L-functions over function fields: A cohomological approach

(report on joint work with Richard Pink)

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Notation
$X/\mathbf{F}_q$ a scheme of finite type
$\sigma = \sigma_X = \mathrm{Frob}_X$
$|X|$ the closed points of X
for $x \in |X|$:
k_x the residue field at x
$d_x = [k_x : \mathbf{F}_q]$ its degree

1 Motivation

The Hasse-Weil zeta-function

$$Z(X,T) = \prod_{x \in |X|} (1 - T^{d_x})^{-1} \ \in \mathbb{Z}[[T]] \tag{1}$$

provides an ingeneous way of encoding $\#X(\mathbf{F}_{q^n}), n \in \mathbb{N}$ because of the

Weil conjecture (1949). *There exists a cohomology theory $X \mapsto H_c^*(X)$ such that H_c^i are finite-dimensional vector spaces over a field of characteristic zero carrying an endomorphism* Frob_q*; $H_c^i(X) = 0$ for $i > \dim X$ and*

$$Z(X,T) = \prod_{i=0}^{2\dim X} \det(1 - T\mathrm{Frob}_q | H_c^i(X))^{(-1)^{i+1}}.$$

In particular, $Z(X,T)$ is a rational function of T.

A **rationality proof** of $Z(X,T)$ was given by Hasse in '33 for elliptic curves, by Weil in '40 for all curves, by Dwork in '60 in general. Artin and Grothendieck gave a general proof in the sixties along the lines foreseen by Weil using the formalism of l-adic cohomology.

For a constructible $\mathbb{Q}_l$-sheaf $\mathcal{F}$ one obtains a morphism

$$\phi : \sigma_X^* \mathcal{F} \to \mathcal{F},$$

This is used to define the L-function of $\mathcal{F}$. Formula (1) is then a consequence of the Lefschetz-Verdier trace formula.

Global L-functions. For any global scheme $\mathcal{X}$ of finite type over Spec($\mathbb{Z}$) its fibers above p are defined by

$$\begin{array}{ccc} \mathcal{X} & \longleftarrow & \mathcal{X}_p \\ {\scriptstyle \text{fin. type}}\Big\downarrow & & \Big\downarrow \\ \operatorname{Spec}\mathbb{Z} & \longleftarrow & \operatorname{Spec}\mathbf{F}_q \end{array}$$

One then combines all the local data in the following way to obtain a global zeta-function

$$Z^{\mathrm{gl}}(\mathcal{X},s) = \prod_{x\in|\mathcal{X}|}(1-(\#k_x)^{-s})^{-1} = \prod_{p \text{ prime}} Z(\mathcal{X}_p,T)|_{T=p^{-s}}.$$

(For $\mathcal{X} = \operatorname{Spec}\mathbb{Z}$, the global zeta-function is the Riemann zeta-function.) For global zeta-functions one expects a meromorphic continuation to all of $\mathbb{C}$ (and more).

2 A (global) function field analogue

More notation.

A a normal domain (or a finite ring, or ...)
$C = \operatorname{Spec}(A)$ the "coefficient scheme"
$K = \operatorname{Frac}(A)$

Definition 2.1 A τ-sheaf $\underline{\mathcal{F}}$ on X over A consists of

(i) a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X \otimes A$-modules

(ii) a morphism $\tau := \tau_{\underline{\mathcal{F}}} \;:\; (\sigma_X \times \mathrm{id}_C)^*\mathcal{F} \to \mathcal{F}$.

The category of τ-sheaves on X over A is denoted by $\operatorname{Coh}_\tau(X,A)$.

Examples 2.2 (i) If $\mathcal{C}$ is a smooth projective curve, ∞ a closed point of $\mathcal{C}$, and A the coordinate ring of $C := \mathcal{C} - \{\infty\}$ then there is a natural map

$$\left\{\begin{array}{c}\text{Families of Drinfeld}\\ A-\text{modules param. by } X\end{array}\right\} \to \left\{\begin{array}{c}\tau\text{-sheaves } \underline{\mathcal{F}} \text{ on } X \text{ over } A\\ (\underline{\mathcal{F}} \text{ locally free})\\ \text{and a map "char": } X \to \operatorname{Spec} A\end{array}\right\}$$

where the map "char" is the analogue of $\mathcal{X} \to \operatorname{Spec}\mathbb{Z}$.

(ii) For $\mathfrak{m} \in \operatorname{MaxSpec}(A)$, there is an equivalence of categories

$$\left\{\begin{array}{c}\text{Continuous } \hat{A}_{\mathfrak{m}}\text{-adic Galois}\\ \text{representation of } \pi_1(X)\end{array}\right\} \to \left\{\begin{array}{c}\text{lisse formal } \tau\text{-sheaves}\\ \text{with } \hat{A}_{\mathfrak{m}}\text{-coefficients}\end{array}\right\}$$

$$= \left\{\begin{array}{c}\text{Inverse system } (\underline{\mathcal{F}_n})\\ \underline{\mathcal{F}} \text{ locally free}/X\otimes A/\mathfrak{m}^n,\\ \text{and an iso.} \tau_n : (\sigma\times\mathrm{id})^*\underline{\mathcal{F}_n} \to \underline{\mathcal{F}_n}\end{array}\right\}$$

David Goss has made two conjectures, wher he assumes that $\underline{\mathcal{F}}$ and A are as in Example 2.2.(i) above. (At least for the first conjecture this is not necessary.)

Define, for A normal,

$$L(\underline{\mathcal{F}},X,\tau) := \prod_{x\in|X|}\det(1-T^{d_x}\tau_x^{d_x}|\mathcal{F}_x\otimes_A K) \in A[[T]],$$

where $\mathcal{F}_x$ is the fiber of $\mathcal{F}$ at x.

Conjecture 2.3 (Local) $L(\underline{\mathcal{F}}, X\tau)$ is a rational function in T.

For the global conjecture, assume that $\underline{\mathcal{F}}$ and char : $X \to A$ are as in the example above. Let K_∞ be the completion of K w.r.t. the ∞-valuation and A_∞ its ring of integers. Let $\mathbb{C}_\infty$ be a completion of an algebraic closure of K_∞.

We now want to make sense out of $\mathfrak{p}^{-s}$ for $\mathfrak{p}$ a prime ideal in A and s in some suitable domain. For simplicity, assume that $A = \mathbf{F}_q[T]$, let $\mathfrak{p} \in A$ be generated by a monic element $f_\mathfrak{p}$ of A. For $s \in S_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p$ define

$$\mathfrak{p}^s := x^{\deg \mathfrak{p}} \cdot \left(\frac{f_\mathfrak{p}(t)}{t^{d_x}}\right)^y \in \mathbb{C}_\infty^* \cdot A_\infty^* = \mathbb{C}_\infty^*,$$

so that $\mathfrak{p}^s / x^{\deg \mathfrak{p}}$ is a 1-unit in A_∞.

Consider the pullback map $i_\mathfrak{p}$ given by

$$\begin{array}{ccc} X & \xleftarrow{i_\mathfrak{p}} & X_\mathfrak{p} \\ {\scriptstyle \text{char}}\downarrow & & \downarrow \\ \text{Spec } A & \longleftarrow & \mathfrak{p} \end{array}$$

Define for $|x| > c$ (c sufficiently large) and all $y \in \mathbb{Z}_p$

$$L^{\mathrm{gl}}(\underline{\mathcal{F}}, X, s) := \prod_{\mathfrak{p}} L(i_\mathfrak{p}^* \underline{\mathcal{F}}, X_\mathfrak{p}, T)|_{T = \mathfrak{p}^{-s}}.$$

Conjecture 2.4 (Global) $L^{\mathrm{gl}}(\underline{\mathcal{F}}, X, s)$ is a "continuous family of meromorphic functions on $\mathbb{C}_\infty$ parametrized by $\mathbb{Z}_p$"

The local and global conjecture for $A = \mathbf{F}_q[T]$ were proved in '96 by Taguchi and Wan using Dwork-type methods.

3 A cohomology theory of 'τ-sheaves': Crystals

We now want to construct a cohomology theory with 'τ-sheaves' as coefficients with as many of the usual six functors $f_*, f^*, f_!, f^!, \otimes, \mathrm{Hom}$ as possible, as well as a trace formula. We will be able to construct $f_!, f^*, \otimes$, i.e., half of the usual six functors, and prove a trace formula. The absence of half of the functors should not come as too much of a surprise, as there are no known functional equations for the L-functions our theory will describe, and thus one cannot hope for a duality theory.

Known results for τ-sheaves:
For a smooth projective curve X over $\mathbf{F}_q$, one has the Woodshole fixed point formula for coherent cohomology which gives a trace formula for τ-sheaves. Furthermore Anderson proves, by elementary means, a trace formula for X smooth (not necessarily proper), and $\underline{\mathcal{F}}$ locally free, however without any cohomological interpretation.

Missing for τ-sheaves:
$j_!$: A definition of $j_!$ for an open immersion j. (The usual $j_!$ of a coherent sheaf is in general no longer coherent.)
Rigidity: Example 2.2.(ii) above suggests that τ-sheaves should behave like étale sheaves, and in particular they should posess no non-trivial infinitesimal deformations:

$$\{\text{'}\tau\text{-sheaves' on } X\} \cong \{\text{'}\tau\text{-sheaves' on } X_{red}\}$$

3.1 Crystals

The 'τ-sheaves', which from now on are called crystals, are defined as follows.

Definition 3.1 (i) $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X, A)$ is called *nilpotent* iff

$$\exists n : \tau^n : (\sigma^n \times \mathrm{id})^* \mathcal{F} \to \mathcal{F} \text{ is zero.}$$

(ii) $\phi : \underline{\mathcal{F}} \to \underline{\mathcal{G}}$ is a *nil-isomorphism* (we write $\underline{\mathcal{F}} \Rightarrow \underline{\mathcal{G}}$) iff $\ker(\phi)$, $\mathrm{coker}(\phi)$ are nilpotent.
Fact: The nil-isomorphism form a saturated multiplicative system $\mathcal{S}_{\mathrm{nil}}$ in $\mathbf{Coh}_\tau(X, A)$.

(iii) $\mathbf{Crys}(X, A)$ is defined to be the localization of $\mathbf{Coh}_\tau(X, A)$ at $\mathcal{S}_{\mathrm{nil}}$. The objects of $\mathbf{Crys}(X, A)$ are called crystals.

$\mathbf{Crys}(X, A)$ has the same objects as $\mathbf{Coh}_\tau(X, A)$, however, the morphism sets are different. For crystals $\underline{\mathcal{F}}, \underline{\mathcal{G}}$, thought of as τ-sheaves, a morphism is a diagram

$$\underline{\mathcal{F}} \Leftarrow \underline{\mathcal{F}}' \to \underline{\mathcal{G}}.$$

Fact: $\underline{\mathcal{F}} \Rightarrow \underline{\mathcal{G}}$ implies that $L(\underline{\mathcal{F}}, X, T) = L(\underline{\mathcal{G}}, X, T)$.

Theorem 3.2 *If A is finite, then $\mathbf{Crys}(X, A)$ is equivalent to the category of constructible étale A-torsion sheaves on X.*

3.2 Construction of $j_!$

Suppose we are given $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Y$, j an open immersion, i the embedding of a closed complement whose ideal sheaf is given by $\mathcal{J}$. Let $\underline{\mathcal{F}}$ be a crystal on U (over A).

Let $\tilde{\mathcal{F}}$ be any coherent extension of $\mathcal{F}$ to $X \otimes C$. Then for $n \gg 0$ there exists

$$\tilde{\tau} : (\sigma \times \mathrm{id})^* (\mathcal{J}^n \tilde{\mathcal{F}}) \cong \mathcal{F}^{qn} (\sigma \times \mathrm{id})^* \tilde{\mathcal{F}} \to \mathcal{J}^n \tilde{\mathcal{F}}.$$

Thus $\tilde{\tau}$ is nilpotent on the restriction of $\mathcal{J}^{n+1}\tilde{\mathcal{F}}$ to Y. We find:

Theorem 3.3 *Given $\underline{\mathcal{F}}$, a crystal on U (over A), there exists a crystal $j_!\underline{\mathcal{F}}$ on X unique up to unique isomorphism, such that $j^* j_! \underline{\mathcal{F}} \cong \underline{\mathcal{F}}$ and $i^* j_! \underline{\mathcal{F}} = 0$. Furthermore, $j_! : \mathbf{Crys}(U, A) \to \mathbf{Crys}(X, A)$ is an exact functor.*

Proposition 3.4 *With the above notation, the following is a short exact sequence of functors*

$$0 \to j_! j^* \to \mathrm{id} \to i_* i^* \to 0.$$

3.3 Functors on Crystals

We fix $f : X \to Y$. To define f^* on crystals, we first define it on τ-sheaves using the usual formalsm of f^* for coherent sheaves. One easily checks that f^* preserves nil-isomorphisms, and thus one has defined f^* on crystals.

Theorem 3.5 *f^* is exact on crystals.*

The main point of the proof is that $L_1 f^* \underline{\mathcal{F}}$ is nilpotent for any τ-sheaf $\underline{\mathcal{F}}$. A useful tool is the following pointwise characterization of nilpotency.

Theorem 3.6 *$\underline{\mathcal{F}}$ is nilpotent iff for all $x \in X$ (or even for all $x \in |X|$), $i_x^*\underline{\mathcal{F}}$ is nilpotent ($i_x : \{x\} \to X$).*

Theorem 3.7 (Rigidity) *If f is radicial surjective, then f^* and f_* are mutually inverse isomorphisms between $\mathbf{Crys}(X, A)$ and $\mathbf{Crys}(Y, A)$.*

In particular $\mathbf{Crys}(X, A) \cong \mathbf{Crys}(X_{\mathrm{red}}, A)$

Theorem 3.8 *If f is proper, then $f_* \colon \mathbf{Crys}(X, A) \to \mathbf{Crys}(Y, A)$ is right adjoint to f^*.*

Properness is needed so that f^* maps coherent sheaves to coherent ones. We could use coherent cohomology to define higher derived functors of f^*. However we would really like to have Rf_* in the derived context.

3.4 Derived categories

Let $\mathbf{QCrys}(X, A)$ be the localization of the category of quasi-coherent τ-sheaves at nil-isomorphisms. (In the quasi-coherent context, a τ-sheaf is nilpotent iff it is the union of coherent nilpotent subsheaves.) As in general not all quasi-coherent τ-sheaves can be represented as filtered direct limits of coherent ones, the following theorem requires some work.

Theorem 3.9 $\mathbf{D}^b(\mathbf{Crys}(X, A)) \xrightarrow{\cong} \mathbf{D}^b_{\mathrm{crys}}(\mathbf{QCrys}(X, A))$.

Theorem 3.10 *For f proper, there exists a right derived functor*

$$Rf_* \colon \mathbf{D}^b_{\mathrm{crys}}(\mathbf{QCrys}(X, A)) \to \mathbf{D}^b_{\mathrm{crys}}(\mathbf{QCrys}(Y, A)),$$

which can be computed in via Čech-resolutions.

As $j_!$ and f^* are exact on crystals, they obviously give rise to fonctors in the derived setting. Furthermore, for f compactifiable, we define $f_! := R\bar{f}_* \circ j_!$, where $f = \bar{f} \circ j$, j is an open immersion and $\bar{f}$ is proper.

In the derived setting, one can prove the proper base change theorem, the projection formula, and various other expected compatibilities.

4 Trace formula

For $(\underline{\mathcal{F}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(X, A))$, we define

$$L(X, (\underline{\mathcal{F}}^\bullet), T) := \prod_i L(X, \underline{\mathcal{F}}^i, T)^{(-1)^i} \in A[[T]].$$

Theorem 4.1 *For $f \colon X \to Y$ compactifiable (which is always the case, if X, Y are of finite type over $\mathbf{F}_q$):*

$$L(X, (\underline{\mathcal{F}}^\bullet), T) = L(Y, f_!(\underline{\mathcal{F}}^\bullet), T).$$

Proof: We may assume that $A = K$, X smooth, affine, quasi-projective, $(\underline{\mathcal{F}}^\bullet)$ concentrated in degree zero, $\underline{\mathcal{F}}^0$ locally free. Then we can apply Anderson's trace formula, and the above result follows upon properly interpreting Anderson's nucleus. ∎

Corollary 4.2 *Conjecture 2.3 holds for any $X/\mathbf{F}_q$ of finite type, any normal domain A and any crystal on X over A.*

5 Global L-functions

Let $A = \mathbf{F}_q[t]$, let $\underline{\mathcal{F}}$ be as in Example 2.2.(ii). Thus $A_\infty = \mathbf{F}_q[[1/T]]$ and $K_\infty = \mathbf{F}_q((1/T))$.

Using our global theory, we can define 'motivic pieces' of $\underline{\mathcal{F}}$:

$$\mathcal{M}^{(i)}\underline{\mathcal{F}} := R^i\mathrm{char}_!\underline{\mathcal{F}} \in \mathbf{Crys}(\mathrm{Spec}A, A).$$

For $|x| > c$ one finds $L^{\mathrm{gl}}(X, \underline{\mathcal{F}}, s) = \prod_i L^{\mathrm{gl}}(\mathrm{Spec}A, \mathcal{M}^{(i)}\underline{\mathcal{F}}, s)$. Thus to prove 'meromorphy' it suffices to understand the $\mathcal{M}^{(i)}\underline{\mathcal{F}}$. One may further reduce this to understanding $L^{\mathrm{gl}}(X, \underline{\mathcal{G}}, s)$ for $\underline{\mathcal{G}}$ locally free in $\mathbf{Crys}(\mathrm{Spec}A, A)$.

Let $\underline{\mathcal{C}}$ be the Carlitz module (see Goss' book on function fields). This is a particular locally free rank one τ-sheaf on $\mathrm{Spec}A$ over A. Then for $|x| > c|\pi|^{-n}$, one has

$$L^{\mathrm{gl}}(\mathrm{Spec}A, \underline{\mathcal{G}}, (xT^{-n}, -n)) = L(\mathrm{Spec}A, \underline{\mathcal{G}} \otimes \underline{\mathcal{C}}^n, T)_{|T=x^{-1}}.$$

The cohomology theory of crystals shows that the right hand side is a polynomial in $A[x^{-1}]$ AND it can be used to show that for $n \to \infty$ the degree of this polynomial in the variable x^{-1} has order of magnitude $O(\log_q n)$. Thus one has good control over special values of Goss' global L-function.

The latter together with Goss' interpolation methods (modelled at those used in the construction of p-adic L-functions), suffices to prove the holomorphy on S_∞ of $L^{\mathrm{gl}}(\mathrm{Spec}A, \underline{\mathcal{G}}, s)$.

Corollary 5.1 *Conjecture 2.4 holds for $A = \mathbf{F}_q[t]$ and any $X/\mathbf{F}_q$ of finite type.*

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