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Integrable Structure of the Dirichlet Boundary Problem in Multiply-Connected Domains

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Abstract

We study the integrable structure of the Dirichlet boundary problem in two dimensions and extend the approach to the case of planar multiply-connected domains. The solution to the Dirichlet boundary problem in multiply-connected case is given through a quasiclassical tau-function, which generalizes the tau-function of the dispersionless Toda hierarchy. It is shown to obey an infinite hierarchy of Hirota-like equations which directly follow from properties of the Dirichlet Green function and from the Fay identities. The relation to multi-support solutions of matrix models is briefly discussed.

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1 Introduction

The Dirichlet boundary problem [1] is to reconstruct a harmonic function in a bounded domain from its values on the boundary. Remarkably, this standard problem of complex analysis, related however to string theory and matrix models, possesses a hidden integrable structure [2], which we clarify further in this paper. It turns out that variation of a solution to the Dirichlet problem under variation of the domain is described by an infinite hierarchy of non-linear partial differential equations known (in the simply-connected case) as dispersionless Toda hierarchy. It is a particular example of the universal hierarchy of quasiclassical or Whitham equations introduced in [3, 4].

The quasiclassical tau-function, or, more precisely, its logarithm F, is the main new object associated with a family of domains in the plane. Any domain in the complex plane with sufficiently smooth boundary can be parametrized by its moments with respect to a basis of harmonic functions. The F-function is a function of the full infinite set of the moments. The first order derivatives of F are then moments of the complementary domain. This gives a formal solution to the inverse potential problem, considered for a simply-connected case in [5, 6]. The second order derivatives are coefficients of the Taylor expansion of the Dirichlet Green function and therefore they solve the Dirichlet boundary problem. These coefficients are constrained by infinite number of universal (i.e. domain-independent) relations which, unified in a generating form, just constitute the dispersionless Hirota equations. For the third order derivatives (their role in problems of complex analysis is not yet quite clear) there is a nice "residue formula" which allows one to prove [7] that F obeys the WDVV equaitons.

Below we are going to demonstrate that for planar multiply-connected domains the solution to the Dirichlet boundary problem can be performed in a similar way. Specifically, we consider domains which are obtained by cutting several "holes" in the complex plane. Boundaries of the holes are assumed to be smooth simple non-intersecting curves. In this case, the complete set of independent variables can again be chosen to be the set of harmonic moments. However, the basis of harmonic functions in a domain with non-trivial cycles (in our case the cycles are homologically equivalent to different components of the boundary) has to be enlarged to include functions whose analytic part is not single-valued. This results in an additional finite set of extra variables, one for each hole. We construct the F-function and prove that its second order derivatives satisfy non-linear relations, which generalize the Hirota equations of the dispersionless Toda hierarchy. These relations are derived from the Fay identities [8] for the Riemann theta functions on the Jacobian of Riemann surface obtained as the Schottky double of the plane with given holes.

We note that the extra variables, specific for the multiply-connected case, can be chosen in different ways having different geometric interpretations, depending on the choice of a basis of homologically non-trivial cycles on the Schottky double. The corresponding F-functions are shown to be connected by a (partial) Legendre transform, with the generalized Hirota relations being the same.

Now let us give a bit more expanded description of the Dirichlet problem in planar domains. Let D^{c} be a domain in the complex plane bounded by one or several non-intersecting curves. It will be convenient for us to realize the D^{c} as a complement to another domain D, having one or more connected components. We consider the Dirichlet problem in D^{c}. The problem is to find a harmonic function $u(z)$ in D^{c}, such that it is continuous up to the boundary $\partial\mathsf{D}^{\mathsf{c}}$ and equals a given function $u_0(\xi)$ on the boundary. The problem has a unique solution written in terms of the Dirichlet Green function $G(z, \xi)$:

$$u(z) = -\frac{1}{2\pi}\oint_{\partial\mathsf{D}^{\mathsf{c}}} u_0(\xi)\partial_n G(z,\xi)|d\xi| \tag{1.1}$$

where ∂_n is the normal derivative on the boundary with respect to the second variable, and the normal vector $\vec{n}$ is directed inward $\mathsf{D^c}$, $|d\xi| := dl(\xi)$ is an infinitesimal element of the length of the boundary $\partial\mathsf{D^c}$.

The main object to study is, therefore, the Dirichlet Green function. It is uniquely determined by the following properties [1]:

($G1$) The function $G(z, z')$ is symmetric and harmonic everywhere in $\mathsf{D^c}$ (including ∞ if $\mathsf{D^c} \ni \infty$) in both arguments except $z = z'$ where $G(z, z') = \log|z - z'| + \ldots$ as $z \to z'$;

($G2$) $G(z, z') = 0$ if any of the variables z, z' belongs to the boundary.

Note that the definition implies that $G(z, z') < 0$ inside $\mathsf{D^c}$. In particular, $\partial_n G(z, \xi)$ is strictly negative for all $\xi \in \partial\mathsf{D^c}$.

If $\mathsf{D^c}$ is simply-connected (the boundary has only one component), the Dirichlet problem is equivalent to finding a bijective conformal map from $\mathsf{D^c}$ onto the unit disk or any other reference domain for which the Green function is known explicitly. Such a conformal map exists by virtue of the Riemann mapping theorem. Let $w(z)$ be such a bijective conformal map of $\mathsf{D^c}$ onto the complement to the unit disk, then

$$G(z, z') = \log\left|\frac{w(z) - w(z')}{w(z)\overline{w(z')} - 1}\right| \tag{1.2}$$

where bar means complex conjugation. It connects the Green function at two points with the conformal map normalized at some third point (say at $z = \infty$: $w(\infty) = \infty$). It is this formula which allows one to derive the Hirota equations for the tau-function of the Dirichlet problem in the most economic and transparent way [2].

For multiply-connected domains, formulas of this type do not exist. In general, there is no canonical choice of the reference domain, and, moreover, the shape of a reference domain depends on $\mathsf{D^c}$ itself. In fact, as we demonstrate in the paper, the correct extension of (1.2) needed for derivation of the generalized Hirota equations follows from a different direction which is no longer related to bijective conformal maps. Namely, the logarithm of conformal map $\log w(z)$ should be replaced now by the Abel map from the Schottky double of the plane with holes $\mathsf{D^c}$ to the Jacobi variety of this Riemann surface, while the rational function under the logarithm should be replaced by a ratio of the prime forms or Riemann theta-functions.

We show that the Green function of multiply-connected domains admits a representation through the logarithm of the tau-function of the form

$$G(z, z') = \log\left|\frac{1}{z} - \frac{1}{z'}\right| + \frac{1}{2}\nabla(z)\nabla(z')F \tag{1.3}$$

Here $\nabla(z)$ (see (2.5) below) is certain differential operator with constant coefficients (depending, however, on the point z as a parameter) in the space of harmonic moments. Existence of such a representation of the Green function was first conjectured by Takhtajan. For the simply-connected case, this formula was obtained in [9] (see also [10] for a detailed proof and discussion). In this paper we also obtain similar formulas for the harmonic measures of the boundary components and for the Abel map. A combination of these formulas with the Fay identity yields the generalized Hirota-like equations for F.

Our main tool is the Hadamard variational formula [11] which gives variation of the Dirichlet Green function under small deformations of the domain in terms of the Green function itself:

$$\delta G(z, z') = \frac{1}{2\pi}\oint_{\partial\mathsf{D^c}} \partial_n G(z, \xi)\partial_n G(z', \xi)\delta n(\xi)|d\xi|. \tag{1.4}$$

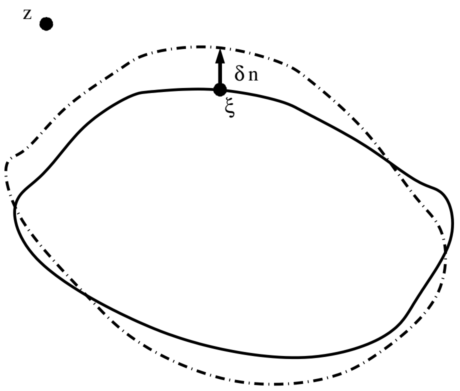

Figure 1: *A "pictorial" derivation of the Hadamard formula. We consider a small deformation of the domain, with the new boundary being depicted by the dashed line. According to (G2) the Dirichlet Green function vanishes $G(z,\xi) = 0$ if ξ belongs to the old boundary. Then the variation $\delta G(z,\xi)$ is simply equal to the new value, i.e. in the leading order $\delta G(z,\xi) = -\delta n(\xi)\partial_n G(z,\xi)$. Now notice that $\delta G(z,\xi)$ is a harmonic function (the logarithmic singularity cancels since it is the same for both old and new functions) with the boundary value $-\delta n(\xi)\partial_n G(z,\xi)$. Applying (1.1) one obtains (1.4). The argument is exactly the same for multiply-connected domains.*

Here $\delta n(\xi)$ is the normal displaycement (with sign) of the boundary under the deformation, counted along the normal vector at the boundary point ξ. It was shown in [2] that this remarkable formula reflects all integrable properties of the Dirichlet problem. An extremely simple "pictorial" derivation of the formula (1.4) is presented in fig. 1.

We start with a brief recollection of the results for the simply-connected case in sect. 2. However, instead of "bump" deformations used in [2] we work here with a family of infinitesimal deformations which we call elementary ones. This approach is basically motivated by the theory of interface dynamics in viscous fluids, which is known to be closely connected with the formalism developed in [2] and in the present paper (see [5] for details).

In sect. 3 we define the class of multiply-connected domains under consideration and introduce various analytic and algebro-geometric objects associated with such a domain. Next, we introduce local coordinates in the space of algebraic domains and express the elementary deformations in these coordinates. Using the Hadamard formula, we then observe remarkable "exchange relations" (3.14) which connect elementary deformations of the Green function and harmonic measures. The existence of the tau-function and the formula (1.3) for the Green function directly follow from these relations. In sect. 4 we make a Legendre transform to another set of local coordinates in the space of algebraic multiply-connected domains, which is in a sense dual to the original one. In these coordinates, eq. (1.3) gives another version of the Green function which solves the so-called modified Dirichlet problem. We also discuss the relation to multi-support solutions of matrix models in the planar large N limit.

In sect. 5 we combine the results outlined above with the representation of the Green function in terms of the prime form on the Schottky double. This allows us to obtain an infinite system of partial differential equations on the tau-function which generalize the dispersionless Hirota equations.

2 The Dirichlet problem for simply-connected domains and dispersionless Hirota equations

In this section we rederive the results from [2] for the simply-connected case in a slightly different manner, more suitable for further generalizations.

Let D be a connected domain in the complex plane bounded by a simple analytic curve. We consider the exterior Dirichlet problem in $\mathsf{D}^{\mathsf{c}} = \mathbf{C} \setminus \mathsf{D}$ which is the complement of D in the whole (extended) complex plane. Without loss of generality, we assume that D is compact and contains the point $z = 0$. Then D^{c} is an unbounded simply-connected domain containing ∞.

Harmonic moments and deformations of the boundary. Let t_k be moments of the domain $\mathsf{D}^{\mathsf{c}} = \mathbf{C} \setminus \mathsf{D}$ defined with respect to the harmonic functions (we use the basis $\{z^{-k}/k\}$, $\{\bar{z}^{-k}/k\}$, $k > 0$):

$$t_k = -\frac{1}{\pi k}\int_{\mathsf{D}^{\mathsf{c}}} z^{-k}\, d^2 z\,, \qquad k = 1, 2, \ldots \tag{2.1}$$

and $\bar{t}_k$ be the complex conjugated moments. The Stokes formula represents the harmonic moments as contour integrals $t_k = \frac{1}{2\pi i k}\oint_{\partial\mathsf{D}} z^{-k}\bar{z}dz$, providing, in particular, a regularization of possibly divergent integrals (2.1). Besides, we denote by t_0 the area (divided by π) of the domain D:

$$t_0 = \frac{1}{\pi}\int_D d^2 z \tag{2.2}$$

The harmonic moments of D^{c} are coefficients of the Taylor expansion of the potential

$$\Phi(z) = -\frac{2}{\pi}\int_{\mathsf{D}} \log|z - z'| d^2 z' \tag{2.3}$$

induced by the domain D filled by two-dimensional Coulomb charges with the uniform density $\rho = -1$. Clearly, $\partial_z\partial_{\bar{z}}\Phi(z) = -1$ if $z \in \mathsf{D}$ and vanishes otherwise, so around the origin (recall that $\mathsf{D} \ni 0$) the potential is $-|z|^2$ plus a harmonic function:

$$\Phi(z) - \Phi(0) = -|z|^2 + \sum_{k\geq 1}\left(t_k z^k + \bar{t}_k \bar{z}^k\right) \tag{2.4}$$

A simple calculation shows that t_k are just given by (2.1).

The basic fact of the theory of deformations of closed analytic curves is that the (in general complex) moments $\{t_k, \bar{t}_k\} \equiv \{t_{\pm k}\}$ supplemented by the real variable t_0 form a set of local coordinates in the "moduli space" of smooth closed curves [12] (see also [10]). This means that under any small deformation of the domain the set $\mathbf{t} = \{t_0, t_{\pm k}\}$ is subject to a small change and vice versa.

Elementary deformations and the operator $\nabla(z)$. The differential operators

$$\nabla(z) = \partial_{t_0} + \sum_{k\geq 1}\left(\frac{z^{-k}}{k}\partial_{t_k} + \frac{\bar{z}^{-k}}{k}\partial_{\bar{t}_k}\right) \tag{2.5}$$

span the complexified tangent space to the space of curves. The operator $\nabla(z)$ has a clear geometrical meaning.

Fix a point $z \in \mathsf{D}^{\mathsf{c}}$ and consider a special infinitesimal deformation of the domain such that the normal displaycement of the boundary is proportional to the gradient of the Green function at the boundary point (fig. 2):

$$\delta n(\xi) = -\frac{\epsilon}{2}\partial_n G(z, \xi) \tag{2.6}$$

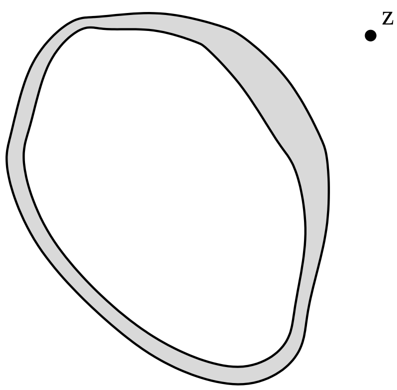

Figure 2: *The elementary deformation with the base point z.*

For any sufficiently smooth initial boundary this deformation is well-defined as $\epsilon \to 0$. We call infinitesimal deformations from this family, parametrized by $z \in \mathsf{D}^c$, *elementary deformations.* The point z is refered to as the *base point* of the deformation. Note that since $\partial_n G < 0$ (see the remark after the definition of the Green function in the Introduction), δn for the elementary deformations is either strictly positive or strictly negative depending of the sign of the ϵ.

Let δ_z be variation of any quantity under the elementary deformation with the base point z. It is easy to see that $\delta_z t_0 = \epsilon$, $\delta_z t_k = \epsilon z^{-k}/k$. Indeed,

$$\delta_z t_k = \frac{1}{\pi k} \oint \xi^{-k} \delta n(\xi) |d\xi| = -\frac{\epsilon}{2\pi k} \oint \xi^{-k} \partial_n G(z, \xi) |d\xi| = \frac{\epsilon}{k} \, z^{-k} \tag{2.7}$$

by virtue of the Dirichlet formula (1.1).

Let $X = X(\mathbf{t})$ be any functional of our domain that depends on the harmonic moments only (in what follows we are going to consider only such functionals). The variation $\delta_z X$ in the leading order in ϵ is then given by

$$\delta_z X = \sum_k \frac{\partial X}{\partial t_k} \delta_z t_k = \epsilon \nabla(z) X \tag{2.8}$$

The right hand side suggests that for functionals X such that the series $\nabla(z)X$ converges everywhere in D^c up to the boundary, $\delta_z X$ is a harmonic function of the base point z.

Note that in [2] we used the "bump" deformation and continued it harmonically to D^c. So it was the elementary deformation (2.7) $\delta_z \propto \oint |d\xi| \partial_n G(z, \xi) \delta^{\mathrm{bump}}(\xi)$ that was really used. The "bump" deformation should be understood as a (carefully taken) limit of δ_z when the point z tends to the boundary.

The Hadamard formula as integrability condition. Variation of the Green function under small deformations of the domain is known due to Hadamard, see eq. (1.4). To find how the Green function changes under small variations of the harmonic moments, we fix three points $a, b, c \in \mathbf{C} \setminus \mathsf{D}$ and compute $\delta_c G(a, b)$ by means of the Hadamard formula (1.4). Using (2.8), one can identify the result with the action of the vector field $\nabla(c)$ on the Green function:

$$\nabla(c) G(a, b) = -\frac{1}{4\pi} \oint_{\partial \mathsf{D}} \partial_n G(a, \xi) \partial_n G(b, \xi) \partial_n G(c, \xi) |d\xi| \tag{2.9}$$

Remarkably, the r.h.s. of (2.9) is *symmetric* in all three arguments:

$$\nabla(a) G(b, c) = \nabla(b) G(c, a) = \nabla(c) G(a, b) \tag{2.10}$$

This is the key relation, which allows one to represent the Dirichlet problem as an integrable hierarchy of non-linear differential equations [2]. This relation is the integrability condition of the hierarchy.

It follows from (2.10) (see [2] for details) that there exists a function $F = F(\mathbf{t})$ such that

$$G(z, z') = \log \left| \frac{1}{z} - \frac{1}{z'} \right| + \frac{1}{2} \nabla(z) \nabla(z') F \tag{2.11}$$

The function F is (logarithm of) the tau-function of the integrable hierarchy. In [9] it was called the tau-function of the (real analytic) curves.

Dispersionless Hirota equations for F. Combining (2.11) and (1.2), we obtain the relation

$$\log \left| \frac{w(z) - w(z')}{w(z)\overline{w(z')} - 1} \right|^2 = \log \left| \frac{1}{z} - \frac{1}{z'} \right|^2 + \nabla(z) \nabla(z') F \tag{2.12}$$

which implies an infinite hierarchy of differential equations on the function F. It is convenient to normalize the conformal map $w(z)$ by the conditions that $w(\infty) = \infty$ and $\partial_z w(\infty)$ is real, so that

$$w(z) = \frac{z}{r} + O(1) \quad \text{as } z \to \infty \tag{2.13}$$

where the real number $r = \lim_{z \to \infty} dz/dw(z)$ is called the (external) conformal radius of the domain D (equivalently, it can be defined through the Green function as $\log r = \lim_{z \to \infty} (G(z, \infty) + \log |z|)$, see [13]). Then, tending $z' \to \infty$ in (2.12), one gets

$$\log |w(z)|^2 = \log |z|^2 - \partial_{t_0} \nabla(z) F \tag{2.14}$$

The limit $z \to \infty$ of this equality yields a simple formula for the conformal radius:

$$\log r^2 = \partial_{t_0}^2 F \tag{2.15}$$

Let us now separate holomorphic and antiholomorphic parts of these equations. To do that it is convenient to introduce holomorphic and antiholomorphic parts of the operator $\nabla(z)$ (2.5):

$$D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{t_k}, \qquad \bar{D}(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \partial_{\bar{t}_k}, \tag{2.16}$$

Rewrite (2.12) in the form

$$\log \left(\frac{w(z) - w(z')}{w(z)\overline{w(z')} - 1} \right) - \log \left(\frac{1}{z} - \frac{1}{z'} \right) - \left(\frac{1}{2} \partial_{t_0} + D(z) \right) \nabla(z') F =$$

$$= \; -\log \left(\frac{\overline{w(z)} - \overline{w(z')}}{w(z')\overline{w(z)} - 1} \right) + \log \left(\frac{1}{\bar{z}} - \frac{1}{\bar{z}'} \right) + \left(\frac{1}{2} \partial_{t_0} + \bar{D}(\bar{z}) \right) \nabla(z') F$$

The l.h.s. is a holomorphic function of z while the r.h.s. is antiholomorphic. Therefore, both are equal to a z-independent term which can be found from the limit $z \to \infty$. As a result, we obtain the equation

$$\log \left(\frac{w(z) - w(z')}{w(z) - (\overline{w(z')})^{-1}} \right) = \log \left(1 - \frac{z'}{z} \right) + D(z) \nabla(z') F \tag{2.17}$$

which, as $z' \to \infty$, turns into the formula for the conformal map $w(z)$:

$$\log w(z) = \log z - \frac{1}{2}\partial_{t_0}^2 F - \partial_{t_0} D(z) F \tag{2.18}$$

(here we used (2.15)). Proceeding in a similar way, one can rearrange (2.17) in order to write it separately for holomorphic and antiholomorphic parts in z':

$$\log \frac{w(z) - w(z')}{z - z'} = -\frac{1}{2}\partial_{t_0}^2 F + D(z)D(z')F \tag{2.19}$$

$$-\log\left(1 - \frac{1}{w(z)\overline{w(z')}}\right) = D(z)\bar{D}(\bar{z}')F \tag{2.20}$$

Writing down eqs. (2.19) for the pairs of points (a, b), (b, c) and (c, a) and summing up the exponentials of the both sides of each equation one arrives at the relation

$$(a-b)e^{D(a)D(b)F} + (b-c)e^{D(b)D(c)F} + (c-a)e^{D(c)D(a)F} = 0 \tag{2.21}$$

which is the dispersionless Hirota equation (for the KP part of the two-dimensional Toda lattice hierarchy) written in the symmetric form. This equation can be regarded as a very degenerate case of the trisecant Fay identity. It encodes the algebraic relations between the second order derivatives of the function F. As $c \to \infty$, we get these relations in a more explicit but less symmetric form:

$$1 - e^{D(a)D(b)F} = \frac{D(a) - D(b)}{a - b}\,\partial_{t_1} F \tag{2.22}$$

which makes it clear that the totality of second derivatives $F_{ij} := \partial_{t_i}\partial_{t_j} F$ are expressed through the derivatives with one of the indices equal to unity.

More general equations of the dispersionless Toda hierarchy obtained in a similar way by combining eqs. (2.18), (2.19) and (2.20) include derivatives w.r.t. t_0 and $\bar{t}_k$:

$$(a-b)e^{D(a)D(b)F} = ae^{-\partial_{t_0} D(a)F} - be^{-\partial_{t_0} D(b)F} \tag{2.23}$$

$$1 - e^{-D(z)\bar{D}(\bar{z})F} = \frac{1}{z\bar{z}} e^{\partial_{t_0}\nabla(z)F} \tag{2.24}$$

These equations allow one to express the second derivatives $\partial_{t_m}\partial_{t_n} F$, $\partial_{t_m}\partial_{\bar{t}_n} F$ with $m, n \geq 1$ through the derivatives $\partial_{t_0}\partial_{t_k} F$, $\partial_{t_0}\partial_{\bar{t}_k} F$. In particular, the dispersionless Toda equation,

$$\partial_{t_1}\partial_{\bar{t}_1} F = e^{\partial_{t_0}^2 F} \tag{2.25}$$

which follows from (2.24) as $z \to \infty$, expresses $\partial_{t_1}\partial_{\bar{t}_1} F$ through $\partial_{t_0}^2 F$.

For a comprehensive exposition of Hirota equations for dispersionless KP and Toda hierarchies we refer the reader to [14, 15].

Integral representation of the tau-function. Eq. (2.11) allows one to obtain a representation of the tau-function as a double integral over the domain D. Set $\tilde{\Phi}(z) := \nabla(z)F$. One is able to determine this function via its variation under the elementary deformation:

$$\delta_a \tilde{\Phi}(z) = -2\epsilon \log\left|a^{-1} - z^{-1}\right| + 2\epsilon G(a, z) \tag{2.26}$$

which is read from eq. (2.11) by virtue of (2.8). This allows one to identify $\tilde{\Phi}$ with the "modified potential" $\tilde{\Phi}(z) = \Phi(z) - \Phi(0) + t_0 \log|z|^2$, where Φ is given by (2.3). Thus we can write

$$\nabla(z)F = \tilde{\Phi}(z) = -\frac{2}{\pi}\int_{\mathsf{D}} \log|z^{-1} - \zeta^{-1}|\, d^2\zeta = v_0 + 2\mathrm{Re}\sum_{k>0}\frac{v_k}{k} z^{-k} \tag{2.27}$$

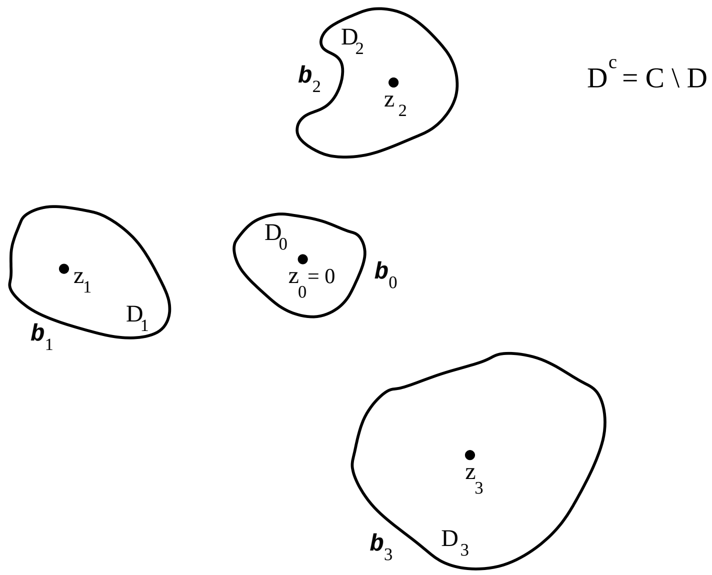

Figure 3: *A multiply-connected domain* $\mathsf{D^c} = \mathbf{C} \setminus \mathsf{D}$ *for* $g = 3$*. The domain* $\mathsf{D} = \bigcup_{\alpha=0}^{3} \mathsf{D}_\alpha$ *consists of* $g + 1 = 4$ *disconnected parts* D_α *with the boundaries* b_α*. To define the complete set of harmonic moments, we also need the auxiliary points* $z_\alpha \in \mathsf{D}_\alpha$ *which should be always located inside the corresponding domains.*

The last equality is to be understood as the Taylor expansion around infinity. The coefficients v_k are moments of the interior domain (the "dual" harmonic moments) defined as

$$v_k = \frac{1}{\pi} \int_{\mathsf{D}} z^k \, d^2 z \quad (k > 0) \,, \qquad v_0 = -\Phi(0) = \frac{2}{\pi} \int_{\mathsf{D}} \log |z| d^2 z \tag{2.28}$$

From (2.27) it is clear that

$$v_k = \partial_{t_k} F \,, \quad k \geq 0 \tag{2.29}$$

In a similar manner, one arrives at the integral representation of the tau-function

$$F = -\frac{1}{\pi^2} \int_{\mathsf{D}} \int_{\mathsf{D}} \log |z^{-1} - \zeta^{-1}| d^2 z d^2 \zeta \tag{2.30}$$

or

$$F = \frac{1}{2\pi} \int_{\mathsf{D}} \tilde{\Phi}(z) d^2 z \tag{2.31}$$

As we see below, the main formulas from this paragraph remain intact in the multiply-connected case.

3 The Dirichlet problem and the tau-function in the multiply-connected case

Let now D_α, $\alpha = 0, 1, \ldots, g$, be a *collection* of $g+1$ non-intersecting bounded connected domains in the complex plane with smooth boundaries $\partial \mathsf{D}_\alpha$. Set $\mathsf{D} = \cup_{\alpha=0}^{g} \mathsf{D}_\alpha$, so that the complement $\mathsf{D^c} = \mathbf{C} \setminus \mathsf{D}$ becomes a multiply-connected unbounded domain in the complex plane (see fig. 3). Let b_α be the boundary curves. The are assumed to be positively oriented as boundaries of $\mathsf{D^c}$, so that $\cup_{\alpha=0}^{g} \mathsf{b}_\alpha = \partial \mathsf{D^c}$ while $\mathsf{b}_\alpha = -\partial \mathsf{D}_\alpha$ has the clockwise orientation.

Comparing to the simply-connected case, nothing is changed in posing the standard Dirichlet problem. The definition of the Green function and the formula (1.1) for the solution of the Dirichlet problem through the Green function are the same.

A difference is in the nature of harmonic functions. Any harmonic function is still the real part of an analytic function but in the multiply-connected case these analytic funstions are not necessarily single-valued (only their real parts have to be single-valued). In other words, the harmonic functions may have non-zero "periods" over non-trivial cycles[1]. In our case, the non-trivial cycles are the boundary curves b_α. In general, the Green function has non-zero "periods" over all boundary contours. Whence it is natural to introduce new objects, specific to the multiply-connected case, which are defined as "periods" of the Green function.

First, the *harmonic measure* $\omega_\alpha(z)$ of the boundary component b_α is the harmonic function in D^c such that it is equal to 1 on b_α and vanishes on the other boundary curves. Thus the harmonic measure is the solution to the particular Dirichlet problem. From the general formula (1.1) we conclude that

$$\omega_\alpha(z) = -\frac{1}{2\pi}\oint_{\mathsf{b}_\alpha} \partial_n G(z,\zeta)|d\zeta|, \qquad \alpha = 1,\dots,g \tag{3.1}$$

so the harmonic measure is the period of the Green function w.r.t. one of its arguments. From the maximum principle for harmonic functions it follows that $0 < \omega_\alpha(z) < 1$ in internal points. Obviously, $\sum_{\alpha=0}^{g}\omega_\alpha(z) = 1$. In what follows we consider the linear independent functions $\omega_\alpha(z)$ with $\alpha = 1,\dots,g$.

Further, taking "periods" in the remaining variable, we define

$$\Omega_{\alpha\beta} = -\frac{1}{2\pi}\oint_{\mathsf{b}_\beta} \partial_n \omega_\alpha(\zeta)|d\zeta|, \qquad \alpha,\beta = 1,\dots,g \tag{3.2}$$

The matrix $\Omega_{\alpha\beta}$ is known to be symmetric, non-degenerate and positively-definite. The matrix $T_{\alpha\beta} = i\pi\Omega_{\alpha\beta}$ is called the *period matrix*. It is the matrix of periods of holomorphic differentials on the Schottky double of the domain D^c (see formula (3.8) below). For brevity, we also refer to $\Omega_{\alpha\beta}$ as to period matrix.

For the harmonic measure and the period matrix there are variational formulas similar to the Hadamard formula (1.4). They can be derived either by a direct variation of (3.1) and (3.2) using the Hadamard formula or (much easier) by a "pictorial" argument like in fig. 1. The formulas are:

$$\delta\omega_\alpha(z) = \frac{1}{2\pi}\oint_{\partial\mathsf{D}} \partial_n G(z,\xi)\,\partial_n\omega_\alpha(\xi)\,\delta n(\xi)\,|d\xi| \tag{3.3}$$

$$\delta\Omega_{\alpha\beta} = \frac{1}{2\pi}\oint_{\partial\mathsf{D}} \partial_n\omega_\alpha(\xi)\,\partial_n\omega_\beta(\xi)\,\delta n(\xi)\,|d\xi| \tag{3.4}$$

Algebraic domains. In what follows we restrict our analysis by the class of *algebraic domains*. In the simply-connected case, the algebraic domains we dealt with in the previous section were simply images of the exterior of the unit disk under one-to-one conformal maps given by *rational functions* whose singularities are all in the other "half" of the plane, i.e., inside the unit circle. Note that the unit circle is the set of fixed points of the inversion $w \to -1/\bar{w}$ which is an antiholomorphic involution of the w-plane compactified by a point at infinity (the Riemann sphere). The notion of planar algebraic domains with holes is similar, with the Riemann sphere being substituted by a higher genus Riemann surface (a complex algebraic curve) with antiholomorphic involution.

[1]Here, by "periods" of a harmonic function f we mean the integrals $\oint \partial_n f\, dl$ over non-trivial cycles.

More precisely, let R be a *real* Riemann surface by which we mean a (non-singular) complex algebraic curve of genus g with an antiholomorphic involution such that the set of fixed points consists of exactly $g+1$ closed contours (such curves are sometimes called M-curves). Then R can be naturally divided in two "halves" (say upper and lower sheets) which are interchanged by the involution. Algebraic domains with g holes in the plane can be defined as images of the upper half of the real Riemann surface under bijective conformal maps given by rational (meromorphic) functions on R. The definition can be made more precise (see, e.g. [16]) but we do not need it here. For the purpose of this paper, it is convenient to use another, more direct characterization of algebraic domains, which appears to be equivalent to the one above.

A more practical way to introduce the algebraic domains is as follows. Let us begin with the potential $\Phi(z)$ (2.3) created by the two-dimensional Coulomb charge uniformly distributed in each D_α with equal constant density, and define the "complimentary" potential created by the same uniform charge density in $\mathsf{D^c}$:

$$\Phi^c(z) = -\frac{2}{\pi}\int_{\mathsf{D^c}} \log|z-z'|d^2z' \tag{3.5}$$

Note that $\Phi(z) + \Phi^c(z) = -|z|^2$ up to an inessential (infinite) constant. Assuming that the system of charges is bounded by a very big circle centered at the origin, the potential generated by the uniform charge inside the circle is, in our normalization, $-|z|^2$. Consider

$$J(z) = -\partial_z \Phi^c(z) = \frac{1}{\pi}\int_{\mathsf{D^c}} \frac{d^2z'}{z-z'} = \frac{1}{2\pi i}\oint_{\partial\mathsf{D^c}} \frac{\bar{\xi}d\xi}{z-\xi}\,, \qquad z\in\mathsf{D} \tag{3.6}$$

It is clear from the definition that it is holomorphic in each hole D_α.

It can be proven that the domain $\mathsf{D^c}$ is algebraic if and only if $J(z)$ is extendable to a rational (meromorphic) function on the whole complex plane with a marked point at infinity. In other words, the domain $\mathsf{D^c}$ is algebraic if and only if there exists a meromorphic function on the plane such that its restrictions to the holes D_α coincide with the functions defined by the integral in (3.6), (of course all poles of this function must be in $\mathsf{D^c}$). For a proof of this statement and for a more precise definition of the algebraic domains see [16]. Note also that in [17, 18] the same class of domains was studied under the name *quadrature domains*.

In what follows we use

$$V(z) = \int_0^z J(z)dz \tag{3.7}$$

which is multi-valued, if $J(z)$ has simple poles. To fix a single-valued branch, we make cuts from ∞ to all simple poles of $J(z)$. In fact we shall need only the real part of $V(z)$. If residues at the poles of $J(z)$ are real, it is single-valued even without cuts.

The Schottky double. It is customary to associate with a planar multiply-connected domain its *Schottky double*, a compact Riemann surface without boundary endowed with an antiholomorpic involution, the boundary of the initial domain being the set of fixed points of the involution. The Schottky double of the multiply-connected domain $\mathsf{D^c}$ can be thought of as two copies of $\mathsf{D^c}$ ("upper" and "lower" sheets of the double) glued along the boundaries $\cup_{\alpha=0}^{g}\mathsf{b}_\alpha = \partial\mathsf{D^c}$, with points at infinity added (∞ and $\bar{\infty}$). In this set-up the holomorphic coordinate on the upper sheet is z inherited from $\mathsf{D^c}$, while the holomorphic coordinate[2] on the other sheet is $\bar{z}$. The Schottky double of D_c with two infinities added is a compact Riemann surface of genus $g = \#\{\mathsf{D}_\alpha\}-1$. In fact it is conformally equivalent to the real Riemann surface

[2]More precisely, the proper coordinates should be $1/z$ (and $1/\bar{z}$), which have first order zeros instead of poles at $z=\infty$ (and $\bar{z}=\bar{\infty}$).

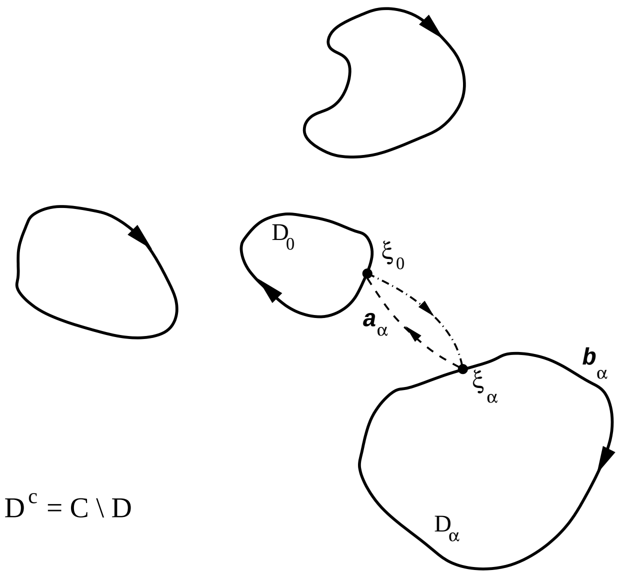

Figure 4: *The domain* $\mathsf{D^c}$ *with the* a_α*-cycle, going one way along the "upper sheet" and back along the "lower sheet" of the Shottky double of* $\mathsf{D^c}$*. For such choice one clearly gets an intersection form* $\mathsf{a}_\alpha \circ \mathsf{b}_\beta = \delta_{\alpha\beta}$ *for* $\alpha, \beta = 1, \dots, g$*.*

R used above for the characterization of the algebraic domains. A meromorphic function on the double is a pair of meromorphic functions $f, \tilde{f}$ on $\mathsf{D^c}$ such that $f(z) = \tilde{f}(\bar{z})$ on the boundary. Similarly, a meromorphic differential on the double is a pair of meromorphic differentials $f(z)dz$ and $\tilde{f}(\bar{z})d\bar{z}$ such that $f(z)dz = \tilde{f}(\bar{z})d\bar{z}$ along the boundary curves.

On the double, one may choose a canonical basis of cycles. The b-cycles are just boundaries of the holes b_α for $\alpha = 1, \dots, g$. Note that regarded as the oriented boundaries of $\mathsf{D^c}$ (not D) they have the *clockwise* orientation. The a_α-cycle connects the α-th hole with the 0-th one. To be more precise, fix points ξ_α on the boundaries, then the a_α cycle starts from ξ_0, goes to ξ_α on the "upper" (holomorphic) sheet of the double and goes back the same way on the "lower" sheet, where the holomorphic coordinate is $\bar{z}$, see fig. 4.

Being harmonic, ω_α can be represented as the real part of a holomorphic function:

$$\omega_\alpha(z) = W_\alpha(z) + \overline{W_\alpha(z)}$$

where $W_\alpha(z)$ are holomorphic multivalued functions in $\mathsf{D^c}$. The differentials dW_α are holomorphic in $\mathsf{D^c}$ and purely imaginary on all boundary contours. So they can be extended holomorphically to the lower sheet as $-d\overline{W_\alpha(z)}$. In fact this is the canonically normalized basis of holomorphic differentials on the double. Indeed, according to the definitions,

$$\oint_{\mathsf{a}_\alpha} dW_\beta = \int_{\xi_0}^{\xi_\alpha} dW_\beta(z) + \int_{\xi_\alpha}^{\xi_0} \left(-d\overline{W_\beta(z)}\right) = 2\mathrm{Re} \int_{\xi_0}^{\xi_\alpha} dW_\beta(z) = \omega_\beta(\xi_\alpha) - \omega_\beta(\xi_0) = \delta_{\alpha\beta}$$

Then the matrix of b-periods of these differentials reads (cf. (3.2))

$$T_{\alpha\beta} = \oint_{\mathsf{b}_\alpha} dW_\beta = -\frac{i}{2} \oint_{\mathsf{b}_\alpha} \partial_n \omega_\beta dl = i\pi\Omega_{\alpha\beta} \tag{3.8}$$

The period matrix $T_{\alpha\beta}$ is purely imaginary non-degenerate matrix with positively definite imaginary part.

The Schwarz function. A useful object is the *Schwarz function* [18, 19] which is defined as the analytic continuation of the function $\bar{z}$ from the boundary. In other words, the Schwarz function $S(z)$ is a function analytic in some neighbourhood of the boundary such that

$$S(z) = \bar{z} \quad \text{on the boundary curves} \tag{3.9}$$

For algebraic domains, one may decompose the Schwarz function as

$$S(z) = J(z) + \text{ regular in } \mathsf{D}^{\mathsf{c}}$$

where J is the rational (regular in D) function defined in (3.6). This follows from the contour integral representation of J given by the last expression in (3.6).

One may treat $S(z)$ as a function on the Schottky double extending it to the lower sheet as $\bar{z}$. It can be proven that the domain D^{c} is algebraic if and only if the so extended Schwarz function is meromorphic on the double. The latter simply means that $S(z)$ is meromorphic in D^{c}. For the particular case when the Schwarz function in D^{c} is allowed to have a high order pole at infinity only (i.e. when $J(z)$ is a polynomial) the equation of the corresponding algebraic curve was proposed in [20].

Harmonic moments. One may still use harmonic moments to characterize the shape of a multiply-connected domain. However, the set of linearly independent harmonic functions should be extended by adding functions whose holomorphic parts are not single-valued.

We choose the basis of harmonic functions in D^{c} as follows: z^{-k}, $\bar{z}^{-k}$, $k \geq 1$, together with the constant function (these have zero "periods" and are the same as in the simply-connected case) and g functions of the form

$$\nu_\alpha(z) = \log \left| 1 - \frac{z_\alpha}{z} \right|^2 , \qquad \alpha = 1, \dots, g$$

where $z_\alpha \in \mathsf{D}_\alpha$ are some marked points, one in each hole (see fig. 3). Without loss of generality, it is convenient to put $z_0 = 0$. The "periods" of these functions are: $\oint_{\mathsf{b}_\alpha} \partial_n \nu_\beta(z) |dz| = 4\pi\delta_{\alpha\beta}$. The independent variables are moments of D^{c} with respect to this basis:

$$\begin{aligned}
t_0 &= \frac{1}{\pi} \int_{\mathsf{D}} d^2 z = \frac{\text{Area}(\mathsf{D})}{\pi} \\
t_k &= -\frac{1}{\pi k} \int_{\mathsf{D}^{\mathsf{c}}} z^{-k} d^2 z , \quad k \geq 1 \\
\phi_\alpha &= -\frac{1}{\pi} \int_{\mathsf{D}^{\mathsf{c}}} \log \left| 1 - \frac{z_\alpha}{z} \right|^2 d^2 z , \quad \alpha = 1, \dots, g
\end{aligned} \tag{3.10}$$

(t_k are complex numbers while t_0 and ϕ_α are real). The moments ϕ_α are essentially values of the potentials at the points z_α:

$$\phi_\alpha = \Phi^c(z_\alpha) - \Phi^c(0) = \Phi(0) - \Phi(z_\alpha) - |z_\alpha|^2 \tag{3.11}$$

As is seen from (3.6) and (3.10), $J(z)$ and $V(z)$ are expanded around the origin as follows:

$$J(z) = \sum_{k \geq 1} k t_k z^{k-1} , \qquad V(z) = \sum_{k \geq 1} t_k z^k$$

In the same manner as for the case of one boundary curve it can be proven that the harmonic moments are local coordinates on the space of algebraic multiply-connected domains (see [16] for the proof).

The variables Π_α. For algebraic domains, there is a better set of variables to use instead of ϕ_α. From the fact that $J(z)$ is globally defined as a rational function on the plane, one may write the following expansion of the potential $\Phi(z)$ for $z \in \mathsf{D}_\alpha$:

$$\Phi(z) - \Phi(0) = -|z|^2 - \Pi_\alpha + V(z) + \overline{V(z)} \tag{3.12}$$

Here $\mathrm{Re}\, V(z)$ is understood as a globally defined function on the plane (with cuts if necessary) such that $V(0) = 0$ (see (3.7)). The constant term Π_α may be different in different connected components. In our normalization, $\Pi_0 \equiv 0$. The other Π_α's (along with t_k's) will serve as local coordinates in the space of algebraic multiply-connected domains. Their advantage is that they do not depend on the choice of the auxiliary points z_α. They are related to ϕ_α's by $\Pi_\alpha = \phi_\alpha - 2\,\mathrm{Re}\, V(z_\alpha)$, which is a non-degenerate change of variables.

Elementary deformations. In addition to the set of elementary deformations δ_z introduced in sect. 2, we need extra deformations $\delta^{(\alpha)}$ ($\alpha = 1, \ldots, g$), which are specific for the multiply-connected case. Consider the infinitesimal deformation

$$\delta n(\xi) = -\frac{\epsilon}{2} \partial_n \omega_\alpha(\xi)$$

where $\omega_\alpha(z)$ is the harmonic measure of the boundary component b_α (see (3.1)). By $\delta^{(\alpha)} X$ we denote variation of any functional $X = X(\Pi_\alpha, \mathbf{t})$ under this elementary deformation.

This class of deformations was considered in [17] in connection with quadrature domains. It is important for us that the elementary deformations $\{\delta_z, \delta^{(\alpha)}\}$ preserve the space of algebraic domains. Indeed, the variation of the function $J(z)$ under these deformations is a rational function, this is easy to verify directly.

Calculations similar to performed in sect. 2 yield the following formulas for variations of the local coordinates:

$$\delta_z t_k = \epsilon \frac{z^{-k}}{k}, \quad \delta^{(\alpha)} t_k = 0, \quad \delta_z \Pi_\alpha = 0, \quad \delta^{(\alpha)} \Pi_\beta = -2\epsilon \delta_{\alpha\beta} \tag{3.13}$$

For example, the second relation follows from

$$\delta^{(\alpha)} t_k \propto \oint_{\partial \mathsf{D}^{\mathsf{c}}} |d\xi| \xi^{-k} \partial_n \omega_\alpha(\xi) = \oint_{\partial \mathsf{D}_\alpha} |d\xi| \partial_n \xi^{-k} = 0$$

since ξ^{-k} is harmonic in D_α. To find $\delta_z \Pi_\alpha$ we write:

$$\delta_z \Pi_\alpha = -\delta_z(\Phi(\zeta) - \Phi(0)) + 2\,\mathrm{Re}\, \delta_z V(\zeta)$$

where ζ is any point in D_α. In complete analogy with the calculation from sect. 2 we find $\delta_z(\Phi(\zeta) - \Phi(0)) = -2\epsilon \log \left|1 - \frac{\zeta}{z}\right|$ while $\delta_z V(\zeta) = -\epsilon \log \left(1 - \frac{\zeta}{z}\right)$ by the definition of $V(z)$, and these two contributions cancel each other. Let us also show how to find $\delta^{(\alpha)} \Pi_\beta$. In this case $\delta^{(\alpha)} \Pi_\beta = -\delta^{(\alpha)}(\Phi(\zeta) - \Phi(0))$ for any point ζ in D_α (the moments t_k do not change and $\delta^{(\alpha)} V(\zeta) = 0$). We have:

$$\delta^{(\alpha)} \Pi_\beta = -\frac{\epsilon}{\pi} \oint_{\partial \mathsf{D}} \log \left|1 - \zeta \xi^{-1}\right| \partial_n \omega_\alpha(\xi) |d\xi|$$

Since both functions in the integrand are harmonic in $\mathbf{C} \setminus \mathsf{D}$, the Green formula allows one to switch the normal derivative from one function to another. Using then that the harmonic measure ω_α is non-zero only on b_α and $\zeta \in \mathsf{D}_\beta$, we get the result:

$$\delta^{(\alpha)} \Pi_\beta = -\frac{\epsilon}{\pi} \oint_{\partial \mathsf{D}} \omega_\alpha(\xi) \partial_n \log \left|1 - \zeta \xi^{-1}\right| |d\xi| = -\frac{\epsilon}{\pi} \oint_{\mathsf{b}_\alpha} \partial_n \log \left|1 - \zeta \xi^{-1}\right| |d\xi| = -2\epsilon \delta_{\alpha\beta}$$

Applying the variational formulas (1.4), (3.3), (3.4), we can find variations of the Green function, harmonic measure and period matrix under the elementary deformations. In this way we obtain a number of important relations which connect elementary deformations of these objects:

$$\begin{aligned}
&\delta_a G(b,c) = \delta_b G(c,a) = \delta_c G(a,b) \\
&\delta_a \omega_\alpha(b) = \delta^{(\alpha)} G(a,b) \\
&\delta^{(\alpha)} \omega_\beta(z) = \delta^{(\beta)} \omega_\alpha(z) \\
&\delta_z \Omega_{\alpha\beta} = \delta^{(\alpha)} \omega_\beta(z) \\
&\delta^{(\alpha)} \Omega_{\beta\gamma} = \delta^{(\beta)} \Omega_{\gamma\alpha} = \delta^{(\gamma)} \Omega_{\alpha\beta}
\end{aligned} \tag{3.14}$$

The F-function. From (3.13) we conclude that for any functional X on the space of algebraic domains it holds

$$\begin{aligned}
\delta_z X &= \sum_k \frac{\partial X}{\partial t_k}\, \delta_z t_k + \sum_\beta \frac{\partial X}{\partial \Pi_\beta}\, \delta_z \Pi_\beta = \epsilon \nabla(z) X \\
\delta^{(\alpha)} X &= \sum_k \frac{\partial X}{\partial t_k}\, \delta^{(\alpha)} t_k + \sum_\beta \frac{\partial X}{\partial \Pi_\beta}\, \delta^{(\alpha)} \Pi_\beta = -2\epsilon \frac{\partial X}{\partial \Pi_\alpha}
\end{aligned} \tag{3.15}$$

so the formulas (3.14) can be rewritten in terms of the differential operators $\nabla(z)$ and ($\partial_\alpha := \partial/\partial\Pi_\alpha$):

$$\begin{aligned}
&\nabla(a) G(b,c) = \nabla(b) G(c,a) = \nabla(c) G(a,b) \\
&\nabla(a) \omega_\alpha(b) = -2\partial_\alpha G(a,b) \\
&\partial_\alpha \omega_\beta(z) = \partial_\beta \omega_\alpha(z) \\
&\nabla(z) \Omega_{\alpha\beta} = -2\partial_\alpha \omega_\beta(z) \\
&\partial_\alpha \Omega_{\beta\gamma} = \partial_\beta \Omega_{\gamma\alpha} = \partial_\gamma \Omega_{\alpha\beta}
\end{aligned} \tag{3.16}$$

These relations generalize (2.10). The first line just coincides with (2.10) while the other ones extend the symmetry of the derivatives to the harmonic measure and the period matrix.

Again, (3.16) can be regarded as a set of compatibility conditions of an infinite hierarchy of differential equations. They imply that there exists a function $F = F(\Pi_\alpha, \mathbf{t})$ such that

$$G(a,b) = \log\left|a^{-1} - b^{-1}\right| + \frac{1}{2}\nabla(a)\nabla(b)F \tag{3.17}$$

$$\omega_\alpha(z) = -\,\partial_{\Pi_\alpha} \nabla(z) F \tag{3.18}$$

$$T_{\alpha\beta} = i\pi\Omega_{\alpha\beta} = 2\pi i\, \frac{\partial^2 F}{\partial\Pi_\alpha \partial\Pi_\beta} \tag{3.19}$$

This is the tau-function of multiply-connected domains.

Dual moments and integral representation of the tau-function. To obtain the integral representation of the function F, we proceed exactly in the same manner as in sect. 2 (see also [2] for more details).

Set $\tilde{\Phi}(z) = \nabla(z)F$. Eqs. (3.17) determine the function $\tilde{\Phi}(z)$ for $z \in \mathsf{D}^{\mathsf{c}}$ via its variations under the elementary deformations:

$$\begin{aligned} &\delta_a \tilde{\Phi}(z) = -2\epsilon \log \left| a^{-1} - z^{-1} \right| + 2\epsilon G(a, z) \\ &\delta^{(\alpha)} \tilde{\Phi}(z) = 2\epsilon \omega_\alpha(z) \end{aligned} \tag{3.20}$$

It is easy to verify that the function

$$\tilde{\Phi}(z) = -\frac{2}{\pi} \int_{\mathsf{D}} \log |z^{-1} - \zeta^{-1}| d^2\zeta = \Phi(z) - \Phi(0) + t_0 \log |z|^2 \tag{3.21}$$

has the required variation. We see that this formula coincides with $\tilde{\Phi}$ given by (2.27), where D is understood as the union of all D_α's. Correspondingly, the formulas (2.29) for the dual moments given by (2.28) remain intact.

Let us repeat this argument for

$$s_\alpha := -\frac{\partial F}{\partial \Pi_\alpha} \tag{3.22}$$

Eqs. (3.18), (3.19) determine these quantities via their variations: $\delta_a s_\alpha = \epsilon \omega_\alpha(a)$, $\delta^{(\beta)} s_\alpha = \epsilon \Omega_{\alpha\beta}$. An immediate check shows that the solution for s_α is

$$s_\alpha = \frac{1}{\pi} \int_{\mathsf{D}_\alpha} d^2\zeta = \frac{\mathrm{Area}(\mathsf{D}_\alpha)}{\pi} \tag{3.23}$$

To summurize:

$$\nabla(z)F = -\frac{2}{\pi} \int_{\mathsf{D}} \log \left| \frac{1}{z} - \frac{1}{\zeta} \right| d^2\zeta \,, \qquad \frac{\partial F}{\partial \Pi_\alpha} = -\frac{\mathrm{Area}(\mathsf{D}_\alpha)}{\pi} \tag{3.24}$$

The areas of the holes provide an alternative ("dual") set of extra variables characterizing the domain (see below).

The integral representation of F is found in the same way through its variations which are read from (3.24). The result is given by the same formula (2.30) as in the simply-connected case with the understanding that D is now the union of D_α's.

Finally, eq. (2.31) allows one to obtain an analog of formula (5.11) from [2]:

$$2F = -\frac{1}{2} t_0^2 + t_0 v_0 + \frac{1}{2} \sum_{k \geq 1} (2 - k) t_k v_k + \bar{t}_k \bar{v}_k) - \sum_{\alpha=1}^{g} \Pi_\alpha s_\alpha \tag{3.25}$$

which is the quasi-homogeneity condition obeyed by F.

$\{\Pi_\alpha\}$ and $\{s_\alpha\}$ as periods of the generating differential on the Schottky double. Let us show that the dual sets of variables, $\{\Pi_\alpha\}$ and $\{s_\alpha\}$, are obtained as periods of one and the same meromorphic differential over dual cycles of the Schottky double.

We begin with Π_α. The derivation is similar to the one proposed in [20] for the case of finite number of nonvanishing moments $\{t_k\}$. Fix the points $\xi_\alpha \in \partial \mathsf{D}_\alpha$, like in fig. 4, then, subtracting

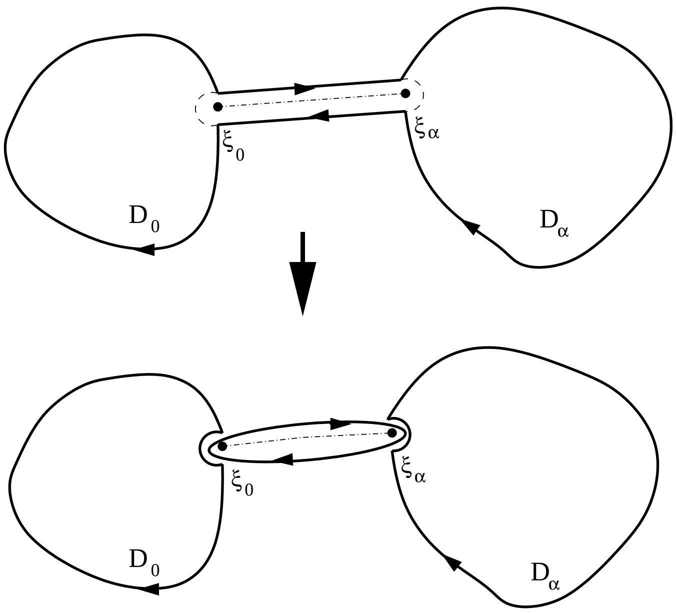

Figure 5: *The integration along the boundary $\partial \mathsf{D}^{\mathsf{c}}_{\mathsf{cut}}$ can be split into the integral along $\partial \mathsf{D}^{\mathsf{c}}$ and integral (in the clockwise direction) along the cut as shown on the figure. In this way one gets (3.29) from (3.28). Then both contributions combine into the integral along the cut.*

eqs. (3.12) at $z = \xi_\alpha$ and $z = \xi_0$, and representing the result in terms of the potential Φ^c, we get:

$$\Pi_\alpha = 2\,\mathrm{Re}\int_{\xi_0}^{\xi_\alpha} J(z)dz + \Phi^c(\xi_\alpha) - \Phi^c(\xi_0) \tag{3.26}$$

where Φ^c is defined in (3.5). We thus can write:

$$\Pi_\alpha = 2\,\mathrm{Re}\left(\int_{\xi_0}^{\xi_\alpha} J(z)dz - \frac{1}{\pi}\int_{\mathsf{D}^{\mathsf{c}}} \log\left(\frac{z-\xi_\alpha}{z-\xi_0}\right) d^2 z\right) \tag{3.27}$$

Let us transform the second integral:

$$\frac{1}{\pi}\int_{\mathsf{D}^{\mathsf{c}}} \log\left(\frac{z-\xi_\alpha}{z-\xi_0}\right) d^2 z = \frac{1}{2\pi i}\oint_{\partial \mathsf{D}^{\mathsf{c}}_{\mathsf{cut}}} \bar z \log\left(\frac{z-\xi_\alpha}{z-\xi_0}\right) dz \tag{3.28}$$

where we used the Stokes theorem. Here $\mathsf{D}^{\mathsf{c}}_{\mathsf{cut}}$ is the domain D^{c} cut along a path between the points ξ_0 and ξ_α (say, along the upper half of the a_α-cycle, see fig. 4). In this domain, one can choose a single-valued branch of the logarithmic function. The integration goes along all the boundary of D^{c}, where we can substitute $\bar z$ by the Schwarz function, and also goes around the cut (see fig. 5):

$$(3.28) = \frac{1}{2\pi i}\oint_{\partial \mathsf{D}^{\mathsf{c}}} S(z) \log\left(\frac{z-\xi_\alpha}{z-\xi_0}\right) dz - \frac{1}{2\pi i}\oint_{\mathsf{cut}} \bar z \log\left(\frac{z-\xi_\alpha}{z-\xi_0}\right) dz \tag{3.29}$$

Here the contour in $\oint_{\mathsf{cut}}$ first goes from ξ_0 to ξ_α along the right hand side of the cut and then goes back along its left hand side, so its orientation is positive (i.e. *anti*-clockwise), this orientation is opposite to the one shown in fig. 5.

Recall that the Schwarz function can be represented as $S(z) = J(z) + S^{\mathrm{reg}}(z)$, where S^{reg} is a function holomorphic in D^{c} and vanishing at infinity while J is a rational function in the plane regular in each D_α. Therefore J does not contribute to the integral. As far as the integral with $S^{\rm reg}$ is concerned, the only singularity of the integrand in $\mathsf{D^c}$ is the logarithmic cut (see fig. 5), so the integral can be taken over the cut:

$$(3.28) = \frac{1}{2\pi i}\oint_{\mathsf{cut}} (S^{\rm reg}(z) - \bar z) \log \frac{z-\xi_\alpha}{z-\xi_0} dz = -\int_{\xi_0}^{\xi_\alpha} (S^{\rm reg}(z) - \bar z)\, dz$$

Plugging this into (3.27), we finally obtain:

$$\Pi_\alpha = 2{\rm Re} \int_{\xi_0}^{\xi_\alpha} (S(z) - \bar z) dz = \int_{\xi_0}^{\xi_\alpha} \left(S(z)dz - \bar z d\overline{S(z)} \right) \tag{3.30}$$

This expression does not depend on the choice of ξ_0, ξ_α and on a particular integration path unless it intersects the cuts used for defining a single-valued branch of the function $\int^z J(\zeta)d\zeta$. The integral in the r.h.s. is exactly the a-period of the differential Sdz extended meromorphically to the lower sheet of the double:

$$\Pi_\alpha = \oint_{\mathsf{a}_\alpha} S(z)dz \tag{3.31}$$

It is also clear that the dual parameters s_α (3.23) are given by b-periods of the same differential,

$$s_\alpha = \frac{1}{2\pi i}\oint_{\partial \mathsf{D}_\alpha} \bar z dz = -\frac{1}{2\pi i}\oint_{\mathsf{b}_\alpha} S(z)dz \tag{3.32}$$

Note also that

$$t_k = -\frac{1}{2\pi i k}\oint_{\partial \mathsf{D^c}} z^{-k} S(z)dz\,, \quad k\geq 1 \tag{3.33}$$

$$v_k = -\frac{1}{2\pi i}\oint_{\partial \mathsf{D^c}} z^{k} S(z)dz\,, \quad k\geq 1 \tag{3.34}$$

Extra parameters. Counting the total number of parameters in multiply-connected case one finds that we have introduced, for the algebraic case, only g new independent parameters ($\{\Pi_\alpha\}$ or $\{s_\alpha\}$) while the total dimension of moduli space is $3g-3$ and the general tau-function of the Whitham hierarchy depends nontrivially on all these variables. Indeed, one may introduce more parameters which would describe the extra moduli explicitly.

Let us outline how to do this. Instead of (3.9) one can write

$$S(z) = \bar z + K_\alpha \quad \text{on the boundary component} \quad \mathsf{b}_\alpha \tag{3.35}$$

where K_α are some complex constants, thus incorporating $2g$ real parameters. The role of these variables is similar to t_1 which is responsible for the "shift of the origin" in the simply-connected case. The simplest example is the case of two boundary curves ($g=1$) described by the "conformal map" $z(W) = r_1\zeta(W-W_\infty) - r_2\zeta(W+W_\infty) - \frac{\eta}{\omega}(r_1-r_2)W$ with $r_1\neq r_2$, where we use the standard notation from the theory of elliptic functions (in particular, $\zeta(W)$ is the Weierstrass ζ-function).

The dependence of the tau-function F on the new variables is determined by [4]

$$\frac{\partial F}{\partial K_\alpha} \propto \oint_{\mathsf{b}_\alpha} z\bar z dz \tag{3.36}$$

For algebraic domains, one can put $K_\alpha = 0$ for all $\alpha = 0,1,\dots,g$. The case $K_\alpha\neq 0$ will be discussed elsewhere.

4 The duality transformation

The "physical" interpretation of the independent variables Π_α used in the previous section is not as transparent as the meaning of the dual variables s_α, which are simply areas of the holes D_α. In this section we show how to pass to the set of independent variables $s_1, \ldots, s_g$ (3.23) (together with the infinite set of t_k's). This transformation is similar to the passing from "external" to "internal" moments in the simply-connected case (see sect. 5 of [2]). The difference is that only a finite number of times are subject to the transformation while the infinite set of t_k's remains the same.

The Legendre transform. Passing from Π_α to s_α is a kind of duality transformation which is equivalent to the interchanging of the a and b cycles on the Schottky double. This is achieved by the (partial) Legendre transform $F(\Pi_\alpha, \mathbf{t}) \longrightarrow \tilde{F}(s_\alpha, \mathbf{t})$, where

$$\tilde{F} = F + \sum_{\alpha=1}^{g} \Pi_\alpha s_\alpha \tag{4.1}$$

The function $\tilde{F}$ is the "dual" tau-function. Below in this section we see that $\tilde{F}$ solves the modified Dirichlet problem and can be identified with the free energy of the two matrix model in the planar large N limit in the case when the support of eigenvalues consists of a few disconnected domains (a so called multi-support solution, see e.g. [20]).

The main properties of $\tilde{F}$ follow from those of F. According to (2.29), (3.24) we have $dF = -\sum_\alpha s_\alpha d\Pi_\alpha + \sum_k v_k dt_k$ (for brevity, k is assumed to run over all integer values, $t_{-k} \equiv \bar{t}_k$, etc.), so $d\tilde{F} = \sum_\alpha \Pi_\alpha ds_\alpha + \sum_k v_k dt_k$. This gives the first order derivatives:

$$\Pi_\alpha = \frac{\partial \tilde{F}}{\partial s_\alpha}\,, \qquad v_k = \frac{\partial \tilde{F}}{\partial t_k} \tag{4.2}$$

The second order derivatives are transformed as follows (see e.g. [21]): Set

$$F_{\alpha\beta} = \frac{\partial^2 F}{\partial \Pi_\alpha \partial \Pi_\beta}\,, \quad F_{\alpha k} = \frac{\partial^2 F}{\partial \Pi_\alpha \partial t_k}\,, \quad F_{ik} = \frac{\partial^2 F}{\partial t_i \partial t_k}$$

and similarly for $\tilde{F}$. Then

$$\begin{aligned} F_{\alpha\beta} &= -(\tilde{F}^{-1})_{\alpha\beta} \\ F_{\alpha k} &= -\sum_{\gamma=1}^{g} (\tilde{F}^{-1})_{\alpha\gamma} \tilde{F}_{\gamma k} \\ F_{ik} &= \tilde{F}_{ik} - \sum_{\gamma,\gamma'=1}^{g} \tilde{F}_{i\gamma} (\tilde{F}^{-1})_{\gamma\gamma'} \tilde{F}_{\gamma' k} \end{aligned} \tag{4.3}$$

Here $(\tilde{F}^{-1})_{\alpha\beta}$ means the matrix element of the matrix inverse to the $g \times g$ matrix $\tilde{F}_{\alpha\beta}$.

Using these formulas, it is easy to see that the main property (3.17) of the tau-function are translated to the dual tau-function as follows:

$$\tilde{G}(a,b) = \log|a^{-1} - b^{-1}| + \frac{1}{2} \nabla(a) \nabla(b) \tilde{F} \tag{4.4}$$

$$2\pi i\, \tilde{\omega}_\alpha(z) = -\, \partial_{s_\alpha} \nabla(z) \tilde{F} \tag{4.5}$$

$$2\pi i \, \tilde{T}_{\alpha\beta} = -2(\Omega^{-1})_{\alpha\beta} = \frac{\partial^2 \tilde{F}}{\partial s_\alpha \partial s_\beta} \tag{4.6}$$

where t_k-derivatives in $\nabla(z)$ are taken at fixed s_α. The objects in the left hand sides of these relations are:

$$\tilde{G}(a,b) = G(a,b) - \sum_{\alpha,\beta=1}^{g} \omega_\alpha(a)(\Omega^{-1})_{\alpha\beta}\, \omega_\beta(b) \tag{4.7}$$

$$\tilde{\omega}_\alpha(z) = -\sum_{\beta=1}^{g} (T^{-1})_{\alpha\beta}\, \omega_\beta(z) \tag{4.8}$$

$$\tilde{T} = -T^{-1} \tag{4.9}$$

The function $\tilde{G}$ is the modified Green function (see below). The matrix $\tilde{T}$ is the period matrix of holomorphic differentials on the double normalized with respect to the b-cycles (more precisely, the change of cycles is $\mathsf{a}_\alpha \to \mathsf{b}_\alpha$, $\mathsf{b}_\alpha \to -\mathsf{a}_\alpha$).

An important remark is in order. By a simple rescaling of the independent variables one is able to write the group of relations (4.4)–(4.6) for the function $\tilde{F}$ in exactly the same form as the ones for the function F (3.17)–(3.19), so that they differ merely by the notation. We use this fact in sect. 5.

The modified Dirichlet problem. The modified Green function (4.7) solves the modified Dirichlet problem which is described below.

One may modify the Green function G to eliminate all but one periods, thus making it similar, in this respect, to the Green function of a simply-connected domain. (Recall that the latter has the non-zero period 2π over the only boundary curve b_0.) This leads to the following *modified Dirichlet problem* (see e.g. [23]): to find a harmonic function $u(z)$ in D^{c} such that it is continuous up to the boundary and equals $u_0(z) + C_\alpha$ on the α's boundary component. Here, C_α's are some constants. It is important to stress that they are not given *a priori* but have to be determined from the condition that the solution $u(z)$ has vanishing periods over the boundaries $\mathsf{b}_1, \ldots, \mathsf{b}_g$. One of these constants can be put equal to zero. We set $C_0 = 0$. This problem also has a unique solution. It is given by the same formula (1.1) in terms of the *modified Green function* $\tilde{G}(z,\zeta)$. The definition of the latter is similar to that of the $G(z,\zeta)$ but differs in two respects:

($\tilde{G}$1) $\tilde{G}(z,\zeta)$ is required to have zero periods over the boundaries $\mathsf{b}_1, \ldots, \mathsf{b}_g$;

($\tilde{G}$2) not $\tilde{G}(z,\zeta)$ itself but its derivative along the boundary $\partial_z \tilde{G}(z,\zeta)$ vanishes on the boundary.

Under the condition that $\tilde{G}(z,\zeta) = 0$ on b_0 such a function is unique. The function given by (4.7) just meets these requirements. We conclude that the modified Green function is expressed through the dual tau-function.

Note that variations of the modified Green function under small deformations of the domain are described by the same Hadamard formula (1.4), where each Green function is replaced by $\tilde{G}$. This follows, after some algebra, from the formula for $\tilde{G}$ in terms of G, ω_α and $\Omega_{\alpha\beta}$. Therefore, all the arguments of sect. 3 could be repeated in a completely parallel way starting from the modified Dirichlet problem. One may also say that the functions G and $\tilde{G}$ differ merely by a prefered basis of cycles on the double: the differential $\partial_z G dz$ has zero periods over the a-cycles while $\partial_z \tilde{G} dz$ has zero periods over the b-cycles.

The function $f_a(z)$ such that $\tilde{G}(z,a) = -\log|f_a(z)|$ maps D^{c} onto the exterior of the unit circle which is slit along g concentric circular arcs (see fig. 6). Since the periods of the function

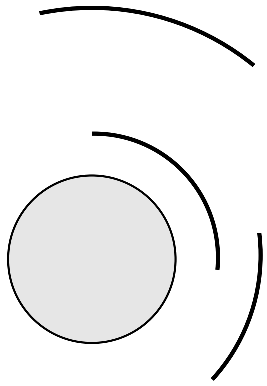

Figure 6: *The image of a triply-connected domain under the conformal map $f_a(z)$ such that $\tilde{G}(z,a) = -\log|f_a(z)|$. The function f_a maps the domain onto the exterior of the unit (dashed) circle with $g = 3$ concentric circular cuts. Positions and lengths of the arcs depend on the shape of D^{c} in fig. 3 and depend also on the normalization point $z = a$ which is mapped to ∞.*

$\tilde{G}$ vanish, the function w_a is single-valued. Positions of the arcs depend on the shape of D^{c} as well as on the point a which is mapped to ∞. The radii of the arcs, R_α, are expressed through the dual tau-function as

$$\log R_\alpha^2 = \partial_{s_\alpha} \nabla(a) \tilde{F} \tag{4.10}$$

In particular, for $a = \infty$ we have $\log R_\alpha^2 = \partial_{s_\alpha} \partial_{t_0} \tilde{F}$ (cf. eq. (2.15) for the conformal radius).

Relation to multi-support solutions of matrix models. The partition function of the model[3], written as an integral over eigenvalues, reads:

$$Z_N = \frac{1}{N!} \int \exp\left(\sum_{i<j}^{N} \log|z_i - z_j|^2 + \frac{1}{\hbar} \sum_{i=1}^{N} U(z_i) \right) \prod_{j=1}^{N} d^2 z_j \tag{4.11}$$

where

$$U(z) = -|z|^2 + \sum_{k \geq 1} \left(t_k z^k + \bar{t}_k \bar{z}^k \right) \tag{4.12}$$

and $\hbar$ is a parameter which, in the large N limit, tends to 0 simultaneously with $N \to \infty$ in such a way that $t_0 = N\hbar$ is kept finite and fixed. In the leading order, one can apply the saddle point method. The saddle point condition is

$$\hbar \sum_{j=1, \neq i}^{N} \frac{1}{z_i - z_j} + \partial_{z_i} U(z_i) = 0$$

In the large N limit, one treats $\rho(z) = \hbar \sum_i \delta^{(2)}(z - z_i)$ as a continuous function normalized as $\int \rho(z) d^2 z = t_0$. In terms of the continuous density the saddle point equation reads

$$\frac{\rho(\zeta) d^2 \zeta}{z - \zeta} + \partial_z U(z) = 0 \tag{4.13}$$

The solution is well known and easy to obtain: the extremal density $\rho_0(z)$ is constant in some domain D (the support of eigenvalues which can be a union of disconnected domains D_α) and zero otherwise. More precisely,

$$\rho_0(z) = \begin{cases} 1/\pi & z \in \mathsf{D} \\ 0 & z \in \mathbf{C} \setminus \mathsf{D} \equiv \mathsf{D}^{\mathsf{c}} \end{cases}$$

[3]One may have in mind the model of all complex or mutually hermitean conjugated or normal matrices.

Then the saddle point equation just means that the "effective potential" $\frac{1}{\pi}\int_{\mathsf{D}} \log|z-\zeta|^2 d^2\zeta + U(z)$ is constant in each D_α, i.e. for $z \in \mathsf{D}_\alpha$ it holds:

$$\frac{2}{\pi}\int_{\mathsf{D}} \log|z-\zeta| d^2\zeta + U(z) = v_0 + \Pi_\alpha \tag{4.14}$$

where, in our normalization, $\Pi_0 = 0$.

Let N_α be the number of eigenvalues in D_α, then $\lim_{N\to\infty} \hbar N_\alpha = s_\alpha$. First we find the free energy with some fixed s_α:

$$\lim_{N\to\infty} \left(\hbar^2 \log Z_N\right)\Big|_{s_\alpha \text{ fixed}} = \int\!\!\int \rho_0(z) \log|z-\zeta| \rho_0(\zeta) d^2z d^2\zeta + \int \rho_0(z) U(z) d^2z$$

where one should substitute (4.14). The result is

$$\lim_{N\to\infty} \left(\hbar^2 \log Z_N\right)\Big|_{s_\alpha \text{ fixed}} = -\frac{1}{\pi^2}\int_{\mathsf{D}}\int_{\mathsf{D}} \log|z^{-1}-\zeta^{-1}| d^2z d^2\zeta + \sum_{\alpha=1}^{g} \Pi_\alpha s_\alpha = \tilde{F}(s_\alpha, \mathbf{t}) \tag{4.15}$$

This quantity depends on s_α's. It is given by the value of the integrand in (4.11) at the saddle point with fixed s_α.

If one wants to take into account the "tunneling" of eigenvalues between different components of the support, s_α are no longer free parameters, and the free energy $F^{(0)}$ of the "planar limit" of the matrix model should be obtained by extremizing $\tilde{F}$ with respect to s_α (with fixed t_0). It follows from the above that $\partial_{s_\alpha}\tilde{F} = \Pi_\alpha$ and so the extremum is at $\Pi_\alpha = 0$. Therefore,

$$F^{(0)} = F\Big|_{\Pi_\alpha=0} \tag{4.16}$$

where F is given by (2.30).

When studying matrix models, one usually restricts oneself to the case of finite amount of nonvanishing moments $\{t_k, \bar{t}_k\}$ playing the role of the coefficients of the matrix model potential. In such case the corresponding complex curve can be described by an algebraic equation [20], which can be thought of as auxiliary constraint to the second derivatives of F. These auxiliary constraints look similar to the reduction conditions in the case of Landau-Ginzburg topological theories (see, for example, discussion of such conditions in the context of dispersionless Hirota equations and WDVV equations in [7]). Their sense is that the derivatives w.r.t. the times $\{t_k\}$ for $k > g+1$ (where g is genus of corresponding algebraic complex curve) can be expressed through the derivatives restricted to $k \le g+1$.

5 Green function on the Schottky double and generalized Hirota equations

Let us now turn to the generalization of the dispersionless Hirota equations to the multiply-connected case.

For a unified treatment of the two "dual" representations of the Dirichlet problem discussed in the two previous sections, we make a simple change of variables. Namely, let us introduce the generic "period" variables x_α which are identified either with Π_α or $2\pi i\, s_\alpha$ dependently on the choice of the set of cycles (see (3.31), (3.32)), and the function $\mathcal{F}(x_\alpha, \mathbf{t})$ equal to $F(x_\alpha, \mathbf{t})$ or $\tilde{F}(\frac{x_\alpha}{2\pi i}, \mathbf{t})$ respectively. Then the main relations (3.17) – (3.19) and (4.4) – (4.6) acquire the form

$$G(a,b) = \log|a^{-1}-b^{-1}| + \frac{1}{2}\nabla(a)\nabla(b)\mathcal{F} \tag{5.1}$$

$$\omega_\alpha(z) = -\,\partial_\alpha \nabla(z)\mathcal{F} \tag{5.2}$$

$$T_{\alpha\beta} = 2\pi i\,\partial_\alpha\partial_\beta\mathcal{F} \tag{5.3}$$

where $\partial_\alpha := \partial_{x_\alpha}$ and G, ω and T stand for the corresponding objects with or without tilde, depending on the chosen basis of cycles.

We will see below that, in analogy to the simply-conected case, any second order derivative of the function $\mathcal{F}$ w.r.t. t_k (and $\bar{t}_k$), $\mathcal{F}_{ik}$, will be expressed through the derivatives $\{\mathcal{F}_{\alpha\beta}\}$ where $\alpha,\beta = 0,\dots,g$ together with $\{\mathcal{F}_{\alpha t_i}\}$ and their complex conjugated. To be more precise, one can consider all second derivatives as functions of $\{\mathcal{F}_{\alpha\beta},\mathcal{F}_{\alpha k}\}$ modulo certain relations on the latter, like the relation (5.14) to be discussed below. Sometimes on this "small phase space" more extra constraints arise, which can be written in the form similar to the Hirota or WDVV equations [22]; we are not going to discuss this issue here, restricting ourselves to the generic situation.

The Abel map. To derive equations for the function $F = \mathcal{F}$ in sect. 2, we used the representation (2.18) for the conformal map $w(z)$ in terms of $\mathcal{F}$ and eq. (1.2) relating the conformal map to the Green function which, in its turn, is expressed through the second order derivatives of $\mathcal{F}$. In the multiply-connected case, our strategy is basically the same, with the suitable analog of the conformal map $w(z)$ (or rather of $\log w(z)$) being the embedding of D^{c} into the g-dimensional complex torus **Jac**, the Jacobi variety of the Schottky double. This embedding is given, up to an overall shift in **Jac**, by the Abel map $z \mapsto \mathbf{W}(z) := (W_1(z),\dots,W_g(z))$ where

$$W_\alpha(z) = \int_{\xi_0}^{z} dW_\alpha \tag{5.4}$$

is the holomorphic part of the harmonic measure ω_α. By virtue of (5.2), the Abel map is represented through the second order derivatives of the function $\mathcal{F}$:

$$W_\alpha(z) - W_\alpha(\infty) = \int_{\infty}^{z} dW_\alpha = -\partial_\alpha D(z)\mathcal{F} \tag{5.5}$$

$$2\,\mathrm{Re}\,W_\alpha(\infty) = \omega_\alpha(\infty) = -\partial_{t_0}\partial_\alpha\mathcal{F} \tag{5.6}$$

The last formula immediately follows from (5.2).

The Green function and the prime form. The Green function of the Dirichlet boundary problem, appearing in (5.1), can be written in terms of the prime form (B.4) on the Schottky double (cf. (1.2)):

$$G(z,\zeta) = \log\left|\frac{E(z,\zeta)}{E(z,\bar{\zeta})}\right| \tag{5.7}$$

Here by $\bar{\zeta}$ we mean the (holomorphic) coordinate of the "mirror" point on the Schottky double, i.e. the "mirror" of ζ under the antiholomorphic involution. The pairs of such mirror points satisfy the condition $W_\alpha(\zeta) + \overline{W_\alpha(\bar{\zeta})} = 0$ in the Jacobian (i.e., the sum should be zero modulo the lattice of periods). The prime form[4] is written through the Riemann theta functions and the Abel map as follows:

$$E(z,\zeta) = \frac{\theta_*(\mathbf{W}(z) - \mathbf{W}(\zeta))}{h(z)\,h(\zeta)} \tag{5.8}$$

[4]Given a Riemann surface with local the coordinates $1/z$ and $1/\bar{z}$ we trivialize the bundle of $-\frac{1}{2}$-differentials and "redefine" the prime form $E(z,\zeta) \to E(z,\zeta)(dz)^{1/2}(d\zeta)^{1/2}$ so that it becomes a function. However for different coordinate patches (the "upper" and "lower" sheets of the Schottky double) one gets different functions, see, for example, formulas (5.8) and (5.9) below.

when the both points are on the upper sheet and

$$E(z,\bar{\zeta}) = \frac{\theta_*(\mathbf{W}(z) + \overline{\mathbf{W}(\zeta)})}{ih(z)\,\overline{h(\zeta)}} \tag{5.9}$$

when z is on the upper sheet and $\bar{\zeta}$ is on the lower one (for other cases we define $E(\bar{z},\bar{\zeta}) = \overline{E(z,\zeta)}$, $E(\bar{z},\zeta) = \overline{E(z,\bar{\zeta})}$). Here $\theta_*(\mathbf{W}) \equiv \theta_{\boldsymbol{\delta}^*}(\mathbf{W}|T)$ is the Riemann theta function (B.2) with the period matrix $T_{\alpha\beta} = 2\pi i\,\partial_\alpha\partial_\beta\mathcal{F}$ and any odd characteristics $\boldsymbol{\delta}^*$, and

$$h^2(z) = z^2\sum_{\alpha=1}^{g}\theta_{*,\alpha}(0)\partial_z W_\alpha(z) = \sum_{\alpha=1}^{g}\theta_{*,\alpha}(0)\sum_{k\geq 1} z^{-k+1}\partial_\alpha\partial_{t_k}\mathcal{F} \tag{5.10}$$

Note that in the l.h.s. of (5.9) the bar means the reflection in the double while in the r.h.s. the bar means complex conjugation. The notation is consistent since the local coordinate in the lower sheet is just the complex conjugate one. However, one should remember that $E(z,\bar{\zeta})$ is *not* obtained from (5.8) by a simple substitution of the complex conjugated argument. On different sheets so defined prime "form" E is represented by different functions. In our normalization (5.9) $iE(z,\bar{z})$ is real (see also Appendix C) and $\lim_{z\to\infty}(zE(z,\infty)) = 1$.

The prime form and the tau-function. In (5.7), the h-functions in the prime forms cancel, so the analog of (2.12) reads

$$\log\left|\frac{\theta_*(\mathbf{W}(z) - \mathbf{W}(\zeta))}{\theta_*(\mathbf{W}(z) + \overline{\mathbf{W}(\zeta)})}\right|^2 = \log\left|\frac{1}{z} - \frac{1}{\zeta}\right|^2 + \nabla(z)\nabla(\zeta)\mathcal{F} \tag{5.11}$$

This equation already explains the claim made in the beginning of this section. Indeed, the r.h.s. is the generating function for the derivatives $\mathcal{F}_{ik}$ while the l.h.s. is expressed through derivatives of the form $\mathcal{F}_{\alpha k}$ and $\mathcal{F}_{\alpha\beta}$ only. The expansion in powers of z,ζ allows one to express the former through the latter.

The analogs of eqs. (2.14), (2.15) are, respectively:

$$\log\left|\frac{\theta_*(\mathbf{W}(z) - \mathbf{W}(\infty))}{\theta_*(\boldsymbol{\omega}(\infty))}\right|^2 = -\log|z|^2 + \partial_{t_0}\nabla(z)\mathcal{F} \tag{5.12}$$

$$\log\left|\frac{h^2(\infty)}{\theta_*(\boldsymbol{\omega}(\infty))}\right|^2 = \partial_{t_0}^2\mathcal{F} \tag{5.13}$$

Here $\boldsymbol{\omega}(z) \equiv 2\,\mathrm{Re}\,\mathbf{W}(z) = (\omega_1(z),\ldots,\omega_g(z))$ and

$$h^2(\infty) = \lim_{z\to\infty} z\,\theta_*\left(\int_\infty^z d\mathbf{W}\right) = \sum_{\alpha=1}^{g}\theta_{*,\alpha}(0)\partial_\alpha\partial_{t_1}\mathcal{F}$$

A simple check shows that the l.h.s. of (5.13) can be written as $-2\log(iE(\infty,\bar{\infty}))$. As is seen from the expansion $G(z,\infty) = -\log|z| - \log(iE(\infty,\bar{\infty})) + O(z^{-1})$ as $z\to\infty$, $(iE(\infty,\bar{\infty}))^{-1}$ is a natural analog of the conformal radius, and (5.13) indeed turns to (2.15) in the simply-connected case (see also Appendix C for explicit illustrative example). However, now it provides a nontrivial relation on $\mathcal{F}_{\alpha\beta}$'s and $\mathcal{F}_{\alpha i}$'s:

$$\left(\sum_\alpha \theta_{*,\alpha}\partial_\alpha\partial_{t_1}\mathcal{F}\right)\left(\sum_\beta \theta_{*,\beta}\partial_\beta\partial_{\bar{t}_1}\mathcal{F}\right) = \theta_*^2(\boldsymbol{\omega}(\infty))e^{\partial_{t_0}^2\mathcal{F}} \tag{5.14}$$

so that the "small phase space" contains the derivatives modulo this relation.

The next steps are exactly the same as in sect. 2: we are going to decompose these equalities into holomorphic and antiholomorphic parts. The results are conveniently written in terms of the prime form. The counterpart of (2.17) is

$$\log \frac{E(\zeta, z)E(\infty, \bar{\zeta})}{E(\zeta, \infty)E(z, \bar{\zeta})} = \log\left(1 - \frac{\zeta}{z}\right) + D(z)\nabla(\zeta)\mathcal{F} \tag{5.15}$$

Tending $\zeta \to \infty$, we get:

$$\log \frac{E(z, \bar{\infty})}{E(z, \infty)} = \log z + \log E(\infty, \bar{\infty}) - \partial_{t_0} D(z)\mathcal{F} \tag{5.16}$$

Separating holomorphic and antiholomorphic parts of (5.15) in ζ, we get analogs of (2.19) and (2.20):

$$\log \frac{E(z, \zeta)}{E(z, \infty)E(\infty, \zeta)} = \log(z - \zeta) + D(z)D(\zeta)\mathcal{F} \tag{5.17}$$

$$-\log \frac{E(z, \bar{\zeta})E(\infty, \bar{\infty})}{E(z, \bar{\infty})E(\infty, \bar{\zeta})} = D(z)\bar{D}(\bar{\zeta})\mathcal{F} \tag{5.18}$$

Combining these equalities (with merging points $z \to \zeta$ in particular), one is able to obtain the following representations of the prime form itself:

$$E(z, \zeta) = (z^{-1} - \zeta^{-1})e^{-\frac{1}{2}(D(z) - D(\zeta))^2 \mathcal{F}} \tag{5.19}$$

$$iE(z, \bar{\zeta}) = e^{-\frac{1}{2}(\partial_{t_0} + D(z) + \bar{D}(\bar{\zeta}))^2 \mathcal{F}} \tag{5.20}$$

Note also the nice formula

$$iE(z, \bar{z}) = e^{-\frac{1}{2}\nabla^2(z)\mathcal{F}} \tag{5.21}$$

Generalized Hirota relations. For higher genus Riemann surfaces there are no simple universal relations connecting values of prime forms at different points, which, via (5.19), (5.20), could be used to generate equations on $\mathcal{F}$. The best available relation [8] is the celebrated Fay identity (B.5). Although it contains not only prime forms but Riemann theta functions themselves, it is really a source of closed equations on $\mathcal{F}$, since all the ingredients are in fact representable in terms of second order derivatives of $\mathcal{F}$ in different variables.

An analog of the KP version of the Hirota equation (2.21) for the function $\mathcal{F}$ can be obtained by plugging eqs. (5.5) and (5.19) into the Fay identity (B.5). As a result, one obtains a closed equation which contains second order derivatives of the $\mathcal{F}$ only (recall that the period matrix in the theta-functions is essentially the matrix of the derivatives $\mathcal{F}_{\alpha\beta}$). A few equivalent forms of this equation are available. First, shifting $\mathbf{Z} \to \mathbf{Z} - \mathbf{W}_3 - \mathbf{W}_4$ in (B.5) and putting $z_4 = \infty$, one gets the relation

$$\begin{aligned}
&(a-b)e^{D(a)D(b)\mathcal{F}}\ \theta\left(\int_\infty^a d\mathbf{W} + \int_\infty^b d\mathbf{W} - \mathbf{Z}\right)\ \theta\left(\int_\infty^c d\mathbf{W} - \mathbf{Z}\right) + \\
&+ (b-c)e^{D(b)D(c)\mathcal{F}}\ \theta\left(\int_\infty^b d\mathbf{W} + \int_\infty^c d\mathbf{W} - \mathbf{Z}\right)\ \theta\left(\int_\infty^a d\mathbf{W} - \mathbf{Z}\right) + \\
&+ (c-a)e^{D(c)D(a)\mathcal{F}}\ \theta\left(\int_\infty^c d\mathbf{W} + \int_\infty^a d\mathbf{W} - \mathbf{Z}\right)\ \theta\left(\int_\infty^b d\mathbf{W} - \mathbf{Z}\right) = 0
\end{aligned} \tag{5.22}$$

The vector $\mathbf{Z}$ is arbitrary (in particular, zero). We see that (2.21) gets "dressed" by the theta-factors. Each theta-factor is expressed through $\mathcal{F}$ only. For example,

$$\theta\left(\int_{\infty}^{z} d\mathbf{W}\right) = \sum_{n_\alpha \in \mathbf{Z}} \exp\left(-2\pi^2 \sum_{\alpha\beta} n_\alpha n_\beta \partial^2_{\alpha\beta}\mathcal{F} - 2\pi i \sum_{\alpha} n_\alpha \partial_\alpha D(z)\mathcal{F}\right)$$

Another form of this equation, obtained from (B.5) for a particular choice of $\mathbf{Z}$, reads

$$(a-b)c^{-1}e^{\frac{1}{2}(D(a)+D(b))^2\mathcal{F}}h(c)\theta_*\left(\int_{\infty}^{a} d\mathbf{W}+\int_{\infty}^{b} d\mathbf{W}\right) + [\text{cyclic per-s of } a,b,c] = 0 \tag{5.23}$$

Taking the limit $c \to \infty$ in (5.22), one gets an analog of (2.22):

$$\begin{aligned} 1 \; - \; & \frac{\theta(\int_\infty^a d\mathbf{W} + \int_\infty^b d\mathbf{W} - \mathbf{Z}) \;\; \theta(\mathbf{Z})}{\theta(\int_\infty^a d\mathbf{W} - \mathbf{Z}) \; \theta(\int_\infty^b d\mathbf{W} - \mathbf{Z})} \, e^{D(a)D(b)\mathcal{F}} \; = \\ = \; & \frac{D(a)-D(b)}{a-b}\partial_{t_1}\mathcal{F} + \frac{1}{a-b}\sum_{\alpha=1}^{g} \frac{\partial}{\partial Z_\alpha} \log \frac{\theta(\int_\infty^a d\mathbf{W} - \mathbf{Z})}{\theta(\int_\infty^b \mathbf{W} - \mathbf{Z})} \, \partial_\alpha \partial_{t_1}\mathcal{F} \end{aligned} \tag{5.24}$$

which also follows from another Fay identity (B.6).

Equations on $\mathcal{F}$ with $\bar{t}_k$-derivatives follow from the general Fay identity (B.5) with some points on the lower sheet. Besides, many other equations can be derived as various combinations and specializations of the ones mentioned above. Altogether, they form an infinite hierarchy of consistent differential equations of a very complicated structure which deserves further investigation. The functions $\mathcal{F}$ corresponding to different choices of independent variables (i.e., to different bases in homology cycles on the Schottky double) provide different solutions to this hierarchy.

Higher genus analogs of the dispersionless Toda equation. Let us show how the simplest equation of the hierarchy, the dispersionless Toda equation (2.25), is modified in the multiply-connected case. Applying $\partial_z \partial_{\bar{\zeta}}$ to both sides of (5.18) and setting $\zeta = z$, we get:

$$(\partial D(z))(\bar{\partial}\bar{D}(\bar{z}))\mathcal{F} = -\partial_z \partial_{\bar{z}} \log E(z,\bar{z})$$

Here $\partial D(z)$ is the z-derivative of the operator $D(z)$: $\partial D(z) = -\sum_k z^{-k-1}\partial_{t_k}$. To transform the r.h.s., we use the identity (B.8) (Appendix B) and specialize it to the particular local parameters on the two sheets:

$$\partial_z \partial_{\bar{z}} \log E(z,\bar{z}) = \frac{\theta(\boldsymbol{\omega}(z)+\mathbf{Z})\theta(\boldsymbol{\omega}(z)-\mathbf{Z})}{\theta^2(\mathbf{Z})E^2(z,\bar{z})} + |z|^4 \sum_{\alpha,\beta} (\log\theta(\mathbf{Z}))_{,\alpha\beta}\, \partial_z W_\alpha(z) \partial_{\bar{z}} \overline{W_\beta(z)}$$

Tending z to ∞, we obtain a family of equations (parametrized by an arbitrary vector $\mathbf{Z}$) which generalize the dispesrionless Toda equation for the tau-function:

$$\partial_{t_1}\partial_{\bar{t}_1}\mathcal{F} = \frac{\theta(\boldsymbol{\omega}(\infty)+\mathbf{Z})\theta(\boldsymbol{\omega}(\infty)-\mathbf{Z})}{\theta^2(\mathbf{Z})}\, e^{\partial^2_{t_0}\mathcal{F}} + \sum_{\alpha,\beta=1}^{g} (\log\theta(\mathbf{Z}))_{,\alpha\beta}\, (\partial_\alpha \partial_{t_1}\mathcal{F})(\partial_\beta \partial_{\bar{t}_1}\mathcal{F}) \tag{5.25}$$

(Here we used the $z \to \infty$ limits of (5.5) and (5.21).) The following two equations correspond to special choices of the vector $\mathbf{Z}$:

$$\partial_{t_1}\partial_{\bar{t}_1}\mathcal{F} - \sum_{\alpha,\beta=1}^{g} (\log\theta(0))_{,\alpha\beta}\, (\partial_\alpha \partial_{t_1}\mathcal{F})(\partial_\beta \partial_{\bar{t}_1}\mathcal{F}) = \frac{\theta^2(\boldsymbol{\omega}(0))}{\theta^2(0)}\, e^{\partial^2_{t_0}\mathcal{F}} \tag{5.26}$$

$$\partial_{t_1}\partial_{\bar t_1}\mathcal{F} = \sum_{\alpha,\beta=1}^{g} \left[\log\theta_*(\boldsymbol{\omega}(\infty))\right]_{,\alpha\beta} (\partial_\alpha\partial_{t_1}\mathcal{F})(\partial_\beta\partial_{\bar t_1}\mathcal{F}) \tag{5.27}$$

Finally, let us specify the equation (5.25) for the genus $g = 1$ case. In this case there is only one extra variable (Π_1 or $2\pi i s_1$) $x := x_1$. The Riemann theta-function $\theta(\boldsymbol{\omega} + \mathbf{Z})$ is then replaced by the Jacobi theta-function $\vartheta\left(\partial_x\partial_{t_0}\mathcal{F} + Z|\,\tau\right) \equiv \vartheta_3\left(\partial_x\partial_{t_0}\mathcal{F} + Z|\,\tau\right)$, where the elliptic modular parameter is $\tau = 2\pi i\,\partial_x^2\mathcal{F}$, and the vector $Z \equiv \mathbf{Z}$ has only one component. The equation has the form:

$$\begin{aligned}\partial_{t_1}\partial_{\bar t_1}\mathcal{F} = {} & \frac{\vartheta_3\left(\partial_x\partial_{t_0}\mathcal{F}+Z|\,2\pi i\,\partial_x^2\mathcal{F}\right)\vartheta_3\left(\partial_x\partial_{t_0}\mathcal{F}-Z|\,2\pi i\,\partial_x^2\mathcal{F}\right)}{\vartheta_3^2\left(Z|\,2\pi i\,\partial_x^2\mathcal{F}\right)}\, e^{\partial_{t_0}^2\mathcal{F}} + \\ & + \partial_Z^2\log\vartheta_3\left(Z|\,2\pi i\,\partial_x^2\mathcal{F}\right)\,(\partial_x\partial_{t_1}\mathcal{F})(\partial_x\partial_{\bar t_1}\mathcal{F})\end{aligned} \tag{5.28}$$

Note also that equation (5.14) acquires the form

$$(\partial_x\partial_{t_1}\mathcal{F})(\partial_x\partial_{\bar t_1}\mathcal{F}) = \left(\frac{\vartheta_1\left(\partial_x\partial_{t_0}\mathcal{F}|\,2\pi i\,\partial_x^2\mathcal{F}\right)}{\vartheta_1'\left(0|\,2\pi i\,\partial_x^2\mathcal{F}\right)}\right)^2 e^{\partial_{t_0}^2\mathcal{F}} \tag{5.29}$$

where $\vartheta_* \equiv \vartheta_1$ is the only odd Jacobi theta-function. Combining (5.28) and (5.29) one may write

$$\begin{aligned}\partial_{t_1}\partial_{\bar t_1}\mathcal{F} = {} & \Bigg[\frac{\vartheta_3\left(\partial_x\partial_{t_0}\mathcal{F}+Z|\,2\pi i\,\partial_x^2\mathcal{F}\right)\vartheta_3\left(\partial_x\partial_{t_0}\mathcal{F}-Z|\,2\pi i\,\partial_x^2\mathcal{F}\right)}{\vartheta_3^2\left(Z|\,2\pi i\,\partial_x^2\mathcal{F}\right)} + \\ & + \left(\frac{\vartheta_1\left(\partial_x\partial_{t_0}\mathcal{F}|\,2\pi i\,\partial_x^2\mathcal{F}\right)}{\vartheta_1'\left(0|\,2\pi i\,\partial_x^2\mathcal{F}\right)}\right)^2 \partial_Z^2\log\vartheta_3\left(Z|\,2\pi i\,\partial_x^2\mathcal{F}\right)\Bigg]\, e^{\partial_{t_0}^2\mathcal{F}}\end{aligned} \tag{5.30}$$

which has mostly close to (2.25) form with nontrivial "coefficient" in square brackets instead of unity.

6 Discussion

In this paper we have considered the Dirichlet boundary problem in planar multiply-connected domains. A planar multiply-connected domain D^{c} is the complex plane with several holes. We study how the solution of the Dirichlet problem depends on small deformations of boundaries of the holes.

General properties of such deformations allow us to introduce the quasiclassical tau-function associated to the variety of planar multiply-connected domains. By the tau-function, we actually mean its logarithm, which only makes sense for the quasiclassical or Whitham-type integrable hierarchies. Namely, the key properties are the specific "exchange relations" (3.14) which follow from the Hadamard variational formula for the Green function and the harmonic measure. They have the form of integrability conditions and thus ensure the existence of the tau-function. The tau-function corresponds to a particular solution of the universal Whitham hierarchy [4] and generalizes the dispersionless tau-function which describes deformations of simply-connected domains.

The algebro-geometric data associated with the multiply-connected geometry include a Riemann surface with antiholomorphic involution, the Schottky double of the domain $\mathsf{D}^{\mathsf{c}} = \mathbf{C} \setminus \mathsf{D}$ endowed with particular holomorphic coordinates z and $\bar z$ on the two sheets of the double, respecting the involution. This Riemann surface has genus $g = \#\{\text{holes}\} - 1$. The (logarithm of) tau-function $\mathcal{F}$, describes small deformations of these data as functions of an infinite set of independent deformation parameters which are basically harmonic moments of the domain. These variables can be equivalently redefined as periods of the generating one-form $S(z)dz$ over non-trivial cycles on the double and the residues of the one-forms $z^{-k}S(z)dz$.

We have obtained simple expressions for the period matrix, the Abel map and the prime form on the Schottky double in terms of the function $\mathcal{F}$. Specifically, all these objects are expressed through second order derivatives of the $\mathcal{F}$ in its independent variables.

The generalized dispersionless Hirota equations on $\mathcal{F}$ for the multiply-connected case (equivalent to the Whitham hierarchy) are obtained by incorporating the above mentioned expressions into the Fay identities. As a result, one comes to a series of quite non-trivial equations for (second derivatives of) the function F, which have not been written before (except for certain relations for the second derivatives of the Seiberg-Witten prepotential [24]). When the Riemann surface degenerates to the Riemann sphere with two marked points, they turn into Hirota equations of the dispersionless Toda hierarchy.

In this paper our analysis was restricted to the class of domains which can be obtained as conformal images of a "half" of a complex algebraic curve with the antiholomorphic involution under conformal maps given by rational functions on the curve. In particular, all domains having only a finite amount of non-vanishing harmonic moments, are in this class. In this case one can define the curve by a polynomial equation written explicitly in [20]. This situation is an analog of the (Laurent-) polynomial conformal maps in the simply-connected case and literally corresponds to multi-support solutions of matrix models with polynomial potentials. The definition of the tau-function for multiply-connected domains proposed above holds in a broader set-up of general algebraic domains. It does not rely on the finitness of the amount of non-vanishing moments. In general any effective way to describe the complex curve associated to a multiply-connected domain by a system of polynomial or algebraic equations is not known. The curve may be thought of as a spectral curve corresponding to a generic finite-gap solution to the Toda lattice hierarchy.

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Appendix A. The tau-function in the variables ϕ_α

Here we demonstrate the existence of the tau-function for multiply-connected domains in a somewhat different way, via direct representation of the bump deformations through the variation of harmonic moments, which is closer to the one adopted in [2]. As the independent variables, we choose the harmonic moments (3.10). Recall that to define the moments ϕ_α, one has to fix an auxiliary point $z_\alpha \in \mathsf{D}_\alpha$ in each domain D_α.

Let us introduce the differential operator

$$\tilde{\nabla}(z) = \nabla_\phi(z) + \nabla(z) \tag{A.1}$$

where $\nabla(z) = \partial_{t_0} + D(z) + \bar{D}(\bar{z})$ is the same operator (2.5) as in the simply-connected case and

$$\nabla_\phi(z) = \sum_{\alpha=1}^{g} \nu_\alpha(z) \frac{\partial}{\partial \phi_\alpha}\,, \qquad \nu_\alpha(z) = \log \left| 1 - \frac{z_\alpha}{z} \right|^2 \tag{A.2}$$

It is the operator $\tilde{\nabla}(z)$ (A.1) that still possesses the key property: for any $\xi \in \partial D$ the bump deformation (see [2]) of any functional X is given by

$$\delta^{\rm bump} X := \delta_{\epsilon(\xi)} X = \frac{\epsilon}{\pi} \tilde{\nabla}(\xi) X \tag{A.3}$$

or, equivalently, $\tilde{\nabla}(z)X$ is the harmonic extension of the function $\frac{\epsilon}{\pi}\tilde{\nabla}(z)X$ to $\mathsf{D^c}$ from the boundary. This extension is given by the solution of the Dirichlet problem in $\mathsf{D^c}$, and, in fact, coincides with the elementary deformation (2.8).

Suppose that each of the domains D_α is filled with an electric charge with uniform density $\rho = -1$. The potential created by this charge is still given by the formula (2.3), where now $\mathsf{D} = \cup_{\alpha=0}^{g} \mathsf{D}_\alpha$. The tau-function can be introduced along the lines of [9], using an electrostatic variational principle. The result is given by the same formula (2.30) as in the simply-connected case. However, the set of independent variables is different. Note that the integral (2.30) does not depend on the choice of the points $\{z_\alpha\}$ but F regarded as a function of $\{\phi_\alpha\}$ does. (This dependence, however, disappears after passing to the variables $\{\Pi_\alpha\}$.)

Let us apply the operator $\tilde{\nabla}$ to the F given by (2.30). We still have, like in (2.27):

$$\tilde{\nabla}(z) F = \tilde{\Phi}(z) = -\frac{2}{\pi} \int_{\mathsf{D}} \log |z^{-1} - \zeta^{-1}| d^2\zeta = \Phi(z) + v_0 + t_0 \log |z|^2 \tag{A.4}$$

In the multiply-connected case it is a harmonic function in $\mathsf{D^c}$ with non-zero "periods"

$$\oint_{\mathsf{b}_\alpha} \partial_n \tilde{\Phi}(z) |dz| = \int_{\mathsf{D}_\alpha} \Delta \tilde{\Phi} d^2 z = -4\pi s_\alpha$$

where $\Delta = 4\partial_z \partial_{\bar{z}}$ is the Laplace operator and $\{s_\alpha\}$ are defined in (3.23).

We can write the expansion:

$$\tilde{\Phi}(z) = -t_0 \log |z|^2 - \sum_{\alpha=1}^{g} s_\alpha \log \left| 1 - \frac{z_\alpha}{z} \right|^2 + 2{\rm Re} \sum_{k \geq 1} \tilde{v}_k \frac{z^{-k}}{k} \tag{A.5}$$

where the logarithmic terms ensure the right values of the periods and

$$\tilde{v}_k = v_k - \sum_{\alpha=1}^{g} s_\alpha z_\alpha^k = \frac{1}{\pi} \int_{\mathsf{D}} z^k d^2 z \, - \sum_{\alpha=1}^{g} s_\alpha z_\alpha^k \tag{A.6}$$

Comparing with

$$\tilde{\nabla}(z) F = \sum_{\alpha=1}^{g} \nu_\alpha(z) \partial_{\phi_\alpha} F + \partial_{t_0} F + \sum_{k \geq 1} \left(\frac{z^{-k}}{k} \partial_{t_k} F + \text{c.c.} \right)$$

we find:

$$s_\alpha = -\frac{\partial F}{\partial \phi_\alpha} \quad (\alpha = 1, \ldots, g)\,, \qquad \tilde{v}_k = \frac{\partial F}{\partial t_k} \tag{A.7}$$

Using exactly the same reasoning as in the simply-connected case, we obtain the formula which represents the Green function of the Dirichlet boundary problem through the second order derivatives of the function F:

$$G(z,\zeta) = \log|z^{-1} - \zeta^{-1}| + \frac{1}{2}\tilde{\nabla}(z)\tilde{\nabla}(\zeta)F \tag{A.8}$$

Taking "periods" over the non-trivial cycles, one may extract from it similar formulas for the harmonic measure and for the period matrix. Integrating normal derivatives of the both sides over b_α, one has

$$\frac{1}{2\pi}\oint_{\mathsf{b}_\alpha} \partial_n G(z,\zeta)|d\zeta| = \frac{1}{4\pi}\oint_{\mathsf{b}_\alpha} \partial_n \nabla_\phi(\zeta)\tilde{\Phi}(z)|d\zeta|$$

where it is taking into account that in $\tilde{\nabla}(\zeta)$ only $\nabla_\phi(\zeta)$ contributes to the period. The right hand side is equal to $\partial_{\phi_\alpha}\tilde{\Phi}(z)$. The left hand side is the harmonic measure (3.1) of the contour b_α with the sign minus. Therefore,

$$\omega_\alpha(z) = -\partial_{\phi_\alpha}\tilde{\nabla}(z)F \tag{A.9}$$

Further, taking "periods" of the both sides of (A.9), we get, in a similar way:

$$-\frac{1}{2\pi}\oint_{\mathsf{b}_\beta} \partial_n \omega_\alpha(z)|dz| = \Omega_{\alpha\beta} = 2\frac{\partial^2 F}{\partial\phi_\alpha \partial\phi_\beta} \tag{A.10}$$

Using (A.10), we can rewrite (A.9) in the form

$$\omega_\alpha(z) = -\frac{1}{2}\sum_{\beta=1}^{g} \Omega_{\alpha\beta}\nu_\beta(z) + g_\alpha(z) \tag{A.11}$$

where

$$g_\alpha(z) = -\partial_{\phi_\alpha}\nabla(z)F \tag{A.12}$$

is a harmonic function in D^{c} with zero "periods". After some algebra, one can represent the relation (A.8) in the form

$$\begin{aligned} G(z,\zeta) = \log|z^{-1} - \zeta^{-1}| + \frac{1}{2}\nabla(z)\nabla(\zeta)F - \sum_{\alpha,\beta=1}^{g} g_\alpha(z)(\Omega^{-1})_{\alpha\beta}\, g_\beta(\zeta) + \\ + \sum_{\alpha,\beta=1}^{g} \omega_\alpha(z)(\Omega^{-1})_{\alpha\beta}\, \omega_\beta(\zeta) \end{aligned} \tag{A.13}$$

Here only the terms in the second line have non-vanishing "periods".

Appendix B. Theta functions and Fay identities

Here we present some definitions and useful formulas from [8]. The Riemann theta function $\theta(\mathbf{W}) \equiv \theta(\mathbf{W}|T)$ is defined as

$$\theta(\mathbf{W}) = \sum_{\mathbf{n}\in\mathbf{Z}^g} e^{i\pi\mathbf{n}\cdot T\cdot\mathbf{n} + 2\pi i\mathbf{n}\cdot\mathbf{W}} \tag{B.1}$$

The theta function with a (half-integer) characteristics $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)$, where $\delta_\alpha = \delta_{1,\alpha} + T_{\alpha\beta}\delta_{2,\beta}$ and $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \in \frac{1}{2}\mathbf{Z}^g$ is

$$\theta_{\boldsymbol{\delta}}(\mathbf{W}) = e^{i\pi\boldsymbol{\delta}_1\cdot T\cdot\boldsymbol{\delta}_1 + 2\pi i\boldsymbol{\delta}_1\cdot(\mathbf{W}+\boldsymbol{\delta}_2)}\theta(\mathbf{W}+\boldsymbol{\delta}) = \sum_{\mathbf{n}\in\mathbf{Z}^g} e^{i\pi(\mathbf{n}+\boldsymbol{\delta}_1)\cdot T\cdot(\mathbf{n}+\boldsymbol{\delta}_1) + 2\pi i(\mathbf{n}+\boldsymbol{\delta}_1)\cdot(\mathbf{W}+\boldsymbol{\delta}_2)} \tag{B.2}$$

Under shifts by a period of the lattice, it transforms according to

$$\begin{aligned} \theta_{\boldsymbol{\delta}}(\mathbf{W}+\mathbf{e}_\alpha) &= e^{2\pi i\delta_{1,\alpha}}\theta_{\boldsymbol{\delta}}(\mathbf{W}) \\ \theta_{\boldsymbol{\delta}}(\mathbf{W}+T_{\alpha\beta}\mathbf{e}_\beta) &= e^{-2\pi i\delta_{2,\alpha}-i\pi T_{\alpha\alpha}-2\pi i W_\alpha}\theta_{\boldsymbol{\delta}}(\mathbf{W}) \end{aligned} \tag{B.3}$$

The prime form $E(z,\zeta)$ is defined as

$$E(z,\zeta) = \frac{\theta_*(\mathbf{W}(z)-\mathbf{W}(\zeta))}{\sqrt{\sum_\alpha \theta_{*,\alpha}dW_\alpha(z)}\sqrt{\sum_\beta \theta_{*,\beta}dW_\beta(\zeta)}} \tag{B.4}$$

where θ_* is any odd theta function, i.e., the theta function with *any odd* characteristic $\boldsymbol{\delta}^*$ (the characteristics is odd if $4\boldsymbol{\delta}_1^*\cdot\boldsymbol{\delta}_2^* = \text{odd}$). The prime form does not depend on the particular choice of the odd characteristics. In the denominator,

$$\theta_{*,\alpha} = \theta_{*,\alpha}(0) = \left.\frac{\partial\theta_*(\mathbf{W})}{\partial W_\alpha}\right|_{\mathbf{W}=0}$$

is the set of θ-constants.

The data we use in the main text contain also a distinguished coordinates on a Riemann surface: the holomorphic co-ordinates z and $\bar{z}$ on two different sheets, and we do not distinguish, unless it is nessesary between the prime form (B.4) and a *function* $E(z,\zeta) \equiv E(z,\zeta)(dz)^{1/2}(d\zeta)^{1/2}$ "normalized" onto the differentials of distinguished co-ordinate.

Let us now list the Fay identities [8] used in the paper. The basic one is Fay's trisecant formula (equation (45) from p. 34 of [8])

$$\begin{aligned} &\theta(\mathbf{W}_1-\mathbf{W}_3-\mathbf{Z})\ \theta(\mathbf{W}_2-\mathbf{W}_4-\mathbf{Z})\ E(z_1,z_4)E(z_3,z_2)+ \\ &+\theta(\mathbf{W}_1-\mathbf{W}_4-\mathbf{Z})\ \theta(\mathbf{W}_2-\mathbf{W}_3-\mathbf{Z})\ E(z_1,z_3)E(z_2,z_4) = \\ &= \theta(\mathbf{W}_1+\mathbf{W}_2-\mathbf{W}_3-\mathbf{W}_4-\mathbf{Z})\ \theta(\mathbf{Z})\ E(z_1,z_2)E(z_3,z_4) \end{aligned} \tag{B.5}$$

Here $\mathbf{W}_i \equiv \mathbf{W}(z_i)$. This identity holds for *any* four points $z_1,\ldots,z_4$ on a Riemann surface and *any* vector $\mathbf{Z}\in\mathbf{Jac}$. In the limit $z_3\to z_4\equiv\infty$ one gets (formula (38) from p. 25 of [8])

$$\begin{aligned} &\frac{\theta(\int_\infty^{z_1}d\mathbf{W}+\int_\infty^{z_2}d\mathbf{W}-\mathbf{Z})\ \theta(\mathbf{Z})}{\theta(\int_\infty^{z_1}d\mathbf{W}-\mathbf{Z})\ \theta(\int_\infty^{z_2}d\mathbf{W}-\mathbf{Z})}\frac{E(z_1,z_2)}{E(z_1,\infty)E(z_2,\infty)} = \\ &= \Omega_{z_1-z_2}(\infty) + \sum_{\alpha=1}^{g} dW_\alpha(\infty)\ \partial_{Z_\alpha}\log\ \frac{\theta(\int_\infty^{z_1}d\mathbf{W}-\mathbf{Z})}{\theta(\int_\infty^{z_2}d\mathbf{W}-\mathbf{Z})} \end{aligned} \tag{B.6}$$

where

$$\Omega_{z_1-z_2}(z) = d_z\log\ \frac{E(z,z_1)}{E(z,z_2)} \tag{B.7}$$

is the normalized Abelian differential of the third kind with simple poles at z_1 and z_2.

Another relation from [8] we use (see e.g. (29) on p.20 and (39) on p.26) is

$$\frac{\theta(\mathbf{W}_1-\mathbf{W}_2-\mathbf{Z})\theta(\mathbf{W}_1-\mathbf{W}_2+\mathbf{Z})}{\theta^2(\mathbf{Z})E^2(z_1,z_2)} = \omega(z_1,z_2) + \sum_{\alpha,\beta=1}^{g}(\log\theta(\mathbf{Z}))_{,\alpha\beta}\,dW_\alpha(z_1)dW_\beta(z_2) \tag{B.8}$$

where

$$(\log\theta(\mathbf{Z}))_{,\alpha\beta} = \frac{\partial^2\log\theta(\mathbf{Z})}{\partial Z_\alpha\partial Z_\beta}$$

and

$$\omega(z_1,z_2) = d_{z_1}d_{z_2}\log E(z_1,z_2) \tag{B.9}$$

is the canonical bi-differential of the second kind with the double pole at $z_1=z_2$.

Appendix C. Degenerate Schottky double

For an illustrative purpose we would like to adopt some of the above formulas to the simplest possible case, which is the Riemann sphere realized as the Schottky double of the complement to the disk of radius r. In this case

$$dW = \frac{dz}{z} = id\varphi = -\frac{d\bar{z}}{\bar{z}} \tag{C.1}$$

is purely imaginary on the circle and obviously satisfies the condition $dW(z) + \overline{dW(\bar{z})} = 0$. Further,

$$W(z) = \int_r^z dW = \log\frac{z}{r} \tag{C.2}$$

and

$$w(z) = e^{W(z)} = \frac{z}{r} \tag{C.3}$$

which is nothing but the conformal map of the exterior of the circle $|z| \geq r$ onto exterior of the unit circle $|w| \geq 1$. Note that on the "lower" sheet of the double

$$W(\bar{z}) = \int_r^{\bar{z}} dW = \log\frac{r}{\bar{z}} \tag{C.4}$$

and instead of (C.3) one gets

$$w(\bar{z}) = \frac{r}{\bar{z}} \tag{C.5}$$

which is the conformal map of the exterior of the disk $|\bar{z}| \geq r$ on the lower sheet onto the *interior* of the unit circle $|w| \leq 1$. Now, the prime form on the genus zero Riemann surface is (cf. (B.4))

$$E(z_1, z_2) = \frac{w_1 - w_2}{\sqrt{dw_1}\sqrt{dw_2}} \tag{C.6}$$

where $w_i \equiv w(z_i)$. Let us compute $E(\infty, \bar{\infty})$, which is understood in the main text, as "normalized" on the values of the local coordinates $z_\infty = 1/z$ and $\bar{z}_\infty = 1/\bar{z}$ in the points ∞ and $\bar{\infty}$ on two sheets of the double. One gets (cf. with (5.9))

$$E^2(\infty, \bar{\infty}) = -\frac{(w(\infty) - w(\bar{\infty}))^2}{(dw(\infty)/dz_\infty)(dw(\bar{\infty})/dz_{\bar{\infty}})} \tag{C.7}$$

Substituting into (C.7) the formulas (C.3), (C.5) and

$$\begin{gathered} dw(\infty) = \lim_{z\to\infty}\frac{dz}{r} = -\lim_{z\to\infty}\frac{z^2}{r}\,dz_\infty \\ dw(\bar{\infty}) = r\,dz_{\bar{\infty}} \end{gathered} \tag{C.8}$$

one finally gets

$$E(\infty, \bar{\infty})^2 = \lim_{z\to\infty}\frac{z^2/r^2}{(z^2/r)r} = \frac{1}{r^2} \tag{C.9}$$

and this demostrates that (5.14) indeed turns in the limit into (2.15).

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