# On the Stickelberger Ideal of a Composite Field of Some Quadratic Fields

by

**Noboru Aoki**

Max-Planck-Institut für Mathematik
Gottfried-Claren-Strasse 26
5300 Bonn 3, BRD

and

Department of Mathematics
Rikkyo University
Nishi-Ikebukuro
Tokyo 171, Japan

MPI/90-16

# ON THE STICKELBERGER IDEAL OF A COMPOSITE FIELD OF SOME QUADRATIC FIELDS

by

**Noboru Aoki**

**Max-Planck-Institut für Mathematik**

and

**Rikkyo University**

## Introduction.

Let $K$ be the cyclotomic field of $m$-th roots of unity and $G$ the Galois group of $K$ over the rational number field. The stickelberger ideal $S_K$ of K, which is an ideal of the group ring $\mathbf{Z}[G]$, is a quite interesting object in number theory in view of the following tow points, both of which are closely related. The first point is that $S_K$ annihilates the ideal class group of $K$ (Stickelberger's theorem). If we denote by $A_K$ the set of elements $\eta \in \mathbf{Z}[G]$ such that $(1+j)\eta \in s(G)\mathbf{Z}$, where $j$ is the complex conjugation and $s(G)$ denotes the sum in $\mathbf{Z}[G]$ of the elements of $G$, then $S_K$ is contained in $A_K$. One may expect that the index $[A_K : S_K]$ carries some information of the class number of K. In fact, when $m$ is a power of a prime number, Iwasawa **[I1]** showed that $[A_K : S_K]$ is precisely equal to $h_K^-$, the relative class number of $K/K^+$, where $K^+$ denotes the maximal real subfield of K. Later, Sinnott **[Sin1]** extended Iwasawa's results to general cyclotomic fields. In **[I2]** Iwasawa defined the Stickelberger ideal $S_k$ for arbitrary abelian field $k$, and Sinnott **[Sin2]** and Kimura-Horie **[K-H]** calculated the index $[A_k : S_k]$ in some cases.(See Theorem 1.1.) However, the precise formula of the index for general cases is not known. Our first result (Theorem 3.1) gives an explicit formula for the index when $k$ is a composite field of some quadratic fields.

The second point is that every element of $S_K$ appears as the infinity type of a Jacobi sum Hecke character of $K$. In §2 we define an index $\nu(\xi) = [\mathbf{Z}\xi : S \cap \mathbf{Z}\xi]$ for each element $\xi$ of $A_K$. It follows easily from the Iwasawa's finiteness theorem for the index $[A_K : S_K]$ (see Theorem 1.1) that $\nu(\xi)$ is also finite. By definition $\nu = \nu(\xi)$ is the smallest positive integer such that, for any algebraic Hecke character $\chi$ of a finite extension of K of infinity type $\xi$, $\chi^\nu$ is a twist of a Jacobi sum Hecke character of $K$. Our second result (Theorem 4.5) gives a formula for $\nu(\xi)$ for any element of $(A_K)^{Gal(K/k_0)}$, where $k_0$ is the composite field of all quadratic fields contained in $K$.

The contents of this paper is as follows. In §1 we will briefly review some fundamental properties of the Stickelberger ideal of abelian fields. In §2 we will review algebraic Hecke characters and Jacobi sum Hecke characters and study a certain relation between $\nu(\xi)$ and those characters. §3 and §4 will be devoted to the proof of Theorem 3.1 and 4.5 respectively.

## §1. The Stickelberger ideal.

In this section we recall mainly from [Sin1] and [Sin2] the definition and some fundamental properties of the Stickelberger ideal of an abelian field. Let $K = \mathbf{Q}(\zeta_m)$ be the cyclotomic field of $m$-th roots of unity and $G$ the Galois group $Gal(K/\mathbf{Q})$. For any $t \in (\mathbf{Z}/m\mathbf{Z})^\times$, we denote by $\sigma_t$ the element of $G$ characterized by $\zeta_m^{\sigma_t} = \zeta_m^t$. We identify $G$ with $(\mathbf{Z}/m\mathbf{Z})^\times$ via this correspondence.

Let $R'$ be a free abelian group generated by the elements of $\mathbf{Z}/m\mathbf{Z} \setminus \{0\}$:

$$R' = \mathbf{Z}[\mathbf{Z}/m\mathbf{Z} \setminus \{0\}].$$

Then $R'$ is a $G$-module via the natural action of $(\mathbf{Z}/m\mathbf{Z})^\times$ on $\mathbf{Z}/m\mathbf{Z} \setminus \{0\}$. Moreover we can regard it as a commutative ring: For any $a, b \in \mathbf{Z}/m\mathbf{Z} \setminus 0$, define $[a][b]$ to be $[ab]$ if $ab \neq 0$, and 0 otherwise. If we extend linearly this multiplication law to $R'$, then it becomes a commutative ring. Define

$$R = \{\sum c_a[a] \in R' | \sum c_a a = 0\}.$$

Then $R$ is a subring of $R'$ and stable under the action of $G$.

For any element $a \in \mathbf{Z} \setminus \{0\}$, we define a Stickelberger element $\theta(a) \in \mathbf{Q}[G]$ by

$$\theta(a) = \sum_{t \in (\mathbf{Z}/m\mathbf{Z})^\times} \langle ta/m \rangle \sigma_t^{-1},$$

where $\langle ta/m \rangle$ denotes the element of $\frac{1}{m}\mathbf{Z}$ such that $0 < \langle ta/m \rangle < 1$ and $m\langle ta/m \rangle \equiv ta \pmod{m}$. If $\alpha = \sum c_a[a]$ is an element of $R'$, we set

$$\theta(\alpha) = \sum c_a \theta(a).$$

Then $\theta$ is a $G$-homomorphism from $R'$ to $\mathbf{Q}[G]$. Let $S'_K = \theta(R')$. The Stickelberger ideal $S_K$ of $K$ is defined by

$$S_K = S'_K \cap \mathbf{Z}[G].$$

It is easy to see that $S_K = \theta(R)$. Let $k$ be a subfield of $K$ and $\Gamma$ its Galois group over $\mathbf{Q}$. In [**I2**] Iwasawa defined the Stickelberger ideal $S_k$ of $k$ by

$$S_k = res_{K/k}(S_K),$$

where $res_{K/k} : \mathbf{Z}[G] \longrightarrow \mathbf{Z}[\Gamma]$ denotes the restriction map. If we set

$$S'_k = res_{K/k}(S'_K),$$

then it is easy to see that $S_k = S'_k \cap \mathbf{Z}[\Gamma]$. Moreover the definition of $S_k$ and $S'_k$ do not depnd on the choice of the cyclotomic field $K$. The ideal $S_k$ has the following remarkable property which is often called *Stickelberger's relation*. (See [**L1**], [**We2**] and [**Sin2**].)

**Theorem 1.1.** *The Stickelberger ideal $S_k$ annihilates the ideal class group of $k$. That is, for any ideal $\mathfrak{a}$ of $k$ and for any element $\eta$ of $S_k$, the ideal $\mathfrak{a}^\eta$ is a principal ideal.*

For any finite Galois extension $L$ of $\mathbf{Q}$, we denote by $A_L$ the set of element $\xi \in \mathbf{Z}[Gal(L/\mathbf{Q})]$ such that $(1+j)\xi = ws(Gal(L/\mathbf{Q}))$ with an integer $w$, where $s(Gal(L/\mathbf{Q}))$ is the summation in $\mathbf{Z}[Gal(L/\mathbf{Q})]$ of all the elements óf $Gal(L/\mathbf{Q})$ and $j$ denotes the complex conjugation. It is known that $S_k$ is a $G$-submodule of $A_k$ ([**Sin2**], Lemma 2.1). The integer $w$ is called the *weight* of $\xi$. In [**I1**], Iwasawa calculated the index $[A_K : S_K]$ when $m$ is a power of a prime number. Sinnott ([**Sin1**],[**Sin2**] and [**Sin3**]) extended Iwasawa's results to more general cases. (See also [**K-H**].) To state the results we need some notation. Let $E_k$ and $W_k$ be the group of units of $k$ and the group of roots of unity in $k$ respectively. Let $k^+$ be the maximal real field in $k$, and set $E_k^+ = E_k \cap k^+$. Let $Q_k = [E_k : W_k E_k^+]$, and let $h_k^-$ be the relative class number of $k/k^+$. Then their results may be sammerized as follows. (For more precise statements and further results, see the references in the theorem.)

**Theorem 1.2.** *The index $[A_k : S_k]$ is finite and of the following form:*

$$[A_k : S_k] = \frac{h_k^-}{Q_k} \cdot c_k,$$

*where $c_k$ is a positive integer divisible by only the primes dividing the order $|\Gamma|$ of $\Gamma$. Let $r$ be the number of primes which ramifies in $k$. Then the following assertions hold.*

(1) *If $k = K$ and $r \leq 2$, then $c_k = Q_k$. (Iwsawa[**I1**])*

(2) *If $k = K$ and $r > 2$, then $c_k = 2^{2^{r-2}}$. (Sinnott [**Sin1**])*

(3) *If $r \leq 2$, then $c_k = 1$ or $2$. (Sinnott [**Sin2**], Kimura-Horie [**K-H**])*

(4) *If $r = 3$, then $c_k = 2^n$ for some $n \geq 0$. (Kimura-Horie [**K-H**], Sinnott [**Sin3**])*

(5) *If $\Gamma$ is cyclic, then $c_k = 1$. (Sinnott [**Sin2**])*

(6) *If $\Gamma$ is the direct product of its inertia groups, then $c_k = 2^n$ for some $n \geq 0$. (Sinnott [**Sin2**])*

**Remark 1.3.** Although $c_k$ is a power of 2 in all cases listed above, this is in general not the case. For detail, see [**Sin2, Sin3**], [**K-H**].

## §2. Algebraic Hecke characters and Jacobi sum Hecke characters.

In this section we recall some basic facts about algebraic Hecke characters and Jacobi sum Hecke characters. For the detail, see **[D]**, **[L2]** or **[Scha]**. Let $L$ and $E$ be two number fields and $\mathfrak{f}$ a non-zero integral ideal of $L$. Let $Hom(L, \bar{E})$ be the set of embeddins of $L$ into a fixed algebraic closure $\bar{E}$ of $E$. A group homomomrphism

$$\chi : I_L(\mathfrak{f}) \longrightarrow E^{\times}$$

from the group $I_L(\mathfrak{f})$ of the ideals of $L$ prime to $\mathfrak{f}$ to the multiplicative group of $E$ is called an *Algebraic Hecke character of $L$ with values in $E$*, if

$$\chi((\alpha)) = \prod_{\sigma \in Hom(L,\bar{E})} (\alpha^{\sigma})^{n_{\sigma}},$$

for any $\alpha \in K^{\times}$ with $\alpha \equiv 1 (mod. \mathfrak{f})$. The elemnt $\xi = \sum n_{\sigma}\sigma$ of $\mathbf{Z}[Hom(L, \bar{E})]$ is called ***the infinity type*** of $\chi$ and will be denoted by $u(\chi)$ in this paper. We denote by $\mathcal{G}_L(E)$ the group of algebraic Hecke characters of $L$ with values in $E$.

In what follows we assume that $E = K$ and $L$ is a finite Galois extension of $\mathbf{Q}$ containing $K$. In this case we have a isomorphism $\mathbf{Z}[Hom(L, \bar{E})] \cong \mathbf{Z}[Gal(L/\mathbf{Q})]$. Let $A_L$ be the set of element $\xi \in \mathbf{Z}[Gal(L/\mathbf{Q})]$ such that $(1 + j)\xi \in s(Gal(L/\mathbf{Q}))\mathbf{Z}$. It is well known that $u(\chi)$ lies in $A_L$ for any algebraic Hecke character of $L$. The correspondence $u$ which associates $\chi$ with $u(\chi)$ defines a homomorphism

$$u : \mathcal{G}_L(K) \longrightarrow A_L.$$

If $\varepsilon \in Hom(G(L_{ab}/L), \mathbf{C}^{\times})$, then by class field theory $\varepsilon$ can be regarded as an algebraic Hecke character of $L$ with the trivial infinity type. Conversely we have

**Proposition 2.1.** $Ker(u) = Hom(G(L_{ab}/L), \mathbf{C}^{\times})$.

**Proof:** See **[Iw2]**, **[Schm]**.

Among algebraic Hecke characters of $K$ there are specially interesting characters, called *Jacobi sum Hecke cheracters*. We recall the definition in what follows. Let $p$ be a prime number which does not divide $m$. Let $\mathfrak{p}$ be a prime ideal of $K$ lying above $p$, and let $\mathbf{F}_q$ be the residue field at $\mathfrak{p}$. Let $\chi_\mathfrak{p}$ be the character of $\mathbb{F}_q^\times$, with values in the group of $m$-th roots of unity, characterized by

$$\chi_\mathfrak{p}(x) \equiv x^{\frac{q-1}{m}} \pmod{\mathfrak{p}},\ x \in \mathbf{F}_q^\times.$$

For any element $a_1, ..., a_n \in \mathbf{Z}/m\mathbf{Z} \setminus \{0\}$ such that $a_1 + ... + a_n = 0$, we set

$$J_{a_1,...,a_n}(\mathfrak{p}) = (-1)^n \sum \chi_\mathfrak{p}(x_1)^{a_1} ... \chi_\mathfrak{p}(x_{n-1})^{a_{n-1}}$$

where the summation runs over (n-1)-tuples $(x_1, ..., x_{n-1}) \in (\mathbb{F}_q^\times)^{n-1}$ such that $1 + x_1 + ... + x_{n-1} = 0$. If $\alpha \in R$, then there exist elements $a_1, ..., a_r, b_1, ..., b_s \in \mathbf{Z}/m\mathbf{Z} \setminus \{0\}$ such that $a_1 + ... + a_r = b_1 + ... + b_s = 0$ and $\alpha = \sum[a_i] - \sum[b_j]$. We set

$$J_\alpha(\mathfrak{p}) = J_{a_1,...,a_r}(\mathfrak{p})/J_{b_1,...,b_s}(\mathfrak{p}),$$

and extend it multiplicatively to get a homomorphism from $I_K$ to $K^\times$. This definition depends only on $\alpha$ but not on the expression of $\alpha$.

**Theorem 2.2.** *(Weil [We2]) For any $\alpha \in R$, $J_\alpha$ is an algebraic Hecke character of $K$. Moreover the infinity type of $J_\alpha$ is given by $\theta(\alpha)$.*

Now for any element $\xi$ of $A_K$ we set

$$\nu(\xi) = [\mathbf{Z}\xi : S_K \cap \mathbf{Z}\xi].$$

This index is finite since the inclusion map $\mathbf{Z}\xi \hookrightarrow A_K$ induces an injection $\mathbf{Z}\xi/(S_K \cap \mathbf{Z}\xi) \hookrightarrow A_K/S_K$. In particular $\nu(\xi)$ divides the index $[A_K : S_K]$. If $\chi \in \mathcal{G}_L(K)$ and

$u(\chi) = cor_{L/K}(\xi)$, then $u(\chi^{\nu(\xi)}) = cor_{L/K}(\nu(\xi)\xi) \in cor_{L/K}(S_K)$ by definition, hence $\chi^{\nu(\xi)} = \varepsilon J_\alpha \circ N_{L/K}$ for a character $\varepsilon \in Hom(Gal(L^{ab}/L), \mathbf{C}^\times)$ by Prposition 2.2 . Thus $\nu(\xi)$ measures the difference between $\chi$ and Jacobi sum Hecke characters.

Fix an element $\xi$ of $A_K$. By the general theory of algebraic Hecke chracters there exists a finite extension $L$ of $K$ for which the following condition holds:

$$u(\chi) = cor_{L/K}(\xi) \quad \text{for some} \quad \chi \in \mathcal{G}_L(K).$$

Let $L_\xi$ be the smallest field among such $L's$. Then the theory of complex multiplication for CM-motives (see **[D]**, **[DMOS]**, **[Scha]**, **[B]**), which genererlize the complex multiplication theory of abelian varieties of CM-type due to Shimura and Taniyama (**[S-T]**), says that $L_\xi$ is the unramified abelian extension of $K$ corresponding via class field theory to the following subgroup

$$P_K(\xi) = \{ \ \mathfrak{a} \in I_K \mid \mathfrak{a}^\xi = (\mu), \ N(\mathfrak{a})^w = \mu\bar{\mu} \ \text{ for some } \ \mu \in K^\times \}$$

of the ideal group $I_K$ of $K$, where $\mathfrak{a}^\xi = \prod_\sigma (\mathfrak{a}^\sigma)^{n_\sigma}$ if $\xi = \sum n_\sigma \sigma$ and $w$ is the weight of $\xi$. We define the annihilator of the ideal class group $Cl_K$ of $K$ by

$$\tilde{S}_K = \{\eta \in A_K \mid \mathfrak{a}^\eta \sim 1 \ \text{ for any } \ \mathfrak{a} \in I_K\}.$$

Then, by the Stickelberger's relation (Theorem 1.1), $S_K$ is contained in $\tilde{S}_K$. In general, the structure of $\tilde{S}_K/S_K$ is not known. Let

$$\tilde{\nu}(\xi) = [\mathbf{Z}\xi : \tilde{S}_K \cap \mathbf{Z}\xi].$$

Obviously $\tilde{\nu}(\xi)$ is a divisor of $\nu(\xi)$. The following proposition is not difficult, and we leave it to the reader.

**Proposition 2.3.** *The quotient group $\tilde{S}_K/S_K$ contains a cyclic group of order $\nu(\xi)/\tilde{\nu}(\xi)$.*

Recall that the exponent of a finite abelian group $X$ is defined to be the smallest integer $n$ such that $nx = 0$ for all $x \in X$.

**Proposition 2.4.** *The exponent of $I_K/P_K(\xi)$ is $\tilde{\nu}(\xi)$. In particular $\tilde{\nu}(\xi)$ divides $[L_\xi : K]$.*

**Proof:** We consider a paring

$$I_K \times \mathbf{Z}\xi \longrightarrow I_K^\xi, \qquad (\mathfrak{a}, n\xi) \longmapsto \mathfrak{a}^{n\xi},$$

where $I_K^\xi = \{\mathfrak{a}^\xi | \mathfrak{a} \in I_K\}$. This pairing induces a non-degenerate pairing

$$I_K/P_K(\xi) \times \mathbf{Z}\xi/(\tilde{S}_K \cap \mathbf{Z}\xi) \longrightarrow I_K^\xi.$$

Since $I_K^\xi \cong I_K/P_K(\xi)$ and $\mathbf{Z}\xi/(\tilde{S}_K \cap \mathbf{Z}\xi) \cong \mathbf{Z}/\tilde{\nu}(\xi)\mathbf{Z}$, we get an isomorphism

$$I_K/P_K(\xi) \cong Hom(\mathbf{Z}/\tilde{\nu}(\xi)\mathbf{Z}, I_K/P_K(\xi)).$$

This proves the first statement. The second statement follows from this and the isomorphism $I_K/P_K(\xi) \cong Gal(L_\xi/K)$. Q.E.D.

As an illustration of the above proposition, we consider the case where $K$ contains an imaginary qudratic field $k = \mathbf{Q}(\sqrt{-m})$ with the discriminant $-m$. Let $H = G(K/k)$ and $\xi = s(H) \in A_K$ the sum of elements of $H$. Then the above proposition says that $\tilde{\nu}(\xi)$ divides $h_k/2^{r-1}$, where $h_k$ denotes the class number of $k$ and $r$ is the number of prime number dividing $m$. Indeed, if we denotes by $Cl_K$ and $Cl_k$ the ideal class group of $K$ and $k$ respectively, then the subgroup of $Cl_K$ which corresponds to $L_\xi$ is the kernel of the norm map $N_{K/k} : Cl_K \longrightarrow Cl_k$. Therefore we have

$$[L_\xi : K] = |N_{K/k}(Cl_K)| = [k^{ur} : k^{ur} \cap K],$$

where $k^{ur}$ denotes the Hilbert class field of $k$. The genus theory of quadratic fields implies that the last index is $h_k/2^{r-1}$.

## §3. The structure of $S_k$ and the index $[A_k : S_k]$.

In this section and next section we will assume that $ord_2(m) = 0, 2$ or $3$ and $ord_p(m) = 0$ or $1$ for any odd prime number $p$. Let $k_0$ be the composite field of all quadratic fields in $K$ and put $H_0 = Gal(K/k_0)$. Let $k$ be a subfield of $k_0$, which will be assumed to be imaginary throughout this section. Thus the degree $[k : \mathbb{Q}] = 2^n$ for an integer $n$ such that $1 \leq n \leq r$, where $r$ is the number of prime factors of $m$. We denote by $k^+$ the maximal real subfield of $k$. Let $D_k$ and $D_{k^+}$ be the discriminants of $k$ and $k^+$ respectively. We set

$$D_k^- = D_k/D_{k^+}.$$

Let $\Gamma = Gal(k/\mathbb{Q})$ and $\hat{\Gamma}$ the character group of $\Gamma$. We denote by $\hat{\Gamma}^-$ the set of odd characters of $\Gamma$, i.e.

$$\hat{\Gamma}^- = \{\chi \in \hat{\Gamma} \mid \chi(j) = -1\},$$

which is non-empty since $k$ is imaginary. For each character $\chi \in \hat{\Gamma}$, let $d_\chi$ be the conductor of $\chi$. Then $ord_2(d_\chi) = 0, 2$ or $3$, and $ord_p(d_\chi) = 0$ or $1$ for any odd prime $p$. By the conductor-discriminant formula (see [**Wa**], Theorem 3.11), we find

$$D_k^- = \pm \prod_{\chi \in \hat{\Gamma}^-} d_\chi. \tag{1}$$

If $a$ is an integer, we define a non-negative intger $v(a)$ by

$$v(a) = \sum_{p|a} ord_p(a).$$

Now the main theorem in this section can be stated as follows.

**Theorem 3.1.** *Let $h_k^-$ be the relative class number of $k/k^+$ and $Q_k$ the unit index of $k$ defined in §1. Let $a_k$ be the number of odd character with odd conductor if $m$*

*is even, and* $a_k = 0$ *otherwise. Then*

$$[A_k : S_k] = \frac{h_k^-}{Q_k} \cdot 2^{(2v(m)+1-n)2^{n-2} - v(D_k^-) - a_k}.$$

Let $\Lambda_k = \bigoplus_{\chi \in \hat{\Gamma}^-} \mathbf{Z}$, then we have a ring homomrphism

$$\psi_k : \mathbf{Q}[\Gamma] \longrightarrow \Lambda_k \otimes \mathbf{Q}$$

which sends $[\sigma]$ to $(..., \chi(\sigma), ...)_{\chi \in \hat{\Gamma}} \in \Lambda_k$ for any $\sigma \in \Gamma$. Let $e^- = (1-j)/2 \in \mathbf{Q}[\Gamma]$. Then $\psi_k$ induces an injection from $e^-\mathbf{Q}[\Gamma]$ into $\Lambda_k \otimes \mathbf{Q}$.

**Proposition 3.2.** *The image* $\psi_k(e^- A_k)$ *of* $e^- A_k$ *is a sublattice of* $\Lambda_k$. *The index is given by*

$$[\Lambda_k : \psi_k(e^- A_k)] = 2^{(n-1)2^{(n-2)}}.$$

.

**Proof:** The first statement is clear since $\chi(e^-) = 1$ for any $\chi \in \hat{\Gamma}^-$. To compute the index we define a integral matrix $M$ of size $2^{n-1}$ by

$$M = (\chi(\sigma))_{\chi \in \hat{\Gamma}^-, \sigma \in \Gamma/<j>} \cdot$$

Then it follows immediately from the definition of $\psi_k$ that $\psi_k(e^- A_k) = M\Lambda_k$. Therefore the index $[\Lambda_k : \psi_k(e^- A_k)]$ equals $|det(M)|$. Since $M^t M = 2^{n-1} I$, we have $det(M) = \pm 2^{(n-1)2^{(n-2)}}$. This completes the proof. Q.E.D.

Recall that $S_k$ is an ideal of $A_k$, hence $e^- S_k \subset e^- A_k$. We want to know the image of $e^- S_k$ by $\psi_k$. For each $\chi \in \hat{\Gamma}$, we denote by $B_{1,\chi}$ the generelized Bernoulli number. Then it is well known that $B_{1,\chi}$ equals the class number of the quadratic field corresponding to $\chi$ if $\chi \in \hat{\Gamma}^-$. The following proposition is fundamental in the proof of Therem 3.1.

**Proposition 3.3.** *For each $\chi \in \hat{\Gamma}^-$, let $\varepsilon_\chi = 1$ if $m$ is even and $d_\chi$ is odd, and $\varepsilon_\chi = 0$ othewise. Then*

$$\psi_k(e^- S'_k) = \bigoplus_{\chi \in \hat{\Gamma}^-} 2^{v(m/d_\chi)-\varepsilon_\chi} B_{1,\chi} \mathbf{Z}. \tag{2}$$

**Proof:** If we denote by $proj_k$ the projection map from $\Lambda_{k_0} \otimes \mathbb{Q}$ to $\Lambda_k \otimes \mathbb{Q}$, then $\psi_k(e^- S_k) = proj_k(\psi_{k_0}(e^- S_{k_0}))$. It therefore suffices to show the proposition for $k = k_0$. The idea of the proof is to construct an element $\alpha_\chi$ of $R'^{H_0}$ for each $\chi \in \hat{\Gamma}$, which satisfies the following condition.

$$\chi'(\theta(\alpha_\chi)) = \begin{cases} |H_0| 2^{v(m/d_\chi)-\varepsilon_\chi} B_{1,\chi}, & \text{if } \chi' = \chi, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

If $\chi \in \hat{\Gamma}$, then $\chi$ has the decomposition $\chi = \chi_1 \ldots \chi_s$, where $\chi_i$'s $\in \hat{\Gamma}$ are the characters uniquely determined by the following property:

$$d_{\chi_i} = 4, 8 \ \text{ or an odd prime,}$$

$$(d_{\chi_i}, d_{\chi_j}) = 1, \quad i \neq j.$$

For each $\chi \in \hat{\Gamma}$, we define a subgroup $H_\chi$ of $(\mathbf{Z}/m\mathbf{Z})^\times$ by

$$H_\chi = \left\{ t \in H_0 \;\middle|\; \begin{array}{rcl} t & \equiv & 1 (mod. m/d_\chi), \text{ and} \\ \chi_i(t) & = & 1 \text{ for all } i \end{array} \right\},$$

and set

$$\gamma_\chi = \sum_{t \in H_\chi} [t].$$

Then $\gamma_\chi$ is an element of $R'$, and clearly $[m/d_\chi]\gamma_\chi \in R'^{H_0}$. For any divisor $d$ of $m$, we denote by $Q_d$ (resp. $Q_d^\circ$) the submodule of $R'^{H_0}$ generated by $[m/d_\chi]\gamma_\chi$ for all $\chi \in \hat{\Gamma}$ with $d_\chi | d$ (resp. $d_\chi | d$ and $d_\chi < d$).

We now need two lemmas below.

**Lemma 3.4.** *Let $d$ be any divisor of $m$ and $t$ any element of $(\mathbf{Z}/m\mathbf{Z})^\times$. Let $\chi$ be any character of $\Gamma$. Then we have $\chi(\theta([m/d]\,[t])) = 0$ unless $d|d_\chi$ and $\chi \in \tilde{\Gamma}^-$, in which case we have*

$$\chi(\theta(\left[\frac{m}{d}\right][t])) = \frac{\varphi(m)}{\varphi(d)}\chi(t) \prod_{p|d_\chi/d} (1-\chi(p)) \cdot B_{1,\chi}.$$

**Proof:** See for example [**L1**] or [**A**].

**Lemma 3.5.** *Let $\chi_0 \in \hat{\Gamma}$ be any character with an odd conductor $d := d_{\chi_0}$. Let $\beta$ be any element of $R'$ such that, for any $\chi \in \hat{\Gamma}$, $\chi(\theta(\beta)) = 0$ if $d_\chi \nmid d$ and $\chi(\theta(\alpha)) \in |H_0|2^{v(m/d_\chi)-\varepsilon}B_{1,\chi}\mathbf{Z}$ if $d_\chi|d$, where $\varepsilon = 1$ if $m$ is even, and 0 otherwise. Then there exists an element $\gamma \in Q_d^\circ$ such that*

$$\chi(\theta(\beta+\gamma)) = 0 \tag{4}$$

*for any $\chi \neq \chi_0$.*

**Proof:** Put

$$\gamma = \sum_{\substack{\chi' \in \hat{\Gamma} \\ d_{\chi'}|d,\, d_{\chi'}<d}} c_{\chi'} \left[\frac{m}{d_{\chi'}}\right] \gamma_{\chi'} \quad \in Q_d^\circ.$$

We want to show that we can take integers $c_{\chi'}$'s so that $\gamma$ has the property (4). It follows from Lemma 3.4 that $\chi(\theta(\gamma)) = 0$ if $d_\chi \nmid d$, hence (4) holds in this case. If $d_\chi|d$, then by the same lemma

$$\chi(\theta(\gamma)) = \sum_{\substack{\chi' \in \hat{\Gamma} \\ d_\chi|d_{\chi'}|d}} c_{\chi'} \frac{\varphi(m)}{\varphi(d_{\chi'})}|H_{\chi'}| \prod_{p|d_{\chi'}/d_\chi} (1-\chi(p)) \cdot B_{1,\chi}. \tag{5}$$

Since $|H_{\chi'}| = \varphi(d_{\chi'})/2^{v(d_{\chi'})}$ and $|H_0| = \varphi(m)/2^{v(m)-\epsilon}$, we have

$$\frac{\varphi(m)}{\varphi(d_{\chi'})}|H_{\chi'}| = |H_0|2^{v(m/d_\chi)-\epsilon},$$

hence the right hand side of (5) is equal to

$$|H_0|2^{v(m/d_\chi)-\epsilon}B_{1,\chi} \sum_{\substack{\chi'\in\hat{\Gamma} \\ d_\chi | d_{\chi'} | d}} c_{\chi'} \prod_{p | d_{\chi'}/d_\chi} \frac{1-\chi(p)}{2}.$$

Hence (4) is equivalent to the following equality

$$b_\chi + c_\chi + \sum_{\substack{\chi'\in\hat{\Gamma} \\ d_\chi | d_{\chi'} | d \\ d_\chi < d_{\chi'} < d}} c_{\chi'} \prod_{p | d_{\chi'}/d_\chi} \frac{1-\chi(p)}{2} = 0,$$

where $b_\chi$ is an integer determined by $\chi(\theta(\beta)) = |H_0|2^{v(m/d_\chi)-\epsilon}B_{1,\chi}b_\chi$. Since $c_{\chi'}$ and $(1-\chi'(p))/2$ are integers, we can take integers $c_{\chi'}$ inductively. Q.E.D.

We continue the proof of Proposition 3.3. Take a character $\chi$ and fix it. First suppose that $d := d_\chi$ is odd. Put

$$\beta = \left[\frac{m}{d}\right]\gamma_\chi.$$

Then one can easily check that $\beta$ satisfies the condition in Lemma 3.5. Let $\gamma \in Q_d^\circ$ be the element obtained by applying that lemma to $\beta$, and put

$$\alpha_\chi = \beta + \gamma$$

Then $\chi(\theta(\alpha_\chi)) = \chi((\theta(\beta))) = |H_0|2^{v(m/d)-\epsilon_\chi}$ and $\chi'(\theta(\alpha_\chi)) = 0$ for any $\chi' \neq \chi$, hence $\alpha_\chi$ satisfies (3).

Next consider the case where $d$ is even, say $e = 2^{ord_2(d)} = 4$ or $8$. Let $\chi_1$ be the unique element of $\hat{\Gamma}^-$ with $d_{\chi_1} = d/e$. We put

$$\beta' = \begin{cases} \left[\frac{m}{d}\right]\gamma_\chi, & \text{if } \chi_1(2) = 1 \\ \left[\frac{m}{d}\right]\gamma_\chi + \left[-\frac{m}{d/e}\right]\gamma_{\chi_1}, & \text{if } \chi_1(2) = -1. \end{cases}$$

Then it is easy to see that $\chi(\theta(\beta')) = |H_0|2^{v(m/d)}$ and $\chi'(\theta(\beta')) = 0$ if $\frac{d}{e}|\delta|d$ and $\delta < d$. Let $\gamma' \in Q^{\circ}_{d/e}$ be the element obtained by applying Lemma 3.5 to $\beta'$. If we put

$$\alpha_{\chi} = \beta' + \gamma',$$

then $\alpha_{\chi}$ satisfies the condition (3).

Now we note that there exists an element $\eta_{\chi} \in S'_{k_0}$ such that $cor_{K/k_0}(\eta_{\chi}) = \theta(\alpha_{\chi})$. Indeed this follows from the fact that $\alpha_{\chi} \in R'^{H_0}$ and the relation

$$cor_{K/k_0}(S'_{k_0}) = cor_{K/k_0}(res_{K/k_0}(S'_K)) = \theta(R'^{H_0}). \tag{6}$$

Since $cor_{K/k_0}$ is a $G$-module homomorphism, we find $[H_0]\chi'(\eta_{\chi}) = \chi'(\theta(\alpha_{\chi}))$, hence

$$\chi'(\eta_{\chi}) = \begin{cases} 2^{v(m/d_{\chi})-e_{\chi}}B_{1,\chi}, & \text{if } \chi' = \chi \\ 0, & \text{otherwise.} \end{cases}$$

The proof of Proposition 3.3 is complete if we show that $\eta_{\chi}$'s generate $S'_{k_o}$ as a $G/H_0$-module. But this is clear from (6) since $R'^{H_0}$ is generated by $\alpha_{\chi}$'s as a $G/H_0$-module. Q.E.D.

Let $U_k$ a the submodule of $\mathbf{Q}[\Gamma]$ defined in [**Sin2**], Corollarly to Proposition 2.2. We do not give the definition in this paper. What we need here is the following relation between $S'_k$ and $U_k$:

$$\chi(S'_k) = \chi(U_k)B_{1,\chi}\mathbf{Z} \ \text{ for any } \ \chi \in \hat{\Gamma}.$$

From this and Proposition 3.3 we have

**Corollarly 3.6.** *Notation being as above, we have*

$$\psi_k(e^- U_k) = \bigoplus_{\chi \in \hat{\Gamma}^-} 2^{v(m/d_{\chi})-e_{\chi}}\mathbf{Z}.$$

**Proof of Theorem 3.1:** For any two submodules $X, Y$ of $A_k$ we denote by $(X : Y)$ the generalized index. (See [**Sin1**] for the definition.) By [**Sin2**], Theorem 2.2, we have

$$(7) \qquad [A_k : S_k] = \frac{h_k^-}{Q_k} \cdot (e^- A_k : e^- U_k).$$

If we recall that the map $\psi_k$ is injective on $e^-\mathbf{Q}[\Gamma]$, we can easily see that

$$(e^- A_k : e^- U_k) = \frac{[\Lambda_k : \psi_k(e^- U_k)]}{[\Lambda_k : \psi_k(e^- A_k)]}.$$

In Proposition 3.3 we have already calculated the denominator. As for the numerator, by Corollarly 3.6, we have

$$[\Lambda_k : \psi_k(e^- U_k)] = \prod_{\chi \in \hat{\Gamma}^-} 2^{v(m/d_\chi) - \epsilon_\chi} = 2^{v(m)2^{n-1} - v(D_k^-) - a_k}.$$

Here we have used the following relation:

$$\sum_{\chi \in \hat{\Gamma}^-} v(d_\chi) = v(D_k^-),$$

which is clear from (1). Hence

$$(e^- A_k : e^- U_k) = 2^{(2v(m) - n + 1)2^{n-2} - v(D_k^-) - a_k}.$$

Combining this and (7), we obtain the desired formula. Q.E.D.

## §4. Calculation of $\nu(\xi)$.

Let $K$ and $k_0$ be as in §3. We denote by $\Gamma_0$ the Galois group $Gal(k_0/\mathbf{Q})$. For each $a \in \mathbf{Z}/m\mathbf{Z}$, define $\tilde{\theta}(a) \in \mathbf{Q}[G]$ by

$$\tilde{\theta}(a) = \sum_{t \in (\mathbf{Z}/m\mathbf{Z})^\times} (\left\langle \frac{ta}{m} \right\rangle - \frac{1}{2})\sigma_t^{-1}.$$

Note that $\tilde{\theta}(-a) = -\tilde{\theta}(a)$, hence $\tilde{\theta}(a) \in e^- S'_K$. Extending it linearly, we obtain a $G$-module homomorphism

$$\tilde{\theta} : R' \longrightarrow e^- S'_K.$$

Clearly $\tilde{\theta}$ is surjective. Let $B$ be the kernel of $\tilde{\theta}$. Thus we have the following short exact sequence of $G$-modules

$$0 \longrightarrow B \longrightarrow R' \xrightarrow{\tilde{\theta}} e^- S'_K \longrightarrow 0.$$

Taking the cohomology groups $H^*(H_0, -)$, we obtain a long exact sequence

$$0 \longrightarrow B^{H_0} \longrightarrow R'^{H_0} \xrightarrow{\tilde{\theta}} (e^- S'_K)^{H_0} \xrightarrow{\delta} H^1(H_0, B) \longrightarrow H^1(H_0, R') \longrightarrow .$$

From this and the next lemma we obtain the following exact sequence

$$0 \longrightarrow \tilde{\theta}(R'^{H_0}) \longrightarrow (e^- S'_K)^{H_0} \xrightarrow{\delta} H^1(H_0, B) \longrightarrow 0. \tag{1}$$

**Lemma 4.1.** $H^1(H_0, R') = 0$.

**Proof:** For each divisor $d$ of $m$, let $G_d = Gal(K/\mathbf{Q}(\zeta_d))$. Then $R'$ is isomorphic to

$$\bigoplus_{\substack{d|m \\ d<m}} \mathbf{Z}[G]^{G_d}$$

as a $G$-module, hence

$$H^1(H_0, R') \cong \bigoplus_{\substack{d|m \\ d<m}} H^1(H_0, \mathbf{Z}[G]^{G_d}).$$

The inflation-restriction exact sequence shows that the sequence

$$0 \longrightarrow H^1(H_0/H_0 \cap G_d, \mathbf{Z}[G]^{G_d}) \longrightarrow H^1(H_0, \mathbf{Z}[G]^{G_d}) \longrightarrow H^1(H_0 \cap G_d, \mathbf{Z}[G]^{G_d})$$

is exact. The first group is trivial since $\mathbf{Z}[G]^{G_d}$ is a free $H_0/H_0 \cap G_d$-module, and the last one is also trivial since $H_0 \cap G_d$ acts trivially on $\mathbf{Z}[G]^{G_d}$. Therefore $H^1(H_0, \mathbf{Z}[G]^{G_d}) = 0$ for any $d$. This proves the lemma. Q.E.D.

Now, for any $\xi \in A_K^{H_0}$ with weight $w$, let $V_\xi$ be the image of $(e^- S'_K)^{H_0} \cap \mathbf{Z}\xi'$ under the map $\delta$ in (1), where $\xi' = \xi - \frac{w}{2}s(G)$. We then have an exact sequence with ovbious maps

$$0 \longrightarrow V_\xi \longrightarrow \mathbf{Z}\xi'/(\tilde{\theta}(R'^{H_0}) \cap \mathbf{Z}\xi) \longrightarrow \mathbf{Z}\xi'/(e^- S'_K \cap \mathbf{Z}\xi') \longrightarrow 0.$$

Since $\mathbf{Z}\xi'/(e^- S'_K \cap \mathbf{Z}\xi') \cong \xi\mathbf{Z}/(S_K \cap \mathbf{Z}\xi)$, we have

$$\nu(\xi) = \frac{[\mathbf{Z}\xi' : \tilde{\theta}(R'^{H_0}) \cap \mathbf{Z}\xi']}{|V_\xi|}.$$

**Proposition 4.2.** *Let $\varepsilon_\chi$ be as in Proposition 3.3. Let $\xi_0$ be an element of $\mathbf{Z}[\Gamma_0]$ such that $\xi = cor_{K/k_0}(\xi_0)$. Then*

$$[\mathbf{Z}\xi' : \tilde{\theta}(R'^{H_0}) \cap \mathbf{Z}\xi'] = \underset{\chi \in \hat{\Gamma}_0^-, \chi(\xi_0) \neq 0}{L.C.M.} \left\{ \frac{2^{v(m/d_\chi) - \varepsilon_\chi} B_{1,\chi}}{(2^{v(m/d_\chi) - \varepsilon_\chi} B_{1,\chi},\ \chi(\xi_0))} \right\}.$$

Before going into the proof of the proposition, we state an elementary lemma. We leave the proof to the reader.

**Lemma 4.3.** *Let $a_1, ..., a_n, b_1, ..., b_n$ be positive integers and put $\alpha = (a_1, ..., a_n) \in \mathbf{Z}^n$. Let $X = \mathbf{Z}\alpha$ and $Y = b_1\mathbf{Z} \oplus ... \oplus b_n\mathbf{Z}$. Then*

$$[X : X \cap Y] = L.C.M.\left\{\frac{b_1}{(a_1, b_1)}, ..., \frac{b_n}{(a_n, b_n)}\right\}.$$

**Proof of Proposition 4.2:** For any element $\eta$ of $\mathbf{Z}[G]^{H_0} = cor_{K/k_0}(\mathbf{Z}[\Gamma_0])$, take any element $\eta_0$ of $\mathbf{Z}[\Gamma_0]$ such that $\eta = cor_{K/k_0}(\eta_0)$. Let $\psi_0(\eta) = \psi_{k_0}(\eta_0)$, where $\psi_{k_0}$ is the map defined in §3. Then $\psi_0$ defines an injection

$$\psi_0 : e^-\mathbf{Q}[G]^{H_0} \hookrightarrow \bigoplus_{\chi \in \hat{\Gamma}_0^-} \mathbf{Q}.$$

Note that both $\tilde{\theta}(R'^{H_0})$ and $\mathbf{Z}\xi'$ are contained in $e^-\mathbf{Q}[G]^{H_0}$. Hence $\psi_0$ induces the isomorphism

$$\tilde{\theta}(R'^{H_0}) \cap \mathbf{Z}\xi' \xrightarrow{\sim} \psi_0(R'^{H_0}) \cap \mathbf{Z}\psi_0(\xi').$$

By Proposition 3.3 we have

$$\psi_0(\tilde{\theta}R'^{H_0}) = \psi_0(S'_{k_0}) = \bigoplus_{\chi \in \hat{\Gamma}_0^-} 2^{v(m/d_\chi)-\varepsilon_\chi} B_{1,\chi}\mathbf{Z}.$$

On the other hand, by definition, we have

$$\psi_0(\xi') = (..., \chi(\xi_0), ...)_{\chi \in \hat{\Gamma}_0^-}.$$

Then, by applying Lemma 4.3 to $X = \psi_0(\mathbf{Z}\xi'), Y = \psi_0(\tilde{\theta}(R'^{H_0}))$, we get the desired formula. Q.E.D.

It seems difficult to determine the order $|V_\xi|$ exactly in general. In what follows we consider the following condition on $m$.

(2) $\qquad p \equiv 3 \pmod{4}$ for any odd prime divisor $p$ of $m$.

**Proposition 4.4.** *If $m$ satisfies the condition (2), then $H^1(H_0, B) = 0$. In particular, $V_\xi = 0$.*

**Proof:** Clearly it suffices to show the first statement. Let $B^*$ be the submodule of $B$ generated by "standard elements":

$$\sum_{i=0}^{p-1}[a + \frac{im}{p}] + [-pa], \qquad p|m,\ p = \text{odd},\ pa \neq 0,$$
$$[a] + [a + \frac{m}{2}] + [-2a] + [\frac{m}{2}], \qquad 2|m,\ 2a \neq 0.$$

and $[a] + [-a]$ for all $a \in \mathbf{Z}/m\mathbf{Z} \setminus \{0\}$. Then it is known that $B/B^*$ is an elementary abelian group of exponent 2. (See **[Y]**, **[Ku]** or **[A]**.) From the exact sequence

$$0 \longrightarrow B^* \longrightarrow B \longrightarrow B/B^* \longrightarrow 0,$$

we have an exact sequence

$$H^1(H_0, B^*) \longrightarrow H^1(H_0, B) \longrightarrow H^1(H_0, B/B^*)$$

The last group is zero since the order of $H_0$ is prime to 2 by our assumption and $B/B^*$ is a 2-group. We must show that the first group is also zero. For that purpose let $D$ be the submodule of $B$ generated by elements of the form $[a]+[-a]$. Then it can be shown without difficulty that $B^*/D$ is a free $H_0$-module and so $H^1(H_0, D) = 0$. Hence from the exact sequence

$$H^1(H_0, D) \longrightarrow H^1(H_0, B^*) \longrightarrow H^1(H_0, B^*/D),$$

we find that $H^1(H_0, B^*) = 0$. This completes the proof. Q.E.D.

Combining the results obtained so far, we have

**Theorem 4.5.** *For any $\xi \in A_K$ such that $\xi = cor_{K/k_0}(\xi_0)$ for some $\xi_0 \in \mathbf{Z}[\Gamma_0]$, we have*

$$\nu(\xi) = \frac{1}{|V_\xi|} \cdot \underset{\chi \in \hat{\Gamma}_0^-, \chi(\xi_0) \neq 0}{L.C.M.} \left\{ \frac{2^{v(m/d_\chi) - \epsilon_\chi} B_{1,\chi}}{(2^{v(m/d_\chi) - \epsilon_\chi} B_{1,\chi}, \ \chi(\xi_0))} \right\}.$$

*Moreover, if $m$ satisfies the condition (2), then*

$$\nu(\xi) = \underset{\chi \in \hat{\Gamma}_0^-, \chi(\xi_0) \neq 0}{L.C.M.} \left\{ \frac{2^{v(m/d_\chi) - \epsilon_\chi} B_{1,\chi}}{(2^{v(m/d_\chi) - \epsilon_\chi} B_{1,\chi}, \ \chi(\xi_0))} \right\}.$$

**Remark 4.6.** If $k = \mathbb{Q}(\sqrt{-m})$ is an imaginary quadratic field and $\xi = s(H)$, then the first statement of Theorem 4.4 implies that

$$\nu(\xi) = \frac{1}{|V_\xi|} \cdot \frac{h_k}{(h_k, \chi(\xi_0))} = \frac{h_k}{2^{r-1}|V_\xi|}$$

since $\chi(\xi_0) = 2^{r-1}$ for the unique nontrivial character $\chi \in \hat{\Gamma}$ and $h_k$ is divisible by $2^{r-1}$. In particular $\nu(\xi)$ divides $h_k/2^{r-1}$. This is also a consequence of Proposition 2.4 if $\nu(\xi) = \tilde{\nu}(\xi)$. (See the discussion at the end of §2.) Moreover, if $m$ satisfies the condition (2), then $\nu(\xi) = h_k/2^{r-1}$. But, if $m$ does not satisfy (2), then $V_\xi$ is not necessarily zero. For example suppose that $m$ is of the form

$$m = p_1 \ldots p_{r-1} q, \quad p_i \equiv 3 \pmod 4, \ q \equiv 5 \pmod 8.$$

Then we can show that $\nu(\xi) = h_k/2^r$, hence $|V_\xi| = 2$. This, in particular, implies that $N_{K/k}(Cl_K)$ is not a cyclic group.(See Proposition 2.4.)

**References.**

[**A**] Aoki, N., *On Some Arithmetic Problems Related to the Hodge Cycles on the Fermat Varieties*, Math. Ann **266** (1983), 23-54. (Erratum : Math. Ann. **267**, 572 (1984)).

[**B**] Blasius, D., *On the critical values of Hecke L-series*, Ann. of Math **124** (1986), 23-63.

[**D**] Deligne, P., *Valeurs de Fonction L et périodes d'intégrales*, Proc. Sym. Pure Math. **33** (1979), 313-346.

[**DMOS**] Deligne, P., Milne, J., Ogus, A., Shih, K., "Hodge Cycles, Motives and Shimura Varieties," Springer Lect. Notes in Math, **900**, 1982.

[**I1**] Iwasawa, K., *A class number formula for cyclotomic fields*, Ann. of Math. **76** (1962), 171-179.

[**I2**] ———, *Some remarks on Hecke characters*, Algebraic Number Theory (Kyoto Int. Sympos.,1976) (1977), 99-108.

[**K-H**] Kimura, T. and Horie, K., *On the Stickelberger Ideal and Relative Class Number*, Trans. Amer. Math. Soc. **302** (1987), 727-739.

[**Ku**] Kubert, D., *The universal ordinary distribution*, Bull. Soc. Math. France **107** (1979), 179-202.

[**L1**] Lang, S., "Cyclotomic fields," Springer, 1978.

[**L2**] ———, "Complex Multiplication," Springer, 1983.

[**Scha**] Schappacher, N., "Periods of Hecke characters," Springer Lect. Notes in Math, **1031**, 1988.

[**Schm**] Schmidt, C.-G., "Zur Arithmetik abelscher Varietäten mit komplexer Multiplikation," Springer Lect. Notes in Math, **1082**, 1984.

[**S-T**] Shimura, G. and Taniyama, Y., "Complex Multiplication of Abelian Varieties and its Applications to Number Theory," Publ. Math. Soc. Japan, 1961.

[**Si1**] Sinnott, W., *On the Stickelberger ideal and the circular units of a cyclotomic field*, Ann. of Math. **108** (1978), 107-134.

[**Si2**] ________, *On the Stickelberger ideal and the circular units of an abelian field*, Inv. Math **62** (1980), 181-234.

[**Si3**] ________, *On the Stickelberger ideal and the circular units of an abelian field*, Séminaire de Théorie des Nombres, Paris (1981), 277-286.

[**Wa**] Wsahington, W., "Introduction to Cyclotomic Fields," Springer, 1982.

[**We1**] Weil, A., *Jacobi sums as "Grössencharactere"*, Trans. Amer. Math. Soc. **73** (1952), 487-495.

[**We2**] ______, *Sommes de Jacobi et caractères de Hecke*, Nachr. Akad. Wiss. Göttingen, Math.-Phys.Kl (1974), 1-14.

[**Y**] Yamamoto, K., *The gap group of multiplicative relationship of Gaussian sums*, Symp. Math. **XV** (1975), 427-440.

Max-Planck-Institut für Mathematik

Gottfried-Claren-strasse 26

5300 Bonn 3, BRD

and

Department of Mathematics

Rikkyo University

Nishi-ikebukuro, Tokyo, 171 Japan