ARITHMETIC FAKE PROJECTIVE SPACES AND ARITHMETIC FAKE GRASSMANNIANS

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Dedicated to Robert P. Langlands on his 70th birthday

1. Introduction

Let n be an integer > 1. A compact Kähler manifold of dimension $n-1$ is called a *fake projective space*, or a *fake* $\mathbf{P}_{\mathbb{C}}^{n-1}$, if it is not isomorphic to $\mathbf{P}_{\mathbb{C}}^{n-1}$ but it has the same Betti numbers as $\mathbf{P}_{\mathbb{C}}^{n-1}$. We will call a fake projective space of dimension $n-1$ an *arithmetic fake projective space*, or an *arithmetic fake* $\mathbf{P}_{\mathbb{C}}^{n-1}$, if it is the quotient of the open unit ball B^{n-1} in $\mathbb{C}^{n-1}$ by a torsion-free cocompact *arithmetic* subgroup of $\mathrm{PU}(n-1,1)$. Note that B^{n-1} is the symmetric space of $\mathrm{PU}(n-1,1)$, $\mathbf{P}_{\mathbb{C}}^{n-1}$ is its compact dual, and the Euler-Poincaré characteristic of $\mathbf{P}_{\mathbb{C}}^{n-1}$, and so also of any arithmetic fake $\mathbf{P}_{\mathbb{C}}^{n-1}$, is n.

We observe that the Grassmann space $\mathbf{Gr}_{m,n}$ of m-dimensional subspaces in $\mathbb{C}^n$ is the compact dual of the symmetric space of the group $\mathrm{PU}(n-m,m)$, and so we will call the quotient of the symmetric space of $\mathrm{PU}(n-m,m)$ by a cocompact torsion-free arithmetic subgroup (of $\mathrm{PU}(n-m,m)$), whose Betti numbers are same as that of $\mathbf{Gr}_{m,n}$, an *arithmetic fake* $\mathbf{Gr}_{m.n}$.

It is an immediate consequence of the Hirzebruch proportionality principle, see [Se1], Proposition 23, that the orbifold Euler-Poincaré characteristic (i.e., the Euler-Poincaré characteristic in the sense of C.T.C. Wall, see [Se1]) of any cocompact discrete subgroup of $\mathrm{PU}(n-1,1)$, for n even, is negative. This implies that if there exists an arithmetic fake $\mathbf{P}_{\mathbb{C}}^{n-1}$, then n is necessarily *odd*. The purpose of this paper is to determine all irreducible cocompact torsion-free arithmetic subgroups $\overline{\Gamma}$ of a product $\overline{\mathcal{G}}$ of r groups of the form $\mathrm{PU}(n-m,m)$, $m<n$, $n>3$ odd, whose Euler-Poincaré characteristic $\chi(\overline{\Gamma})$ is equal to the Euler-Poincaré characteristic $\chi(X_u)$ of the compact dual X_u of the symmetric space X of $\overline{\mathcal{G}}$.

Let $\overline{\Gamma}$ be an irreducible cocompact torsion-free arithmetic subgroup of $\overline{\mathcal{G}}$ with $\chi(\overline{\Gamma}) = \chi(X_u)$. Let $\mathcal{G}$ be the connected semi-simple Lie group obtained by replacing each of the r factors $\mathrm{PU}(n-m,m)$ of $\overline{\mathcal{G}}$ by $\mathrm{SU}(n-m,m)$. As the kernel of the natural surjective homomorphism $\mathcal{G} \to \overline{\mathcal{G}}$ is a group of order n^r, if $\widetilde{\Gamma}$ is the full inverse image of $\overline{\Gamma}$ in $\mathcal{G}$, then $\widetilde{\Gamma}$ is an arithmetic subgroup whose orbifold Euler-Poincaré characteristic is $\chi(X_u)/n^r$. Therefore, the orbifold Euler-Poincaré characteristic of any arithmetic subgroup of $\mathcal{G}$, which contains $\widetilde{\Gamma}$, is a submultiple of $\chi(X_u)/n^r$. Assume, if possible, that $\mathcal{G}$ contains an irreducible maximal arithmetic subgroup

Γ whose orbifold Euler-Poincaré characteristic $\chi(\Gamma)$ is a submultiple of $\chi(X_u)/n^r$. As Γ is an irreducible maximal arithmetic subgroup of $\mathcal{G}$, there exist a totally real number field k, an absolutely simple simply connected group G defined over k, r archimedean places of k, say v_j, $j = 1, \ldots, r$, such that $\mathcal{G} \cong \prod_{j=1}^{r} G(k_{v_j})$, and for all other archimedean places v of k, $G(k_v)$ is isomorphic to the compact real Lie group $\mathrm{SU}(n)$, and a "principal" arithmetic subgroup Λ of $G(k)$ such that Γ is the normalizer of Λ in $\mathcal{G}$ (we identify $\mathcal{G}$ with $\prod_{j=1}^{r} G(k_{v_j})$ and use this identification to realize $G(k)$ as a subgroup of $\mathcal{G}$), see Proposition 1.4(iv) of [BP].

From the description of absolutely simple groups of type ${}^2A_{n-1}$ (see, for example, [T1]), we know that there exists a quadratic extension ℓ of k, a division algebra $\mathcal{D}$ with center ℓ and of degree $s = \sqrt{[\mathcal{D} : \ell]}$, $s|n$, $\mathcal{D}$ given with an involution σ of the second kind such that $k = \{x \in \ell \,|\, x = \sigma(x)\}$, and a nondegenerate hermitian form h on $\mathcal{D}^{n/s}$ defined in terms of the involution σ so that G is the special unitary group $\mathrm{SU}(h)$ of h. It is obvious that ℓ is totally complex.

In terms of the normalized Haar-measure μ on $\mathcal{G} = \prod_{j=1}^{r} G(k_{v_j})$ used in [P] and [BP], and to be used in this paper, $\chi(\Gamma) = \chi(X_u)\mu(\mathcal{G}/\Gamma)$ (see [BP], 4.2). Thus the condition that $\chi(\Gamma)$ is a submultiple of $\chi(X_u)/n^r$ is equivalent to the condition that $\mu(\mathcal{G}/\Gamma)$ is a submultiple of $1/n^r$. We shall prove that if $n > 7$, there does not exist an arithmetic subgroup whose covolume is $\leqslant 1/n^r$, and if $n = 5$ or 7, there does not exist an arithmetic subgroup whose covolume is a submultiple of $1/n^r$.

The main result of this paper implies that *arithmetic fake* $\mathbf{P}_{\mathbb{C}}^{n-1}$ *can exist only if* $n = 3$ *or* 5, and an arithmetic fake $\mathbf{Gr}_{m,n}$ exists, with $n > 3$ odd, only if $n = 5$. The first fake projective plane was constructed by David Mumford in [M] using p-adic uniformization. In [PY] we have constructed seventeen distinct (finite) classes of arithmetic fake projective planes, and have proved that there can exist at most four more. In §5 of this paper we have constructed four distinct 4-dimensional arithmetic fake projective spaces and four distinct fake $\mathbf{Gr}_{2,5}$. We will also use certain results and computations of [PY] to exhibit five smooth complex projective varieties ("fake $\mathbf{P}_{\mathbb{C}}^2 \times \mathbf{P}_{\mathbb{C}}^2$") which are not isomorphic to, but have the same Betti numbers as, $\mathbf{P}_{\mathbb{C}}^2 \times \mathbf{P}_{\mathbb{C}}^2$, and which are not isomorphic to the product of a fake projective plane with either $\mathbf{P}_{\mathbb{C}}^2$ or with a fake projective plane. Al! l these are connected smooth (complex projective) Shimura varieties, and these are the first examples of fake $\mathbf{P}_{\mathbb{C}}^4$, fake Grassmannians, and irreducible fake $\mathbf{P}_{\mathbb{C}}^2 \times \mathbf{P}_{\mathbb{C}}^2$.

It was proved by Bruno Klingler and the second author independently that any fake projective plane is arithmetic. It has just been shown by the second author in [Y] that any fake projective space of dimension 4 is arithmetic. According to Proposition 5 of this paper, the first integral homology group of an arithmetic fake $\mathbf{P}_{\mathbb{C}}^4$ is always nonzero. This leads to the following very interesting result: *A compact Kähler manifold of dimension* 4 *is isomorphic to* $\mathbf{P}_{\mathbb{C}}^4$ *if it has the same integral homology as* $\mathbf{P}_{\mathbb{C}}^4$.

§2. Preliminaries

A comprehensive survey of the basic definitions and the main results of the Bruhat–Tits theory of reductive groups over nonarchimedean local fields is given in [T2].

2.1. Throughout this paper we will use the notations introduced in the introduction. n will always be an odd integer > 3. All unexplained notations are as in [BP] and [P]. Thus for a number field K, D_K denotes the absolute value of its discriminant, h_K its class number, i.e., the order of its class group $Cl(K)$. We shall denote by $h_{K,n}$ the order of the subgroup (of $Cl(K)$) consisting of the elements of order dividing n. Then $h_{K,n} \leqslant h_K$. We shall denote by U_K the multiplicative-group of units of K, and by K_n the subgroup of $K^\times$ consisting of the elements x such that for every normalized valuation v of K, $v(x) \in n\mathbb{Z}$.

We will denote the degree $[k:\mathbb{Q}]$ of k by d, V_f (resp. V_∞) will denote the set of nonarchimedean (resp. archimedean) places of k. As k admits at least r distinct real places, see the introduction, $d \geqslant r$. For $v \in V_f$, q_v will denote the cardinality of the residue field $\mathfrak{f}_v$ of k_v.

For a parahoric subgroup P_v of $G(k_v)$, we define $e(P_v)$ by the following formula (cf. Theorem 3.7 of [P]):

$$e(P_v) = \frac{q_v^{(\dim \overline{M}_v + \dim \overline{\mathcal{M}}_v)/2}}{\#\overline{M}_v(\mathfrak{f}_v)}. \tag{1}$$

If v splits in ℓ, let

$$e'(P_v) = e(P_v)\prod_{j=1}^{n-1}(1-\frac{1}{q_v^{j+1}}),$$

and if v does not split in ℓ, then

$$e'(P_v) = e(P_v)\prod_{j=1}^{(n-1)/2}(1-\frac{1}{q_v^{2j}})(1+\frac{1}{q_v^{2j+1}}),$$

or

$$e'(P_v) = e(P_v)\prod_{j=1}^{(n-1)/2}(1-\frac{1}{q_v^{2j}})$$

according as v *does not* or *does* ramify in ℓ. It can be seen that for all $v \in V_f$, $e'(P_v)$ is an integer. It is obvious that $e'(P_v) < e(P_v)$.

2.2. We note that if P_v is a *hyperspecial* parahoric subgroup of $G(k_v)$, then the $\mathfrak{f}_v$-group $\overline{M}_v$, which in this case is just the "reduction mod $\mathfrak{p}$" of P_v, is either SL_n or SU_n according as v does or does not split in ℓ, and $\overline{\mathcal{M}}_v = \overline{M}_v$. If v ramifies in ℓ, then G is quasi-split over k_v, and if P_v is *special*, then $\overline{M}_v$ is isogenous to either SO_n or Sp_{n-1}, and so is $\overline{\mathcal{M}}_v$. Now by a direct computation we find that $e'(P_v) = 1$ if either P_v is hyperspecial, or v ramifies in ℓ and P_v is special.

2.3. Let v be a nonarchimedean place of k which splits in ℓ and G splits at v. Then $\overline{\mathcal{M}}_v$ is $\mathfrak{f}_v$-isomorphic to SL_n. It can be seen by a direct computation that for any nonhyperspecial parahoric subgroup P_v of $G(k_v)$, $e'(P_v)$ is an integer greater than n.

Let now v be a nonarchimedean place of k which splits in ℓ but G does not split at v. Then $k_v \otimes_k \mathcal{D} = (k_v \otimes_k \ell) \otimes_\ell \mathcal{D} = M_{n/d_v}(\mathfrak{D}_v) \oplus \sigma(M_{n/d_v}(\mathfrak{D}_v))$, where $\mathfrak{D}_v$ is a division algebra with center k_v, of degree $d_v > 1$, $d_v|n$. Hence, G is k_v-isomorphic to $\mathrm{SL}_{n/d_v,\mathfrak{D}_v}$. Let P_v be a maximal parahoric subgroup of $G(k_v)$. Then $\overline{\mathcal{M}}_v$ is $\mathfrak{f}_v$-isomorphic to SL_n, and $\overline{M}_v$ is isogenous to the product of the norm-1 torus $R^{(1)}_{F_v/\mathfrak{f}_v}(\mathrm{GL}_1)$ and the semi-simple group $R_{F_v/\mathfrak{f}_v}(\mathrm{SL}_{n/d_v})$, where F_v is the field extension of $\mathfrak{f}_v$ of degree d_v. So

$$\#\overline{M}_v(\mathfrak{f}_v) = \frac{q_v^{n^2/d_v}}{q_v - 1} \prod_{j=1}^{n/d_v} \big(1 - \frac{1}{q_v^{jd_v}}\big),$$

and hence,

$$\begin{aligned} e'(P_v) &= q_v^{n^2(d_v-1)/2d_v} \prod_{j=1}^{n/d_v} \big(1 - \frac{1}{q_v^{jd_v}}\big)^{-1} \prod_{j=1}^{n} \big(1 - \frac{1}{q_v^{j}}\big) \\ &= \frac{\prod_{j=1}^{n}(q_v^j - 1)}{\prod_{j=1}^{n/d_v}(q_v^{jd_v} - 1)} > q_v^{(n^2-2n)(d_v-1)/2d_v} > n. \end{aligned}$$

It is obvious that for any parahoric subgroup P'_v contained in P_v, $e'(P'_v)$ is an integral multiple of $e'(P_v)$.

2.4. Let Γ be a maximal arithmetic subgroup of $\mathcal{G} = \prod_{j=1}^{r} G(k_{v_j})$ such that $n^r\mu(\mathcal{G}/\Gamma) \leqslant 1$, see the introduction. Let $\Lambda = \Gamma \cap G(k)$. Then Γ is the normalizer of Λ in $\mathcal{G}$, and Λ is a *principal* arithmetic subgroup (see [BP], Proposition 1.4(iv)), i.e., if for a nonarchimedean place v of k, P_v is the closure of Λ in $G(k_v)$, then P_v is a parahoric subgroup, and $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$. Let $\mathcal{T}$ be the set of v ($\in V_f$) which split in ℓ and P_v is not a hyperspecial parahoric subgroup of $G(k_v)$. Let $\mathcal{T}'$ be the set of v ($\in V_f$) which do not split in ℓ, and either P_v is not a hyperspecial parahoric subgroup of $G(k_v)$ but a hyperspecial parahoric exists (which is the case if, and only if, v is unramified over ℓ), or v is ramified in ℓ and P_v is not a special parahoric subgroup.

2.5. Let μ_n be the kernel of the endomorphism $x \mapsto x^n$ of GL_1. Then the center C of G is k-isomorphic to the kernel of the norm map $N_{\ell/k}$ from the algebraic group $R_{\ell/k}(\mu_n)$, obtained from μ_n by Weil's restriction of scalars, to μ_n.

As n is odd, the norm map $N_{\ell/k} : \mu_n(\ell) \to \mu_n(k)$ is onto, $\mu_n(k)/N_{\ell/k}(\mu_n(\ell))$ is trivial, and hence, the Galois cohomology group $H^1(k, C)$ is isomorphic to the kernel of the homomorphism $\ell^\times/\ell^{\times n} \to k^\times/k^{\times n}$ induced by the norm map. We shall denote this kernel by $(\ell^\times/\ell^{\times n})_\bullet$ in the sequel.

By Dirichlet's unit theorem, $U_k \cong \{\pm 1\} \times \mathbb{Z}^{d-1}$, and $U_\ell \cong \mu(\ell) \times \mathbb{Z}^{d-1}$, where $\mu(\ell)$ is the finite cyclic group of roots of unity in ℓ. Hence, $U_k/U_k^n \cong (\mathbb{Z}/n\mathbb{Z})^{d-1}$, and $U_\ell/U_\ell^n \cong \mu(\ell)_n \times (\mathbb{Z}/n\mathbb{Z})^{d-1}$, where $\mu(\ell)_n$ is the group of n-th roots of unity in ℓ. Now we observe that $N_{\ell/k}(U_\ell) \supset N_{\ell/k}(U_k) = U_k^2$, which implies that, as n is odd, the homomorphism $U_\ell/U_\ell^n \to U_k/U_k^n$, induced by the norm map, is onto. Therefore, the order of the kernel $(U_\ell/U_\ell^n)_\bullet$ of this homomorphism equals $\#\mu(\ell)_n$.

The short exact sequence (4) in the proof of Proposition 0.12 of [BP] gives us the following exact sequence:

$$1 \to (U_\ell/U_\ell^n)_\bullet \to (\ell_n/\ell^{\times n})_\bullet \to (\mathcal{P}\cap\mathcal{I}^n)/\mathcal{P}^n,$$

where $(\ell_n/\ell^{\times n})_\bullet = (\ell_n/\ell^{\times n}) \cap (\ell^\times/\ell^{\times n})_\bullet$, $\mathcal{P}$ is the group of all fractional principal ideals of ℓ, and $\mathcal{I}$ the group of all fractional ideals (we use multiplicative notation for the group operation in both $\mathcal{I}$ and $\mathcal{P}$). Since the order of the last group of the above exact sequence is $h_{\ell,n}$, see (5) in the proof of Proposition 0.12 of [BP], we conclude that

$$\#(\ell_n/\ell^{\times n})_\bullet \leqslant \#\mu(\ell)_n \cdot h_{\ell,n}.$$

Now we note that the order of the first term of the short exact sequence of Proposition 2.9 of [BP], for $G' = G$ and $S = V_\infty$, is $n^r/\#\mu(\ell)_n$.

Using the above observations, together with Proposition 2.9 and Lemma 5.4 of [BP], and a close look at the arguments in 5.3 and 5.5 of [BP] for $S = V_\infty$ and G of type ${}^2A_{n-1}$, we can derive the following upper bound:

$$[\Gamma:\Lambda] \leqslant n^{r+\#\mathcal{T}} h_{\ell,n}. \tag{2}$$

From this we obtain

$$1 \geqslant n^r \mu(\mathcal{G}/\Gamma) \geqslant \frac{\mu(\mathcal{G}/\Lambda)}{n^{\#\mathcal{T}} h_{\ell,n}}. \tag{3}$$

2.6. Now we will use the volume formula of [P] to write down the precise value of $\mu(\mathcal{G}/\Lambda)$. As the Tamagawa number $\tau_k(G)$ of G equals 1, Theorem 3.7 of [P] (recalled in 3.7 of [BP]), for $S = V_\infty$, gives us for n odd,

$$\mu(\mathcal{G}/\Lambda) = D_k^{(n^2-1)/2}(D_\ell/D_k^2)^{(n-1)(n+2)/4}\Big(\prod_{j=1}^{n-1}\frac{j!}{(2\pi)^{j+1}}\Big)^d \mathcal{E}, \tag{4}$$

where $\mathcal{E} = \prod_{v\in V_f} e(P_v)$, with $e(P_v)$ as in 2.1.

2.7. Let ζ_k be the Dedekind zeta-function of k, and $L_{\ell|k}$ be the Hecke L-function associated to the quadratic Dirichlet character of ℓ/k. Then

$$\zeta_k(j) = \prod_{v\in V_f}(1-\frac{1}{q_v^j})^{-1};$$

$$L_{\ell|k}(j) = \prod{}'(1-\frac{1}{q_v^j})^{-1}\prod{}''(1+\frac{1}{q_v^j})^{-1},$$

where $\prod'$ is the product over those nonarchimedean places of k which split in ℓ, and $\prod''$ is the product over all the other nonarchimedean places v which do not ramify in ℓ. Hence the Euler product $\mathcal{E}$ appearing in (4) can be rewritten as

$$\mathcal{E} = \prod_{j=1}^{(n-1)/2}\big(\zeta_k(2j)L_{\ell|k}(2j+1)\big)\prod_{v\in V_f} e'(P_v).$$

Since $e'(P_v) = 1$, if $v \notin \mathcal{T} \cup \mathcal{T}'$ (see 2.2),

$$\mathcal{E} = \prod_{j=1}^{(n-1)/2} \big(\zeta_k(2j)L_{\ell|k}(2j+1)\big) \prod_{v \in \mathcal{T}\cup\mathcal{T}'} e'(P_v).$$

2.8. Using the functional equations

$$\zeta_k(2j) = D_k^{\frac{1}{2}-2j} \big(\frac{(-1)^j 2^{2j-1}\pi^{2j}}{(2j-1)!}\big)^d \zeta_k(1-2j),$$

and

$$L_{\ell|k}(2j+1) = \big(\frac{D_k}{D_\ell}\big)^{2j+\frac{1}{2}} \big(\frac{(-1)^j 2^{2j}\pi^{2j+1}}{(2j)!}\big)^d L_{\ell|k}(-2j),$$

we find that

$$D_k^{(n^2-1)/2}(D_\ell/D_k^2)^{(n-1)(n+2)/4}\Big(\prod_{j=1}^{n-1}\frac{j!}{(2\pi)^{j+1}}\Big)^d \prod_{j=1}^{(n-1)/2}\big(\zeta_k(2j)L_{\ell|k}(2j+1)\big) \tag{5}$$

$$= \mathcal{R} := 2^{-(n-1)d}\zeta_k(-1)L_{\ell|k}(-2)\zeta_k(-3)L_{\ell|k}(-4)\cdots\zeta_k(2-n)L_{\ell|k}(1-n).$$

2.9. We assume here that the orbifold Euler-Poincaré characteristic $\chi(\Gamma)$ of Γ is a submultiple of $\chi(X_u)/n^r$, see the introduction. Then we have the following as $\chi(\Lambda) = \chi(X_u)\mu(\mathcal{G}/\Lambda)$,

$$\chi(\Gamma) = \frac{\chi(\Lambda)}{[\Gamma:\Lambda]} = \frac{\chi(X_u)\mu(\mathcal{G}/\Lambda)}{[\Gamma:\Lambda]}.$$

On the other hand, Proposition 2.9 of [BP] applied to $G' = G$ and $\Gamma' = \Gamma$, implies that any prime divisor of the integer $[\Gamma:\Lambda]$ divides n. So we conclude that any prime which divides the numerator of the rational number $\mu(\mathcal{G}/\Lambda)$ is a divisor of n.

It is easy to see, using the volume formula of [P], Theorem 3.7, and (5), that $\mu(\mathcal{G}/\Lambda)$ is an *integral* multiple of $\mathcal{R}$; the latter as in (5). Hence we obtain the following proposition.

Proposition 1. *If the orbifold Euler-Poincaré characteristic of* Γ *is a submultiple of* $\chi(X_u)$*, then any prime divisor of the numerator of the rational number* $\mathcal{R}$ *divides* n*.*

2.10. We know (cf. [P], Proposition 2.10(iv), and 2.3 above) that

$$\text{for all } v \in V_f,\quad e(P_v) > 1, \text{ and for all } v \in \mathcal{T},\quad e(P_v) > n. \tag{6}$$

Now combining (3), (4) and (6), we obtain

$$1 \geqslant n^r\mu(\mathcal{G}/\Gamma) > \frac{{D_\ell}^{(n-1)(n+2)/4}}{D_k^{(n-1)/2}h_{\ell,n}}\Big(\prod_{j=1}^{n-1}\frac{j!}{(2\pi)^{j+1}}\Big)^d. \tag{7}$$

It follows from Brauer-Siegel Theorem that for all real $s > 1$,

$$h_\ell R_\ell \leqslant w_\ell s(s-1)\Gamma(s)^d((2\pi)^{-2d}D_\ell)^{s/2}\zeta_\ell(s), \tag{8}$$

where h_ℓ is the class number and R_ℓ is the regulator of ℓ, and w_ℓ is the order of the finite group of roots of unity contained in ℓ. Using the lower bound $R_\ell \geqslant 0.02 w_\ell\, e^{0.1d}$ due to R. Zimmert [Z], we get

$$\frac{1}{h_{\ell,n}} \geqslant \frac{1}{h_\ell} \geqslant \frac{0.02}{s(s-1)}\Big(\frac{(2\pi)^s e^{0.1}}{\Gamma(s)}\Big)^d \frac{1}{D_\ell^{s/2}\zeta_\ell(s)}. \tag{9}$$

Now from bound (7) we obtain

$$1 > \frac{D_\ell{}^{(n-1)(n+2)/4}}{D_k^{(n-1)/2} D_\ell^{s/2}\zeta_\ell(s)} \cdot \frac{0.02}{s(s-1)}\Big(\frac{(2\pi)^s e^{0.1}}{\Gamma(s)}\Big)^d \Big(\prod_{j=1}^{n-1} \frac{j!}{(2\pi)^{j+1}}\Big)^d. \tag{10}$$

Letting $s = 1+\delta$, with δ in the interval $[1, 10]$, and using $D_\ell \geqslant D_k^2$, and the obvious bound $\zeta_\ell(1+\delta) \leqslant \zeta(1+\delta)^{2d}$, we get

$$D_k^{1/d} \leqslant D_\ell^{1/2d} < \Big[\Big\{\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}} \prod_{j=1}^{n-1} \frac{(2\pi)^{j+1}}{j!}\Big\} \cdot \{50\delta(1+\delta)\}^{1/d}\Big]^{2/(n^2-2\delta-3)}. \tag{11}$$

We will now prove the following simple lemma.

Lemma 1. *For every integer $j \geqslant 2$, $\zeta_k(j)^{1/2} L_{\ell|k}(j+1) > 1$.*

Proof. The lemma follows from the product formula for $\zeta_k(j)$ and $L_{\ell|k}(j+1)$ and the following observation.

For any positive integer $q \geqslant 2$,

$$(1 - \frac{1}{q^j})(1+\frac{1}{q^{j+1}})^2 = 1 - \frac{q-2}{q^{j+1}} - \frac{2q-1}{q^{2j+2}} - \frac{1}{q^{3j+2}} < 1.$$

The above lemma implies that for every integer $j \geqslant 2$, $\zeta_k(j)L_{\ell|k}(j+1) > \zeta_k(j)^{1/2} > 1$. Also we have the following obvious bounds for any number field k of degree d over $\mathbb{Q}$, where, as usual, $\zeta(j)$ denotes $\zeta_{\mathbb{Q}}(j)$. For every positive integer j,

$$1 < \zeta(dj) \leqslant \zeta_k(j) \leqslant \zeta(j)^d.$$

From this we obtain the following:

Lemma 2. *Let $\mathcal{E}_0 = \prod_{j=1}^{(n-1)/2} \big(\zeta_k(2j) L_{\ell|k}(2j+1)\big)$. Then $\mathcal{E}_0 > E_0 := \prod_{j=1}^{(n-1)/2} \zeta(2dj)^{1/2}$.*

2.11. To find restrictions on n and d, we will use the following three bounds for the relative discriminant D_ℓ/D_k^2 obtained from bounds (3), (4), (8) and (9), and Lemma 2.

$$D_\ell/D_k^2 < \mathfrak{p}_1(n, d, D_k, \delta) \tag{12}$$

$$:= \Big[\frac{50\delta(1+\delta)}{E_0 D_k^{(n^2-2\delta-3)/2}} \cdot \Big\{\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}} \prod_{j=1}^{n-1} \frac{(2\pi)^{j+1}}{j!}\Big\}^d\Big]^{4/(n^2+n-2\delta-4)}.$$

$$D_\ell/D_k^2 < \mathfrak{p}_2(n, d, D_k, R_\ell/w_\ell, \delta) \tag{13}$$

$$:= \Big[\frac{\delta(1+\delta)}{(R_\ell/w_\ell)E_0 D_k^{(n^2-2\delta-3)/2}}\cdot\Big\{\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}}\cdot\prod_{j=1}^{n-1}\frac{(2\pi)^{j+1}}{j!}\Big\}^d\Big]^{4/(n^2+n-2\delta-4)}.$$

$$D_\ell/D_k^2 < \mathfrak{p}_3(n,d,D_k,h_{\ell,n}) \tag{14}$$

$$:= \Big[\frac{h_{\ell,n}}{E_0}\cdot\Big\{\prod_{j=1}^{n-1}\frac{(2\pi)^{j+1}}{j!}\Big\}^d D_k^{-(n^2-1)/2}\Big]^{4/(n-1)(n+2)}.$$

Similarly, from bounds (3), (4), (8), and Lemma 2 we obtain the following:

$$D_k^{1/d} \leqslant D_\ell^{1/2d} < \varphi(n,d,R_\ell/w_\ell,\delta) \tag{15}$$

$$:= \Big[\Big\{\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}}\prod_{j=1}^{n-1}\frac{(2\pi)^{j+1}}{j!}\Big\}\cdot\Big\{\frac{\delta(1+\delta)}{(R_\ell/w_\ell)E_0}\Big\}^{1/d}\Big]^{2/(n^2-2\delta-3)}.$$

3. Determining k

3.1. We define $M_r(d) = \min_K D_K^{1/d}$, where the minimum is taken over all totally real number fields K of degree d. Similarly, we define $M_c(d) = \min_K D_K^{1/d}$, by taking the minimum over all totally complex number fields K of degree d.

The precise values of $M_r(d), M_c(d)$ for low values of d are given in the following table (cf. [N]).

d	2	3	4	5	6	7	8
$M_r(d)^d$	5	49	725	14641	300125	20134393	282300416
$M_c(d)^d$	3		117		9747		1257728

We also need the following proposition which provides lower bounds for the discriminant of a totally real number field in terms its degree.

Proposition 2. (a) *Let k be a totally real number field of degree d, $k \neq \mathbb{Q}$. Then* $D_k^{1/d} \geqslant \sqrt{5} > 2.23$.
(b) *If $d \geqslant 3$, then* $D_k^{1/d} \geqslant 49^{1/3} > 3.65$.
(c) *If $d \geqslant 4$, then* $D_k^{1/d} \geqslant 725^{1/4} > 5.18$.
(d) *If $d \geqslant 5$, then* $D_k^{1/d} \geqslant 14641^{1/5} > 6.8$.

Proof. Let $g(x,d)$ and x_0 be as in 6.2 of [PY]. Let $\mathfrak{N}(d) = \limsup_{x\geqslant x_0} g(x,d)$. It has been observed in [PY], Lemma 6.3, that $\mathfrak{N}(d)$ is an increasing function of d, and it follows from the estimates of Odlyzko [O] that $M_r(d) \geqslant \mathfrak{N}(d)$. We see by a direct computation that $g(2,9) > 9.1$. Hence, $M_r(d) \geqslant \mathfrak{N}(d) \geqslant \mathfrak{N}(9) \geqslant g(2,9) > 9.1$, for all $d \geqslant 9$. For $1 \leqslant d \leqslant 7$, the explicit values of $M_r(d)$ listed above satisfies $M_r(d) \leqslant M_r(d+1)$.

(a)–(d) now follow from the explicit values of $M_r(d)$, for $d \leqslant 8$, and the above bound for $M_r(d)$ for $d \geqslant 9$.

3.2. We note here for latter use that except for the totally complex sextic fields with discriminants

$$-9747,\ -10051,\ -10571,\ -10816,\ -11691,\ -12167,$$

and the totally complex quartic fields with discriminants

$$117,\ 125,\ 144,$$

$R_\ell/w_\ell > 1/8$ for every number field ℓ, see [F], Theorem B′.

For $r_2 = d = 2$, we have the unconditional bound $R_\ell/w_\ell \geqslant 0.09058$, see Theorem B′ and Table 3 in [F].

3.3. For d a positive integer, n an odd positive integer, and $\delta \geqslant 0.02$, denote by $f(n,d,\delta)$ the expression on the extreme right of bounds (11) i.e.,

$$f(n,d,\delta) = \Big[\Big\{\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}}\prod_{j=1}^{n-1}\frac{(2\pi)^{j+1}}{j!}\Big\}\cdot\{50\delta(1+\delta)\}^{1/d}\Big]^{2/(n^2-2\delta-3)}.$$

For fixed n and δ, $f(n,d,\delta)$ clearly decreases as d increases. Furthermore, for a given d and δ, $f(n,d,\delta)$ decreases as n increases provided $n \geqslant a$, where a is any positive integer such that $(a-1)! > (2\pi)^a$.

It is easy to see that $(a-1)! > (2\pi)^a$ for all $a \geqslant 19$. So we conclude that $f(n,d,\delta)$ is a decreasing function of d for fixed n and δ, and a decreasing function of n for $n \geqslant 19$ if d and δ are fixed.

We obtain by a direct computation the following upper bound for the value of $f(n,2,3)$ for small n.

n	$f(n,2,3) <$
19	1.5
17	1.7
15	1.9
13	2.2

From the above table, Proposition 2, and the fact that $f(n,2,3)$ is decreasing in n for $n \geqslant 19$, we conclude that *if* $n \geqslant 13$, *then* $d = 1$, *i.e.,* $k = \mathbb{Q}$.

3.4. Now we will investigate the restriction on the degree d of k for $n \leqslant 11$ imposed by bound (11). We get the following table by evaluating $f(n,d,\delta)$, with n given in the first column, d given in the second column, and δ given in the third column

n	d	δ	$f(n,d,\delta) <$
11	3	2	2.6
9	3	1.7	3.2
7	4	1.5	4.1
5	5	1.2	6.2

Taking into account the upper bound in the last column of the above table, Proposition 2 implies the following:

If $n = 11$, $d \leqslant 2$.

If $n = 9$, $d \leqslant 2$.
If $n = 7$, $d \leqslant 3$.
If $n = 5$, $d \leqslant 4$.

We will now prove the following theorem by a case-by-case analysis.

Theorem 1. *If $n > 7$ and the orbifold Euler-Poincaré characteristic of Γ is $\leqslant \chi(X_u)/n^r$, then $d = 1$, i.e., $k = \mathbb{Q}$. If $n = 7$ or 5, and the orbifold Euler-Poincaré characteristic of Γ is a submultiple of $\chi(X_u)/n^r$, then again $k = \mathbb{Q}$.*

Proof. (*i*) First of all, we will show that if $n = 11$, then d cannot be 2. A direct computation shows that $f(11, 2, 1.8) < 2.6$. Hence, if $n = 11$ and $d = 2$, then $D_\ell < 2.6^4 < 46$. However, from the table in 3.1, we see that the smallest discriminant of a totally complex quartic is 117. So we conclude that if $n = 11$, then $d = 1$.

(*ii*) Let us now consider the case $n = 9$. We will rule out the possibility that $d = 2$ using bound (15). Note that we can use the lower bound $R_\ell/w_\ell \geqslant 0.09058$, see 3.2. We see by a direct computation that $\varphi(9, 2, 0.09058, 1.5)^4 < 97$. Hence, $D_\ell < 97$ from bound (15). As $M_c(4)^4 = 117$, $d = 2$ cannot occur. Hence, if $n = 9$, then $d = 1$.

(*iii*) We now consider the case $n = 7$. We need to rule out the possibilities that d is either 3 or 2. We see from a direct computation that $f(7, 2, 1.2) < 4.3$ and $f(7, 3, 1.4) < 4.14$, where $f(n, d, \delta)$ is as in 3.3.

Consider first the case where $d = 3$ (and $n = 7$). As $D_\ell^{1/6} < f(7, 3, 1.4) < 4.14$, $D_\ell < 4.14^6 < 5036$. This leads to a contradiction since according to the table in 3.1, a lower bound for the absolute value of the discriminant of all totally complex sextic fields is 9747. Hence, it is impossible to have $d = 3$ if $n = 7$.

Consider now the case where $n = 7$ and $d = 2$. As mentioned above, $f(7, 2, 1.2) < 4.3$, and hence,

$$D_k^{1/2} \leqslant D_\ell^{1/4} < f(7, 2, 1.2) < 4.3.$$

It follows that $D_k < 4.3^2 < 18.5$. There are then the following five cases to discuss.
(a) $D_k = 5$, $k = \mathbb{Q}(\sqrt{5})$
(b) $D_k = 8$, $k = \mathbb{Q}(\sqrt{2})$
(c) $D_k = 12$, $k = \mathbb{Q}(\sqrt{3})$
(d) $D_k = 13$, $k = \mathbb{Q}(\sqrt{13})$
(e) $D_k = 17$, $k = \mathbb{Q}(\sqrt{17})$.

Case (e): We will use bound (13). As $R_\ell/w_\ell \geqslant 0.09058$ (see 3.2),

$$D_\ell/D_k^2 < \mathfrak{p}_2(7, 2, 17, 0.09058, 1.26) < 1.1,$$

which implies that $D_\ell = D_k^2 = 17^2$. From the table of totally complex quartics in [1], we find that there does not exist a totally complex quartic with discriminant 17^2.

Case (d): $D_\ell/D_k^2 < 4.3^4/13^2 < 2.1$. Hence, $D_\ell/D_k^2 = 1$ or 2. So $D_\ell = 169$ or 338. From the table of totally complex quartics in [1], we see that neither of these two numbers occurs as the discriminant of such a field. Therefore we conclude that case (d) does not occur.

Case (c): $D_\ell/D_k^2 < 4.3^4/12^2 < 2.4$. Hence $D_\ell/D_k^2 = 1$ or 2 and $D_\ell = 144$ or 288. Again, from the table of totally complex quartics in [1], we know that there is no complex quartic with discriminant 288. Moreover, there is a unique totally complex quartic ℓ, namely $\ell = \mathbb{Q}[x]/(x^4-x^2+1) = \mathbb{Q}(\sqrt{-1}, \sqrt{3})$, whose discriminant equals 144. It clearly contains $k = \mathbb{Q}(\sqrt{3})$. We will now eliminate this case using Proposition 1 (whenever we use Proposition 1 in the sequel, we will assume that the orbifold Euler-Poincaré characteristic of Γ is a submultiple of $\chi(X_u)/n^r$).

In this case, we have the following data.

$$\zeta_k(-1) = 1/6, \quad \zeta_k(-3) = 23/60, \quad \zeta_k(-5) = 1681/126,$$

$$L_{\ell|k}(-2) = 1/9, \quad L_{\ell|k}(-4) = 5/3, \quad L_{\ell|k}(-6) = 427/3.$$

(Observe that for a positive integer j, $\zeta_k(-(2j-1))$ and $L_{\ell|k}(-2j)$ are rational numbers according to well-known results of Siegel and Klingen. The denominators of these rational numbers can be estimated. In this paper, we have used the software PARI together with their functional equations to obtain the actual values of the Dedekind zeta and Hecke L-functions. These values have been rechecked using MAGMA. This software provides precision up to more than 40 decimal places!) Therefore, $\mu(G(k_{v_o})/\Lambda)$ is an integral multiple of

$$2^{-12}\zeta_k(-1)L_{\ell|k}(-2)\zeta_k(-3)L_{\ell|k}(-4)\zeta_k(-5)L_{\ell|k}(-6) = 23 \cdot 41^2 \cdot 61/2^{16} \cdot 3^8.$$

As the numerator of this number is not a power of 7, according to Proposition 1 this case cannot occur.

Case (b): $D_\ell/D_k^2 < 4.3^4/8^2 < 5.4$. Hence, $D_\ell/D_k^2 = c$ and $D_\ell = 64c$, where c is a positive integer $\leqslant 5$. As $D_\ell \geqslant M_c(4)^4 \geqslant 117$, the possible values of D_ℓ are $128, 192, 256, 320$. According to the tables in [1], the only possibilities are:

$D_\ell = 256$: ℓ is obtained by adjoining a primitive 8-th root of unity to $\mathbb{Q}$; the class number of this field is 1.

$D_\ell = 320$: ℓ is obtained by adjoining a root of the polynomial $x^4 - 2x^3 + 2$ to $\mathbb{Q}$, the class number of this field is also 1.

Now, as $\mathfrak{p}_3(7, 2, 8, 1) < 3.1$, from bound (14) we find that $D_\ell \leqslant 3 \times 8^2 = 192$. So neither of the above two cases can occur.

Case (a): As $D_k = 5$, D_ℓ is an integral multiple of 25. We will now use bound (13) to find an upper bound for D_ℓ/D_k^2, making use of the estimate of Friedman [F] mentioned in 3.2 that $R_\ell/w_\ell > 1/8$ if $D_\ell \neq 125$. We find that $D_\ell/D_k^2 < \mathfrak{p}_2(7, 2, 5, 1/8, 1.3) < 8.7$. So $D_\ell = 25c$, where c is a positive integer $\leqslant 8$. Since the smallest discriminant of a totally complex quartic is 117, $c \geqslant 5$. Hence, $5 \leqslant c \leqslant 8$. The possible values of D_ℓ are therefore 125, 150, 175, 200. From the tables in [1] we see that there is no totally complex quartic field with discriminant 150, 175 or 200, whereas the field ℓ obtained by adjoining a primitive 5th root of unity to $\mathbb{Q}$ is the unique totally complex quartic field with $D_\ell = 125$. It is a cyclic extension of $\mathbb{Q}$, and it contains $k = \mathbb{Q}(\sqrt{5})$. We will use Proposition 1 to eliminate this case. In this case, we have the following data.

$$\zeta_k(-1) = 1/30, \quad \zeta_k(-3) = 1/60, \quad \zeta_k(-5) = 67/630,$$

$$L_{\ell|k}(-2) = 4/5, \quad L_{\ell|k}(-4) = 1172/25, \quad L_{\ell|k}(-6) = 84676/5.$$

Hence $\mu(G(k_{v_o})/\Lambda)$ is an integral multiple of

$$2^{-12}\zeta_k(-1)L_{\ell|k}(-2)\zeta_k(-3)L_{\ell|k}(-4)\zeta_k(-5)L_{\ell|k}(-6) = 67 \cdot 293 \cdot 21169/2^{10} \cdot 3^4 \cdot 5^7 \cdot 7.$$

Again, as the numerator of this number is not a power of 7, according to Proposition 1 this case cannot occur.

(*iv*) Finally we take-up the case $n = 5$. We will rule out the possibilities that $d = 4$, 3 or 2.

(**1**) Consider first the case where $n = 5$ and $d = 4$. Bound (11) with $\delta = 1$ leads to $D_\ell^{1/8} < f(5,4,1) < 6.4$. Now from Table 2 of [F] we find that $R_\ell/w_\ell \geqslant 0.1482$. Next we use bound (15) to conclude that $D_k^{1/4} \leqslant D_\ell^{1/8} < \varphi(5,4,0.1482,1.2) < 6.05$. As $6.05^4 < 1340$, $D_k < 1340$. From the list of quartics with small discriminants given in [1], we see that the only integers smaller than 1340 which are the discriminant of a totally real quartic k are 725 and 1125. Moreover, for any of these two integers, there is a unique totally real quartic field k whose discriminant is that integer. Each of these fields has class number 1.

If $D_k = 1125$,

$$D_\ell/D_k^2 < 6.05^8/1125^2 < 2.$$

So $D_\ell/D_k^2 = 1$. This implies that $D_\ell = 1125^2 = 1265625$.

If $D_k = 725$,

$$D_\ell/D_k^2 < 6.05^8/725^2 < 4.$$

Hence $D_\ell = c725^2$ with $c \leqslant 3$. In particular, $D_\ell \leqslant 1576875$.

At our request, Gunter Malle has shown by explicit computation[1] that there is exactly one pair of number fields (k,ℓ) with (D_k, D_ℓ) among the four possiblities above. k (resp., ℓ) is obtained by adjoining a root of $x^4 - x^3 - 4x^2 + 4x + 1$ (resp., a primitive 15th root of unity which is a root of $x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$) to $\mathbb{Q}$. For this pair $D_k = 1125$, $D_\ell = 1125^2 = 1265625$, and the class number of ℓ is 1. We will now employ Proposition 1 to eliminate this case. We have the following values of the Dedekind zeta and Dirichlet L-functions for this pair (k,ℓ).

$$\zeta_k(-1) = 4/15,\ \zeta_k(-3) = 2522/15,\ L_{\ell|k}(-2) = 128/45,\ L_{\ell|k}(-4) = 2325248/75.$$

From which we conclude that $\mu(G(k_{v_o})/\Lambda)$ is an integral multiple of

$$2^{-16}\zeta_k(-1)L_{\ell|k}(-2)\zeta_k(-3)L_{\ell|k}(-4) = 2^2 \cdot 13 \cdot 31 \cdot 97 \cdot 293/3^5 \cdot 5^5.$$

As the numerator of this number is not a power of 5, Proposition 1 rules out this case.

(**2**) We will consider now the case where $n = 5$ and $d = 3$. As ℓ is a totally complex sextic field, from 3.2 we know that $R_\ell/w_\ell > 1/8$ unless ℓ is a totally

[1]Malle used the following procedure in his computation. Any quadratic extension of k is of the form $k(\sqrt{\alpha})$, with α in the ring of integers $\mathfrak{o}_k$ of k. As the class number of any totally real quartic k presently under consideration is 1, $\mathfrak{o}_k$ is a unique factorization domain. Now using factorization of small primes and explicit generators of the group of units of k, he listed all possible α modulo squares, and then for each of the α, the discriminant of $k(\sqrt{\alpha})$ could be computed.

complex sextic field whose discriminant equals one of the six negative integers listed in 3.2. Now using this lower bound for R_ℓ/w_ℓ, we deduce from (15) that $D_k \leqslant D_\ell^{1/2} < \varphi(5,3,1/8,1)^3 < 6.24^3 < 243$. On the other hand, if ℓ is a totally complex sextic field whose discriminant equals one of the six negative integers listed in 3.2, then $D_k \leqslant 12167^{1/2} < 111$. Hence, if $n=5$, $d=3$, then $D_k < 243$. From Table B.4 in [Co] of discriminants of totally real cubic number fields we infer that D_k must equal one of the following five integers: 49, 81, 148, 169, and 229.

• If $D_k = 229$, $D_\ell/D_k^2 < 6.24^6/229^2 < 1.2$. Hence, $D_\ell = 229^2 = 52441$. There are however no such totally complex sextic fields according to [1].

• If $D_k = 169$ or 148, $D_\ell \geqslant D_k^2 \geqslant 148^2 > 12167$, and hence $R_\ell/w_\ell > 1/8$, see 3.2. We will now use bound (13). As $\mathfrak{p}_2(5,3,169,1/8,1.1) < 1.9$, and $\mathfrak{p}_2(5,3,148,1/8,1.1) < 2.3$, D_ℓ must equal cD_k^2 for some $c \leqslant 2$. None of these appear in the table t60.001 of totally complex sextics in [1].

• If $D_k = 81$, then $81^2|D_\ell$, but none of the six negative integers listed in 3.2 are divisible by 81^2. Hence, $R_\ell/w_\ell > 1/8$. We will again use bound (13). We see by a direct computation that $\mathfrak{p}_2(5,3,81,1/8,1.1) < 6.2$. Therefore, if $D_k = 81$, then $D_\ell = cD_k^2$, with $1 \leqslant c \leqslant 6$. But from the table t60.001 in [1] we see that there is no totally complex sextic field the absolute value of whose discriminant equals $c81^2$, with $1 \leqslant c \leqslant 6$, except for $c = 3$. Thus D_ℓ can only be $3 \times 81^2 = 19683$.

Let k be the field obtained by adjoining a root of $x^3 - 3x - 1$ to $\mathbb{Q}$, and ℓ its totally complex quadratic extension obtained by adjoining a primitive 9th root of unity to $\mathbb{Q}$. Then k (resp., ℓ) is the unique totally real cubic (resp., totally complex sextic) field with $D_k = 81$ (resp., $D_\ell = 19683$). In this case, we have the following data on the values of the zeta and L-functions.

$$\zeta_k(-1) = -1/9,\ \zeta(-3) = 199/90,\ L_{\ell|k}(-2) = -104/27,\ L_{\ell|k}(-4) = 57608/9.$$

From which we conclude that $\mu(G(k_{v_o})/\Lambda)$ is an integral multiple of

$$2^{-12}\zeta_k(-1)L_{\ell|k}(-2)\zeta_k(-3)L_{\ell|k}(-4) = 13\cdot 19\cdot 199\cdot 379/2^7\cdot 3^9\cdot 5.$$

As the numerator of this rational number is not a power of 5, according to Proposition 1 this case cannot occur.

• If $D_k = 49$, then D_ℓ is divisible by 49^2, but none of the six negative integers listed in 3.2 are divisible by 49^2. So $R_\ell/w_\ell > 1/8$. We apply bound (13) to obtain $D_\ell/D_k^2 < \mathfrak{p}_2(5,3,49,1/8,1.2) < 14.3$. Hence $D_\ell = c49^2$, with $1 \leqslant c \leqslant 14$. On the other hand, the table in 3.1 implies that $c > 9747/49^2 > 4$. Therefore, we need only consider $5 \leqslant c \leqslant 14$. From the table t60.001 in [1] we see that among these ten integers, $7\times 49^2 = 16807$ is the only one which equals D_ℓ of a totally complex sextic ℓ. This ℓ is obtained by adjoining a primitive 7th root of unity to $\mathbb{Q}$ and it contains the totally real cubic field k obtained by adjoining a root of $x^3 - x^2 - 2x + 1$ to $\mathbb{Q}$. It is easy to see that $D_k = 49$ in this case. We have the following data on the values of the zeta and L-functions.

$$\zeta_k(-1) = -1/21,\ \zeta(-3) = 79/210,\ L_{\ell|k}(-2) = -64/7,\ L_{\ell|k}(-4) = 211328/7.$$

From which we conclude that $\mu(G(k_{v_o})/\Lambda)$ is an integral multiple of

$$2^{-12}\zeta_k(-1)L_{\ell|k}(-2)\zeta_k(-3)L_{\ell|k}(-4) = 13\cdot 79\cdot 127/3^2\cdot 5\cdot 7^4.$$

Again, as the numerator of this rational number is not a power of 5, according to Proposition 1 this case cannot occur.

(**3**) We will consider now the case $n = 5$, $d = 2$. We recall the lower bound $R_\ell/w_\ell \geqslant 0.09058$ from 3.2. From bound (15) we obtain that $D_k^{1/2} \leqslant D_\ell^{1/4} < \varphi(5, 2, 0.09058, 1) < 6.7$. Since $6.7^2 < 45$, $D_k \leqslant 44$. It follows that the discriminant D_k of the real quadratic field k can only be one of the following integers,

$$5,\ 8,\ 12,\ 13,\ 17,\ 21,\ 24,\ 28,\ 29,\ 33,\ 37,\ 40,\ 41,\ 44.$$

• If $D_k \geqslant 37$, $D_\ell/D_k^2 < 6.7^4/37^2 < 2$. In these cases, $D_\ell = D_k^2$ is one of the following integers 1369, 1600, 1681, 1936. Of these only 1600 and 1936 appear as the discriminant of a totally complex quartic ℓ, check [1]. Moreover, there is a unique totally complex quartic ℓ with $D_\ell = 1600$ (resp., $D_\ell = 1936$). The class number of both of these quartics is 1. Now we will use bound (14). Since $\mathfrak{p}_3(5, 2, 40, 1) < 0.6 < 1$ and $\mathfrak{p}_3(5, 2, 44, 1) < 0.5 < 1$, if either $D_k = 40$ or 44, $D_\ell/D_k^2 < 1$, which is impossible.

• If $D_k = 33$, then $D_\ell \geqslant 33^2 = 1089$, and hence $R_\ell/w_\ell > 1/8$, see 3.2. Now from bound (13) we obtain that $D_\ell/D_k^2 < \mathfrak{p}_2(5, 2, 33, 1/8, 1) < 2$. Hence, $D_\ell = D_k^2 = 1089$. There is a unique totally complex quartic ℓ whose discriminant is 1089. Its class number is 1. Now we apply bound (14), $1 \leqslant D_\ell/D_k^2 < \mathfrak{p}_3(5, 2, 33, 1) < 0.77$, to reach a contradiction.

• If $D_k = 29$, $D_\ell/D_k^2 < 6.7^4/29^2 < 3$. Hence, $D_\ell/D_k^2 = 1$ or 2 . Therefore, $D_\ell = 29^2 = 841$ or 1682. None of these integers is the discriminant of a totally complex quartic ([1]).

• If $D_k = 17$ or 13, then $D_\ell \geqslant 169$, and hence $R_\ell/w_\ell > 1/8$ from 3.2. Now we will use bound (13). As $\mathfrak{p}_2(5, 2, 17, 1/8, 1) < 4.7$, and $\mathfrak{p}_2(5, 2, 13, 1/8, 1) < 7.2$, $D_\ell = c17^2$, with $1 \leqslant c \leqslant 4$, or $D_\ell = c13^2$, with $1 \leqslant c \leqslant 7$. But of these eleven integers none appears as the discriminant of a totally complex quartic field.

• To eliminate the remaining cases (namely, where $D_k = 5,\ 8,\ 12,\ 21,\ 24$ or 28), we will use Proposition 1. Let us assume in the rest of this section that D_k is one of the following six integers: 5, 8, 12, 21, 24, 28. As D_ℓ is an integral multiple of D_k^2, we conclude from 3.2 that unless $D_\ell = 125$ or 144, $R_\ell/w_\ell > 1/8$. We will now use upper bounds (13) and (14) for D_ℓ/D_k^2 to make a list of the pairs (k, ℓ) which can occur.

(i) As $\mathfrak{p}_2(5, 2, 28, 1/8, 1) < 2.1$, if $D_k = 28$, $D_\ell = c28^2$, with $c = 1$ or 2. We see from [1] that the class number of any totally complex quartic ℓ with $D_\ell = 28^2$ or 2×28^2 is 1. Now we note that $\mathfrak{p}_3(5, 2, 28, 1) < 1.1$. Hence D_ℓ can only be $28^2 = 784$. The corresponding quartic field is $\ell = \mathbb{Q}[x]/(x^4 - 3x^2 + 4) = \mathbb{Q}(\sqrt{-1}, \sqrt{7})$, which contains $k = \mathbb{Q}(\sqrt{7})$. We shall denote this pair (k, ℓ) by $\mathfrak{C}_1$.

(ii) As $\mathfrak{p}_2(5, 2, 24, 1/8, 1) < 2.6$, if $D_k = 24$, $D_\ell = c24^2$, with $1 \leqslant c \leqslant 2$. Of these integers, only $24^2 = 576$ is the discriminant of a totally complex quartic. There are two totally complex quartics with discriminant 576, but only one of them contains $k = \mathbb{Q}(\sqrt{6})$. This quartic is $\ell = \mathbb{Q}[x]/(x^4 - 2x^2 + 4) = \mathbb{Q}(\sqrt{-3}, \sqrt{6})$. We shall denote this pair (k, ℓ) by $\mathfrak{C}_2$.

(iii) As $\mathfrak{p}_2(5, 2, 21, 1/8, 1) < 3.3$, if $D_k = 21$, $D_\ell = c21^2$, with $1 \leqslant c \leqslant 3$. Of these three integers, only $21^2 = 441$ is the discriminant of a totally complex quartic ℓ. This quartic is $\ell = \mathbb{Q}[x]/(x^4 - x^3 - x^2 - 2x + 4) = \mathbb{Q}(\sqrt{-3}, \sqrt{-7})$, and it clearly contains $k = \mathbb{Q}(\sqrt{21})$. We shall denote this pair (k, ℓ) by $\mathfrak{C}_3$.

(iv) As $\mathfrak{p}_2(5, 2, 12, 1/8, 1) < 8.3$, if $D_k = 12$, $D_\ell = c12^2$, with $1 \leqslant c \leqslant 8$. Among these, only for $c = 1$, 3, 4, and 7, there exists a totally complex quartic ℓ with $D_\ell = c12^2$, and all these quartics have the class number 1. Now we note that $\mathfrak{p}_3(5, 2, 12, 1) < 4.4$, which implies that $c \leqslant 4$; i.e., $c = 1$, 3, or 4. The quartics corresponding to $c = 3$ and 4 do not contain $\mathbb{Q}(\sqrt{3})$. As we observed while dealing with Case (c) in (*iii*) above, there is a unique totally complex quartic ℓ, namely $\ell = \mathbb{Q}[x]/(x^4 - x^2 + 1) = \mathbb{Q}(\sqrt{-1}, \sqrt{3})$, whose discriminant equals $12^2 = 144$. It contains $k = \mathbb{Q}(\sqrt{3})$. The pair $\big(\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{-1}, \sqrt{3})\big)$ will be denoted by $\mathfrak{C}_4$.

(v) As $\mathfrak{p}_2(5, 2, 8, 1/8, 1) < 16.2$, if $D_k = 8$, $D_\ell = c8^2$, with $1 \leqslant c \leqslant 16$. Among these, only for $c = 4$, 5, 8, 9, and 13, there exists a totally complex quartic field, and all these quartics have the class number 1. Now we observe that $\mathfrak{p}_3(5, 2, 8, 1) < 8.7$, which implies that $c = 4$, 5 or 8. There is only one totally complex quartic field ℓ containing $k = \mathbb{Q}(\sqrt{2})$, with discriminant as above. This is $\ell = \mathbb{Q}[x]/(x^4 + 1) = \mathbb{Q}(\sqrt{-1}, \sqrt{2})$ (with $D_\ell = 256$). The corresponding pair (k, ℓ) will be denoted by $\mathfrak{C}_5$.

(vi) As $\mathfrak{p}_2(5, 2, 5, 1/8, 1) < 35.5$, and $D_\ell \geqslant 117$, see 3.1, if $D_k = 5$, $D_\ell = c25$, with $5 \leqslant c \leqslant 35$. Among these, only for $c = 5$, 9 and 16, there exists a totally complex quartic field. Thus the possible values of D_ℓ are 125, 225 and 400. There are precisely three totally complex quartic fields containing $k = \mathbb{Q}(\sqrt{5})$ and with discriminant in $\{125, 225, 400\}$. These are $\ell = \mathbb{Q}[x]/(x^4 - x^3 + x^2 - x + 1)$ (= the field obtained by adjoining a primitive 5th root of unity to $\mathbb{Q}$, its discriminant is 125), $\ell = \mathbb{Q}[x]/(x^4 - x^3 + 2x^2 + x + 1) = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$ (with discriminant 225), and $\ell = \mathbb{Q}[x]/(x^4 + 3x^2 + 1) = \mathbb{Q}(\sqrt{-1}, \sqrt{5})$ (with discriminant 400). The corresponding pairs (k, ℓ) will be denoted by $\mathfrak{C}_6$, $\mathfrak{C}_7$ and $\mathfrak{C}_8$ respectively.

We observe that in all the above cases, the conclusion of Proposition 1 is violated, see the last column of the table below, where $\mathcal{R} = 2^{-8}\zeta_k(-1)L_{\ell|k}(-2)\zeta_k(-3)L_{\ell|k}(-4)$ is as in (5) for $n = 5$ and $d = 2$. Hence none of these cases can occur. We have thus completely proved Theorem 1.

(k,ℓ)	$\zeta_k(-1)$	$\zeta_k(-3)$	$L_{\ell\|k}(-2)$	$L_{\ell\|k}(-4)$	$\mathcal{R}$
$\mathfrak{C}_1$	$2/3$	$113/15$	$8/7$	80	$113/3^2\cdot 7$
$\mathfrak{C}_2$	$1/2$	$87/20$	$2/3$	38	$19\cdot 29/2^9\cdot 5$
$\mathfrak{C}_3$	$1/3$	$77/30$	$32/63$	$64/3$	$2^2\cdot 11/3^5\cdot 5$
$\mathfrak{C}_4$	$1/6$	$23/60$	$1/9$	$5/3$	$23/2^{11}\cdot 3^5$
$\mathfrak{C}_5$	$1/12$	$11/120$	$3/2$	$285/2$	$11\cdot 19/2^{15}$
$\mathfrak{C}_6$	$1/30$	$1/60$	$4/5$	$1172/25$	$293/2^7\cdot 3^2\cdot 5^5$
$\mathfrak{C}_7$	$1/30$	$1/60$	$32/9$	$1984/3$	$31/3^5\cdot 5^2$
$\mathfrak{C}_8$	$1/30$	$1/60$	15	8805	$587/2^{11}$

4. Restrictions on ℓ and the main result

4.1. We shall assume in the sequel that $k=\mathbb{Q}$. (We have proved in the preceding section that this is the case if $n>7$, or if $n=7$ or 5 and the orbifold Euler-Poincaré characteristic of Γ is a submultiple of $\chi(X_u)/n^r$.) Hence, $\ell=\mathbb{Q}(\sqrt{-a})$ for some square-free positive integer a. By setting $d=1$ and $D_k=1$ in bound (10) we obtain

$$1>\frac{D_\ell^{(n-1)(n+2)/4}}{D_\ell^{s/2}\zeta_\ell(s)}\cdot\frac{0.02}{s(s-1)}\cdot\frac{(2\pi)^s e^{0.1}}{\Gamma(s)}\cdot\prod_{j=1}^{n-1}\frac{j!}{(2\pi)^{j+1}}.$$

Using the obvious bound $\zeta_\ell(s)\leqslant\zeta(s)^2$, and by setting $s=1+\delta$, we derive from the above that

$$D_\ell<\{50\delta(1+\delta)e^{-0.1}\Gamma(1+\delta)(2\pi)^{-1-\delta}\zeta(1+\delta)^2\prod_{j=1}^{n-1}\frac{(2\pi)^{j+1}}{j!}\}^{4/(n^2+n-2\delta-4)}. \tag{16}$$

4.2. Denote by $\kappa(n,\delta)$ the right hand side of the above bound. We see, as in 3.3, that for a fixed value of δ, $\kappa(n,\delta)$ decreases as n increases provided $n\geqslant 19$. We obtain the following upper bound for $\kappa(n,\delta)$ for n listed in the first column and δ listed in the second column of the following table:

n	δ	$\kappa(n,\delta)<$
19	2	2.2
17	2	2.7
15	2	3.4
13	2	4.5
11	2	6.2
9	2	9.4
7	1	15.7
5	0.5	37.4

The bound for D_ℓ given by the bound for $\kappa(n,\delta)$ in the above table restricts the possibilities for n and ℓ. In particular, since an imaginary quadratic field has discriminant at least 3, we deduce from the above table and the monotonicity of $\kappa(n,\delta)$ for a fixed δ that *it is impossible for n to be larger than* 15.

We recall that for $\ell = \mathbb{Q}(\sqrt{-a})$, where a is a square-free positive integer, $D_\ell = a$ if $a \equiv 3 \pmod 4$, and $D_\ell = 4a$ otherwise. Now we see that the following proposition enumerates all the possible n and ℓ.

Proposition 3. (a) $n \leqslant 15$.
(b) *The only possibilities for the number fields k and ℓ are $k = \mathbb{Q}$ and $\ell = \mathbb{Q}(\sqrt{-a})$, where, for a given $n \leqslant 15$, the possible a are listed in the following table:*

n	a
15	3
13	$1, 3$
11	$1, 3$
9	$1, 2, 3, 7$
7	$1, 2, 3, 7, 11, 15$
5	$1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 35$

4.3. It is known that the class number of the fields ℓ appearing in the above table is 1, except when $a = 5$, 6, 15, or 35, in which cases ℓ has the class number 2, or $a = 23$, or 31, in which cases ℓ has the class number 3. Hence from (7) we get the following bound:

$$D_\ell < [h_{\ell,n} \prod_{j=1}^{n-1} \frac{(2\pi)^{j+1}}{j!}]^{4/(n-1)(n+2)}, \tag{17}$$

where $h_{\ell,n}$ can only be 1 or 3 since n is odd. Let $\lambda(n, h_{\ell,n})$ be the function on the right hand side of the above bound. Direct computation yields the following table.

n	15	13	11	9	7	5
$\lambda(n, 3) <$	3.3			8.1		
$\lambda(n, 1) <$	3.3	4.2	5.5	7.7	11.2	17.6

Using the above table, and upper bound (17) for D_ℓ, we conclude the following.

Proposition 4. *The only possibilities for the number field $\ell = \mathbb{Q}(\sqrt{-a})$ are those listed in the following table.*

n	a
15	3
13	$1, 3$
11	$1, 3$
9	$1, 3, 7$
7	$1, 2, 3, 7, 11$
5	$1, 2, 3, 7, 11, 15$

4.4. In the considerations so far we did not have to use that Γ is cocompact. Now we will assume that Γ is cocompact and use the description of G given in the introduction. Let ℓ, the division algebra $\mathcal{D}$, and the hermitian form h be as in there.

If $\mathcal{D} = \ell$, then h is an hermitian form on ℓ^n and its signature over $\mathbb{R}$ is $(n-1,1)$. The hermitian form h gives us a quadratic form q on the $2n$-dimensional $\mathbb{Q}$-vector space $V = \ell^n$ defined as follows:

$$q(v) = h(v,v) \quad \text{for} \quad v \in V.$$

The quadratic form q is isotropic over $\mathbb{R}$, and hence by Meyer's theorem it is isotropic over $\mathbb{Q}$ (cf. [Se2]). This implies that h is isotropic, and hence so is $G/\mathbb{Q}$, and then by Godement's compactness criterion, Γ is noncocompact, which is contrary to our hypothesis. We conclude therefore that $\mathcal{D} \neq \ell$, and so it is a nontrivial central simple division algebra over ℓ.

From the classification of central simple division algebras over ℓ, which admit an involution of the second kind, we know that the set $\mathcal{T}_0$ of rational primes p which split in ℓ, but the group G does not split over $\mathbb{Q}_p$, is nonempty. Note that $\mathcal{T}_0 \subset \mathcal{T}$, where $\mathcal{T}$ is as in 2.4, and $p \in \mathcal{T}_0$ if, and only if, $\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathcal{D} = (\mathbb{Q}_p \otimes_{\mathbb{Q}} \ell) \otimes_{\ell} \mathcal{D}$ is isomorphic to $M_r(\mathfrak{D}_p) \oplus M_r(\mathfrak{D}_p^o)$, where $\mathfrak{D}_p$ is a nontrivial central division algebra over $\mathbb{Q}_p$, $\mathfrak{D}_p^o$ is its opposite, and r is a positive integer. We shall denote the degree of $\mathfrak{D}_p$ by d_p in the sequel.

4.5. Now we will use the Euler product $\mathcal{E}$ appearing in the volume formula (4) to eliminate all but the pair $(n,a) = (5,7)$ appearing in Proposition 4. Recall from 2.7 that

$$\begin{aligned} \mathcal{E} &= \prod_{j=1}^{(n-1)/2} \left(\zeta(2j) L_{\ell|\mathbb{Q}}(2j+1)\right) \prod_{p \in \mathcal{T} \cup \mathcal{T}'} e'(P_p) \\ &= \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3, \end{aligned}$$

where

$$\begin{aligned} \mathcal{E}_1 &= \prod_{j=1}^{(n-1)/2} \frac{\zeta(2j)}{\zeta(2j+1)}, \\ \mathcal{E}_2 &= \prod_{j=1}^{(n-1)/2} \zeta_\ell(2j+1), \\ \mathcal{E}_3 &= \prod_{p \in \mathcal{T} \cup \mathcal{T}'} e'(P_p). \end{aligned}$$

In the above we have used the simple fact that $L_{\ell|\mathbb{Q}}(j) = \zeta_\ell(j)/\zeta(j)$.

4.6. Clearly, $\mathcal{E}_2 > 1$ since each factor in the product formula for $\zeta_\ell(2j+1)$, for $j > 0$, is greater than 1. From (3), (4) and (6) we obtain

$$\begin{aligned} D_\ell &\leqslant \Big(h_{\ell,n}\frac{n^{\#\mathcal{T}}}{\mathcal{E}}\prod_{j=1}^{n-1}\frac{(2\pi)^{j+1}}{j!}\Big)^{4/(n-1)(n+2)} \\ &< \Big(h_{\ell,n}\frac{n^{\#\mathcal{T}}}{\mathcal{E}_1\mathcal{E}_3}\prod_{j=1}^{n-1}\frac{(2\pi)^{j+1}}{j!}\Big)^{4/(n-1)(n+2)} \\ &\leqslant \Big(h_{\ell,n}\cdot\prod_{p\in\mathcal{T}_0}\frac{n}{e'(P_p)}\cdot\prod_{j=1}^{(n-1)/2}\frac{\zeta(2j+1)}{\zeta(2j)}\cdot\prod_{j=1}^{n-1}\frac{(2\pi)^{j+1}}{j!}\Big)^{4/(n-1)(n+2)}. \end{aligned}$$

We now recall from 2.3 that for $p \in \mathcal{T}_0$, $e'(P_p)$ is an integral multiple of

$$\frac{\prod_{j=1}^{n}(p^j-1)}{\prod_{j=1}^{n/d_p}(p^{jd_p}-1)},$$

where $d_p > 1$ and $d_p|n$. Let q be the largest prime belonging to $\mathcal{T}_0$. Then

$$\prod_{p\in\mathcal{T}_0}\frac{e'(P_p)}{n} \geqslant \frac{1}{n}\cdot\frac{\prod_{j=1}^{n}(q^j-1)}{\prod_{j=1}^{n/d_q}(q^{jd_q}-1)},$$

which implies that

$$D_\ell < L(n, d_q, q, h_{\ell,n}),$$

where for any divisor d of n,

$$L(n,d,q,h_{\ell,n}) = \Big(nh_{\ell,n}\cdot\frac{\prod_{j=1}^{n/d}(q^{jd}-1)}{\prod_{j=1}^{n}(q^j-1)}\cdot\prod_{j=1}^{(n-1)/2}\frac{\zeta(2j+1)}{\zeta(2j)}\cdot\prod_{j=1}^{n-1}\frac{(2\pi)^{j+1}}{j!}\Big)^{4/(n-1)(n+2)}.$$

Note that $L(n,d,q,h_{\ell,n})$ is decreasing in q if the other three arguments are fixed. Also note that $L(n,d_q,q,h_{\ell,n}) \leqslant L(n,d,q,h_{\ell,n})$, where d is any divisor of d_q.

Let a be a square-free positive integer. We recall now the following well-known fact (cf. [BS]).

Lemma 3. *(a) An odd prime p splits in $\ell = \mathbb{Q}(\sqrt{-a})$ if, and only if, p does not divide a, and $-a$ is a square modulo p.*
(b) 2 splits in ℓ if, and only if, $a \equiv -1 \pmod 8$.
(c) A prime p ramifies in ℓ if, and only if, $p|D_\ell$.

As $q \in \mathcal{T}_0$, q splits in ℓ. Thus if $p = p_a$ is the smallest prime splitting in $\ell = \mathbb{Q}(\sqrt{-a})$, then $q \geqslant p$. Hence, $D_\ell < L(n,d_q,q,h_{\ell,n}) \leqslant L(n,d_q,p,h_{\ell,n})$.

We easily see using Lemma 3 that the smallest prime splitting in $\ell = \mathbb{Q}(\sqrt{-a})$ for $a = 1$, 2, 3, 7, 11 and 15 are respectively 5, 3, 7, 2, 3 and 2. The class number h_ℓ of $\ell = \mathbb{Q}(\sqrt{-a})$, for $a = 1$, 2, 3, 7, 11 and 15 are 1, 1, 1, 1, 1 and 2 respectively. Now we see by a simple computation that for the pairs (n,a) appearing in Proposition 4, $L(n,d,p,h_{\ell,n}) < D_\ell$, for any prime divisor d of n, except for $(n,a) = (5,7)$.

Moreover, $L(5,5,2,1) > D_{\mathbb{Q}(\sqrt{-7})} = 7$, but for any $q > 2$, $L(5,5,q,1) < 7$. We conclude therefore the following.

Theorem 2. *The only possibilities for ℓ, n and $\mathcal{T}_0$ are $\ell = \mathbb{Q}(\sqrt{-7})$, $n = 5$ and $\mathcal{T}_0 = \{2\}$.*

In particular, $\mathrm{PU}(n-m,m)$ *can contain a cocompact arithmetic subgroup whose orbifold Euler-Poincaré characteristic is $\chi(X_u)/n$, where X_u is the compact dual of the symmetric space of* $\mathrm{PU}(n-m,m)$*, only if $n = 3$ or 5.*

5. Four arithmetic fake $\mathbf{P}^4_{\mathbb{C}}$ and four arithmetic fake $\mathbf{Gr}_{2,5}$

5.1. Let now $\ell = \mathbb{Q}(\sqrt{-7})$ and $\mathcal{D}$ be a division algebra with center ℓ and of degree 5 such that for every place v of ℓ not lying over 2, $\ell_v \otimes_\ell \mathcal{D}$ is the matrix algebra $M_5(\ell_v)$, and the invariant of $\mathcal{D}$ at v' is $a/5$ and at v'' it is $-a/5$, where v' and v'' are the places of ℓ lying over 2, and a is a positive integer less than 5. Let $m = 1$ or 2. Then $\mathcal{D}$ admits an involution σ of the second kind such that if G is the simply connected simple algebraic $\mathbb{Q}$-group with

$$G(\mathbb{Q}) = \{x \in \mathcal{D}^\times \mid x\sigma(x) = 1 \ \text{ and } \ \mathrm{Nrd}\, x = 1\},$$

then $G(\mathbb{R})$ is isomorphic to $\mathrm{SU}(5-m,m)$. We note that by varying $\mathcal{D}$, and for a given m, varying σ, we get exactly *two* distinct groups G up to $\mathbb{Q}$-isomorphism.

Now in the group $G(A_f)$ of finite adèles of G, we fix a maximal compact-open subgroup $P = \prod P_q$, where for all $q \neq 2$, 7, P_q is a hyperspecial parahoric subgroup of $G(\mathbb{Q}_q)$, $P_2 = G(\mathbb{Q}_2)$, and P_7 is a *special* maximal parahoric subgroup of $G(\mathbb{Q}_7)$ (we note that there are exactly *two* such parahoric subgroups containing a given Iwahori subgroup of $G(\mathbb{Q}_7)$ and they are nonisomorphic as topological groups, cf. [T2]). Let $\Lambda = G(\mathbb{Q}) \cap P$. Then Λ, considered as a subgroup of $G(\mathbb{R})$, is a principal arithmetic subgroup. The following lemma implies that Λ *is torsion-free.*

Lemma 4. *Let $\mathfrak{D}$ be a division algebra of degree 5 with center $\ell = \mathbb{Q}(\sqrt{-a})$, where a is a square-free positive integer different from 11. Let τ be an involution of $\mathfrak{D}$ of the second kind. Then the subgroup H of $\mathfrak{D}^\times$ consisting of the elements x such that $x\tau(x) = 1$, and* $\mathrm{Nrd}\,(x) = 1$*, is torsion-free.*

Proof. Let $x \in H$ be a nontrivial element of finite order. Since the reduced norm of -1 in $\mathfrak{D}$ is -1, $x \neq -1$, and therefore the $\mathbb{Q}$-subalgebra $K := \mathbb{Q}[x]$ of $\mathfrak{D}$ generated by x is a nontrivial field extension of $\mathbb{Q}$. If $K = \ell$, then x lies in the center of $\mathfrak{D}$, and hence it is of order 5. However, a nontrivial fifth-root of unity cannot be contained in a quadratic extension of $\mathbb{Q}$ and so we conclude that $K \neq \ell$. Then K is an extension of $\mathbb{Q}$ of degree 5 or 10. As no extension of $\mathbb{Q}$ of degree 5 contains a root of unity other than -1, K must be of degree 10, and hence, in particular, it contains $\ell = \mathbb{Q}(\sqrt{-a})$. Now we note that the only roots of unity which can be contained in an extension of $\mathbb{Q}$ of degree 10 are the 11th and 22nd roots of unity. But the only quadratic extension contained in the field extension generated by either of these roots of unity is $\mathbb{Q}(\sqrt{-11})$. Since $K \supset \mathbb{Q}(\sqrt{-a})$, and, by hypothesis, $a \neq 11$, we have arrived at a contradiction.

5.2. We shall now compute the covolume and the Euler-Poincaré characteristic of the principal arithmetic subgroup Λ.

Let X be the symmetric space of $G(\mathbb{R})$, X_u be the compact dual of X, and $\mathfrak{F} = X/\Lambda$. We note that if $m = 1$, $G(\mathbb{R}) = \mathrm{SU}(4,1)$ and $X_u = \mathbf{P}^4_{\mathbb{C}}$; if $m = 2$, $G(\mathbb{R}) = \mathrm{SU}(3,2)$ and $X_u = \mathbf{Gr}_{2,5}$; $\mathfrak{F}$ is a connected smooth complex projective variety.

As $k = \mathbb{Q}$, $\ell = \mathbb{Q}(\sqrt{-7})$, the volume formula of [P], Theorem 3.7, taking into account the value of the Euler-product $\mathcal{E}$ determined in 2.7, gives us

$$\begin{aligned}\mu(G(\mathbb{R})/\Lambda) &= D_\ell^7 \cdot \prod_{j=1}^{4} \frac{j!}{(2\pi)^{j+1}} \cdot \mathcal{E} \\ &= D_\ell^7 \cdot \prod_{j=1}^{4} \frac{j!}{(2\pi)^{j+1}} \cdot \zeta(2)L_{\ell|\mathbb{Q}}(3)\zeta(4)L_{\ell|\mathbb{Q}}(5) \cdot \prod_{v\in\mathcal{T}} e'(P_v).\end{aligned}$$

From the functional equation for the L-function we obtain

$$\begin{aligned}L_{\ell|\mathbb{Q}}(3) &= -2\pi^3 D_\ell^{-5/2} L_{\ell|\mathbb{Q}}(-2), \\ L_{\ell|\mathbb{Q}}(5) &= \frac{2\pi^5}{3} D_\ell^{-9/2} L_{\ell|\mathbb{Q}}(-4).\end{aligned}$$

The following values of zeta and L-functions have been obtained using the software PARI.

$$\zeta(2) = \frac{\pi^2}{6},\quad \zeta(4) = \frac{\pi^4}{90},\quad L_{\ell|\mathbb{Q}}(-2) = -\frac{16}{7},\quad L_{\ell|\mathbb{Q}}(-4) = 32.$$

Note also that for the subgroup Λ under consideration, $\mathcal{T} = \{2\}$, and $d_2 = 5$, so that (2.3)

$$\prod_{v\in\mathcal{T}} e'(P_v) = \frac{\prod_{j=1}^{5}(2^j - 1)}{(2^5 - 1)} = \prod_{j=1}^{4}(2^j - 1).$$

Substituting all this in the above, we obtain

$$\begin{aligned}\mu(G(\mathbb{R})/\Lambda) &= \prod_{j=1}^{4} \frac{j!}{(2\pi)^{j+1}} \cdot \Big(\frac{(-4\pi^{14})}{3\cdot 6\cdot 90} \cdot L_{\ell|\mathbb{Q}}(-2)L_{\ell|\mathbb{Q}}(-4)\Big) \cdot \prod_{j=1}^{4}(2^j - 1) \\ &= \frac{9}{512\pi^{14}} \cdot \frac{(-4\pi^{14})}{3\cdot 6\cdot 90} \cdot (-\frac{16}{7}) \cdot 32 \cdot 315 \ = 1.\end{aligned}$$

Therefore, $\chi(\Lambda) = \chi(X_u)$. Theorem 3.2 of [Cl] implies that $H^j(\Lambda,\mathbb{R})$ vanishes for all odd j. Also, there is a natural embedding of $H^*(X_u,\mathbb{R})$ in $H^*(\Lambda,\mathbb{R})$, see, for example, [B], 3.1 and 10.2. Now since $\chi(\mathfrak{F}) = \chi(\Lambda) = \chi(X_u)$, and for all odd j, $H^j(\mathfrak{F},\mathbb{R})$ $(= H^j(\Lambda,\mathbb{R}))$ vanishes, we conclude that $\mathfrak{F}$ is an *arithmetic fake* $\mathbf{P}^4_{\mathbb{C}}$ if $m = 1$, and is an *arithmetic fake* $\mathbf{Gr}_{2,5}$ if $m = 2$. Thus we have proved the following.

Theorem 3. *There are at least four arithmetic fake $\mathbf{P}^4_{\mathbb{C}}$, and at least four arithmetic fake $\mathbf{Gr}_{2,5}$. There does not exist any arithmetic fake projective space of dimension > 4, or an arithmetic fake $\mathbf{Gr}_{m,n}$, with $n > 5$ odd.*

We next prove the following interesting proposition.

Proposition 5. *The first integral homology group of any arithmetic fake* $\mathbf{P}^4_{\mathbb{C}}$*, or an arithmetic fake* $\mathbf{Gr}_{2,5}$*, is nonzero.*

Proof. Let $\mathfrak{F}$ be either an arithmetic fake $\mathbf{P}^4_{\mathbb{C}}$, or an arithmetic fake $\mathbf{Gr}_{2,5}$. Let Π be its fundamental group. Then $H_1(\mathfrak{F}, \mathbb{Z}) = \Pi/[\Pi, \Pi]$.

It follows from Theorem 2 that Π is a cocompact torsion-free arithmetic subgroup of $\overline{G}(\mathbb{R})$, where G is as in 5.1, and $\overline{G}$ is its adjoint group. Proposition 1.2 of [BP] implies that Π is actually contained in $\overline{G}(\mathbb{Q})$. We will view it as a subgroup of $\overline{G}(\mathbb{Q}_2)$.

Let $\mathcal{D}$ and σ be as in 5.1. Since $\mathbb{Q}_2 \otimes_{\mathbb{Q}} \mathcal{D} = (\mathbb{Q}_2 \otimes_{\mathbb{Q}} \ell) \otimes_{\ell} \mathcal{D} = \mathfrak{D} \oplus \mathfrak{D}^o$, where $\mathfrak{D}$ is a division algebra with center $\mathbb{Q}_2$, of degree 5, $\mathfrak{D}^o$ is its opposite, and $\sigma(\mathfrak{D}) = \mathfrak{D}^o$, $G(\mathbb{Q}_2)$ equals the group $\mathrm{SL}_1(\mathfrak{D})$ of elements of reduced norm 1 in $\mathfrak{D}$, and $\overline{G}(\mathbb{Q}_2)$ equals $\mathfrak{D}^\times/\mathbb{Q}_2^\times$. We now observe that $\overline{G}(\mathbb{Q}_2) = \mathfrak{D}^\times/\mathbb{Q}_2^\times$ is a pro-solvable group, i.e., if we define the decreasing sequence $\{\mathcal{G}_i\}$ of subgroups of $\mathcal{G} := \overline{G}(\mathbb{Q}_2)$ inductively as follows: $\mathcal{G}_0 = \mathcal{G}$, and $\mathcal{G}_i = [\mathcal{G}_{i-1}, \mathcal{G}_{i-1}]$, then $\bigcap \mathcal{G}_i$ is trivial; to see this use Theorem 7(i) of [Ri]. This implies that for any subgroup $\mathcal{H}$ of $\mathcal{G}$, $[\mathcal{H}, \mathcal{H}]$ is a proper subgroup of $\mathcal{H}$. In particular $[\Pi, \Pi]$ is a proper subgroup of Π. This proves the proposition.

6. Five arithmetic fake $\mathbf{P}^2_{\mathbb{C}} \times \mathbf{P}^2_{\mathbb{C}}$

We will now use certain results and computations of [PY] to construct five irreducible fake $\mathbf{P}^2_{\mathbb{C}} \times \mathbf{P}^2_{\mathbb{C}}$. Let ζ_3 be a primitive cube-root of unity, and let the pair (k, ℓ) of number fields be one of the following three:

$$\mathcal{C}_2 = (\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}, \zeta_3)),$$
$$\mathcal{C}_{10} = (\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-7+4\sqrt{2}}\,)),$$
$$\mathcal{C}_{18} = (\mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{6}, \zeta_3)).$$

Let $\mathfrak{v}$ be the unique place of k lying over 2 if the pair is $\mathcal{C}_2$ or $\mathcal{C}_{10}$, and the unique place of k lying over 3 if the pair is $\mathcal{C}_{18}$. For a given pair (k, ℓ), let $q_{\mathfrak{v}}$ be the cardinality of the residue field of the completion $k_{\mathfrak{v}}$ of k at $\mathfrak{v}$.

Let $\mathcal{D}$ be a cubic division algebra with center ℓ whose local invariants at the places of ℓ lying over $\mathfrak{v}$ are nonzero and negative of each other, and whose local invariants at all the other places of ℓ is zero. Then $k_{\mathfrak{v}} \otimes_k \mathcal{D} = (k_{\mathfrak{v}} \otimes_k \ell) \otimes_\ell \mathcal{D} = \mathfrak{D} \oplus \mathfrak{D}^o$, where $\mathfrak{D}$ is a cubic division algebra with center $k_{\mathfrak{v}}$, and $\mathfrak{D}^o$ is its opposite. $\mathcal{D}$ admits an involution of the second kind with k being the fixed field in ℓ. We fix such an involution σ so that if G is the simple simply connected k-group with

$$G(k) = \{z \in \mathcal{D}^\times \mid z\sigma(z) = 1 \ \text{ and } \ \mathrm{Nrd}(z) = 1\},$$

then $G(k_v) \cong \mathrm{SU}(2,1)$ for any real place v of k.

As $\sigma(\mathfrak{D}) = \mathfrak{D}^o$, $G(k_{\mathfrak{v}})$ is the compact group $\mathrm{SL}_1(\mathfrak{D})$ of elements of reduced norm 1 in $\mathfrak{D}$. Let $(P_v)_{v \in V_f}$, be a coherent collection of maximal parahoric subgroups P_v of $G(k_v)$, $v \in V_f$, such that P_v is hyperspecial whenever $G(k_v)$ contains such a subgroup. Let $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$. Let v' and v'' be the two real places of k and let $\mathcal{G} = G(k_{v'}) \times G(k_{v''})$. Then $\mathcal{G} \cong \mathrm{SU}(2,1) \times \mathrm{SU}(2,1)$. Let $\overline{\mathcal{G}}$ be the adjoint group of $\mathcal{G}$. Let X be the symmetric space of $\mathcal{G}$ and X_u its compact dual. Then $X_u = \mathbf{P}^2_{\mathbb{C}} \times \mathbf{P}^2_{\mathbb{C}}$, and hence, $\chi(X_u) = 9$.

We will view Λ as a diagonally embedded arithmetic subgroup of $\mathcal{G}$. Then, in terms of the normalized Haar measure μ on $\mathcal{G}$ used in [P], we see using the volume formula given in that paper (cf. also [PY]) that $\mu(\mathcal{G}/\Lambda) = \mu e'(P_{\mathfrak{v}}) = \mu(q_{\mathfrak{v}} - 1)^2(q_{\mathfrak{v}} + 1)$, where the values of μ and $q_{\mathfrak{v}}$ are as given in the table in section 9.1 of [PY]. Moreover, according to the result in section 4.2 of [BP], the orbifold Euler-Poncaré characteristic $\chi(\Lambda)$ of Λ equals $\chi(X_u)\mu(\mathcal{G}/\Lambda) = 9\mu(q_{\mathfrak{v}} - 1)^2(q_{\mathfrak{v}} + 1)$. Now using the values of μ and $q_{\mathfrak{v}}$ given in the table in section 9.1 of [PY] we find that $\chi(\Lambda) = 3$ if (k, ℓ) is either $\mathcal{C}_2$ or $\mathcal{C}_{18}$, and $\chi(\Lambda) = 9$ if $(k, \ell) = \mathcal{C}_{10}$.

We now observe that Lemma 9.2 of [PY] holds for the group $G(k)$ described above (the proof of this lemma given in [PY] remains valid), i.e., $G(k)$ is torsion-free if $(k, \ell) = \mathcal{C}_{10}$, and in case (k, ℓ) is either $\mathcal{C}_2$ or $\mathcal{C}_{18}$, the only nontrivial elements of finite order of $G(k)$ are central, and hence are of order 3. Let $\overline{\Lambda}$ be the image of Λ in $\overline{\mathcal{G}}$. Then $\overline{\Lambda}$ is a torsion-free cocomact irreducible arithmetic subgroup of $\overline{\mathcal{G}}$. Moreover, the natural homomorphism $\Lambda \to \overline{\Lambda}$ is an isomorphism if $(k, \ell) = \mathcal{C}_{10}$, and its kernel is of order 3 if (k, ℓ) is either $\mathcal{C}_2$ or $\mathcal{C}_{18}$. Hence, for each of the three pairs

(k,ℓ), $\chi(\overline{\Lambda}) = 9 = \chi(X_u)$. Let $\mathfrak{P} = X/\overline{\Lambda}$. Then $\mathfrak{P}$ is a smooth projective variety, and $\chi(\mathfrak{P}) = \chi(\overline{\Lambda}) = \chi(X_u)$. It is known (see the remark following Theorem 15.3.1 in [Ro]) that $H^j(\overline{\Lambda},\mathbb{R})$ vanishes for all odd j. Also, there is a natural embedding of $H^*(X_u,\mathbb{R})$ in $H^*(\overline{\Lambda},\mathbb{R})$, [B], 3.1 and 10.2. As $\chi(\mathfrak{P}) = \chi(X_u)$, we conclude that the Betti numbers of $\mathfrak{P}$ are same as that of $X_u = \mathbf{P}^2_{\mathbb{C}} \times \mathbf{P}^2_{\mathbb{C}}$, and hence $\mathfrak{P}$ is an irreducible arithmetic fake $\mathbf{P}^2_{\mathbb{C}} \times \mathbf{P}^2_{\mathbb{C}}$.

If (k,ℓ) is either $\mathcal{C}_2$ or $\mathcal{C}_{10}$, there is a unique nonarchimedean place, say v_o, of k, which ramifies in ℓ. In $G(k_{v_o})$, up to cojugacy, there are two maximal parahoric subgroups, and thus we get four distinct Λs. On the other hand, if $(k,\ell) = \mathcal{C}_{18}$, as $D_\ell = D_k^2$, every place of k is unramified in ℓ, and so, up to conjugacy, we get only one Λ. Thus all together we obtain five distinct $\mathfrak{P}$ from the above construction.

Remark. Theorem 10.1 of [PY] holds for $\mathfrak{P}$ (with the same proof as in [PY]), i.e., $H_1(\mathfrak{P},\mathbb{Z})$ is nontrivial.

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