Moduli Spaces of Harmonic and Holomorphic Mappings and the Diophantus Geometry

by

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Introduction. In this paper we first show some new results on the structure of the moduli space of harmonic (resp. holomorphic) mappings into a Riemannian (resp. Kähler) manifold with non–positive sectional curvature, and then, applying these results, give a survey on recent developments in the theory of the Diophantus geometry. Let N be a compact analytic Riemannian manifold with non–positive sectional curvatures and M a compact Riemannian manifold. We denote by $\mathrm{Harm}\,(M, N)$ the space of all harmonic mappings of M into N endowed with compact–open topology, which is called the *moduli space of harmonic mappings* of M into N. Let $f\colon M \longrightarrow N$ be a smooth mapping and put

$$\mathrm{Harm}(M, N; f) = \{g \in \mathrm{Harm}(M, N),\ g \sim f(\text{homotopic})\}\ .$$

Then Schoen–Yau [SY] proved that $\mathrm{Harm}(M, N; f)$ carries a structure of a compact Riemannian manifold such that the evaluation mapping

$$\Phi_{1p} : g \in \mathrm{Harm}(M, N; f) \longrightarrow g\,(p) \in N$$

with an arbitrarily fixed point $p \in M$ is an isometric immersion onto a totally geodesic submanifold of N (see also $[S_2]$ for the case of locally symmetric N). We put $X = \mathrm{Harm}(M, N; f)$ and consider X as a domain and M as a parameter space. That is, we put

$$Y = \mathrm{Harm}(M, N; \Phi_{1,p})\ ,$$

$$\Phi_2 : (x, y) \in X \times Y \longrightarrow y(x) \in N\,.$$

These Y and $\Phi_2(x, \cdot\ )$ have properties similar to X and Φ_{1p}. Naturally, we have a smooth mapping $p \in M \longrightarrow \Phi_{1p} \in Y$ and the following commutative diagram:

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$$\begin{tikzcd}
p \in M \arrow[r, "x"] & x(p) \arrow[d, equal] \\
(x,\ p) \in X \times M \arrow[u] \arrow[r] \arrow[d] & \Phi_{1p}(x) \in N \arrow[d, equal] \\
(x,\ \Phi_{1p}) \in X \times Y \arrow[r, "\Phi_2"'] & \Phi_2(x,\ \Phi_{1p})
\end{tikzcd}$$

Our first theorem is

Theorem (1.14). *Assume that the groups of isometries of* X *and* Y *are finite (in particular, the Ricci curvature of* N *is negative,* $\dim X \geq 2$, *and* $\dim Y \geq 2$*) . Then the second evaluation mapping*

$$\Phi_2 : (x,\ y) \in X \times Y \longrightarrow y\,(\mathrm{x}) \in N$$

is an immersion onto a totally geodesic submanifold of N *and the pull–backed metric by* Φ_2 *is isometric to the product metric on* $X \times Y$ *.*

In the complex category, we deal with the case where N is not necessarily compact, but a complete Kähler manifold such that the (Riemannian) sectional curvatures are non–positive and the holomorphic sectional curvatures are bounded from above by a negative constant. Moreover, we assume that N is a Zariski open subset of a complex projective variety $\overline{N}$ such that N is hyperbolically imbedded into $\overline{N}$. Let M be a Zariski open subset of a compact Kähler manifold $\overline{M}$. We denote by $\mathrm{Hol}(M,\ N)$ the moduli space of all holomorphic mappings of M into N endowed with compact–open topology. By [N_3] $\mathrm{Hol}(M,\ N)$ has a structure of a Zariski open subset of a compact complex space, and non–singular. Let $X \subset \mathrm{Hol}(M,\ N)$ be a connected component and put

$$\Phi_1 : (x,\ p) \in X \times M \longrightarrow x\,(p) \in N\,.$$

Let $Y \subset \mathrm{Hol}(X,\ N)$ be the connected component containing $\Phi_1(\ \cdot\ ,\ p)(p \in M)$. Then we have

Theorem (2.15). *The second evaluation mapping*

$$\Phi_2 : (x,\ y) \in X \times Y \longrightarrow y(x) \in N$$

is a proper holomorphic immersion onto a complex totally geodesic submanifold of N *, and the pull–backed metric by* Φ_2 *is isometric to the product metric on* $X \times Y$ *.*

It is the most important and interesting case when N is a quotient $\Gamma \backslash \mathbb{D}$ of a bounded symmetric domain $\mathbb{D}$ by an arithmetic discrete subgroup Γ of the holomorphic automorphism group $\mathrm{Aut}(\mathbb{D})$.

In the course of the proof of the above theorem we show

Theorem (2.4). *Let S be a complete hyperbolic manifold such that S is a Zariski open subset of a compact complex space $\bar{S}$ and S is hyperbolically imbedded into $\bar{S}$. Then* $\mathrm{Aut}(S)$ *is a finite group.*

The first and the second sections are devoted to the proof of the above results, and to the preparations for the latter sections.

In § 3 we discuss the higher dimensional Mordell's conjecture over function fields and related topics. In § 4 we deal with the Parshin–Arakelov–type theorems for curves, Abelian varieties and K3–surfaces . We will give a new proof to the Parshin–Arakelov theorem for curves, based on our results.

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§ 1. Moduli space of harmonic mappings

Let M and N be Riemannian manifolds and TM (resp. $T^{*}M$) denote the tangent (resp. cotangent) bundle over M. Let $f: M \longrightarrow N$ be a C^{∞}–mapping . The energy functional $E(f)$ is defined by

$$E(f) = \int_M < df, df > dV_M .$$

We call f a *harmonic* mapping if for any smooth family $f_t : M \longrightarrow N$, $-1 \le t \le 1$, of C^∞–mappings such that $f_0 = f$ on M and $f = f_t$, $-1 \le t \le 1$, outside a compact subset $K \subset M$,

$$\frac{d}{dt}\Big|_{t=0} \int_K < df_t, df_t > dV_M = 0 .$$

It is known that f is harmonic if and only if

$$\text{(1.1)} \qquad \operatorname{trace} {}^{T^*M \otimes f^{-1}TN}\nabla\, df = 0 ,$$

where ${}^{T^*M \otimes f^{-1}TN}\nabla$ is the Riemannian connection of $T^*M \otimes f^{-1}TN$ induced from those on TM and TN.

If $f : M \longrightarrow N$ is an immersion and if $f \circ c : [a,b] \longrightarrow N$ are geodesic for all geodesic curves $c : [a, b] \longrightarrow M$ with $[a, b] \subset \mathbb{R}$, then f is called a *totally geodesic* immersion; an immersion f is totally geodesic if and only if

$$\text{(1.2)} \qquad {}^{T^*M \otimes f^{-1}TN}\nabla\, df = 0 .$$

Hence a totally geodesic immersion is harmonic. We recall the following theorem due to Schoen–Yau [SY, p. 371], which will play a key role.

(1.3) Theorem. *Assume the following conditions:*

a) *M is complete and has a finite volume;*

b) *N is complete and has non–positive sectional curvature.*

Let $f, g : M \longrightarrow N$ be harmonic mappings with finite energies which are mutually homotopic by $H : [0, 1] \times M \longrightarrow N$ with $H(0, \cdot\,) = f$ and $H(1, \cdot\,) = g$. Then there exists a 1–parameter family $F(t, p) = f_t(p)$, $t \in [0, 1]$ of harmonic mappings $f_t : M \longrightarrow N$ with $f_0 = f$ and $f_1 = g$ such that

i) *the homotopy $F(t, \cdot\,)$ is equivalent to $H(t, \cdot\,)$,*

ii) *for any $p \in M$, the curve $\{f_t(p); t \in [0, 1]\}$ is a geodesic with constant speed independent of $p \in M$,*

iii) *the section $p \in M \longrightarrow F_*((\partial/\partial t)_{(t,p)}) \in f_t^{-1}TN$ is parallel.*

Moreover, we obtain the following Lemma from the proof of the above theorem ([SY, p. 370]).

(1.4) Lemma. *Let the notation be as in Theorem* (1.3). *Then $f_{t*}(v)$ is parallel along the geodesic $\{f_t(p); t \in [0, 1]\}$ for all $p \in M$ and $v \in T_pM$.*

(1.5) Corollary. *Let $f, g : M \longrightarrow N$ be as in Theorem (1.3). Then f is an immersion (resp. isometric immersion) if so is g.*

The proof is clear.

(1.6) Proposition. *Let M be a compact Riemannian manifold, N a compact analytic Riemannian manifold with non–positive sectional curvatures, and $f : M \longrightarrow N$ a totally geodesic immersion. Then every harmonic mapping from M to N which is homotopic to f is a totally geodesic immersion.*

Proof. Put $X = \mathrm{Harm}(M, N; f)$ and let $\Phi_1 : (g, p) \in X \times M \longrightarrow g(p) \in N$ be the evaluation mapping. We endow with X the pull–backed Riemannian metric by $\Phi_1(\cdot\ , p)$. We show that the image of any geodesic in M by any $g \in X$ is a geodesic in N. Let $c : t \in [0, \varepsilon] \longrightarrow c(t) \in M (\varepsilon > 0)$ be a geodesic in M. Then $f \circ c : [0, \varepsilon] \longrightarrow N$ is a geodesic in N. We may assume that $\|(f \circ c)_*(\partial/\partial t)\| = 1$ and $\varepsilon < \mathrm{Inj}(N)/2$, where $\mathrm{Inj}(N)$ is the injective radius of N. Note that $\Phi_1(\cdot\ , c(0))$ and $\Phi_1(\cdot\ , c(\varepsilon))$ are isometric immersions from X onto totally geodesic submanifolds of N. Therefore they are harmonic and $\Phi_1(\cdot\ , c(t))$ gives a homotopy between them. Deforming $\Phi_1(\cdot\ , c(t))$, we have by Theorem (1.3) a 1–parameter family

$$\Omega_s : X \longrightarrow N, \ 0 \le s \le 1$$

of harmonic mappings with $\Omega_0 = \Phi_1(\cdot\ , c(0))$ and $\Omega_1 = \Phi_1(\cdot\ , c(\varepsilon))$. Moreover, for any $g \in X$, the curve $\{\Omega_s(g);\ s \in [0, 1]\}$ is a geodesic with constant speed, independent of g. Since $\{\Omega_s(f);\ s \in [0, 1]\}$ is a geodesic which coincides with $\{\Phi_1(f, c(t));\ t \in [0, \varepsilon]\}$, we change the parameter s of Ω_s so that

$$(1.8)\qquad \begin{cases} \Omega_t(f) = \Phi_1(f, c(t)),\ 0 \le t \le \varepsilon, \\ \Omega_0 = \Phi_1(\cdot\ , c(0)),\quad \Omega_\varepsilon = \Phi_1(\cdot\ , c(\varepsilon)). \end{cases}$$

By Lemma (1.4) we see that $\{\Phi_1(g, c(t));\ 0 \le t \le \varepsilon\}$ is a curve with arclength parametrization. Therefore we have

$$(1.9)\qquad d_N(\Phi_1(g, c(t)), \Omega_t(g)) \le 2\varepsilon < \mathrm{Inj}(N).$$

Using Theorem (1.3) again, we see that $d_N(\Phi_1(h, c(t)), \Omega_t(h))$ is a constant $(\le 2\varepsilon)$ in $h \in X$.

Since $\Phi_1(f, c(t)) = \Omega_t(f)$ by (1.8),

$$\Phi_1(g, c(t)) = \Omega_t(g)$$

for all t. Hence $\{\Phi_1(g, c(t));\ 0 \le t \le \varepsilon\}$ is a geodesic. Q.E.D.

Let $f: M \longrightarrow N$ be as in Proposition (1.6), $X = \mathrm{Harm}(M, N; f)$ and

$$\Phi_1 : (g, p) \in X \times M \longrightarrow g(p) \in N .$$

We endow X with the pull–backed Riemannian metric by $\Phi_1(\cdot\ , p)$. Then we have the natural mapping $p \in M \longrightarrow \Phi_1(\cdot\ , p) \in \mathrm{Harm}(X, N)$. Put

$$Y = \mathrm{Harm}(X, N; \Phi_1(\cdot\ , p)) ,$$

$$\Phi_2 : (g, y) \in X \times Y \longrightarrow y(g) \in N .$$

We endow Y with the pull–backed Riemannian metric by $\Phi_2(g, \cdot)$.

(1.10) Corollary. *For any* $y \in Y$

$$\Phi_2(\cdot\ , y) : g \in X \longrightarrow y(g) \in N$$

is a totally geodesic isometric immersion.

This immediately follows from Corollary (1.5) and Proposition (1.6). By the construction we see also that

(1.11) for any $v_g \in T_gX$ (resp. $v_y \in T_yY$) the section $z \in Y \longrightarrow d\Phi_2(g, z)(v_g) \in \Phi_2(g\ , \cdot)^{-1}\ TN$ (resp. $x \in X \longrightarrow d\Phi_2(x, y)(v_y) \in \Phi_2(\cdot\ , y)^{-1} TN$) is parallel.

We denote by $\mathrm{Is}(X)$ the group of all isometries of X .

(1.12) Lemma. *Let* $\Phi_2 : X \times Y \longrightarrow N$ *be as above.*

(i) *If* $\mathrm{Is}(X)$ *or* $\mathrm{Is}(Y)$ *is finite, then* Φ_2 *is an immersion.*
(ii) *If* $\mathrm{Is}(X)$ *and* $\mathrm{Is}(Y)$ *are finite, then the pull–backed metric on* $X \times Y$ *by* Φ_2 *is the product metric of those on* X *and* Y .

Proof. (i) Note that $\Phi_2(\cdot\ , y)$ and $\Phi_2(x, \cdot)$ are totally geodesic isometric immersions for $y \in Y$ and $x \in X$. Suppose that Φ_2 is not an immersion. Then there are non–zero vectors $v_g \in T_gX$ and $v_y \in T_yY$ such that $d\Phi_2(g, y)(v_g + v_y) = 0$, so that

$$d\Phi_2(g, y)(v_g) = -\, d\Phi_2(g, y)(v_y) \neq 0 . \tag{1.13}$$

Now, consider the vector field $\{d\Phi_2(x, y)(v_y);\ x \in X\}$ in $\Phi_2(\cdot\ , y)^{-1}TN$. It follows from (1.11) and (1.13) that $d\Phi_2(x, y)(v_y)$ is paralled and tangent to the image $\Phi_2(X, y)$. Therefore we have a non–zero parallel vector field on X, which generates a 1–parameter subgroup of $\mathrm{Is}(X)$. In the same way, we have a 1–parameter subgroup of $\mathrm{Is}(Y)$. Hence, if one of $\mathrm{Is}(X)$ and $\mathrm{Is}(Y)$ is finite, Φ_2 is an immersion.

(ii) Take an arbitrary $v_x \in T_x X$. Then by (1.11), $y \in Y \longrightarrow d\Phi_2(x, y)(v_x)$ is parallel. Suppose that $d\Phi_2(x, y)(v_x)$ is not perpendicular to $\Phi_2(x, Y)$ at some $y_0 \in Y$. Then $d\Phi_2(x, y)(v_x)$ are not perpendicular to $\Phi_2(x, Y)$ at all $y \in Y$. Let u_y denote the orthogonal projection of $d\Phi_2(x, y)(v_x)$ to $T\Phi_2(x, Y)$ in TN. We get a parallel vector field on Y, which generates a 1–parameter subgroup of $\mathrm{Is}(Y)$. Therefore we obtain our assertion. Q.E.D.

(1.14) Theorem. *Let* $\Phi_2 : X \times Y \longrightarrow N$ *be as above. Assume that* $\mathrm{Is}(X)$ *and* $\mathrm{Is}(Y)$ *are finite (especially, the Ricci curvature of* N *is negative,* $\dim X \geq 2$*, and* $\dim Y \geq 2$*). Then* Φ_2 *is a totally geodesic isometric immersion into* N.

Proof. It remains to show that

$$ {}^{E}\nabla d\Phi_2 \equiv 0 , $$

where $E = T^*(X \times Y) \otimes \Phi_2^{-1} TN$ with the naturally induced metric and connection ${}^{E}\nabla$. Put

$$ r = \dim X , \quad s = \dim Y , \quad n = \dim N . $$

Take an arbitrary point $(x, y) \in X \times Y$. Let $(x^1, \dots , x^r)$ (resp. $(y^1, \dots , y^s)$) be a local coordinate system around x (resp. y). Since Φ_2 is an immersion, we can take a local coordinate system $(w^1, \dots , w^n)$ around $\Phi_2(x, y)$ such that

$$ \Phi_2(x^1, \dots , x^n, y^1, \dots , y^s) = (x^1, \dots , x^n, y^1, \dots , y^s, 0, \dots , 0) . $$

We use indices as follows.

$$ \begin{aligned} i, j, k &= 1, \dots , r , \\ \mu, \nu, \tau &= 1, \dots , s , \\ \alpha, \beta, \gamma &= 1, \dots , n . \end{aligned} $$

Put $\varphi = \Phi_2(x, \cdot) : Y \longrightarrow N$. Then φ is a totally geodesic immersion, so that ${}^{T^* Y \otimes \varphi^{-1} TN}\nabla d\varphi \equiv 0$; in terms of local coordinates,

$$ \frac{\partial^2 \varphi^\alpha}{\partial y^\mu \partial y^\nu} - {}^{Y}\Gamma^\tau_{\mu\nu}(y) \frac{\partial \varphi^\alpha}{\partial y^\tau} + {}^{N}\Gamma^\alpha_{\beta\gamma}(y(x)) \frac{\partial \varphi^\beta}{\partial y^\mu} \frac{\partial \varphi^\gamma}{\partial y^\nu} \equiv 0 , $$

where ${}^{Y}\Gamma^{\tau}_{\mu\nu}$ (resp. ${}^{N}\Gamma^{\alpha}_{\beta\gamma}$) are Christoffel symbols of Y (resp. N) and Einstein's summation convention is used. Note that

$$\frac{\partial \varphi^{\alpha}}{\partial y^{\mu}} = \delta^{\alpha}_{r+\mu} \, .$$

Thus we have

$${}^{N}\Gamma^{\alpha}_{r+\mu\ r+\nu}(y(x)) = 0 \, , \ 1 \le \alpha \le r \, ,$$

$$-\ {}^{Y}\Gamma^{\alpha-r}_{\mu\nu}(x) + {}^{N}\Gamma^{\alpha}_{r+\mu\ r+\nu}(y(x)) = 0 \, , \ r < \alpha \le s \, .$$

Since Φ_2 is isometric, we have

$$(1.15) \qquad \begin{cases} {}^{X\times Y}\Gamma^{k}_{\mu\nu}(x,y) = {}^{N}\Gamma^{k}_{r+\mu\ r+\nu}(y(x)) = 0 \, , \\ {}^{X\times Y}\Gamma^{\tau}_{\mu\nu}(x,y) = {}^{N}\Gamma^{r+\tau}_{r+\mu\ r+\nu}(y(x)) = {}^{Y}\Gamma^{\tau}_{\mu\nu}(y) \, . \end{cases}$$

In the same way, we obtain

$$(1.16) \qquad \begin{cases} {}^{X\times Y}\Gamma^{k}_{ij}(x,\, y) = {}^{N}\Gamma^{k}_{i\,j}(y(x)) = {}^{X}\Gamma^{k}_{ij}(x) \\ {}^{X\times Y}\Gamma^{\tau}_{ij}(x,\, y) = {}^{N}\Gamma^{r+\tau}_{i\,j}(y(x)) = 0 \, . \end{cases}$$

The section $x' \in X \longrightarrow d\Phi_2(x',\, y)(\partial/\partial y^{\mu}) \in \Phi_2(\cdot\,,\, y)^{-1}TN$ is parallel by (1.11), and hence

$${}^{\Phi_2(\cdot\ ,\ y)^{-1}TN}\nabla_{\frac{\partial}{\partial x^{i}}} \left[d\Phi_2(x,\, y)\left[\frac{\partial}{\partial y^{\mu}}\right]\right] = 0 \, .$$

It follows that

$${}^{TN}\nabla_{d\Phi_2(\partial/\partial x^{i})} \left[d\Phi_2(x,\, y)\left[\frac{\partial}{\partial y^{\mu}}\right]\right] = 0 \, .$$

On the other hand

$$\frac{\partial}{\partial w^{i}} = d\Phi_2\left[\frac{\partial}{\partial x^{i}}\right] ,$$

$$\frac{\partial}{\partial w^{r+\mu}} = d\Phi_2\left[\frac{\partial}{\partial y^\mu}\right].$$

Therefore

$$TN_{\nabla_{\frac{\partial}{\partial w^i}}}\left[\frac{\partial}{\partial w^{r+\mu}}\right]_{y(x)} = 0\,.$$

We get

$$(1.17)\qquad {}^N\Gamma^\alpha_{i\,r+\mu}(y(x)) = 0\,,$$

and so

$$(1.18)\qquad \begin{cases} {}^{X\times Y}\Gamma^k_{i\mu}(x) = {}^N\Gamma^k_{i\,r+\mu}(y(x)) = 0\,, \\ {}^{X\times Y}\Gamma^\tau_{i\mu}(x) = {}^N\Gamma^{r+\tau}_{i\,r+\mu}(y(x)) = 0\,. \end{cases}$$

By the choices of local coordinate systems, we have

$$(1.19)\qquad \frac{\partial\Phi_2^\alpha}{\partial x^i} = \delta^\alpha_i\,,\ \frac{\partial\Phi_2^\alpha}{\partial y^\mu} = \delta^\alpha_{r+\mu}.$$

By making use of (1.15) – (1.19), we compute ${}^E\nabla d\Phi_2$ at (x, y). The $dx^i\otimes dx^j\otimes\Phi_2^{-1}(\partial/\partial w^\alpha)$ component of ${}^E\nabla d\Phi_2$ at (x, y) is as follows:

$$\frac{\partial^2\Phi_2^\alpha}{\partial x^i\partial y^j} - {}^{X\times Y}\Gamma^k_{ij}(x,y)\frac{\partial\Phi_2^\alpha}{\partial x^k} - {}^{X\times Y}\Gamma^\tau_{ij}(x,y)\frac{\partial\Phi_2^\alpha}{\partial y^\tau} + {}^N\Gamma^\alpha_{\beta\gamma}(y(x))\frac{\partial\Phi_2^\beta}{\partial x^i}\frac{\partial\Phi_2^\gamma}{\partial x^j}$$

$$= \frac{\partial^2\Phi_2^\alpha}{\partial x^i\partial y^j} - {}^X\Gamma^k_{ij}(x)\frac{\partial\Phi_2^\alpha}{\partial x^k} + {}^N\Gamma^\alpha_{\beta\gamma}(y(x))\frac{\partial\Phi_2^\beta}{\partial x^i}\frac{\partial\Phi_2^\gamma}{\partial x^j} = 0\,.$$

Here one reminds that $\Phi_2(\cdot\,, y)$ is totally geodesic. In the same way as above, we see that $dy^\mu\otimes dy^\nu\otimes\Phi_2^{-1}(\partial/\partial w^\alpha)$ component of ${}^E\nabla d\Phi_2$ at (x, y) vanishes. The cross–term, $dx^i\otimes dy^\mu\otimes\Phi_2^{-1}(\partial/\partial w^\alpha)$ component of ${}^E\nabla d\Phi_2$ at (x, y) is computed as follows:

$$\frac{\partial^2\Phi_2^\alpha}{\partial x^i\partial y^\mu} - {}^{X\times Y}\Gamma^k_{i\mu}(x, y)\frac{\partial\Phi_2^\alpha}{\partial x^k} - {}^{X\times Y}\Gamma^\tau_{i\mu}(x,y)\frac{\partial\Phi_2^\alpha}{\partial y^\tau} + {}^N\Gamma^\alpha_{\beta\gamma}(y(x))\frac{\partial\Phi_2^\beta}{\partial x^i}\frac{\partial\Phi_2^\gamma}{\partial x^\mu}$$

$$= {}^{N}\Gamma^{\alpha}_{i\,r+\mu}(y(x)) = 0\,.$$

Q.E.D.

§ 2. Moduli space of holomorphic mappings

In this section we deal with moduli spaces of holomorphic mappings between complex manifolds, especially, those into complete hyperbolic manifolds. For fundamental facts on hyperbolic manifolds (or hyperbolic complex spaces) we refer to $[K_1, K_3]$ and [NO].

Roughly speaking, we have the following correspondences between the cases of Riemannian and complex manifolds:

Riemannian manifolds with non–positive or negative curvatures	$\longleftrightarrow$	hyperbolic manifolds,
harmonic mappings	$\longleftrightarrow$	holomorphic mappings,
isometries	$\longleftrightarrow$	holomorphic automorphisms,
parallel vector fields	$\longleftrightarrow$	holomorphic vector fields.

There are two theorems which are essential in our arguments. One is Theorem (1.3) due to Schoen–Yau [SY], and another is the following:

(2.1) Theorem ($[N_3]$). *Let N be a complete hyperbolic complex space such that N is hyperbolically imbedded into a compact complex space as a Zariski open subset. Let M be a complex manifold which is a Zariski open subset of a compact complex space. Then the moduli space* $\mathrm{Hol}(M, N)$ *of all holomorphic mappings from M into N endowed with compact–open topology carries a structure of a Zariski open subset of a compact complex space such that*

i) *the evaluation mapping*

$$\Phi_1 : (f, p) \in \mathrm{Hol}(M, N) \times M \longrightarrow f(p) \in N$$

is holomorphic,

ii) *if $F: T \times M \longrightarrow N$ is a holomorphic mapping with a complex space T, then the mapping, $t \in T \longrightarrow F(t, \cdot) \in \mathrm{Hol}(M, N)$ is holomorphic.*

(2.2) Remark. Let $\tilde{M}$ be a compactification of M such that $\partial M = \tilde{M} - M$ is a hypersurface with only normal crossings. Then $\mathrm{Hol}(M, N)$ is imbedded into $\mathrm{Hol}(\tilde{M}, \bar{N})$ by holomorphic extension so that the closure $\overline{\mathrm{Hol}(M, N)}$ in $\mathrm{Hol}(\tilde{M}, \bar{N})$ is a compact complex space and $\mathrm{Hol}(M, N)$ is a Zariski open subset of $\overline{\mathrm{Hol}(M, N)}$. The above evaluation mapping Φ_1 extends to a holomorphic mapping from $\overline{\mathrm{Hol}(M, N)} \times \tilde{M}$ into $\bar{N}$.

For $f \in \mathrm{Hol}(M, N)$ we set

$$\operatorname{rank} f = \sup\{\dim M - \dim_p f^{-1} f(p);\ p \in M\}\ ,$$

$$\mathrm{Hol}(k;\ M,\ N) = \{f \in \mathrm{Hol}(M, N);\ \operatorname{rank} f = k\}\ .$$

The notion, rank f, is similarly defined for holomorphic mappings between complex spaces. We give some general fact:

(2.3) Proposition. *Let* Y_1 *and* Y_2 *be compact complex spaces with reduced structure. Then* $\mathrm{Hol}(k, Y_1, Y_2)$ *are open and closed in* $\mathrm{Hol}(Y_1, Y_2)$ *for all* k.

The proof is quite elementary (see [N_3, Lemma (2.17)]).

(2.4) Theorem. *Let* N *be as in Theorem* (2.1). *Assume that* N *is non-singular. Then the group* $\mathrm{Aut}(N)$ *of holomorphic automorphisms of* N *is finite.*

Remark. In the case where N is compact, this was obtained by Kobayashi ([K_1, p. 70]).

Proof. Let $\tilde{N}$ be a compactification of N such that $\tilde{N} - N$ is a hypersurface with only normal crossings. Put $n = \dim N$. By Remark (2.2) we see that $\mathrm{Hol}(n;\ N,\ N)$ is imbedded into $\mathrm{Hol}(n;\ \tilde{N}, \bar{N})$ and the closure $\overline{\mathrm{Hol}(n;\ N, N)}$ in $\mathrm{Hol}(\tilde{N}, \bar{N})$ is a compact complex space. We first claim that $\mathrm{Hol}(n;\ N,\ N)$ is compact in $\mathrm{Hol}(\tilde{N}, \bar{N})$. Let $f_\nu \in \mathrm{Hol}(n;\ N,\ N)$, $\nu = 1, 2, \dots$, be any sequence. Since $\mathrm{Hol}(N, N)$ is relatively compact in $\mathrm{Hol}(\tilde{N}, \bar{N})$, we may assume that the extended mapping $\bar{f}_\nu \in \mathrm{Hol}(\tilde{N}, \bar{N})$ converges to $\bar{f} \in \mathrm{Hol}(\tilde{N}, \bar{N})$. Since $\operatorname{rank} \bar{f}_\nu = n$, $\operatorname{rank} \bar{f} = n$ by Proposition (2.3). Thus the image $\bar{f}(\tilde{N})$ contains a non-empty open subset. Take a point $p_0 \in N$ such that $\bar{f}(p_0) = q_0 \in N$. For an arbitrary point $p \in N$ we have

$$f_\nu(p) = \bar{f}_\nu(p) \longrightarrow \bar{f}(p)\ (\nu \longrightarrow \infty)\ ,$$

$$d_N(f_\nu(p),\ f_\nu(p_0)) \le d_N(p,\ p_0)\ .$$

Since $f_\nu(p_0) \longrightarrow q_0$ and N is complete hyperbolic, $f_\nu(p)$ stay in a compact subset of N. Therefore we have that $\bar{f}(p) \in N$. The restriction $f = \bar{f} \mid N$ belongs to $\mathrm{Hol}(n; N, N)$ and is the limit of f_ν, $\nu = 1, 2, \dots$. Put

$$X = \{f \in \mathrm{Hol}(n; N, N) ; \bar{f}(\bar{N} - N) \subset \bar{N} - N\} .$$

Then X is an analytic subset of $\mathrm{Hol}(n; N, N)$. Since N is measure–hyperbolic (cf. [K_1, Chapter IX]) and its total measure is positive and finite, we easily see that $\deg \bar{f} = 1$ for all $f \in \mathrm{Hol}(n; N, N)$. Let $f \in X$. Then the inverse $\bar{f}^{-1} : \bar{N} \longrightarrow \bar{N}$ of $\bar{f} : \bar{N} \longrightarrow \bar{N}$ is meromorphic and satisfies

$$g = \bar{f}^{-1} \mid N : N \longrightarrow N .$$

Since N is complete hyperbolic, g must be holomorphic (see [K_1, p. 90]), so that $g = f^{-1}$. Therefore $X = \mathrm{Aut}(N)$ and it follows that $\mathrm{Aut}(N)$ is a compact complex Lie group. Since N is hyperbolic, there is no 1–parameter subgroup of $\mathrm{Aut}(N)$. Thus $\mathrm{Aut}(N)$ is finite.

Q.E.D.

In the rest of this section we assume the following conditions for N:

(2.5) (i) N is a complete Kähler manifold with non–positive (Riemannian) sectional curvatures and with negative holomorphic sectional curvatures bounded away from 0,

(ii) N is quasi–projective algebraic and carries a projective compactification $\bar{N}$ such that N is hyperbolically imbedded into $\bar{N}$.

(2.6) Remark. It follows from (2.5), (i) that N is complete hyperbolic; moreover, there is a constant $C > 0$ such that

$$\sqrt{h(v, v)} \leq C\, F_N(v) , \quad v \in TN ,$$

where h denotes Kähler metric on N and F_N the infinitesimal Kobayashi metric. Therefore we have

$$\mathrm{Vol}_h(N) = \int_N dV_h < \infty .$$

Now, let $\overline{M}$ be a compact Kähler manifold and M a Zariski open subset of $\overline{M}$.

(2.7) Lemma. *Let M be as above. Then there is a complete Kähler metric g on M with finite volume such that*

$$F_M \leq C_1 \sqrt{g},$$

where $C_1 > 0$ is a constant.

Proof. Take a modification $\tilde{M} \longrightarrow \overline{M}$ with center contained in $\overline{M} - M$ so that $\tilde{M}$ is Kähler and $\tilde{M} - M$ is a hypersurface with only simple normal crossings. Then as in the proof of Proposition (6.2) of [GK], one can construct such a metric g. Q.E.D.

By Theorem (2.1) we see that $\mathrm{Hol}(M, N)$ carries a structure of a Zariski open subset of a compact complex space, so that

(2.8) $\mathrm{Hol}(M, N)$ is locally arcwise connected.

We endow M with a Kähler metric in Lemma (2.7). The choices of the metrics on M and N, and the decreasing principle $f^* F_N \leq F_M$ for $f \in \mathrm{Hol}(M, N)$ imply that

(2.9) $E(f) < \infty$ for all $f \in \mathrm{Hol}(M, N)$.

Since M and N are Kähler,

(2.10) all $f \in \mathrm{Hol}(M, N)$ are harmonic.

By making use of Theorem (1.3), we have the following lemma (cf. [N_3, p. 29]):

(2.11) Lemma. *Let $f : M \longrightarrow N$ be a harmonic mapping, homotopic to a holomorphic mapping from M into N. Then f is holomorphic, too.*

Remark. In the case where M is compact, this lemma holds without the assumption of non–positive curvatures on N, and is called the Lichnerowicz theorem. In the present case, we have to use that curvature assumption to get some estimate on the boundary behavior of the homotopy between f and the holomorphic mapping.

Even though the global injectivity radius of N is zero, by making use of (2.8), (2.9) and Lemma (2.11), we can apply the arguments of Schoen–Yau [SY, p. 372] to infer that the evaluation mapping $\Phi_1 : \mathrm{Hol}(M, N) \times M \longrightarrow N$ has the following properties:

(2.12) Theorem. i) *For an arbitrary point* $p \in M$, *the holomorphic mapping* $\Phi_1(\cdot\,, p)$: $f \in \mathrm{Hol}(M, N) \longrightarrow f(p) \in N$ *is a proper immersion onto a complex totally geodesic submanifold of* N .

ii) *The pull backed metric* $\Phi_1(\cdot\,, p)^* h$ *is independent of* p .

Thus, $\mathrm{Hol}(M, N)$ is non–singular. We take a connected component X of $\mathrm{Hol}(M, N)$. We may assume that $\partial M = \overline{M} - M$ is a hypersurface with only normal corssings. Then we have the natural imbedding

$$f \in \mathrm{Hol}(M, N) \longrightarrow \bar{f} \in \mathrm{Hol}(\overline{M}, \overline{N})$$

which is an into–homeomorphism (cf. [N_3, Theorem (1.19)]). Let $\overline{X}$ denote the closure of X in $\mathrm{Hol}(\overline{M}, \overline{N})$. By Remark (2.2) $\Phi(\cdot\,, p)(p \in M)$ naturally extends to a holomorphic mapping

$$\overline{\Phi}(\cdot\,, p) : \overline{X} \longrightarrow \overline{N}\,.$$

(2.13) Lemma. i) X *is complete hyperbolic and hyperbolically imbedded into* $\overline{X}$.

ii) X *is quasi–projective.*

Proof. i) It is easy to show that X is complete hyperbolic. Take distinct F_1 , $F_2 \in \overline{X} - X$. Take any sequences $\{f_{1\nu}\}_{\nu=1}^{\infty}$, $\{f_{2\nu}\}_{\nu=1}^{\infty}$ in X such that $\bar{f}_{1\nu} \longrightarrow F_1$ and $\bar{f}_{2\nu} \longrightarrow F_2$ in $\mathrm{Hol}(\overline{M}, \overline{N})$. Since $F_1 \neq F_2$, there is a point $p \in M$ such that $F_1(p) \neq F_2(p)$ and $F_i(p) \in N$, $i = 1, 2$. Then

$$f_{1\nu}(p) \longrightarrow F_1(p) \in N\,,$$

$$f_{2\nu}(p) \longrightarrow F_2(p) \in N\,.$$

Therefore we have

$$d_X(f_{1\nu}, f_{2\nu}) \geq d_N(f_{1\nu}(p), f_{2\nu}(p) > \tfrac{1}{2}\, d_N(F_1(p), F_2(p)) > 0$$

for all large ν . This shows that $X \hookrightarrow \overline{X}$ is a hyperbolic imbedding.

ii) This follows from the fact that $\Phi_1(\cdot\,, p) : X \longrightarrow \Phi_1(X, p)(\subset N)$ with $p \in M$ is finite and $\overline{\Phi_1(X, p)}(\subset N)$ is projective. Q.E.D.

(2.14) Corollary. Aut(X) *is finite.*

This follows from Lemma (2.13) and Theorem (2.4).

As in § 1, we now consider X as a domain space and M as a parameter space. Note that by Lemma (2.13) X satisfies the conditions put on M. Let $Y \subset \mathrm{Hol}(X, N)$ be the connected component containing $\Phi_1(\cdot\,, p)(p \in M)$. Then we have the evalutation mapping

$$\Phi_2 : X \times Y \longrightarrow N.$$

By Corollary (2.14), Aut(Y) is finite, too. To obtain a complex analytic version of Theorem (1.14), we first note that the topology of X (resp. Y) is defined by the compact–open topology on the mapping space from M (resp. X) into N, and that X and Y are locally arcwise connected. These makes us possible to carry out the same arguments as in the proof of Theorem (1.14) in compact subsets of N, where, of course, their injectivity radii remain positive. We endow X (resp. Y) with the pull–backed Kähler metric $\Phi_1(\cdot\,, p)^* h$ (resp. $\Phi_2(x\,, \cdot)^* h$) with $p \in M$ (resp. $x \in X$), and $X \times Y$ with the product metric.

(2.15) Theorem. *Let* $\Phi_2 : X \times Y \longrightarrow N$ *be as above. Then* Φ_2 *is a proper, holomorphic, totally geodesic and isometric immersion.*

Remark. X is a connected component of Hol(Y, N) which contains $\Phi_2(x\,, \cdot)(x \in X)$. For the composition of the holomorphic mapping $p \in M \longrightarrow \Phi_1(\cdot\,, p) \in Y$ and any $g \in \mathrm{Hol}(Y, N)$ is a holomorphic mapping from M into N which is homotopic to $\Phi_1(x\,, \cdot) : M \longrightarrow N$.

§ 3. Higher dimensional Mordell's conjecture over function fields and related topics

By making use of the results obtained in § 2, we discuss the Diophantus geometry in the present and the next sections. We start with the following theorem due to Manin [M] and Grauert [G].

(3.1) Theorem (Mordell's conjecture over function fields). *Let* K *be a function field over an algebraically closed field* k *with* char $k = 0$. *Let* C *be a smooth curve defined over* K, *of which genus is not less than* 2. *Then either the set* $C(K)$ *of* K*–rational points of* C *is finite, or* C *is* K*–isomorphic to a curve* C_0 *defined over* k *and* $C_0(K) - C_0(k)$ *is finite.*

Remark. The original conjecture over number fields was solved by G. Faltings $[F_2]$.

In geometric terms, the above Theorem (3.1) is equivalent to the following.

(3.2) Theorem. *Let* $X \xrightarrow{\pi} R$ *be a smooth fiber space over* k *such that the genus of the fibers* $X_t\,(t \in R)$ *is not less than* 2. *Then the set* Σ *of rational sections* $\sigma : R \longrightarrow X\,(\pi \circ \sigma = id)$ *is finite, or* X *is isomorphic to the product* $R \times X_{t_0}$ *and there are only finitely many non-constant rational sections (non-constant mappings from* R *to* X_{t_0}*).*

Related to this theorem, S. Lang $[L_1]$ gave two conjectures.

(3.3) Conjecture. *Let* $X \longrightarrow R$ *be an algebraic fiber space over* $\mathbb{C}$ *with hyperbolic fibers* X_t. *If the set* Σ *of rational sections of* $X \longrightarrow R$ *is infinite, then* $X \longrightarrow R$ *contains a splitting fiber subspace.*

(3.4) Conjecture. *Let* N *be a complex projective manifold such that* N *is hyperbolic. Let* M *be another complex projective manifold. Then there are only finitely many surjective holomorphic mappings from* M *onto* N.

For a moment we discuss the latter conjecture. Instead of the hyperbolicity assumption on N, Kobayashi–Ochiai $[KO_2]$ assumed that N is of general type, and showed a finiteness theorem:

(3.5) Theorem ($[KO_2]$). *Let* N *be a complex manifold of general type. Then there are only finitely many surjective meromorphic mappings from* M *onto* N.

It is natural to expect

(3.6) Conjecture. *A complex projective algebraic hyperbolic manifold is of general type.*

This is true for curves, and for surfaces by Mori–Mukai [MM]; in fact, they proved that a complex projective algebraic surface is of general type if and only if it is measure–hyperbolic.

Let N be a complex projective manifold. According to the theory of Mori [Mo], if there is a curve $C \subset N$ such that the intersection number $C \cdot K_N$ of C and the canonical bundle K_N of N is negative, then C is deformed to a sum of curves, which contains a rational curve. Thus, if N is hyperbolic, then $C \cdot K_N \geq 0$ for all curves $C \subset N$. Making use of this fact, Horst $[H_1, H_2]$ solved Conjecture (3.4):

(3.7) Theorem ($[H_1, H_2]$). *Let* N *be a hyperbolic Kähler manifold and* M *a complex manifold. Then there are only finitely many surjective holomorphic mappings from* M *onto* N.

Remark. Assuming additionally that K_N carries a metric with non–positive curvature form, Noguchi [N_2] proved the above finiteness theorem.

There are many finiteness theorems of various types. Cf. [S_1], [S_2], [I], [NS], [BN] and [KSW].

As for Conjecture (3.3), Noguchi first proved the following by employing the idea of Grauert [G]:

(3.8) Theorem ([N_1]). *Let* $X \longrightarrow R$ *be an algebraic smooth fiber space such that the holomorphic tangent bundle* $\mathrm{T}(X_{t_0})$ *is negative* ($\mathrm{T}^*(X_{t_0})$ *is ample) for some point* $t_0 \in R$. *Assume that* $\Sigma(t_0) = \{\sigma(t_0); \sigma \in \Sigma\}$ *is Zariski dense in* X_{t_0}. *Then there is a Zariski open neighborhood* $R' \subset R$ *of* t_0 *such that* $X|R' \cong R' \times X_{t_0}$, *and there are only finitely many non–constant rational sections.*

If $\mathrm{T}(X_{t_0})$ is negative, then X_{t_0} is hyperbolic ([K_2]); but the converse is not true. For instance, if C_i, $i = 1, 2,$ are smooth algebraic curves with genus ≥ 2, then $C_1 \times C_2$ is hyperbolic, but $\mathrm{T}(C_1 \times C_2)$ is not negative.

In general, let $\pi : X \longrightarrow R$ be a proper fiber space, of which general fibers are irreducible. We call (X, π, R) a *hyperbolic fiber space* if X_t are hyperbolic complex spaces for all $t \in R$. Assume that (X, π, R) has a compactification $(\overline{X}, \overline{\pi}, \overline{R})$; that is, $\overline{\pi} : \overline{X} \longrightarrow \overline{R}$ is a compact fiber space such that R is a Zariski open subset of $\overline{R}$, $X = \overline{X}|R$ and $\overline{\pi}|X = \pi$. Now we consider the relative version of the notion of hyperbolic imbedding.

(3.9) Definition. Let (X, π, R) be a hyperbolic fiber space and $(\overline{X}, \overline{\pi}, \overline{R})$ its compactification. We say that (X, π, R) is hyperbolically imbedded into $(\overline{X}, \overline{\pi}, \overline{R})$ along $\partial R = \overline{R} - R$ if for any point $t \in \partial R$ there are small neigborhoods U and V of t in $\overline{R}$ such that $V \subset U$, V is relatively compact in U and $X|(V - \partial R)$ is hyperbolically imbedded into $\overline{X}|U$.

(3.10) Theorem. *Let* (X, π, R) *be a hyperbolic fiber space with a compactification* $(\overline{X}, \overline{\pi}, \overline{R})$ *such that* (X, π, R) *is hyperbolically imbedded into* $(\overline{X}, \overline{\pi}, \overline{R})$ *along* ∂R. *Assume that* R *is smooth and that there is a point* $t_0 \in R$ *such that* $\Sigma(t_0)$ *is Zariski dense in* X_{t_0}. *Then*

i) *there is a finite Galois covering* $\tilde{R} \longrightarrow R$ *such that* $\tilde{R} \times_R X \cong \tilde{R} \times X_{t_0}$;

ii) *if* X_{t_0} *is a Kähler manifold, then* $X \cong R \times X_{t_0}$ *(i.e.,* $\tilde{R} = R$*)*.

The assertion i) was proved by $[N_2]$. Combining the argument of the proof of i) with Theorem (3.7), we have ii) (see $[N_2, \S\, 3]$).

(3.11) Theorem. *Let* (X, π, R) *be a smooth fiber space of curves with genus* $g \geq 2$ *and* $\dim R = 1$. *Then there is a compactification* $(\overline{X}, \overline{\pi}, \overline{R})$ *of* (X, π, R) *such that* (X, π, R) *is hyperbolically imbedded into* $(\overline{X}, \overline{\pi}, \overline{R})$.

Combining Theorem (3.11) with Theorem (3.10), we have Theorem (3.2) in the case of $\dim R = 1$; this is an essential case. For the proof of Theorem (3.11), see $[N_2, \S\, 5]$. It has been informed to the authors that there is some incomplete part in the proof of Theorem (3.11). We here make the point clear. Since the construction of $(\overline{X}, \overline{\pi}, \overline{R})$ is local around points of ∂R, we assume that $\overline{R} = \Delta$ (the unit disc in $\mathbb{C}$) and $R = \Delta^*(= \Delta - \{0\})$. It was first showed that there is a finite Galois covering $S \longrightarrow \Delta^*$ with covering group G such that (X_S, η, S) is hyperbolically imbedded into some compactification $(\overline{X}_S, \overline{\eta}, \overline{S})$ along $\overline{S} - S$, where $X_S = S \times_{\Delta^*} X$ and $\eta : X_S \longrightarrow S$ is the projection. The group G acts holomorphically on X_S and the action extends meromorphically on $\overline{X}_S$. The point is that this extended action is not necessarily holomorphic. But we proceed as follows. Consider the following imbedding*)

$$\alpha : x \in X_S \longrightarrow (\ldots, g(x), \ldots)_{g \in G} \in \prod_{\# G} X_S \subset \prod_{\# G} \overline{X}_S .$$

Note that $\prod_{\# G} X_S$ is hyperbolically imbedded into $\prod_{\# G} \overline{X}_S$ along $\overline{S} - S$. The action of G on X_S is transformed to the exchanges of variables of $\prod_{\# G} X_S$, which extend holomorphically on $\prod_{\# G} \overline{X}_S$. Put $Y = \alpha(X_S)$. Then the closure $\overline{Y}$ in $\prod_{\# G} \overline{X}_S$ is an analytic subspace which is invariant by the action of G. Identify X_S with Y through α. Then the quotient $G \backslash \overline{Y}$ provides the desired compactification.

Now we assume that $X \xrightarrow{\pi} R$ is a proper smooth fiber space. Brody [B] proved that if a fiber X_{t_0} is hyperbolic, then there is a neighborhood U of t_0 (with respect to the differential

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*) This trick was suggested by C.T.C. Wall.

topology) such that $X|U$ is hyperbolic, and so X_t are hyperbolic for all $t \in U$. In connection with Theorem (3.10), it is interesting to ask

(3.12) Question (Lang). *Let $X \xrightarrow{\pi} R$ be an algebraic fiber space. Then does the set $\{t \in R; X_t$ is hyperbolic$\}$ form a Zariski open subset of R?*

Lang has asked also

(3.13) Question. *If (X, π, R) is a hyperbolic algebraic fiber space, then does there exist a compactification $(\overline{X}, \overline{\pi}, \overline{R})$ of (X, π, R) such that (X, π, R) is hyperbolically imbedded into $(\overline{X}, \overline{\pi}, \overline{R})$ along ∂R?*

It is also interesting to point out that Parshin $[P_2]$ gave a proof of the following theorem due to Raynaud [R] which is based on the Kobayashi distance.

(3.14) Theorem ($[R, P_2]$). *Let $X \subset A$ be a subvariety of an Abelian variety A, defined over a function field $K = \mathbb{C}(R)$ of a curve R. If X does not contain any translation of a non–trivial Abelian subvariety, then the set $X(K)$ of K–rational points on X is finite modulo the $(K/\mathbb{C}-)$ trace A_0.*

Here it is known that there exists a unique maximal Abelian subvariety A_0 of A defined over $\mathbb{C}$, and A_0 is called the $(K/\mathbb{C}-)$ trace (see $[L_2]$).

On the hyperbolicity of a subvariety of an Abelian variety or of a complex torus we know ([Gr])

(3.15) Proposition. *Let T be a complex torus and X an analytic subspace of T. Then X is hyperbolic if and only if X does not contain any translation of a positive dimensional subtorus of T.*

Lately, Faltings $[F_3]$ proved a very surprising result:

(3.16) Theorem. *Let A be an Abelian variety over a number field and $X \subset A$ a subvariety. If X does not contain any translation of a positive dimensional Abelian subvariety, then X contains only finitely many k–rational points.*

Now it is interesting to recall the following conjecture of Lang:

(3.17) Conjecture. *Let X be a projective algebraic variety defined over a number field k. Assume that for some imbedding $k \subset \mathbb{C}$, X is hyperbolic as a complex manifold. Then the number of* *k–rational points of X is finite.*

It is also interesting to ask

(3.18) Question. *Let X be a projective algebraic variety defined over a number field k. Assume that for some imbedding $k \subset \mathbb{C}$, X is hyperbolic. Then is X hyperbolic for any other imbedding $k \subset \mathbb{C}$?*

§ 4. Parshin–Arakelov–type theorems

Let $\overline{R}$ be a smooth compact Riemann surface, $S \subset \overline{R}$ a finite set of points of $\overline{R}$ and put $R = \overline{R} - S$. Let $g \geq 0$ be an integer and $\mathbb{M}(\overline{R}, S, g)$ denote the set of all fiber spaces $\overline{X} \xrightarrow{\overline{\pi}} \overline{R}$ of compact Riemann surfaces such that

i) $\pi : X \longrightarrow R$ are smooth, where $X = \overline{X} | R$ and $\pi = \overline{\pi} | X$,
ii) the genus of $X_t \,(t \in R)$ are g,
iii) $\pi : X \longrightarrow R$ are not locally trivial.

In the case of $S = \phi$ Parshin $[P_1]$, and in general Arakelov [A] proved the following theorem which had been conjectured by Shafarevich.

(4.1) Theorem. *If* $g \geq 1$, *then* $\mathbb{M}(\overline{R}, S, g)$ *is finite.*

The case of $g = 1$ is not difficult and was somehow already known. The case of $g \geq 2$ is of our interest. Parshin $[P_1]$ proved that the finiteness of $\mathbb{M}(\overline{R}, S, g)\ (g \geq 2)$ implies Mordell's conjecture over function fields (Theorem (3.1)), and observed that if the same holds over the ring of integers of an algebraic number field, Mordell's conjecture follows. Falting's solution of Mordell's conjecture was carried out along this line.

Imayoshi and Shiga [IS] lately proved Theorem (4.1) by a purely function theoretic method. The proof of such a finiteness theorem is, in general, divided into two parts, boundedness and rigidity. They first proved the compactness of $\mathbb{M}(\overline{R}, S, g)$, of which proof is rather hard. Combining our elementary result, Theorem (3.1) with their easier part of rigidity, we here give a proof of Theorem (4.1).

Let $\mathbf{T}_g$ be the Teichmüller space of compact Riemann surfaces with genus $g \geq 1$, and Π_g the Teichmüller modular group. Royden [Ro] proved that the Teichmüller distance on $\mathbf{T}_g$ coincides with Kobayashi distance, so that

(4.2) $$\mathbf{T}_g \text{ is complete hyperbolic.}$$

It is known that Π_g contains a normal subgroup Π'_g of finite index which freely acts on $\mathbf{T}_g$. By making use of the Torelli mappings, we have (see [N_3])

(4.3) Lemma. *The quotient* $\Pi'_g\backslash\mathbf{T}_g$ *has a projective compactification* $\overline{\Pi'_g\backslash\mathbf{T}_g}$ *such that* $\Pi'_g\backslash\mathbf{T}_g$ *is hyperbolically imbedded into* $\overline{\Pi'_g\backslash\mathbf{T}_g}$.

Every element $\alpha = (\overline{X}, \overline{\pi}, \overline{R}) \in \mathbf{M}(\overline{R}, S, g)$ naturally defines a monodromy representation

$$\chi_\alpha : \pi_1(R) \longrightarrow \Pi_g ,$$

which induces

$$[\chi_\alpha] : \pi_1(R) \longrightarrow \Pi_g/\Pi'_g .$$

Put $\pi_1(R)' = \mathrm{Ker}[\chi_\alpha]$ and let $R'_{[\chi_\alpha]} \longrightarrow R$ be a finite Galois covering with group $\pi_1(R)/\pi_1(R)'$. Then α naturally defines a non-constant holomorphic mapping

$$f_\alpha : R'_{[\chi_\alpha]} \longrightarrow \Pi'_g\backslash\mathbf{T}_g .$$

Since $\pi_1(R)$ is finitley generated, $\mathrm{Hom}(\pi_1(R), \Pi_g\backslash\Pi'_g)$ is finite. Note that for distinct α, $\beta \in \mathbf{M}(\overline{R}, S, g)$ with $[\chi_\alpha] = [\chi_\beta]$, $f_\alpha \not\equiv f_\beta$. Therefore the proof of Theorem (4.1) is reduced to

(4.4) Theorem. *The space* $\mathrm{Hol}'(R, \Pi'_g\backslash\mathbf{T}_g)$ *of all non-constant holomorphic mappings from* R *into* $\Pi'_g\backslash\mathbf{T}_g$ *is finite for* $g \geq 1$.

Proof. It follows from Theorem (2.1), (4.2) and Lemma (4.3) that $\mathrm{Hol}'(R, \Pi'_g\backslash\mathbf{T}_g)$ is a Zariski open subset of a compact complex space. Therefore there are only finitely many homotopy types of $f \in \mathrm{Hol}'(R, \Pi'_g\backslash\mathbf{T}_g)$. Let Δ be the unit disc in $\mathbb{C}$, $\Delta \longrightarrow R$ the universal covering and $\Gamma = \pi_1(R)$. Let f, $g \in \mathrm{Hol}'(R, \Pi'_g\backslash\mathbf{T}_g)$ belong to the same connected component. Then f and g are mutually homotopic. We claim

(4.5) $$f \equiv g .$$

There is a homomorphism $\chi \in \mathrm{Hom}(\Gamma, \Pi'_g)$ such that

$$\tilde{f} \circ \gamma = \chi(\gamma) \circ \tilde{f}$$

$$\tilde{g} \circ \gamma = \chi(\gamma) \circ \tilde{g}$$

for all $\gamma \in \Gamma$, where $\tilde{f} : \Delta \longrightarrow \mathbf{T}_g$ (resp. $\tilde{g} : \Delta \longrightarrow \mathbf{T}_g$) is a suitable lifting of f (resp. g). By Ber's imbedding, $\mathbf{T}_g$ is realized as a bounded domain of $\mathbb{C}^{3g-3}$. Then $\tilde{f}$ and $\tilde{g}$ are represented by $(3g-3)$ bounded holomorphic functions on Δ. By Fatou's theorem, $\tilde{f}$ and $\tilde{g}$ have non-tangential boundary values at almost all points of $\partial\Delta$. Assume that there is a subset $E \subset \partial\Delta$ with positive measure such that $\tilde{f}(z) \neq \tilde{g}(z)$ for $z \in E$. By the idea of the rigidity part of [IS] we see that for all most all $z \in E$, there are $\gamma_n \in \Gamma$, $n = 1, 2, \ldots$, and $z_0 \in \Delta$ such that

$$\gamma_n(z_0) \longrightarrow z\text{, non-tangentially}$$

$$d_{\mathbf{T}_g}(\tilde{f}(\gamma_n(z_0)), \tilde{g}(\gamma_n(z_0)) \longrightarrow +\infty .$$

Since the Kobayashi hyperbolic distance $d_{\mathbf{T}_g}$ is invariant by holomorphic automorphisms,

$$\begin{aligned} d_{\mathbf{T}_g}(\tilde{f}(\gamma_n(z_0)), \tilde{g}(\gamma_n(z_0)) &= d_{\mathbf{T}_g}(\chi(\gamma_n) \circ \tilde{f}(z_0), \chi(\gamma_n) \circ \tilde{g}(z_0)) \\ &= d_{\mathbf{T}_g}(\tilde{f}(z_0), \tilde{g}(z_0)) < \infty . \end{aligned}$$

This is a contradiction. Q.E.D.

In $[F_1]$ Faltings dealt with Parshin–Arakelov–type theorem for principally polarized Abelian varieties. Let $\overline{R}$, S and g be as above and $A(\overline{R}, S, g)$ denote the set of all fiber spaces $\overline{A} \longrightarrow \overline{R}$ such that $A(= \overline{A} \mid R) \longrightarrow R$ are smooth, locally–nontrivial fiber spaces of g–dimensional Abelian varieties $A_t(t \in R)$ with principal polarizations. Faltings proved that $A(\overline{R}, S, g)$ forms a scheme of finite type over $\mathbb{C}$. Applying Theorem (2.12), we have the following theorem.

(4.6) Theorem ([N, p. 32]). $A(\overline{R}, S, g)$ *is quasi–projective and every connected component* Z *of* $A(\overline{R}, S, g)$ *is a quotient of a symmetric bounded domain such that there is a proper, holomorphic, totally geodesic, isometric immersion* $\varphi : Z \longrightarrow Sp(2g, \mathbb{Z})\backslash \mathbf{H}_g$, *where* $\mathbf{H}_g$ *is the Siegel upper–half* *space of rank g .*

From now on we call such φ a *Kuga–Satake immersion.* Faltings $[F_1]$ gave also a criterion of the rigidity of an element of $A(\overline{R}, S, g)$. The both, rigid and non–rigid cases can happen. This contrasts to the arithmetic case where the rigidity always holds ($[F_1]$). His criterion was described in terms of Hodge structure, and was generalized by Peters [Pe] to more general Hodge structures (see also Saitoh–Zucker [SZ] for K3–surfaces) .

In general, let $\mathbb{D}$ be a symmetric bounded domain and $\Gamma \subset \mathrm{Aut}_0(\mathbb{D})$ an arithmetic discrete subgroup of the identity component $\mathrm{Aut}_0(\mathbb{D})$ of $\mathrm{Aut}(\mathbb{D})$. Let $\overline{M}$ be a compact Kähler manifold and M a Zariski open subset of $\overline{M}$.

Let $\mathrm{Hol}_{\mathrm{lift}}(M, \Gamma\backslash\mathbb{D})$ be the space of all "liftable" holomorphic mapping $f: M \longrightarrow \Gamma\backslash\mathbb{D}$; that is, there are holomorphic mapping $\tilde{f}$ from the universal covering space $\tilde{M}$ of M into $\mathbb{D}$ and a homomorphism $\chi : \pi_1(M) \longrightarrow \Gamma$ such that

$$\tilde{f} \circ \alpha = \chi(\alpha) \circ \tilde{f} \ , \ \alpha \in \pi_1(M)$$

and $\tilde{f}$ induces f. If Γ is torsion free, then

$$\mathrm{Hol}_{\mathrm{lift}}(M, \Gamma\backslash\mathbb{D}) = \mathrm{Hol}(M, \Gamma\backslash\mathbb{D}) \ .$$

For simplicity, we assume in the sequel that Γ is torsion free, but remark that the same results hold with a slight modification even in the case where Γ contains a torsion element. We recall the following facts:

(4.7) i) The Bergman metric on $\Gamma\backslash\mathbb{D}$ is complete, of finite volume and has non–positive sectional curvature.

ii) $\Gamma\backslash\mathbb{D}$ is complete hyperbolic and hyperbolically imbedded into the Satake compactification $\overline{\Gamma\backslash\mathbb{D}}$ which is projective.

For ii), cf. Kobayashi–Ochiai $[KO_1]$. Hence we can apply the results in § 2 to $\mathrm{Hol}(M, \Gamma\backslash\mathbb{D})$. Let X be a connected component of $\mathrm{Hol}(M, \Gamma\backslash\mathbb{D})$. Then X is a non–singular quasi–projective manifold and represented by $\Gamma_1\backslash\mathbb{D}_1$, where $\mathbb{D}_1$ is a symmetric bounded domain. Let

$$\Phi_1 : (\Gamma_1\backslash\mathbb{D}_1) \times M \longrightarrow \Gamma\backslash\mathbb{D}$$

be the evalution mapping. Then for any $p \in M$

$$\Phi_1(\cdot \ , p) : \Gamma_1\backslash\mathbb{D}_1 \longrightarrow \Gamma\backslash\mathbb{D}$$

is a Kuga–Satake immersion. Let $\ell(\mathbb{D})$ (resp. $\ell(\Gamma)$) denote the maximum dimension of proper (resp. Γ–rational) boundary components of $\mathbb{D}$. Note that $\operatorname{rank} f$ is constant in $f \in X$ (cf. Proposition (2.3)).

(4.8) Theorem (**[N_3, §§ 3 and 4]**). *Let the notation be as above.*

i) *If* $\operatorname{rank} f > \ell(\Gamma)$ *for* $f \in X$, X *is compact.*

ii) *If* $\operatorname{rank} f > \ell(\mathbb{D})$ *for* $f \in X$, *then* $\dim X = 0$.

iii) *If* $\overline{f(M)} \subset \Gamma\backslash\mathbb{D}$ *for one (and all)* $f \in X$, *then* $\dim X \leq \ell(\Gamma)$, *where* $\overline{f(M)}$ *denote the closure of* $f(M)$ *in* $\overline{\Gamma\backslash\mathbb{D}}$.

iv) $\dim \operatorname{Hol}(M, \Gamma\backslash\mathbb{D}) \leq \ell(\mathbb{D})$.

Now, suppose that $\dim X > 0$. Let Y be the connected component of $\operatorname{Hol}(X, \Gamma\backslash\mathbb{D})$ containing $\Phi_1(\cdot\,, p)$, $p \in M$. Then $Y = \Gamma_2\backslash\mathbb{D}_2$ as in the case of X. We have the evaluation mapping

$$\Phi_2 : (\Gamma_1\backslash\mathbb{D}_1) \times (\Gamma_2\backslash\mathbb{D}_2) \longrightarrow \Gamma\backslash\mathbb{D}.$$

(4.9) Theorem. *Let the notation be as above. Then* Φ_2 *is a Kuga–Satake immersion.*

While this is just a special case of Theorem (2.15), it would be worth to state it separately, as it is one of our goals in this directions. It is interesting to note that starting from any M, we come to a Kuga–Satake immersion, provided that the moduli has a positive dimension.

Now we consider the Parshin–Arakelov–type theorem for polarized algebraic K3–surfaces. It is known that the moduli space of polarized algebraic K3–surfaces is represented by the quotient $\Gamma\backslash\mathbb{D}_{IV}$ of a symmetric bounded domain $\mathbb{D}_{IV}$ of type IV (cf., e.g. [SZ]):

$$(4.10)\qquad \begin{cases} D_{IV} = SO(2,19)/SO(2) \times SO(19), \\ \dim_{\mathbb{C}} D_{IV} = 19, \\ \ell(\mathbb{D}_{IV}) = \ell(\Gamma) = 1. \end{cases}$$

Let $X \xrightarrow{\pi} M$ be a smooth fiber space of locally non–trivial polarized algebraic K3–surfaces, and $f \in \operatorname{Hol}_{\mathrm{lift}}(M, \Gamma\backslash\mathbb{D})$ the corresponding holomorphic mapping. Since $\operatorname{Hol}_{\mathrm{lift}}(M, \Gamma\backslash\mathbb{D})$ is a finite sum of quasi–projective varieties,

(4.11) there are only finitely many rigid (X, π, M).

Moreover,

(4.12) if $\operatorname{rank} f \geq 2$, then (X, π, M) is rigid.

Assume that rank $f=1$ and (X, π, M) is not rigid (there is such an example). Then by Theorems (4.8), (4.9) and (4.10) the problem is reduced to investigate a Kuga–Satake immersion

$$\Phi : (\Gamma_1\backslash \mathbb{H}) \times (\Gamma_2\backslash \mathbb{H}) \longrightarrow \Gamma\backslash \mathbb{D}_{IV} ,$$

where $\mathbb{H}$ is the upper–half plane of $\mathbb{C}$. Saitoh–Zucker [SZ] classified all possible such Φ .

References

[A] S.Ju. Arakelov, Families of algebraic curves with fixed degeneracies, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1277–1302.

[B] R. Brody, Compact manifolds and hyperbolicity, Trans. Amer. Math.. Soc. **235** (1978), 213–219.

[BN] A. Borel and R. Narasimhan, Uniqueness conditions for certain holomorphic mappings, Invent. Math. **2** (1967), 247–255.

[F_1] G. Faltings, Arakelov's theorem for Abelian varieties, Invent. Math. **73** (1983), 337–347.

[F_2] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, Invent. Math. **73** (1983), 349–366.

[F_3] G. Faltings, Diophantine approximation on Abelian varieties, preprint.

[G] H. Grauert, Mordells Vermutung über rationale Punkte auf Algebraischen Kurven und Funktionenkörper, Publ. Math. IHES **25** (1965), 131–149.

[GK] P. Griffiths and J. King, Nevanlinna theory and holomorphic mappings between algebraic varieties, Acta Math. **130** (1973), 145–220.

[Gr] M. Green, Holomorphic maps to complex tori, Amer. J. Math. **100** (1978), 615–620.

[H_1] C. Horst, Compact varieties of surjective holomorphic mappings, Math. Z. **196** (1987), 259–269.

[H_2] C. Horst, A finiteness criterion for compact varieties of surjective holomorphic mappings, preprint.

[I] Y. Imayoshi, Generalizations of de Franchis theorem, Duke Math. J. **50** (1983), 393–408.

[IS] Y. Imayoshi and H. Shiga, A finiteness theorem for holomorphic families of Riemann surfaces. Holomorphic Functions and Moduli II, pp. 207–219, Springer–Verlag, New York–Berlin, 1988.

[K_1] S. Kobayashi, Hyperbolic Manifolds and Holomorphic Mappings, Marcel Dekker, New York 1970.

[K_2] S. Kobayashi, Negative vector bundles and complex Finsler structures, Nagoya Math. J. **57** (1975), 153–166.

[K_3] S. Kobayashi, Intrinsic distances, measures, and geometric function theory, Bull. Amer. Math. Soc. **82** (1976), 357–416.

[KO_1] S. Kobayashi and T. Ochiai, Satake compactification and the great Picard theorem, J. Math. Soc. Japan **23** (1971), 340–350.

[KO_2] S. Kobayashi and T. Ochiai, Meromorphic mappings onto compact complex spaces of general type, Invent. Math. **31** (1975), 7–16.

[KSW] M. Kalka, B. Schiffman and B. Wang, Finiteness and rigidity theorems for holomorphic mappings, Michigan Math. J. **28** (1981), 289–295.

[L_1] S. Lang, Higher dimensional Diophantine problems, Bull. Amer. Math. Soc. **80** (1974), 779–787.

[L_2] S. Lang, Fundamentals of Diophantine Geometry, Springer–Verlag, Berlin, 1983.

[M] Ju. Manin, Rational points of algebraic curves over function fields, Izv. Akad. Nauk. SSSR. Ser. Mat. **27** (1963), 1395–1440.

[Mo] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. Math. **116** (1982), 133–176.

[MM] S. Mori and S. Mukai, The uniruledness of the moduli space of curves of genus 11, Algebraic Geometry, Proc. Japan–France Conf. Tokyo and Kyoto 1982, Lecture Notes in Math. 1016, Springer–Verlag, Berlin–Heidelberg–New York, 1983.

[N_1] J. Noguchi, A higher dimensional analogue of Mordell's conjecture over function fields, Math. Ann. **258** (1981), 207–212.

[N_2] J. Noguchi, Hyperbolic fibre spaces and Mordell's conjecture over function fields, Publ. RIMS, Kyto University **21** (1985), 27–46.

[N_3] J. Noguchi, Moduli spaces of holomorphic mappings into hyperbolically imbedded complex spaces and locally symmetric spaces, Invent. Math. **93** (1988), 15–34.

[NO] J. Noguchi and T. Ochiai, Geometric Function Theory in Several Complex Variables, to appear from A.M.S. Monograph Series (the Japanese edition, Iwanami Shoten, Tokyo, 1984).

[NS] J. Noguchi and T. Sunada, Finiteness of the family of rational and meromorphic mappings into algebraic varieties, Amer. J. Math. **104** (1982), 887–900.

[P_1] A.N. Parshin, Algebraic curves over function fields, I, Izv. Akad. Nauk SSSR Ser. Mat., **32** (1968), 1145–1170.

[P_2] A.N. Parshin, Finiteness theorems and hyperbolic manifolds, preprint.

[Pe] C. Peters, Rigidity for variations of Hodge structure and Arakelov–type finiteness theorems, preprint.

[R] M. Raynoud, Around the Mordell conjecture for function fields and a conjecture of Serge Lang, Lecture Notes in Math., vol. 1016, pp. 1–19, Springer–Verlag, Berlin–New York, 1983.

[Ro] H.L. Royden, Automorphisms and isometries of Teichmüller spaces, Advances in the Theory of Riemann Surfaces, pp. 369–383, Ann. of Math. Studies **66**, Princeton Univ. Press, Princeton, New Jersey, 1971.

[SY] R. Schoen and S.–T. Yau, Compact group actions and the topology of manifolds with non–positive curvature, Topology **18** (1979), 361–380.

[S_1] T. Sunada, Holomorphic mappings into a compact quotient of symmetric bounded domain, Nagoya Math. J. **64** (1976), 159–175.

[S_2] T. Sunada, Rigidity of certain harmonic mappings, Invent. Math. **51** (1979), 297–307.

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