An I-category structure for crossed chain algebras

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MPI/95-135

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1 Crossed chain algebras and their pushouts

Let $(\mathbf{Crs}, \otimes)$ be the monoidal category of crossed (chain) complexes (of groupoids), with terminal object $*$ and 'interval' object $\mathcal{I}$ given by the fundamental crossed complex of the zero- and one-simplex respectively. The initial object in $\mathbf{Crs}$ is the empty crossed complex $\emptyset$. A *homotopy* in $\mathbf{Crs}$ is a homomorphism $\mathcal{I} \otimes A \to B$.

A *crossed chain algebra* $\mathbf{X}$ consists of a crossed complex X with unit and multiplication given by homomorphisms $0: * \to X$ and $\mu: X \otimes X \to X$ satisfying the usual identity and associativity laws. Morphisms of crossed chain algebras are crossed complex homomorphisms respecting the extra structure. The category so formed is denoted $\mathbf{CrsAlg}$. The crossed complex $*$ with unit and multiplication structures given by the isomorphisms $* \cong *$ and $* \otimes * \cong *$ is both initial and terminal in $\mathbf{CrsAlg}$.

The forgetful functor $U: \mathbf{CrsAlg} \to \mathbf{Crs}$ given by $\mathbf{X} \mapsto X$ has a left adjoint F, where FA has underlying crossed complex $\coprod_{n \geq 0} A^{\otimes n}$ with unit given by the inclusion of $*$ as the 0-fold tensor product and multiplication given by the isomorphisms $A^{\otimes p} \otimes A^{\otimes q} \cong A^{\otimes (p+q)}$. Given a crossed chain algebra $\mathbf{X}$ and a crossed complex homomorphism $f: A \to X$, we write f^{T} for the corresponding morphism $FA \to \mathbf{X}$.

Suppose B is a crossed complex and $\mathbf{X}$ a crossed chain algebra. Then the free product of algebras $\mathbf{X} \amalg FB$ has underlying crossed complex $\coprod_{n \geq 0} X \otimes (B \otimes X)^{\otimes n}$ and multiplication given by μ_X on the inner factors:

$$X \otimes (B \otimes X)^{\otimes p} \;\otimes\; X \otimes (B \otimes X)^{\otimes q} \xrightarrow{1 \otimes \mu_X \otimes 1} X \otimes (B \otimes X)^{\otimes (p+q)}$$

The algebra maps from $\mathbf{X}$, FB to $\mathbf{X} \amalg FB$ are defined using the isomorphisms $X \cong X \otimes (B \otimes X)^{\otimes 0}$, $FB \cong \coprod_{n \geq 0} * \otimes (B \otimes *)^{\otimes n}$ respectively.

Given also a crossed complex A and homomorphisms $k: A \to B$, $f: A \to X$, we can take the pushout of algebras $\mathbf{Y} = \mathbf{X} \amalg_{FA} FB$:

$$\begin{array}{ccc} FA & \xrightarrow{Fk} & FB \\ {\scriptstyle f^{\mathrm{T}}}\downarrow & & \downarrow \\ \mathbf{X} & \longrightarrow & \mathbf{Y} \end{array}$$

Let $C = \coprod_{n \geq 0} X \otimes (B \otimes X)^{\otimes n}$, the underlying crossed complex of the free product, and consider the homomorphisms $a, b : C \otimes A \otimes C \longrightarrow C$ defined by k and by f and μ:

$$\begin{array}{ccc} C \otimes A \otimes C & \xrightarrow{1 \otimes f \otimes 1} & C \otimes X \otimes C \\ {\scriptstyle 1 \otimes k \otimes 1}\downarrow & & \downarrow{\scriptstyle 1 \otimes \mu_X^2 \otimes 1} \\ C \otimes B \otimes C & \hookrightarrow & C \end{array}$$

Then the pushout $\mathbf{Y}$ has underlying crossed complex given by the coequaliser of a and b:

$$C \otimes A \otimes C \underset{b}{\overset{a}{\rightrightarrows}} C \xrightarrow{\ q\ } Y$$

and the unit and multiplication structures on Y are induced by those on C given in the free product case. Suppose given a crossed chain algebra $\mathbf{Z}$, a crossed complex homomorphism $r: B \to Z$ and a morphism $s: \mathbf{X} \to \mathbf{Z}$, such that the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{Fk} & FB \\ {\scriptstyle f^{\mathrm{T}}}\downarrow & & \downarrow{\scriptstyle r^{\mathrm{T}}} \\ \mathbf{X} & \xrightarrow{\ s\ } & \mathbf{Z} \end{array}$$

Then the morphism $t: \mathbf{Y} \to \mathbf{Z}$ corresponding to (r, s) may be defined via the homomorphisms

$$X \otimes (B \otimes X)^{\otimes n} \xrightarrow{s \otimes (r \otimes s)^{\otimes n}} Z^{\otimes(2n+1)} \xrightarrow{\mu_Z{}^{2n}} Z$$

Recall that the diagonal approximation map of the Eilenberg-Zilber theorem [4] gives homomorphisms $\mathcal{I} \to \mathcal{I}^{\otimes n}$ which satisfy the obvious associativity laws and commute with the canonical map $p: \mathcal{I} \to *$. Using the symmetry of $\otimes$, these give homomorphisms

$$\mathcal{I} \otimes A_1 \otimes A_2 \otimes \ldots \otimes A_n \xrightarrow{d^{(n)}} \mathcal{I} \otimes A_1 \otimes \mathcal{I} \otimes A_2 \otimes \ldots \otimes \mathcal{I} \otimes A_n$$

for any crossed complexes $A_1, A_2, \ldots, A_n$. Given crossed chain algebras $\mathbf{X}, \mathbf{X}'$ we will say that a homotopy $\sigma: \mathcal{I} \otimes X \to X'$ in **Crs** *respects the multiplication* if the following diagrams commute:

$$\begin{array}{ccc} \mathcal{I} \otimes * & \xrightarrow{\ p\ } & * \\ {\scriptstyle 1 \otimes 0_X}\downarrow & & \downarrow{\scriptstyle 0_{X'}} \\ \mathcal{I} \otimes X & \xrightarrow{\ \sigma\ } & X' \end{array} \qquad \begin{array}{ccccc} \mathcal{I} \otimes X \otimes X & \xrightarrow{d^{(2)}} & \mathcal{I} \otimes X \otimes \mathcal{I} \otimes X & \xrightarrow{\sigma \otimes \sigma} & X' \otimes X' \\ {\scriptstyle 1 \otimes \mu_X}\downarrow & & & & \downarrow{\scriptstyle \mu_{X'}} \\ \mathcal{I} \otimes X & & \xrightarrow{\qquad \sigma \qquad} & & X' \end{array}$$

The following proposition will be used inductively in the constructions of section 2:

Proposition 1 (a) *Suppose given algebras $\mathbf{X}$, $\mathbf{X}'$, homomorphisms $k: A \to B$, $f: A \to X$, a homotopy $\sigma_X: \mathcal{I} \otimes X \to X'$ which respects the multiplication, and a morphism $\pi_X: \mathbf{X}' \to \mathbf{X}$ satisfying $\pi_X \circ \sigma_X = p: \mathcal{I} \otimes X \to X$. Let $\mathbf{Y}$, $\mathbf{Y}'$ be the algebras given by the pushouts*

$$\begin{array}{ccc} FA & \xrightarrow{Fk} & FB \\ {\scriptstyle f^{\mathrm{T}}}\downarrow & & \downarrow{\scriptstyle \overline{f}^{\mathrm{T}}} \\ \mathbf{X} & \xrightarrow{\ i\ } & \mathbf{Y} \end{array} \qquad \begin{array}{ccc} F(\mathcal{I} \otimes A) & \xrightarrow{F(1 \otimes k)} & F(\mathcal{I} \otimes B) \\ {\scriptstyle g^{\mathrm{T}}}\downarrow & & \downarrow{\scriptstyle \overline{g}^{\mathrm{T}}} \\ \mathbf{X}' & \xrightarrow{\ i'\ } & \mathbf{Y}' \end{array}$$

where $g = \sigma_X \circ (1 \otimes f)$. Then there exist a homotopy $\sigma_Y: \mathcal{I} \otimes Y \to Y'$ which respects the multiplication and a morphism $\pi_Y: \mathbf{Y}' \to \mathbf{Y}$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{I} \otimes B & & \\ {\scriptstyle 1 \otimes \overline{f}}\downarrow & \searrow^{\overline{g}} & \\ \mathcal{I} \otimes Y & \xrightarrow{\sigma_Y} & Y' \end{array} \qquad \begin{array}{ccc} \mathcal{I} \otimes Y & \xrightarrow{\sigma_Y} & Y' \\ & \searrow_{p} & \downarrow{\scriptstyle \pi_Y} \\ & & Y \end{array} \qquad \begin{array}{ccc} \mathcal{I} \otimes X & \xrightarrow{1 \otimes i} & \mathcal{I} \otimes Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X' & \xrightarrow{\ i'\ } & Y' \end{array} \qquad \begin{array}{ccc} \mathbf{X}' & \xrightarrow{\ i'\ } & \mathbf{Y}' \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ \mathbf{X} & \xrightarrow{\ i\ } & \mathbf{Y} \end{array}$$

(b) *Given also algebras $\mathbf{Z}$, $\mathbf{Z}'$, a homotopy $\sigma_Z: \mathcal{I} \otimes Z \to Z'$ which respects the multiplication, and a morphism $\pi_Z: \mathbf{Z}' \to \mathbf{Z}$ such that $\pi_Z \circ \sigma_Z = p$, together with morphisms $s: \mathbf{X} \to \mathbf{Z}$, $t: \mathbf{Y} \to \mathbf{Z}$, $s': \mathbf{X}' \to \mathbf{Z}'$ such that the diagrams*

$$
\begin{array}{ccc}
\mathbf{X} & & \\
{\scriptstyle i}\downarrow & \searrow^{s} & \\
\mathbf{Y} & \xrightarrow[t]{} & \mathbf{Z}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{I} \otimes X & \xrightarrow{1 \otimes s} & \mathcal{I} \otimes Z \\
{\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Z} \\
X' & \xrightarrow{s'} & Z'
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{X}' & \xrightarrow{s'} & \mathbf{Z}' \\
{\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Z} \\
\mathbf{X} & \xrightarrow{s} & \mathbf{Z}
\end{array}
$$

commute, then there exists a unique morphism of algebras $t': \mathbf{Y}' \to \mathbf{Z}'$ making the following diagrams commute:

$$
\begin{array}{ccc}
\mathbf{X}' & & \\
{\scriptstyle i'}\downarrow & \searrow^{s'} & \\
\mathbf{Y}' & \xrightarrow[t']{} & \mathbf{Z}'
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{I} \otimes Y & \xrightarrow{1 \otimes t} & \mathcal{I} \otimes Z \\
{\scriptstyle \sigma_Y}\downarrow & & \downarrow{\scriptstyle \sigma_Z} \\
Y' & \xrightarrow{t'} & Z'
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{Y}' & \xrightarrow{t'} & \mathbf{Z}' \\
{\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_Z} \\
\mathbf{Y} & \xrightarrow{t} & \mathbf{Z}
\end{array}
$$

Proof: (a) Let $\mathbf{C}$, $\mathbf{C}'$ be the free products $\mathbf{X} \amalg FB$, $\mathbf{X}' \amalg F(\mathcal{I} \otimes B)$ and consider Y, Y' in terms of coequalisers in **Crs** as above. Since $\mathcal{I} \otimes (-)$ preserves colimits, we may specify σ_Y by giving homotopies σ', σ'' as in the following diagram:

$$
\begin{array}{ccccc}
\mathcal{I} \otimes C \otimes A \otimes C & \underset{1 \otimes b}{\overset{1 \otimes a}{\rightrightarrows}} & \mathcal{I} \otimes C & \xrightarrow{q} & \mathcal{I} \otimes Y \\
{\scriptstyle \sigma''}\downarrow & & \downarrow{\scriptstyle \sigma'} & & \vdots{\scriptstyle \sigma_Y} \\
C' \otimes \mathcal{I} \otimes A \otimes C' & \underset{b'}{\overset{a'}{\rightrightarrows}} & C' & \xrightarrow{q'} & Y'
\end{array}
$$

We define σ' via the homomorphisms $\sigma_X \otimes (1 \otimes \sigma_X)^{\otimes n} \circ d^{(2n+1)}$:

$$
\mathcal{I} \otimes X \otimes (B \otimes X)^{\otimes n} \xrightarrow{d^{(2n+1)}} \mathcal{I} \otimes X \otimes (\mathcal{I} \otimes B \otimes \mathcal{I} \otimes X)^{\otimes n} \xrightarrow{\sigma_X \otimes (1 \otimes \sigma_X)^{\otimes n}} X' \otimes (\mathcal{I} \otimes B \otimes X')^{\otimes n}
$$

Note that σ' respects the multiplication since σ_X does, and that the following diagrams commute:

$$
\begin{array}{ccccc}
\mathcal{I} \otimes B & \xrightarrow{\cong} & \mathcal{I} \otimes * \otimes B \otimes * & \xrightarrow{1 \otimes 0_X \otimes 1 \otimes 0_X} & \mathcal{I} \otimes C \\
& {\scriptstyle \cong}\searrow & \cong\downarrow{\scriptstyle p \otimes 1 \otimes p \circ d^{(3)}} & & \downarrow{\scriptstyle \sigma'} \\
& & * \otimes \mathcal{I} \otimes B \otimes * & \xrightarrow{0_{X'} \otimes 1 \otimes 0_{X'}} & C'
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{I} \otimes X & \hookrightarrow & \mathcal{I} \otimes C \\
{\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma'} \\
X' & \hookrightarrow & C'
\end{array}
$$

Thus the relations $\overline{g} = \sigma_Y \circ (1 \otimes \overline{f})$ and $i' \circ \sigma_X = \sigma_Y \circ (1 \otimes i)$ will follow.

Now consider the diagrams

$$\begin{array}{ccccc}
\mathcal{I}\otimes C\otimes A\otimes C & \xrightarrow{1\otimes k\otimes 1} & \mathcal{I}\otimes C\otimes B\otimes C & \hookrightarrow & \mathcal{I}\otimes C \\
\downarrow{\scriptstyle d^{(3)}} & & \downarrow{\scriptstyle d^{(3)}} & & \\
\mathcal{I}\otimes C\otimes\mathcal{I}\otimes A\otimes\mathcal{I}\otimes C & \xrightarrow{1\otimes k\otimes 1} & \mathcal{I}\otimes C\otimes\mathcal{I}\otimes B\otimes\mathcal{I}\otimes C & & \downarrow{\scriptstyle \sigma'} \\
\downarrow{\scriptstyle \sigma'\otimes 1\otimes\sigma'} & & \downarrow{\scriptstyle \sigma'\otimes 1\otimes\sigma'} & & \\
C'\otimes\mathcal{I}\otimes A\otimes C' & \xrightarrow{1\otimes k\otimes 1} & C'\otimes\mathcal{I}\otimes B\otimes C' & \hookrightarrow & C'
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{I}\otimes C\otimes A\otimes C & \xrightarrow{1\otimes f\otimes 1} & \mathcal{I}\otimes C\otimes X\otimes C & \xrightarrow{1\otimes\mu_X^2\otimes 1} & \mathcal{I}\otimes C \\
\downarrow{\scriptstyle d^{(3)}} & & \downarrow{\scriptstyle d^{(3)}} & & \\
\mathcal{I}\otimes C\otimes\mathcal{I}\otimes A\otimes\mathcal{I}\otimes C & \xrightarrow{1\otimes f\otimes 1} & \mathcal{I}\otimes C\otimes\mathcal{I}\otimes X\otimes\mathcal{I}\otimes C & & \downarrow{\scriptstyle \sigma'} \\
\downarrow{\scriptstyle \sigma'\otimes 1\otimes\sigma'} & & \downarrow{\scriptstyle \sigma'\otimes\sigma_X\otimes\sigma'} & & \\
C'\otimes\mathcal{I}\otimes A\otimes C' & \xrightarrow{1\otimes g\otimes 1} & C'\otimes X'\otimes C' & \xrightarrow{1\otimes\mu_{X'}^2\otimes 1} & C'
\end{array}$$

The commutativity of the first of these is clear; the second requires the fact that σ_X respects the multiplication on X, X'. Thus we put $\sigma'' = (\sigma'\otimes 1\otimes\sigma')\circ d^{(3)}$ and σ_Y is well defined.

By definition of $\mathbf{Y}$, the relation $\pi_X\circ\sigma_X = p$ and naturality we have the following diagram:

$$\begin{array}{ccccc}
F(\mathcal{I}\otimes A) & \xrightarrow{F(1\otimes k)} & F(\mathcal{I}\otimes B) & & \\
\downarrow{\scriptstyle F(1\otimes f)} & & \downarrow{\scriptstyle F(1\otimes\bar{f})} & \searrow{\scriptstyle Fp} & \\
F(\mathcal{I}\otimes X) & \xrightarrow{F(1\otimes i)} & F(\mathcal{I}\otimes Y) & & FB \\
\downarrow{\scriptstyle \sigma_X^{\mathrm{T}}} & \searrow{\scriptstyle p^{\mathrm{T}}} & & \searrow{\scriptstyle p^{\mathrm{T}}} & \downarrow{\scriptstyle \bar{f}^{\mathrm{T}}} \\
\mathbf{X}' & \xrightarrow{\pi_X} & \mathbf{X} & \xrightarrow{i} & \mathbf{Y}
\end{array}$$

Then $\pi_Y\colon\mathbf{Y}'\to\mathbf{Y}$ is defined as the canonical morphism from the pushout, and the relation $\pi_Y\circ i' = i\circ\pi_X$ is clear. Explicitly π_Y may be written in terms of the homomorphisms:

$$X'\otimes(\mathcal{I}\otimes B\otimes X')^{\otimes n}\xrightarrow{\pi_X\otimes(1\otimes\pi_X)^{\otimes n}} X\otimes(\mathcal{I}\otimes B\otimes X)^{\otimes n}\xrightarrow{1\otimes(p\otimes 1)^{\otimes n}} X\otimes(B\otimes X)^{\otimes n}$$

and thus the relation $\pi_{Y'}\circ\sigma_Y = p$ follows from the diagram below.

$$\begin{array}{ccccc}
\mathcal{I}\otimes X\otimes(B\otimes X)^{\otimes n} & \xrightarrow{d^{(2n+1)}} & \mathcal{I}\otimes X\otimes(\mathcal{I}\otimes B\otimes\mathcal{I}\otimes X)^{\otimes n} & \xrightarrow{\sigma_X\otimes(1\otimes\sigma_X)^{\otimes n}} & X'\otimes(\mathcal{I}\otimes B\otimes X')^{\otimes n} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p\otimes(1\otimes p)^{\otimes n}} & \swarrow{\scriptstyle \pi_X\otimes(1\otimes\pi_X)^{\otimes n}} & \\
X\otimes(B\otimes X)^{\otimes n} & \xleftarrow{1\otimes(p\otimes 1)^{\otimes n}} & X\otimes(\mathcal{I}\otimes B\otimes X)^{\otimes n} & &
\end{array}$$

(b) For the second part, we note we have the commutative diagram

$$\begin{array}{ccccc}
F(\mathcal{I}\otimes A) & \xrightarrow{F(1\otimes k)} & F(\mathcal{I}\otimes B) & & \\
\Big\downarrow{\scriptstyle F(1\otimes f)} & & \Big\downarrow{\scriptstyle F(1\otimes \overline{f})} & & \\
F(\mathcal{I}\otimes X) & \xrightarrow{F(1\otimes i)} & F(\mathcal{I}\otimes Y) & \xrightarrow{F(1\otimes t)} & F(\mathcal{I}\otimes Z) \\
\Big\downarrow{\scriptstyle \sigma_X{}^{\mathrm{T}}} & & & & \Big\downarrow{\scriptstyle \sigma_Z{}^{\mathrm{T}}} \\
\mathbf{X}' & & \xrightarrow{\qquad s' \qquad} & & \mathbf{Z}'
\end{array}$$

by our hypotheses and the definition of Y. Thus we have a canonical morphism $t'\colon \mathbf{Y}' \to \mathbf{Z}'$, with $s' = t' \circ i'$, by the definition of $\mathbf{Y}'$ as the pushout. If we put $r = t \circ \overline{f}$ (and recall that $s = t \circ i$) we note that t, t' are given by the homomorphisms

$$X \otimes (B \otimes X)^{\otimes n} \xrightarrow{\quad s\otimes(r\otimes s)^{\otimes n} \quad} Z^{\otimes(2n+1)} \xrightarrow{\quad \mu_Z{}^{2n} \quad} Z$$

$$X' \otimes (\mathcal{I} \otimes B \otimes X')^{\otimes n} \xrightarrow{s'\otimes(1\otimes r\otimes s')^{\otimes n}} Z' \otimes (\mathcal{I} \otimes Z \otimes Z')^{\otimes n} \xrightarrow{1\otimes(\sigma_Z\otimes 1)^{\otimes n}} Z'^{\otimes(2n+1)} \xrightarrow{\mu_{Z'}{}^{2n}} Z'$$

respectively. Recalling the descriptions of σ_Y, π_Y above, the required relations $\sigma_Z \circ (1 \otimes t) = t' \circ \sigma_Y$, $\pi_Z \circ t' = t \circ \pi_Y$ thus follow from the diagrams

$$\begin{array}{ccccc}
\mathcal{I}\otimes X\otimes(B\otimes X)^{\otimes n} & \xrightarrow{1\otimes s\otimes(r\otimes s)^{\otimes n}} & \mathcal{I}\otimes Z^{\otimes(2n+1)} & \xrightarrow{1\otimes\mu_Z{}^{2n}} & \mathcal{I}\otimes Z \\
\Big\downarrow{\scriptstyle d^{(2n+1)}} & & \Big\downarrow{\scriptstyle d^{(2n+1)}} & & \Big| \\
\mathcal{I}\otimes X\otimes(\mathcal{I}\otimes B\otimes\mathcal{I}\otimes X)^{\otimes n} & \xrightarrow{1\otimes s\otimes(1\otimes r\otimes 1\otimes s)^{\otimes n}} & (\mathcal{I}\otimes Z)^{\otimes(2n+1)} & & \Big| \\
\Big\downarrow{\scriptstyle \sigma_X\otimes(1\otimes\sigma_X)^{\otimes n}} & & \Big\downarrow{\scriptstyle \sigma_Z\otimes(1\otimes\sigma_Z)^{\otimes n}} & & \Big|{\scriptstyle \sigma_Z} \\
X'\otimes(\mathcal{I}\otimes B\otimes X')^{\otimes n} & \xrightarrow{s'\otimes(1\otimes r\otimes s')^{\otimes n}} & Z'\otimes(\mathcal{I}\otimes Z\otimes Z')^{\otimes n} & & \Big| \\
 & & \Big\downarrow{\scriptstyle 1\otimes(\sigma_Z\otimes 1)^{\otimes n}} & & \Big\downarrow \\
 & & Z'^{\otimes(2n+1)} & \xrightarrow{\mu_{Z'}{}^{2n}} & Z'
\end{array}$$

$$\begin{array}{ccccccc}
X'\otimes(\mathcal{I}\otimes B\otimes X')^{\otimes n} & \xrightarrow{s'\otimes(1\otimes r\otimes s')^{\otimes n}} & Z'\otimes(\mathcal{I}\otimes Z\otimes Z')^{\otimes n} & \xrightarrow{1\otimes(\sigma_Z\otimes 1)^{\otimes n}} & Z'^{\otimes(2n+1)} & \xrightarrow{\mu_{Z'}{}^{2n}} & Z' \\
\Big\downarrow{\scriptstyle \pi_X\otimes(p\otimes\pi_X)^{\otimes n}} & & \Big\downarrow{\scriptstyle \pi_Z\otimes(p\otimes\pi_Z)^{\otimes n}} & & \Big\downarrow{\scriptstyle \pi_Z{}^{\otimes(2n+1)}} & & \Big\downarrow{\scriptstyle \pi_Z} \\
X\otimes(B\otimes X)^{\otimes n} & \xrightarrow{s\otimes(r\otimes s)^{\otimes n}} & Z\otimes(Z\otimes Z)^{\otimes n} & \xrightarrow{\cong} & Z^{\otimes(2n+1)} & \xrightarrow{\mu_Z{}^{2n}} & Z
\end{array}$$

which commute by the naturality of the diagonal approximation, by the relations $\sigma_Z \circ (1 \otimes s) = s' \circ \sigma_X$, $\pi_Z \circ s' = s \circ \pi_X$, $\pi_Z \circ \sigma_Z = p$, and since σ_Z respects the multiplication and π_Z is an algebra morphism.

For uniqueness, suppose $t''\colon \mathbf{Y}' \to \mathbf{Z}'$ is another morphism satisfying the required relations. Then $t'' \circ \overline{g}^{\mathrm{T}} = t'' \circ \sigma_Y^{\mathrm{T}} \circ F(1 \otimes \overline{f}) = \sigma_Z^{\mathrm{T}} \circ F(1 \otimes t) \circ F(1 \otimes \overline{f})$ and $t'' \circ i' = s'$, so $t'' = t'$ by the universal property of the pushout. □

2 Cofibrations and cylinders in CrsAlg

We first recall the notion of a crossed complex homomorphism of *relative free type* [2]. Let E^r be the free crossed complex on one generator in dimension r, and let $x_r: S^{r-1} \to E^r$ be the inclusion into E^r of its $(r-1)$-truncation. We write $\mathcal{Z}$ for the class of arbitrary coproducts of the homomorphisms x_r. Then a homomorphism $k: C \to D$ in **Crs** is said to be of relative free type if there exists a sequence of pushouts

$$\begin{array}{ccc} A_n & \xrightarrow{\;z_n\;} & B_n \\ {\scriptstyle y_n}\downarrow & & \downarrow \\ D_n & \xrightarrow{\;\overline{z_n}\;} & D_{n+1} \end{array}$$

for $n \geq 0$, with $D_0 = C$, y_n arbitrary, $z_n \in \mathcal{Z}$, such that k is given by the canonical homomorphism

$$C \longrightarrow \operatorname{colim}\left(D_0 \xrightarrow{\overline{z_0}} D_1 \xrightarrow{\overline{z_1}} D_2 \xrightarrow{\overline{z_2}} D_3 \longrightarrow \cdots \right)$$

A crossed complex D is termed *free* if the homomorphism $\emptyset \to D$ is of relative free type.

We define a cofibration in **CrsAlg** to be any transfinite composite of pushouts of morphisms of the form Fg for g of relatively free type. Clearly the class of cofibrations is closed under pushouts, composition and isomorphism. For $\mathbf{X}$ an arbitrary crossed chain algebra, we will write $\mathbf{X}/\mathbf{CrsAlg}_c$ for the category with objects the cofibrations with domain $\mathbf{X}$ and with arrows $i \to i'$ the algebra morphisms j which satisfy $j \circ i = i'$. An arrow of $\mathbf{X}/\mathbf{CrsAlg}_c$ is termed a cofibration if the underlying algebra morphism is a cofibration.

A crossed chain algebra $\mathbf{X}$ is termed *cofibrant* if the unique morphism $* \to \mathbf{X}$ is a cofibration. Note that $*/\mathbf{CrsAlg}_c$ is just the full subcategory of **CrsAlg** on the cofibrant objects.

Suppose $i: \mathbf{X} \to \mathbf{Y}$ is a cofibration in **CrsAlg** given by a sequence of pushouts $\psi = (\psi_\kappa)_{\kappa\in\lambda}$ for some infinite regular cardinal λ, as follows:

$$\begin{array}{ccc} FA_\kappa & \xrightarrow{\;Fk_\kappa\;} & FB_\kappa \\ {\scriptstyle f_\kappa^{\mathrm{T}}}\downarrow & \psi_\kappa & \downarrow{\scriptstyle \overline{f_\kappa}^{\mathrm{T}}} \\ \mathbf{Y}_\kappa & \xrightarrow{\;i_\kappa\;} & \mathbf{Y}_{\kappa+1} \end{array}$$

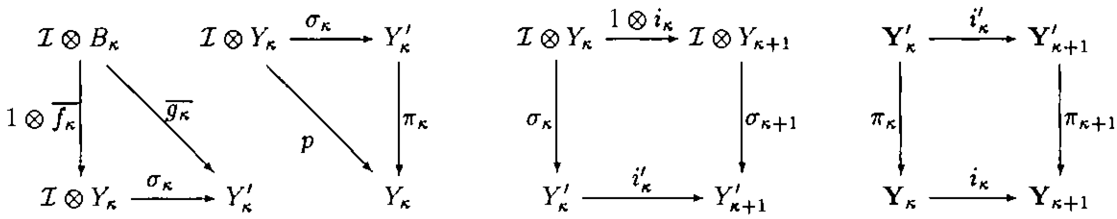
where $\mathbf{Y}_0 = \mathbf{X}$, each k_κ is a homomorphism of relative free type, and i is the canonical morphism $\mathbf{X} \to \varinjlim_{\lambda} Y_\kappa$.

Suppose also we are given a crossed chain algebra $\mathbf{X}'$ together with a homotopy $\sigma_X: \mathcal{I} \otimes X \to X'$ which respects the multiplication and a morphism $\pi_X: \mathbf{X}' \to \mathbf{X}$ satisfying $\pi_X \circ \sigma_X = p$. We use transfinite induction to define for each ordinal $\kappa \in \lambda$ a pushout

$$\begin{array}{ccc} F(\mathcal{I} \otimes A_\kappa) & \xrightarrow{\;F(1\otimes k_\kappa)\;} & F(\mathcal{I} \otimes B_\kappa) \\ {\scriptstyle g_\kappa^{\mathrm{T}}}\downarrow & \psi'_\kappa & \downarrow{\scriptstyle \overline{g_\kappa}^{\mathrm{T}}} \\ \mathbf{Y}'_\kappa & \xrightarrow{\;i'_\kappa\;} & \mathbf{Y}'_{\kappa+1} \end{array}$$

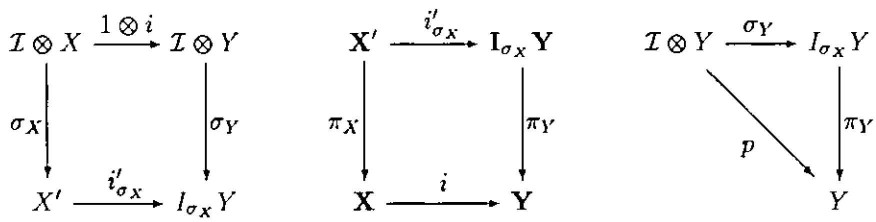
together with a homotopy $\sigma_\kappa: \mathcal{I} \otimes Y_\kappa \to Y'_\kappa$ which respects the multiplication and a morphism

$\pi_\kappa\colon \mathbf{Y}'_\kappa \to \mathbf{Y}_\kappa$ which make the following diagrams commute:



Let $\sigma_0 = \sigma_X$ and $\pi_0 = \pi_X$. Having defined σ_κ and π_κ, we give ψ'_κ by putting $g_\kappa = \sigma_\kappa \circ (1 \otimes f_\kappa)$ and then $\sigma_{\kappa+1}$ and $\pi_{\kappa+1}$ are defined by applying proposition 1(a) to ψ_κ, ψ'_κ. For a limit ordinal $\kappa \leq \lambda$, σ_κ and π_κ are those induced by the $\sigma_{\kappa'}$ and $\pi_{\kappa'}$ for $\kappa' \in \kappa$.

From [2] we know that homomorphisms of relative free type are closed under tensoring with free objects and in particular with $\mathcal{I}$. Thus $\psi' = (\psi'_\kappa)_{\kappa\in\lambda}$ generates a cofibration in **CrsAlg**, termed the relative cylinder on (i, σ_X) and written $i'_{\sigma_X}\colon \mathbf{X}' \to \mathbf{I}_{\sigma_X}\mathbf{Y}$. Note that the construction respects the identity and composition of cofibrations. Also we have a homotopy $\sigma_Y = \sigma_\lambda\colon \mathcal{I} \otimes Y \to I_{\sigma_X} Y$ which respects the multiplication and a morphism $\pi_Y = \pi_\lambda\colon \mathbf{I}_{\sigma_X}\mathbf{Y} \to \mathbf{Y}$ such that the following diagrams commute:

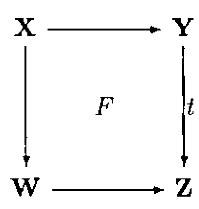


These are termed the *shift* and *projection* maps respectively.

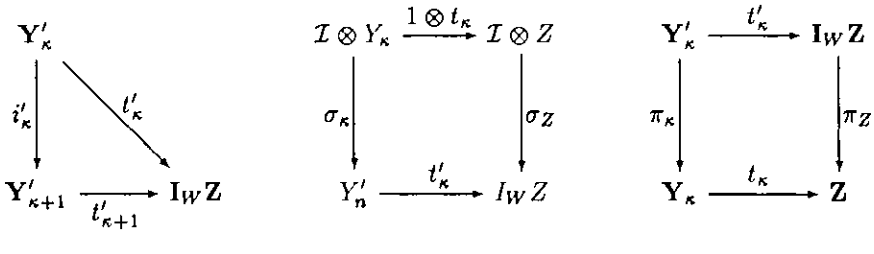
In the special case $\mathbf{X}' = \mathbf{X}$, with $\sigma_X\colon \mathcal{I} \otimes X \to X$ and $\pi_X\colon \mathbf{X} \to \mathbf{X}$ given by p and the identity respectively, the cofibration i'_{σ_X} is termed the relative cylinder on i written $i'\colon \mathbf{X} \to \mathbf{I}_X\mathbf{Y}$.

Let $\alpha_0, \alpha_1\colon Y \to \mathcal{I} \otimes Y$ be the homomorphisms given by the two inclusions $* \to \mathcal{I}$. Note that the homomorphisms $\alpha_r{}^{\otimes n}$ may be written as $d^{(n)} \circ \alpha_r\colon Y^{\otimes n} \to \mathcal{I} \otimes Y^{\otimes n} \to (\mathcal{I} \otimes Y)^{\otimes n}$ and that $p \circ \alpha_r = 1_Y$ for $r = 0, 1$. It follows that composing α_0, α_1 with the shift map gives morphisms of crossed chain algebras $\iota_0, \iota_1\colon \mathbf{Y} \to \mathbf{I}_X\mathbf{Y}$ such that $\pi_Y \circ \iota_r$ is the identity on $\mathbf{Y}$ for $r = 0, 1$.

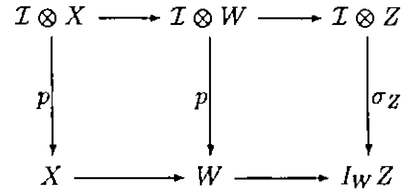
Suppose we have another cofibration $\mathbf{W} \to \mathbf{Z}$ and a commutative diagram F as below.



To define the relative cylinder on the morphism of cofibrations F, write t_κ for the composite $\mathbf{Y}_\kappa \to \mathbf{Y} \to \mathbf{Z}$ and note that $t_\kappa = t_{\kappa+1} \circ i_\kappa$ for each ordinal $\kappa \in \lambda$. We use transfinite induction to define morphisms $t'_\kappa\colon \mathbf{Y}'_\kappa \to \mathbf{I}_W\mathbf{Z}$, $\kappa \leq \lambda$, which satisfy

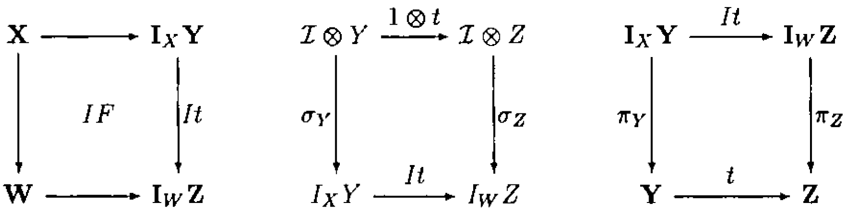


Let t'_0 be the composite $\mathbf{X} \to \mathbf{W} \to \mathbf{I}_W\mathbf{Z}$; the relation $t'_0 \circ \sigma_X = \sigma_Z \circ (1 \otimes t_0)$ follows from the diagram



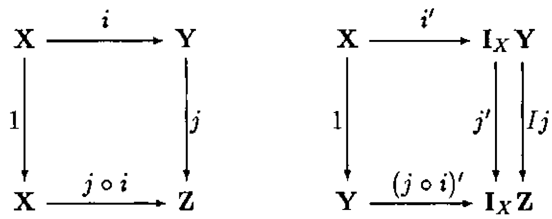
together with F. Having defined t'_κ for $\kappa \in \lambda$, we let $t'_{\kappa+1}$ be the unique morphism satisfying the required relations given by proposition 1(b). For $\kappa \leq \lambda$ a limit ordinal, t'_κ is that induced by the $t'_{\kappa'}$ for $\kappa' \in \kappa$.

Writing It for t_λ, we have a morphism of cofibrations IF satisfying $It \circ \sigma_Y = \sigma_Z \circ (1 \otimes t)$ and $\pi_Z \circ It = t \circ \pi_Y$:

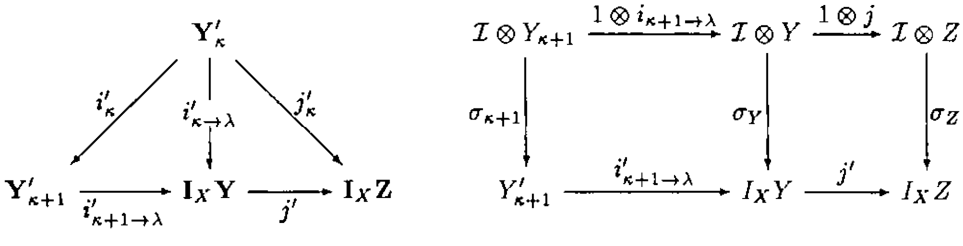


In certain situations the above constructions coincide:

Proposition 2 *Suppose $i: \mathbf{X} \to \mathbf{Y}$, $j: \mathbf{Y} \to \mathbf{Z}$ are cofibrations and let i', $(j \circ i)'$ be the relative cylinders on i, $j \circ i$ with corresponding shift maps σ_Y, σ_Z. Then the morphism Ij obtained from $(1, j)$ regarded as a morphism of cofibrations $i \to (j \circ i)$ is itself a cofibration, given by the relative cylinder j' on (j, σ_Y).*



Proof: Suppose i is given by the sequence of pushouts $(\psi_\kappa)_{\kappa \in \lambda}$ as above, and let $i_{\kappa\to\lambda}$, $i'_{\kappa\to\lambda}$ be the canonical morphisms $\mathbf{Y}_\kappa \to \mathbf{Y}$, $\mathbf{Y}'_\kappa \to \mathbf{I}_X\mathbf{Y}$ for each ordinal $\kappa \leq \lambda$. Then the morphisms $j'_\kappa: \mathbf{Y}'_\kappa \to \mathbf{I}_X\mathbf{Z}$ in the construction of Ij are defined via proposition 1(b) from the morphisms $j_\kappa = j \circ i_{\kappa\to\lambda}: \mathbf{Y}_\kappa \to \mathbf{Z}$. We will show by transfinite induction that $j'_\kappa = j' \circ i'_{\kappa\to\lambda}$ for all $\kappa \leq \lambda$. For $\kappa = 0$ this is just $(j \circ i)' = j' \circ i'$. If the result holds for $\kappa \in \lambda$ then the following diagrams commute:

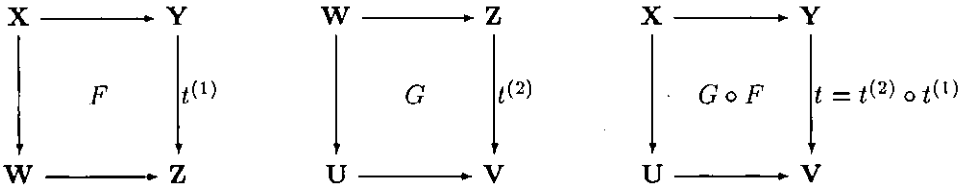


(as does the corresponding diagram for π instead of σ) and so $j'_{\kappa+1} = j' \circ i'_{\kappa+1\to\lambda}$ by the uniqueness part of proposition 1(b). For a limit ordinal $\kappa \leq \lambda$ the result follows from the result for all $\kappa' \in \kappa$. □

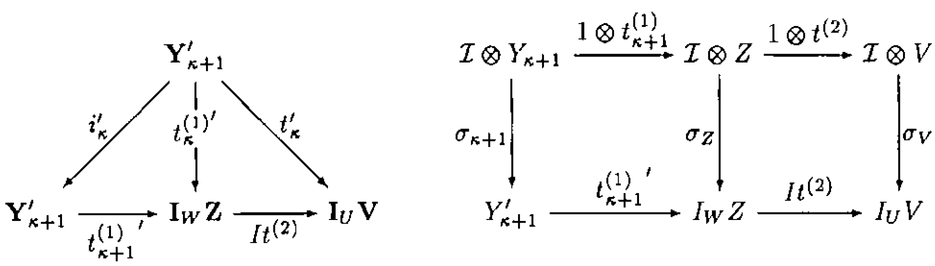
Proposition 3 *The relative cylinder construction is functorial.*

Proof: Putting $j = 1$ in the previous proposition gives $Ij = j' = 1$, so I preserves identity morphisms between cofibrations.

Suppose given cofibrations $\mathbf{X} \to \mathbf{Y}$, $\mathbf{W} \to \mathbf{Z}$, $\mathbf{U} \to \mathbf{V}$ and morphisms of cofibrations F, G, $G \circ F$:



Suppose $\mathbf{X} \to \mathbf{Y}$ is given by $(\psi_\kappa)_{\kappa \in \lambda}$ and write $t^{(1)}_\kappa = t^{(1)} \circ i_{\kappa\to\lambda}$, $t_\kappa = t \circ i_{\kappa\to\lambda}$ as usual. Note that $t^{(2)} \circ t^{(1)}_\kappa = t_\kappa$ for each ordinal $\kappa \leq \lambda$. We show inductively $It^{(2)} \circ t^{(1)'}_\kappa = t'_\kappa$ for all $\kappa \leq \lambda$, which gives $IG \circ IF = I(G \circ F)$. For $\kappa = 0$, this follows from the commutativity of the diagram IG. If the result holds for $\kappa \in \lambda$, we have the following commutative diagrams:



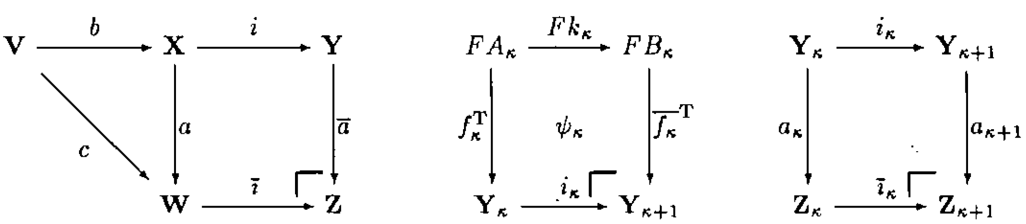
plus a corresponding diagram for π. Thus the result holds for $\kappa + 1$ by the uniqueness in the definition of $t'_{\kappa+1}$. The result for a limit ordinal $\kappa \leq \lambda$ follows from the result for all $\kappa' \in \kappa$ as usual. □

Corollary 4 *The relative cylinder* $\mathbf{X} \to \mathbf{I}_X\mathbf{Y}$ *on a cofibration* $\mathbf{X} \to \mathbf{Y}$ *is well defined up to isomorphism.* □

Let Cof(**CrsAlg**) be the category whose objects are cofibrations of crossed chain algebras and whose morphisms are commutative diagrams F as above. Then we have a relative cylinder functor I on Cof(**CrsAlg**), together with natural transformations $\iota_0, \iota_1 : \mathrm{id} \to I$ and $\pi : I \to \mathrm{id}$ such that $\pi \circ \iota_r = 1$ for $r = 0, 1$. In particular, we have I, ι_r, π on $\mathbf{V}/\mathbf{CrsAlg_c}$, for any algebra $\mathbf{V}$, and by proposition 2 I takes cofibrations to cofibrations in $\mathbf{V}/\mathbf{CrsAlg_c}$.

Proposition 5 *The relative cylinder functor preserves pushouts of cofibrations in* $\mathbf{V}/\mathbf{CrsAlg_c}$.

Proof: Given an arrow $a: b \to c$ and a cofibration $i: b \to i \circ b$, the pushout of i along a in $\mathbf{V}/\mathbf{CrsAlg_c}$ is given by $\bar{\imath}: c \to \bar{\imath} \circ c$ where $\bar{\imath}$ is the pushout of i along a in **CrsAlg**. Suppose i is given by the sequence of pushouts $(\psi_\kappa)_{\kappa\in\lambda}$ as usual, and define morphisms $a_\kappa: \mathbf{Y}_\kappa \to \mathbf{Z}_\kappa$ inductively by setting $a_{\kappa+1}$ to be the pushout of a_κ along i_κ, with $a_0 = a$.



Then $\bar{\imath}$ is the composite of the pushouts of Fk_κ along $a_\kappa \circ f_\kappa^{\mathrm{T}}$, and so by proposition 2 Ii and $I\bar{\imath}$ are given by sequences of pushouts ϕ and φ as below.

$$\begin{array}{ccc} F(\mathcal{I}\otimes A_\kappa) & \xrightarrow{F(1\otimes k_\kappa)} & F(\mathcal{I}\otimes B_\kappa) \\ \downarrow{\scriptstyle (\sigma_{Y_\kappa}\circ 1\otimes f_\kappa)^{\mathrm{T}}} & \phi_\kappa & \downarrow{\scriptstyle (\sigma_{Y_{\kappa+1}}\circ 1\otimes \overline{f_\kappa})^{\mathrm{T}}} \\ \mathbf{Y}'_\kappa & \xrightarrow{Ii_\kappa} & \mathbf{Y}'_{\kappa+1} \end{array} \qquad \begin{array}{ccc} F(\mathcal{I}\otimes A_\kappa) & \xrightarrow{F(1\otimes k_\kappa)} & F(\mathcal{I}\otimes B_\kappa) \\ \downarrow{\scriptstyle (\sigma_{Z_\kappa}\circ 1\otimes (a_\kappa\circ f_\kappa))^{\mathrm{T}}} & \varphi_\kappa & \downarrow{\scriptstyle (\sigma_{Z_{\kappa+1}}\circ 1\otimes (a_{\kappa+1}\circ\overline{f_\kappa}))^{\mathrm{T}}} \\ \mathbf{Z}'_\kappa & \xrightarrow{I\bar{\imath}_\kappa} & \mathbf{Z}'_{\kappa+1} \end{array}$$

Now consider the following diagrams for $\kappa \in \lambda$:

$$\begin{array}{ccc} F(\mathcal{I}\otimes A_\kappa) & \xrightarrow{F(1\otimes k_\kappa)} & F(\mathcal{I}\otimes B_\kappa) \\ {\scriptstyle (\sigma_{Y_\kappa}\circ 1\otimes f_\kappa)^{\mathrm{T}}}\downarrow & & \downarrow{\scriptstyle (\sigma_{Y_{\kappa+1}}\circ 1\otimes \overline{f_\kappa})^{\mathrm{T}}} \\ \mathbf{Y}'_\kappa & \xrightarrow{Ii_\kappa} & \mathbf{Y}'_{\kappa+1} \\ {\scriptstyle Ia_\kappa}\downarrow & & \downarrow{\scriptstyle Ia_{\kappa+1}} \\ \mathbf{Z}'_\kappa & \xrightarrow{I\bar{\imath}_\kappa} & \mathbf{Z}'_{\kappa+1} \end{array}$$

From the relations $Ia_\kappa \circ \sigma_{Y_\kappa} = \sigma_{Z_\kappa} \circ (1 \otimes a_\kappa)$ the outer rectangles are just the pushouts φ_κ, and the upper squares are the pushouts ϕ_κ. Thus the lower squares are also pushout squares and we are done. □

Given a cofibration $i: \mathbf{X} \to \mathbf{Y}$ as usual, and homotopies $\sigma_X: \mathcal{I} \otimes X \to X'$, $\sigma'_X: \mathcal{I} \otimes X' \to X''$ which respect the multiplication, we can define the *double relative cylinder* on (i, σ_X, σ'_X) as the relative cylinder on $(i'_{\sigma_X}, \sigma'_X)$. As a cofibration this is given by the sequence of pushouts ψ'' as follows:

$$\begin{array}{ccc} F(\mathcal{I}\otimes\mathcal{I}\otimes A_\kappa) & \xrightarrow{F(1\otimes 1\otimes k_\kappa)} & F(\mathcal{I}\otimes\mathcal{I}\otimes B_\kappa) \\ {\scriptstyle (\sigma^2_\kappa\circ 1\otimes 1\otimes f_\kappa)^{\mathrm{T}}}\downarrow & \psi''_\kappa & \downarrow{\scriptstyle (\sigma^2_{\kappa+1}\circ 1\otimes 1\otimes \overline{f_\kappa}^{\mathrm{T}}} \\ \mathbf{Y}''_\kappa & \xrightarrow{i''_\kappa} & \mathbf{Y}''_{\kappa+1} \end{array}$$

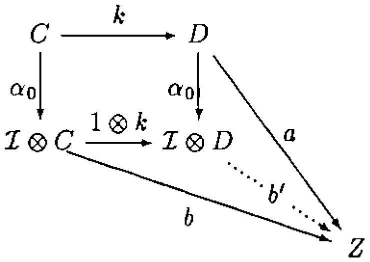
where $\sigma^2_\kappa = \sigma'_\kappa \circ (1 \otimes \sigma_\kappa)$. If $t: \mathcal{I} \otimes \mathcal{I} \cong \mathcal{I} \otimes \mathcal{I}$ is given by the symmetry of $\otimes$, and τ_X is an endomorphism of $\mathbf{X}''$ satisfying $\sigma^2_0 \circ (\tau \otimes 1) = \tau_0 \circ \sigma^2_0$, we can define inductively from t, τ_X endomorphisms τ_κ of $\mathbf{Y}''_\kappa$ satisfying $\sigma^2_\kappa \circ (\tau \otimes 1) = \tau_\kappa \circ \sigma^2_\kappa$.

$$\begin{array}{ccc} \mathcal{I}\otimes\mathcal{I}\otimes Y_\kappa & \xrightarrow{\tau\otimes 1} & \mathcal{I}\otimes\mathcal{I}\otimes Y_\kappa \\ {\scriptstyle \sigma^2_\kappa}\downarrow & & \downarrow{\scriptstyle \sigma^2_\kappa} \\ Y''_\kappa & \xrightarrow{\tau_\kappa} & Y''_\kappa \end{array}$$

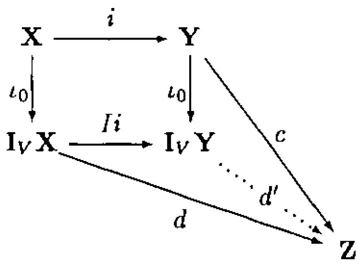
The induced morphism τ_Y on $\mathbf{I}_{\sigma'_X}\mathbf{I}_{\sigma_X}\mathbf{Y}$ is termed the *interchange* map, and taking $\sigma_X = \sigma'_X = 1_X$ we get a natural transformation $\tau: II \to II$ in $\mathbf{X}/\mathbf{CrsAlg}_c$. Since the morphisms $\iota_r, I\iota_r: \mathbf{I}_X\mathbf{Y} \to \mathbf{I}^2_X\mathbf{Y}$ can be defined via the homomorphisms $* \otimes \mathcal{I} \otimes Y_\kappa \to \mathcal{I} \otimes \mathcal{I} \otimes Y_\kappa \to Y''_\kappa$, $\mathcal{I} \otimes * \otimes Y_\kappa \to \mathcal{I} \otimes \mathcal{I} \otimes Y_\kappa \to Y''_\kappa$ respectively, we have $\tau \circ I\iota_r = \iota_r$ and $\tau \circ \iota_r = I\iota_r$ for $r = 0, 1$.

3 The homotopy extension property in Crs and CrsAlg

A homomorphism $k: C \to D$ is said to have the homotopy extension property (HEP) in **Crs** if, given homomorphisms $a: D \to Z$, $b: \mathcal{I} \otimes C \to Z$ such that $b \circ \alpha_0 = a \circ k$, there exists a homomorphism $b': \mathcal{I} \otimes D \to Z$ satisfying $b' \circ (1 \otimes k) = b$ and $b' \circ \alpha_0 = a$.



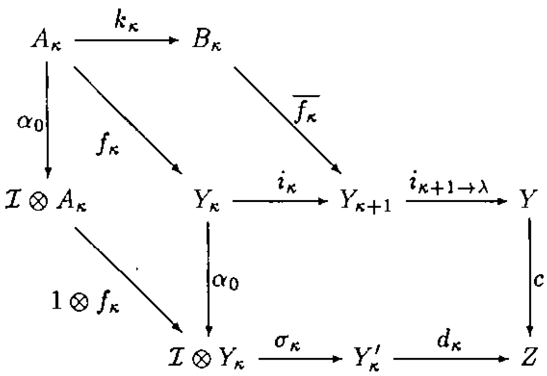
Suppose $x: \mathbf{V} \to \mathbf{X}$, $y: \mathbf{V} \to \mathbf{Y}$ are cofibrations in **CrsAlg**, and write x', y' for the corresponding relative cylinders. Then an arrow $i: x \to y$ is said to have the homotopy extension property (HEP) in $\mathbf{V}/\mathbf{CrsAlg}_c$ if, given arrows $c: y \to z$, $d: x' \to z$ such that $d \circ \iota_0 = c \circ i$, there exists an arrow $d': y' \to z$ satisfying $d' \circ Ii = d$ and $d' \circ \iota_0 = c$.

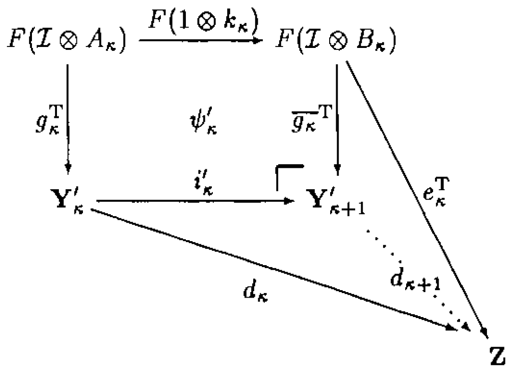


Note that by the symmetry of $\mathcal{I}$, it is equivalent to use α_1, ι_1 instead of α_0, ι_0 in the above definitions.

Proposition 6 *Suppose every homomorphism of relative free type has the HEP in* **Crs**. *Then all cofibrations have the HEP in* $\mathbf{V}/\mathbf{CrsAlg}_c$.

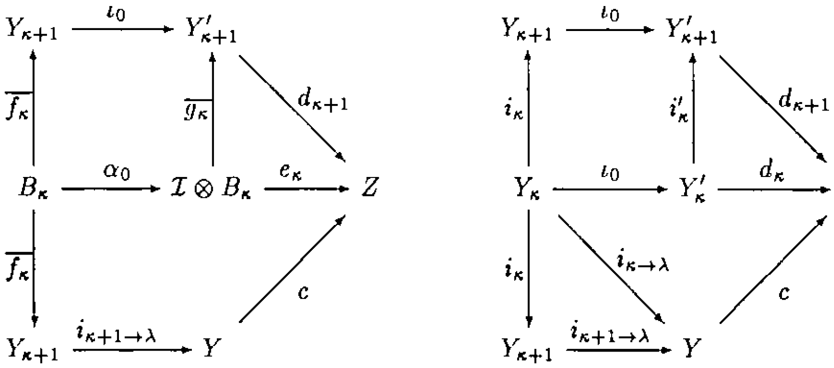
Proof: Suppose i, c, d are as above, with i a cofibration given by a pushout sequence ψ as usual. We define inductively morphisms $d_\kappa: \mathbf{Y}'_\kappa \to \mathbf{Z}$ satisfying $d_\kappa \circ \iota_0 = c \circ i_{\kappa \to \lambda}$ and $d_\kappa = d_{\kappa+1} \circ i'_\kappa$. Let $d_0 = d$. Given d_κ, we define a homomorphism $e_\kappa: \mathcal{I} \otimes B_\kappa \to Z$ by the HEP for k_κ and hence a morphism $d_{\kappa+1}: \mathbf{Y}'_{\kappa+1} \to \mathbf{Z}$ by the definition of $\mathbf{Y}'_\kappa$ as a pushout:





It remains to show that $d_{\kappa+1} \circ \iota_0 = c \circ i_{\kappa+1 \to \lambda}$. On precomposing each side with $\overline{f_\kappa}$ and with i_κ

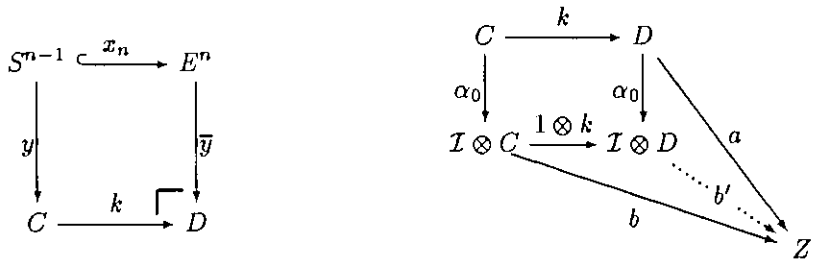
we have the following commutative diagrams:



Thus the result follows. □

Proposition 7 *All homomorphisms of relative free type have the HEP in* **Crs**.

Proof: We prove the result for the addition of a single generator e_n to a crossed complex C.



For $n = 0$ we have $S^{n-1} = \emptyset$, so b' may be defined via $b\colon \mathcal{I}\otimes C \to Z$ and $a\circ\overline{y}\circ p\colon \mathcal{I}\otimes E^0 \to E^0 \to D \to Z$.

For $n = 1$, let $s(e^1), t(e^1) \in C_0$ be the images under y of $0, 1 \in S^0$. Then $\mathcal{I}\otimes D$ is the crossed complex with generators $0\otimes e^1$, $1\otimes e^1$, $e^1\otimes e^1$ and $i\otimes c \in \mathcal{I}\otimes C$, subject to the following relations:

$$\begin{aligned}
s(r\otimes e^1) &= r\otimes se^1\\
t(r\otimes e^1) &= r\otimes te^1\\
t(e^1\otimes e^1) &= 1\otimes te^1\\
\delta_2(e^1\otimes e^1) &= (1\otimes e^1)^{-1}\circ(e^1\otimes se^1)^{-1}\circ 0\otimes e^1\circ e^1\otimes te^1
\end{aligned}$$

for $r = 0, 1$, together with the standard relations in $\mathcal{I}\otimes C$ [4]. Note that $sa(e^1) = b(0\otimes se^1) = sb(e^1\otimes se^1)$ and $ta(e^1) = sb(e^1\otimes te^1)$ similarly. Then b' is defined on the generators as follows:

$$\begin{aligned}
b'(0\otimes e^1) &= a(e^1)\\
b'(1\otimes e^1) &= b(e^1\otimes se^1)^{-1}\circ a(e^1)\circ b(e^1\otimes te^1)\\
b'(e^1\otimes e^1) &= \mathrm{id}_{b(1\otimes te^1)}\\
b'(i\otimes c) &= b(i\otimes c)
\end{aligned}$$

The boundary relations are clear, as are $b'\circ\alpha_0 = a$ and $b'\circ k = b$.

For $n = 2$, let $s^1 \in C_1$ be the image under y of the generator of S^1, and write e^0 for $t(s^1)$. Then $\mathcal{I}\otimes D$ is the crossed complex with generators $0\otimes e^2$, $1\otimes e^2$, $e^1\otimes e^2$ and $i\otimes c\in\mathcal{I}\otimes C$, subject to the relations

$$\begin{aligned}
t(r\otimes e^2) &= r\otimes e^0\\
\delta_2(r\otimes e^2) &= r\otimes s^1\\
t(e^1\otimes e^2) &= 1\otimes e^0\\
\delta_3(e^1\otimes e^2) &= (1\otimes e^2)^{-1}\circ(0\otimes e^2)^{e^1\otimes e^0}\circ(e^1\otimes s^1)^{-1}
\end{aligned}$$

for $r = 0, 1$, plus the relations in $\mathcal{I} \otimes C$. Then b' is defined by:

$$\begin{aligned} b'(0 \otimes e^2) &= a(e^2) \\ b'(1 \otimes e^2) &= a(e^2)^{b(e^1 \otimes e^0)} \circ b(e^1 \otimes s^1)^{-1} \\ b'(e^1 \otimes e^2) &= \mathrm{id}_{b(1 \otimes e^0)} \\ b'(i \otimes c) &= b(i \otimes c) \end{aligned}$$

The boundary relation $\delta_2 b'(1 \otimes e^2) = b'\delta_2(1 \otimes e_2)$ follows from

$$\delta_2 b(e^1 \otimes s^1) = b((1 \otimes s^1)^{-1} \circ (e^1 \otimes e^0)^{-1} \circ 0 \otimes s^1 \circ e^1 \otimes e^0) = b(1 \otimes s^1)^{-1} \circ \delta_2(a(e^2)^{b(e^1 \otimes e^0)})$$

since $\delta_2 a(e^2) = a(\delta_2 e^2) = b(0 \otimes s^1)$. The other relations are clear.

For $n \geq 3$ the constructions differ from the $n = 2$ case only in very minor ways. For example, the triviality of $\delta_{n-1} s^{n-1}$ is used instead of $ss^1 = ts^1 = e^0$ in showing $\delta_n b(e^1 \otimes s^{(n-1)}) = b(1 \otimes s^{(n-1)})^{-1} \circ \delta_n(a(e^n)^{b(e^1 \otimes e^0)})$. □

A similar technique can be used to prove the relative cylinder axiom for crossed chain algebras. First we need the fact that for any homomorphism $k: C \to D$, the canonical homomorphism

$$D \amalg_C (\mathcal{I} \otimes C) \amalg_C D \longrightarrow \mathcal{I} \otimes D$$

is also of relative free type; this is quite straightforward. Now suppose $i: x \to y$ is a cofibration in $\mathbf{V}/\mathbf{CrsAlg_c}$ given by a pushout sequence ψ as usual. Then the canonical morphism

$$\mathbf{Y} \amalg_X \mathbf{I}_V \mathbf{X} \amalg_X \mathbf{Y} \longrightarrow \mathbf{I}_V \mathbf{Y}$$

is a cofibration also, given by the pushout sequence ϕ as below.

$$\begin{array}{ccc} F(B_\kappa \amalg_{A_\kappa} (\mathcal{I} \otimes A_\kappa) \amalg_{A_\kappa} B_\kappa) & \longrightarrow & F(\mathcal{I} \otimes B_\kappa) \\ \downarrow & \phi_\kappa & \downarrow \\ \mathbf{Y} \amalg_{Y_\kappa} \mathbf{Y}'_\kappa \amalg_{Y_\kappa} \mathbf{Y} & \longrightarrow & \mathbf{Y} \amalg_{Y_{\kappa+1}} \mathbf{Y}'_{\kappa+1} \amalg_{Y_{\kappa+1}} \mathbf{Y} \end{array}$$

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