# DOUBLE SPACE WITH DOUBLE LINE

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Abstract. We describe birational maps of a singular double cover of $\mathbb{P}^3$ ramified in sextic with a double line into Fano 3-folds with canonical singularities, elliptic fibrations and fibrations on surfaces of Kodaira dimension zero.

## §1. Introduction.

Let $\gamma : V \to \mathbb{P}^3$ be a double cover ramified in sextic $S \subset \mathbb{P}^3$, such that $S$ is smooth outside of some line $L \subset S$ and $S$ has singularities of type $x^2 + y^2 = 0 \subset \mathbb{C}^3$ in the generic point of line $L$. Then $V$ is smooth outside of proper transform $\tilde{L} \subset V$ of line $L$ and has singularities of type $\mathbb{A}_1 \times \mathbb{C}$ in generic point of $\tilde{L}$. Variety $V$ can be given by equation

$$u^2 = x^2 p_4(x, y, z, t) + y^2 q_4(x, y, z, t)$$

in weighted projective space $\mathbb{P}(1, 1, 1, 3)$, where $p_4$ and $q_4$ are homogeneous polynomials of degree four, $x$, $y$, $z$ and $t$ are homogeneous coordinates of weight one and $u$ is a homogeneous coordinate of weight three. Pencil of planes in $\mathbb{P}^3$ passing through $L$ defines a pencil $\mathcal{P}$, such that normalization of general surface in pencil $\mathcal{P}$ is a smooth del Pezzo surfaces of degree two. Consider a restriction $f : X \to V$ on $V$ of blow up of smooth curve $\tilde{L} \subset \mathbb{P}(1, 1, 1, 3)$.

*Remark 1.1.* We will assume that $X$ is smooth and $p_4(0, 0, z, t) = q_4(0, 0, z, t) = 0$ has eight different homogeneous solutions, which defines eight points $O_i \in \tilde{L}$ for $i = 1, \ldots, 8$.

Hence, $V$ is a Fano 3-fold with $\mathbb{Q}$-factorial canonical singularities and Picard group $\mathbb{Z}$, divisor $K_V$ is Cartier, $-K_V^3 = 2$, linear system $|-K_V|$ is free, $\phi_{|-K_V|} = \gamma$, $K_X = f^*(K_V)$, pencil $|-K_X - E|$ gives a fibration $\tau : X \to \mathbb{P}^1$ on del Pezzo surface of degree two, $f$ has eight reducible fibers $Z_i = Z_i^0 \cup Z_i^1$ over points $O_i$, where $E$ is an exceptional divisor of $f$, curves $Z_i^0$ and $Z_i^1$ are smooth and intersect each other transversally in one point.

*Remark 1.2.* Blowing up curve $Z_i^k$ and making Francia antiflip in proper transform of curve $Z_i^{1-k}$ we get a birational map from $V$ to a 3-fold with nef and big anticanonical

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All varieties are assumed to be projective, normal and defined over the field $\mathbb{C}$.

divisor, which gives a birational map $\rho_{i,j} : V \dashrightarrow V_{i,k}$, where $V_{i,k}$ is a Fano 3-fold with canonical singularities and $-K^3_{V_{i,k}} = \frac{1}{2}$ (for details see §4).

The purpose of this paper is to prove the following two results.

THEOREM 1.3. *Let $\rho : V \dashrightarrow \mathbb{P}^2$ be a dominant rational map, whose general fiber is an elliptic curve. Then there is a rational map $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, such that $\alpha \circ \rho = \tau \circ f^{-1}$.*

THEOREM 1.4. *Fano 3-fold $V$ is not birationally isomorphic to any Fano 3-fold with canonical singularities except $V$ itself and sixteen 3-folds $V_{i,k}$ for $i = 1, \ldots, 8$ and $k = 0, 1$.*

Note, that the claim of Theorem 1.3 can not be improved. Indeed, let $C$ be a section of fibration $\tau$ and $\mathcal{H}_C \subset |-K_X + \tau^*(\mathcal{O}_{\mathbb{P}^1}(n))|$ be a linear system of surfaces passing though section $C$ for $n \gg 0$. Then the general fiber of rational map $\phi_{\mathcal{H}_C} : X \dashrightarrow Z_C$ is an elliptic curve, surface $Z_C$ is rational, resolution of indeterminacy of map $\phi_{\mathcal{H}_C}$ gives an elliptic fibration with a section and there is a natural rational projection $\alpha_C : Z_C \dashrightarrow \mathbb{P}^1$, such that $\alpha_C \circ \phi_{\mathcal{H}_C} = \tau$. On the other hand, the following result was proved in [1].

THEOREM 1.5. *Let $Y$ be a projective variety, $g : Y \to R$ be a morphism with a section onto a smooth curve, $r_1, \ldots, r_k \in R$ be closed points, such that $Y_i = g^{-1}(r_i)$ are smooth and separably rationally connected, $y_i \in Y_i$ be closed points. Then there is a section $C \subset Y$ of morphism $g$, such that $y_i \in C$ for $i = 1, \ldots, k$.*

Moreover, fibration $\tau$ always has a section due to the following result in [2].

THEOREM 1.6. *Let $Y$ be a smooth, proper and geometrically irreducible surface over a field $\mathbb{F}$, where $\mathbb{F}$ is a $C_1$-field and $Y$ is rational over an algebraic closure of $\mathbb{F}$. Then $Y$ has a point in $\mathbb{F}$.*

Therefore, fibration $\tau : X \to \mathbb{P}^1$ has huge set of sections. Moreover, for two general enough sections $C_1$ and $C_2$ of $\tau$ rational maps $\phi_{\mathcal{H}_{C_1}}$ and $\phi_{\mathcal{H}_{C_2}}$ give non-equivalent[1] elliptic fibrations. Therefore, the claim of Theorem 1.3 can not be improved, but it can be clarified by means of the following result of [3], because the structure of the group of birational automorphisms of a del Pezzo surface of degree two was described in [4-5].

THEOREM 1.7. *Let $Y$ be a smooth del Pezzo surface of degree two with $Pic(Y) \cong \mathbb{Z}$ defined over a field $\mathbb{F}$, $\rho : Y \dashrightarrow V$ be a birational map and $\omega : V \to R$ be a fibration on elliptic curves. Then there is a birational automorphism $\sigma$ of surface $Y$ and birational morphism $\alpha : W \to X$, such that $K^2_W = 0$, divisor $-K_W$ is nef, linear system $|-nK_W|$ is free, $\phi_{|-nK_W|}$ is an elliptic fibration for some $n \in \mathbb{N}$ and $\phi_{|-nK_W|} \circ \sigma$ is equivalent to $\omega$.*

Birational transformations into elliptic fibrations were used in [6-7] during the proof of potential density[2] of rational points on Fano 3-folds, where the following was proved.

[1]Fibrations $\tau : U \to Z$ and $\bar{\tau} : \bar{U} \to \bar{Z}$ are equivalent in case there are two birational maps $\alpha : U \dashrightarrow \bar{U}$ and $\beta : Z \dashrightarrow \bar{Z}$, such that $\bar{\tau} \circ \alpha = \beta \circ \tau$ and $\alpha$ induces an isomorphism of the generic fibers of $\tau$ and $\bar{\tau}$.

[2]Rational points of a variety $X$ defined over a number field $\mathbb{F}$ are potentialy dense if for some finite extension $\mathbb{K}/\mathbb{F}$ set of $\mathbb{K}$-rational points $X(\mathbb{K})$ are Zariski dense in $X$.

THEOREM 1.8. *Rational points are potentially dense on all smooth Fano 3-folds, with the possible exception of the family of double covers of $\mathbb{P}^3$ ramified in a smooth sextic surface.*

The possible exception appears in Theorem 1.8 due to the following result of [3].

THEOREM 1.9. *Let $\bar{\gamma}:\bar{V}\to\mathbb{P}^3$ ramified in smooth sextic. Then $\bar{V}$ is not birational to elliptic fibrations and canonical Fano 3-folds except itself, all birational maps of $\bar{V}$ into fibrations on surfaces of Kodaira dimension zero are induced by pencils in $|-K_{\bar{V}}|$.*

Other rationally connected varieties that can not be birationally transformed into elliptic fibrations were described in [3] and [8-10]. The following result was proved in [11].

THEOREM 1.10. *Let $\tilde{\gamma}:\tilde{V}\to\mathbb{P}^3$ ramified in sextic $\tilde{S}\subset\mathbb{P}^3$, such that $\tilde{S}$ is smooth outside of one point $O\in\tilde{S}$ and has singularities of type $\mathbb{A}_1$ in $O$. Then $\tilde{V}$ is not birationally isomorphic to any Fano 3-fold with canonical singularities except 3-fold $\tilde{V}$, every birational transformation of $\tilde{V}$ into a fibration on surfaces of Kodaira dimension zero is given by a pencil of K3 surfaces in $|-K_{\tilde{V}}|$, 3-fold $\tilde{V}$ is birationally isomorphic to a unique elliptic fibration, which is given by lines in $\mathbb{P}^3$ passing through point $O$.*

Birational geometry of $V$ was studied in [12], where the following result was proved.

THEOREM 1.11. *Fano 3-fold $V$ is not birationally isomorphic to a fibration on rational surfaces different from $\tau:X\to\mathbb{P}^1$ and sequence $1\to\Gamma\to Bir(V)\to\mathbb{G}\to 1$ is exact, where $\Gamma$ is a free product of Bertini involutions of a generic fiber of $\tau$ considered as a smooth del Pezzo surface of degree two with Picard group $\mathbb{Z}$ over the field $\mathbb{C}(x)$, group $\mathbb{G}$ is a group of fiberwise birational automorphisms of $\tau$ acting biregularly on generic fiber of $\tau$.*

In particular, 3-fold $V$ is not rational. The following result was proved in [13].

THEOREM 1.12. *Suppose that every fiber $F$ of $\tau$ is smooth along every curve $C\subset F$, such that $\gamma\circ f(C)$ is a line and $\gamma\circ f|_C$ is an isomorphism. Then $V$ is not birationally isomorphic to any Fano 3-fold with terminal $\mathbb{Q}$-factorial singularities and Picard group $\mathbb{Z}$ and there is an exact sequence of groups $1\to\Gamma\to Bir(V)\to Aut(V)\to 1$, where $\Gamma$ is a free product of Bertini involutions of a generic fiber of $\tau$ considered as a smooth del Pezzo surface of degree two with Picard group $\mathbb{Z}$ over the field $\mathbb{C}(x)$.*

The following result was proved in [14].

THEOREM 1.13. *Let $\tau_1:X_1\to\mathbb{P}^1$ and $\tau_2:X_2\to\mathbb{P}^1$ be fibrations on del Pezzo surfaces of degree two, $\alpha:X_1\dashrightarrow X_2$ and $\beta:\mathbb{P}^1\to\mathbb{P}^1$ be birational maps, such that $\beta\circ\tau_1=\tau_2\circ\alpha$, birational map $\alpha$ induces an isomorphism of generic fibers of fibrations $\tau_1$ and $\tau_2$, both varieties $X_1$ and $X_2$ are smooth. Then birational map $\alpha$ is biregular.*

COROLLARY 1.14. *Sequence $1\to\Gamma\to Bir(V)\to Aut(V)\to 1$ is exact, where $\Gamma$ is a free product of Bertini involutions of a generic fiber of $\tau$ considered as a smooth del Pezzo surface of degree two with Picard group $\mathbb{Z}$ over the field $\mathbb{C}(x)$.*

We will prove Corollary 1.14 in §6 and we will also prove the following result.

THEOREM 1.15. *Let $E \subset X$ be a smooth surface, $L$ be the only line on sextic $S$ passing through one of points $\theta(O_i) \in \mathbb{P}^3$, $\rho : V \dashrightarrow \mathbb{P}^1$ be a rational map, whose general fiber is an irreducible surface of Kodaira dimension zero. Then there are birational automorphism $\sigma$ of 3-fold $V$ and pencil $\mathcal{P} \subset |-K_V|$, such that $\rho \circ \sigma = \phi_{\mathcal{P}}$.*

Proofs of Theorems 1.3-1.4 and 1.15 use *method of super-maximal singularity* of [15].

*Remark 1.16.* Linear system $|-(k+1)K_X - kE|$ is free for $k \gg 0$ and gives a double cover of $Proj(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ ramified in surface $R_X \sim 4M + 2H$, where $M$ and $H$ are tautological sheaf and fiber of projection to $\mathbb{P}^1$ of $Proj(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ correspondingly.

*Example 1.17.* Hypersurface $u^2 = x^2(x^4+z^4+t^4)+y^2(y^4+z^4+2t^4)$ in $\mathbb{P}(1,1,1,3)$ fits all conditions of Theorems 1.3-1.4 and 1.15, where $x$, $y$, $z$ and $t$ are homogeneous coordinates of weight one and $u$ is a homogeneous coordinate of weight three.

## §2. MOVABLE LOG PAIRS.

In this chapter we will consider properties of movable log pairs introduced in [16].

*Definition 2.1.* Movable log pair $(X, M_X)$ is a variety $X$ together with a movable boundary $M_X$, where $M_X = \sum_{i=1}^{n} a_i \mathcal{M}_i$ is a formal finite linear combination of linear systems $\mathcal{M}_i$ on variety $X$ without fixed components, such that all $a_i \in \mathbb{Q}_{\geq 0}$.

Every movable log pair can be considered as a usual log pair by replacing every linear system with either its general element or with the apropriate weighted sum of its general elements. In particular, for a given movable log pair $(X, M_X)$ we may handle movable boundary $M_X$ as an effective divisor and we will say that $K_X + M_X$ is a log canonical divisor of movable log pair $(X, M_X)$. Moreover, in the rest of this chapter we will assume that log canonical divisors of all considered log pairs are $\mathbb{Q}$-Cartier divisors.

*Remark 2.2.* For a movable log pair $(X, M_X)$ we can condsider $M_X^2$ as a well defined effective codimension-two cycle in case $X$ is a $\mathbb{Q}$-factorial.

Strict transform of a movable boundary is well defined for any birational map.

*Definition 2.3.* Movable log pairs $(X, M_X)$ and $(Y, M_Y)$ are birationally equivalent if there is a birational map $\rho : X \dashrightarrow Y$, such that $M_Y = \rho(M_X)$.

Discrepancies, terminality, canonicity, log terminality and log canonicity can be defined for movable log pairs in the same way as for usual ones (see [17]).

*Remark 2.4.* Application of Log Minimal Model Program to a canonical and terminal movable log pairs preserves their canonicity and terminality correspondingly.

Every movable log pair is birationally equivalent to a log pair with canonical singularities and singularities of a movable log pair coincide with singularities of the variety outside of the base locuses of the components of the boundary.

*Definition 2.5.* Proper irreducible subvariety $Y \subset X$ is called a center of canonical singularities of movable log pair $(X, M_X)$, if there is a birational morphism $f : W \to X$ and such $f$-exceptinal divisor $E_1 \subset W$, that

$$K_W + f^{-1}(M_X) \sim_{\mathbb{Q}} f^*(K_X + M_X) + \sum_{i=1}^{k} a(X, M_X, E_i)E_i,$$

inequality $a(X, M_X, E_1) \leq 0$ holds and $f(E_1) = Y$, where $a(X, M_X, E_i) \in \mathbb{Q}$ and $E_i$ is an exceptional divisor of birational morphism $f$.

*Definition 2.6.* $\mathbb{CS}(X, M_X)$ will denote a set of all centers of canonical singularities of movable log pair $(X, M_X)$ and $CS(X, M_X)$ will denote a locus of all centers of canonical singularities of movable log pair $(X, M_X)$.

In particular, movable log pair $(X, M_X)$ is terminal $\iff \mathbb{CS}(X, M_X) = \emptyset$.

*Definition 2.7.* Proper irreducible subvariety $Y \subset X$ is called a proper center of canonical singularities of movable log pair $(X, M_X)$, if there is a birational morphism $f : W \to X$ and such $f$-exceptinal divisor $E_1 \subset W$, that

$$K_W + f^{-1}(M_X) \sim_{\mathbb{Q}} f^*(K_X + M_X) + \sum_{i=1}^{k} a(X, M_X, E_i)E_i,$$

inequality $a(X, M_X, E_1) \leq 0$ holds, $f(E_1) = Y$ and $Y$ is contained in the base locus of some component of movable boundary $M_X$, where $a(X, M_X, E_i) \in \mathbb{Q}$ and $E_i$ is an exceptional divisor of birational morphism $f$.

*Definition 2.8.* $\overline{\mathbb{CS}}(X, M_X)$ will denote a set of all proper centers of canonical singularities of movable log pair $(X, M_X)$ and $\overline{CS}(X, M_X)$ will denote a locus of all proper centers of canonical singularities of movable log pair $(X, M_X)$.

*Remark 2.9.* $\mathbb{CS}(X, M_X) = \overline{\mathbb{CS}}(X, M_X) \cup \mathbb{CS}(X, \emptyset)$, in particular, in case when $X$ has terminal singularities $\mathbb{CS}(X, M_X) = \overline{\mathbb{CS}}(X, M_X)$.

*Definition 2.10.* Singularities of log pair $(X, M_X)$ is semi-terminal if $\overline{\mathbb{CS}}(X, M_X) = \emptyset$.

*Remark 2.11.* Suppose singularities of movable log pair $(X, M_X)$ is semi-terminal, then for small enough $\epsilon \in \mathbb{Q}_{>1}$ singularities of movable log pair $(X, \epsilon M_X)$ is still semi-terminal.

*Definition 2.12.* Number

$$\kappa(X, M_X) = \begin{cases} dim(\phi_{|nm(K_W+M_W)|}(W)) \text{ for such } n \gg 0, \text{ that } |n(K_W + M_W)| \neq \emptyset, \\ -\infty \text{ when } |nm(K_W + M_W)| = \emptyset \text{ for all natural numbers } n. \end{cases}$$

is called Kodaira dimension of movable log pair $(X, M_X)$, where $(W, M_W)$ is a movable log pair with canonical singularities, which is birationally equivalent to $(X, M_X)$, and $m \in \mathbb{N}$, such that divisor $m(K_W + M_W)$ is Cartier.

LEMMA 2.13. *Kodaira dimension of a movable log pair is well defined and does not depend on the choice of the birationally equivalent canonical movable log pair.*

*Proof.* Suppose two movable log pairs $(X, M_X)$ and $(Y, M_Y)$ have canonical singularities and $M_X = \rho(M_Y)$ for some birational map $\rho : Y \dashrightarrow X$. Consider natural $m_X$ and $m_Y$, such that divisors $m(K_X + M_X)$ and $m(K_Y + M_Y)$ are Cartier. To prove the claim we should show that either $|nm_X(K_X + M_X)| = |nm_Y(K_Y + M_Y)| = \emptyset$ for all natural $n$, or

$$\phi_{|nm_X(K_X+M_X)|}(X) = \phi_{|nm_Y(K_Y+M_Y)|}(Y)$$

for big enough natural numbers $n$. Let $W$ be a variety and $g : W \to X$ and $f : W \to Y$ be birational morphisms, such that $\rho = g \circ f^{-1}$. Then

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \Sigma_X \sim_{\mathbb{Q}} f^*(K_Y + M_Y) + \Sigma_Y,$$

where $M_W = g^{-1}(M_X)$, $\Sigma_X$ and $\Sigma_Y$ are exceptional divisors of $g$ and $f$ correspondingly and canonicity of $(X, M_X)$ and $(Y, M_Y)$ implies effectivity of $\Sigma_X$ and $\Sigma_Y$. Let $k$ be big enough and divisible enough natural number. Then effectivity of $\Sigma_X$ and $\Sigma_V$ implies that dimensions of linear systems $|k(K_W + M_W)|$, $|g^*(k(K_X + M_X))|$ and $|f^*(k(K_Y + M_Y))|$ are the same and in case they are not empty

$$\phi_{|k(K_W+M_W)|} = \phi_{|g^*(k(K_X+M_X))|} = \phi_{|f^*(k(K_Y+M_Y))|},$$

which implies the claim. $\square$

By definition Kodaira dimension of a movable log pair is a birational invariant and a non-decreasing function of the coefficients of the movable boundary.

*Definition 2.14.* Movable log pair $(V, M_V)$ is called a canonical model of a movable log pair $(X, M_X)$, if there is a birational map $\psi : X \dashrightarrow V$, such that $M_V = \psi(M_X)$, log canonical divisor $K_V + M_V$ is ample and $(V, M_V)$ has canonical singularities.

THEOREM 2.15. *Canonical model of a movable log pair is unique if it exists.*

*Proof.* Suppose that both movable log pairs $(X, M_X)$ and $(V, M_V)$ are canonical models and $M_X = \rho(M_V)$ for some birational map $\rho : V \dashrightarrow X$. Let $W$ be a variety and $g : W \to X$ and $f : W \to V$ be birational morphisms, such that $\rho = g \circ f^{-1}$. Then

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \Sigma_X \sim_{\mathbb{Q}} f^*(K_V + M_V) + \Sigma_V,$$

where $M_W = g^{-1}(M_X) = f^{-1}(M_V)$ and $\Sigma_X$ and $\Sigma_V$ are exceptional divisors of birational morphisms $g$ and $f$ correspondingly. Canonicity of log pairs $(X, M_X)$ and $(V, M_V)$ implies effectivity of divisors $\Sigma_X$ and $\Sigma_V$. Let $n$ be big enough and divisible enough natural number, such that divisors $n(K_W + M_W)$, $n(K_X + M_X)$ and $n(K_V + M_V)$ are Cartier divisors. Then effectivity of divisors $\Sigma_X$ and $\Sigma_V$ implies that

$$\phi_{|n(K_W+M_W)|} = \phi_{|g^*(n(K_X+M_X))|} = \phi_{|f^*(n(K_V+M_V))|}$$

and $\rho$ is an isomorphism, because $K_X + M_X$ and $K_V + M_V$ are ample. $\square$

Kodaira dimension of the canonical model equals to the dimension of the variety.

## §3. Preliminary results.

As we already mentioned in the previous chapter, movable boundaries can be considered as effective divisors and movable log pairs can be considered as usual ones. Therefore, we may use log pairs containing movable components. We will not assume any restrictions on the coefficients of the boundaries, in particular, boundaries may not be effective unless effectiveness is explicitly stated.

We will assume that log canonical divisors of all log pairs are $\mathbb{Q}$-Cartier divisors.

*Definition 3.1.* For a log pair $(X, B_X)$ and birational map $f : V \to X$ log pair $(V, B^V)$ is called a log pull back of log pair $(X, B_X)$ if $B^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i$ and equivalence $K_V + B^V \sim_{\mathbb{Q}} f^*(K_X + B_X)$ holds, where $a(X, B_X, E_i) \in \mathbb{Q}$ and $E_i$ is an exceptional divisor of birational map $f$.

*Definition 3.2.* Proper irreducible subvariety $Y \subset X$ is called a center of log canonical singularities of $(X, B_X)$, if there are birational morphism $f : W \to X$ and divisor $E \subset W$, such that $E$ is contained in the support of effective part of divisor $\lfloor B^Y \rfloor$.

*Definition 3.3.* $\mathbb{LCS}(X, B_X)$ will denote a set of all centers of log canonical singularities of log pair $(X, B_X)$ and $LCS(X, B_X)$ will denote a locus of all centers of canonical singularities of log pair $(X, M_X)$.

Consider log pair $(X, B_X)$, where $B_X = \sum_{i=1}^{k} a_i B_i$, $B_i$ are effective and prime divisors on variety $X$, $a_i \in \mathbb{Q}$. Choose a birational morphism $f : Y \to X$, such that $W$ is smooth and union of all proper transforms of divisors $B_i$ on $Y$ and $f$-exceptional divisors forms divisor with simple normal crossing. Morphism $f$ is called a log resolution of $(X, B_X)$ and we have $K_Y + B^Y \sim_{\mathbb{Q}} f^*(K_X + B_X)$ for a log pull back $(Y, B^Y)$ of $(X, B_X)$.

*Definition 3.4.* Subscheme associated with ideal sheaf $\mathcal{I}(X, B_X) = f_*(\lceil -B^Y \rceil)$ is called a log canonical singularities subscheme of log pair $(X, B_X)$ and denoted as $\mathcal{L}(X, B_X)$.

Support of subscheme $\mathcal{L}(X, B_X)$ is exactly the locus $LCS(X, B_X)$. The following result is a famous vanishing theorem of V.Shokurov.

Theorem 3.5. *Let $(X, B_X)$ be a log pair, $B_X$ be an effective, $H$ be a nef and big divisor on $X$, such that $D = K_X + B_X + H$ is Cartier. Then $H^i(X, \mathcal{I}(X, B_X) \otimes D) = 0$ for $i > 0$.*

*Proof.* $R^i f_*(f^*(K_X + B_X + H) + \lceil -B^W \rceil) = 0$ for $i > 0$ by relative Kawamata-Viehweg vanishing (see [17]). Degeneration of local-to-global spectral sequence and

$$R^0 f_*(f^*(K_X + B_X + H) + \lceil -B^W \rceil) = \mathcal{I}(X, B_X) \otimes D$$

imply that for all $i$

$$H^i(X, \mathcal{I}(X, B_X) \otimes D) = H^i(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil),$$

but $H^i(W, f^*(K_X+B_X+H)+\lceil -B^W \rceil) = 0$ for $i > 0$ by Kawamata-Viehweg vanishing. $\square$

For any Cartier divisor $D$ on $X$ we have the exact sequence

$$0 \to \mathcal{I}(X, B_X) \otimes D \to \mathcal{O}_X(D) \to \mathcal{O}_{\mathcal{L}(X,B_X)}(D) \to 0$$

and Theorem 3.5 implies the following two connectedness theorems of V.Shokurov.

THEOREM 3.6. *Let $(X, B_X)$ be a log pair, $B_X$ be an effective, divisor $-(K_X + B_X)$ be nef and big. Then locus $LCS(X, B_X)$ is connected.*

THEOREM 3.7. *Let $(X, B_X)$ be a log pair, $B_X$ be an effective, divisor $-(K_X + B_X)$ be $g$-nef and $g$-big for some morphism $g : X \to Z$ with $g_*(\mathcal{O}_X) = \mathcal{O}_Z$. Then $LCS(X, B_X)$ is connected in the neighbourhood of any fiber of $g$.*

The following result is Theorem 17.4 of [18].

THEOREM 3.8. *Let $g : X \to Z$ be a morphism of normal varieties, $D_X = \sum_{i\in I} d_i D_i$ be a divisor on $X$, $h : V \to X$ be a resolution of singularities of $X$, such that $g_*(\mathcal{O}_X) = \mathcal{O}_Z$, divisor $-(K_X + D_X)$ is $g$-nef and $g$-big, codimension of every subvariety $g(D_i) \subset Z$ is greater or equal then $2$ in case $d_i < 0$, union of all divisors $h^{-1}(D_i)$ and all $h$-exceptional divisors forms a divisor with simple normal crossing. Then locus $\cup_{a_E\leq -1} E$ is connected in a neighbourhood of every fiber of morphism $g \circ h$, where rational numbers $a_E$ are given by $\mathbb{Q}$-rational equivalence $K_V \sim_{\mathbb{Q}} f^*(K_X + D_X) + \sum_{E\subset V} a_E E$.*

*Proof.* Let $f = g \circ h$, $A = \sum_{a_E > -1} E$ and $B = \sum_{a_E\leq -1} E$. Then

$$\lceil A\rceil - \lfloor B\rfloor \sim_{\mathbb{Q}} K_V - h^*(K_X + D_X) + \{-A\} + \{B\}$$

and $R^1 f_*\mathcal{O}_V(\lceil A\rceil - \lfloor B\rfloor) = 0$ by relative Kawamata-Viehweg vanishing (see [17]). Hence, natural map $f_*\mathcal{O}_V(\lceil A\rceil) \to f_*\mathcal{O}_{\lfloor B\rfloor}(\lceil A\rceil)$ is surjective. On the other hand, every irreducible component of divisor $\lceil A\rceil$ is either $h$-exceptional or a proper transform of some $D_j$ with negative $d_j$. Therefore, $h_*(\lceil A\rceil)$ is $g$-exceptional. Thus $f_*\mathcal{O}_V(\lceil A\rceil) = \mathcal{O}_Z$ and there is a surjection $\mathcal{O}_Z \to f_*\mathcal{O}_{\lfloor B\rfloor}(\lceil A\rceil)$, which implies that $\lfloor B\rfloor$ is connected in a neighbourhood of every fiber of morphism $f$, because divisor $\lceil A\rceil$ is effective and has no common component with divisor $\lfloor B\rfloor$. □

In the previous chapter in Definitions 2.5 and 2.6 we defined the center of canonical singularities of movable log pair, set of centers of canonical singularities of movable log pair and locus of canonical singularities of movable log pair. Actually, movability of a boundary has nothing to do with all these notions, but we defined them for movable log pairs on purpose, because they are used mostly for movable log pairs and they arrise naturaly in certain constructions related with movable log pairs. Therefore, we will use these notions for regular log pairs as well.

THEOREM 3.9. *Let $(X, B_X)$ be a log pair, $B_X$ be an effective boundary, $Z$ be a center of canonical singularities of log pair $(X, B_X)$ and $H$ be an effective irreducible Cartier divisor on variety $X$, such that $Z \subset H$, $H$ is not a component of $B_X$, both $X$ and $H$ are smooth in the generic point of $Z$. Then $Z \in \mathbb{LCS}(H, B_X|_H)$.*

*Proof.* Let $f : W \to X$ be log resolution of $(X, B_X + H)$ and $\hat{H} = f^{-1}(H)$. Then

$$K_W + \hat{H} \sim_{\mathbb{Q}} f^*(K_X + B_X + H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E$$

and by assumption $\{Z, H\} \subset LCS(X, B_X + H)$. Therefore, application Theorem 3.8 to the log pulback of $(X, B_X + H)$ on $W$ gives $\hat{H} \cap E \neq \emptyset$ for some divisor $E \neq \hat{H}$ on $W$, such that $f(E) = Z$ and $a(X, B_X, E) \leq -1$. Now equivalence

$$K_{\hat{H}} \sim (K_W + \hat{H})|_{\hat{H}} \sim_{\mathbb{Q}} f|_{\hat{H}}^*(K_H + B_X|_H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E|_{\hat{H}}$$

implies the claim. □

Next result is Theorem 3.1 of [19], it is so useful that we will provide a proof from [19].

THEOREM 3.10. *Let $H$ be a smooth surface, $O$ be a point on surface $H$, $M_H$ be an effective movable boundary on surface $H$, $a_1$ and $a_2$ be non-negative rational numbers, $\Delta_1$ and $\Delta_2$ be irreducible and reduced curves on surface $H$, which are intersecting normally at point $O$. Then $O \in \mathbb{LCS}(H, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H)$ implies inequality*

$$mult_O(M_H^2) \geq \begin{cases} 4a_1a_2 \text{ in case } a_1 \leq 1 \text{ or } a_2 \leq 1 \\ 4(a_1 + a_2 - 1) \text{ in case } a_1 > 1 \text{ and } a_2 > 1 \end{cases}$$

*and inequality is strict if log pair $(H, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H)$ is not log canonical in the neighbourhood of point $O$.*

*Proof.* Let $D = (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H$ and $f : S \to H$ be a birational morphism, such that surface $S$ is smooth and

$$K_S + f^{-1}(D) \sim_{\mathbb{Q}} f^*(K_H + D) + \sum_{i=1}^{k} a(H, D, E_i)E_i$$

where $E_i$ is $f$-exceptional curve, $a(H, D, E_i) \in \mathbb{Q}$ and $a(H, D, E_1) \leq -1$. Then birational morphism $f$ is a composition of $k$ blow ups of smooth points.

Suppose that the claim is proved when $a_1 \leq 1$ or $a_2 \leq 1$. We may assume that $a_1 > 1$ and $a_2 > 1$. Define rational numbers $a(H, E_i)$, $m(H, M_H, E_i)$ and $m(H, \Delta_j, E_i)$ by means of $\sum_{i=1}^k a(H, E_i)E_i \sim_{\mathbb{Q}} K_S - f^*(K_H)$, $\sum_{i=1}^k m(H, M_H, E_i)E_i \sim_{\mathbb{Q}} f^{-1}(M_H) - f^*(M_H)$ and also $\sum_{i=1}^k m(H, \Delta_j, E_i)E_i \sim_{\mathbb{Q}} f^{-1}(\Delta_j) - f^*(\Delta_j)$. Then

$$a(H, D, E_i) = a(H, E_i) - m(H, M_H, E_i) + m(H, \Delta_1, E_i)(a_1 - 1) + m(H, \Delta_2, E_i)(a_2 - 1)$$

and we may assume that $m(H, \Delta_1, E_1) \geq m(H, \Delta_2, E_1)$. Therefore,

$$-1 \geq a(H, D, E_1) \geq a(H, E_i) - m(H, M_H, E_i) + m(H, \Delta_2, E_i)(a_1 + a_2 - 2)$$

and $O \in \mathbb{LCD}(H, (2-a_1-a_2)\Delta_2 + M_H)$. This implies $mult_O(M_H^2) \geq 4(a_1+a_2-1)$, because we assumed that the claim holds for log pair $(H, (2-a_1-a_2)\Delta_2 + M_H)$.

We may assume that $a_1 \leq 1$. Let $h : T \to H$ be a blow up of point $O$ and $E$ be an exceptional curve of $h$. Then $f$ factors through $h$, such that $f = g \circ h$ for some birational morphism $g : S \to T$, which is a composition of $k-1$ blow ups of smooth points. Then

$$K_T + (1-a_1)\bar{\Delta}_1 + (1-a_2)\bar{\Delta}_2 + (1-a_1-a_2+m)E + M_T \sim_{\mathbb{Q}} h^*(K_H + D),$$

where $\bar{\Delta}_j = h^{-1}(\Delta_j)$, $m = mult_O(M_H)$ and $M_T = h^{-1}(M_H)$.

In case $k=1$ we have $S=T$, $E_1 = E$ and $a(H, D, E_1) = a_1 + a_2 - m - 1 \leq -1$. Thus

$$mult_O(M_H^2) \geq m^2 \geq (a_1+a_2)^2 \geq 4a_1a_2$$

and we are done. Therefore, we may assume that $k > 1$ and $g(E_1)$ is a point $P \in E$.

We have $P \in \mathbb{LCD}(T, (1-a_1)\bar{\Delta}_1 + (1-a_2)\bar{\Delta}_2 + (1-a_1-a_2+m)E + M_T)$ and there are three possible cases: $P \in E \cap \bar{\Delta}_1$, $P \in E \cap \bar{\Delta}_2$ and $P \notin \bar{\Delta}_1 \cup \bar{\Delta}_2$. Moreover, by induction we may assume that the claim holds for log pair $(T, (1-a_1)\bar{\Delta}_1 + (1-a_1-a_2+m)E + M_T)$ in case $P \in E \cap \bar{\Delta}_1$, for log pair $(T, (1-a_2)\bar{\Delta}_2 + (1-a_1-a_2+m)E + M_T)$ in case $P \in E \cap \bar{\Delta}_2$ or for log pair $(T, (1-a_1-a_2+m)E + M_T)$ in case $P \notin \bar{\Delta}_1 \cup \bar{\Delta}_2$, because all conditions of the theorem are satisfied in on of these cases and $g$ consists of $k-1$ blow ups of smooth points. Moreover, $mult_O(M_H^2) \geq m^2 + mult_P(M_T^2)$.

Consider case $P \in E \cap \bar{\Delta}_1$. Then by induction

$$mult_O(M_H^2) \geq m^2 + 4a_1(a_1+a_2-m) = (2a_1-m)^2 + 4a_1a_2 \geq 4a_1 4a_2.$$

Consider case $P \in E \cap \bar{\Delta}_2$. If either $a_2 \leq 1$ or $a_1 + a_2 - m \leq 1$, then we can proceed as in previous case. Thus, we may assume that $a_2 < 1$ and $a_1 + a_2 - m < 1$. Then

$$mult_O(M_H^2) \geq m^2 + 4(a_1 + 2a_2 - m - 1) > 4a_2 \geq 4a_1 4a_2.$$

Consider case $P \notin \bar{\Delta}_1 \cup \bar{\Delta}_2$. Then by induction

$$mult_O(M_H^2) \geq m^2 + 4(a_1+a_2-m) > m^2 + 4a_1(a_1+a_2-m) \geq 4a_1 4a_2,$$

which completes the proof. □

Most applications use the following simplified version of Theorem 3.11.

LEMMA 3.11. *Let $H$ be a smooth surface, $O$ be a point on $H$, $M_H$ be an effective movable boundary on surface $H$ and $O \in \mathbb{LCS}(H, M_H)$. Then $mult_O(M_H^2) \geq 4$ and equality holds only if $mult(M_H) = 2$.*

The following result is a Corollary 7.3 of [20].

THEOREM 3.12. *Let $X$ be a smooth 3-fold, $O$ be a point on $X$, $M_X$ be an effective movable boundary on $X$ and $O \in \mathbb{CS}(X, M_X)$. Then $mult_O(M_X^2) \geq 4$ and equality holds only if $mult_O(M_X) = 2$.*

*Proof.* Let $H$ be a general very ample divisor on $X$ containing $O$. Then $O$ is a center of log canonical singularities of log pair $(H, M_X|_H)$ by Theorem 3.9. On the other hand,

$$mult_O(M_X) = mult_O(M_X|_H) \text{ and } mult_O(M_X^2) = mult_O((M_X|_H)^2)$$

and the claim follows from Theorem 3.11. $\square$

Actually, statement of Theorem 3.12 can be explained in much more geometric way.

LEMMA 3.13. *Let $X$ be a smooth 3-fold, $O$ be a point on $X$ and $M_X$ be an effective movable boundary on 3-fold $X$, such that singularities of log pair $(X, M_X)$ are canonical and $O \in \mathbb{CS}(X, M_X)$. Then there is a birational map $f : V \to X$, such that $V$ has terminal $\mathbb{Q}$-factorial singularities and morphism $f$ contracts exactly one exceptional divisor $E$ to point $O$ and $K_V + M_V \sim_{\mathbb{Q}} f^*(K_X + M_X)$, where $M_V = f^{-1}(M_X)$.*

*Proof.* Number of different divisorial discrete valuations $\nu$ of the field of rational functions of $X$, whose center on $X$ is point $O$ and whose discripancy $a(X, M_X, \nu)$ is non-positive, is finite due to canonicity of log pair $(X, M_X)$. Therefore, we may consider birational morphism $g : W \to X$, such that 3-fold $W$ is smooth, $g$ contracts $k$ divisors,

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \sum_{i=1}^{k} a_i E_i,$$

movable log pair $(W, M_W)$ has canonical singularities and set $\mathbb{CS}(W, M_W)$ does not contain subvarieties of $\cup_{i=1}^{k} E_i$, where $M_W = g^{-1}(M_X)$, $g(E_i) = O$ and $a_i \in \mathbb{Q}$. Applying relative version of Log Minimal Model Program to movable log pair $(W, M_W)$ over $X$ we may assume that $W$ has terminal $\mathbb{Q}$-factorial singularities $K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X)$ due to canonicity of $(X, M_X)$. Applying relative Minimal Model Program to $W$ over $X$ we got the neccesary 3-fold and birational morphism. $\square$

The following result was conjectured in [21] and proved in [22].

THEOREM 3.14. *Let $X$ be a smooth 3-fold, $O$ be a point on $X$, $f : V \to X$ be a birational morphism, such that $V$ has terminal $\mathbb{Q}$-factorial singularities and morphism $f$ contracts exactly one exceptional divisor $E$ to point $O$. Then $f$ is a weighted blow up of $O$ with weights $(1, K, N)$ in suitable local coordinates on $X$, where natural $K$ and $N$ are coprime.*

Moreover, Theorem 3.12 was proved in [21] up to the existence of Theorem 3.14 in the following way, which explains the geometrical nature of the inequality in Theorem 3.12.

PROPOSITION 3.15. *Let $X$ be a smooth 3-fold, $O$ be a point on $X$, $M_X$ be an effective movable boundary on $X$, $O \in \mathbb{CS}(X, M_X)$, $f : V \to X$ be a weighted blow up of point $O$ with weights $(1, K, N)$ in suitable local coordinates on $X$, such that $K_V + M_V \sim_{\mathbb{Q}} f^*(K_X + M_X)$, where naturals $L$ and $N$ are coprime and $M_V = f^{-1}(M_X)$. Then the following holds*

$$mult_O(M_X^2) \geq \frac{(K+N)^2}{KN} = 4 + \frac{(K-N)^2}{KN} \geq 4$$

*and $K=N$ implies that $f$ is a regular blow up of $O$ and $mult_O(M_X)=2$.*

*Proof.* On $V$ we have $K_V \sim_{\mathbb{Q}} f^*(K_X)+(N+K)E$ and $M_V \sim_{\mathbb{Q}} f^*(M_X)+mE$ for $f$-exceptional divisor $E$ and $m \in \mathbb{Q}_{>0}$. Thus, $m=K+N$ and $mult_O(M_X^2) \geq m^2E^3 = \frac{(K+N)^2}{KN}$. □

## §4. Construction of birational map $\rho_{i,k}$.

In the notations of §1 let $g: W \to X$ be a blow up of $Z_{1,0}$, $G$ be a $g$-exceptional divisor and $\bar{Z}_{1,1}$ be a proper transform of $Z_{1,1}$ on $W$. Then $K_W \cdot \bar{Z}_{1,1} = 1$.

Lemma 4.1. *$\mathcal{N}_{\bar{Z}_{1,1}/W} \cong \mathcal{O}_{\bar{Z}_{1,1}}(-1) \oplus \mathcal{O}_{\bar{Z}_{1,1}}(-2)$.*

*Proof.* Let $\mathcal{N}_{\bar{Z}_{1,1}/W} \cong \mathcal{O}_{\bar{Z}_{1,1}}(m) \oplus \mathcal{O}_{\bar{Z}_{1,1}}(n)$ for $m \geq n$ and $H$ be a general enough surface in linear system $|-K_W|$ passing through curve $\bar{Z}_{1,1}$. Then $H$ is a smooth K3 surface, $m+n = -K_W \cdot \bar{Z}_{1,1} - 2 = -3$ and exact sequence

$$0 \to \mathcal{N}_{\bar{Z}_{1,1}/H} \to \mathcal{N}_{\bar{Z}_{1,1}/W} \to \mathcal{N}_{H/W}|_{\bar{Z}_{1,1}} \to 0$$

implies $n \geq -2$. Thus, $n=-2$ and $m=-1$. □

Let $h: U \to W$ be a blow up of curve $\bar{Z}_{1,1}$, $F$ be a $h$-exceptional divisor and $G_U$ be a proper transform of divisor $G$ on 3-fold $U$. Then $F \cong \mathbb{F}_1$, $K_U \sim (g \circ h)^*(K_X) + F + G_U$, anticanonical linear system $|-K_U|$ is free and morphism $\phi_{|-K_U|}: U \to \mathbb{P}^2$ is an elliptic fibration with quasi-sections $F$ and $G_U$.

*Remark 4.2.* Elliptic curves in fibers of $\phi_{|-K_U|}$ are mapped to lines on $\mathbb{P}^3$ passing through point $O_1$ by morphism $\theta \circ f \circ g \circ h$ and $\phi_{|-K_U|}$ is induced by projection from point $O_1$.

Let $C \subset U$ be an exceptional section of $F \cong \mathbb{F}^1$. Then $C$ lies in fibers of $\phi_{|-K_U|}$.

Lemma 4.3. *$\mathcal{N}_{C/U} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$.*

*Proof.* Let $\mathcal{N}_{C/U} \cong \mathcal{O}_C(m) \oplus \mathcal{O}_C(n)$ for $m \geq n$. Then $m+n = -K_U \cdot C - 2$, but exact sequence $0 \to \mathcal{N}_{C/F} \to \mathcal{N}_{C/U} \to \mathcal{N}_{F/U}|_C \to 0$ implies $n \geq -1$. Hence, $m=n=-1$. □

Let $\hat{p}: \hat{U} \to U$ be a blow up of $C$, $R$ be a $\hat{p}$-exceptional divisor, $\check{p}: \hat{U} \to \check{U}$ be a blow down of $R \cong \mathbb{P}^1 \times \mathbb{P}^1$ to a curve $\check{C}$, such that $\check{p} \circ \hat{p}^{-1}$ is not biregular. Then $\check{p} \circ \hat{p}^{-1}$ is a flop in curve $C$, linear system $|-K_{\check{U}}|$ is free and $\phi_{|-K_{\check{U}}|} \circ \check{p} = \phi_{|-K_U|} \circ \hat{p}$.

*Remark 4.4.* Proper transform $\check{F} \subset \check{U}$ of $F$ is $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$.

Therefore, we can contract $\check{h}: \check{U} \to \check{W}$ divisor $\check{F}$ to a singular point $P \cong \frac{1}{2}(1,1,1)$.

Lemma 4.5. *$\check{W}$ is projective.*

*Proof.* Antiflip $\check{h} \circ \check{p} \circ \hat{p}^{-1} \circ h^{-1}$ is a log flip for log terminal log pair $(W, \epsilon|-K_W|)$ for small enough rational $\epsilon > 1$, which implies the claim. □

Lemma 4.6. *Base locus of* $|-K_{\check{W}}|$ *consists of* $P$ *and* $-K_{\check{W}}^3 = \frac{1}{2}$.

*Proof.* Linear system $|-K_{\check{W}}|$ is a proper transform of free linear system $|-K_{\check{U}}|$, which implies that base locus of linear system $|-K_{\check{W}}|$ consists of point $P$. Equivalence

$$K_{\check{U}} \sim_{\mathbb{Q}} \check{h}^*(K_{\check{W}}) + \frac{1}{2}\check{F}$$

and equalities $K_{\check{U}}^3 = 0$ and $\check{F}^3 = 4$ imply $-K_{\check{W}}^3 = \frac{1}{2}$. $\square$

Hence, divisor $-K_{\check{W}}$ is nef and big, linear system $|-nK_{\check{W}}|$ is free for $n \gg 0$ and gives birational morphism $\phi_{|-nK_{\check{W}}|} : \check{W} \to V_{1,0}$. Therefore, we constructed a birational map

$$\rho_{1,0} = \phi_{|-nK_{\check{W}}|} \circ \check{h} \circ \check{p} \circ \hat{p}^{-1} \circ h^{-1} \circ g^{-1} \circ f^{-1} : V \dashrightarrow V_{1,0},$$

where $V_{1,0}$ is a Fano 3-fold with canonical singularities and $-K_{V_{1,0}}^3 = \frac{1}{2}$.

*Remark 4.7.* Actually, singularities of $V_{1,0}$ are terminal and non-$\mathbb{Q}$-factorial.

Constructions of $\rho_{i,k} : V \dashrightarrow V_{i,k}$ are identical to the construction of $\rho_{1,0}$.

## §5. Proof of Theorem 1.3.

In the notations of §1 suppose the existence of birational transformation $\beta : V \dashrightarrow Y$ and fibration $\pi : Y \to \mathbb{P}^2$, whose general fiber is connected smooth elliptic curve, and suppose there is no rational map $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, such that $\alpha \circ \rho = \tau \circ f^{-1}$. In the rest of this chapter we will derive a contradiction. Put $\rho = \pi \circ \beta$ and consider linear system $|\pi^*(D)|$, where $D$ is a very ample divisor on $\mathbb{P}^2$. Let $\mathcal{D}_V = \beta^{-1}(|\pi^*(D)|)$. Then $\mathcal{D}_V \subset |-nK_V|$ for some natural number $n$. Consider movable boundary $\frac{1}{n}\mathcal{D}_V$.

Lemma 5.1. $\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V) \neq \emptyset$.

*Proof.* Suppose $(V, \frac{1}{n}\mathcal{D}_V)$ is semi-terminal. Then for some $\epsilon \in \mathbb{Q}_{>\frac{1}{n}}$ log pair $(V, \epsilon\mathcal{D}_V)$ is a canonical model and $\kappa(V, \epsilon\mathcal{D}_V) = 3$, but $\kappa(V, \epsilon\mathcal{D}_V) = 2$ by construction of $\mathcal{D}_V$. $\square$

Therefore, set $\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ contains either a point or a curve on $V$.

*Remark 5.2.* $\mathcal{D}_V$ is not contained in the fibers of any dominant map $\chi : V \dashrightarrow \mathbb{P}^1$.

Lemma 5.3. $\mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$ *does not contain smooth points of* $V$.

*Proof.* Suppose $\mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$ contains a point $O \notin \tilde{L}$. Take general $H_O \in |-K_V|$ passing through $O$. Then $2n^2 = H_O \cdot \mathcal{D}_V^2 \geq n^2 mult_O(\mathcal{D}_V^2) \geq 4n^2$ by Theorem 3.12. $\square$

Lemma 5.4. *Suppose* $\mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$ *contains a curve* $C \neq \tilde{L}$. *Then* $-K_V \cdot C = 1$.

*Proof.* Inequality $mult_C(\mathcal{D}_V) \geq n$ is equivalent to $C \in \mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$. Thus, for general enough divisor $H$ in the base free linear system $|-K_V|$ inequalities

$$2n^2 = H \cdot \mathcal{D}_V^2 \geq mult_C(\mathcal{D}_V^2) H \cdot C \geq mult_C^2(\mathcal{D}_V) H \cdot C \geq n^2 H \cdot C$$

imply $-K_V \cdot C \leq 2$. Suppose $-K_V \cdot C = 2$. Then $mult_C(\mathcal{D}_V^2) = mult_C^2(\mathcal{D}_V) = n^2$ and support of one-dimensional cycle $\mathcal{D}_V^2$ coincide with $C$. By construction $\phi_{\mathcal{D}_V}(V) = \mathbb{P}^2$ and there is a point $P \in V$, such that $P \notin C$ and linear subsystem $\mathcal{D}_P \subset \mathcal{D}_V$ of surfaces in linear system $\mathcal{D}_V$ passing through $P$ has no fixed components. Hence, in set-theoretic sense we have $P \in \mathcal{D}_P^2 \subset \mathcal{D}_V^2 = C$, which is a contradiction. □

LEMMA 5.5. *$\mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$ does not contain a curve $C$ with $C \cap \tilde{L} = \emptyset$.*

*Proof.* Let $C \subset V$ be a curve, $C \in \mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$ and $C \cap \tilde{L} = \emptyset$. Then $\gamma(C) \subset \mathbb{P}^3$ is a line and $\gamma|_C$ is an isomorphism by Lemma 5.4.

Suppose that $\gamma(C) \not\subset S$. Let $\mathcal{H}_C \subset |-K_V|$ be a linear system of surfaces passing through curve $C$, $H_C$ be a general enough surface in $\mathcal{H}_C$ and $\tilde{C}$ be curve on 3-fold $V$, such that $\tilde{C} \neq C$ and $\gamma(\tilde{C}) = \gamma(C)$. Then $H_C$ is smooth K3 surface containing $C$ and $\tilde{C}$, on surface $H_C$ we have $C^2 = \tilde{C}^2 = -2$ and $C \cdot \tilde{C} = 3$, base locus of $\mathcal{H}_C$ consists of $C \cup \tilde{C}$ and

$$\mathcal{D}_V|_{H_C} = mult_C(\mathcal{D}_V) C + mult_{\tilde{C}}(\mathcal{D}_V) \tilde{C} + R_{H_C},$$

where $R_{H_C}$ is a movable boundary on $H_C$. Hence, equalities

$$n = \mathcal{D}_V|_{H_C} \cdot \tilde{C} = 3 mult_C(\mathcal{D}_V) - 2 mult_{\tilde{C}}(\mathcal{D}_V) + R_{H_C} \cdot \tilde{C}$$

imply $mult_{\tilde{C}}(\mathcal{D}_V) = mult_C(\mathcal{D}_V) = n$ and $R_{H_C} = \emptyset$. Therefore, set $\mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$ contains curve $\tilde{C}$ as well and $R_{H_C} = \emptyset$ implies that $\mathcal{D}_V$ lies in the fibers of $\phi_{\mathcal{H}_C}$, which is impossible.

Suppose $\gamma(C) \subset S$. Let $g : W \to V$ be a blow up of curve $C$, $G$ be a $g$-exceptional divisor and $\mathcal{D}_W = g^{-1}(\mathcal{D}_V)$. Then $|-K_W|$ is a pencil and its base locus consists of smooth rational curve $\bar{C}$, which is a section of ruled surface $g|_G : G \to C$. Take general enough divisor $H_{\bar{C}}$ in linear system $|-K_W|$. Then $H_{\bar{C}}$ is smooth K3 surface and

$$\mathcal{D}_W|_{H_{\bar{C}}} = mult_{\bar{C}}(\mathcal{D}_W) \bar{C} + R_{H_{\bar{C}}},$$

where $R_{H_{\bar{C}}}$ is a movable boundary on $H_{\bar{C}}$. On surface $H_{\bar{C}}$ we have $\bar{C}^2 = -2$ and

$$\mathcal{D}_W|_{H_{\bar{C}}} \sim_{\mathbb{Q}} n\bar{C} + (n - mult_C(\mathcal{D}_V)) G|_{H_{\bar{C}}},$$

which implies $mult_{\bar{C}}(\mathcal{D}_W) = mult_C(\mathcal{D}_V) = n$ and $R_{H_{\bar{C}}} = \emptyset$. Now $R_{H_{\bar{C}}} = \emptyset$ implies that linear system $\mathcal{D}_W$ lies in the fibers of map $\phi_{|-K_W|} : W \dashrightarrow \mathbb{P}^1$, which is impossible. □

LEMMA 5.6. *Suppose $\tilde{L} \notin \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$. Then $\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ does not contain curves.*

*Proof.* Curve $\tilde{L}$ is not contained in the base locus of $\mathcal{D}_V$, because set $\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ does not contain curve $\tilde{L}$. In particular, proper transform $\mathcal{D}_X$ of linear system $\mathcal{D}_V$ on $X$ is a linear subsystem of $|-nK_X|$. We must show that set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ does not contain curves on 3-fold $X$ not contained in $E$. Suppose $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains a curve $C \subset X$, which is not contained in $E$. Then Lemmas 5.4 and 5.5 imply $\gamma \circ f(C)$ is a line, $\gamma \circ f|_C$ is an isomorphism and $C \cap E \neq \emptyset$.

Suppose that $\gamma \circ f(C) \not\subset S$. Let $\mathcal{H}_C \subset |-K_X|$ be a linear system of surfaces passing through $C$, $H_C$ be a general element in $\mathcal{H}_C$, $Z$ be a fiber of $f$ over $f(C) \cap \tilde{L}$ and $\tilde{C}$ be curve on $X$, such that $\tilde{C} \neq C$ and $\gamma \circ f(\tilde{C}) = \gamma \circ f(C)$. Then $H_C$ is smooth K3 surface containing curves $C \cap \tilde{C} \cap Z$, where $Z = Z_i^0 \cup Z_i^1$ in case $f(C) \cap \tilde{L} = O_i$ and $Z$ is irreducible otherwise. On surface $H_C$ we have $Z^2 = C^2 = \tilde{C}^2 = -2$, $Z \cdot C = Z \cdot \tilde{C} = 1$ and $C \cdot \tilde{C} = 2$,

$$\mathcal{D}_X|_{H_C} = mult_C(\mathcal{D}_X)C + mult_{\tilde{C}}(\mathcal{D}_X)\tilde{C} + \Delta_Z + R_{H_C},$$

because base locus of pencil $\mathcal{H}_C$ consists of $Z \cup C \cup \tilde{C}$, where $\Delta_Z$ is an effective divisor, whose support is $Z$, and $R_{H_C}$ is a movable boundary on $H_C$. Therefore, on surface $H_C$

$$R_{H_C} \sim_{\mathbb{Q}} (n - mult_C(\mathcal{D}_X))C + (n - mult_{\tilde{C}}(\mathcal{D}_X))\tilde{C} + nZ - \Delta_Z$$

and intersection form of curve in $Z \cup \tilde{C}$ is negatively defined. Therefore, boundary $R_{H_C}$ is empty and $mult_{\tilde{C}}(\mathcal{D}_X) = mult_{\tilde{Z}}(\mathcal{D}_X) = n$. On the other hand, $R_{H_C} = \emptyset$ implies that linear system $\mathcal{D}_X$ lies in the fibers of $\phi_{\mathcal{H}_C}$, which is impossible.

Suppose that $\gamma \circ f(C) \not\subset S$. Let $g : W \to X$ be a blow up of curve $C$ and $G$ be a $g$-exceptional divisor and $\mathcal{D}_W = g^{-1}(\mathcal{D}_X)$. Then $|-K_W|$ is a pencil and its base locus consists of $\bar{Z}$ and $\bar{C}$, where $\bar{C}$ is a section of ruled surface $g|_G : G \to C$ and $\bar{Z}$ is a proper transform on $W$ of the fiber of $f$ over $f(C) \cap \tilde{L}$, which consists of two smooth rational curves in case $f(C) \cap \tilde{L} = O_i$. Take general $H_{\bar{C}}$ in $|-K_W|$. Then $H_{\bar{C}}$ is smooth and

$$\mathcal{D}_W|_{H_{\bar{C}}} = mult_{\bar{C}}(\mathcal{D}_W)\bar{C} + \Delta_{\bar{Z}} + R_{H_{\bar{C}}},$$

where $\Delta_{\bar{Z}}$ is effective divisor,whose support is $\bar{Z}$, and $R_{H_{\bar{C}}}$ is a movable boundary. On $H_{\bar{C}}$

$$\mathcal{D}_W|_{H_{\bar{C}}} \sim_{\mathbb{Q}} n\bar{Z} + n\bar{C} + (n - mult_C(\mathcal{D}_X))G|_{H_{\bar{C}}}$$

and intersection form of curves in $\bar{C} \cup \bar{Z}$ is negatively defined. Therefore, $R_{H_{\bar{C}}}$ is empty and equalities $mult_{\bar{Z}}(\mathcal{D}_W) = mult_{\bar{C}}(\mathcal{D}_W) = mult_C(\mathcal{D}_X) = n$ hold. However, $R_{H_{\bar{C}}} = \emptyset$ implies that linear system $\mathcal{D}_W$ lies in the fibers of map $\phi_{|-K_W|} : W \dashrightarrow \mathbb{P}^1$, which is impossible. $\square$

LEMMA 5.7. *Suppose $\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ does not contain $\tilde{L}$. Then set $\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ does not contain points on 3-fold $V$ different from points $O_i$.*

*Proof.* Let $O$ be a point on $V$, which is a center in $\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$. Then $O$ is contained in curve $\tilde{L}$ by Lemma 5.3. On the other hand, curve $\tilde{L}$ does not belong to the base locus of linear system $\mathcal{D}_V$ due to $\tilde{L} \notin \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$. In particular, proper transform $\mathcal{D}_X$ of linear system $\mathcal{D}_V$ on $X$ is a linear subsystem of $|-nK_X|$ and movable log pair $(X, \frac{1}{n}\mathcal{D}_X)$ is a log pull back of movable log pair $(V, \frac{1}{n}\mathcal{D}_V)$. Therefore, there is an irreducible fiber $Z$ of ruled surface $f|_E : E \to \tilde{L}$, such that set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains either $Z$ or a point $P \in Z$.

Suppose set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains curve $Z$. Let $g : W \to X$ be a blow up of $Z$, linear system $\mathcal{D}_W$ be a proper transform of $\mathcal{D}_X$ on $W$ and $G$ be a $g$-exceptional divisor. Then

$$\mathcal{D}_W \sim g^*(-nK_X) - mult_Z(\mathcal{D}_X)G$$

and $mult_Z(\mathcal{D}_X) \geq n$. On the other hand, anticanonical linear system $|-K_W|$ is free and gives an elliptic fibration $\phi_{|-K_W|} : W \to \mathbb{P}^2$. Let $C$ be a general enough fiber of elliptic fibration $\phi_{|-K_W|}$. Then $\mathcal{D}_W \cdot C = 2n - 2mult_Z(\mathcal{D}_X)$. So, $mult_Z(\mathcal{D}_X) = n$ and linear system $\mathcal{D}_W$ lies in the fibers of elliptic fibration $\phi_{|-K_W|}$. Thus, fibration $\pi$ and $\phi_{|-K_W|}$ are equivalent elliptic fibrations. Moreover, there is a projection from a point $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, such that $\alpha \circ \phi_{|-K_W|} \circ (f \circ g)^{-1} = \tau \circ f^{-1}$, which contradicts our initial assumption.

Suppose $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains a point $P \in Z$. Then $\mathcal{D}_X^2 = mult_Z(\mathcal{D}_X^2)Z + C_X$, where support of one-dimensional effective cycle $C_X$ does not contain curve $Z$ and

$$mult_P(\mathcal{D}_X^2) = mult_Z(\mathcal{D}_X^2) + mult_P(C_X) \geq 4n^2$$

by Theorem 3.12. On the other hand, $Z \cdot F = 2$ and $\mathcal{D}_X^2 \cdot F = 2n^2$, where $F$ is a fiber of fibration on del Pezzo surfaces of degree two $\tau$. In particular, $mult_Z(\mathcal{D}_X^2) \leq n^2$ and inequality $mult_P(C_X) \geq 3n^2$ holds. Let $H$ be a general enough divisor in $|-K_X|$ passing through curve $Z$. Then $H$ does not contain any irreducible component of $C_X$ and

$$2n^2 = H \cdot C_X \geq mult_P(C_X) \geq 3n^2,$$

which is a contradiction. $\square$

Let $\mathcal{D}_X$ be a proper transform of linear system $\mathcal{D}_V$ on 3-fold $X$, $F$ be a fiber of del Pezzo fibration $\tau : X \to \mathbb{P}^1$. Then $K_X \sim f^*(K_V)$, $F \sim -K_X - E$ and $\mathcal{D}_X \sim f^*(-nK_V) - mE$, where $m = mult_{\tilde{L}}(\mathcal{D}_V)$.

Lemma 5.8. *$m < n$.*

*Proof.* Inequality $m \geq n$ implies $m = n$ and $\phi_{\mathcal{D}_X} = \tau$, which is impossible. $\square$

Let $\mu = \frac{1}{n-m}$ and $r = \frac{m}{n-m}$. Then $K_X + \mu\mathcal{D}_X \sim_{\mathbb{Q}} rF$.

*Remark 5.9.* $r > 0 \iff m > 0 \iff \tilde{L} \in \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$.

Lemma 5.10. *$\mathbb{CS}(X, \mu\mathcal{D}_X) \neq \emptyset$.*

*Proof.* Suppose $(X, \mu\mathcal{D}_X)$ is terminal. In case $r > 0$ log pair $(X, \epsilon\mathcal{D}_X)$ is a canonical model for some $\epsilon \in \mathbb{Q}_{>\mu}$, but $\kappa(X, \epsilon\mathcal{D}_X) \leq 2$ by construction of $\mathcal{D}_V$. In case $r = 0$ movable log pair $(V, \frac{1}{n}\mathcal{D}_V)$ is terminal, which contradicts Lemma 5.1. $\square$

The following result is well known (see [15]).

Lemma 5.11. *Let $C$ be a curve in $\mathbb{CS}(X, \mu\mathcal{D}_X)$, such that $C$ is contained in the fibers of del Pezzo fibration $\tau$. Then log pair $(X, \mu\mathcal{D}_X)$ is canonical in the generic point of $C$.*

*Proof.* Let $F$ be a fiber of $\tau$ containing $C$ and $L_F$ be a general enough curve on $F$, such that $\gamma \circ f(L_F)$ is a line in $\mathbb{P}^3$, which is contained in the plane $\gamma \circ f(F) \supset L$. Then

$$2(n-m) = \mathcal{D}_X \cdot L_F \geq mult_C(\mathcal{D}_X)(L_F \cdot C)_F \geq (n-m)(L_F \cdot C)_F,$$

which implies that either $\gamma \circ f(C)$ is a line in $\mathbb{P}^3$ and $\gamma \circ f|_C : C \to \gamma \circ f(C)$ is an isomorphism or inequality $mult_C(\mathcal{D}_X) = n-m$ holds and log pair $(X, \mu\mathcal{D}_X)$ is canonical in the generic point of $C$. Hence, we will assume $\gamma \circ f(C)$ is a line in $\mathbb{P}^3$ and $\gamma \circ f|_C$ is an isomorphism.

Suppose that $F$ has singular point $O \in C$. Let $L_O$ be a general enough curve on $F$, such that $\gamma \circ f(L_O)$ is a line in $\mathbb{P}^3$, which passes through the point $\gamma \circ f(O)$ and is contained in the plane $\gamma \circ f(F)$ passing through the line $L$. Then curve $L_O$ is singular in point $O$ and

$$2(n-m) = \mathcal{D}_X \cdot L_O \geq mult_C(\mathcal{D}_X) mult_O(L_O) \geq 2(n-m),$$

which implies $mult_C(\mathcal{D}_X) = n-m$ and $(X, \mu\mathcal{D}_X)$ is canonical in the generic point of $C$.

Suppose that fiber $F$ is non-singular along the curve $C$. Let $\tilde{C}$ be a smooth rational curve in fiber $F$, such that $\tilde{C} \neq C$ and $\gamma \circ f(\tilde{C}) = \gamma \circ f(C)$. Then

$$\mathcal{D}_X|_F = m_C C + m_{\tilde{C}} \tilde{C} + R_F$$

for such natural numbers $m_C \geq mult_C(\mathcal{D}_X)$ and $m_{\tilde{C}} \geq mult_{\tilde{C}}(\mathcal{D}_X)$, that $R_F$ is an effective divisor on surface $F$, whose support does not contain curves $C$ and $\tilde{C}$. On surface $F$

$$\mathcal{D}_X|_F \sim (n-m)C + (n-m)\tilde{C}$$

and $C^2 = \tilde{C}^2 = -1$. Morover, on surface $F$ inequalities $R_F \cdot C \geq 0$ and $R_F \cdot \tilde{C} \geq 0$ hold, which imply $m_C \leq n-m$ and $m_{\tilde{C}} \leq n-m$. Therefore, log pair $(X, \mu\mathcal{D}_X)$ is canonical in the generic point of curve $C$. $\square$

The following result is classical and due to [4-5].

Lemma 5.12. *There is a composition $\sigma$ of Bertini involutions of generic fiber of $\tau$, such that log pair $(X, \mu_\sigma \sigma(\mathcal{D}_X))$ has canonical singularities in codimension one, where $\mu_\sigma \in \mathbb{Q}_{>0}$ and equivalence $K_X + \mu_\sigma \sigma(\mathcal{D}_X) \sim_{\mathbb{Q}} r_\sigma F$ holds for some non-negative rational number $r_\sigma$.*

*Proof.* Suppose $mult_C(\mathcal{D}_X) > n-m$ for some curve $C \subset X$, which is not contained in the fibers of $\tau$ due to Lemma 5.11. Let $F$ be a general enough fiber of $\tau$. Then

$$2(n-m)^2 = \mathcal{D}_X^2 \cdot F \geq mult_C(\mathcal{D}_X^2) C \cdot F \geq mult_C^2(\mathcal{D}_X) C \cdot F > (n-m)^2 C \cdot F,$$

which implies $C \cdot F = 1$. Therefore, we may consider $X$ to be a smooth del Pezzo surface of degree two with Picard group $\mathbb{Z}$ defined over an algebraically non-closed field $\mathbb{F} = \mathbb{C}(x)$, consider $C$ to be $\mathbb{F}$-point of $X$ and consider $\mathcal{D}_X$ to be a linear system without fixed curves on $X$, such that $\mathcal{D}_X \subset |(m-n)K_X|$ and $mult_C(\mathcal{D}_X) > n-m$.

Linear system $|-K_X|$ does not contain curve $Z$, such that $Z$ is singular in $C$, because otherwise $2(n-m) = Z \cdot \mathcal{D}_X > mult_C(Z)(n-m)$. Therefore, point $C$ is not contained in ramification divisor of double cover $\phi_{|-K_X|} : X \to \mathbb{P}^2$. Let $g : W \to X$ be a blow up of $\mathbb{F}$-point $C$ and $E$ be an exceptional divisor of $g$. Then morphism $\phi_{|-2K_W|}$ is a double cover. Consider involution $\chi$ of $W$ interchanging fibers of $\phi_{|-2K_W|}$. Easy calculations show that $\chi^*(g^*(-K_X)) \sim 3g^*(-K_X) - 4E$ and $\chi^*(E) \sim 2g^*(-K_X) - 3E$. Involution $\chi$ induces birational and non-biregular involution $\psi_C$ of $X$ known as Bertini involution.

Let $\mu_{\psi_C}$ be a positive rational number, such that $K_X + \mu_{\psi_C}\psi_C(\mathcal{D}_X) \sim_{\mathbb{Q}} 0$. Then

$$\mu_{\psi_C} = \frac{\mu}{3 - 2mult_C(\mathcal{D}_X)\mu},$$

which implies $\mu_{\psi_C} > \mu = \frac{1}{n-m}$. Therefore, we may repeat construction of Bertini involution for log pair $(X, \mu_{\psi_C}\psi_C(\mathcal{D}_X))$ in case log pair $(X, \mu_{\psi_C}\psi_C(\mathcal{D}_X))$ is not canonical. Thus, for a composition $\sigma$ of Bertini involutions of surface $X$ we obtain a log pair $(X, \mu_\sigma\sigma(\mathcal{D}_X))$, such that $K_X + \mu_\sigma\sigma(\mathcal{D}_X) \sim_{\mathbb{Q}} 0$ and $\frac{1}{\mu_\sigma} \in \mathbb{N}$. Hence, we may assume that rational number $\mu_\sigma$ is minimal for all possible combinations of Bertini involutions and $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ is canonical.

Consider $X$ as a 3-fold. We obtained a composition $\sigma$ of Bertini involutions of generic fiber of $\tau$, such that $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ has canonical singularities in generic fibers of curves not contained in the fibers of $\tau$, where $\mu_\sigma \in \mathbb{Q}_{>0}$ and $K_X + \mu_\sigma\sigma(\mathcal{D}_X) \sim_{\mathbb{Q}} r_\sigma F$ for some rational number $r_\sigma$. Now we may apply Lemmas 5.1-5.6 and Lemmas 5.10-5.11 to linear system $f \circ \sigma(\mathcal{D}_X)$ instead of linear system $\mathcal{D}_V$ to get inequality $r_\sigma \geq 0$ and to prove that singularities of log pair $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ are canonical in codimension one. □

Birational automorphism $\sigma$ commutes with $\tau$, therefore, we may substitute birational map $\beta$ with $\beta \circ \sigma^{-1}$ and assume that $(X, \mu\mathcal{D}_X)$ is canonical in codimension one.

LEMMA 5.13. *$r > 0$.*

*Proof.* Suppose that $r = 0$. Then $\tilde{L} \notin \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ and set $\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ contains some point $O_i$ by Lemma 5.7. We may assume $i = 1$. Therefore, set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains either a point $P \in Z_1$ or a component of $Z_1 = Z_1^0 \cup Z_1^1$, where $Z_1$ is a redicible fiber of birational morphism $f$ over $O_1$. Let $D_1$ and $D_2$ be general enough surfaces in $\mathcal{D}_X$. Then

$$n^2(Z_1 + 2C) \equiv n^2K_X^2 \equiv D_1 \cdot D_2 = mult_{Z_1^0}(\mathcal{D}_X^2)Z_1^0 + mult_{Z_1^1}(\mathcal{D}_X^2)Z_1^1 + C_X + R_X,$$

where $C$ is a curve in fibers of $\tau$ with $-K_X \cdot C = 1$, $C_X$ is an effective one-cycle on $X$ contained in the fibers of $\tau$ and $R_X$ is an effective one-cycle on $X$, whose components are not contained in the fibers of $\tau$.

Suppose that set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains some point $P \in Z_1$. Then Theorem 3.12 implies inequality $mult_P(D_1 \cdot D_2) \geq 4n^2$. On the other hand, $Z_1^0 \cdot F = Z_1^1 \cdot F = 2$ and

$$2n^2 = D_1 \cdot D_2 \cdot F = mult_{Z_1^0}(\mathcal{D}_X^2) + mult_{Z_1^1}(\mathcal{D}_X^2) + R_X \cdot F,$$

where $F$ is a fiber of $\tau$. In particular, $mult_{Z_1^0}(\mathcal{D}_X^2) + mult_{Z_1^1}(\mathcal{D}_X^2) \leq 2n^2$ and equality holds in case $R_X = \emptyset$. Let $H_{Z_1}$ be a general enough surface in linear system $|-K_X|$ passing through $Z_1$. Then $H_{Z_1}$ does not contain irreducible components of $C_X$ and $R_X$. Thus,

$$H_{Z_1} \cdot (C_X + R_X) \geq mult_P(C_X) + mult_P(R_X) \geq 4n^2 - mult_{Z_i^0}(\mathcal{D}_X^2)\delta_P^0 + mult_{Z_1^1}(\mathcal{D}_X^2)\delta_P^1,$$

where $\delta^i_P = mult_P(Z^i_1)$. However, $H_{Z_1} \cdot (C_X + R_X) = 2n^2$. Hence, cycle $R_X$ is empty, equality $mult_P(C_X) = H_{Z_1} \cdot C_X = 2n^2$ holds and either point $P$ is a point of intersection of curves $Z^0_1$ and $Z^1_1$ and equality $mult_{Z^0_1}(\mathcal{D}^2_X) + mult_{Z^1_1}(\mathcal{D}^2_X) = 2n^2$ holds or point $P$ is contained only in one component $Z^k_1$ of $Z_1$, $mult_{Z^k_1}(\mathcal{D}^2_X) = 2n^2$ and $mult_{Z^{1-k}_1}(\mathcal{D}^2_X) = 0$, but both multiplicities $mult_{Z^0_1}(\mathcal{D}^2_X)$ and $mult_{Z^1_1}(\mathcal{D}^2_X)$ are always positive, because $D_1 \cdot Z^0_1 = 0$ and $D_1 \cdot Z^1_1 = 0$. Thus, point $P$ is an intersection $Z^0_1 \cap Z^1_1$ and

$$mult_{Z^0_1}(\mathcal{D}^2_X) + mult_{Z^1_1}(\mathcal{D}^2_X) = 2n^2,$$

but equalities $mult_P(C_X) = H_{Z_1} \cdot C_X = 2n^2$ imply that effective cycle $C_X$ is contained in the fiber $F_P$ of $\tau$, which passes through point $P$. By construction $\phi_{\mathcal{D}_X}(X) = \mathbb{P}^2$ and there is a point $O \in X$, such that $O \notin F_P \cup E$ and linear subsystem $\mathcal{D}_O \subset \mathcal{D}_X$ of surfaces in linear system $\mathcal{D}_X$ passing through point $O$ has no fixed components. Hence, in set-theoretic sense we have $O \in \mathcal{D}^2_O \subset \mathcal{D}^2_V \subset F_P \cup E$, which is a contradiction.

Suppose set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains both curves $Z^0_1$ and $Z^1_1$. Let $g : W \to X$ be a blow up of curve $Z^0_1$, and $h : U \to W$ be a blow ups of proper transform of curve $Z^1_1$, linear system $\mathcal{D}_U$ be a proper transform of $\mathcal{D}_X$ on $U$. Then $|-K_U|$ is free and gives an elliptic fibration $\phi_{|-K_U|} : U \to \mathbb{P}^2$. Let $C$ be a general enough fiber of $\phi_{|-K_U|}$. Then

$$\mathcal{D}_U \cdot C = 2n - mult_{Z^0_1}(\mathcal{D}_X) - mult_{Z^1_1}(\mathcal{D}_X),$$

which implies that $\mathcal{D}_U$ lies in the fibers of $\phi_{|-K_U|}$. Thus, fibration $\pi$ and $\phi_{|-K_U|}$ are equivalent and there is a map $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, such that $\alpha \circ \phi_{|-K_U|} \circ (f \circ h \circ g)^{-1} = \tau \circ f^{-1}$, which contradicts our initial assumption.

Suppose $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains either $Z^0_1$ or $Z^1_1$. We may assume $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains curve $Z^0_1$. Let $g : W \to X$ be a blow up of curve $Z^0_1$, $G$ be an exceptional divisor of $g$ and linear system $\mathcal{D}_W$ be a proper transform of $\mathcal{D}_X$ on $W$. Then log pair $(W, \frac{1}{n}\mathcal{D}_W)$ is a log pull back of log pair $(X, \frac{1}{n}\mathcal{D}_X)$. Therefore, singularities of log pair $(W, \frac{1}{n}\mathcal{D}_W)$ are canonical. Moreover, in case when singularities of log pair $(W, \frac{1}{n}\mathcal{D}_W)$ are not terminal there is a curve $C_W \subset W$, such that $C_W \in \mathbb{CS}(W, \frac{1}{n}\mathcal{D}_W)$ and $C_W$ is not contracted by $g$, because set $\mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$ does not contain smooth points of $V$ due to Lemma 5.3, set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ does not contain points in irreducible fibers of $f$ by Lemma 5.7 and we already proved that set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ does not contain points in $Z_i$. Moreover, set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ does not contain curves different from $Z^k_i$ by Lemmas 5.6-5.7 and we assumed $C_W \neq Z^1_1$. Therefore, either curve $C_W$ dominates $Z^0_1$ or $C_W = Z^k_{j\neq 1}$. On the other hand, $D_1|_{H_{Z_1}} \cdot Z^1_1$, where $H_{Z_1}$ is a general surface in $|-K_X|$ passing through $Z_1$. Therefore, $mult_{Z^1_1}(\mathcal{D}^2_X) > 0$. Hence,

$$2n^2 = D_1 \cdot D_2 \cdot F \geq mult^2_{C_W}(\mathcal{D}_W) + mult^2_{Z^0_1}(\mathcal{D}_X) + mult^2_{Z^1_1}(\mathcal{D}_X) > 2n^2,$$

which is impossible and log pair $(W, \frac{1}{n}\mathcal{D}_W)$ is terminal. Let $\check{h} \circ \check{p} \circ \hat{p}^{-1} \circ h^{-1} : W \dashrightarrow \check{W}$ be an antiflip in curve $Z^1_1$ in the notations of §4 and $\mathcal{D}_{\check{W}}$ be a proper transform of linear system $\mathcal{D}_W$ on $\check{W}$. Then singularities of movable log pair $(\check{W}, \frac{1}{n}\mathcal{D}_{\check{W}})$ are terminal as well, because $(K_W + \frac{1}{n}\mathcal{D}_W) \cdot Z_1^1 = 0$ and antiflip in $Z_1^1$ is a log flop for $(W, \frac{1}{n}\mathcal{D}_W)$. Thus, for small enough $\epsilon \in \mathbb{Q}_{>\frac{1}{n}}$ singularities of $(\check{W}, \epsilon\mathcal{D}_{\check{W}})$ are still terminal and equivalence

$$K_{\check{W}} + \epsilon\mathcal{D}_{\check{W}} \sim_{\mathbb{Q}} (\frac{1}{n} - \epsilon)K_{\check{W}}$$

holds. Divisor $-K_{\check{W}}$ is nef and big by Lemma 4.6. Therefore, $|-nK_{\check{W}}|$ is free and gives birational morphism $\phi_{|-nK_{\check{W}}|} : \check{W} \to V_{1,0}$ for $n \gg 0$, where $V_{1,0}$ is a Fano 3-fold with canonical singularities and $-K^3_{V_{1,0}} = \frac{1}{2}$. Morphism $\phi_{|-nK_{\check{W}}|}$ contracts curves having trivial intersection with $-K_{\check{W}}$. Thus, every curve contracted by $\phi_{|-nK_{\check{W}}|}$ has trivial intersection with divisor $K_{\check{W}} + \epsilon\mathcal{D}_{\check{W}}$ as well and log pair $(\check{W}, \epsilon\mathcal{D}_{\check{W}})$ is a log pull back of $(V_{1,0}, \epsilon\mathcal{D}_{V_{1,0}})$, where $\mathcal{D}_{V_{1,0}}$ is a proper transform of $\mathcal{D}_{\check{W}}$ on $V_{1,0}$. In particular, log pair $(V_{1,0}, \epsilon\mathcal{D}_{V_{1,0}})$, is canonical and $\kappa(V_{1,0}, \epsilon\mathcal{D}_{V_{1,0}}) = 3$, but $\kappa(V_{1,0}, \epsilon\mathcal{D}_{V_{1,0}}) \leq 2$ by construction of $\mathcal{D}_V$. □

Let $h : U \to X$ be a birational morphism with smooth $U$ and regular $\beta \circ f \circ h$. Then

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(rF) + \sum_{i=1}^{k} a_i E_i,$$

where $\mathcal{D}_U$ is a proper transform of linear system without fixed components $\mathcal{D}_X$ on $U$, divisor $E_i$ is $h$-exceptional and $a_i$ is rational. Consider finite or empty subset $\mathcal{J} \subset \mathbb{P}^1$, such that for one of divisors $E_i$ with $a_i < 0$ its image on $X$ is a point in the fiber of $\tau$ over the point in $\mathcal{J}$. For every point $\lambda$ in $\mathcal{J}$ we have rational equivalence

$$h^*(F_\lambda) \sim h^{-1}(F_\lambda) + \sum_{j=1}^{k_\lambda} b_j E_j,$$

where $b_i$ is a natural number and $F_\lambda$ is a fiber of $\tau$ over $\lambda$. Finally, let $\mathcal{I} = \cup_{\lambda\in\mathcal{J}}\mathcal{I}_\lambda$, where for every $\lambda \in \mathcal{J}$ let $\mathcal{I}_\lambda \subset \{1, \ldots, k\}$ be a finite subset, such that $i \in \mathcal{I}_\lambda$ if and only if $h(E_i)$ is a point in fiber $F_\lambda$ and $a_i < 0$.

The following result is famous *super-maximal singularity* of [15].

Proposition 5.14. *$r + \sum_{\lambda\in\mathcal{J}} min\{\frac{a_i}{b_i} |\ h(E_i) \in F_\lambda$ and $a_i < 0\} \leq 0$.*

*Proof.* Suppose the claim is false. Then there are positive rational numbers $\epsilon$ and $c_\lambda$, such that $r = \epsilon + \sum_{\lambda\in\mathcal{J}} c_\lambda$ and $c_\lambda + min\{\frac{a_i}{b_i} |\ h(E_i) \in F_\lambda$ and $a_i < 0\} > 0$. Then

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(\epsilon F) + \sum_{\lambda\in\mathcal{J}} \Big(h^*(c_\lambda F_\lambda) + \sum_{i\in\mathcal{I}_\lambda} a_i E_i\Big) + \sum_{i\notin\mathcal{I}} a_i E_i$$

and for every $\lambda \in \mathcal{J}$ divisor $h^*(c_\lambda F_\lambda) + \sum_{i\in\mathcal{I}_\lambda} a_i E_i$ is effective by the choice of positive rational numbers $c_i$ and divisor $\sum_{i\notin\mathcal{I}} a_i E_i$ is effective, because $(X, \mu\mathcal{D}_X)$ is canonical in codimension one. Let $C$ be a general fiber of $\rho \circ f \circ h$. Then $K_U \cdot C = 0$, because $C$ is an elliptic curve, and $\mathcal{D}_U \cdot C = 0$ by construction of linear system $\mathcal{D}_V$. Thus, we have

$$(K_U + \mu\mathcal{D}_U) \cdot C = h^*(\epsilon F) \cdot C + \sum_{\lambda\in\mathcal{J}} \Big(h^*(c_\lambda F_\lambda) + \sum_{i\in\mathcal{I}_\lambda} a_i E_i\Big) \cdot C + \sum_{i\notin\mathcal{I}} a_i E_i \cdot C = 0$$

and, therefore, $h^*(\epsilon F)\cdot C=0$ due to generality in the choice of $C$. Hence, there is a rational map $\alpha:\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, such that $\alpha\circ\rho=\tau\circ f^{-1}$, which is impossible by our assumption. $\square$

COROLLARY 5.15. *Singularities of log pair $(X,\mu\mathcal{D}_X)$ are not canonical.*

*Remark 5.16.* Geometric meaning of Proposition 5.14 is simple: for any set of positive rational numbers $\{c_\lambda\}_{\lambda\in\mathcal{J}}$ with $r=\sum_{\lambda\in\mathcal{J}}c_\lambda$ we have $\mathbb{CS}(X,\mu\mathcal{D}_X-\sum_{\lambda\in\mathcal{J}}c_\lambda F_\lambda)\neq\emptyset$.

Let $Z$ be a fiber of $f|_E:E\to\tilde{L}$ and $C$ be a curve in fibers of $\tau$ with $-K_X\cdot C=1$.

*Remark 5.17.* $\overline{\mathbb{NE}}(X)=\mathbb{R}_{\geq 0}Z\oplus\mathbb{R}_{\geq 0}C$ and $F\cdot Z=E\cdot C=2$ and $K_X^2\equiv Z+2C$.

Equivalence $\mu\mathcal{D}_X\sim -K_X+rF$ implies $\mu^2\mathcal{D}_X^2\equiv Z+(2+4r)C$. For two general enough divisors $D_1$ and $D_2$ in linear system $\mathcal{D}_X$ consider effective one-cycle

$$T_0=\mu^2 D_1\cdot D_2=Z_X+\sum_{\lambda\in\mathbb{P}^1}C_\lambda,$$

where all components of effective one-cycle $Z_X$ do not lie in the fibers of $\tau$ and every component of effective one-cycle $C_\lambda$ lies in the fiber $F_\lambda$ of $\tau$ over the point $\lambda\in\mathbb{P}^1$.

*Remark 5.18.* $-K_X\cdot\sum_{\lambda\in\mathbb{P}^1}C_\lambda=2+4r\leq 2+4\sum_{\lambda\in\mathcal{J}}max\{\frac{-a_i}{b_i}|\ h(E_i)\in F_\lambda \text{ and } a_i<0\}$.

COROLLARY 5.19. *There is $\omega\in\mathbb{P}^1$ and a $h$-exceptional divisor $E_t$, such that $h(E_t)$ is a point $O$ in the fiber $F_\omega$ of $\tau$ over $\omega$, inequalities $-K_X\cdot C_\omega\leq 2-4a_t/b_t$ and $a_t<0$ hold, where $a_t/b_t=min\{\frac{a_i}{b_i}|\ h(E_i)\in F_\omega$ and $a_i<0\}$.*

*Remark 5.20.* Inequality $-K_X\cdot C_\omega\leq 2/p-4a_t/b_t$ holds, where $p$ is a cardinality of $\mathcal{J}$.

*Remark 5.21.* Log pair $(X,\mu\mathcal{D}_X+\frac{a_t}{b_t}F_\lambda)$ is canonical in $O$ and in equivalence

$$K_U+\mu\mathcal{D}_U+\frac{a_t}{b_t}h^{-1}(F_\omega)\sim_{\mathbb{Q}} h^*((r+\frac{a_t}{b_t})F)+\sum_{i=1}^{k}a(X,\mu\mathcal{D}_X+\frac{a_t}{b_t}F_\lambda,E_i)E_i.$$

log discripancy $a(X,\mu\mathcal{D}_X+\frac{a_t}{b_t}F_\lambda,E_t)$ vanishes, so, we have $O\in\mathbb{CS}(X,\mu\mathcal{D}_X+\frac{a_t}{b_t}F_\lambda)$ and there are only finite number of divisorial discrete valuations with center $O$ on $X$, whose corresponding log discrepances vanishes, and all of them are associated to $h$-exceptional divisors $E_j$, such that $a_j/b_j=a_t/b_t$, and instead of $E_t$ we always can take one of them.

Therefore, we have a point $\omega$ on $\mathbb{P}^1$, fiber $F_\omega$ of fibration on del Pezzo surfaces of degree two $\tau:X\to\mathbb{P}^1$ over $\omega$, birational morphism $h:U\to X$, $h$-exceptional divisor $E_t$, whose image $h(E_t)$ is a point $O$ on fiber $F_\omega$ and inequalities $-K_X\cdot C_\omega\leq 2-4a_t/b_t$ and $a_t<0$ hold, where $C_\omega$ is a component of cycle $T_0=\mu^2D_1\cdot D_2$ lying in $F_\omega$, divisors $D_1$ and $D_2$ are two general elements in linear system $\mathcal{D}_X$, rational number $a_t$ is defined through

$$K_U+\mu\mathcal{D}_U\sim_{\mathbb{Q}} h^*(rF)+\sum_{i=1}^{k}a_iE_i$$

and natural number $b_t$ is defined through $h^*(F_\omega) \sim h^{-1}(F_\omega) + \sum_{i=1}^{k} b_i E_i$, where $\mathcal{D}_U$ is a proper transform of $\mathcal{D}_X$ on $U$, divisor $E_i$ is $h$-exceptional, $a_i \in \mathbb{Q}$ and $b_i \in \mathbb{N}$.

LEMMA 5.22. *Fiber $F_\omega$ is non-singular in point $O$.*

*Proof.* Suppose $F_\omega$ is singular in $O$. Let $L_O$ be a general curve on $F_\omega$, such that $\gamma \circ f(L_O)$ is a line in $\mathbb{P}^3$, which passes through $\gamma \circ f(O)$. Then $L_O$ is singular in $O$ and

$$2(n-m) = \mathcal{D}_X \cdot L_O \geq mult_O(\mathcal{D}_X) mult_O(L_O) \geq 2 mult_O(\mathcal{D}_X),$$

but $a_t < 0$ implies $mult_O(\mathcal{D}_X) > \frac{1}{\mu} = n - m$. $\square$

LEMMA 5.23. $mult_O(\mathcal{D}_X) \leq \frac{2}{\mu} = 2(n-m)$.

*Proof.* Let $O \in L_O$ be a general curve on $F_\omega$, such that $\gamma \circ f(L_O)$ is a line. Then

$$2(n-m) = \mathcal{D}_X \cdot L_O \geq mult_O(\mathcal{D}_X) mult_O(L_O) \geq mult_O(\mathcal{D}_X),$$

which implies the claim. $\square$

Numbers $a_t$ and $b_t$ depend only on divisorial discrete valuation associated to divisor $E_t$, they are log discrepancy of $(X, \mu\mathcal{D}_X)$ and multiplicity of $F_\omega$ correspondingly. In particular, they do not depend on $h$. Therefore, we may choose $h$ to be a composition $\psi_{1,0} \circ \cdots \circ \psi_{N,N-1}$ of birational morphisms $\psi_{i,i-1} : X_i \to X_{i-1}$ for $i = 1, \ldots, N$ and $X = X_0$, which have the following properties: morphism $\psi_{1,0}$ is a blow up of point $P_0 = O$; morphism $\psi_{i,i-1}$ is a blow up of a point $P_{i-1} \in G_{i-1}$ for $i = 2, \ldots, K$, where $G_i = \psi_{i,i-1}^{-1}(P_{i-1})$; birational morphism $\psi_{K+1,K}$ is a blow up of a curve $P_K \subset G_K$; birational morphism $\psi_{i,i-1}$ is a blow up of a curve $P_{i-1} \subset G_{i-1}$ for $i = K+2, \ldots, N$, which does not lie in the fibers of morphism $\psi_{i-1,i-2}|_{G_{i-1}}$; divisorial discrete valuation associated to divisor $G_N$ coincide with divisorial discrete valuation associated to $E_t$.

Put $\psi_{j,i} = \psi_{i+1,i} \circ \ldots \circ \psi_{j,j-1} : X_j \to X_i$ for $j > i$ and $\psi_{j,j} = id_{X_j}$. Consider proper transforms $G_i^j = \psi_{j,i}^{-1}(G_i) \subset X_j$ for $j \geq i$, linear systems $\mathcal{D}_{X_i} = \psi_{i,0}^{-1}(\mathcal{D}_X)$ on $X_i$ and proper transforms $F_\omega^i = \psi_{i,0}^{-1}(F_\omega) \subset X_i$ for $i = 0, \ldots, N$. We have equivalences

$$K_{X_N} + \mu\mathcal{D}_{X_N} \sim_{\mathbb{Q}} \psi_{N,0}^*(rF_\omega) + \sum_{i=1}^{N} a(X, \mu\mathcal{D}_X, G_i) G_i^N$$

and $\psi_{N,0}^*(F_\omega) \sim F_\omega^N + \sum_{i=1}^{N} b(X, F_\omega, G_i) G_i^N$, where number $a(X, \mu\mathcal{D}_X, G_i)$ is rational and number $b(X, F_\omega, G_i)$ is natural. By construction $a(X, \mu\mathcal{D}_X, G_N) = a_t$, $b(X, F_\omega, G_N) = b_t$,

$$-K_X \cdot C_\omega \leq 2 - 4a(X, \mu\mathcal{D}_X, G_N)/b(X, F_\omega, G_N)$$

and $a(X, \mu\mathcal{D}_X, G_N) < 0$. Moreover, due to Remark 5.21 we always may assume that inequality $a(X, \mu\mathcal{D}_X, G_N)/b(X, F_\omega, G_N) < a(X, \mu\mathcal{D}_X, G_i)/b(X, F_\omega, G_i)$ holds for $i < N$.

*Remark 5.24.* Apriori, we should consider 3-fold $X_i$ to be either quasiprojective or singular for $i > K$, but inequality $a(X, \mu\mathcal{D}_X, G_N) < 0$, Remark 5.21 and Lemma 5.23 implicitly imply that curve $P_K$ is a line in $G_K \cong \mathbb{P}^2$ and curves $P_i$ are sections of ruled surface $\psi_{i-1,i-2}|_{G_{i-1}} : G_{i-1} \to P_{i-1} \cong \mathbb{P}^1$ for $i > K$.

The main object of the rest of the chapter is effective one-cycle

$$T_0 = \mu^2 D_1 \cdot D_2 = Z_X + C_\omega + \sum_{\omega \neq \lambda \in \mathbb{P}^1} C_\lambda,$$

where all components of $Z_X$ do not lie in the fibers of $\tau$, every component of $C_\lambda$ lies in the fiber $F_\lambda$ of $\tau$ over the point $\lambda \neq \omega$ and $C_\omega \subset F_\omega$. Inequality $a(X, \mu\mathcal{D}_X, G_K) < 0$ implies inequality $mult_O(T_0) > 4$ by Theorem 3.12, but $mult_O(Z_X) \leq F_\omega \cdot Z_X = 2$. Therefore, we have inequality $mult_O(C_\omega) > 2$, which is rather strong condition, but unfortunately it seems not to be strong enough to get a contradiction.

*Remark 5.25.* $2 < mult_O(C_\omega) \leq -K_X \cdot C_\omega \leq 2 - 4a(X, \mu\mathcal{D}_X, G_N)/b(X, F_\omega, G_N)$.

The following construction is due to [15] and [20]. Let every exceptional divisor $G_i$ be a vertex of an oriented graph $\Gamma$ and arrow $G_j \to G_i$ means that $G_j$ is joined with $G_i$ by an oriented edge, which goes from vertex $G_j$ to vertex $G_i$, and let

$$G_j \to G_i \iff j > i \text{ and } P_{j-1} \subset G_i^{j-1} \subset X_{j-1},$$

thus, in particular, we always have $G_i \to G_{i-1}$ for $i = 1, \ldots, N$.

*Remark 5.26.* In case when discrete valuation of $G_N$ coincide with discrete valuation of exceptional divisor of a weighted blow up of point $O$ with weights $(1, K, N)$ in suitable local coordinates on $X$ and coprime natural $K$ and $N$ graph $\Gamma$ is a chain due to [22], which simply means that $G_j \to G_i \iff j = i + 1$.

Let $P(i)$ be a number of paths in $\Gamma$ from $G_N$ to $G_i$ for $i \neq N$ and $P(N) = 1$. Then

$$a(X, \mu\mathcal{D}_X, G_N) = \sum_{i=1}^{K} P(i)(2 - \nu_i) + \sum_{i=K+1}^{N} P(i)(1 - \nu_i),$$

where $\nu_i = mult_{P_{i-1}}(\mu\mathcal{D}_X^{i-1})$. Put $T_0^j = \psi_{j,0}^{-1}(T_0)$, $Z_X^j = \psi_{j,0}^{-1}(Z_X)$, $C_\omega^j = \psi_{j,0}^{-1}(C_\omega)$ and

$$m_{T_0}^i = mult_{P_{i-1}}(T_0^{i-1}), m_{Z_X}^i = mult_{P_{i-1}}(Z_X^{i-1}), m_{C_\omega}^i = mult_{P_{i-1}}(C_\omega^{i-1}),$$

where $i = 1, \ldots, K$. Let $D_1^i = \psi_{i,0}^{-1}(D_1)$, $D_2^i = \psi_{i,0}^{-1}(D_2)$ and $T_i \subset G_i \cong \mathbb{P}^2$ be an effective one-cycle on 3-fold $X_i$ for $i = 1, \ldots, N$ given by relation

$$\mu^2 \mathcal{D}_{X_i}^2 \equiv \mu^2 D_1^i \cdot D_2^i = \psi_{i,i-1}^{-1}(\mu^2 D_1^{i-1} \cdot D_2^{i-1}) + T_i$$

and $T_i^j = \psi_{j,i}^{-1}(T_i)$ for $j \geq i$. Then $\mu^2 D_1^i \cdot D_2^i = T_0^i + T_1^i + \cdots + T_{i-1}^i + T_i$, for $i = 1, \ldots, K$.

Let $m_{i,j} = mult_{P_{j-1}}(T_i^{j-1})$ and $d_i$ be a degree of $T_i$ in $\mathbb{P}^2 \cong G_i$ for $j > i = 1, \ldots, K$.

LEMMA 5.27. $\sum_{j=0}^{i-1} m_{j,i} = \nu_i^2 + d_i$ *for* $i = 1, \ldots, K$.

*Proof.* Taking general hyperplane section of $X_{i-1}$ through point $P_{i-1}$ we reduce the claim to the well known result on multiplicity of intersection on smooth surface. □

LEMMA 5.28. $d_K \geq \sum_{i=K+1}^{N} \nu_i^2 deg(\psi_{i-1,L}|_{P_{i-1}})$.

*Proof.* For $i > K$ consider $T_i$ as a subscheme in $G_i$. Let $d_i$ be an intersection of $T_i$ in $G_i$ with a general enough fiber of $\psi_{i,i-1}|_{G_i}$. Then $d_i \geq mult_{P_i}(\mathcal{D}^2_{X_i}) deg(\psi_{i,i-1}|_{P_i})$ for $i > K$ and $mult_{P_i}(\mathcal{D}^2_{X_i}) = \nu_{i+1}^2 + d_{i+1}$ for $i \geq K$. Therefore, we have inequalities

$$d_K \geq mult_{P_K}(\mathcal{D}^2_{X_L}) \geq \sum_{i=K+1}^{N} \nu_i^2 \prod_{j=K}^{i-1} deg(\psi_{j+1,j}|_{P_{j+1}}) = \sum_{i=K+1}^{N} \nu_i^2 deg(\psi_{i-1,K}|_{P_{i-1}}),$$

which is what we need. □

One can prove that curve $P_K$ is a line in $G_K \cong \mathbb{P}^2$ and $deg(\psi_{i-1,L}|_{P_{i-1}}) = 1$ for $i > K$.

COROLLARY 5.29. $d_K \geq \sum_{i=K+1}^{N} \nu_i^2$.

Consider a restriction of $P(i)$ to $[1, K] \subseteq [1, N]$. Let $Q : [1, K] \to \mathbb{N}$ be a function, such that $Q(i)$ a number of oriented paths in $\Gamma$, which goes from $G_N$ to $G_i$, where $i = 1, \ldots, K$.

*Remark 5.30.* $Q(i) = \sum_{G_j \to G_i} P(j) = \sum_{G_j \to G_i} Q(j) + \sum_{G_j \to G_i}^{j>K} P(j) \geq \sum_{G_j \to G_i} Q(j)$.

LEMMA 5.31. $\sum_{i=1}^{K} Q(i) m_{0,i} \geq \sum_{i=1}^{K} Q(i)\nu_i^2 + Q(K) d_K$.

*Proof.* For $j = 1, \ldots, K$ multiply $\sum_{i=0}^{j-1} m_{i,j} = \nu_j^2 + d_j$ of by $Q(j)$. Take a sum

$$\sum_{j=1}^{K} \sum_{i=0}^{j-1} Q(j) m_{i,j} = \sum_{j=1}^{K} Q(j)\nu_j^2 + \sum_{j=1}^{K} Q(j) d_j$$

and change the order of summation

$$\sum_{j=1}^{K} \sum_{i=0}^{j-1} Q(j) m_{i,j} = \sum_{i=0}^{K-1} \sum_{j=i+1}^{K} Q(j) m_{i,j},$$

where $m_{i,j} \leq d_i$ in case $1 \leq i < j \leq K$. Moreover, $m_{i,j} > 0 \iff G_j \to G_i$. Hence,

$$\sum_{i=0}^{K-1} \sum_{j=i+1}^{K} Q(j) m_{i,j} \leq \sum_{j=1}^{K} Q(j) m_{0,j} + \sum_{i=1}^{K-1} \sum_{G_j \to G_i} Q(j) d_i \leq \sum_{j=1}^{K} Q(j) m_{0,j} + \sum_{i=1}^{K-1} Q(i) d_i,$$

therefore,

$$\sum_{j=1}^{K} Q(j)m_{0,j} + \sum_{i=1}^{K-1} Q(i)d_i \geq \sum_{j=1}^{K} Q(j)\nu_j^2 + \sum_{j=1}^{K} Q(j)d_j$$

and finally

$$\sum_{j=1}^{K} Q(j)m_{0,j} \geq \sum_{j=1}^{K} Q(j)\nu_j^2 + Q(K)d_K,$$

which is what we need. $\square$

Corollary 5.32. $\sum_{i=1}^{K} P(i)m_{0,i} \geq \sum_{i=1}^{N} P(i)\nu_i^2$.

Let $\Sigma_0 = \sum_{i=1}^{K} P(i)$ and $\Sigma_1 = \sum_{i=K+1}^{N} P(i)$. Then elementary calculus implies

$$\sum_{i=1}^{K} P(i)m_{0,i} \geq \sum_{i=1}^{N} P(i)\nu_i^2 \geq \frac{(2\Sigma_0 + \Sigma_1 - a(X, \mu\mathcal{D}_X, G_N))^2}{(\Sigma_0 + \Sigma_1)},$$

because $\sum_{i=1}^{N} P(i)\nu_i = 2\Sigma_0 + \Sigma_1 - a(X, \mu\mathcal{D}_X, G_N)$.

By definition $m_{0,i} = m_{T_0}^i = m_{Z_X}^i + m_{C_\omega}^i$ for $i = 1, \ldots, K$ and $m_{C_\omega}^i = 0$ for $i > M$, where natural number $M \leq K$ is the biggest with $P_{M-1} \in F_\omega^{M-1}$. Moreover, $mult_O(Z_X) \geq m_{Z_X}^i$ for all $i = 1, \ldots, K$ and $mult_O(C_\omega) \geq m_{C_\omega}^i$ for all $i = 1, \ldots, M$. Therefore, we have

$$mult_O(Z_X)\Sigma_0 + mult_O(C_\omega)\Sigma_0' \geq \frac{(2\Sigma_0 + \Sigma_1 - a(X, \mu\mathcal{D}_X, G_N))^2}{(\Sigma_0 + \Sigma_1)},$$

where $\Sigma_0' = \sum_{i=1}^{M} P(i)$. On the other hand, we have $mult_O(Z_X) \leq F_\omega \cdot Z_X = 2$ and

$$mult_O(C_\omega) \leq -K_X \cdot C_\omega \leq 2 - 4a(X, \mu\mathcal{D}_X, G_N)/b(X, F_\omega, G_N)$$

and $b(X, F_\omega, G_N) \geq \sum_{i=1}^{M} q_i P(i) \geq \Sigma_0'$, where $q_i = mult_{P_{i-1}}(F_\omega^{i-1})$ for $i = 1, \ldots, M$.

Combining all inequalities together we get

$$2(\Sigma_0 - \Sigma_0')(\Sigma_0 + \Sigma_1) + (\Sigma_1 + a(X, \mu\mathcal{D}_X, G_N))^2 \leq 0,$$

which implies $a(X, \mu\mathcal{D}_X, G_N)) = -\Sigma_1$ and $\Sigma_0 = \Sigma_0'$. Moreover, equality

$$\sum_{i=1}^{N} P(i)\nu_i^2 = \frac{(2\Sigma_0 + \Sigma_1 - a(X, \mu\mathcal{D}_X, G_N))^2}{(\Sigma_0 + \Sigma_1)}$$

holds, which is possible only when $\nu_1 = \nu_2 = \cdots = \nu_N = \nu$. Hence, we have

$$\nu(\Sigma_0 + \Sigma_1) = \sum_{i=1}^{N} P(i)\nu_i = 2\Sigma_0 + \Sigma_1 - a(X, \mu\mathcal{D}_X, G_N) = 2(\Sigma_0 + \Sigma_1),$$

which implies $\nu = 2$. Every inequality above must be an equality. In particular, cardinality of set $\mathcal{J}$ is one by Remark 5.20. Thus, fiber $F_\omega$ is the only fiber of $\tau$, such that singularities of log pair $(X, \mu\mathcal{D}_X)$ are not canonical in points of fiber $F_\omega$. Moreover, equality

$$-K_X \cdot C_\omega = 2 + 4r = -K_X \cdot \sum_{\lambda \in \mathbb{P}^1} C_\lambda$$

implies that all effective one-cycles $C_\lambda$ are empty for $\lambda \neq \omega$. Equivalence

$$T_0 = Z_X + C_\omega \equiv Z + (2 + 4r)C,$$

where $Z$ is a fiber of $f|_E : E \to \tilde{L}$ and $C$ is a curve in $F_\omega$ with $-K_X \cdot C = 1$, implies that support of cycle $Z_X$ are contained in fibers of $f|_E$. In particular, $O \in F_\omega \cap Z_\delta$, where $Z_\delta$ is a fiber of $f|_E : E \to \tilde{L}$ over some point $\delta \in \tilde{L}$. However, equality $2 = F_\omega \cdot Z_X = mult_O(Z_X)$ implies that support of cycle $Z_X$ is contained in $Z_\delta$, which must be reducible. Thus, point $\delta$ is one of the points $O_i$ and $Z_\delta = Z_i = Z_i^0 \cup Z_i^1$. On the other hand, $\mathcal{D}_X$ is not composed from a pencil, because $\phi_{\mathcal{D}_X}(X) = \mathbb{P}^2$. Therefore, there is a point $P \in X$, such that $P \notin F_\omega \cup E$ and linear subsystem $\mathcal{D}_P \subset \mathcal{D}_X$ of surfaces in linear system $\mathcal{D}_X$ passing through point $P$ has no fixed components. Hence, we have $P \in A \cap B$ for two general enough divisors $A$ and $B$ in $\mathcal{D}_P$. However, we can substitute divisors $D_1$ and $D_2$ in linear system $\mathcal{D}_X$ by $A$ and $B$ correspondingly. Therefore, in set-theoretic sense

$$P \in A \cap B \subset F_\omega \cup E,$$

which is a contradiction. Thus, we proved Theorem 1.3.

## §6. Proof of Theorem 1.4.

In the notations of §1 suppose the existence of birational transformation $\beta : V \dashrightarrow Y$, such that $Y$ is canonical Fano 3-fold. In this chapter we will prove the following.

PROPOSITION 6.1. *There is birational automorphism $\sigma$ of $V$, such that $\sigma$ is a uniquely defined composition of Bertini involutions of a generic fiber of del Pezzo fibration $\tau$ and either $\beta \circ \sigma$ is biregular or $\beta \circ \sigma = \alpha \circ \rho_{i,k}$ for some biregular automorphism $\alpha$ of $V_{i,k}$.*

COROLLARY 6.2. *Either $Y \cong V$ or $Y \cong V_{i,k}$.*

COROLLARY 6.3. *Sequence $1 \to \Gamma \to Bir(V) \to Aut(V) \to 1$ is exact, where $\Gamma$ is a free product of Bertini involutions of a generic fiber of $\tau$ considered as a smooth del Pezzo surface of degree two with Picard group $\mathbb{Z}$ over the field $\mathbb{C}(x)$.*

Let $\mathcal{D}_V = \beta^{-1}(\mathcal{D}_Y)$, where $\mathcal{D}_Y = |-tK_Y|$ for $t \gg 0$. Then $\mathcal{D}_V \subset |-nK_V|$ for some natural $n$. Let $\mathcal{D}_X$ be a proper transform of $\mathcal{D}_V$ on $X$ and $F$ be a fiber $\tau : X \to \mathbb{P}^1$. Then

$$\mathcal{D}_X \sim f^*(-nK_V) - mE \sim -nK_X - mE \sim f^*(K_V) \sim n(F + E) - mE,$$

where $n > m = mult_{\tilde{L}}(\mathcal{D}_V) \geq 0$. Let $\mu = \frac{1}{n-m}$ and $r = \frac{m}{n-m}$. Then $K_X + \mu\mathcal{D}_X \sim_{\mathbb{Q}} rF$.

*Remark 6.4.* $m = 0 \iff r = 0 \iff \mu = \frac{1}{n} \iff \tilde{L} \notin \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$.

LEMMA 6.5. *$\mathbb{CS}(X, \mu\mathcal{D}_X) = \emptyset$ implies $\beta$ is biregular.*

*Proof.* Suppose $r > 0$. Then log pair $(X, \epsilon\mathcal{D}_X)$ is a canonical model for some small enough rational $\epsilon > \mu$. Therefore, $\kappa(Y, \epsilon\mathcal{D}_Y) = \kappa(X, \epsilon\mathcal{D}_X) = 3$ and $(Y, \epsilon\mathcal{D}_Y)$ is canonical model as well. Hence, birational map $\beta \circ f$ is biregular by Theorem 2.15, which is a contradiction, because $-K_X$ is not ample.

Suppose $r = 0$. Then $\mu = \frac{1}{n}$, log pair $(X, \frac{1}{n}\mathcal{D}_X)$ is a log pull back of log pair $(V, \frac{1}{n}\mathcal{D}_V)$ and $\tilde{L} \notin \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$. Thus, log pair $(V, \frac{1}{n}\mathcal{D}_V)$ is semi-terminal. Then log pair $(V, \epsilon\mathcal{D}_V)$ is a canonical model for small enough $\epsilon \in \mathbb{Q}_{>\frac{1}{n}}$. Therefore, $\kappa(Y, \epsilon\mathcal{D}_Y) = \kappa(V, \epsilon\mathcal{D}_V) = 3$ and log pair $(Y, \epsilon\mathcal{D}_Y)$ is a canonical model. Hence, map $\beta$ is biregular by Theorem 2.15. □

LEMMA 6.6. *Let $C$ be a curve in $\mathbb{CS}(X, \mu\mathcal{D}_X)$, such that $C$ is contained in the fibers of del Pezzo fibration $\tau$. Then log pair $(X, \mu\mathcal{D}_X)$ is canonical in the generic point of $C$.*

*Proof.* See the proof of Lemma 5.11. □

LEMMA 6.7. *There is a uniquely defined composition $\sigma$ of Bertini involutions of generic fiber of $\tau$, such that log pair $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ has canonical singularities in codimension one.*

*Proof.* See the proof of Lemma 5.12. □

*Remark 6.8.* We may replace birational map $\beta$ by a composition $\beta \circ \sigma^{-1}$ and assume that singularities of log pair $(X, \mu\mathcal{D}_X)$ are canonical in codimension one.

LEMMA 6.9. *Suppose singularities of log pair $(X, \mu\mathcal{D}_X)$ are canonical. Then $r = 0$.*

*Proof.* Let $r > 0$. Then $\kappa(X, \mu\mathcal{D}_X) = 1$, but $\kappa(Y, \mu\mathcal{D}_Y) \in \{-\infty, 0, 3\}$ by construction, which contradicts the birational invariance of Kodaira dimension. □

Let $h : U \to X$ be a birational morphism with smooth $U$ and regular $\beta \circ f \circ h$. Then

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(rF) + \sum_{i=1}^{k} a_i E_i,$$

where $\mathcal{D}_U$ is a proper transform of linear system without fixed components $\mathcal{D}_X$ on $U$, divisor $E_i$ is $h$-exceptional and $a_i$ is rational. Consider finite or empty subset $\mathcal{J} \subset \mathbb{P}^1$, such that for one of divisors $E_i$ with $a_i < 0$ its image on $X$ is a point in the fiber of $\tau$ over the point in $\mathcal{J}$. For every point $\lambda$ in $\mathcal{J}$ we have rational equivalence

$$h^*(F_\lambda) \sim h^{-1}(F_\lambda) + \sum_{j=1}^{k_\lambda} b_j E_j,$$

where $b_i$ is a natural number and $F_\lambda$ is a fiber of $\tau$ over $\lambda$. Finally, let $\mathcal{I} = \cup_{\lambda\in\mathcal{J}}\mathcal{I}_\lambda$, where for every $\lambda \in \mathcal{J}$ let $\mathcal{I}_\lambda \subset \{1, \dots, k\}$ be a finite subset, such that $i \in \mathcal{I}_\lambda$ if and only if $h(E_i)$ is a point in fiber $F_\lambda$ and $a_i < 0$.

PROPOSITION 6.10. *$r + \sum_{\lambda\in\mathcal{J}} min\{\frac{a_i}{b_i} |\ h(E_i) \in F_\lambda$ and $a_i < 0\} \le 0$.*

*Proof.* Suppose the claim is false. Then there are positive rational numbers $\epsilon$ and $c_\lambda$, such that $r = \epsilon + \sum_{\lambda\in\mathcal{J}} c_\lambda$ and $c_\lambda + min\{\frac{a_i}{b_i} |\ h(E_i) \in F_\lambda$ and $a_i < 0\} > 0$. Then

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(\epsilon F) + \sum_{\lambda\in\mathcal{J}} \Big( h^*(c_\lambda F_\lambda) + \sum_{i\in\mathcal{I}_\lambda} a_i E_i \Big) + \sum_{i\notin\mathcal{I}} a_i E_i$$

and for every $\lambda \in \mathcal{J}$ divisor $h^*(c_\lambda F_\lambda) + \sum_{i\in\mathcal{I}_\lambda} a_i E_i$ is effective by the choice of positive rational numbers $c_i$ and divisor $\sum_{i\notin\mathcal{I}} a_i E_i$ is effective, because $(X, \mu\mathcal{D}_X)$ is canonical in codimension one. Thus, $\kappa(U, \mu\mathcal{D}_U) \ge 1$, but $\kappa(Y, \mu\mathcal{D}_Y) \in \{-\infty, 0, 3\}$ by construction, which contradicts the birational invariance of Kodaira dimension. □

LEMMA 6.11. *$r = 0$.*

*Proof.* Let $Z$ be a fiber of $f|_E : E \to \tilde{L}$ and $C$ be a curve in fibers of $\tau$ with $-K_X \cdot C = 1$, choose two general enough surfaces $D_1$ and $D_2$ in $\mathcal{D}_X$. Then $\overline{\mathbb{NE}}(X) = \mathbb{R}_{\ge 0}Z \oplus \mathbb{R}_{\ge 0}C$ and

$$\mu^2 D_1 \cdot D_2 = Z_X + \sum_{\lambda\in\mathbb{P}^1} C_\lambda \equiv Z + (2 + 4r)C,$$

where all components of effective cycle $Z_X$ do not lie in the fibers of $\tau$ and every component of effective cycle $C_\lambda$ lies in the fiber $F_\lambda$ of $\tau$ over $\lambda \in \mathbb{P}^1$. Therefore,

$$-K_X \cdot \sum_{\lambda\in\mathbb{P}^1} C_\lambda = 2 + 4r \le 2 - 4 \sum_{\lambda\in\mathcal{J}} min\{\frac{a_i}{b_i} |\ h(E_i) \in F_\lambda \text{ and } a_i < 0\}$$

by Proposition 6.10. Suppose $r > 0$. Then there is $\omega \in \mathbb{P}^1$, such that

$$-K_X \cdot C_\omega \le 2 - 4a_t/b_t,$$

where $a_t/b_t = min\{\frac{a_i}{b_i} |\ h(E_i) \in F_\omega$ and $a_i < 0\}$. Now we can repeat all arguments and constructions of the previous chapter starting from Lemma 5.22 to get a contradiction. □

COROLLARY 6.12. *$\tilde{L} \notin \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$.*

LEMMA 6.13. *$\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V) = \emptyset$ implies $\beta$ is biregular.*

*Proof.* Suppose $(V, \frac{1}{n}\mathcal{D}_V)$ is semi-terminal. Then for some $\epsilon \in \mathbb{Q}_{>\frac{1}{n}}$ log pair $(V, \epsilon\mathcal{D}_V)$ is a canonical model and $\kappa(Y, \epsilon\mathcal{D}_Y) = \kappa(V, \epsilon\mathcal{D}_V) = 3$. Hence, log pair $(Y, \epsilon\mathcal{D}_Y)$ is a canonical model as well. Therefore, rational map $\beta$ is biregular by Theorem 2.15. □

*Remark 6.14.* Linear system $\mathcal{D}_V$ is not contained in the fibers of any dominant rational map $\chi : V \dashrightarrow Z$, where dimension of $Z$ is two or one.

LEMMA 6.15. *$\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ does not contain curves and smooth points of $V$.*

*Proof.* See the proofs of Lemmas 5.3-5.6. $\square$

LEMMA 6.16. *$\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ does not contain points different from $O_i$.*

*Proof.* Let $O$ be a point on 3-fold $V$, such that $O \in \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ and $O \neq O_i$. Then $O$ is contained in curve $\tilde{L}$ by Lemma 6.15, but $\tilde{L} \notin \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$. Hence, $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains either an irreducible fiber $Z$ of $f|_E : E \to \tilde{L}$ over point $O$ or a point $P \in Z$.

Suppose set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains curve $Z$. Let $g : W \to X$ be a blow up of curve $Z$, linear system $\mathcal{D}_W$ be a proper transform of linear system $\mathcal{D}_X$ on 3-fold $W$ and $G$ be an exceptional divisor of $g$. Then linear system $|-K_W|$ is free, morphism $\phi_{|-K_W|}$ is an elliptic fibration and $\mathcal{D}_W \cdot C = 2n - mult_Z(\mathcal{D}_X)$ for general fiber $C$ of $\phi_{|-K_W|}$. Thus, $\mathcal{D}_W$ lies in the fibers of $\phi_{|-K_W|}$, which is impossible.

Suppose $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains a point $P \in Z$. Then $\mathcal{D}_X^2 = mult_Z(\mathcal{D}_X^2)Z + C_X$, where support of one-dimensional effective cycle $C_X$ does not contain curve $Z$ and

$$mult_P(\mathcal{D}_X^2) = mult_Z(\mathcal{D}_X^2) + mult_P(C_X) \geq 4n^2$$

by Theorem 3.12. On the other hand, $Z \cdot F = 2$ and $\mathcal{D}_X^2 \cdot F = 2n^2$, where $F$ is a fiber of fibration on del Pezzo surfaces of degree two $\tau$. In particular, $mult_Z(\mathcal{D}_X^2) \leq n^2$ and inequality $mult_P(C_X) \geq 3n^2$ holds. Let $H$ be a general enough divisor in $|-K_X|$ passing through curve $Z$. Then $H$ does not contain any irreducible component of $C_X$ and

$$2n^2 = H \cdot C_X \geq mult_P(C_X) \geq 3n^2,$$

which is a contradiction. $\square$

LEMMA 6.17. *Suppose $O_i \in \mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$. Then $\beta$ is either $\alpha \circ \rho_{i,0}$ or $\alpha \circ \rho_{i,1}$ for some biregular automorphism $\alpha$ of $V_{i,0}$ or $V_{i,1}$ correspondingly.*

*Proof.* We may assume $i = 1$. Let $Z_1 = Z_1^0 \cup Z_1^1 = f^{-1}(O_1)$. Then $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains either a point $P \in Z_1$ or a component of $Z_1$. Moreover, we may repeat all arguments of the proof of Lemma 5.13 and assume that set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ consists of curve $Z_1^0$ and log pair $(W, \frac{1}{n}\mathcal{D}_W)$ is terminal, where $g : W \to X$ is a blow up of curve $Z_1^0$ and $\mathcal{D}_W$ is a proper transform of linear system $\mathcal{D}_X$ on 3-fold $W$. We will show that $\beta = \alpha \circ \rho_{1,0}$ for some biregular automorphism $\alpha$ of $V_{1,0}$.

Let $\check{h} \circ \check{p} \circ \hat{p}^{-1} \circ h^{-1} : W \dashrightarrow \check{W}$ be an antiflip in curve $Z_1^1$ in the notations of §4 and $\mathcal{D}_{\check{W}}$ be a proper transform of linear system $\mathcal{D}_W$ on $\check{W}$. Then singularities of log pair $(\check{W}, \frac{1}{n}\mathcal{D}_{\check{W}})$ are terminal as well, because $(K_W + \frac{1}{n}\mathcal{D}_W) \cdot Z_1^1 = 0$ and antiflip in $Z_1^1$ is a log flop for log pair $(W, \frac{1}{n}\mathcal{D}_W)$. Thus, for some $\epsilon \in \mathbb{Q}_{>\frac{1}{n}}$ log pair $(\check{W}, \epsilon\mathcal{D}_{\check{W}})$ is terminal and

$$K_{\check{W}} + \epsilon\mathcal{D}_{\check{W}} \sim_{\mathbb{Q}} (\frac{1}{n} - \epsilon)K_{\check{W}},$$

where $-K_{\check{W}}$ is nef and big by Lemma 4.6, linear system $|-nK_{\check{W}}|$ is free and gives birational morphism $\phi_{|-nK_{\check{W}}|} : \check{W} \to V_{1,0}$ for $n \gg 0$. Morphism $\phi_{|-nK_{\check{W}}|}$ contracts curves having trivial intersection with $-K_{\check{W}}$. Therefore, log pair $(V_{1,0},\epsilon\mathcal{D}_{V_{1,0}})$ is a canonical model and equalitites $\kappa(V_{1,0},\epsilon\mathcal{D}_{V_{1,0}})=\kappa(Y,\epsilon\mathcal{D}_{Y})=3$ hold. Hence, $(Y,\mathcal{D}_{Y})$ is also a canonical model, but $\mathcal{D}_{V_{1,0}}=\rho_{1,0}(\mathcal{D}_{V})$ by construction. Thus, $\rho_{1,0}\circ\beta^{-1}$ is biregular by Theorem 2.15. $\square$

Therefore, we proved Theorem 1.4.

## §7. Proof of Theorem 1.15.

In notations of §1 suppose that $E\subset X$ is smooth, line $L$ be the only line on sextic $S$, which passes through one of point $\theta(O_i)\in\mathbb{P}^3$, and there is a birational map $\beta: V\dashrightarrow Y$ and fibration $\pi: Y\to\mathbb{P}^1$, such that general fiber of $\pi$ is connected smooth surface with numerically trivial canonical divisor. We will find birational automorphism $\sigma$ of 3-fold $V$ and pencil $\mathcal{P}\subset|-K_V|$, such that $\rho\circ\sigma=\phi_{\mathcal{P}}$, where $\rho=\pi\circ\beta$.

*Remark 7.1.* Some arguments of §5-6 fail in this chapter.

Let $\mathcal{D}_V=\beta^{-1}(|\tau^*(\mathcal{O}_{\mathbb{P}^1}(1)|)$. Then $\mathcal{D}_V\subset|-nK_V|$ for some $n\in\mathbb{N}$.

Lemma 7.2. $\overline{\mathbb{CS}}(V,\frac{1}{n}\mathcal{D}_V)\neq\emptyset$.

*Proof.* Suppose $(V,\frac{1}{n}\mathcal{D}_V)$ is semi-terminal. Then $\kappa(V,\epsilon\mathcal{D}_V)=3$ for some $\epsilon\in\mathbb{Q}_{>\frac{1}{n}}$, but birational invariance of Kodaira dimension implies $\kappa(V,\epsilon\mathcal{D}_V)=1$. $\square$

Lemma 7.3. *$\mathbb{CS}(V,\frac{1}{n}\mathcal{D}_V)$ does not contain smooth points of $V$.*

*Proof.* Suppose $\mathbb{CS}(V,\frac{1}{n}\mathcal{D}_V)$ contains a point $O\notin\tilde{L}$. Take general $H_O\in|-K_V|$ passing through $O$. Then $2n^2=H_O\cdot\mathcal{D}_V^2\geq n^2 mult_O(\mathcal{D}_V^2)\geq 4n^2$ by Theorem 3.12. $\square$

Lemma 7.4. *Suppose $\mathbb{CS}(V,\frac{1}{n}\mathcal{D}_V)$ contains a curve $C\neq\tilde{L}$. Then $\gamma(C)$ is a line in $\mathbb{P}^3$.*

*Proof.* Inequality $mult_C(\mathcal{D}_V)\geq n$ is equivalent to $C\in\mathbb{CS}(V,\frac{1}{n}\mathcal{D}_V)$. Thus, for general divisor $H$ in linear system $|-K_V|$ inequalities

$$2n^2=H\cdot\mathcal{D}_V^2\geq mult_C(\mathcal{D}_V^2)H\cdot C\geq mult_C^2(\mathcal{D}_V)H\cdot C\geq n^2H\cdot C$$

imply $-K_V\cdot C\leq 2$. In case $-K_V\cdot C=1$ image $\gamma(C)$ is a line in $\mathbb{P}^3$. Thus, we may assume that $-K_V\cdot C=2$, equalities $mult_C(\mathcal{D}_V)=n$ and $mult_C(\mathcal{D}_V^2)=n^2$ hold, curve $\gamma(C)$ is a conic in $\mathbb{P}^3$, map $\gamma|_C: C\to\gamma(C)$ is an isomorphism and support of $\mathcal{D}_V^2$ consists of $C$.

Suppose that $C\cap\tilde{L}=\emptyset$. Let $g: W\to V$ be a blow up of curve $C$, let $G$ be an exceptional divisor of $g$ and $\mathcal{D}_W$ be a proper transform of $\mathcal{D}_V$ on 3-fold $W$. We claim that effective divisor $g^*(-3K_V)-G$ is numerically effective. Suppose $\gamma(C)\not\subset S$. Consider a curve $\tilde{C}\subset W$, such that $\gamma\circ g(\tilde{C})=\gamma(C)$ and $g(\tilde{C})\neq C$. Then $(g^*(-2K_V)-G)\cdot\tilde{C}=-2$ and curve $\tilde{C}$ is the only curve in the base locus of linear system $|g^*(-2K_V)-G|$, which proves that $g^*(-3K_V)-E$ is numerically effective. Suppose that $\gamma(C)\subset S$. Then base locus of linear system $|g^*(-2K_V)-G|$ is contained in $G$. Let $s_\infty$ be an exceptional section of ruled surface $g|_G : G \to C$. Numerical effectiveness of $g^*(-3K_V) - G$ follows from inequality $(g^*(-3K_V) - G)|_G \cdot s_\infty \geq 0$, but elementary properties of blow ups imply

$$(g^*(-3K_V) - G)|_G \cdot s_\infty = 6 + \frac{s_\infty^2}{2}$$

and $G^3 = 0$. We have to show $s_\infty^2 \geq -12$. Let $\mathcal{N}_{C/V} \cong \mathcal{O}_C(m) \oplus \mathcal{O}_C(n)$ for $m \geq n$. Then

$$m + n = deg(\mathcal{N}_{C/V}) = -K_V \cdot C + 2g(C) - 2 = 0$$

and exact sequence

$$0 \to \mathcal{N}_{C/\tilde{S}} \to \mathcal{N}_{C/V} \to \mathcal{N}_{\tilde{S}/V}|_C \to 0$$

imply $n \geq deg(\mathcal{N}_{C/\tilde{S}}) = -6$, where $\tilde{S} = \gamma^{-1}(S)$. So, $s_\infty^2 = n - m = 2n \geq -12$ and effective divisor $g^*(-3K_V) - G$ is numerically effective. Therefore, we have inequality

$$6n^2 - mult_C(\mathcal{D}_V)(6 mult_C(\mathcal{D}_V) + 4n) = (g^*(-3K_V) - G) \cdot \mathcal{D}_W^2 \geq 0,$$

which is a contradiction. Hence, $C \cap \tilde{L} \neq \emptyset$.

Let $\hat{C} = f^{-1}(C)$ and $h : U \to X$ be a blow up of curve $C$, let $H$ be an exceptional divisor of $h$, $\mathcal{D}_U$ be a proper transform of $\mathcal{D}_V$ on $U$, let $D_1$ and $D_2$ be two general enough divisors in linear system $\mathcal{D}_U$ and $r = E \cdot \hat{C} = 2 - F \cdot \hat{C}$, where $F$ is a fiber of $\tau$. Then

$$T_U = D_1 \cdot D_2 = \sum_{O_i \neq \delta \in \tilde{L} \cap C} p_\delta Z_\delta + \sum_{O_i \in \tilde{L} \cap C} (p_{i,0} \bar{Z}_i^0 + p_{i,1} \bar{Z}_i^1) + C_H,$$

where $Z_\delta$ is a proper transform on $U$ of an irreducible fiber of $f$ over point $\delta \in \tilde{L}$, $\bar{Z}_{i,k}$ is a proper transform on $U$ of a component of fiber $Z_i = Z_{i,0} \cup Z_{i,1}$ of $f$ over $O_i$, integers $p_\delta$ and $p_{i,k}$ are non-negative, support of cycle $C_H$ is contained in $H$. Moreover, equality

$$(2 - r)n^2 = h^*(F) \cdot \mathcal{D}_U^2 = \sum 2p_\delta + \sum p_{i,0} + \sum p_{i,1}$$

holds, because cycle $C_H$ is contracted by $h$ due to $mult_C^2(\mathcal{D}_V) = mult_C(\mathcal{D}_V^2)$. However, effective divisor $h^*(-3K_X) - H$ is not numerically effective. Nevertheless, we always have

$$(h^*(-3K_X) - H) \cdot T_U \geq -H \cdot C_H - \sum p_\delta - \sum p_{i,0} - \sum p_{i,1} \geq (r - 2)n^2 \geq -n^2$$

and, therefore, we have

$$6n^2 - mult_C(\mathcal{D}_V)(6 mult_C(\mathcal{D}_V) + 4n) = (h^*(-3K_X) - H) \cdot T_U \geq -n^2,$$

which is a contradiction. $\square$

Lemma 7.5. *Suppose $\mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$ contains a curve $C$ with $-K_V \cdot C = 2$. Then there is a pencil $\mathcal{P}$ of K3 surfaces in linear system $|-K_V|$, such that $\rho = \phi_{\mathcal{P}}$.*

*Proof.* Image of $C$ on $\mathbb{P}^3$ is a line due to Lemma 7.4. Thus, there is a pencil $\mathcal{P} \subset |-K_V|$ of surfaces passing through curve $C$. Let $g : W \to V$ be a resolution of indeterminacy of rational map $\phi_{\mathcal{P}}$, such that $W$ is smooth and there is unique $g$-exceptional divisor $G$, which dominates curve $C$, let $\mathcal{D}_W$ be a proper transform of $\mathcal{D}_V$ on $W$ and $D_W$ be a general enough fiber of morphism $\phi_{\mathcal{P}} \circ g : W \to \mathbb{P}^1$. Then equality $mult_C(\mathcal{D}_V) = n$ implies

$$\mathcal{D}_W|_{D_W} \sim \sum_{i=1}^{k} a_i G_i,$$

where $a_i$ are non-negative integer numbers and every divisor $G_i$ is $g$-exceptional, whose image on $V$ is a point on curve $C$. On the other hand, linear system $\mathcal{D}_W$ has no fixed components, which implies that $\phi_{\mathcal{D}_W} = \phi_{\mathcal{P}} \circ g$. Therefore, $\rho = \phi_{\mathcal{P}}$. $\square$

LEMMA 7.6. *Suppose that $\mathbb{CS}(V, \frac{1}{n}\mathcal{D}_V)$ contains a curve $C$ with $C \cap \tilde{L} = \emptyset$. Then there is a pencil $\mathcal{P}$ in linear system $|-K_V|$, such that $\rho = \phi_{\mathcal{P}}$.*

*Proof.* See the proof of Lemma 5.5. $\square$

LEMMA 7.7. *Suppose that set $\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ does not contain curve $\tilde{L}$ and contains some curve $C$ on 3-fold $V$. Then there is a pencil $\mathcal{P} \subset |-K_V|$, such that $\rho = \phi_{\mathcal{P}}$.*

*Proof.* See the proof of Lemma 5.6. $\square$

LEMMA 7.8. *$\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ contains either a curve on $V$ or some point $O_i$.*

*Proof.* Suppose the claim is false. Then Lemmas 7.3-7.7 imply existence of point $O \neq O_i$, such that $O \in \tilde{L}$ and $O \in \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$. Proper transform $\mathcal{D}_X$ of linear system $\mathcal{D}_V$ on $X$ is a linear subsystem of linear system $|-nK_X|$ and set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains either an irreducible fiber $Z$ of birational morphism $f$ over point $O$ or a point $P \in Z$.

Suppose $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains a point $P \in Z$. Then $\mathcal{D}_X^2 = mult_Z(\mathcal{D}_X^2)Z + C_X$, where support of one-dimensional effective cycle $C_X$ does not contain curve $Z$ and

$$mult_P(\mathcal{D}_X^2) = mult_Z(\mathcal{D}_X^2) + mult_P(C_X) \geq 4n^2$$

by Theorem 3.12. On the other hand, $Z \cdot F = 2$ and $\mathcal{D}_X^2 \cdot F = 2n^2$, where $F$ is a fiber of fibration on del Pezzo surfaces of degree two $\tau$. In particular, $mult_Z(\mathcal{D}_X^2) \leq n^2$ and inequality $mult_P(C_X) \geq 3n^2$ holds. Let $H$ be a general enough divisor in $|-K_X|$ passing through curve $Z$. Then $H$ does not contain any irreducible component of $C_X$ and

$$2n^2 = H \cdot C_X \geq mult_P(C_X) \geq 3n^2,$$

which is a contradiction.

Suppose $Z \in \mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$. Let $g : W \to X$ be a blow up of $Z$, let $\mathcal{D}_W$ be a proper transform of $\mathcal{D}_X$ on $W$ and $G$ be a $g$-exceptional divisor. Then $|-K_W|$ is free and $\phi_{|-K_W|}$ is an elliptic fibration. Let $C$ be a general fiber of $\phi_{|-K_W|}$. Then $\mathcal{D}_W \cdot C = 2n - mult_Z(\mathcal{D}_X)$, which implies that $\mathcal{D}_W$ lies in the fibers of $\phi_{|-K_W|}$. Moreover, set $\mathbb{CS}(W, \frac{1}{n}\mathcal{D}_W)$ does not contain curves, which are not contracted by $\tau \circ g$, because otherwise

$$2n^2 = \mathcal{D}_X^2 \cdot F \geq 3n^2,$$

where $F$ is a fiber of $\tau$. We already proved that set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ can not contain points in irreducible fibers of $f$. Suppose that set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains some point $P_i$ on reducible fiber $Z_i = Z_i^0 \cup Z_i^1$ of $f$. Let $H_{Z_i}$ be a general element in $|-K_X|$ passing through $Z_i$ and

$$n^2(Z_i + 2C) \equiv D_1 \cdot D_2 = mult_{Z_i^0}(\mathcal{D}_X^2)Z_i^0 + mult_{Z_1^1}(\mathcal{D}_X^2)Z_i^1 + C_X + R_X,$$

where $D_1$ and $D_2$ are general divisors in $\mathcal{D}_P$, $C_X$ is an effective one-cycle on $X$ contained in the fibers of $\tau$, $R_X$ is an effective one-cycle on $X$, whose components are not contained in the fibers of $\tau$, $C$ is a curve in fibers of $\tau$ with $-K_X \cdot C = 1$. Then

$$2n^2 = H_{Z_1} \cdot (C_X + R_X) \geq mult_P(C_X) + mult_P(R_X) > 2n^2,$$

because $mult_{Z_i^0}(\mathcal{D}_X^2) + mult_{Z_i^1}(\mathcal{D}_X^2) + R_X \cdot F = 2n^2$, cycle $R_X$ is not empty by assumption and inequality $mult_{P_i}(D_1 \cdot D_2) \geq 4n^2$ holds by Theorem 3.12. Therefore, singularities of log pair $(W, \frac{1}{n}\mathcal{D}_W)$ are terminal. Hence, for small enough $\epsilon \in \mathbb{Q}_{>\frac{1}{n}}$ log pair $(W, \epsilon\mathcal{D}_W)$ is terminal as well and $\kappa(W, \epsilon\mathcal{D}_W) = 2$, but $\kappa(W, \epsilon\mathcal{D}_W) = 1$ by construction of $\mathcal{D}_V$. $\square$

Let $\mathcal{D}_X$ be a proper transform of $\mathcal{D}_V$ on $X$ and $F$ be a fiber of $\tau : X \to \mathbb{P}^1$. Then

$$\mathcal{D}_X \sim f^*(-nK_V) - mE \sim -nK_X - mE \sim f^*(K_V) \sim n(F + E) - mE,$$

where $n > m = mult_{\tilde{L}}(\mathcal{D}_V)$. Let $\mu = \frac{1}{n-m}$ and $r = \frac{m}{n-m}$. Then $K_X + \mu\mathcal{D}_X \sim_{\mathbb{Q}} rF$.

*Remark 7.9.* $r > 0 \iff m > 0 \iff \tilde{L} \in \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$.

LEMMA 7.10. *$\mathbb{CS}(X, \mu\mathcal{D}_X) \neq \emptyset$.*

*Proof.* Suppose $(X, \mu\mathcal{D}_X)$ is terminal. In case $r > 0$ log pair $(X, \epsilon\mathcal{D}_X)$ is a canonical model for some $\epsilon \in \mathbb{Q}_{>\mu}$, but $\kappa(X, \epsilon\mathcal{D}_X) \leq 1$ by construction of $\mathcal{D}_V$. In case $r = 0$ movable log pair $(V, \frac{1}{n}\mathcal{D}_V)$ is terminal, which contradicts Lemma 7.1. $\square$

LEMMA 7.11. *Let $C$ be a curve in $\mathbb{CS}(X, \mu\mathcal{D}_X)$, such that $C$ is contained in the fibers of del Pezzo fibration $\tau$. Then log pair $(X, \mu\mathcal{D}_X)$ is canonical in the generic point of $C$.*

*Proof.* See the proof of Lemma 5.11. $\square$

LEMMA 7.12. *There is a composition $\sigma$ of Bertini involutions of generic fiber of $\tau$, such that log pair $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ has canonical singularities in codimension one, where $\mu_\sigma \in \mathbb{Q}_{>0}$ and equivalence $K_X + \mu_\sigma\sigma(\mathcal{D}_X) \sim_{\mathbb{Q}} r_\sigma F$ holds for some positive rational number $r_\sigma$.*

*Proof.* See the proof of Lemma 5.12. $\square$

Composing $\beta$ with $\sigma^{-1}$ we may assume that $(X, \mu\mathcal{D}_X)$ is canonical in codimension one.

Lemma 7.13. *$\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ contains some curve on $V$.*

*Proof.* Suppose the claim is false. Then $\tilde{L} \notin \overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ and set $\overline{\mathbb{CS}}(V, \frac{1}{n}\mathcal{D}_V)$ contains some point $O_i$ by Lemma 7.8. We may assume that $i = 1$. Therefore, Lemma 7.10 implies that set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains either a point $P \in Z_1$ or a component of $Z_1 = Z_1^0 \cup Z_1^1$, where $Z_1$ is a redicible fiber of birational morphism $f$ over $O_1$. Let $D_1$ and $D_2$ be general enough surfaces in $\mathcal{D}_X$. Then

$$n^2(Z_1 + 2C) \equiv n^2 K_X^2 \equiv D_1 \cdot D_2 = mult_{Z_1^0}(\mathcal{D}_X^2)Z_1^0 + mult_{Z_1^1}(\mathcal{D}_X^2)Z_1^1 + C_X + R_X,$$

where $C$ is a curve in fibers of $\tau$ with $-K_X \cdot C = 1$, $C_X$ is an effective one-cycle on $X$ contained in the fibers of $\tau$ and $R_X$ is an effective one-cycle on $X$, whose components are not contained in the fibers of $\tau$.

Suppose that set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ contains some point $P \in Z_1$. Then Theorem 3.12 implies inequality $mult_P(D_1 \cdot D_2) \geq 4n^2$. On the other hand, $Z_1^0 \cdot F = Z_1^1 \cdot F = 2$ and

$$2n^2 = D_1 \cdot D_2 \cdot F = mult_{Z_1^0}(\mathcal{D}_X^2) + mult_{Z_1^1}(\mathcal{D}_X^2) + R_X \cdot F,$$

where $F$ is a fiber of $\tau$. In particular, $mult_{Z_1^0}(\mathcal{D}_X^2) + mult_{Z_1^1}(\mathcal{D}_X^2) \leq 2n^2$ and equality holds in case $R_X = \emptyset$. Let $H_{Z_1}$ be a general enough surface in linear system $|-K_X|$ passing through $Z_1$. Then $H_{Z_1}$ does not contain irreducible components of $C_X$ and $R_X$. Thus,

$$H_{Z_1} \cdot (C_X + R_X) \geq mult_P(C_X) + mult_P(R_X) \geq 4n^2 - mult_{Z_i^0}(\mathcal{D}_X^2)\delta_P^0 + mult_{Z_1^1}(\mathcal{D}_X^2)\delta_P^1,$$

where $\delta_P^i = mult_P(Z_1^i)$. However, $H_{Z_1} \cdot (C_X + R_X) = 2n^2$. Hence, cycle $R_X$ is empty, equality $mult_P(C_X) = H_{Z_1} \cdot C_X = 2n^2$ holds and either point $P$ is a point of intersection of curves $Z_1^0$ and $Z_1^1$ and equality $mult_{Z_1^0}(\mathcal{D}_X^2) + mult_{Z_1^1}(\mathcal{D}_X^2) = 2n^2$ holds or point $P$ is contained only in one component $Z_1^k$ of $Z_1$, $mult_{Z_1^k}(\mathcal{D}_X^2) = 2n^2$ and $mult_{Z_1^{1-k}}(\mathcal{D}_X^2) = 0$, but equality $D_1 \cdot Z_{1-k}^0 = 0$ implies $mult_{Z_1^{1-k}}(\mathcal{D}_X^2) > 0$. Thus, $P = Z_1^0 \cap Z_1^1$ and

$$mult_{Z_1^0}(\mathcal{D}_X^2) + mult_{Z_1^1}(\mathcal{D}_X^2) = 2n^2,$$

but $mult_P(C_X) = H_{Z_1} \cdot C_X$ implies that $C_X$ is contained in the fiber $F_P$ of $\tau$ passing through $P$. Moreover, $mult_P(\mathcal{D}_X) = 2n$ by Theorem 3.12. Consider $V$ as a hypersurface

$$u^2 = x^2 \sum_{i=0}^{4} \bar{p}_i(x, y, z)t^{4-i} + y^2 \sum_{i=0}^{4} \bar{q}_i(x, y, z)t^{4-i}$$

in $\mathbb{P}(1, 1, 1, 3)$, where $\bar{p}_i$ and $\bar{q}_i$ are homogeneous polynomials of degree $i$, $x$, $y$, $z$ and $t$ are coordinates of weight one and $u$ is a coordinate of weight three, such that curve $\tilde{L}$ is given by $x = y = 0$ and point $O_1$ is given by $x = y = z = 0$. Then either $\bar{q}_0$ or $\bar{p}_0$ vanishes by definition of $O_1$ and due to our initial assumption that there are exactly eight different points $O_i$. Suppose $\bar{q}_0 = 0$. Then linear form $\bar{q}_1(x, y, z)$ does not vanish due to smoothness of 3-fold $X$. Moreover, $\bar{q}_1(0, 0, z)$ does not vanish due to smoothness of $E$. Therefore, pencil $\mathcal{P}$ in linear system $|-K_V|$ given by equation $Ax + B\bar{q}_1(x, y, z) = 0$ does not have base components, where $A$ and $B$ are complex numbers. Let $\mathcal{P}_X$ be a proper transform of pencil $\mathcal{P}$ on 3-fold $X$ and $D_X$ be a general element in $\mathcal{P}_X$. Then

$$mult_P(D_X) = 2, D_X \sim -K_X, mult_P(D_1 \cdot D_X) = 4n$$

and we can apply previous calculations to effective one-cycle $D_1 \cdot D_X$ instead of effecive one-cycle $D_1 \cdot D_2$. Therefore, $D_1 \cdot D_X \subset E \cup F_P$. Let $O \notin E \cup F_P$ be a general enough point on surface $D_1$ and $D_X$ be a surface in pencil $\mathcal{P}_X$ passing through $P$. Then $D_1 = D_X$, because surfaces $D_1$ and $D_X$ are irreducible. Therefore, $\mathcal{D}_V = \mathcal{P}$, but set $\overline{\mathbb{CS}}(V, \mathcal{P})$ contains some curve, which contradicts our assumption.

Hence, we may assume that set $\mathbb{CS}(X, \frac{1}{n}\mathcal{D}_X)$ does not contain points on $X$ and contains either both curves $Z_1^0$ and $Z_1^1$ or only one of them. However, in all these cases we can apply arguments from the proofs of Lemmas 7.8 and 5.13 to derive a contradiction. □

*Remark 7.14.* We may assume that $r > 0$.

Let $h : U \to X$ be a birational morphism with smooth $U$ and regular $\beta \circ f \circ h$. Then

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(rF) + \sum_{i=1}^{k} a_i E_i,$$

where $\mathcal{D}_U$ is a proper transform of linear system without fixed components $\mathcal{D}_X$ on $U$, divisor $E_i$ is $h$-exceptional and $a_i$ is rational. Consider finite or empty subset $\mathcal{J} \subset \mathbb{P}^1$, such that for one of divisors $E_i$ with $a_i < 0$ its image on $X$ is a point in the fiber of $\tau$ over the point in $\mathcal{J}$. For every point $\lambda$ in $\mathcal{J}$ we have rational equivalence

$$h^*(F_\lambda) \sim h^{-1}(F_\lambda) + \sum_{j=1}^{k_\lambda} b_j E_j,$$

where $b_i$ is a natural number and $F_\lambda$ is a fiber of $\tau$ over $\lambda$. Finally, let $\mathcal{I} = \cup_{\lambda \in \mathcal{J}} \mathcal{I}_\lambda$, where for every $\lambda \in \mathcal{J}$ let $\mathcal{I}_\lambda \subset \{1, \ldots, k\}$ be a finite subset, such that $i \in \mathcal{I}_\lambda$ if and only if $h(E_i)$ is a point in fiber $F_\lambda$ and $a_i < 0$.

Proposition 7.15. *$r + \sum_{\lambda \in \mathcal{J}} min\{\frac{a_i}{b_i} | \ h(E_i) \in F_\lambda$ and $a_i < 0\} \leq 0$.*

*Proof.* Suppose the claim is false. Then there are positive rational numbers $\epsilon$ and $c_\lambda$, such that $r = \epsilon + \sum_{\lambda \in \mathcal{J}} c_\lambda$ and $c_\lambda + min\{\frac{a_i}{b_i} | \ h(E_i) \in F_\lambda$ and $a_i < 0\} > 0$. Then

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(\epsilon F) + \sum_{\lambda \in \mathcal{J}} \Big( h^*(c_\lambda F_\lambda) + \sum_{i \in \mathcal{I}_\lambda} a_i E_i \Big) + \sum_{i \notin \mathcal{I}} a_i E_i$$

and for every $\lambda \in \mathcal{J}$ divisor $h^*(c_\lambda F_\lambda) + \sum_{i \in \mathcal{I}_\lambda} a_i E_i$ is effective by the choice of positive rational numbers $c_i$ and divisor $\sum_{i \notin \mathcal{I}} a_i E_i$ is effective, because $(X, \mu\mathcal{D}_X)$ is canonical in codimension one. Let $O$ be a general enough point in general enough fiber $D_O$ of $\rho \circ f \circ h$ and $C$ be a proper transfom on $U$ of a general enough irreducible curve in general enough fiber of $\pi$, such that $C$ contains point $O$. Then $K_U \cdot C = 0$, because canonical divisor of the general enough fiber of $\pi$ is numerically trivial by assumption, and $\mathcal{D}_U \cdot C = 0$ by construction of linear system $\mathcal{D}_V$. Thus, we have

$$(K_U + \mu\mathcal{D}_U) \cdot C = h^*(\epsilon F) \cdot C + \sum_{\lambda \in \mathcal{J}} \Big(h^*(c_\lambda F_\lambda) + \sum_{i \in \mathcal{I}_\lambda} a_i E_i\Big) \cdot C + \sum_{i \notin \mathcal{I}} a_i E_i \cdot C = 0$$

and, therefore, $h^*(\epsilon F) \cdot C = 0$ due to generality in the choice of curve $C$. Let $F_C$ be a fiber of $\tau \circ h$ passing through $O$. Then $h^*(F) \cdot C = 0$ implies $C \subset F_C$. On the other hand, both surfaces $D_O$ and $F$ are irreducible due to generality of the choice of $O$. Hence, generality in the choice of $C$ implies $F = D_0$, which is impossible, because Kodaira dimension of surface $F$ is $-\infty$ and Kodaira dimension of surface $D_0$ is zero. $\square$

Let $Z$ be a fiber of $f|_E : E \to \tilde{L}$ and $C$ be a curve in fibers of $\tau$ with $-K_X \cdot C = 1$, choose two general enough surfaces $D_1$ and $D_2$ in $\mathcal{D}_X$. Then $\overline{\mathbb{NE}}(X) = \mathbb{R}_{\geq 0} Z \oplus \mathbb{R}_{\geq 0} C$ and

$$T_0 = \mu^2 D_1 \cdot D_2 = Z_X + \sum_{\lambda \in \mathbb{P}^1} C_\lambda \equiv Z + (2 + 4r)C,$$

where all components of effective cycle $Z_X$ do not lie in the fibers of $\tau$ and every component of effective cycle $C_\lambda$ lies in the fiber $F_\lambda$ of $\tau$ over $\lambda \in \mathbb{P}^1$. Therefore,

$$-K_X \cdot \sum_{\lambda \in \mathbb{P}^1} C_\lambda = 2 + 4r \leq 2 - 4 \sum_{\lambda \in \mathcal{J}} min\{\frac{a_i}{b_i} |\ h(E_i) \in F_\lambda \text{ and } a_i < 0\}$$

and there is $\omega \in \mathbb{P}^1$ and a $h$-exceptional divisor $E_t$, such that $h(E_t)$ is a point $O$ in the fiber $F_\omega$ of $\tau$ over $\omega$, and for $a_t/b_t = min\{\frac{a_i}{b_i} |\ h(E_i) \in F_\omega \text{ and } a_i < 0\}$ inequality

$$-K_X \cdot C_\omega \leq 2 - 4a_t/b_t$$

holds. Thus, we can repeat all arguments of §5 starting from Lemma 5.22 except the last one, because linear system $\mathcal{D}_X$ is a pencil. Therefore, in notations of §5

$$T_0 = Z_X + C_\omega, a(X, \mu\mathcal{D}_X, G_N)) = -\Sigma_1, \Sigma_0 = \Sigma_0', b(X, F_\omega, G_N) = \Sigma_0, r = \frac{\Sigma_1}{4\Sigma_0}, \nu_i = 2,$$

support of cycle $Z_X$ is contained in some reducible fiber $Z_i = Z_i^0 \cup Z_i^1$ of $f$ and

$$mult_O(Z_X) = 2n^2, mult_O(C_\omega) = 2 + 4r, -K_X \cdot C_\omega = mult_O(C_\omega), m_{T_0}^i = m_{T_0}^0.$$

*Remark 7.16.* Fiber $F_\omega$ is a double cover of $\theta \circ f(F_\omega) \cong \mathbb{P}^2$ ramified in possible singular quartic curve, therefore, smoothness of $F_\omega$ in $O$ and $-K_X \cdot C_\omega = mult_O(C_\omega)$ imply that support of cycle $\theta \circ f(C_\lambda)$ is a unique line in $\mathbb{P}^3$ passing through point $\theta(O_i) \in L$.

LEMMA 7.17. $O = Z_i^0 \cap Z_i^1$.

*Proof.* Suppose $O \neq Z_i^0 \cap Z_i^1$ and $O \in Z_i^k$. Then support of $C_\omega$ consists of smooth rational curve $C \subset X$, such that $-K_X \cdot C = 1$ and $O \in C$. However, we have

$$\mathcal{D}_X|_{F_\omega} \sim (m-n)K_{F_\omega} \sim (m-n)K_X|_{F_\omega}$$

and $mult_O(\mathcal{D}_X|_{F_\omega}) \geq 2(n-m)$, where $F_\omega$ is considered as double cover $\theta_{F_\omega} : F_\omega \to \mathbb{P}^2$ ramified in quartic curve $S_{F_\omega} \subset \mathbb{P}^2$, which is smooth in point $\theta_{F_\omega}(O)$. Therefore,

$$\mathcal{D}_X|_{F_\omega} = 2(n-m)C,$$

which implies $\theta_{F_\omega}(C) \subset S_{F_\omega}$. In particular, $\theta_{F_\omega}(O) \in S_{F_\omega}$, which implies $O = Z_i^0 \cap Z_i^1$. $\square$

LEMMA 7.18. $K = 1$.

*Proof.* Let $K \neq 1$. Then $P_1$ is a point. Equality $m_{T_0}^0 = m_{T_0}^1$ implies that $P_1$ is contained in proper transform of $Z_i^k$ on $X_1$ and support of $Z_X$ consists of $Z_i^k$. Therefore, proper transform of cycle $C_\omega$ on $X_1$ can not pass through point $P_1$, because curve $Z_i^k$ intersects any curve in the support of cycle $C_\omega$ transversally due to smoothness of $E$ and due to

$$E \cdot C_\omega = -K_X \cdot C_\omega = mult_O(C_\omega),$$

which implies $m_{T_0}^0 > m_{T_0}^1$. $\square$

*Remark 7.19.* Equalities $\nu_i = 2$ imply that graph $\Gamma$ is a chain, $\Sigma_0 = 1$ and $\Sigma_1 = N-1$.

Restriction of $\mathcal{D}_{X_1}$ on $G_1 \cong \mathbb{P}^2$ is rationally equivalent to $\mathcal{O}_{\mathbb{P}^2}(2(n-m))$. On the other hand, equality $mult_{P_1}(\mathcal{D}_{X_1}) = 2(n-m)$ holds. Therefore, $\mathcal{D}_{X_1}|_{G_1} = 2(n-m)P_1$ and $P_1$ is a line on $G_1 \cong \mathbb{P}^2$. Equality $b(X, F_\omega, G_N) = \Sigma_0$ implies $P_1 \not\subset F_\omega^1$. Therefore, restriction of linear system $\mathcal{D}_{X_1}$ on surface $F_\omega^1$ has multiplicity at least $2(n-m)$ in point $P_1 \cap F_\omega^1$, which implies that $\mathcal{D}_X|_{F_\omega} = 2(n-m)C_0$, where $C_0 \subset F_\omega$ is a smooth rational curve passing through $O$, such that equality $-K_X \cdot C_0 = 1$ holds. Thus, curve $\theta \circ f(C_0)$ is a line on $S$, which pass through point $O_i$. Hence, $\theta \circ f(C_0) = L$ and $C_0 = F_\omega \cap E$ due to our initial assumption, which contradicts smoothness of $E$. Therefore, we proved Theorem 1.15.

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