W_{A_1}^{1,q}(X)}, \quad s \in \mathbb{R}, \tag{2.54}$$

for $C_0 = C_0(A_0, g, q, t_0) \in [1, \infty)$. In particular, $b_s \to 0$ strongly in $W_{A_1}^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P)$ as $s \to 0$.

The gauge fixing equation (2.51) takes the form

$$\begin{aligned} e^{\chi_s} u_{t_0}(A_{t_0+s}) - A_0 &= e^{\chi_s}(A_0 + a + b_s) - A_0 \\ &= e^{-\chi_s}(a + b_s)e^{\chi_s} + e^{-\chi_s} d_{A_0}(e^{\chi_s}). \end{aligned}$$

The equation to be solved is then $H(\chi_s, b_s) = 0$, where

$$H(\chi, b) := d_{A_0}^* \left(e^{-\chi}(a+b)e^{\chi} + e^{-\chi} d_{A_0}(e^{\chi})\right). \tag{2.55}$$

For any $q > d/2$, the expression (2.55) for $H$ defines a smooth map,

$$H : (\mathrm{Ker}\,\Delta_{A_0})^\perp \cap W_{A_1}^{2,q}(X; \mathrm{ad}P) \times W_{A_1}^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P) \to (\mathrm{Ker}\,\Delta_{A_0})^\perp \cap L^q(X; \mathrm{ad}P). \tag{2.56}$$

Here, we note that if $\xi = d_{A_0}^* a_\xi$ for some $a_\xi \in W_{A_1}^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P)$, then $\xi \perp \mathrm{Ker}\,\Delta_{A_0}$, as implied by the preceding expression for $H$. Indeed, for any $\gamma \in \mathrm{Ker}\,\Delta_{A_0}$,

$$(\xi, \gamma)_{L^2(X)} = (d_{A_0}^* a_\xi, \gamma)_{L^2(X)} = (a_\xi, d_{A_0}\gamma)_{L^2(X)} = 0.$$

Hence, the image of $H$ is contained in $(\mathrm{Ker}\,\Delta_{A_0})^\perp \cap L^q(X; \mathrm{ad}P)$.

The Implicit Function Theorem asserts that if the partial derivative,

$$(D_1 H)_{(0,0)} : (\mathrm{Ker}\,\Delta_{A_0})^\perp \cap W_{A_1}^{2,q}(X; \mathrm{ad}P) \to (\mathrm{Ker}\,\Delta_{A_0})^\perp \cap L^q(X; \mathrm{ad}P),$$

is surjective, then for small $b \in W_{A_1}^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P)$ there is a small solution $\chi \in W_{A_1}^{2,q}(X; \mathrm{ad}P)$ to $H(\chi, b) = 0$ and that is $L^2$-orthogonal to $\mathrm{Ker}\,\Delta_{A_0}$.

Now the linearization, $(D_1 H)_{(0,0)}$, of the map $H$ at the origin $(0,0)$ with respect to variations in $\chi$ is given by

$$(D_1 H)_{(0,0)}\chi = d_{A_0}^* d_{A_0+a}\chi.$$

But $\|a\|_{W^{1,p}_{A_1}(X)} < 2N\zeta$ by (2.50) and because of the continuous Sobolev embeddings provided by [3, Theorem 4.12],

$$W^{1,p}(X) \subset L^d(X), \quad \text{for } d \geq 3 \text{ and } p \geq d/2,$$
$$W^{1,p}(X) \subset L^4(X), \quad \text{for } d = 2 \text{ and } p \geq 2,$$

we obtain,

$$\|a\|_{L^d(X)} < 2C_1 N\zeta \quad \text{when } d \geq 3 \quad \text{and} \quad \|a\|_{L^4(X)} < 2C_1 N\zeta \quad \text{when } d = 2,$$

where $C_1 = C_1(g,p) \in [1,\infty)$ is the norm of the Sobolev embedding employed. By the hypothesis of Theorem 2, we can choose $\zeta \in (0,1]$ as small as desired. Hence, the operator $d^*_{A_0} d_{A_0+a}$ is surjective by Lemma 2.9.

To summarize, we have shown that if $|s|$ is small and $t_0 \in S$, then there exists $u_{t_0+s} \in \mathrm{Aut}^{2,q}(P)$ such that

$$d^*_{A_0}(u_{t_0+s}(A_{t_0+s}) - A_0) = 0.$$

It remains to check that the following norm condition holds,

$$\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\|A_{t_0+s} - A_0\|_{W^{1,p}_{A_1}(X)}, \tag{2.57}$$

for small enough $|s|$, to conclude that $t_0 + s \in S$. To see this, we first note that since $A_{t_0+s} = A_0 + (t_0+s)(A - A_0)$, we have

$$\|A_{t_0+s} - A_0\|_{W^{1,p}_{A_1}(X)} = (t_0+s)\|A - A_0\|_{W^{1,p}_{A_1}(X)} < (t_0+s)\zeta \quad \text{(by (1.3))},$$

and thus, for $t_0 + s \leq 1$ and $\zeta \leq \varepsilon/C_1$,

$$\|A_{t_0+s} - A_0\|_{W^{1,p}_{A_1}(X)} \leq \varepsilon/C_1,$$

and thus,

$$\|A_{t_0+s} - A_0\|_{L^d(X)} \leq \varepsilon, \quad \text{if } p < d,$$
$$\|A_{t_0+s} - A_0\|_{L^{d+\delta}(X)} \leq \varepsilon, \quad \text{if } p = d,$$
$$\|A_{t_0+s} - A_0\|_{L^p(X)} \leq \varepsilon, \quad \text{if } p > d,$$

where $\varepsilon$ is the constant in Proposition 2.11 and $C_1 = C_1(g,p)$ or $C_1(\delta,p) \in [1,\infty)$ is the norm (provided by [3, Theorem 4.12]) of the continuous Sobolev embedding, $W^{1,p}(X) \subset L^d(X)$ when $d/2 \leq p < d$ and $W^{1,p}(X) \subset L^{d+\delta}(X)$ when $p = d$. This verifies the hypotheses (2.22) and (2.25) of Proposition 2.11 for $A_{t_0+s} - A_0$ (in place of $A - A_0$ in the statement of that proposition).

On the other hand,

$$\begin{aligned}
&\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{W^{1,p}_{A_1}(X)} \\
&\quad = \|e^{\chi_s} u_{t_0}(A_{t_0+s}) - A_0\|_{W^{1,p}_{A_1}(X)} \\
&\quad \leq \|e^{\chi_s} u_{t_0}(A_{t_0+s}) - u_{t_0}(A_{t_0+s})\|_{W^{1,p}_{A_1}(X)} + \|u_{t_0}(A_{t_0+s}) - A_0\|_{W^{1,p}_{A_1}(X)} \\
&\quad = \|e^{\chi_s} u_{t_0}(A_{t_0+s}) - u_{t_0}(A_{t_0+s})\|_{W^{1,p}_{A_1}(X)} + \|a + b_s\|_{W^{1,p}_{A_1}(X)} \quad \text{(by (2.52))}.
\end{aligned}$$

The inequalities (1.3), (2.50), and (2.54) (which also holds with $p$ in place of $q$) yield the bound

$$\begin{aligned}\|a+b_s\|_{W^{1,p}_{A_1}(X)} &\leq \|a\|_{W^{1,p}_{A_1}(X)} + \|b_s\|_{W^{1,p}_{A_1}(X)} \\ &< 2N\zeta + |s|C_0\|A-A_0\|_{W^{1,p}_{A_1}(X)} \\ &\leq 2N\zeta + |s|C_0\zeta \\ &\leq \varepsilon/(2C_1),\end{aligned}$$

for small enough $\zeta$. (Note that we could also have used $\|b_s\|_{W^{1,q}_{A_1}(X)} \leq C_0\|A-A_0\|_{W^{1,q}_{A_1}(X)}$ and the continuous embedding $W^{1,q}(X) \subset W^{1,p}(X)$ and rely on our freedom to also choose $|s|$ small.) But if $|s|$ is small then so is $\|b_s\|_{W^{1,q}_{A_1}(X)}$ by (2.54) and hence $\|\chi_s\|_{W^{2,q}_{A_1}(X)}$ is small by the Implicit Function Theorem[4] and so we may assume that

$$\|e^{\chi_s}u_{t_0}(A_{t_0+s}) - u_{t_0}(A_{t_0+s})\|_{W^{1,p}_{A_1}(X)} \leq \varepsilon/(2C_1),$$

for small enough $|s|$. Collecting the preceding inequalities gives

$$\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{W^{1,p}_{A_1}(X)} \leq \varepsilon/C_1,$$

and thus,

$$\begin{aligned}\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{L^d(X)} &\leq \varepsilon, \quad \text{if } p<d, \\ \|u_{t_0+s}(A_{t_0+s}) - A_0\|_{L^{d+\delta}(X)} &\leq \varepsilon, \quad \text{if } p=d, \\ \|u_{t_0+s}(A_{t_0+s}) - A_0\|_{L^p(X)} &\leq \varepsilon, \quad \text{if } p>d,\end{aligned}$$

verifying the hypotheses (2.22) and (2.25) of Proposition 2.11 for $u_{t_0+s}(A_{t_0+s}) - A_0$ (in place of $u(A) - A_0$ in the statement of that proposition).

Hence, Proposition 2.11 yields the bound

$$\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{W^{1,p}_{A_1}(X)} \leq N\|A_{t_0+s} - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\|A_{t_0+s} - A_0\|_{W^{1,p}_{A_1}(X)}.$$

This verifies the norm condition (2.57) and we conclude that $t_0+s \in S$ and so $S$ is open.

**Step 2** ($S$ is closed)**.** For closedness, we adapt the argument in [10, Section 2.3.7]. Set $B_t := u_t(A_t)$ for $t \in S$ and observe that the inequality (2.48) yields

$$\|F(B_t)\|_{L^q(X)} = \|F(u_t(A_t))\|_{L^q(X)} = \|u_t(F(A_t))\|_{L^q(X)} = \|F(A_t)\|_{L^q(X)} \leq K, \quad \forall\, t \in S.$$

Let $\{t_m\}_{m\in\mathbb{N}} \subset S$ be a sequence and suppose that $t_m \to t_\infty \in [0,1]$ as $m \to \infty$. Since $q > d/2$ by hypothesis, the Uhlenbeck Weak Compactness [54, Theorem 1.5 = 3.6] (see also [58, Theorem 7.1] for a recent exposition) implies that there exists a subsequence $\{m'\} \subset \{m\}$ and, after relabeling, a sequence of $W^{2,q}$ gauge transformations, $\{u_{t_m}\}_{m\in\mathbb{N}} \subset \mathrm{Aut}(P)$ and a $W^{1,q}$ connection $B_\infty$ on $P$ such that, as $m \to \infty$, we have

$$B_{t_m} \rightharpoonup B_\infty \quad \text{weakly in } W^{1,q}_{A_1}(X;\Lambda^1 \otimes \mathrm{ad}P).$$

Hence, for $1 \leq r < q* := dq/(d-q)$ and $W^{1,q}(X) \Subset L^r(X)$, there exists a subsequence $\{m''\} \subset \{m\}$ such that, after again relabeling, as $m \to \infty$ we have

$$B_{t_m} \to B_\infty \quad \text{strongly in } L^r(X;\Lambda^1 \otimes \mathrm{ad}P).$$

[4]The Quantitative Inverse Function Theorem — see, for example, [1, Proposition 2.5.6] — can be used to give a precise bound on $\chi_s$ given a bound on $b_s$.

The $W^{2,q}$ gauge transformations $u_t$ intertwine the $W^{1,q}$ connections $A_t$ and $B_t$ via the relation,

$$B_t = A_0 + u_t^{-1}(A - A_0)u_t + u_t^{-1}d_{A_0}u_t,$$

and thus

$$d_{A_0}u_t = u_tB_t + u_tA_0 + (A - A_0)u_t, \quad \forall\, t \in [0,1].$$

Because $A_{t_m} \to A_\infty := A_0 + t_\infty(A - A_0)$ strongly in $W^{1,q}$ and $B_{t_m} \rightharpoonup B_\infty$ weakly in $W^{1,q}$ and $B_{t_m} \to B_\infty$ strongly in $L^r$ as $m \to \infty$, there exists a $W^{2,q}$ gauge transformation $u_\infty \in \operatorname{Aut}(P)$ such that, as $m \to \infty$,

$$u_{t_m} \rightharpoonup u_\infty \quad \text{weakly in } W^{2,q}_{A_1}(X;\operatorname{Ad}P) \quad \text{and} \quad u_{t_m} \to u_\infty \quad \text{strongly in } W^{1,r}_{A_1}(X;\operatorname{Ad}P).$$

In particular,

$$B_\infty = u_\infty(A_\infty) \quad \text{and} \quad d^*_{A_0}(u_\infty(A_\infty) - A_0) = 0.$$

The Coulomb gauge condition follows from the fact that, for any $\xi \in C^\infty(X;\operatorname{ad}P)$, we have

$$\begin{aligned}
0 &= \lim_{m\to\infty}(d^*_{A_0}(u_{t_m}(A_{t_m}) - A_0), \xi)_{L^2(X)} \\
&= \lim_{m\to\infty}(u_{t_m}(A_{t_m}) - A_0, d_{A_0}\xi)_{L^2(X)} \\
&= (u_\infty(A_\infty) - A_0, d_{A_0}\xi)_{L^2(X)} \\
&= (d^*_{A_0}(u_\infty(A_\infty) - A_0), \xi)_{L^2(X)}.
\end{aligned}$$

Similarly, for any $a \in \Omega^1(X;\operatorname{ad}P)$,

$$\begin{aligned}
\lim_{m\to\infty}(B_{t_m} - A_0, a)_{L^2(X)} &= \lim_{m\to\infty}(u_{t_m}(A_{t_m}) - A_0, a)_{L^2(X)} \\
&= \lim_{m\to\infty}(u_{t_m}^{-1}(A_{t_m} - A_0)u_{t_m} + u_{t_m}^{-1}d_{A_0}u_{t_m}, a)_{L^2(X)} \\
&= \lim_{m\to\infty}((A_{t_m} - A_0)u_{t_m} + d_{A_0}u_{t_m}, u_{t_m}a)_{L^2(X)} \\
&= ((A_\infty - A_0)u_\infty + d_{A_0}u_\infty, u_\infty a)_{L^2(X)} \\
&= (u_\infty^{-1}(A_\infty - A_0)u_\infty + u_\infty^{-1}d_{A_0}u_\infty, a)_{L^2(X)} \\
&= (u_\infty(A_\infty) - A_0, a)_{L^2(X)}.
\end{aligned}$$

We now wish to apply Proposition 2.11 to bound $\|u_{t_\infty}(A_{t_\infty}) - A_0\|_{W^{1,p}_{A_1}(X)}$ and establish the remaining norm condition required to show that $t_\infty \in S$. First, we note that

$$\begin{aligned}
\|A_{t_m} - A_0\|_{W^{1,p}_{A_1}(X)} &= \|A_0 + t_m(A - A_0) - A_0\|_{W^{1,p}_{A_1}(X)} \\
&= t_m\|A - A_0\|_{W^{1,p}_{A_1}(X)} \\
&< t_m\zeta \quad \text{(by hypothesis (1.3))} \\
&\le \zeta, \quad \forall\, m \in \mathbb{N}.
\end{aligned}$$

Because $t_m \in S$, we have

$$\|u_{t_m}(A_{t_m}) - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\|A_{t_m} - A_0\|_{W^{1,p}_{A_1}(X)}, \quad \forall\, m \in \mathbb{N},$$

and combining this inequality with the preceding inequality yields

$$\|u_{t_m}(A_{t_m}) - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\zeta, \quad \forall\, m \in \mathbb{N}.$$

Since $u_\infty(A_\infty)$ is the weak limit of $u_{t_m}(A_{t_m})$ in $W^{1,q}_{A_1}(X;\Lambda^1 \otimes \operatorname{ad}P)$, we have

$$\|u_\infty(A_\infty) - A_0\|_{W^{1,p}_{A_1}(X)} \le \liminf_{m\to\infty} 2N\|A_{t_m} - A_0\|_{W^{1,p}_{A_1}(X)}.$$

But $A_{t_m} \to A_{t_\infty}$ strongly in $W^{1,q}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)$ by construction of the path $A_t$ and as $p \leq q$ by hypothesis and $\|A_{t_m} - A_0\|_{W^{1,p}_{A_1}(X)} < \zeta$ for all $m \in \mathbb{N}$, then

$$\|A_{t_\infty} - A_0\|_{W^{1,p}_{A_1}(X)} = \lim_{m\to\infty} \|A_{t_m} - A_0\|_{W^{1,p}_{A_1}(X)} \leq \zeta.$$

Combining the preceding inequalities yields

$$\|u_\infty(A_\infty) - A_0\|_{W^{1,p}_{A_1}(X)} \leq 2N\zeta.$$

We choose $\zeta \in (0,1]$ small enough that $\zeta \leq \varepsilon/C_1$ and $2N\zeta \leq \varepsilon/C_1$, where $C_1$ is the norm of the Sobolev embedding $W^{1,p}(X) \subset L^d(X)$ when $p \neq d$ or $L^{d+\delta}(X)$ when $p = d$ as in Step 1, and then observe that

$$\|A_{t_\infty} - A_0\|_{W^{1,p}_{A_1}(X)} \leq \varepsilon/C_1 \quad \text{and} \quad \|u_\infty(A_\infty) - A_0\|_{W^{1,p}_{A_1}(X)} < \varepsilon/C_1.$$

The first inequality above verifies the hypothesis (2.25) of Proposition 2.11. Moreover,

$$\begin{aligned}
\|A_{t_\infty} - A_0\|_{L^d(X)} \leq \varepsilon \quad &\text{and} \quad \|u_\infty(A_{t_\infty}) - A_0\|_{L^d(X)} \leq \varepsilon, \quad \text{if } p < d,\\
\|A_{t_\infty} - A_0\|_{L^{d+\delta}(X)} \leq \varepsilon \quad &\text{and} \quad \|u_\infty(A_{t_\infty}) - A_0\|_{L^{d+\delta}(X)} \leq \varepsilon, \quad \text{if } p = d,\\
\|A_{t_\infty} - A_0\|_{L^p(X)} \leq \varepsilon \quad &\text{and} \quad \|u_\infty(A_{t_\infty}) - A_0\|_{L^p(X)} \leq \varepsilon, \quad \text{if } p > d,
\end{aligned}$$

which verifies the hypothesis (2.22) of Proposition 2.11 on norms (for $A_{t_\infty} - A_0$ in place of $A - A_0$ and $u_\infty(A_{t_\infty}) - A_0$ in place of $u(A) - A_0$ in the statement of that proposition). Since $d^*_{A_0}(u_\infty(A_\infty) - A_0) = 0$, as required by (2.21), Proposition 2.11 implies that

$$\|u_\infty(A_\infty) - A_0\|_{W^{1,p}_{A_1}(X)} \leq N\|A_{t_\infty} - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\|A_{t_\infty} - A_0\|_{W^{1,p}_{A_1}(X)}.$$

Thus, $t_\infty \in S$ and so $S$ is closed.

Consequently, $S \subset [0,1]$ is non-empty and open and closed by the preceding two steps, so $S = [0,1]$ and this completes the proof of Theorem 2. ☐

2.7. **Real analytic Banach manifold structure on the quotient space of connections.** The statements and proofs of Lemmata 2.13 and 2.14 would follow standard lines (see Gilkey [25, Theorem 1.5.2], for example) if the operators

$$d_A : \Omega^l(X; \mathrm{ad}P) \to \Omega^{l+1}(X; \mathrm{ad}P), \quad l \geq 0,$$

had $C^\infty$ coefficients, rather than Sobolev coefficients as we allow here, and formed an elliptic complex, rather than only satisfying $d_A \circ d_A = F_A$.

**Lemma 2.13** (Continuous operators on $L^p$ spaces and $L^2$-orthogonal decompositions)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$. If $A$ is a connection on $P$ of class $W^{1,q}$ with $q \geq d/2$, and $A_1$ is a $C^\infty$ is reference connection on $P$, and $l \geq 1$ is an integer, and $p$ obeys $d/2 \leq p \leq q$, then the operator*

$$d^*_A : W^{1,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P) \to L^p(X; \Lambda^{l-1} \otimes \mathrm{ad}P),$$

*is continuous and, if in addition $q > d/2$, then the operator*

$$d_A : W^{2,p}_{A_1}(X; \Lambda^{l-1} \otimes \mathrm{ad}P) \to W^{1,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P),$$

*is also continuous, and there is an $L^2$-orthogonal decomposition,*

$$W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P) = \mathrm{Ker}\left(d_A^*: W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)\to L^p(X;\Lambda^{l-1}\otimes \mathrm{ad}P)\right)$$
$$\oplus\, \mathrm{Ran}\left(d_A: W^{2,p}_{A_1}(X;\Lambda^{l-1}\otimes \mathrm{ad}P)\to W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)\right).$$

*Proof.* If $\xi\in W^{2,p}_{A_1}(X;\Lambda^{l-1}\otimes \mathrm{ad}P)$ and we write $A=A_1+a$, with $a\in W^{1,q}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P)$, then $d_A\xi = d_{A_1}\xi+[a,\xi]$ and using the fact that $W^{1,p}(X)\subset L^{2p}(X)$ for any $p\geq d/2$ by [3, Theorem 4.12] and applying the Kato Inequality [22, Equation (6.20)],

$$\begin{aligned}\|d_A\xi\|_{L^p(X)} &\leq z\left(\|\nabla_{A_1}\xi\|_{L^p(X)} + \|a\|_{L^{2p}(X)}\|\xi\|_{L^{2p}(X)}\right)\\ &\leq z\left(\|\nabla_{A_1}\xi\|_{L^p(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\|\xi\|_{W^{1,p}_{A_1}(X)}\right)\\ &\leq z\left(1+\|a\|_{W^{1,q}_{A_1}(X)}\right)\|\xi\|_{W^{1,p}_{A_1}(X)},\end{aligned}$$

where $z=z(g,G,p,q)\in[1,\infty)$ and we use the fact that $q\geq p$. Similarly,

$$\|d_A^*\xi\|_{L^p(X)} \leq z\left(1+\|a\|_{W^{1,q}_{A_1}(X)}\right)\|\xi\|_{W^{1,p}_{A_1}(X)}$$

and so the operator $d_A^*: W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)\to L^p(X;\Lambda^{l-1}\otimes \mathrm{ad}P)$ is continuous.

Moreover, defining $r\in[p,\infty]$ by $1/p=1/q+1/r$, we recall that by [3, Theorem 4.12] we have *i)* $W^{2,p}(X)\subset L^r(X)$ for any $r\in[1,\infty)$ when $p=d/2$, and *ii)* $W^{2,p}(X)\subset L^\infty(X)$ when $p>d/2$. Thus, using

$$\nabla_{A_1}d_A\xi = \nabla_{A_1}d_{A_1}\xi + \nabla_{A_1}a\times\xi + a\times\nabla_{A_1}\xi,$$

we see that

$$\begin{aligned}\|\nabla_{A_1}d_A\xi\|_{L^p(X)} &\leq z\left(\|\nabla^2_{A_1}\xi\|_{L^p(X)} + \|\nabla_{A_1}a\|_{L^q(X)}\|\xi\|_{L^r(X)} + \|a\|_{L^{2p}(X)}\|\nabla_{A_1}\xi\|_{L^{2p}(X)}\right)\\ &\leq z\left(\|\nabla^2_{A_1}\xi\|_{L^p(X)} + \|\nabla_{A_1}a\|_{L^q(X)}\|\xi\|_{W^{2,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{W^{1,p}_{A_1}(X)}\right)\\ &\leq z\left(1+\|a\|_{W^{1,q}_{A_1}(X)}\right)\|\xi\|_{W^{2,p}_{A_1}(X)}.\end{aligned}$$

We conclude that the operator $d_A: W^{2,p}_{A_1}(X;\Lambda^{l-1}\otimes \mathrm{ad}P)\to W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)$ is also continuous.

Note that $W^{1,p}(X)\subset L^2(X)$ when $p\geq 2$ or, when $1\leq p<2$, if $p^*:=dp/(d-p)\geq 2$, that is, $dp\geq 2d-2p$ or $p\geq 2d/(d+2)$. But $p\geq d/2$ by hypothesis and $d/2\geq 2d/(d+2$ for all $d\geq 2$, so we have $W^{1,p}(X)\subset L^2(X)$ for all $p\geq d/2$ and $d\geq 2$. Using $\perp$ to denote $L^2$-orthogonal complement and $\oplus$ to denote $L^2$-orthogonal decomposition, we have

$$W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P) = \left(\mathrm{Ran}\left(d_A: W^{2,p}_{A_1}(X;\Lambda^{l-1}\otimes \mathrm{ad}P)\to W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)\right)\right)^\perp$$
$$\oplus\, \mathrm{Ran}\left(d_A: W^{2,p}_{A_1}(X;\Lambda^{l-1}\otimes \mathrm{ad}P)\to W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)\right).$$

For all $\eta\in W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)$ and $\xi\in W^{2,p}_{A_1}(X;\Lambda^{l+1}\otimes \mathrm{ad}P)$ we have

$$(\eta, d_A\xi)_{L^2(X)} = (d_A^*\eta,\xi)_{L^2(X)}$$

and so $\eta\perp \mathrm{Ran}(d_A: W^{2,p}_{A_1}(X;\Lambda^{l-1}\otimes \mathrm{ad}P)\to W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P))$ if and only if $\eta\in \mathrm{Ker}(d_A^*: W^{1,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)\to L^p(X;\Lambda^{l-1}\otimes \mathrm{ad}P))$. This concludes the proof of the lemma. □

Although not required by the proofs of Lemma 2.13 or Corollary 4, it is useful to note that the operator $d_A$ in that statement has closed range.

**Lemma 2.14** (Closed range operators on $L^p$ spaces)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$. If $A$ is a connection on $P$ of class $W^{1,q}$ with $d/2 < q < \infty$, and $A_1$ is a $C^\infty$ is reference connection on $P$, and $p$ obeys $d/2 \leq p \leq q$, then the operator*

$$d_A : W^{2,p}_{A_1}(X; \mathrm{ad}P) \to W^{1,p}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P),$$

*have closed range.*

*Proof.* Note that the two operators in the statement of the lemma are bounded by Lemma 2.13. Let $\{\chi_n\}_{n\in\mathbb{N}} \subset W^{2,p}_{A_1}(X;\mathrm{ad}P)$ and suppose that $d_A\chi_n \to \xi \in W^{1,p}_{A_1}(X;\Lambda^1 \otimes \mathrm{ad}P)$ as $n \to \infty$. Thus $d_A^* d_A \chi_n = \Delta_A \chi_n \to d_A^*\xi \in L^p(X;\mathrm{ad}P)$ as $n \to \infty$. We may assume without loss of generality that $\{\chi_n\}_{n\in\mathbb{N}} \subset (\operatorname{Ker}\Delta_A)^\perp$, where $\perp$ denotes $L^2$-orthogonal complement, and so the *a priori* estimate (2.5) in Corollary 2.5 then implies that

$$\|\chi_n - \chi_m\|_{W^{2,p}_{A_1}(X)} \leq C\|\Delta_A(\chi_n - \chi_m)\|_{L^p(X)}, \quad \forall\, n,m \in \mathbb{N}.$$

Hence, the sequence $\{\chi_n\}_{n\in\mathbb{N}}$ is Cauchy in $W^{2,p}_{A_1}(X;\mathrm{ad}P)$ and thus $\chi_n \to \chi \in W^{2,p}_{A_1}(X;\mathrm{ad}P)$ and $d_A\chi_n \to d_A\chi \in W^{1,p}_{A_1}(X;\Lambda^1 \otimes \mathrm{ad}P)$ as $n \to \infty$. Therefore, $d_A$ on $W^{1,p}_{A_1}(X;\Lambda^1 \otimes \mathrm{ad}P)$ has closed range. □

We are now ready to complete the

*Proof of Corollary 4.* Every compact Lie group has a compatible structure of a real analytic manifold [8, Section III.4, Exercise 1] and this structure is unique by [56, Theorem 2.11.3]. In particular, the exponential map is a real analytic diffeomorphism from an open neighborhood of the origin in the Lie algebra $\mathfrak{g}$ onto an open neighborhood of the identity in $G$. We recall from [22, Proposition A.2] that $\mathrm{Aut}^{k+1,2}(P)$ may be given the structure of a Hilbert Lie group when $k \geq 2$ and, because $W^{2,q}(X)$ (with $q > d/2$) and $H^{k+1}(X) = W^{k+1,2}(X)$ are Banach algebras and contained in $C(X)$ (the Banach algebra of continuous functions on $X$), the same arguments shows that $\mathrm{Aut}^{2,q}(P)$ may be given the structure of a $C^\infty$ Banach Lie group and that both $\mathrm{Aut}^{2,q}(P)$ and $\mathrm{Aut}^{k,2}(P)$ may be given the structure of real analytic manifolds.

According to [22, Proposition A.3], the (right) action of $\mathrm{Aut}^{k+1,2}(P)$ on $\mathscr{A}^{k,2}(P)$ is $C^\infty$ when $k \geq 2$ and the same proof applies *mutatis mutandis* to show that this action is real analytic and that the action (2.1) of $\mathrm{Aut}^{2,q}(P)$ on $\mathscr{A}^{1,q}(P)$ is not only $C^\infty$ but also real analytic.

The only additional ingredient one needs to show that $\mathscr{B}^*(P)$ is real analytic is the observation that the map $H$ defined in (2.55) and (2.56) is real analytic and thus, rather than apply the customary $C^\infty$ Inverse Function Theorem one can instead apply its real analytic counterpart [19, Section 2.2.1] to show that for each $A_0 \in \mathscr{A}^{1,q}(P)$, the map defined in the statement of [10, Theorem 3.2],

$$\begin{aligned}(2.58)\quad \Xi_{A_0} : \mathscr{A}^{1,q}(P) \supset \mathscr{O}_{A_0} \ni A \mapsto (u(A) - A_0, A)& \\ \in \operatorname{Ker}\left\{d_{A_0}^* : W^{1,q}_{A_1}(X;\Lambda^1\otimes\mathrm{ad}P) \to L^q(X;\mathrm{ad}P)\right\} &\times \mathrm{Aut}^{2,q}(P),\end{aligned}$$

is a real analytic diffeomorphism onto an open neighborhood of $(0, \mathrm{id}_P)$, for a small enough open neighborhood $\mathscr{O}_{A_0}$ of a $W^{1,q}$ connection $A$ on $P$ and the gauge transformation $u$ is produced by Theorem 2, so $u(A)$ is in Coulomb gauge with respect to $A_0$. The open neighborhood $\mathscr{O}_{A_0}$ may be chosen to be $\mathrm{Aut}^{2,q}(P)$-invariant and the map $\Xi_{A_0}$ is $\mathrm{Aut}^{2,q}(P)$-equivariant. The proof that the quotient $\mathscr{B}(P)$ is a Hausdorff topological space follows *mutatis mutandis* either by adapting the proof of [22, Corollary, p. 50] or by adapting the proof of [10, Lemma 4.2.4], using the observation the $L^2$ distance function,

$$\operatorname{dist}_{L^2}([A],[B]) := \inf_{u\in \operatorname{Aut}^{2,q}(P)} \|u(A) - B\|_{L^2(X)}, \tag{2.59}$$

is a metric on $\mathscr{B}(P)$ and, in particular, that the quotient topology is metrizable. This completes the proof of Corollary 4. □

2.8. **Existence of Coulomb gauge transformations for pairs.** We now adapt the construction of Section 2.6 to the case of pairs. In [15, p. 280], we employed a left action of $\operatorname{Aut}(P)$ on the affine space of pairs, $\mathscr{A}(P)\times C^\infty(X;E)$, so $\operatorname{Aut}(P)$ acts on $\mathscr{A}(P)$ by pushforward (consistent with Donaldson and Kronheimer [10]) and on $C^\infty(X;E)$ in the usual way, which is a left action. Here, to be consistent with Section 2.6 we shall use the opposite convention and continue to let $\operatorname{Aut}(P)$ act on $\mathscr{A}(P)$ by pullback (consistent with Freed and Uhlenbeck [22] and Uhlenbeck [54]) and use inversion to define a right action on $C^\infty(X;E)$, so that

$$u(A,\Phi) := (u^*A, u^{-1}\Phi), \quad \forall\, A\in\mathscr{A}(P), \Phi\in C^\infty(X;E), \text{ and } u\in\operatorname{Aut}(P),$$

giving a smooth (affine) right action,

$$\mathscr{A}(P)\times C^\infty(X;E)\times \operatorname{Aut}(P)\to \mathscr{A}(P)\times C^\infty(X;E).$$

Passing to Banach space completions, but temporarily suppressing the $W^{1,q}$ reference connection $A_0$ (for $q>d/2$) from our notation, the differential of the smooth map,

$$\operatorname{Aut}^{2,q}(P)\ni u\mapsto u(A,\Phi)\in \mathscr{A}^{1,q}(P)\times W^{1,q}(X;E),$$

at $\operatorname{id}_P\in \operatorname{Aut}^{2,q}(P)$ is given by

$$W^{2,q}_{A_1}(X;\operatorname{ad}P)\ni\xi\mapsto d_{A,\Phi}\xi := (d_A\xi, -\xi\Phi)\in W^{1,q}_{A_1}(X;\operatorname{ad}P)\oplus W^{1,q}_{A_1}(X;E), \tag{2.60}$$

using $u=e^\xi$ for $u$ near $\operatorname{id}_P$; compare [15, Proposition 2.1]. We say that a $W^{1,q}$ pair $(A,\Phi)$ is in *Coulomb gauge relative to* $(A_0,\Phi_0)$ if

$$d^*_{A_0,\Phi_0}((A,\Phi)-(A_0,\Phi_0)) = 0, \tag{2.61}$$

As in the case of Theorem 2, the proof of Theorem 3 is facilitated by preparatory lemmata and a proposition, which we now state. For convenience, we define

$$\Delta_{A_0,\Phi_0} := d^*_{A_0,\Phi_0} d_{A_0,\Phi_0} \quad \text{on } W^{2,q}_{A_1}(X;\operatorname{ad}P).$$

The proofs of Propositions 2.1 and 2.3 and Corollary 2.5 adapt *mutatis mutandis* to establish the following analogues for pairs, specialized to the case $l=0$.

**Proposition 2.15** (*A priori* $L^p$ estimate for a Laplace operator with Sobolev coefficients)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d\geq 2$, and $G$ be a compact Lie group and $P$ be a smooth principal $G$-bundle over $X$, and $E=P\times_\varrho\mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, and $\varrho: G\to\operatorname{Aut}_{\mathbb{C}}(\mathbb{E})$. If $(A,\Phi)$ is a $W^{1,q}$ pair on $(P,E)$ with $q>d/2$, and $A_1$ is a $C^\infty$ connection on $P$, and $p$ obeys $d/2\leq p\leq q$, then*

$$\Delta_{A,\Phi}: W^{2,p}_{A_1}(X;\operatorname{ad}P)\to L^p(X;\operatorname{ad}P) \tag{2.62}$$

*is a bounded operator. If in addition $p\in(1,\infty)$, then there is a constant $C=C(A,\Phi,A_1,g,G,p,q)\in[1,\infty)$ such that*

$$\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C\left(\|\Delta_{A,\Phi}\xi\|_{L^p(X)} + \|\xi\|_{L^p(X)}\right), \quad \forall\,\xi\in W^{2,p}_{A_1}(X;\operatorname{ad}P). \tag{2.63}$$

Again, it is convenient to abbreviate $\mathfrak{H} = W^{1,2}_{A_1}(X;\mathrm{ad}P)$ and let

$$I : \mathfrak{H} \to \mathfrak{H}^*, \quad u \mapsto (\cdot, u)_{L^2(X)}$$

denote the compact embedding.

**Proposition 2.16** (Spectral properties of a Laplace operator with Sobolev coefficients)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group and $P$ be a smooth principal $G$-bundle over $X$, and $E = P\times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, and $\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$. If $(A,\Phi)$ is a $W^{1,q}$ pair on $(P,E)$ with $d/2 < q < \infty$, and $A_1$ is a $C^\infty$ connection on $P$, then the operator*

$$\Delta_{A,\Phi} : \mathfrak{H} \to \mathfrak{H}^*$$

*is bounded and there is a countable subset $\Sigma \subset [0,\infty)$ without accumulation points and having the following significance. If $\lambda \in \mathbb{R} \setminus \Sigma$, then the equation*

$$(\Delta_{A,\Phi} - \lambda I)\xi = \mathfrak{f}$$

*has a unique solution $\xi \in \mathfrak{H}$ for each $\mathfrak{f} \in \mathfrak{H}^*$. If $\lambda \in \Sigma$, then $\mathrm{Ker}(\Delta_{A,\Phi} - \lambda I) \cap \mathfrak{H}$ has finite, positive dimension.*

**Corollary 2.17** (*A priori* $L^p$ estimate for a Laplace operator with Sobolev coefficients)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group and $P$ be a smooth principal $G$-bundle over $X$, and $E = P\times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, and $\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$. If $(A,\Phi)$ is a $W^{1,q}$ pair on $(P,E)$ with $d/2 < q < \infty$, and $A_1$ is a $C^\infty$ connection on $P$, and $p \in (1,\infty)$ obeys $d/2 \leq p \leq q$, then the kernel $\mathrm{Ker}\,\Delta_{A,\Phi} \cap W^{2,p}_{A_1}(X;\mathrm{ad}P)$ of the operator (2.62) is finite-dimensional. Moreover,*

$$\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C\|\Delta_{A,\Phi}\xi\|_{L^p(X)}, \quad \forall\, \xi \in (\mathrm{Ker}\,\Delta_{A,\Phi})^\perp \cap W^{2,p}_{A_1}(X;\mathrm{ad}P), \tag{2.64}$$

*where $\perp$ denotes $L^2$-orthogonal complement and $C = C(A,\Phi,A_1,g,G,p,q) \in [1,\infty)$.*

The proof of Lemma 2.9 adapts *mutatis mutandis* to establish the following analogue for pairs.

**Lemma 2.18** (Surjectivity of a perturbed Laplace operator)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group and $P$ be a smooth principal $G$-bundle over $X$, and $E = P\times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$. Let $A_1$ be a $C^\infty$ connection on $P$ and $(A,\Phi)$ be a $W^{1,q}$ pair on $(P,E)$ with $d/2 < q < \infty$. Then there is a constant $\delta = \delta(A,\Phi,g) \in (0,1]$ with the following significance. If $(a,\phi) \in W^{1,q}_{A_1}(X;\Lambda^1 \otimes \mathrm{ad}P \oplus E)$ obeys*

$$\|(a,\phi)\|_{L^d(X)} < \delta \quad \text{when } d \geq 3 \quad \text{or} \quad \|(a,\phi)\|_{L^4(X)} < \delta \quad \text{when } d = 2, \tag{2.65}$$

*then the operator,*

$$d^*_{A,\Phi} d_{A+a,\Phi+\phi} : (\mathrm{Ker}\,\Delta_{A,\Phi})^\perp \cap W^{2,q}_{A_1}(X;\mathrm{ad}P) \to (\mathrm{Ker}\,\Delta_{A,\Phi})^\perp \cap L^q(X;\mathrm{ad}P),$$

*is surjective.*

Finally, the proof of Proposition 2.11 adapts *mutatis mutandis* to establish the following (simplified) analogue for pairs.

**Proposition 2.19** (*A priori* $W^{1,p}$ estimate for $u(A,\Phi)-(A_0,\Phi_0)$ in terms of $(A,\Phi)-(A_0,\Phi_0)$)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d\geq 2$, and $G$ be a compact Lie group and $P$ be a smooth principal $G$-bundle over $X$, and $E=P\times_\varrho\mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho: G\to \operatorname{Aut}_{\mathbb{C}}(\mathbb{E})$. Let $A_1$ be a $C^\infty$ connection on $P$, and $(A_0,\Phi_0)$ be a $W^{1,q}$ pair on $(P,E)$ with $d/2<q<\infty$ and $p\in(1,\infty)$ obey $d/2\leq p\leq q$. Then there are constants $N=N(A_1,A_0,\Phi_0,g,G,p,q)\in[1,\infty)$ and $\varepsilon=\varepsilon(A_1,A_0,\Phi_0,g,G,p,q)\in(0,1]$ with the following significance. If $(A,\Phi)$ is a $W^{1,q}$ pair on $(P,E)$ and $u\in\operatorname{Aut}(P)$ is a gauge transformation of class $W^{2,q}$ such that*

$$d^*_{A_0,\Phi_0}(u(A,\Phi)-(A_0,\Phi_0))=0, \tag{2.66}$$

*then the following hold. If $(A,\Phi)$ and $u(A,\Phi)$ obey*

$$\|(A,\Phi)-(A_0,\Phi_0)\|_{W^{1,p}_{A_1}(X)}\leq\varepsilon \quad\text{and}\quad \|u(A,\Phi)-(A_0,\Phi_0)\|_{W^{1,p}_{A_1}(X)}\leq\varepsilon, \tag{2.67}$$

*then*

$$\|u(A,\Phi)-(A_0,\Phi_0)\|_{W^{1,p}_{A_1}(X)}\leq N\|(A,\Phi)-(A_0,\Phi_0)\|_{W^{1,p}_{A_1}(X)}. \tag{2.68}$$

*Proof of Theorem 3.* Given these preliminaries, Corollary 2.17, Lemma 2.18, and Proposition 2.19, the proof of Theorem 3 follows *mutatis mutandis* from that of Theorem 2. ☐

The proof of Proposition 2.19 yields the following analogue of Lemma 2.12 for pairs.

**Lemma 2.20** (*A priori* $W^{2,p}$ estimate for a $W^{2,q}$ gauge transformation $u$ intertwining two $W^{1,q}$ pairs)**.** *Assume the hypotheses of Proposition 2.19, excluding those on the pair $(A,\Phi)$. Then there is a constant $C=C(A_0,\Phi_0,A_1,g,G,p,q)\in[1,\infty)$ with the following significance. If $(A,\Phi)$ obeys the hypotheses of Proposition 2.19 and $u\in\operatorname{Aut}^{2,q}(P)$ is the resulting gauge transformation, depending on $(A,\Phi)$ and $(A_0,\Phi_0)$, such that*

$$d^*_{A_0,\Phi_0}(u(A,\Phi)-(A_0,\Phi_0))=0,$$

*then*

$$\|u\|_{W^{2,p}_{A_1}(X)}\leq C.$$

While not required for the proof of Theorem 3, this is a convenient point at which to note that the proof of Lemma 2.13 (specialized to the case $l=1$) adapts *mutatis mutandis* to give the following analogue for pairs.

**Lemma 2.21** (Continuous operators on $L^p$ spaces and $L^2$-orthogonal decompositions)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d\geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E=P\times_\varrho\mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho: G\to\operatorname{Aut}_{\mathbb{C}}(\mathbb{E})$. If $(A,\Phi)$ is a Sobolev pair on $(P,E)$ of class $W^{1,q}$ with $q\geq d/2$, and $A_1$ is a $C^\infty$ reference connection on $P$, and $p$ obeys $d/2\leq p\leq q$, then the operator*

$$d^*_{A,\Phi}: W^{1,p}_{A_1}(X;\Lambda^1\otimes\operatorname{ad}P\oplus E)\to L^p(X;\operatorname{ad}P\oplus E),$$

*is continuous and, if in addition $q>d/2$, then the operator*

$$d_{A,\Phi}: W^{2,p}_{A_1}(X;\operatorname{ad}P\oplus E)\to W^{1,p}_{A_1}(X;\Lambda^1\otimes\operatorname{ad}P\oplus E),$$

*is also continuous and there is an $L^2$-orthogonal decomposition,*

$$\begin{aligned}
W^{1,p}_{A_1}&(X;\Lambda^1\otimes \operatorname{ad}P\oplus E)\\
&= \operatorname{Ker}\left(d^*_{A,\Phi}: W^{1,p}_{A_1}(X;\Lambda^1\otimes \operatorname{ad}P\oplus E)\to L^p(X;\operatorname{ad}P\oplus E)\right)\\
&\qquad \oplus \operatorname{Ran}\left(d_{A,\Phi}: W^{2,p}_{A_1}(X;\operatorname{ad}P\oplus E)\to W^{1,p}_{A_1}(X;\Lambda^1\otimes \operatorname{ad}P\oplus E)\right).
\end{aligned}$$

2.9. **Regularity for solutions to the Yang-Mills and coupled Yang-Mills equations.** It is well-known that techniques due to Uhlenbeck [55, 54] can be used to show that, given a weak solution to the Yang-Mills equation, there exists a gauge transformation such that the gauge-transformed solution is smooth. We give a proof of a similar fact here that generalizes easily to the case of coupled Yang-Mills equations.

We have the following generalization of [45, Theorem 5.3], due to Parker, and [15, Proposition 3.7], due to the author and Leness, from the case of $d = 4$ to arbitrary $d \geq 2$.

**Theorem 2.22** (Regularity for solutions to the Yang-Mills equation)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$. If $d/2 < q < \infty$ and $A$ is a $W^{1,q}$ connection on $P$ that is a weak solution to the Yang-Mills equation with respect to the Riemannian metric $g$, then there exists a $W^{2,q}$ gauge transformation $u \in \operatorname{Aut}(P)$ such that $u(A)$ is a $C^\infty$ Yang-Mills connection on $P$.*

*Proof.* We proceed as in the proof of [15, Proposition 3.7] and note that the affine space $\mathscr{A}(P)$ of $C^\infty$ connections on $P$ is dense in the affine space $\mathscr{A}^{1,q}(P)$ of $W^{1,q}$ connections on $P$ and so there exists a $C^\infty$ connection $A_0$ on $P$ such that

$$\|A - A_0\|_{W^{1,q}_{A_1}(X)} < \zeta,$$

where $\zeta = \zeta(A_0, A_1, g, G, q) \in (0, 1]$ is the constant in Theorem 2 and $A_1$ is any fixed $C^\infty$ reference connection on $P$. Hence, there is a $W^{2,q}$ gauge transformation $u \in \operatorname{Aut}(P)$ such that $u(A)$ obeys

$$d^*_{A_0}(u(A) - A_0) = 0,$$

and

$$\|u(A) - A_0\|_{W^{1,q}_{A_1}(X)} < 2N\|A - A_0\|_{W^{1,q}_{A_1}(X)} < 2N\zeta,$$

where $N = N(A_0, A_1, g, G, q) \in [1, \infty)$ is the constant in Proposition 2.11. Hence, we may assume without loss of generality that $A$ is in Coulomb gauge with respect to $A_0$ and that $a := A - A_0 \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \operatorname{ad}P)$ is a weak solution to

$$d^*_{A_0+a}F_{A_0+a} = 0 \quad\text{and}\quad d^*_{A_0}a = 0,$$

and thus a weak solution to the quasi-linear, second-order elliptic system,

$$(\Delta_{A_0} + \lambda_0)a + a\times\nabla_{A_0}a + a\times a\times a = \lambda a - F_{A_0}, \tag{2.69}$$

where $\Delta_{A_0} = d^*_{A_0}d_{A_0} + d_{A_0}d^*_{A_0}$ is the usual Hodge Laplacian on $\Omega^1(X; \operatorname{ad}P)$ and $\lambda_0 > 0$ is any positive constant. We recall from [13] that, for any $q \in (1, \infty)$ and integer $k \geq 0$, the operator

$$\Delta_{A_0} + \lambda_0 : W^{k+2,q}_{A_1}(X;\Lambda^1\otimes \operatorname{ad}P) \to W^{k,q}_{A_1}(X;\Lambda^1\otimes \operatorname{ad}P) \tag{2.70}$$

is invertible. The right-hand side of Equation (2.69) belongs to $W^{1,q}_{A_1}(X;\Lambda^1\otimes\operatorname{ad}P)$ by hypothesis on $A$ and the fact that $A_0$ is $C^\infty$. If $q > d$, then $W^{1,q}(X) \subset C(X)$ by [3, Theorem 4.12] and so

$$a\times\nabla_{A_0}a + a\times a\times a \in L^q(X;\Lambda^1\otimes\operatorname{ad}P).$$

But then existence and uniqueness of solutions in $W^{2,q}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P)$ to (2.69), given a source term in $L^q(X;\Lambda^1\otimes \mathrm{ad}P)$, implies that $a\in W^{2,q}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P)$. Using the fact that $W^{k,q}(X)$ is a Banach algebra when $kq>d$ and invertibility of (2.70), we can iterate in the usual way to show that $a\in W^{k+2,q}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P)$ for all integers $k\geq 0$ and hence that $a\in C^\infty(X;\Lambda^1\otimes \mathrm{ad}P)$.

Therefore, it suffices to consider the case $d/2<q<d$. If $q^*=dq/(d-q)$ and $r\in[1,q)$ obeys $1/q^*+1/q\leq 1/r$, then there is a continuous multiplication map $L^{q^*}(X)\times L^q(X)\to L^r(X)$ and a continuous Sobolev embedding $W^{1,q}(X)\subset L^{q^*}(X)$ by [3, Theorem 4.12]. Thus, $a\times\nabla_{A_0}a\in L^r(X;\Lambda^1\otimes \mathrm{ad}P)$ if

$$1/r\geq (d-q)/(dq)+1/q=(2d-q)/(dq),$$

that is, $r\leq dq/(2d-q)$.

Similarly, we have a continuous multiplication map $L^{3s}(X)\times L^{3s}(X)\times L^{3s}(X)\to L^s(X)$ for any $s\in[1,\infty]$ and a continuous Sobolev embedding $W^{1,q}(X)\subset L^{3s}(X)$ provided $3s\leq q^*$, that is, $s\leq dq/(3(d-q))$ and for this choice of $s$, we have $a\times a\times a\in L^s(X;\Lambda^1\otimes \mathrm{ad}P)$.

We now observe that $r\leq s$ when $q\geq d/2$, as we assume by hypothesis, if we choose $r=dq/(2d-q)$ and $s=dq/(3(d-q))$. Indeed, we then have

$$r\leq s \iff dq/(2d-q)\leq dq/(3(d-q)) \iff 3d-3q\leq 2d-q \iff d\leq 2q.$$

Hence, for $r=dq/(2d-q)$, we have

$$a\times\nabla_{A_0}a+a\times a\times a\in L^r(X;\Lambda^1\otimes \mathrm{ad}P),$$

and thus elliptic regularity theory for (2.69) implies that $a\in W^{2,r}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P)$, similar to the case $q>d$. By [3, Theorem 4.12], we have a continuous Sobolev embedding, $W^{2,r}(X)\subset W^{1,r^*}$, where $r^*=dr/(d-r)$, and thus we obtain

$$a\in W^{1,r^*}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P).$$

In the limiting case $q=d/2$ we would have $r=(d^2/2)/(2d-d/2)=d/3$ and in the limiting case $q=d$ we would have $r=d$, so $r\in(d/3,d)$ and thus $r^*\in(d/2,\infty)$. In particular,

$$r^*=\frac{dr}{d-r}=\frac{d^2q/(2d-q)}{d-dq/(2d-q)}=\frac{dq/(2d-q)}{1-q/(2d-q)}=\frac{dq}{2(d-q)}.$$

We may write $r^*=q+\delta$, where $\delta=\delta(d,q)$ is defined by

$$\delta:=r^*-q=\frac{dq}{2(d-q)}-q=\frac{dq-2(d-q)q}{2(d-q)}=\frac{2q^2-dq}{2(d-q)}=\frac{q(2q-d)}{2(d-q)},$$

and thus $\delta(d,q)>0$ for $d/2<q<d$. Consequently, we see a regularity improvement and because $\delta(d,q)$ is an increasing function of $q$, only finitely many iterations of this regularity improvement are required to give $r^*>d$, at which point we can apply the regularity argument for the case $q>d$ to again conclude that $a\in C^\infty(X;\Lambda^1\otimes \mathrm{ad}P)$. □

The proof of Theorem 2.22 adapts *mutatis mutandis* to give the following generalization of [45, Theorem 5.3], due to Parker, and [15, Proposition 3.7], due to the author and Leness, from the case of $d=4$ to arbitrary $d\geq 2$.

**Theorem 2.23** (Regularity for solutions to the boson coupled Yang-Mills equations)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d\geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E=P\times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho: G\to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$. If $d/2<q<\infty$ and $(A_\infty,\Phi_\infty)$ is a $W^{1,q}$ pair on $(P,E)$ that is a critical point of the boson coupled Yang-Mills* *energy functional* (1.5)*, then there exists a* $W^{2,q}$ *gauge transformation* $u \in \operatorname{Aut}(P)$ *such that* $u(A_\infty, \Phi_\infty)$ *is a* $C^\infty$ *pair on* $(P, E)$*.*

*Proof.* We proceed as in the proofs of [45, Theorem 5.3] and [15, Proposition 3.7] and note that the affine space $\mathscr{A}(P) \times C^\infty(E)$ of $C^\infty$ pairs is dense in the affine space $\mathscr{A}^{1,q}(P) \times W^{1,q}(X; E)$ of $W^{1,q}$ pairs and so there exists a $C^\infty$ pair $(A_0, \Phi_0)$ on $(P, E)$ such that

$$\|(A_\infty, \Phi_\infty) - (A_0, \Phi_0)\|_{W^{1,q}_{A_1}(X)} < \zeta,$$

where $\zeta = \zeta(A_1, A_0, \Phi_0, g, G, q) \in (0, 1]$ is the constant in Theorem 3 and $A_1$ is any fixed $C^\infty$ reference connection on $P$. Hence, there is a gauge transformation $u \in \operatorname{Aut}(P)$ of class $W^{2,q}$ such that $u(A_\infty, \Phi_\infty)$ obeys

$$d^*_{A_0,\Phi_0}\left(u(A_\infty, \Phi_\infty) - (A_0, \Phi_0)\right) = 0,$$

and

$$\|u(A_\infty, \Phi_\infty) - (A_0, \Phi_0)\|_{W^{1,p}_{A_1}(X)} < 2N\|(A_\infty, \Phi_\infty) - (A_0, \Phi_0)\|_{W^{1,p}_{A_1}(X)} < 2N\zeta,$$

where $N = N(A_1, A_0, \Phi_0, g, G, q) \in [1, \infty)$ is the constant in Proposition 2.19. Hence, we may assume without loss of generality that $(A_\infty, \Phi_\infty)$ is in Coulomb gauge with respect to $(A_0, \Phi_0)$ and that $(a, \phi) := (A_\infty, \Phi_\infty) - (A_0, \Phi_0) \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \operatorname{ad}P \oplus E)$ is a weak solution to the quasi-linear, second-order elliptic system,

$$\begin{aligned}
&d^*_{A_0} d_{A_0} a + \nabla^*_{A_0}\nabla_{A_0}\phi + a \times \nabla_{A_0} a + a \times \nabla_{A_0}\phi + \phi \times \nabla_{A_0}\phi \\
&\quad + a \times a \times a + a \times a \times \phi + a \times \phi \times \phi \\
&\quad + m\phi + s\Phi_0 \times \Phi_0 \times \phi + s\Phi_0 \times \phi \times \phi + s\phi \times \phi \times \phi = f(m, s, A_0, \Phi_0), \\
&d_{A_0,\Phi_0} d^*_{A_0,\Phi_0}(a, \phi) = 0,
\end{aligned}$$

where we employ the expression (1.9) for the gradient of $\mathscr{E}'(A, \Phi)$. Ellipticity of the preceding system follows by expanding the expression (2.60) for $d_{A_0,\Phi_0}$ on $\Omega^0(X; \operatorname{ad}P)$ to extract the second-order term $d_{A_0} d^*_{A_0} a$ and recalling that the operator $d^*_{A_0} d_{A_0} + d_{A_0} d^*_{A_0}$ is clearly elliptic, being the usual Hodge Laplacian on $\Omega^1(X; \operatorname{ad}P)$. The remainder of the proof now follows the pattern of the proof of Theorem 2.22. □

## 3. Łojasiewicz-Simon gradient inequalities for coupled Yang-Mills energy functionals

In this section, we apply our Coulomb-gauge transformation result, Theorem 3, and abstract Łojasiewicz-Simon gradient inequality, Theorems 1, to prove the corresponding Łojasiewicz-Simon gradient inequalities for the boson and fermion coupled Yang-Mills $L^2$-energy functionals, Theorem 5 in Section 3.1, and Theorem 7 in Section 3.2.

3.1. **Łojasiewicz-Simon gradient inequality for boson coupled Yang-Mills energy functional.** We assume the notation and conventions of Section 1.3.1. For any $C^\infty$ reference connection $A_1$ on $P$, let

$$\mathscr{P}^{1,p}(P, E) := \mathscr{A}^{1,p}(P) \times W^{1,p}_{A_1}(X; E),$$

denote the affine space of $W^{1,p}$ pairs on $(P, E)$. Our first task is to establish the

**Proposition 3.1** (Analyticity of the boson coupled Yang-Mills $L^2$-energy functional on the affine space of $W^{1,p}$ pairs)**.** *Let* $(X, g)$ *be a closed, Riemannian, smooth manifold of dimension* $d \geq 2$*, and* $G$ *be a compact Lie group,* $P$ *be a smooth principal* $G$*-bundle over* $X$*, and* $E = P \times_\varrho \mathbb{E}$ *be a smooth Hermitian vector bundle over* $X$ *defined by a finite-dimensional unitary representation,*

*$\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, and $A_1$ be a $C^\infty$ connection on $P$, and $m, s \in C^\infty(X)$. If $4d/(d+4) \le p < \infty$, then the function,*

$$\mathscr{E} : \mathscr{P}^{1,p}(P, E) \to \mathbb{R},$$

*is real analytic, where $\mathscr{E}$ is as in* (1.5).

*Proof.* We fix a pair $(A, \Phi) \in \mathscr{P}^{1,p}(P, E)$ and write $(A, \Phi) = (A_1, \Phi_1) + (a_1, \phi_1)$, where $(a_1, \phi_1) \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P \oplus E)$. We will show that $\mathscr{E}$ is analytic at $(A, \Phi)$. For $(a, \phi) \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P \oplus E)$, we write $A = A_1 + a_1 + a$ and expand

$$F_{A+a} = F_{A_1+a_1+a} = F_{A_1} + d_{A_1}(a_1 + a) + (a_1 + a) \times (a_1 + a)$$

and

$$\nabla_{A+a}(\Phi + \phi) = \nabla_{A+a_1+a}(\Phi + \phi) = \nabla_{A_1}(\Phi + \phi) + \varrho(a_1 + a)(\Phi + \phi).$$

Using the definition (1.5) of $\mathscr{E}$, we compute

$$2\mathscr{E}(A + a, \Phi + \phi) = T_1 + T_2 + T_3,$$

where the terms $T_i := T_i(a, \phi)$, for $i = 1, 2, 3$, are given by

$$\begin{aligned} T_1 := {} & \|F_{A_1}\|^2_{L^2(X)} + \|d_{A_1}(a_1 + a)\|^2_{L^2(X)} \\ & + \|(a_1 + a) \times (a_1 + a)\|^2_{L^2(X)} + 2(F_{A_1}, d_{A_1}(a_1 + a))_{L^2(X)} \\ & + 2(F_{A_1}, (a_1 + a) \times (a_1 + a))_{L^2(X)} \\ & + 2(d_{A_1}(a_1 + a), (a_1 + a) \times (a_1 + a))_{L^2(X)}, \end{aligned}$$

and

$$\begin{aligned} T_2 := {} & \|\nabla_{A_1}(\Phi + \phi)\|^2_{L^2(X)} + \|\varrho(a_1 + a)(\Phi + \phi)\|^2_{L^2(X)} \\ & + (\nabla_{A_1}(\Phi + \phi), \varrho(a_1 + a)(\Phi + \phi))_{L^2(X)} + (\varrho(a_1 + a)(\Phi + \phi), \nabla_{A_1}(\Phi + \phi))_{L^2(X)}, \\ = {} & \|\nabla_{A_1}(\Phi + \phi)\|^2_{L^2(X)} + \|\varrho(a_1 + a)(\Phi + \phi)\|^2_{L^2(X)} \\ & + 2\,\mathrm{Re}(\nabla_{A_1}(\Phi + \phi), \varrho(a_1 + a)(\Phi + \phi))_{L^2(X)}, \end{aligned}$$

and

$$T_3 := -\int_X \left(m|\Phi + \phi|^2 + s|\Phi + \phi|^4\right) \, d\,\mathrm{vol}_g\,.$$

Hence, we can write the difference as

$$2\mathscr{E}(A + a, \Phi + \phi) - 2\mathscr{E}(A, \Phi) = T_1' + T_2' + T_3',$$

where the difference terms $T_i' := T_i(a, \phi) - T_i(0, 0)$, for $i = 1, 2, 3$, are given by

$$\begin{aligned} T_1' = {} & \|d_{A_1}a\|^2_{L^2(X)} + 2(d_{A_1}a_1, d_{A_1}a)_{L^2(X)} + (a \times (a_1 + a), (a_1 + a) \times (a_1 + a))_{L^2(X)} \\ & + 2(F_{A_1}, d_{A_1}a)_{L^2(X)} + (F_{A_1}, a \times (a_1 + a))_{L^2(X)} \\ & + (d_{A_1}a, (a_1 + a) \times (a_1 + a))_{L^2(X)} + (d_{A_1}a_1, a \times (a_1 + a))_{L^2(X)}, \end{aligned}$$

and

$$\begin{aligned} T_2' = {} & \|\nabla_{A_1}\phi\|^2_{L^2(X)} + 2\,\mathrm{Re}(\nabla_{A_1}\Phi, \nabla_{A_1}\phi)_{L^2(X)} + \|\varrho(a)\Phi\|^2_{L^2(X)} + \|\varrho(a_1 + a)\phi\|^2_{L^2(X)} \\ & + 2\,\mathrm{Re}(\varrho(a_1)\Phi, \varrho(a)\Phi)_{L^2(X)} + 2\,\mathrm{Re}(\varrho(a_1 + a)\Phi, \varrho(a_1 + a)\phi)_{L^2(X)} \\ & + 2\,\mathrm{Re}(\nabla_{A_1}\phi, \varrho(a_1 + a)(\Phi + \phi)_{L^2(X)} + 2\,\mathrm{Re}(\nabla_{A_1}\Phi, \varrho(a)\Phi + \varrho(a_1 + a)\phi)_{L^2(X)}, \end{aligned} \tag{3.1}$$

and

$$
\begin{aligned}
T_3' = &\int_X \left(m|\phi|^2 \,+\, 2m\operatorname{Re}\langle\Phi,\phi\rangle + s|\phi|^4 + 4s(\operatorname{Re}\langle\Phi,\phi\rangle)^2 + 4s(|\Phi|^2+|\phi|^2)\operatorname{Re}\langle\Phi,\phi\rangle\right)\,d\operatorname{vol}_g \\
&+ \int_X 2s|\Phi|^2|\phi|^2\,d\operatorname{vol}_g\,.
\end{aligned}
$$

To see the origin of the expression (3.1) for $T_2'$, we observe that

$$
\begin{aligned}
T_2' &:= \|\nabla_{A_1+a_1+a}(\Phi+\phi)\|_{L^2(X)}^2 - \|\nabla_{A_1+a_1}\Phi\|_{L^2(X)}^2 \\
&= \|\nabla_{A_1}(\Phi+\phi) + \varrho(a_1+a)(\Phi+\phi)\|_{L^2(X)}^2 - \|\nabla_{A_1}\Phi + \varrho(a_1)\Phi)\|_{L^2(X)}^2 \\
&= \|\nabla_{A_1}(\Phi+\phi)\|_{L^2(X)}^2 - \|\nabla_{A_1}\Phi\|_{L^2(X)}^2 \\
&\quad + \|\varrho(a_1+a)(\Phi+\phi)\|_{L^2(X)}^2 - \|\varrho(a_1)\Phi\|_{L^2(X)}^2 \\
&\quad + 2\operatorname{Re}(\nabla_{A_1}(\Phi+\phi), \varrho(a_1+a)(\Phi+\phi))_{L^2(X)} - 2\operatorname{Re}(\nabla_{A_1}\Phi, \varrho(a_1)\Phi)_{L^2(X)} \\
&= T_{21}' + T_{22}' + T_{23}'.
\end{aligned}
$$

For the first term, we have

$$
T_{21}' := \|\nabla_{A_1}(\Phi+\phi)\|_{L^2(X)}^2 - \|\nabla_{A_1}\Phi\|_{L^2(X)}^2 = \|\nabla_{A_1}\phi\|_{L^2(X)}^2 + 2\operatorname{Re}(\nabla_{A_1}\Phi, \nabla_{A_1}\phi)_{L^2(X)}.
$$

For the second term, we see that

$$
\begin{aligned}
T_{22}' &:= \|\varrho(a_1+a)(\Phi+\phi)\|_{L^2(X)}^2 - \|\varrho(a_1)\Phi\|_{L^2(X)}^2 \\
&= \|\varrho(a_1)\Phi + \varrho(a)\Phi + \varrho(a_1+a)\phi\|_{L^2(X)}^2 - \|\varrho(a_1)\Phi\|_{L^2(X)}^2 \\
&= \|\varrho(a)\Phi\|_{L^2(X)}^2 + \|\varrho(a_1+a)\phi\|_{L^2(X)}^2 + 2\operatorname{Re}(\varrho(a_1)\Phi, \varrho(a)\Phi)_{L^2(X)} \\
&\quad + 2\operatorname{Re}(\varrho(a_1)\Phi, \varrho(a_1+a)\phi\rangle + 2\operatorname{Re}(\varrho(a)\Phi, \varrho(a_1+a)\phi)_{L^2(X)} \\
&= \|\varrho(a)\Phi\|_{L^2(X)}^2 + \|\varrho(a_1+a)\phi\|_{L^2(X)}^2 + 2\operatorname{Re}(\varrho(a_1)\Phi, \varrho(a)\Phi)_{L^2(X)} \\
&\quad + 2\operatorname{Re}(\varrho(a_1+a)\Phi, \varrho(a_1+a)\phi)_{L^2(X)}.
\end{aligned}
$$

For the third term, we have

$$
\begin{aligned}
T_{23}' &:= 2\operatorname{Re}(\nabla_{A_1}(\Phi+\phi), \varrho(a_1+a)(\Phi+\phi))_{L^2(X)} - 2\operatorname{Re}(\nabla_{A_1}\Phi, \varrho(a_1)\Phi) \\
&= 2\operatorname{Re}(\nabla_{A_1}\Phi, \varrho(a_1)\Phi + \varrho(a)\Phi + \varrho(a_1+a)\phi)_{L^2(X)} + 2\operatorname{Re}(\nabla_{A_1}\phi, \varrho(a_1+a)(\Phi+\phi))_{L^2(X)} \\
&\quad - 2\operatorname{Re}(\nabla_{A_1}\Phi, \varrho(a_1)\Phi)_{L^2(X)} \\
&= 2\operatorname{Re}(\nabla_{A_1}\Phi, \varrho(a)\Phi + \varrho(a_1+a)\phi)_{L^2(X)} + 2\operatorname{Re}(\nabla_{A_1}\phi, \varrho(a_1+a)(\Phi+\phi))_{L^2(X)}.
\end{aligned}
$$

By adding the preceding terms, we obtain the expression (3.1) for $T_2'$.

To complete the proof of analyticity of $\mathscr{E}$ at $(A,\Phi)$, we observe that there is a continuous Sobolev embedding, $W^{1,p}(X) \subset L^4(X)$, when $p^* = dp/(d-p) \geq 4$ by [3, Theorem 4.12], that is, $dp \geq 4(d-p)$ or $p(d+4) \geq 4d$ or $p \geq 4d/(d+4)$ (note that we assume $d \geq 2$). Hence, we obtain a continuous multilinear map, $\otimes_{i=1}^4 W^{1,p}(X) \to L^1(X)$, by combining the Sobolev embedding $W^{1,p}(X) \to L^4(X)$ with the continuous multiplication map, $\otimes_{i=1}^4 L^4(X) \to L^1(X)$. Combining these observations with the Kato Inequality [22, Equation (6.20)], we obtain an estimate of the form,

$$
|\mathscr{E}(A+a, \Phi+\phi) - \mathscr{E}(A,\Phi)| \leq |T_1'| + |T_2'| + |T_3'|,
$$

where, for a constant $C = C(g, G) \in [1, \infty)$,

$$
\begin{aligned}
C^{-1}|T_1'| \leq \|a\|^2_{W^{1,p}_{A_1}(X)} &+ \|a_1\|_{W^{1,p}_{A_1}(X)}\|a\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\left(\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\right)^3 \\
&+ \|F_{A_1}\|_{L^2(X)}\|a\|_{W^{1,p}_{A_1}(X)} + \|F_{A_1}\|_{L^2(X)}\|a\|_{W^{1,p}_{A_1}(X)}\left(\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\right) \\
&+ \|a\|_{W^{1,p}_{A_1}(X)}\left(\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\right)^2 \\
&+ \|a_1\|_{W^{1,p}_{A_1}(X)}\|a\|_{W^{1,p}_{A_1}(X)}\left(\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\right),
\end{aligned}
$$

noting that $W^{1,p}(X) \subset L^2(X)$ if $dp/(d-p) \geq 2$, that is, $dp \geq 2(d-p)$ or $p(d+2) \geq 2d$ or $p \geq 2d/(d+2)$, and

$$
\begin{aligned}
C^{-1}|T_2'| \leq \|\phi\|^2_{W^{1,p}_{A_1}(X)} &+ \|\Phi\|_{W^{1,p}_{A_1}(X)}\|\phi\|_{W^{1,p}_{A_1}(X)} + \|a\|^2_{W^{1,p}_{A_1}(X)}\|\Phi\|^2_{W^{1,p}_{A_1}(X)} \\
&+ \left(\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\right)^2\|\phi\|^2_{W^{1,p}_{A_1}(X)} + \|a_1\|_{W^{1,p}_{A_1}(X)}\|a\|_{W^{1,p}_{A_1}(X)}\|\Phi\|^2_{W^{1,p}_{A_1}(X)} \\
&+ \left(\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\right)^2_{W^{1,p}_{A_1}(X)}\|\Phi\|_{W^{1,p}_{A_1}(X)}\|\phi\|_{W^{1,p}_{A_1}(X)} \\
&+ \|\phi\|_{W^{1,p}_{A_1}(X)}\left(\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\right)\left(\|\Phi\|_{W^{1,p}_{A_1}(X)} + \|\phi\|_{W^{1,p}_{A_1}(X)}\right) \\
&+ \|\Phi\|^2_{W^{1,p}_{A_1}(X)}\|a\|_{W^{1,p}_{A_1}(X)} + \|\Phi\|_{W^{1,p}_{A_1}(X)}\|\phi\|_{W^{1,p}_{A_1}(X)}\left(\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
C^{-1}|T_3'| \leq \|\phi\|^2_{W^{1,p}_{A_1}(X)} &+ \|\Phi\|_{W^{1,p}_{A_1}(X)}\|\phi\|_{W^{1,p}_{A_1}(X)} + \|\phi\|^4_{W^{1,p}_{A_1}(X)} + \|\Phi\|^2_{W^{1,p}_{A_1}(X)}\|\phi\|^2_{W^{1,p}_{A_1}(X)} \\
&+ \|\Phi\|^3_{W^{1,p}_{A_1}(X)}\|\phi\|_{W^{1,p}_{A_1}(X)} + \|\Phi\|_{W^{1,p}_{A_1}(X)}\|\phi\|^3_{W^{1,p}_{A_1}(X)}.
\end{aligned}
$$

Note that $4d/(d+4) \geq 2d/(d+2)$, so the condition $p \geq 2d/(d+2)$ is assured by the stronger $p \geq 4d/(d+4)$. Thus, $\mathscr{E}(A+a, \Phi+\phi)$ is a polynomial of degree four in the variable $(a, \phi) \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P \oplus E)$ and the conclusion follows. □

We can now verify the formula (1.9) for the gradient $\mathscr{E}'(A, \Phi)$in the direction $(a, \phi)$ by extracting the terms that are linear in $(a, \phi)$ from the expressions for $T_i'$, for $i = 1, 2, 3$, arising in the proof of Proposition 3.1. We compute $\mathscr{E}(A + ta, \Phi + t\phi)$ using the identities,

$$
F_{A+ta} = F_A + td_A a + \frac{t^2}{2}[a, a] \quad \text{and} \quad \nabla_{A+ta}(\Phi + t\phi) = \nabla_A(\Phi + t\phi) + \varrho(ta)(\Phi + t\phi),
$$

to obtain

$$
\begin{aligned}
\mathscr{E}(A + ta, \Phi + t\phi) = \frac{1}{2}\int_X &|F_A + td_A a + \frac{t^2}{2}[a, a]|^2 \, d\,\mathrm{vol}_g \\
&+ \frac{1}{2}\int_X |\nabla_A \Phi + t(\nabla_A \phi + \varrho(a)\Phi) + t^2 \varrho(a)\phi)|^2 \, d\,\mathrm{vol}_g \\
&- \frac{1}{2}\int_X \left(m|\Phi + t\phi|^2 + s|\Phi + t\phi|^4\right) \, d\,\mathrm{vol}_g,
\end{aligned}
$$

that is,

$$\begin{aligned}
\mathscr{E}(A+ta,\Phi+t\phi) &= \frac{1}{2}\int_X \left(|F_A|^2 + 2t\langle F_A, d_Aa\rangle\right)\,d\operatorname{vol}_g \\
&\quad + \frac{1}{2}\int_X \left(|\nabla_A\Phi|^2 + 2t\operatorname{Re}\langle \nabla_A\Phi, \nabla_A\phi + \varrho(a)\Phi\rangle_{L^2}\right)\,d\operatorname{vol}_g \\
&\quad - \frac{1}{2}\int_X m\left(|\Phi|^2 + 2t\operatorname{Re}\langle\Phi,\phi\rangle\right)\,d\operatorname{vol}_g \\
&\quad - \frac{1}{2}\int_X s\left(|\Phi|^4 + 4t|\Phi|^2\operatorname{Re}\langle\Phi,\phi\rangle\right)\,d\operatorname{vol}_g + \text{higher powers of } t.
\end{aligned}$$

Hence, the gradient of $\mathscr{E}$ at $(A,\Phi)$, in the direction $(a,\phi)\in W^{1,p}_{A_1}(X;\Lambda^1\otimes\operatorname{ad}P\oplus E)$ is given by

$$\begin{aligned}
\left(\mathscr{E}'(A,\Phi),(a,\phi)\right)_{L^2(X)} &= (d_A^*F_A, a)_{L^2(X)} + \operatorname{Re}(\nabla_A^*\nabla_A\Phi,\phi)_{L^2(X)} + \operatorname{Re}(\nabla_A\Phi,\varrho(a)\Phi)_{L^2(X)} \\
&\quad - \operatorname{Re}(m\Phi,\phi)_{L^2(X)} - 2\operatorname{Re}\int_X s|\Phi|^2\langle\Phi,\phi\rangle\,d\operatorname{vol}_g,
\end{aligned}$$

as asserted in (1.9).

Before proceeding further, it is convenient to define, for $p\in(1,\infty)$ and dual exponent $p'\in(1,\infty)$ determined by $1/p+1/p'=1$, the reflexive Banach space of pairs and its dual space,

$$\mathfrak{X} := W^{1,p}_{A_1}(X;\Lambda^1\otimes\operatorname{ad}P\oplus E) \quad\text{and}\quad \mathfrak{X}^* = W^{-1,p'}_{A_1}(X;\Lambda^1\otimes\operatorname{ad}P\oplus E). \tag{3.2}$$

The expression (1.9) defines the gradient as a map, $\mathscr{E}'(A,\Phi):\mathfrak{X}\to\mathfrak{X}^*$, where $\mathscr{E}'(A,\Phi)(a,\phi)\in\mathfrak{X}^*$ acts on $(b,\varphi)\in\mathfrak{X}$ by $L^2$ inner product. For the Hessian map, $\mathscr{E}''(A,\Phi):\mathfrak{X}\to\mathfrak{X}^*$, we make the

**Claim 3.2** (Hessian of the boson coupled Yang-Mills $L^2$ energy functional)**.** *For each $(a,\phi)\in\mathfrak{X}$, we have the formal expression,*

$$\begin{aligned}
\mathscr{E}''(A,\Phi)(a,\phi) &= d_A^*d_Aa + \nabla_A^*\nabla_A\phi + F_A\times a + \nabla_A^*(\varrho(a)\Phi) + \Phi\times\nabla_A\phi \\
&\quad - \rho(a)^*\nabla_A\Phi + \nabla_A\Phi\times\phi + \varrho(a)\Phi\times\Phi \\
&\quad - (m+2s|\Phi|^2)\phi - 4s\langle\Phi,\phi\rangle\Phi,
\end{aligned} \tag{3.3}$$

*where $\mathscr{E}''(A,\Phi)(a,\phi)\in\mathfrak{X}^*$ acts on $(b,\varphi)\in\mathfrak{X}$ by $L^2$ inner product.*

*Proof.* Let $(a_i,\phi_i)\in\mathfrak{X}$, for $i=1,2$. We compute the terms in $(\mathscr{E}'(A+ta_2,\Phi+t\phi_2),(a_1,\phi_1))_{L^2(X)}$ that are linear in $t$ using the expression (1.9) for the gradient. First,

$$\begin{aligned}
(F_{A+ta_2}, d_{A+ta_2}a_1)_{L^2(X)} &= \left(F_A + td_Aa_2 + \frac{t^2}{2}[a_2,a_2], d_Aa_1 + t[a_2,a_1]\right)_{L^2(X)} \\
&= (F_A, d_Aa_1)_{L^2(X)} + t(F_A,[a_2,a_1])_{L^2(X)} \\
&\quad + t(d_Aa_2, d_Aa_1)_{L^2(X)} + O(t^2).
\end{aligned}$$

Second,

$$\begin{aligned}
&(\nabla_{A+ta_2}(\Phi+t\phi_2), \nabla_{A+ta_2}\phi_1)_{L^2(X)} \\
&\quad = ((\nabla_A + t\varrho(a_2))(\Phi+t\phi_2), \nabla_A\phi_1 + t\varrho(a_2)\phi_1)_{L^2(X)} \\
&\quad = (\nabla_A\Phi,\nabla_A\phi_1)_{L^2(X)} + t(\nabla_A\phi_2,\nabla_A\phi_1)_{L^2(X)} + t(\varrho(a_2)\Phi,\nabla_A\phi_1)_{L^2(X)} \\
&\qquad + t(\nabla_A\Phi,\varrho(a_2)\phi_1)_{L^2(X)} + O(t^2).
\end{aligned}$$

Third,

$$
\begin{aligned}
&(\nabla_{A+ta_2}(\Phi + t\phi_2), \varrho(a_1)(\Phi + t\phi_2))_{L^2(X)} \\
&= ((\nabla_A + t\varrho(a_2))(\Phi + t\phi_2), \varrho(a_1)(\Phi + t\phi_2))_{L^2(X)} \\
&= (\nabla_A \Phi, \varrho(a_1)\Phi)_{L^2(X)} + t(\nabla_A \phi_2, \varrho(a_1)\Phi)_{L^2(X)} + t(\varrho(a_2)\Phi, \varrho(a_1)\Phi)_{L^2(X)} \\
&\quad + t(\nabla_A \Phi, \varrho(a_1)\phi_2)_{L^2(X)} + O(t^2).
\end{aligned}
$$

Fourth,

$$
(m(\Phi + t\phi_2), \phi_1)_{L^2(X)} = (m\Phi, \phi_1)_{L^2(X)} + t(m\phi_2, \phi_1)_{L^2(X)}.
$$

Fifth,

$$
\begin{aligned}
\int_X s|\Phi + t\phi_2|^2 \langle \Phi + t\phi_2, \phi_1\rangle \, d\operatorname{vol}_g &= \int_X s\left(|\Phi|^2 + 2t\operatorname{Re}\langle \Phi, \phi_2\rangle + t^2|\phi_2|^2\right)\langle \Phi + t\phi_2, \phi_1\rangle \, d\operatorname{vol}_g \\
&= \int_X s|\Phi|^2 \langle \Phi, \phi_1\rangle \, d\operatorname{vol}_g \\
&\quad + t\int_X \left(s|\Phi|^2\langle \phi_2, \phi_1\rangle + 2s\operatorname{Re}\langle \Phi, \phi_2\rangle\langle \Phi, \phi_1\rangle\right) \, d\operatorname{vol}_g + O(t^2).
\end{aligned}
$$

By subtracting $(\mathscr{E}''(A,\Phi)(a_2,\phi_2),(a_1,\phi_1))_{L^2(X)}$, collecting all the first-order terms in $t$, and reversing the roles of $(a_1,\phi_1)$ and $(a_2,\phi_2)$, we see that

$$
\begin{aligned}
\left(\mathscr{E}''(A,\Phi)(a_1,\phi_1),(a_2,\phi_2)\right)_{L^2(X)} &= (d_A a_1, d_A a_2)_{L^2(X)} + 2(F_A, [a_1, a_2])_{L^2(X)} \\
&\quad + \operatorname{Re}(\nabla_A \phi_1, \nabla_A \phi_2)_{L^2(X)} \\
&\quad + \operatorname{Re}\left((\rho(a_1)\Phi, \nabla_A \phi_2)_{L^2(X)} + (\rho(a_2)\Phi, \nabla_A \phi_1)_{L^2(X)}\right) \\
&\quad + \operatorname{Re}(\nabla_A \Phi, \rho(a_1)\phi_2 + \rho(a_2)\phi_1)_{L^2(X)} \\
&\quad + \operatorname{Re}(\rho(a_1)\Phi, \rho(a_2)\Phi)_{L^2(X)} \\
&\quad - \operatorname{Re}\int_X \left((m + 2s|\Phi|^2)\langle \phi_1, \phi_2\rangle + 4s\langle \Phi, \phi_1\rangle\langle \Phi, \phi_2\rangle\right) \, d\operatorname{vol}_g,
\end{aligned}
$$

By now viewing $\mathscr{E}''(a_1,\phi_1)$ as an element of $\mathfrak{X}^*$, we obtain the expression (3.3). □

We shall need to compare $\mathscr{E}''(A,\Phi)$ with the $L^2$ self-adjoint elliptic operator,

$$
\begin{aligned}
(3.4)\quad M_{A,\Phi} : W^{1,p}_{A_1}(X; \Lambda^1 \otimes \operatorname{ad}P \oplus E) &\to W^{-1,p'}_{A_1}(X; \Lambda^1 \otimes \operatorname{ad}P \oplus E), \\
(a,\phi) &\mapsto d_A^* d_A a + d_{A,\Phi} d_{A,\Phi}^*(a,\phi) + \nabla_A^* \nabla_A \phi,
\end{aligned}
$$

in order to prove that $\mathscr{E}''(A,\Phi)$ is Fredholm with index zero upon restriction to

$$
(3.5)\qquad \mathscr{X} := \operatorname{Ker}\left(d_{A,\Phi}^* : W^{1,p}_{A_1}(X; \Lambda^1 \otimes \operatorname{ad}P \oplus E) \to L^p(X; \operatorname{ad}P)\right).
$$

We recall from (2.60) that

$$
d_{A,\Phi}\xi = (d_A \xi, -\xi\Phi), \quad \forall\, \xi \in C^\infty(X; \operatorname{ad}P),
$$

with $L^2$-adjoint,

$$
d_{A,\Phi}^*(a,\phi) = d_A^* a - \langle \phi, \cdot\Phi\rangle^*, \quad \forall\, (a,\phi) \in C^\infty(\Lambda^1 \otimes \operatorname{ad}P \oplus E),
$$

for every $(a,\phi) \in C^\infty(\Lambda^1 \otimes \operatorname{ad}P \oplus E)$, where the section $\langle \phi, \cdot\Phi\rangle^*$ of $\operatorname{ad}P$ is defined by

$$
(\langle \phi, \cdot\Phi\rangle^*, \xi)_{L^2(X)} = (\phi, \xi\Phi)_{L^2(X)}, \quad \forall\, \xi \in C^\infty(X; \operatorname{ad}P).
$$

According to Lemma 2.21, the operator,

$$d_{A,\Phi}^* : W_{A_1}^{1,p}(X; \Lambda^1 \otimes \mathrm{ad}P \oplus E) \to L^p(X; \mathrm{ad}P),$$

is bounded when $(A, \Phi)$ is a $W^{1,q}$ pair with $q \geq d/2$ and $p$ obeys $d/2 \leq p \leq q$; therefore $\mathscr{X}$ in (3.5) is a Banach space as a closed subspace of the Banach space $\mathfrak{X} = W_{A_1}^{1,p}(X; \Lambda^1 \otimes \mathrm{ad}P \oplus E)$.

*Remark* 3.3 (Alternative choice of Laplace operator for comparison with the Hessian). We could alternatively compare $\mathscr{E}''(A, \Phi)$ with the simpler $L^2$ self-adjoint elliptic operator,

$$M_A : W_{A_1}^{1,p}(X; \Lambda^1 \otimes \mathrm{ad}P \oplus E) \to W_{A_1}^{-1,p'}(X; \Lambda^1 \otimes \mathrm{ad}P \oplus E), \quad (a, \phi) \mapsto d_A^* d_A a + \nabla_A^* \nabla_A \phi, \tag{3.6}$$

upon restriction to the alternative closed subspace of $\mathfrak{X}$ given by

$$\mathrm{Ker}\left(d_A^* : W_{A_1}^{1,p}(X; \Lambda^1 \otimes \mathrm{ad}P) \to L^p(X; \mathrm{ad}P)\right) \oplus W_{A_1}^{1,p}(X; E). \tag{3.7}$$

However, it is more natural to restrict to a slice that is appropriate for the quotient space of pairs.

Consequently, when $(a, \phi) \in \mathrm{Ker}\, d_{A,\Phi}^*$, the expressions (3.3) and (3.6) yield, after formally expanding $\nabla_A^*(\varrho(a)\Phi) = \nabla_A a \times \Phi + a \times \nabla_A \Phi + a \times \Phi$,

$$\begin{aligned} \mathscr{E}''(A, \Phi)(a, \phi) - M_{A,\Phi}(a, \phi) &= F_A \times a + \nabla_A a \times \Phi + a \times \nabla_A \Phi + \Phi \times \nabla_A \phi \\ &\quad + \varrho(a)\Phi + \nabla_A \Phi \times \phi + \varrho(a)\Phi \times \Phi \\ &\quad - (m + 2s|\Phi|^2)\phi - 4s\langle \Phi, \phi \rangle \Phi. \end{aligned} \tag{3.8}$$

To determine the Fredholm property and index of $\mathscr{E}''(A, \Phi)$, we shall need the following analogue of Proposition B.1 and whose proof is virtually identical.

**Proposition 3.4** (Fredholm property and index of a Laplace operator). *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, and $A_1$ is a $C^\infty$ reference connection on $P$. If $(A, \Phi)$ is a $W^{1,q}$ pair on $(P, E)$ with $d/2 < q < \infty$ and $2 \leq p < \infty$ and $d/2 \leq p \leq q$, then the operator $M_{A,\Phi}$ in* (3.4) *is Fredholm with index zero.*

Let $\iota_{\mathscr{X}} : \mathscr{X} \subset \mathfrak{X}$ denote the continuous embedding, where $\mathfrak{X}$ is as in (3.2) and $\mathscr{X}$ is as in (3.5). We now apply Proposition 3.4 to prove the Fredholm property and compute the index of the Hessian, $\iota_{\mathscr{X}}^* \circ \mathscr{E}''(A, \Phi) \circ \iota_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}^*$, where the continuous operator $\iota_{\mathscr{X}}^* : \mathfrak{X}^* \subset \mathscr{X}^*$ is the adjoint of the continuous embedding.

**Proposition 3.5** (Fredholm property and index of the restriction of the Hessian of the energy functional to a Coulomb-gauge slice). *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, and $A_1$ is a $C^\infty$ reference connection on $P$, and $m, s \in C^\infty(X)$. If $(A, \Phi)$ is a $W^{1,q}$ pair on $(P, E)$ with $d/2 < q < \infty$ and $2 \leq p < \infty$ and $d/2 \leq p \leq q$ and* one *of the following holds,*

$$\begin{cases} d \geq 6, & \text{or} \\ 3 \leq d \leq 5 \text{ and } q \geq 2d/(d-2), & \text{or} \\ d = 2 \text{ and } p \geq 2 \text{ and } q > 2, \end{cases}$$

*then $\iota_{\mathscr{X}}^* \circ \mathscr{E}''(A_\infty, \Phi_\infty) \circ \iota_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}^*$ is Fredholm with index zero.*

*Proof.* The Laplace operator $M_{A,\Phi}$ in (3.4) is Fredholm of index zero by Proposition 3.4. Since

$$\operatorname{Ker}(M_{A,\Phi} : \mathscr{X} \to \mathscr{X}^*) \subset \operatorname{Ker}(M_{A,\Phi} : \mathfrak{X} \to \mathfrak{X}^*),$$

the kernel of $\iota^*_{\mathscr{X}} \circ M_{A,\Phi} \circ \iota_{\mathscr{X}}$ is finite-dimensional. Also, $M_{A,\Phi} : \mathfrak{X} \to \mathfrak{X}^*$ has closed range, so $\iota^*_{\mathscr{X}} \circ M_{A,\Phi} \circ \iota_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}^*$ has closed range too. Observe that

$$(\iota^*_{\mathscr{X}} \circ M_{A,\Phi} \circ \iota_{\mathscr{X}})^* = \iota^*_{\mathscr{X}} \circ M_{A,\Phi} \circ \iota^{**}_{\mathscr{X}} : \mathscr{X}^{**} \to \mathscr{X}^*,$$

so it remains to show that $\mathscr{X}$ is reflexive. But $\mathscr{X} \subset \mathfrak{X}$ is a closed subspace of a reflexive Banach space and so $\mathscr{X}$ is reflexive by [42, Theorem 1.11.16] and

$$(\iota^*_{\mathscr{X}} \circ M_{A,\Phi} \circ \iota_{\mathscr{X}})^* = \iota^*_{\mathscr{X}} \circ M_{A,\Phi} \circ \iota_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}^*$$

is a symmetric operator, where we use the canonical isometric isomorphisms $\mathfrak{X}^{**} \cong \mathfrak{X}$ and $\mathscr{X}^{**} \cong \mathscr{X}$. Hence, [19, Lemma 2.3] implies that $\iota^*_{\mathscr{X}} \circ M_{A,\Phi} \circ \iota_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}^*$ is Fredholm with index zero. Our goal is to prove that the operator,

$$\iota^*_{\mathscr{X}} \circ (\mathscr{E}''(A,\Phi) - M_{A,\Phi}) \circ \iota_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}^*. \tag{3.9}$$

is compact. The argument is similar to the proof in the case of the corresponding result for the pure Yang-Mills $L^2$ energy functional [13, Chapter 7, Claim 21.11].

By [3, Theorem 6.3], we have a compact embedding $W^{1,p} \Subset L^r(X)$ when

(1) $1 \le p < d$ and $1 \le r \le p^* := dp/(d-p)$, or
(2) $p = d$ and $1 \le r < \infty$, or
(3) $d < p < \infty$ and $1 \le r \le \infty$ — we choose $r < \infty$ to appeal to $(L^r(X))^* = L^{r'}(X)$ (which fails when $r = \infty$).

In each case, we obtain a compact embedding $L^{r'}(X) \Subset W^{-1,p'}(X)$ by duality and density of $W^{1,p}(X) \subset L^r(X)$, where $1/r + 1/r' = 1$. We make the

**Claim 3.6** (Boundedness of the difference between the Hessian and the Laplacian)**.** *If $r$ is chosen as in the preceding paragraph, then the following operator is bounded:*

$$\mathscr{E}''(A,\Phi) - M_{A,\Phi} : W^{1,p}_{A_1}(X; \Lambda^1 \otimes \operatorname{ad}P \oplus E) \to L^{r'}(X; \Lambda^1 \otimes \operatorname{ad}P \oplus E). \tag{3.10}$$

*Proof of Claim 3.6.* We shall compute $L^{r'}(X)$ bounds for each of the nonlinear terms appearing in the expression (3.8) for $(\mathscr{E}''(A,\Phi) - M_{A,\Phi})(a,\phi)$. For each term, we consider the cases $p < d$, $p = d$, and $p > d$ separately.

**Step 1** ($L^{r'}(X)$ estimates for $F_A \times a$ and $\nabla_A \Phi \times a$)**.** We claim that

$$\|F_A \times a\|_{L^{r'}(X)} \le z \|F_A\|_{L^q(X)} \|a\|_{W^{1,p}_{A_1}(X)}, \tag{3.11}$$

$$\|\nabla_A \Phi \times a\|_{L^{r'}(X)} \le z \left(1 + \|a_1\|_{W^{1,q}_{A_1}(X)}\right) \|\Phi\|_{W^{1,q}_{A_1}(X)} \|a\|_{W^{1,p}_{A_1}(X)}, \tag{3.12}$$

where $z = z(g,p) \in [1,\infty)$.

*Case* 1 ($p < d$). We choose $r = p^*$. In order to have a continuous multiplication map, $L^q(X) \times L^{p^*}(X) \to L^{r'}(X)$, we require that the following inequality holds,

$$1/q + 1/p^* \le 1/r' = 1 - 1/r = 1 - 1/p^*,$$

that is, $1/q + 2/p^* \le 1$ or

$$1/q + 2(d-p)/(dp) \le 1.$$

We need to verify which ranges of $p, q$ are allowed by the preceding inequality. By hypothesis, we have $q \geq p$, and so for the worst case for $q$, namely $q = p$, we must have

$$1/p + 2(d-p)/(dp) = (3d-2p)/(dp) \leq 1,$$

or equivalently, $3d - 2p \leq dp$ or $p \geq (d+2)/(3d)$. But $d/2 \geq (d+2)/(3d)$ for all $d \geq 2$ and because $p \geq d/2$ by hypothesis, the condition $p \geq (d+2)/(3d)$ is obeyed also. Thus,

$$\|F_A \times a\|_{L^{r'}(X)} \leq z\|F_A\|_{L^q(X)}\|a\|_{L^{p^*}(X)} \leq z\|F_A\|_{L^q(X)}\|a\|_{W^{1,p}_{A_1}(X)},$$

which yields (3.11) for all $d \geq 2$.

*Case* 2 ($p = d$). We can choose $r \in [1, \infty)$ arbitrarily large in the continuous Sobolev embedding $W^{1,p}(X) \subset L^r(X)$ and so obtain $r' > 1$ arbitrarily small. We replace the role of $p^*$ in the case $p < d$ by a constant $s \in [1, \infty)$ that can be arbitrarily large and employ a continuous multiplication map, $L^q(X) \times L^s(X) \to L^{r'}(X)$, and so we require that the following inequality holds,

$$1/q + 1/s \leq 1/r' = 1 - 1/r,$$

that is, $1/q + 1/r + 1/s \leq 1$. But for any $q > 1$ (which we assume by hypothesis), we can choose $r$ and $s$ large enough that the preceding inequality holds. Thus,

$$\|F_A \times a\|_{L^{r'}(X)} \leq z\|F_A\|_{L^q(X)}\|a\|_{L^s(X)} \leq z\|F_A\|_{L^q(X)}\|a\|_{W^{1,p}_{A_1}(X)},$$

which again yields (3.11) for all $d \geq 2$.

*Case* 3 ($p > d$). The argument for the case $p = d$ also yields (3.11) for all $d \geq 2$.

An argument identical to that for (3.11) gives

$$\|\nabla_A \Phi \times a\|_{L^{r'}(X)} \leq z\|\nabla_A \Phi\|_{L^q(X)}\|a\|_{W^{1,p}_{A_1}(X)}.$$

Writing $A = A_1 + a_1$, where $a_1 \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)$ and noting that

$$\|\nabla_A \Phi\|_{L^q(X)} \leq \|\nabla_{A_1}\Phi\|_{L^q(X)} + \|\varrho(a_1)\Phi\|_{L^q(X)},$$

and

$$\|\varrho(a_1)\Phi\|_{L^q(X)} \leq z\|a_1\|_{L^{2q}(X)}\|\Phi\|_{L^{2q}(X)} \leq z\|a_1\|_{W^{1,q}_{A_1}(X)}\|\Phi\|_{W^{1,q}_{A_1}(X)},$$

where the last inequality follows from the continuous Sobolev embedding, $W^{1,q}(X) \subset L^{2q}(X)$, valid for any $q \geq d/2$ (which we assume by hypothesis), now yields (3.12).

**Step 2** ($L^{r'}(X)$ estimates for $\Phi \times \nabla_A \phi$ and $\Phi \times \nabla_A \phi$)**.** We claim that

$$\|\Phi \times \nabla_A \phi\|_{L^{r'}(X)} \leq z\left(1 + \|a_1\|_{W^{1,q}_{A_1}(X)}\right)\|\Phi\|_{W^{1,q}_{A_1}(X)}\|\phi\|_{W^{1,p}_{A_1}(X)}, \tag{3.13}$$

$$\|\Phi \times \nabla_A a\|_{L^{r'}(X)} \leq z\left(1 + \|a_1\|_{W^{1,q}_{A_1}(X)}\right)\|\Phi\|_{W^{1,q}_{A_1}(X)}\|a\|_{W^{1,p}_{A_1}(X)}, \tag{3.14}$$

$$\|\Phi \times a\|_{L^{r'}(X)} \leq z\|\Phi\|_{W^{1,q}_{A_1}(X)}\|a\|_{L^p(X)}, \tag{3.15}$$

where $z = z(g, p) \in [1, \infty)$.

*Case* 1 ($p < d$). We choose $r = p^*$. For $q < d$, we shall use a continuous multiplication map, $L^{q^*}(X) \times L^p(X) \to L^{r'}(X)$, where $q^* := dq/(d-q)$. Thus, we require that the following inequality holds,

$$1/q^* + 1/p = (d-q)/(dq) + 1/p \leq 1/r' = 1 - 1/r = 1 - 1/p^* = 1 - (d-p)/(dp),$$

that is,

$$(d-q)/(dq) + 1/p + (d-p)/(dp) \leq 1.$$

In the worst case for $q$ we have $q = p$, so $p$ must obey

$$1/p + 2(d-p)/(dp) = (3d-2p)/(dp) \leq 1,$$

giving the same condition on $p$ as in the case $p < d$ of the proof of (3.11). By virtue of the continuous multiplication map, $L^{q^*}(X) \times L^p(X) \to L^{r'}(X)$, and continuous Sobolev embeddings, $W^{1,q}(X) \subset L^{q^*}(X)$ and $W^{1,p}(X) \subset L^{2p}(X)$, and writing $A = A_1 + a_1$, we obtain

$$\begin{aligned}
\|\Phi \times \nabla_A \phi\|_{L^{r'}(X)} &\leq z\|\Phi\|_{L^{q^*}(X)}\|\nabla_A \phi\|_{L^p(X)} \\
&\leq z\|\Phi\|_{W^{1,q}_{A_1}(X)} \left(\|a_1\|_{L^{2p}(X)}\|\phi\|_{L^{2p}(X)} + \|\nabla_{A_1}\phi\|_{L^p(X)}\right) \\
&\leq z\|\Phi\|_{W^{1,q}_{A_1}(X)} \left(\|a_1\|_{W^{1,p}_{A_1}(X)}\|\phi\|_{W^{1,p}_{A_1}(X)} + \|\nabla_{A_1}\phi\|_{L^p(X)}\right)
\end{aligned}$$

which gives (3.13) for this case.

For $q = d$, we replace the role of $q^*$ in the argument for $q < d$ by a constant $s \in [1, \infty)$ which is large enough that the following inequality holds,

$$1/s + 1/p \leq 1/r' = 1 - (d-p)/(dp),$$

that is,

$$1/s + 1/p + (d-p)/(dp) = 1/s + 2/p - 1/d \leq 1.$$

Thus a choice of $s$ is possible when $p \geq d/2$, which we assume by hypothesis. Indeed, in the worst case $p = d/2$, any $s \in [1, \infty)$ is valid. By using the continuous multiplication map, $L^s(X) \times L^p(X) \to L^{r'}(X)$, and continuous Sobolev embedding, $W^{1,q}(X) \subset L^s(X)$, we obtain (3.13) for this case.

For $q > d$, the argument for the case $q = d$ also applies to give (3.13).

*Case* 2 ($p = d$). In the worst case for $q$ we have $q = p = d$, so we choose $r, s \in [1, \infty)$ arbitrarily large (and thus $r' > 1$ arbitrarily small) so the following inequality holds,

$$1/s + 1/p = 1/s + 1/d \leq 1/r'.$$

Thus choices of $r, s$ are possible when $p > 1$, which holds since $p = d \geq 2$ for this case. Applying the continuous multiplication map, $L^s(X) \times L^p(X) \to L^{r'}(X)$, and continuous Sobolev embedding, $W^{1,q}(X) \subset L^s(X)$, we obtain

$$\|\Phi \times \nabla_A \phi\|_{L^{r'}(X)} \leq z\|\Phi\|_{L^s(X)}\|\nabla_A \phi\|_{L^p(X)} \leq z\|\Phi\|_{W^{1,q}_{A_1}(X)}\|\nabla_A \phi\|_{L^p(X)},$$

and now (3.13) follows from the preceding inequality just as in the case $p < d$.

*Case* 3 ($p > d$). The argument for the case $p = d$ also yields (3.13) for all $d \geq 2$.

An argument identical to that for (3.13) also gives (3.14) and (3.15).

**Step 3** ($L^{r'}(X)$ estimate for $m\phi$)**.** We have

$$\|m\phi\|_{L^{r'}(X)} \leq \|m\|_{C(X)}\|\phi\|_{L^{r'}(X)},$$

and because $r' \leq p$ by construction in each of the three cases $p < d$, and $p = d$, and $p > d$, we obtain

$$\|m\phi\|_{L^{r'}(X)} \leq z\|m\|_{C(X)}\|\phi\|_{L^p(X)}, \tag{3.16}$$

as desired.

**Step 4** ($L^{r'}(X)$ estimates for $\Phi\times\Phi\times a$ and $\Phi\times\Phi\times\phi$)**.** We claim that

$$\|\Phi\times\Phi\times a\|_{W_{A_1}^{-1,p'}(X)} \leq z\|\Phi\|_{W_{A_1}^{1,q}(X)}^2\|a\|_{W_{A_1}^{1,p}(X)}, \tag{3.17}$$

$$\|\Phi\times\Phi\times \phi\|_{W_{A_1}^{-1,p'}(X)} \leq z\|\Phi\|_{W_{A_1}^{1,q}(X)}^2\|\phi\|_{W_{A_1}^{1,p}(X)}, \tag{3.18}$$

where $z = z(g,p) \in [1,\infty)$.

*Case* 1 ($p<d$)*.* We choose $r = p^* = dp/(d-p)$ and, to obtain a continuous multiplication map, $L^{2q}(X)\times L^{p^*}(X)\to L^{r'}(X)$, we require that the following inequality holds,

$$2/q + 1/p^* = 2/q + (d-p)/(dp) \leq 1/r' = 1 - 1/r = 1 - 1/p^* = 1-(d-p)/(dp),$$

that is,

$$2/q + 2(d-p)/(dp) \leq 1.$$

In the worst case for $q$ we have $q = p$, giving

$$2/p + 2(d-p)/(dp) = (4d-2p)/(dp) \leq 1,$$

that is, $4d-2p\leq dp$ or $p \geq 4d/(d+2)$. By hypothesis, $p\geq d/2$, and we observe that $d/2 \geq 4d/(d+2)$ if and only if $d+2\geq 8$, that is, $d\geq 6$.

Using the continuous multiplication map, $L^{2q}(X)\times L^{p^*}(X)\to L^{r'}(X)$, and continuous Sobolev embeddings, $W^{1,q}(X)\subset L^{2q}(X)$ and $W^{1,p}(X)\subset L^{p^*}(X)$, we obtain

$$\|\Phi\times\Phi\times\phi\|_{L^{r'}(X)} \leq z\|\Phi\|_{L^{2q}(X)}^2\|\phi\|_{L^{p^*}(X)} \leq z\|\Phi\|_{W_{A_1}^{1,q}(X)}^2\|\phi\|_{W_{A_1}^{1,p}(X)},$$

giving (3.18) for $d\geq 6$.

For $2\leq d\leq 5$, we observe that the worst case for $p$ is $p = d/2$, so we make that choice and select $q$ large enough that the following inequality holds,

$$2/q + 2(d-p)/(dp) = 2/q + d/(d^2/2) = 2/q + 2/d \leq 1,$$

or $2/q \leq 1-2/d = (d-2)/d$. If $d>2$, then we select $q$ large enough that $q\geq 2d/(d-2)$ and observe that the argument for $d\geq 6$ applies to give (3.18). If $d=2$, then we use our hypothesis that $p>1$ in this case and select $q$ large enough that the following inequality holds,

$$2/q + 2(d-p)/(dp) = 2/q + (2-p)/p \leq 1,$$

or $2/q\leq 1-(2-p)/p = 2(p-1)/p$ or $q\geq p/(p-1) = p'$ and again observe that the argument for $d\geq 6$ applies to give (3.18).

*Case* 2 ($p=d$)*.* We replace the role of $p^*$ in the argument for $p<d$ by a constant $s\in[1,\infty)$ and use a continuous multiplication map, $L^{2q}(X)\times L^s(X)\to L^{r'}(X)$, where we choose $s$ and $r\in[1,\infty)$ large enough and $r'>1$ small enough that

$$2/q + 1/s \leq 1/r'.$$

In the worst case for $q$ we have $q=p=d$ and for $d\geq 3$, we see that choices of $r,s$ are possible. Applying the continuous Sobolev embeddings, $W^{1,q}(X)\subset L^{2q}(X)$ and $W^{1,p}(X)\subset L^s(X)$, we see that

$$\|\Phi\times\Phi\times a\|_{L^{r'}(X)} \leq z\|\Phi\|_{L^{2q}(X)}^2\|a\|_{L^s(X)} \leq z\|\Phi\|_{W_{A_1}^{1,q}(X)}^2\|a\|_{W_{A_1}^{1,p}(X)},$$

which gives (3.18). For $d=2$, we require that $q>2$ in our hypotheses and observe that choices of $r,s$ are again possible and proceed exactly as when $d\geq 3$ to obtain (3.18).

*Case* 3 ($p>d$)*.* The argument for the case $p=d$ also yields (3.18) for all $d\geq 2$.

An argument identical to that for (3.18) also gives (3.17).

This completes the proof of Claim 3.6, noting that $\|m\|_{C(X)}$ and $\|s\|_{C(X)}$ are finite by hypothesis. □

Because the embedding,

$$L^{r'}(X;\Lambda^1\otimes \mathrm{ad}P\oplus E) \Subset W^{-1,p'}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P\oplus E),$$

is compact by [3, Theorem 6.3] (for $1\le p<\infty$) and $r$ chosen as in our appeal to the Rellich-Kondrachov Theorem for each case of $p$ prior to the statement of Claim 3.6, then the composition of a compact and a bounded operator,

$$\mathscr{E}''(A,\Phi) - M_{A,\Phi} : W^{1,p}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P\oplus E) \to W^{-1,p'}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P\oplus E), \tag{3.19}$$

is a compact operator by Claim 3.6 and [7, Proposition 6.3]. Therefore, the operator (3.9) is compact as the composition of a compact and a bounded operator and by once again applying [7, Proposition 6.3].

Since $\iota^*_{\mathscr{X}}\circ\mathscr{E}''(A,\Phi)\circ\iota_{\mathscr{X}} : \mathscr{X}\to\mathscr{X}^*$ is a compact perturbation of a Fredholm operator with index zero, namely $\iota^*_{\mathscr{X}}\circ\mathscr{E}''(A,\Phi)\circ\iota_{\mathscr{X}} : \mathscr{X}\to\mathscr{X}^*$, then [31, Corollary 19.1.8] implies that $\iota^*_{\mathscr{X}}\circ\mathscr{E}''(A,\Phi)\circ\iota_{\mathscr{X}} : \mathscr{X}\to\mathscr{X}^*$ is Fredholm with index zero. □

By imposing the stronger $W^{2,q}$ regularity hypothesis on $(A,\Phi)$, we obtain the simpler

**Corollary 3.7** (Fredholm property and index of the restriction of the Hessian of the energy functional to a Coulomb-gauge slice)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d\ge 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E=P\times_\varrho\mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho: G\to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, and $A_1$ is a $C^\infty$ reference connection on $P$. If $(A,\Phi)$ is a $W^{2,q}$ pair on $(P,E)$ with $d/2<q<\infty$ and $2\le p<\infty$ and $d/2\le p\le q$, then $\iota^*_{\mathscr{X}}\circ\mathscr{E}''(A_\infty,\Phi_\infty)\circ\iota_{\mathscr{X}} : \mathscr{X}\to\mathscr{X}^*$ is Fredholm with index zero.*

*Proof.* The only bounds arising in the proof of the proof of Claim 3.6 that require additional conditions on $q$ when $(A,\Phi)$ is only assumed to be $W^{1,q}$ are (3.18) and (3.17) in the cases $p<d$ and $p=d=2$.

For $p<d$, we again choose $r=p^*=dp/(d-p)$ but now simply require that $s\in[1,\infty)$ be large enough that we have a continuous Sobolev multiplication map, $L^{2s}(X)\times L^{p^*}(X)\to L^{r'}(X)$, that is, the following inequality holds,

$$2/s+1/p^* = 2/s+(d-p)/(dp) \le 1/r' = 1-1/r = 1-1/p^* = 1-(d-p)/(dp),$$

or equivalently,

$$2/s+2(d-p)/(dp)\le 1.$$

We can choose $s=\infty$ and then observe that $2(d-p)/(dp)\le 1$ if and only if $2(d-p)\le dp$ or $p(d+2)\ge 2d$ or $p\ge 2d/(d+2)$, which is assured by our hypothesis that $p\ge d/2$ since $d/2\ge 2d/(d+2)$ for all $d\ge 2$. Thus,

$$\|\Phi\times\Phi\times a\|_{L^{r'}(X)} \le z\|\Phi\|^2_{L^\infty(X)}\|a\|_{L^{p^*}(X)},$$

and applying the continuous Sobolev embeddings, $W^{2,q}(X)\subset L^\infty(X)$ and $W^{1,p}(X)\subset L^{p^*}(X)$, we obtain

$$\|\Phi\times\Phi\times a\|_{L^{r'}(X)} \le z\|\Phi\|^2_{W^{2,q}_{A_1}(X)}\|a\|_{W^{1,p}_{A_1}(X)}, \tag{3.20}$$

giving the desired replacement for (3.18).

For the case $p = d$, we may choose $r \in [1, \infty)$ arbitrarily large and $r' > 1$ arbitrarily small and $s \in [1, \infty]$ so that

$$2/s + 1/r \leq 1/r' = 1 - 1/r.$$

Indeed, we can select $s = \infty$ and thus we require $2/r \leq 1$, that is, $r \geq 2$. We now apply the continuous Sobolev multiplication map, $L^{2s}(X) \times L^r(X) \to L^{r'}(X)$, and proceed as in the case $p < d$ to obtain (3.20). An identical argument yields

$$\|\Phi \times \Phi \times a\|_{L^{r'}(X)} \leq z\|\Phi\|^2_{W^{2,q}_{A_1}(X)}\|a\|_{W^{1,p}_{A_1}(X)}. \tag{3.21}$$

This completes the proof of Claim 3.6 in the case where $(A, \Phi)$ is $W^{2,q}$ and under the updated hypotheses on $p, q$. The remainder of the proof of Proposition 3.7 applies without change to complete the proof of Corollary 3.7. □

Before proceeding to the proof of Theorem 5, we shall need the elementary

**Lemma 3.8** (Estimate for the action of a $W^{2,q}$ gauge transformation intertwining two $W^{1,q}$ pairs)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, and $A_1$ is a $C^\infty$ reference connection on $P$, $q > d/2$ and $p$ obeys $d/2 \leq p \leq q$, then there is a constant $C = C(g, p) \in [1, \infty)$ with the following significance. If $(A, \Phi)$ and $(A', \Phi')$ are $W^{1,q}$ pairs on $(P, E)$ and $u \in \mathrm{Aut}^{2,q}(P)$, then*

$$\|u(A, \Phi) - u(A', \Phi')\|_{W^{1,p}_{A_1}(X)} \leq C\left(1 + \|u\|_{W^{2,p}_{A_1}(X)}\right)\|(A, \Phi) - (A', \Phi')\|_{W^{1,p}_{A_1}(X)},$$

$$\|(A, \Phi) - (A', \Phi')\|_{W^{1,p}_{A_1}(X)} \leq C\left(1 + \|u\|_{W^{2,p}_{A_1}(X)}\right)\|u(A, \Phi) - u(A', \Phi')\|_{W^{1,p}_{A_1}(X)}.$$

*Proof.* Write $a := A - A' \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)$ and recall that $u(A) - A_1 = u^{-1}(A - A_1)u + u^{-1}d_{A_1}u$ by (2.27) and similarly for $A'$, so

$$u(A) - u(A') = u^{-1}(A - A_1)u - u^{-1}(A' - A_1)u = u^{-1}(A - A')u = u^{-1}au$$

and thus,

$$u(A, \Phi) - u(A', \Phi') = (u^{-1}au, u(\Phi - \Phi')).$$

Therefore,

$$\nabla_{A_1}(u(A - A')) = -u^{-1}(\nabla_{A_1}u)u^{-1}au + u^{-1}(\nabla_{A_1}a)u + u^{-1}a(\nabla_{A_1}u).$$

By taking $L^p$ norms and using the pointwise bound $|u| \leq 1$, the Sobolev embedding $W^{1,p}(X) \subset L^{2p}(X)$ (valid for $p \geq d/2$), and the Kato Inequality [22, Equation (6.20)], we obtain

$$\begin{aligned}\|\nabla_{A_1}(u(A - A'))\|_{L^p(X)} &\leq 2\|\nabla_{A_1}u\|_{L^{2p}(X)}\|a\|_{L^{2p}(X)} + \|\nabla_{A_1}a\|_{L^p(X)} \\ &\leq C\|\nabla_{A_1}u\|_{W^{1,p}_{A_1}(X)}\|A - A'\|_{W^{1,p}_{A_1}(X)} + \|\nabla_{A_1}a\|_{L^p(X)}.\end{aligned}$$

Similarly, $\nabla_{A_1}(u(\Phi - \Phi')) = (\nabla_{A_1}u)(\Phi - \Phi') + u(\nabla_{A_1}(\Phi - \Phi'))$ and

$$\begin{aligned}\|\nabla_{A_1}(u(\Phi - \Phi'))\|_{L^p(X)} &\leq \|\nabla_{A_1}u\|_{L^{2p}(X)}\|\Phi - \Phi'\|_{L^{2p}(X)} + \|\nabla_{A_1}(\Phi - \Phi')\|_{L^p(X)} \\ &\leq \|\nabla_{A_1}u\|_{W^{1,p}_{A_1}(X)}\|\Phi - \Phi'\|_{W^{1,p}_{A_1}(X)} + \|\nabla_{A_1}(\Phi - \Phi')\|_{L^p(X)}.\end{aligned}$$

By combining the preceding estimates, we obtain the first inequality; the second inequality is proved by a symmetric argument. □

We can now proceed to the

*Proof of Theorem 5.* We first consider the simpler case where the pair $(A,\Phi)$ is in Coulomb gauge relative to the critical point $(A_\infty,\Phi_\infty)$ and then consider the general case

*Case* 1 ($(A,\Phi)$ in Coulomb gauge relative to $(A_\infty,\Phi_\infty)$). By hypothesis, $(A_\infty,\Phi_\infty)$ is a $W^{1,q}$ pair that is a critical point for the functional $\mathscr{E}$ in (1.5). By the regularity Theorem 2.23, there exists a $W^{2,q}$ gauge transformation $u_\infty$ such that $u_\infty(A_\infty,\Phi_\infty)$ is a $C^\infty$ pair. In particular, $u_\infty(A_\infty,\Phi_\infty)$ is a $W^{2,q}$ pair and $u_\infty(A,\Phi)$ is in Coulomb gauge relative to $u_\infty(A_\infty,\Phi_\infty)$. Following (3.5), we define the Banach space,

$$\mathscr{X} := \operatorname{Ker}\left(d^*_{u_\infty(A_\infty,\Phi_\infty)} : W^{1,p}_{A_1}(X;\Lambda^1\otimes \operatorname{ad}P\oplus E)\to L^p(X;\operatorname{ad}P)\right),$$

and Hilbert space,

$$\mathscr{H} := L^2(X;\Lambda^1\otimes \operatorname{ad}P\oplus E),$$

with continuous embedding, $\mathscr{X}\subset\mathscr{H}$, and recall that $\mathfrak{X} = W^{1,p}_{A_1}(X;\Lambda^1\otimes\operatorname{ad}P\oplus E)$ by (3.2). Note that $4d/(d+4) > d/2$ for $d=2,3$ and $4d/(d+4)\leq d/2$ for $d\geq 4$, so Proposition 3.1 implies that the functional

$$\mathscr{E}_{\mathscr{X}} \equiv \mathscr{E}\circ\iota_{\mathscr{X}} : \mathscr{X}\to\mathbb{R}$$

is analytic, where $\iota_{\mathscr{X}} : \mathscr{X}\subset\mathfrak{X}$ denotes the continuous embedding and, setting $x_\infty := u_\infty(A_\infty,\Phi_\infty)$, Corollary 3.7 implies that the Hessian operator,

$$\mathscr{E}''_{\mathscr{X}}(x_\infty) = \iota^*_{\mathscr{X}}\circ\mathscr{E}\circ\iota_{\mathscr{X}} : \mathscr{X}\to\mathscr{X}^*$$

is Fredholm with index zero. Hence, Theorem 1 implies that there exist constants $Z'\in[1,\infty)$ and $\sigma'\in(0,1]$ and $\theta\in[1/2,1)$ (depending on $(A_1,\Phi_1)$, and $u_\infty(A_\infty,\Phi_\infty)$, and $g$, $G$, $p$, $P$) such that if

$$\|u_\infty(A,\Phi) - u_\infty(A_\infty,\Phi_\infty)\|_{W^{1,p}_{A_1}(X)} < \sigma', \tag{3.22}$$

then

$$|\mathscr{E}(u_\infty(A,\Phi)) - \mathscr{E}(u_\infty(A_\infty,\Phi_\infty))|^\theta \leq Z'\|\mathscr{E}'(u_\infty(A,\Phi))\|_{W^{-1,p'}_{A_1}(X)}. \tag{3.23}$$

By Lemma 3.8, there exists $C_1 = C_1(A_1,g,p,u_\infty) = C_1(A_1,A_\infty,\Phi_\infty,g,p)\in[1,\infty)$ so that

$$\|u_\infty(A,\Phi) - u_\infty(A_\infty,\Phi_\infty)\|_{W^{1,p}_{A_1}(X)} \leq C_1\|(A,\Phi)-(A_\infty,\Phi_\infty)\|_{W^{1,p}_{A_1}(X)}.$$

More explicitly, Lemma 3.8 gives $C_1 = C(1+\|u_\infty\|_{W^{2,p}_{A_1}(X)})$, where $C = C(g,p)\in[1,\infty)$. Therefore, setting $\sigma := C_1^{-1}\sigma'$, we see that if $(A,\Phi)$ obeys the Łojasiewicz-Simon neighborhood condition (1.15), namely

$$\|(A,\Phi)-(A_\infty,\Phi_\infty)\|_{W^{1,p}_{A_1}(X)} < \sigma,$$

then (3.22) holds and thus also (3.23). Moreover,

$$\begin{aligned}
&\|\mathscr{E}'(u_\infty(A,\Phi))\|_{W^{-1,p'}_{A_1}(X)} \\
&\quad = \sup\left\{|\mathscr{E}'(u_\infty(A,\Phi))(u_\infty(a,\phi))| : \|u_\infty(a,\phi)\|_{W^{1,p}_{A_1}(X)} \le 1\right\} \\
&\quad = \sup\left\{|\mathscr{E}'(A,\Phi)(a,\phi)| : \|u_\infty(a,\phi)\|_{W^{1,p}_{A_1}(X)} \le 1\right\} \quad \text{(by gauge invariance)} \\
&\quad \le \sup\left\{|\mathscr{E}'(A,\Phi)(a,\phi)| : C_1^{-1}\|(a,\phi)\|_{W^{1,p}_{A_1}(X)} \le 1\right\} \quad \text{(by Lemma 3.8)} \\
&\quad = C_1\|\mathscr{E}'(A,\Phi)\|_{W^{-1,p'}_{A_1}(X)}.
\end{aligned}$$

Substituting the preceding inequality into (3.23) yields

$$\begin{aligned}
|\mathscr{E}(A,\Phi) - \mathscr{E}(A_\infty,\Phi_\infty)|^\theta &= |\mathscr{E}(u_\infty(A,\Phi)) - \mathscr{E}(u_\infty(A_\infty,\Phi_\infty))|^\theta \quad \text{(by gauge invariance)} \\
&\le Z'\|\mathscr{E}'(u_\infty(A,\Phi))\|_{W^{-1,p'}_{A_1}(X)} \quad \text{(by (3.23))} \\
&\le Z'C_1\|\mathscr{E}'(A,\Phi)\|_{W^{-1,p'}_{A_1}(X)},
\end{aligned}$$

that is, the Łojasiewicz-Simon gradient inequality (1.16) holds for the pairs $(A,\Phi)$ and $(A_\infty,\Phi_\infty)$ with constants $(Z,\theta,\sigma)$, where $Z := C_1Z'$.

*Case* 2 ($(A,\Phi)$ not in Coulomb gauge relative to $(A_\infty,\Phi_\infty)$). Let $\zeta = \zeta(A_1,A_\infty,\Phi_\infty,g,G,p,q) \in (0,1]$ and $N = N(A_1,A_\infty,\Phi_\infty,g,G,p,q) \in [1,\infty)$ denote the constants in Theorem 3 and choose $\zeta_1 \in (0,\zeta]$ small enough that $2N\zeta_1 < \sigma_1$, where we now use $\sigma_1$ to denote the Łojasiewicz-Simon constant from Case 1. If $(A,\Phi)$ obeys

$$\|(A,\Phi) - (A_\infty,\Phi_\infty)\|_{W^{1,p}_{A_1}(X)} < \zeta_1,$$

then Theorem 3 provides $u \in \mathrm{Aut}^{2,q}(P)$, depending on the pair $(A,\Phi)$, such that

$$\begin{gathered}
d^*_{A_\infty,\Phi_\infty}(u(A,\Phi) - (A_\infty,\Phi_\infty)) = 0, \\
\|u(A,\Phi) - (A_\infty,\Phi_\infty)\|_{W^{1,p}_{A_1}(X)} < 2N\zeta_1 < \sigma.
\end{gathered}$$

By applying Case 1 to the pairs $u(A,\Phi)$ and $(A_\infty,\Phi_\infty)$, we obtain

$$|\mathscr{E}(u(A,\Phi)) - \mathscr{E}(A_\infty,\Phi_\infty)|^\theta \le C_1Z'\|\mathscr{E}'(u(A,\Phi))\|_{W^{-1,p'}_{A_1}(X)}.$$

Estimating as in Case 1, with $u$ replacing $u_\infty$, we see that

$$\|\mathscr{E}'(u(A,\Phi))\|_{W^{-1,p'}_{A_1}(X)} \le C_2\|\mathscr{E}'(A,\Phi)\|_{W^{-1,p'}_{A_1}(X)},$$

where $C_2 = C(1 + \|u\|_{W^{2,p}_{A_1}(X)})$ and $C = C(g,p) \in [1,\infty)$. According to Lemma 2.20, we have

$$\|u\|_{W^{2,p}_{A_1}(X)} \le C_3,$$

where $C_3 = C_3(A_\infty, \Phi_\infty, A_1, g, G, p, q) \in [1, \infty)$. By combining the preceding inequalities, we obtain

$$\begin{aligned}|\mathscr{E}(A,\Phi) - \mathscr{E}(A_\infty,\Phi_\infty)|^\theta &= |\mathscr{E}(u(A,\Phi)) - \mathscr{E}(A_\infty,\Phi_\infty)|^\theta \quad \text{(by gauge invariance)}\\ &\le C_1 Z' \|\mathscr{E}'(u(A,\Phi))\|_{W^{-1,p'}_{A_1}(X)}\\ &\le C_1 C(1+C_3) Z' \|\mathscr{E}'(A,\Phi)\|_{W^{-1,p'}_{A_1}(X)},\end{aligned}$$

and hence we obtain the Łojasiewicz-Simon gradient inequality (1.16) with constants $(Z, \theta, \sigma)$, where we now choose $Z = C_1 C(1 + C_3)Z'$ and $\sigma = \zeta_1$.

This completes the proof of Theorem 5. □

3.2. **Łojasiewicz-Simon gradient inequality for fermion coupled Yang-Mills energy functional.** We assume the notation and conventions of Section 1.3.1 and, given a $C^\infty$ connection $A_1$ on $P$ and $p \in (1, \infty)$, define a Banach space and its dual by

$$\begin{aligned}\mathfrak{Y} &:= W^{1,p}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P) \oplus W^{1,p}_{A_1}(X; W \otimes E),\\ \mathfrak{Y}^* &= W^{-1,p'}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P) \oplus W^{-1,p'}_{A_1}(X; W \otimes E),\end{aligned}$$

by analogy with Section 3.2 and where we fix a smooth reference pair $(A_1, \Psi_1)$. Recall that with respect to the reference connection $A_1$ on $P$, any other $W^{1,p}$ connection $A$ on $P$ may be expressed as $A = A_1 + a$, where $a \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)$ with $(a, \Psi) \in \mathfrak{Y}$. By analogy with Proposition 3.1 we establish the

**Proposition 3.9** (Analyticity of the fermion coupled Yang-Mills $L^2$-energy functional)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \ge 2$, and $(\rho, W)$ be a spin$^c$ structure on $X$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, and $A_1$ be a smooth reference connection on $P$, and $m \in C^\infty(X)$. If $4d/(d + 4) \le p < \infty$, then the functional* (1.7),

$$\mathscr{F} : \mathscr{A}^{1,p}(P) \times W^{1,p}_{A_1}(X; W \otimes E) \to \mathbb{R},$$

*is real analytic.*

*Proof.* We prove analyticity at a point $(A, \Psi)$ and write $A = A_1 + a_1$, where $a \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)$. For any $(a, \psi) \in \mathfrak{Y}$, we have

$$\begin{aligned}F_{A+a} &= F_{A_1} + d_{A_1}(a_1 + a) + \frac{1}{2}[a_1 + a, a_1 + a],\\ D_{A+a}(\Psi + \psi) &= D_{A_1}(\Psi + \psi) + \rho(a_1 + a)(\Psi + \psi).\end{aligned}$$

By definition (1.7) of the fermion coupled Yang-Mills $L^2$-energy functional, we obtain

$$2\mathscr{F}(A + a, \Psi + \psi) = T_1 + T_2 + T_3,$$

where the curvature term,

$$T_1 := (F_{A+a}, F_{A+a})_{L^2(X)},$$

has the same expansion as the corresponding term $T_1$ in Proposition 3.1 for the boson coupled Yang-Mills $L^2$-energy functional, while

$$\begin{aligned}T_2 := {} & (\Psi, D_{A_1}\Psi)_{L^2(X)} + (\Psi, D_{A_1}\psi)_{L^2(X)} + (\psi, D_{A_1}\Psi)_{L^2(X)} + (\psi, D_{A_1}\psi)_{L^2(X)} \\ & + (\Psi, \rho(a_1+a)\Psi)_{L^2(X)} + (\Psi, \rho(a_1+a)\psi)_{L^2(X)} + (\psi, \rho(a_1+a)\Psi)_{L^2(X)} \\ & + (\psi, \rho(a_1+a)\psi)_{L^2(X)},\end{aligned}$$

and thus,

$$\begin{aligned}T_2 = {} & (\Psi, D_{A_1}\Psi)_{L^2(X)} + 2\operatorname{Re}(\Psi, D_{A_1}\psi)_{L^2(X)} + (\psi, D_{A_1}\psi)_{L^2(X)} \\ & + (\Psi, \rho(a_1+a)\Psi)_{L^2(X)} + 2\operatorname{Re}(\Psi, \rho(a_1+a)\psi)_{L^2(X)} + (\psi, \rho(a_1+a)\psi)_{L^2(X)}, \\ T_3 := {} & \int_X m\left(|\Psi|^2 + \langle\Psi,\psi\rangle + \langle\psi,\Psi\rangle + |\psi|^2\right) d\operatorname{vol}_g.\end{aligned}$$

The terms in the expression for the difference,

$$2\mathscr{F}(A+a, \Psi+\psi) - 2\mathscr{F}(A,\Psi) = T_1' + T_2' + T_3',$$

are organized in such a way that

$$T_1' := (F_{A+a}, F_{A+a})_{L^2(X)} - (F_A, F_A)_{L^2(X)}$$

has the same expansion as the corresponding term $T_1'$ for the boson coupled Yang-Mills $L^2$-energy functional in the proof of Proposition 3.9 and the remaining terms are given by

$$\begin{aligned}T_2' := {} & 2\operatorname{Re}(\Psi, D_{A_1}\psi)_{L^2(X)} + (\psi, D_{A_1}\psi)_{L^2(X)} \\ & + 2\operatorname{Re}(\Psi, \rho(a_1+a)\psi)_{L^2(X)} + (\psi, \rho(a_1+a)\psi)_{L^2(X)}, \\ T_3' := {} & \int_X m\left(2\operatorname{Re}\langle\Psi,\psi\rangle + |\psi|^2\right) d\operatorname{vol}_g.\end{aligned}$$

The proof of analyticity of $\mathscr{F}$ at $(A,\Psi)$ now follows by adapting *mutatis mutandis* the arguments used to prove Proposition 3.1. ☐

We can now verify the formula (1.11) for the gradient $\mathscr{F}'(A,\Psi)$ by extracting the terms that are linear in $(a,\psi)$ from the expressions for

$$T_1' = 2(F_A, d_A a)_{L^2(X)} + (d_A a, d_A a)_{L^2(X)}$$

and $T_2'$ and $T_3'$ arising in the proof of Proposition 3.9 to give

$$\begin{aligned}&\left(\mathscr{F}'(A,\Psi), (a,\psi)\right)_{L^2(X)} \\ &\qquad = (d_A^* F_A, a)_{L^2(X)} + \operatorname{Re}(D_A\Psi - m\Psi, \psi)_{L^2(X)} + \frac{1}{2}(\Psi, \rho(a)\Psi)_{L^2(X)}, \quad \forall\,(a,\psi) \in \mathfrak{Y},\end{aligned}$$

as asserted in (1.11).

*Remark* 3.10 (Pointwise self-adjointness and reality). The fact that the term $\langle\Psi, \rho(a)\Psi\rangle$ appearing in (1.11) is real could be inferred indirectly by noting the origin of this term and the fact that the Dirac operator, $D_A$, is self-adjoint. To see directly that $\langle\Psi,\rho(a)\Psi\rangle$ is real, recall that Clifford multiplication is skew-Hermitian, so $c(\alpha)^* = -c(\alpha) \in \operatorname{End}_{\mathbb{C}}(W)$ for all $\alpha \in \Omega^1(X)$ (for example, see [27, p. 49]) while if $\xi \in \mathfrak{g}$, then $\varrho_*(\xi)^* = -\varrho_*(\xi)$ since we assume that Lie structure group, $G$, of $P$ acts on the complex, finite-dimensional vector space $\mathbb{E}$ via a unitary representation, $\varrho : G \to \operatorname{End}_{\mathbb{C}}(\mathbb{E})$, and $\varrho_* : \mathfrak{g} \to \operatorname{End}_{\mathbb{C}}(\mathbb{E})$ is the induced representation of the Lie algebra, $\mathfrak{g}$.

Hence, given $\alpha \otimes \xi \in C^\infty(T^*X \otimes \mathrm{ad}P) = \Omega^1(X;\mathrm{ad}P)$ and recalling that $E = P \times_\varrho \mathbb{E}$, then $\rho(\alpha \otimes \xi) = c(\alpha) \otimes \varrho_*(\xi) \in \mathrm{End}_{\mathbb{C}}(W \otimes E)$ obeys

$$\rho(\alpha \otimes \xi)^* = c(\alpha)^* \otimes \varrho_*(\xi)^* = c(\alpha) \otimes \varrho_*(\xi) = \rho(\alpha \otimes \xi).$$

In particular, $\rho(a) \in \mathrm{End}_{\mathbb{C}}(W \otimes E)$ satisfies $\rho(a)^* = \rho(a)$ for all $a \in \Omega^1(X;\mathrm{ad}P)$.

We now compute the Hessian $\mathscr{F}''$ at a pair $(A,\Psi) \in \mathscr{A}^{1,q}(P) \times W^{1,q}_{A_1}(X; W \otimes E)$. The gradient $\mathscr{F}'(A,\Psi) \in \mathfrak{Y}^*$ in (1.11) may be written as

$$\mathscr{F}'(A,\Psi) = d_A^* F_A + \frac{1}{2}\left((D_A - m)\Psi \cdot + \cdot (D_A - m)\Psi\right) + \frac{1}{2}\rho^{-1}(\Psi \otimes \Psi^*), \tag{3.24}$$

with the understanding that the terms comprising this expression for $\mathscr{F}'(A,\Psi)$ act on elements $(a,\psi) \in \mathfrak{Y}$ via the $L^2$-pairing by writing

$$\mathrm{Re}(D_A\Psi - m\Psi, \psi)_{L^2(X)} = \frac{1}{2}\left(((D_A - m)\Psi, \psi)_{L^2(X)} + (\psi, (D_A - m)\Psi)_{L^2(X)}\right),$$
$$\frac{1}{2}(\Psi, \rho(a)\Psi)_{L^2(X)} = \frac{1}{2}(\Psi \otimes \Psi^*, \rho(a))_{L^2(X)} = \frac{1}{2}(\rho^{-1}(\Psi \otimes \Psi^*), a)_{L^2(X)}.$$

By virtue of (3.24) we may view the Hessian of $\mathscr{F}$ at $(A,\Psi)$ as a bounded linear map $\mathscr{F}''(A,\Psi) : \mathfrak{Y} \to \mathfrak{Y}^*$. Taking the derivative of the gradient $\mathscr{F}'(A,\Psi)$ in (3.24) with respect to $(A,\Psi)$ in the direction $(a,\psi)$ yields

$$\begin{aligned}\mathscr{F}''(A,\Psi)(a,\psi) &= d_A^* d_A a + \frac{1}{2}\left(D_A\psi \cdot + \cdot D_A\psi\right) - \frac{1}{2}\left(m\psi \cdot + \cdot m\psi\right) \\ &\quad + (a \wedge \cdot)^* F_A + \frac{1}{2}\left(\rho(a)\Psi \cdot + \cdot \rho(a)\Psi\right) + \frac{1}{2}\rho^{-1}(\Psi \otimes \psi^* + \psi \otimes \Psi^*).\end{aligned} \tag{3.25}$$

The differential $d_{A,\Psi} : \Omega^0(X;\mathrm{ad}P) \to \Omega^1(X;\mathrm{ad}P) \otimes C^\infty(X; W \otimes E)$ is defined just as in (2.60) except that we replace $E$ by $W \otimes E$. We set

$$\mathscr{Y} := \mathfrak{Y} \cap \mathrm{Ker}\left\{d_{A,\Psi}^* : W^{1,p}_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P) \oplus W^{1,p}_{A_1}(X; W \otimes E) \to L^p(X;\mathrm{ad}P)\right\}.$$

Let $\iota_{\mathscr{Y}} : \mathscr{Y} \to \mathfrak{Y}$ denote the continuous embedding, with adjoint $\iota_{\mathscr{Y}}^* : \mathfrak{Y}^* \to \mathscr{Y}^*$. By analogy with Corollary 3.7 for the boson coupled Yang-Mills $L^2$-energy functional, we have

**Proposition 3.11** (Fredholm and index zero properties for the Hessian of the fermion coupled Yang-Mills $L^2$-energy functional)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $(\rho, W)$ be a spin$^c$ structure on $X$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times_\varrho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, and $A_1$ is a $C^\infty$ reference connection on $P$, and $m \in C^\infty(X)$. If $(A,\Psi)$ is a $W^{2,q}$ pair on $(P, W \otimes E)$ with $d/2 < q < \infty$ and $2 \leq p < \infty$ and $d/2 \leq p \leq q$, then $\iota_{\mathscr{Y}}^* \circ \mathscr{F}''(A,\Phi) \circ \iota_{\mathscr{Y}} : \mathscr{Y} \to \mathscr{Y}^*$ is Fredholm with index zero.*

*Proof.* We apply the argument used to prove Proposition 3.5 and Corollary 3.7 but we now compare $\mathscr{F}''(A,\Psi)$ in (3.25) with the bounded symmetric operator,

$$M_{A,\Psi} := (d_A^* d_A + d_{A,\Psi} d_{A,\Psi}^*) \oplus \frac{1}{2}(\cdot D_A \cdot + \cdot D_A \cdot) : \mathfrak{Y} \to \mathfrak{Y}^*,$$

where, for all $(a,\psi), (b,\vartheta) \in \mathscr{Y}$,

$$\frac{1}{2}(\cdot D_A \cdot + \cdot D_A \cdot)(a,\psi)(b,\vartheta) := \frac{1}{2}\left((D_A\psi, \vartheta)_{L^2(X)} + (\vartheta, D_A\psi)_{L^2(X)}\right).$$

The proof that the operator

$$\iota_{\mathscr{Y}}^* \circ \left(\mathscr{F}''(A,\Psi) - M_{A,\Psi}\right) \circ \iota_{\mathscr{Y}} : \mathscr{Y} \to \mathscr{Y}^*$$

is compact follows *mutatis mutandis* the proof in Proposition 3.5 and Corollary 3.7 of the corresponding fact for the boson coupled Yang-Mills $L^2$-energy functional. □

*Proof of Theorem 7.* The argument applies *mutatis mutandis* the corresponding steps used to prove Theorem 5, replacing the Banach space $\mathscr{X}$ by $\mathscr{Y}$ and now choosing $\mathscr{H} := L^2(X;\Lambda^1 \otimes \mathrm{ad}P) \oplus L^2(X; W \otimes E)$. □

## Appendix A. Equivalence of Sobolev norms defined by Sobolev and smooth connections

Suppose that $(X,g)$ is a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ is a compact Lie group, and $P$ is a smooth principal $G$-bundle over $X$. In standard references for gauge theory [10, 22], it is generally assumed in the construction of Sobolev completions of spaces such as $\Omega^l(X;\mathrm{ad}P)$ that one defines Sobolev norms using a covariant derivative $\nabla_A$ determined by a connection $A$ on $P$ that is smooth or of class $W^{k,p}$ for $p \geq 1$ and an integer $k \geq 1$ large enough that $kp > d$ or even $kp \gg d$. However, in this article, we often consider connections $A$ with more borderline regularity, for example of class $W^{1,q}$ for $q > d/2$, and in that situation, one must exercise care in the definition of Sobolev spaces using such connections. Lemmas A.1 and A.2 provide some guidance.

**Lemma A.1** (Second-order Kato inequality and second-order Sobolev norms)**.** *Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a Lie group, and $P$ be a smooth principal $G$-bundle over $X$, and $V = P \times_\varrho \mathbb{V}$ be a smooth Riemannian vector bundle over $X$ defined by a finite-dimensional, orthogonal representation, $\varrho : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{V})$. Then there exists a constant $C = C(g,q) \in [1,\infty)$ with the following significance. If $A$ is a $W^{1,q}$ connection on $P$ with $q > d/2$, then for all $v \in C^\infty(X;V)$,*

$$\|v\|_{C(X)} \leq C\|v\|_{W_A^{2,q}(X)}. \tag{A.1}$$

*Proof.* The first-order analogue of (A.1), namely,

$$\|v\|_{C(X)} \leq \kappa_1\|v\|_{W_A^{1,q}(X)},$$

when $q > d$ and $\kappa_1 = \kappa_1(g) \in [1,\infty)$ is the norm of the Sobolev embedding $W^{1,q}(X) \subset C(X)$, is an immediate consequence of the pointwise first-order Kato inequality, $|\nabla|v|| \leq |\nabla_A v|$ from [22, Inequality (6.20)], in turn a consequence of the compatibility of the fiber metric on $V$ with $\nabla_A$.

We first note that, for $f \in C^\infty(X;\mathbb{R})$, the norm

$$\|f\|_{W^{2,q}(X)} = \|\nabla^2 f\|_{L^q(X)} + \|\nabla f\|_{L^q(X)} + \|f\|_{L^q(X)}$$

is equivalent (with respect to a constant depending at most on $(g,q)$) by virtue of [24, Theorem 9.11] to

$$\|f\|_{W^{2,q}(X)} = \|\Delta f\|_{L^q(X)} + \|f\|_{L^q(X)},$$

where $\Delta$ is the Laplace operator defined by the Riemannian metric $g$ on $X$. Now recall the pointwise identity [22, Equation (6.18)],

$$\Delta|v|^2 = 2\langle\nabla_A^*\nabla_A v, v\rangle - 2|\nabla_A v|^2.$$

Hence, letting $\kappa_2 = \kappa_2(g) \in [1, \infty)$ denote the norm of the Sobolev embedding $W^{2,q}(X) \subset C(X)$,

$$\begin{aligned}\|v\|_{C(X)}^2 &= \||v|^2\|_{C(X)} \\ &\leq \kappa_2 \||v|^2\|_{W^{2,q}(X)} \\ &= \kappa_2 \left(\|\Delta|v|^2\|_{L^q(X)} + \||v|^2\|_{L^q(X)}\right) \\ &\leq \kappa_2 \left(2\|v\|_{C(X)}\|\nabla_A^*\nabla_A v\|_{L^q(X)} + 2\|\nabla_A v\|_{L^{2q}(X)}^2 + \|v\|_{C(X)}\|v\|_{L^q(X)}\right).\end{aligned}$$

Recall that $W^{1,q}(X) \subset L^{2q}(X)$, for $q < d$, if and only if $2q \leq q^* = dq/(d-q)$, that is, $2(d-q) \leq d$ or $q \geq d/2$; the embedding is immediate from [3, Theorem 4.12] when $q \geq d$. Thus, applying the first-order Kato Inequality and the preceding Sobolev embedding for functions,

$$\||v|\|_{L^{2q}(X)} \leq \kappa_1 \left(\|\nabla|v|\|_{L^q(X)} + \|v\|_{L^q(X)}\right) \leq \kappa_1 \left(\|\nabla_A v\|_{L^q(X)} + \|v\|_{L^q(X)}\right),$$

we obtain

$$\begin{aligned}\|v\|_{C(X)}^2 \leq \kappa_2 \Big(&2\|v\|_{C(X)}\|\nabla_A^*\nabla_A v\|_{L^q(X)} + 2\kappa_1^2 \left(\|\nabla_A^2 v\|_{L^q(X)} + \|\nabla_A v\|_{L^q(X)}\right)^2 \\ &+ \|v\|_{C(X)}\|v\|_{L^q(X)}\Big).\end{aligned}$$

We now use Young's Inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$ from [24, Inequality (7.8)] and rearrangement with a suitably small and universal $\varepsilon$ to give

$$\|v\|_{C(X)}^2 \leq C^2 \left(\|\nabla_A^*\nabla_A v\|_{L^q(X)}^2 + \|\nabla_A^2 v\|_{L^q(X)}^2 + \|\nabla_A v\|_{L^q(X)}^2 + \|v\|_{L^q(X)}^2\right),$$

where $C = C(g, q) \in [1, \infty)$. We simplify the right-hand side in the preceding inequality via

$$\|\nabla_A^*\nabla_A v\|_{L^q(X)} \leq z \left(\|\nabla_A^2 v\|_{L^q(X)} + \|\nabla_A v\|_{L^q(X)} + \|v\|_{L^q(X)}\right),$$

where $z$ is a constant depending at most on the Riemannian metric on $X$. The desired Sobolev inequality (A.1) now follows by taking square roots. □

**Lemma A.2** (Equivalence of Sobolev norms defined by Sobolev and smooth connections)**.** *Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$, , and $V = P \times_\varrho \mathbb{V}$ be a smooth Riemannian vector bundle over $X$ defined by a finite-dimensional, orthogonal representation, $\varrho : G \to \mathrm{Aut}_\mathbb{R}(\mathbb{V})$, and $q > d/2$ and $p$ obey $d/2 \leq p \leq q$. Let $A_1$ be a $C^\infty$ connection on $P$, and $A_0$ be a Sobolev connection on $P$, and $a_0 := A_0 - A_1$.*

*(1) There exists $C = C(g, p) \in [1, \infty)$ such that, if $a_0 \in W_{A_1}^{1,p}(X; \Lambda^1 \otimes \mathrm{ad}P)$, then*

$$\|\xi\|_{L^r(X)} \leq C\|\xi\|_{W_{A_0}^{1,p}(X)}, \quad \text{for } \begin{cases} 1 \leq r \leq dp/(d-p) & \text{if } p < d, \\ 1 \leq r < \infty & \text{if } p = d, \\ r = \infty & \text{if } p > d, \end{cases}$$

*for all $\xi \in C^\infty(X; V)$; moreover, there exists $C = C(A_1, g, p) \in [1, \infty)$ such that*

$$\|\xi\|_{L^r(X)} \leq C\|\xi\|_{W_{A_1}^{2,p}(X)}, \quad \text{for } \begin{cases} 1 \leq r \leq dp/(d-2p) & \text{if } p < d/2, \\ 1 \leq r < \infty & \text{if } p = d/2, \\ r = \infty & \text{if } p > d/2, \end{cases}$$

*for all $\xi \in C^\infty(X; V)$.*

*(2) If* $a_0 \in W^{1,q}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P)$*, then there exists* $C = C(g,q) \in [1,\infty)$ *such that, for all* $\xi \in C^\infty(X;V)$*,*

$$\|\xi\|_{C(X)} \leq C\|\xi\|_{W^{2,q}_{A_0}(X)}.$$

*(3) If* $a_0 \in W^{1,p}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P)$*, then there exists* $C = C(g,p,\|a_0\|_{W^{1,p}_{A_1}(X)}) \in [1,\infty)$ *so that*

$$C^{-1}\|\xi\|_{W^{1,p}_{A_1}(X)} \leq \|\xi\|_{W^{1,p}_{A_0}(X)} \leq C\|\xi\|_{W^{1,p}_{A_1}(X)}, \quad \forall\, \xi \in C^\infty(X;V).$$

*(4) If* $a_0 \in W^{1,q}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P)$*, then there exists* $C = C(g,p,q,\|a_0\|_{W^{1,q}_{A_1}(X)}) \in [1,\infty)$ *so that*

$$C^{-1}\|\xi\|_{W^{2,q}_{A_1}(X)} \leq \|\xi\|_{W^{2,q}_{A_0}(X)} \leq C\|\xi\|_{W^{2,q}_{A_1}(X)}, \quad \forall\, \xi \in C^\infty(X;V).$$

*(5) If* $a_0 \in W^{1,q}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P)$*, then there exists* $C = C(g,p,q,\|a_0\|_{W^{1,q}_{A_1}(X)}) \in [1,\infty)$ *so that*

$$\|\xi\|_{W^{2,p}_{A_0}(X)} \leq C\|\xi\|_{W^{2,p}_{A_1}(X)}, \quad \forall\, \xi \in C^\infty(X;V).$$

*(6) If* $a_0 \in W^{2,q}_{A_1}(X;\Lambda^1\otimes \mathrm{ad}P)$*, then there exists* $C = C(g,p,q,\|a_0\|_{W^{2,q}_{A_1}(X)}) \in [1,\infty)$ *so that*

$$C^{-1}\|\xi\|_{W^{2,p}_{A_1}(X)} \leq \|\xi\|_{W^{2,p}_{A_0}(X)} \leq C\|\xi\|_{W^{2,p}_{A_1}(X)}, \quad \forall\, \xi \in C^\infty(X;V).$$

*Proof.* Item (1) is a well-known consequence of the Sobolev Embedding [3, Theorem 4.12] for scalar functions and the Kato Inequality [22, Equation (6.20)] in the case of the embedding $W^{1,p}(X) \subset L^r(X)$. Item (2)restates the conclusion of Lemma A.1.

For Item (3), we use $\nabla_{A_0}\xi = \nabla_{A_1}\xi + [a_0,\xi]$ and estimate

$$\begin{aligned}\|\nabla_{A_1}\xi\|_{L^p(X)} &\leq \|\nabla_{A_0}\xi\|_{L^p(X)} + \|[a_0,\xi]\|_{L^p(X)}\\ &\leq \|\xi\|_{W^{1,p}_{A_0}(X)} + z\|a_0\|_{L^{2p}(X)}\|\xi\|_{L^{2p}(X)}\\ &\leq C(1+\|a_0\|_{W^{1,p}_{A_1}(X)})\|\xi\|_{W^{1,p}_{A_0}(X)},\end{aligned}$$

where we used the continuous Sobolev embedding $W^{1,p}(X) \subset L^{2p}(X)$ for $p \geq d/2$ and Item (1) to obtain the last inequality. Here, $z = z(g) \in [1,\infty)]$ and $C \in [1,\infty)$ has the stated dependencies. The analogous estimate with the roles of $A_0$ and $A_1$ reversed follows by a symmetric argument.

For Item (4), we first write

$$\nabla^2_{A_1}\xi = \nabla^2_{A_0}\xi + \nabla_{A_0}a_0 \times \xi + a_0 \times \nabla_{A_0}\xi + a_0 \times a_0 \times \xi. \tag{A.2}$$

Taking $L^q$ norms of both sides of (A.2), we see that

$$\begin{aligned}\|\nabla^2_{A_1}\xi\|_{L^q(X)} \leq \|\nabla^2_{A_0}\xi\|_{L^q(X)} &+ \|\nabla_{A_0}a_0\times\xi\|_{L^q(X)} + \|a_0\times\nabla_{A_0}\xi\|_{L^q(X)}\\ &+ \|a_0\times a_0\times\xi\|_{L^q(X)},\end{aligned}$$

and thus, for $z = z(g) \in [1,\infty)$,

$$\begin{aligned}\|\nabla^2_{A_1}\xi\|_{L^q(X)} \leq \|\nabla^2_{A_0}\xi\|_{L^q(X)} + z\|\nabla_{A_0}a_0\|_{L^q(X)}\|\xi\|_{C(X)} + z\|a_0\|_{L^{2q}(X)}\|\nabla_{A_0}\xi\|_{L^{2q}(X)}\\ + z\|a_0\|^2_{L^{2q}}\|\xi\|_{C(X)}.\end{aligned} \tag{A.3}$$

By Item (3), we have

$$\|\nabla_{A_0}a_0\|_{L^q(X)} \leq \|a_0\|_{W^{1,q}_{A_0}(X)} \leq C\|a_0\|_{W^{1,q}_{A_1}(X)},$$

and by Item (1) and the fact that $W^{1,q}(X) \subset L^{2q}(X)$ for $q > d/2$, we obtain

$$\|\nabla_{A_0}\xi\|_{L^{2q}(X)} \leq C\|\xi\|_{W^{2,q}_{A_0}(X)}.$$

Similarly, Item (2)gives

$$\|\xi\|_{C(X)} \leq C\|\xi\|_{W^{2,q}_{A_0}(X)}.$$

By substituting the preceding inequalities into (A.3), we find that

$$\|\xi\|_{W^{2,q}_{A_1}(X)} \leq C\|\xi\|_{W^{2,q}_{A_0}(X)},$$

where $C \in [1,\infty)$ has the stated dependencies. The analogous inequality with the roles of $A_0$ and $A_1$ reversed follows by a symmetric argument.

For Item (5), define $r \in [p,\infty]$ by $1/p = 1/q + 1/r$, recall that $p = d/2 < q$ or $d/2 < p \leq q$, interchange the roles of $A_0$ and $A_1$ in (A.2), and take $L^p$ norms to give

$$\begin{aligned}
\|\nabla^2_{A_0}\xi\|_{L^p(X)} &\leq \|\nabla^2_{A_1}\xi\|_{L^p(X)} + \|\nabla_{A_1}a_0 \times \xi\|_{L^p(X)} + \|a_0 \times \nabla_{A_1}\xi\|_{L^p(X)} \\
&\quad + \|a_0 \times a_0 \times \xi\|_{L^p(X)} \\
&\leq \|\nabla^2_{A_1}\xi\|_{L^p(X)} + z\|\nabla_{A_1}a_0\|_{L^q(X)}\|\xi\|_{L^r(X)} + z\|a_0\|_{L^{2p}(X)}\|\nabla_{A_1}\xi\|_{L^{2p}(X)} \\
&\quad + z\|a_0\|^2_{L^{2q}}\|\xi\|_{L^r(X)} \\
&\leq \|\nabla^2_{A_1}\xi\|_{L^p(X)} + C\|a_0\|_{W^{1,q}_{A_1}(X)}\|\xi\|_{W^{2,p}_{A_1}(X)} + C\|a_0\|_{W^{1,p}_{A_1}(X)}\|\nabla_{A_1}\xi\|_{W^{1,p}_{A_1}(X)} \\
&\quad + C\|a_0\|^2_{W^{1,q}_{A_1}(X)}\|\xi\|_{W^{2,p}_{A_1}(X)},
\end{aligned}$$

where $z = z(g) \in [1,\infty)$ and, to obtain the last inequality, we use the continuous Sobolev embeddings $W^{1,p}(X) \subset L^{2p}(X)$ and $W^{1,q}(X) \subset L^{2q}(X)$ for $d/2 \leq p \leq q$ and Item (1) together with the continuous Sobolev embedding $W^{2,p}(X) \subset L^r(X)$, for $r \in [1,\infty)$ if $p = d/2$ and $r = \infty$ if $p > d/2$. Therefore, we obtain

$$\|\xi\|_{W^{2,p}_{A_0}(X)} \leq C\|\xi\|_{W^{2,p}_{A_1}(X)},$$

where $C \in [1,\infty)$ has the stated dependencies.

For Item (6), we take $L^p$ norms of (A.2) and use $\nabla_{A_0}a_0 = \nabla_{A_1}a_0 + [a_0, a_0]$ to give

$$\begin{aligned}
\|\nabla^2_{A_1}\xi\|_{L^p(X)} &\leq \|\nabla^2_{A_0}\xi\|_{L^p(X)} + \|\nabla_{A_0}a_0 \times \xi\|_{L^p(X)} + \|a_0 \times \nabla_{A_0}\xi\|_{L^p(X)} \\
&\quad + \|a_0 \times a_0 \times \xi\|_{L^p(X)} \\
&\leq \|\nabla^2_{A_0}\xi\|_{L^p(X)} + z\|\nabla_{A_1}a_0\|_{L^{2p}(X)}\|\xi\|_{L^{2p}(X)} + \|a_0\|_{C(X)}\|\nabla_{A_0}\xi\|_{L^p(X)} \\
&\quad + 2z\|a_0\|^2_{C(X)}\|\xi\|_{L^p(X)}.
\end{aligned}$$

Applying the continuous Sobolev embeddings $W^{1,p}(X) \subset L^{2p}(X)$ and $W^{2,q}(X) \subset C(X)$ and Items (1) and (3), we discover that

$$\begin{aligned}
\|\nabla^2_{A_1}\xi\|_{L^p(X)} &\leq \|\nabla^2_{A_0}\xi\|_{L^p(X)} + C\|\nabla_{A_1}a_0\|_{W^{1,p}_{A_1}(X)}\|\xi\|_{W^{1,p}_{A_0}(X)} + \|a_0\|_{W^{2,q}_{A_1}(X)}\|\nabla_{A_0}\xi\|_{L^p(X)} \\
&\quad + 2z\|a_0\|^2_{W^{2,q}_{A_1}(X)}\|\xi\|_{L^p(X)} \\
&\leq C\|\xi\|_{W^{2,p}_{A_0}(X)},
\end{aligned}$$

where $C \in [1,\infty)$ has the stated dependencies. The analogous inequality with the roles of $A_0$ and $A_1$ reversed follows by a symmetric argument. □

## Appendix B. Fredholm properties of a Hodge Laplacian with Sobolev coefficients

In this section we include proofs of results regarding the Fredholm properties of the Hodge Laplace operators encountered in Sections 2 and 3 that would be standard if the operator had smooth coefficients and acted on $L^2$ rather than $L^p$ Sobolev spaces as we allow here.

**Proposition B.1** (Fredholm property and index of a Laplace operator)**.** *Assume the hypotheses of Proposition 2.3 and let $p$ be such that $d/2 \le p \le q$ and $2 \le p < \infty$. Then the operator*

$$\Delta_A : W^{1,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P) \to W^{-1,p'}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$$

*is Fredholm with index zero and closed range $(K^\perp \cap W^{1,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P))^*$, where $\perp$ denotes $L^2$-orthogonal complement and $K \subset W^{1,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$ is the kernel.*

*Proof.* Denote $\mathscr{X} = W^{1,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$ for brevity and $\mathfrak{H} = W^{1,2}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$ as before and observe that $\mathscr{X} \subset \mathfrak{H}$ and $\mathfrak{H}^* \subset \mathscr{X}^*$ are continuous Sobolev embeddings by our hypothesis that $p \ge 2$; the hypothesis that $p < \infty$ yields the identification of $\mathscr{X}^*$.

According to Proposition 2.3, the operator $\Delta_A : \mathfrak{H} \to \mathfrak{H}^*$ is bounded and because $\Delta_A : \mathscr{X} \to \mathscr{X}^*$ can be viewed as the composition of the bounded operators $\mathscr{X} \subset \mathfrak{H}$ and $\Delta_A : \mathfrak{H} \to \mathfrak{H}^*$ and $\mathfrak{H}^* \to \mathscr{X}^*$, it follows that $\Delta_A : \mathscr{X} \to \mathscr{X}^*$ is bounded. Thus,

$$\mathrm{Ker}\,(\Delta_A : \mathscr{X} \to \mathscr{X}^*) \subset \mathrm{Ker}\,(\Delta_A : \mathfrak{H} \to \mathfrak{H}^*),$$

and so Proposition 2.3 implies that $\mathrm{Ker}\,(\Delta_A : \mathscr{X} \to \mathscr{X}^*)$ is finite-dimensional. We now show that $\mathrm{Ran}(\Delta_A : \mathscr{X} \to \mathscr{X}^*)$ is closed with finite codimension. Note from the proof of Proposition 2.3 that

$$\langle v, \Delta_A u\rangle_{\mathscr{X}\times\mathscr{X}^*} = \langle u, \Delta_A v\rangle_{\mathscr{X}\times\mathscr{X}^*}, \quad \forall\, u, v \in \mathscr{X}. \tag{B.1}$$

Denote $\mathscr{H} := L^2(X; \Lambda^l \otimes \mathrm{ad}P)$ for brevity and observe that $\mathscr{X} \subset \mathscr{H}$ is a continuous Sobolev embedding provided $p^* = dp/(d-p) \ge 2$, that is $dp \ge 2d - 2p$ or $p \ge 2d/(d+2)$, which is implied by our hypothesis that $p \ge d/2$ and $d \ge 2$.

Denote $K := \mathrm{Ker}(\Delta_A : \mathscr{X} \to \mathscr{X}^*) \subset \mathscr{X} \subset \mathscr{H}$, so $K$ is finite-dimensional and closed, and let $\Pi \circ \varepsilon : \mathscr{X} \to \mathscr{X}$ denote the composition of $\mathscr{H}$-orthogonal projection $\Pi : \mathscr{H} \to K$ and the continuous embedding $\varepsilon : \mathscr{X} \subset \mathscr{H}$. Define $(\Pi\circ\varepsilon)^\perp := \mathrm{id}_{\mathscr{X}} - \Pi\circ\varepsilon$ and $\mathscr{X}_0 := (\Pi\circ\varepsilon)^\perp\mathscr{X}$ and note that $\mathscr{X}_0 \subset \mathscr{X}$ is a complement of $K$ and a closed subspace by [49, Definition 4.20 and Lemma 4.21], so $\mathscr{X} = \mathscr{X}_0 \oplus K$ (an $L^2$-orthogonal direct sum).

We defined $\widetilde{\Pi} \equiv (\Pi\circ\varepsilon)^* : \mathscr{X}^* \to \mathscr{X}^*$ in [19, Lemma 2.4]. Plainly, $\Pi\circ\varepsilon : \mathscr{X} \to \mathscr{X}$ is a finite-rank operator and if $\{u_1, \dots, u_k\} \subset \mathscr{X}$ is a basis for $K$, then this operator has a unique representation $\Pi\circ\varepsilon = \sum_{i=1}^k \alpha_i \otimes u_i$ for $\{\alpha_1, \dots, \alpha_k\} \subset \mathscr{X}^*$ by [2, Lemma 4.2]. Moreover, $(\Pi\circ\varepsilon)^* = \sum_{i=1}^k u_i \otimes \alpha_i$ and has the same rank as $\Pi\circ\varepsilon$ by [2, Lemma 4.3]. By hypothesis, we have $p \in (1, \infty)$ and hence the Banach space $\mathscr{X}$ is reflexive and the canonical map, $\mathscr{X} \to \mathscr{X}^{**}$ defined by $u(\alpha) := \alpha(u)$ for all $u \in \mathscr{X}$ and $\alpha \in \mathscr{X}^*$, is an isometric isomorphism. With respect to the canonical isomorphism $\mathscr{X}^{**} \cong \mathscr{X}$ and again applying [2, Lemma 4.3], the operator $(\Pi\circ\varepsilon)^{**} : \mathscr{X}^{**} \to \mathscr{X}^{**}$ is equivalent to

$$\widetilde{\Pi}^* = \sum_{i=1}^k \alpha_i \otimes u_i = \Pi\circ\varepsilon : \mathscr{X} \to \mathscr{X}.$$

Recall that if $M \subset \mathscr{X}$ is a subspace, then $M^\circ := \{\alpha \in \mathscr{X}^* : \langle u, \alpha\rangle_{\mathscr{X}\times\mathscr{X}^*} = 0,\ \forall\, u \in M\}$ denotes the annihilator of $M$ in $\mathscr{X}^*$ [49, Section 4.6]. Because $\mathscr{X} = \mathscr{X}_0 \oplus K$, we have $\mathscr{X}^* = \mathscr{X}_0^\circ \oplus K^\circ$, the direct sum of the annihilators in $\mathscr{X}^*$ of the subspaces $\mathscr{X}_0$ and $K$ of $\mathscr{X}$. Taking duals once more, $\mathscr{X}^{**} \cong \mathscr{X}_0 \oplus K$ via the canonical isomorphism.

Because $K \subset \mathscr{X}$ and $\mathscr{X}_0 \subset \mathscr{X}$ are closed subspaces, we have $K^\circ \cong (\mathscr{X}/K)^*$ and $\mathscr{X}_0^\circ \cong K^*$ by [49, Theorem 4.9]. Since $\mathscr{X} = \mathscr{X}_0 \oplus K$, then $\mathscr{X}/K \cong \mathscr{X}_0$ and $\mathscr{X}/\mathscr{X}_0 \cong K$, so $K^\circ \cong \mathscr{X}_0^*$ and $\mathscr{X}_0^\circ \cong K^* \cong K$, where the final isomorphism follows from the fact that $K$ is finite-dimensional. In particular, $\mathscr{X}^* \cong K^* \oplus \mathscr{X}_0^* \cong K \oplus \mathscr{X}_0^*$. Since $K^*$ is finite-dimensional, then $K^* \subset \mathscr{X}^*$ is a closed subspace by [7, Proposition 11.1] and so the complement $\mathscr{X}_0^* \subset \mathscr{X}^*$ is a closed subspace by [49, Definition 4.20 and Lemma 4.21]. By repeating this argument, we obtain $\mathscr{X}^{**} \cong K^{**} \oplus \mathscr{X}_0^{**}$. There are canonical isomorphisms $\mathscr{X}^{**} \cong \mathscr{X}$ and $K^{**} \cong K$ and thus also $\mathscr{X}_0^{**} \cong \mathscr{X}_0$, that is, $\mathscr{X}_0$ is reflexive.

Denote $\widetilde{\Pi}^\perp := \mathrm{id}_{\mathscr{X}^*} - \widetilde{\Pi}$; we have already seen that $K^* = \widetilde{\Pi}\mathscr{X}^*$ and we now observe that $\mathscr{X}_0^* = \widetilde{\Pi}^\perp \mathscr{X}^*$. Let us consider the composition of the bounded operators $\Delta_A : \mathscr{X} \to \mathscr{X}^*$ and $\widetilde{\Pi}^\perp : \mathscr{X}^* \to \mathscr{X}_0^*$, namely

$$\widetilde{\Pi}^\perp \circ \Delta_A : \mathscr{X} \to \mathscr{X}_0^*.$$

According to [49, Theorem 4.12], we have

$$\operatorname{Ran}\left(\widetilde{\Pi}^\perp \circ \Delta_A : \mathscr{X} \to \mathscr{X}_0^*\right)^\diamond = \operatorname{Ker}\left(\left(\widetilde{\Pi}^\perp \circ \Delta_A\right)^* : \mathscr{X}_0^{**} \to \mathscr{X}^*\right),$$

where here we use the symbol $\diamond$ to denote the annihilator in $\mathscr{X}_0^*$. But

$$\begin{aligned}\operatorname{Ker}\left(\left(\widetilde{\Pi}^\perp \circ \Delta_A\right)^* : \mathscr{X}_0^{**} \to \mathscr{X}^*\right) &= \operatorname{Ker}\left(\Delta_A^* \circ (\widetilde{\Pi}^\perp)^* : \mathscr{X}_0^{**} \to \mathscr{X}^*\right)\\ &\cong \operatorname{Ker}\left(\Delta_A \circ (\Pi \circ \varepsilon)^\perp : \mathscr{X}_0 \to \mathscr{X}^*\right),\end{aligned}$$

where in the final isomorphism we make use of the facts that $\widetilde{\Pi}^\perp = \mathrm{id}_{\mathscr{X}^*} - \widetilde{\Pi}$ and $\widetilde{\Pi}^* = (\Pi \circ \varepsilon)^{**} = \Pi \circ \varepsilon$ (with respect to the canonical isomorphism $\mathscr{X}^{**} \cong \mathscr{X}$), and $(\Pi \circ \varepsilon)^\perp = \mathrm{id}_{\mathscr{X}} - \Pi \circ \varepsilon$, the fact the operator $\Delta_A^* : \mathscr{X}^{**} \to \mathscr{X}^*$ is equivalent to $\Delta_A : \mathscr{X} \to \mathscr{X}^*$ (with respect to the canonical isomorphism) via (B.1), and $\mathscr{X}_0^{**} \cong \mathscr{X}_0$. But $\operatorname{Ker}(\Delta_A \circ (\Pi \circ \varepsilon)^\perp : \mathscr{X}_0 \to \mathscr{X}^*) = 0$ by definition of $(\Pi \circ \varepsilon)^\perp$ and thus we obtain

$$\operatorname{Ran}\left(\widetilde{\Pi}^\perp \circ \Delta_A : \mathscr{X} \to \mathscr{X}_0^*\right)^\diamond = 0.$$

Therefore, $\operatorname{Ran}(\widetilde{\Pi}^\perp \circ \Delta_A : \mathscr{X} \to \mathscr{X}_0^*) = \mathscr{X}_0^*$ by definition of the annihilator in $\mathscr{X}_0^*$. Because $\mathscr{X}_0^*$ has finite codimension in $\mathscr{X}^*$ equal to $\dim K^* = \dim K$, then $\operatorname{Ran}(\Delta_A : \mathscr{X} \to \mathscr{X}^*)$ has finite codimension in $\mathscr{X}^*$ too and thus is also closed by [2, Lemma 4.38]. We conclude that $\Delta_A : \mathscr{X} \to \mathscr{X}^*$ is Fredholm.

We can identify the range of $\Delta_A : \mathscr{X} \to \mathscr{X}^*$ explicitly using

$$\begin{aligned}\operatorname{Ran}(\Delta_A : \mathscr{X} \to \mathscr{X}^*) &= \overline{\operatorname{Ran}}(\Delta_A : \mathscr{X} \to \mathscr{X}^*) \quad \text{(by closed range)}\\ &= (\operatorname{Ker}(\Delta_A^* : \mathscr{X}^{**} \to \mathscr{X}^*))^\circ \quad \text{(by [7, Corollary 2.18 (iv))]),}\\ &= (\operatorname{Ker}(\Delta_A : \mathscr{X} \to \mathscr{X}^*))^\circ \quad \text{(by (B.1) and } \mathscr{X}^{**} \cong \mathscr{X}).\end{aligned}$$

But $\operatorname{Ker}(\Delta_A : \mathscr{X} \to \mathscr{X}^*) = K$ and we have seen already that $K^\circ \cong \mathscr{X}_0^*$, so

$$\operatorname{Ran}(\Delta_A : \mathscr{X} \to \mathscr{X}^*) \cong \mathscr{X}_0^*.$$

Hence, $\operatorname{codim} \operatorname{Ran}(\Delta_A : \mathscr{X} \to \mathscr{X}^*) = \dim \operatorname{Ker}(\Delta_A : \mathscr{X} \to \mathscr{X}^*)$ and thus $\operatorname{Index}(\Delta_A : \mathscr{X} \to \mathscr{X}^*) = 0$. □

We also have the slightly simpler

**Proposition B.2** (Fredholm property and index of a Laplace operator)**.** *Assume the hypotheses of Proposition 2.3 but assume in addition $A$ is a $C^\infty$ connection. If $p$ obeys $d/2 \le p < \infty$, then the operator*

$$\Delta_A : W^{2,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P) \to L^p(X; \Lambda^l \otimes \mathrm{ad}P) \tag{B.2}$$

*is Fredholm with index zero and closed range $K^\perp \cap L^p(X;\Lambda^l \otimes \mathrm{ad}P)$, where $\perp$ denotes $L^2$-orthogonal complement and $K \subset W^{2,p}_{A_1}(X;\Lambda^l \otimes \mathrm{ad}P)$ is the kernel.*

*Proof.* The argument is broadly similar to the proof of Proposition B.1, but we shall appeal to elliptic regularity and rely on the proof of Proposition B.1 for functional analysis details omitted here. Denote $\mathscr{X} := W^{2,p}_{A_1}(X;\Lambda^l \otimes \mathrm{ad}P)$, and $\mathfrak{H} := W^{1,2}_{A_1}(X;\Lambda^l \otimes \mathrm{ad}P)$, and $\mathscr{Y} := L^p(X;\Lambda^l \otimes \mathrm{ad}P)$, and $\mathscr{H} := L^2(X;\Lambda^l \otimes \mathrm{ad}P)$ for brevity. Observe that $\mathscr{X} \subset \mathfrak{H}$ and thus $\mathscr{X} \subset \mathscr{H}$ are continuous Sobolev embeddings by our hypotheses that $p \ge d/2$ and $d \ge 2$. This is clear for $p \ge d$ while for $d/2 \le p < d$, we have $W^{2,p}(X) \subset W^{1,2}(X)$ by [49, Theorem 4.12] if $p^* = dp/(d-p) \ge 2$, that is, $dp \ge 2(d-p)$ or $p \ge 2d/(d+2)$; but $d/2 \ge 2d/(d+2)$ for all $d \ge 2$ and so the condition on $p$ is assured by the hypothesis that $p \ge d/2$.

Since $A$ is $C^\infty$ by hypothesis, the operator $\Delta_A : \mathscr{X} \to \mathscr{Y}$ is clearly bounded. As usual, we denote $\mathfrak{H} := W^{1,2}_{A_1}(X;\Lambda^l \otimes \mathrm{ad}P)$. We have the inclusion[5]

$$K := \mathrm{Ker}\,(\Delta_A : \mathscr{X} \to \mathscr{Y}) \subset \mathrm{Ker}\,(\Delta_A : \mathfrak{H} \to \mathfrak{H}^*)\,,$$

and so Proposition 2.3 implies that $K$ is finite-dimensional. Let $\Pi : \mathscr{H} \to K$ denote $\mathscr{H}$-orthogonal projection and write $\Pi^\perp = \mathrm{id}_{\mathscr{H}} - \Pi$. Abusing notation, we continue to write $\Pi$ and $\Pi^\perp$ for the restrictions of these operators to $\mathscr{X}$ or, when $p \ge 2$ to $\mathscr{Y}$ and, as in the proof of Proposition B.1 and denote the closed complements of $K \subset \mathscr{X} \subset \mathscr{Y} \subset \mathscr{H}$ by $\mathscr{X}_0 = K^\perp \cap \mathscr{X}$, and $\mathscr{Y}_0 = K^\perp \cap \mathscr{Y}$, and $K^\perp = K^\perp \cap \mathscr{H}$. Further abusing notation, we denote the closed complements of $K \subset \mathscr{H}^* \subset \mathscr{Y}^* \subset \mathscr{X}^*$ by $\mathscr{X}_0^* = K^\perp \cap \mathscr{X}^*$, and $\mathscr{Y}_0^* = K^\perp \cap \mathscr{Y}^*$, and $K^\perp = K^\perp \cap \mathscr{H}^*$, and write $\Pi$ and $\Pi^\perp$ for the canonical extensions of these operators from $\mathscr{H}^*$ to $\mathscr{X}^*$ or, when $p \ge 2$, to $\mathscr{Y}^* = L^{p'}(X;\Lambda^l \otimes \mathrm{ad}P)$ with $p' \in (1,2]$.

We now show that $\mathrm{Ran}(\Delta_A : \mathscr{X} \to \mathscr{Y})$ is closed with finite codimension. Let $M^\circ$ denote the annihilator in $\mathscr{Y}^*$ of a subspace $M \subset \mathscr{Y}$ and observe that [49, Theorem 4.12] yields

$$\begin{aligned}
&\left(\mathrm{Ran}\left(\Delta_A : W^{2,p}_{A_1}(X;\Lambda^l \otimes \mathrm{ad}P) \to L^p(X;\Lambda^l \otimes \mathrm{ad}P)\right)\right)^\circ \\
&\quad = \mathrm{Ker}\left(\Delta_A^* : L^{p'}(X;\Lambda^l \otimes \mathrm{ad}P) \to W^{-2,p'}_{A_1}(X;\Lambda^l \otimes \mathrm{ad}P)\right) \\
&\quad = \mathrm{Ker}\left(\Delta_A : L^{p'}(X;\Lambda^l \otimes \mathrm{ad}P) \to W^{-2,p'}_{A_1}(X;\Lambda^l \otimes \mathrm{ad}P)\right) \quad \text{(by $L^2$ self-adjointness)} \\
&\quad = \mathrm{Ker}\left(\Delta_A : W^{2,p}_{A_1}(X;\Lambda^l \otimes \mathrm{ad}P) \to L^p(X;\Lambda^l \otimes \mathrm{ad}P)\right) \quad \text{(by Remark 2.2)} \\
&\quad = K.
\end{aligned}$$

Note that our appeal to elliptic regularity in the preceding application of Remark 2.2 uses our hypothesis that $A$ is a $C^\infty$ connection. We now repeat this argument for the composition $\Pi^\perp \circ \Delta_A$,

[5] In fact, equality by elliptic regularity according to Remark 2.2.

but writing $M^\Diamond$ for the annihilator in $\mathscr{Y}_0^*$ of a subspace $M \subset \mathscr{Y}_0$,

$$\begin{aligned}&\left(\mathrm{Ran}\left(\Pi^\perp \circ \Delta_A : W^{2,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P) \to K^\perp \cap L^p(X;\Lambda^l\otimes \mathrm{ad}P)\right)\right)^\Diamond \\ &\quad = \mathrm{Ker}\left(\Delta_A^* \circ \Pi^\perp : K^\perp \cap L^{p'}(X;\Lambda^l\otimes \mathrm{ad}P) \to W^{-2,p'}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)\right) \\ &\quad = \mathrm{Ker}\left(\Delta_A \circ \Pi^\perp : K^\perp \cap L^{p'}(X;\Lambda^l\otimes \mathrm{ad}P) \to W^{-2,p'}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)\right) \\ &\quad = \mathrm{Ker}\left(\Delta_A \circ \Pi^\perp : K^\perp \cap W^{2,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P) \to L^p(X;\Lambda^l\otimes \mathrm{ad}P)\right) \\ &\quad = 0,\end{aligned}$$

where the final equality follows by definition of $\Pi^\perp$. Therefore,

$$\begin{aligned}\mathrm{Ran}\left(\Pi^\perp \circ \Delta_A : W^{2,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P) \to K^\perp \cap L^p(X;\Lambda^l\otimes \mathrm{ad}P)\right)& \\ = K^\perp \cap L^p(X;\Lambda^l\otimes \mathrm{ad}P) \equiv \mathscr{Y}_0&,\end{aligned}$$

by definition of the annihilator in $\mathscr{Y}_0^*$. Because $\mathscr{Y}_0$ has finite codimension in $\mathscr{Y}$ equal to $\dim K$, then $\mathrm{Ran}(\Delta_A : \mathscr{X} \to \mathscr{Y})$ has finite codimension in $\mathscr{Y}$ too and thus is also closed by [2, Lemma 4.38]. We conclude that $\Delta_A : \mathscr{X} \to \mathscr{Y}$ is Fredholm.

We can identify the range of $\Delta_A : \mathscr{X} \to \mathscr{Y}$ explicitly using the fact that

$$\Pi^\perp \circ \Delta_A = \Delta_A \circ \Pi^\perp \quad \text{on} \quad W^{2,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)$$

and so

$$\mathrm{Ran}(\Delta_A : \mathscr{X} \to \mathscr{Y}) = \mathrm{Ran}(\Delta_A \circ \Pi^\perp : \mathscr{X} \to \mathscr{Y}) = \mathrm{Ran}(\Pi^\perp \circ \Delta_A : \mathscr{X} \to \mathscr{Y}) = \mathscr{Y}_0.$$

Hence, $\operatorname{codim}\mathrm{Ran}(\Delta_A : \mathscr{X} \to \mathscr{Y}) = \dim\mathrm{Ker}(\Delta_A : \mathscr{X} \to \mathscr{Y})$ and thus $\mathrm{Index}(\Delta_A : \mathscr{X} \to \mathscr{Y}) = 0$. □

*Remark* B.3 (Relationship between $L^2$-orthogonal complements and annihilators). Suppose that $\mathscr{X}$ is a Banach space with continuous embedding into a Hilbert space $\mathscr{H}$ and $M \subset \mathscr{X}$ is a subspace. Recall that $M^\circ = \{\alpha \in \mathscr{X}^* : \langle x, \alpha\rangle_{\mathscr{X}\times\mathscr{X}^*} = 0,\ \forall\, x \in M\}$. We have a canonical isomorphism $\mathscr{H} \cong \mathscr{H}^*$ and if $\mathscr{X} \subset \mathscr{H}$ is a dense subspace, a continuous embedding $\mathscr{H}^* \subset \mathscr{X}^*$. Since $M \subset \mathscr{H}$, we may define $M^\perp := \{h \in \mathscr{H} : (x,h)_{\mathscr{H}} = 0,\ \forall\, x \in M\}$ and note that there is a canonical isomorphism,

$$M^\perp \cong \{\alpha \in \mathscr{H}^* : \langle x, \alpha\rangle_{\mathscr{H}\times\mathscr{H}^*} = 0,\ \forall\, x \in M\}.$$

Hence, $M^\perp \subset M^\circ$ and there is a canonical isomorphism $M^\perp \cong M^\circ \cap \mathscr{H}^*$.

We have the following useful generalization of Proposition B.2 from the case of a $C^\infty$ to a $W^{1,q}$ connection $A$.

**Corollary B.4** (Fredholm property and index of a Laplace operator). *Assume the hypotheses of Proposition 2.3, including the hypothesis that $A$ is a $W^{1,q}$ connection with $d/2 < q < \infty$. If in addition $p$ obeys $d/2 \le p \le q$ and $p < \infty$, then the operator*

$$\Delta_A : W^{2,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P) \to L^p(X;\Lambda^l\otimes \mathrm{ad}P) \tag{B.3}$$

*is Fredholm with index zero and closed range $K^\perp \cap L^p(X;\Lambda^l\otimes \mathrm{ad}P)$, where $\perp$ denotes $L^2$-orthogonal complement and $K \subset W^{2,p}_{A_1}(X;\Lambda^l\otimes \mathrm{ad}P)$ is the kernel.*

*Proof.* By hypothesis, $A_1$ is a $C^\infty$ connection on $P$. We write $A = A_1 + a$, for $a \in W^{1,q}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P)$ and proceed by modifying the derivation of the $L^p$-bound (2.10) for $(\Delta_A - \Delta_{A_s})\xi$ in the proof of Proposition 2.1 to show that, for suitable $u \in [1, \infty)$ and choosing $A_s = A_1$, the operator

$$\Delta_A - \Delta_{A_1} : W^{1,u}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P) \to L^p(X; \Lambda^l \otimes \mathrm{ad}P) \tag{B.4}$$

is bounded and, because the Sobolev embedding $W^{2,p}(X) \Subset W^{1,u}(X)$ will be compact by the Rellich-Kondrachov [3, Theorem 6.3], then the following composition of that compact embedding and the preceding bounded operator,

$$\Delta_A - \Delta_{A_1} : W^{2,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P) \to L^p(X; \Lambda^l \otimes \mathrm{ad}P) \tag{B.5}$$

is compact by [7, Proposition 6.3].

By retracing the steps in the proof of Proposition 2.1, we find that, for $r \in [p, \infty]$ defined by $1/p = 1/q + 1/r$,

$$\|(\Delta_A - \Delta_{A_1})\xi\|_{L^p(X)} \leq z \left( \|a\|_{W^{1,q}_{A_1}(X)} + \|a\|^2_{W^{1,q}_{A_1}(X)} \right) \|\xi\|_{L^r(X)} + z\|a\|_{W^{1,q}_{A_1}(X)} \|\xi\|_{W^{1,s}_{A_1}(X)}, \tag{B.6}$$

where *i)* $s = d$ when $q < d$, or *ii)* $s = d$ when $q = d$ and $p < d$, or *iii)* $s = 2d$ when $q = d = p$, or *iv)* $s = p$ when $q > d$. We consider each of these cases separately.

*Case* 1 ($p = d/2 < q < d$ and $s = d$). We have $r \in [p, \infty)$ since $p < q$ for this case and a continuous embedding $W^{1,d}(X) \subset L^r(X)$ by [3, Theorem 4.12], so the operator (B.4) is bounded for $u = d$. Moreover, $W^{2,p}(X) \Subset W^{1,d}(X)$ is compact by [3, Theorem 6.3] provided $p^* = dp/(d-p) \geq d$ or equivalently $p \geq d/2$, which holds by hypothesis and the operator (B.5) is compact for this case.

*Case* 2 ($d/2 < p \leq q < d$ and $s = d$). We have $r \in [p, \infty]$ since $p \leq q$ for this case and a continuous embedding $W^{1,d+\varepsilon}(X) \subset L^r(X)$ by [3, Theorem 4.12] for any $\varepsilon > 0$. Also, we have a compact embedding $W^{2,p}(X) \Subset W^{1,d+\varepsilon}(X)$ by [3, Theorem 6.3] provided $p^* = dp/(d-p) \geq d+\varepsilon$; because $dp/(d-p) > d$ when $p > d/2$, a choice of $\varepsilon \in (0, 1]$ is always possible in this case. Hence, the operator (B.4) is bounded for $u = d + \varepsilon$ and the operator (B.5) is compact for this case.

*Case* 3 ($d/2 \leq p < q = d$ and $s = d$). We have $r \in [p, \infty)$ since $p < q$ for this case and a continuous embedding $W^{1,d}(X) \subset L^r(X)$ by [3, Theorem 4.12]. Also, we have a compact embedding $W^{2,p}(X) \Subset W^{1,d}(X)$ by [3, Theorem 6.3] since $p \geq d/2$. Hence, the operator (B.4) is bounded for $u = d$ and the operator (B.5) is compact for this case.

*Case* 4 ($p = q = d$ and $s = 2d$). We have $r = \infty$ since $p = q$ for this case and a continuous embedding $W^{1,d+\varepsilon}(X) \subset L^\infty(X)$ by [3, Theorem 4.12] for any $\varepsilon > 0$ and in particular for $\varepsilon = d$. Also, we have a compact embedding $W^{2,p}(X) \Subset W^{1,v}(X)$ by [3, Theorem 6.3] for any $v \in [1, \infty)$ since $p = d$ and in particular for $v = 2d$. Hence, the operator (B.4) is bounded for $u = 2d$ and the operator (B.5) is compact for this case.

*Case* 5 ($d/2 \leq p < q$ and $q > d$ and $s = p$). We have $r \in [p, \infty)$ and a continuous embedding $W^{1,d}(X) \subset L^r(X)$ by [3, Theorem 4.12]. Also, we have a compact embedding $W^{2,p}(X) \Subset W^{1,d}(X)$ by [3, Theorem 6.3] since $p \geq d/2d$. Hence, the operator (B.4) is bounded for $u = d$ and the operator (B.5) is compact for this case.

*Case* 6 ($p = q = s > d$). We have $r = \infty$ and a continuous embedding $W^{1,p}(X) \subset L^\infty(X)$ by [3, Theorem 4.12]. Also, we have a compact embedding $W^{2,p}(X) \Subset W^{1,p}(X)$ by [3, Theorem 6.3]. Hence, the operator (B.4) is bounded for $u = p$ and the operator (B.5) is compact for this case.

Proposition B.2 implies that the operator

$$\Delta_{A_1} : W^{2,p}_{A_1}(X; \Lambda^l \otimes \mathrm{ad}P) \to L^p(X; \Lambda^l \otimes \mathrm{ad}P)$$

is Fredholm with index zero while the operator $\Delta_A - \Delta_{A_1}$ in (B.5) is compact from each of the preceding cases, so the operator (B.3) is Fredholm with index zero by [31, Corollary 19.1.8]. The identification of the range in the proof of Proposition B.2 applies without change to the case where $A$ is a $W^{1,q}$ connection. □

## References


[1] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis, and applications*, second ed., Springer, New York, 1988. MR 960687 (89f:58001)

[2] Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory*, Graduate Studies in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 2002. MR 1921782 (2003h:47072)

[3] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, second ed., Elsevier/Academic Press, Amsterdam, 2003. MR 2424078 (2009e:46025)

[4] S. B. Bradlow, *Vortices in holomorphic line bundles over closed Kähler manifolds*, Comm. Math. Phys. **135** (1990), no. 1, 1–17. MR 1086749 (92f:32053)

[5] ______, *Special metrics and stability for holomorphic bundles with global sections*, J. Differential Geom. **33** (1991), 169–213. MR 1085139 (91m:32031)

[6] S. B. Bradlow and O. García-Prada, *Non-abelian monopoles and vortices*, Geometry and physics (Aarhus, 1995), Lecture Notes in Pure and Appl. Math., vol. 184, Dekker, New York, 1997, arXiv:alg-geom/9602010, pp. 567–589. MR 1423193 (97k:53032)

[7] H. Brézis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. MR 2759829 (2012a:35002)

[8] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer, New York, 1995. MR 1410059 (97i:22005)

[9] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, vol. 115, Academic Press Inc., Orlando, FL, 1984. MR 768584 (86g:58140)

[10] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, New York, 1990. MR 1079726 (92a:57036)

[11] P. M. N. Feehan, *Discreteness for energies of Yang-Mills connections over four-dimensional manifolds*, arXiv:1505.06995, 89 pages.

[12] ______, *Energy gap for Yang-Mills connections, II: Arbitrary closed Riemannian manifolds*, arXiv:1502.00668, 31 pages.

[13] ______, *Global existence and convergence of smooth solutions to Yang-Mills gradient flow over compact four-manifolds*, arXiv:1409.1525, xvi+425 pages.

[14] ______, *Critical-exponent Sobolev norms and the slice theorem for the quotient space of connections*, Pacific J. Math. **200** (2001), 71–118, arXiv:dg-ga/9711004.

[15] P. M. N. Feehan and T. G. Leness, PU(2) *monopoles. I. Regularity, Uhlenbeck compactness, and transversality*, J. Differential Geom. **49** (1998), 265–410. MR 1664908 (2000e:57052)

[16] ______, PU(2) *monopoles and links of top-level Seiberg-Witten moduli spaces*, J. Reine Angew. Math. **538** (2001), 57–133, arXiv:math/0007190.

[17] P. M. N. Feehan and M. Maridakis, *Discreteness for energies of solutions to coupled Yang-Mills equations over compact four-manifolds*, in preparation.

[18] ______, *Global existence and convergence of smooth solutions to coupled Yang-Mills gradient flows over compact four-manifolds*, in preparation.

[19] ______, *Łojasiewicz-Simon gradient inequalities for analytic functionals on Banach spaces and applications to harmonic maps*, arXiv preprint, October 13, 2015.

[20] E. Feireisl and P. Takáč, *Long-time stabilization of solutions to the Ginzburg-Landau equations of superconductivity*, Monatsh. Math. **133** (2001), no. 3, 197–221. MR 1861137 (2003a:35022)

[21] G. B. Folland, *Introduction to partial differential equations*, second ed., Princeton University Press, Princeton, NJ, 1995. MR 1357411 (96h:35001)

[22] D. S. Freed and K. K. Uhlenbeck, *Instantons and four-manifolds*, second ed., Mathematical Sciences Research Institute Publications, vol. 1, Springer, New York, 1991. MR 1081321 (91i:57019)

[23] R. Friedman and J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Springer, Berlin, 1994.
[24] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, second ed., Springer, New York, 1983.
[25] P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, second ed., Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1396308 (98b:58156)
[26] D. Groisser and T. H. Parker, *The geometry of the Yang-Mills moduli space for definite manifolds*, J. Differential Geom. **29** (1989), 499–544. MR 992329 (90f:58021)
[27] E. Guentner, *K-homology and the index theorem*, Index theory and operator algebras (Boulder, CO, 1991), Contemp. Math., vol. 148, Amer. Math. Soc., Providence, RI, 1993, pp. 47–66. MR 1228499 (94h:19006)
[28] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126. MR 887284 (89a:32021)
[29] M-C. Hong, *Heat flow for the Yang-Mills-Higgs field and the Hermitian Yang-Mills-Higgs metric*, Ann. Global Anal. Geom. **20** (2001), 23–46. MR 1846895 (2002h:53040)
[30] M-C. Hong and L. Schabrun, *Global existence for the Seiberg-Witten flow*, Comm. Anal. Geom. **18** (2010), 433–473. MR 2747435 (2012b:53139)
[31] L. Hörmander, *The analysis of linear partial differential operators, III. pseudo-differential operators*, Springer, Berlin, 2007. MR 2304165 (2007k:35006)
[32] S.-Z. Huang, *Gradient inequalities*, Mathematical Surveys and Monographs, vol. 126, American Mathematical Society, Providence, RI, 2006. MR 2226672 (2007b:35035)
[33] S.-Z. Huang and P. Takáč, *Convergence in gradient-like systems which are asymptotically autonomous and analytic*, Nonlinear Anal. **46** (2001), 675–698. MR 1857152 (2002f:35125)
[34] J. Jost, X. Peng, and G. Wang, *Variational aspects of the Seiberg-Witten functional*, Calc. Var. Partial Differential Equations **4** (1996), 205–218. MR 1386734 (97d:58055)
[35] T. Kato, *Perturbation theory for linear operators*, second ed., Springer, New York, 1984.
[36] P. B. Kronheimer and T. S. Mrowka, *Monopoles and three-manifolds*, Cambridge University Press, Cambridge, 2007. MR 2388043 (2009f:57049)
[37] H. B. Lawson, *The theory of gauge fields in four dimensions*, Conf. Board Math. Sci., vol. 58, Amer. Math. Soc., Providence, RI, 1985.
[38] H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989. MR 1031992 (91g:53001)
[39] J. Li and X. Zhang, *The gradient flow of Higgs pairs*, J. Eur. Math. Soc. (JEMS) **13** (2011), 1373–1422. MR 2825168 (2012m:53043)
[40] S. Łojasiewicz, *Ensembles semi-analytiques*, (1965), Publ. Inst. Hautes Etudes Sci., Bures-sur-Yvette, preprint, 112 pages, `perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf`.
[41] J. Malý and W. P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs, vol. 51, American Mathematical Society, Providence, RI, 1997. MR 1461542 (98h:35080)
[42] R. E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics, vol. 183, Springer-Verlag, New York, 1998. MR 1650235 (99k:46002)
[43] L. I. Nicolaescu, *Notes on Seiberg-Witten theory*, Graduate Studies in Mathematics, vol. 28, American Mathematical Society, Providence, RI, 2000. MR 1787219 (2001k:57037)
[44] C. Okonek and A. Teleman, *The coupled Seiberg-Witten equations, vortices, and moduli spaces of stable pairs*, Internat. J. Math. **6** (1995), 893–910, arXiv:alg-geom/9505012.
[45] T. H. Parker, *Gauge theories on four-dimensional Riemannian manifolds*, Comm. Math. Phys. **85** (1982), 563–602. MR 677998 (84b:58036)
[46] V. Ya. Pidstrigatch and A. N. Tyurin, *Localisation of Donaldson invariants along the Seiberg-Witten classes*, arXiv:dg-ga/9507004.
[47] M. Reed and B. Simon, *Methods of modern mathematical physics. I*, second ed., Academic Press, New York, 1980, Functional analysis. MR 751959 (85e:46002)
[48] J. Råde, *On the Yang-Mills heat equation in two and three dimensions*, J. Reine Angew. Math. **431** (1992), 123–163. MR 1179335 (94a:58041)
[49] W. Rudin, *Functional analysis*, McGraw-Hill, New York, NY, 1973.
[50] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), 525–571. MR 727703 (85b:58121)

[51] C. T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), no. 4, 867–918. MR 944577 (90e:58026)

[52] I. Stakgold and M. Holst, *Green's functions and boundary value problems*, third ed., Wiley, Hoboken, NJ, 2011.

[53] C. H. Taubes, *A framework for Morse theory for the Yang-Mills functional*, Invent. Math. **94** (1988), 327–402. MR 958836 (90a:58035)

[54] K. K. Uhlenbeck, *Connections with $L^p$ bounds on curvature*, Comm. Math. Phys. **83** (1982), 31–42. MR 648356 (83e:53035)

[55] ______, *Removable singularities in Yang-Mills fields*, Comm. Math. Phys. **83** (1982), 11–29. MR 648355 (83e:53034)

[56] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Graduate Texts in Mathematics, vol. 102, Springer-Verlag, New York, 1984, Reprint of the 1974 edition. MR 746308 (85e:22001)

[57] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer, New York, 1983. MR 722297 (84k:58001)

[58] K. Wehrheim, *Uhlenbeck compactness*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2004. MR 2030823 (2004m:53045)

[59] G. Wilkin, *Morse theory for the space of Higgs bundles*, Comm. Anal. Geom. **16** (2008), 283–332. MR 2425469 (2010f:53115)

[60] E. Zeidler, *Nonlinear functional analysis and its applications. II/A*, Springer-Verlag, New York, 1990, Linear monotone operators. MR 1033497 (91b:47001)

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