# FINITE ISOMETRY GROUPS OF 4-MANIFOLDS WITH POSITIVE SECTIONAL CURVATURE

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Abstract. Let $M$ be an oriented compact positively curved 4-manifold. Let $G$ be a finite subgroup of the isometry group of $M$. Among others, we prove that there is a universal constant $C$ (cf. Corollary 4.3 for the approximate value of $C$), such that if the order of $G$ is odd and at least $C$, then $G$ is either abelian of rank at most 2, or non-abelian and isomorphic to a subgroup of $PU(3)$ with a presentation $\{A, B|A^m = B^n = 1, BAB^{-1} = A^r, (n(r-1), m) = 1, r \neq r^3 = 1(\mathrm{mod}\ m)\}$. Moreover, $M$ is homeomorphic to $\mathbb{C}P^2$ if $G$ is non-abelian, and homeomorphic to $S^4$ or $\mathbb{C}P^2$ if $G$ is abelian of rank 2.

## 1. Introduction

It is one of the most central problem in Riemannian geometry to study manifolds with positive sectional curvature. In dimension three the celebrated work of Hamilton [17] shows that spherical space forms are the only manifolds with positive curvature metrics. The celebrated Hopf problem asks if $S^2 \times S^2$ admits a metric with positive sectional curvature. This problem remains open. In case the manifold has a continous symmetry, i.e., it has a non-zero Killing vector field, by Hsiang-Kleiner [18], this manifold is topologically homeomorphic to $S^4$ or $\mathbb{C}P^2$ (in the orientable case). An important quantity of a Riemannian manifold is its isometry group. Since K. Grove in 1991 proposed to study positively curved manifolds with large symmetry, considerable advancement has been accomplished (cf. [12][14][24][13][27][8]). In Shankar [24], Grove-Shankar [13] and Bazaĭkin [4], free isometric actions of rank 2 abelian groups on positively curved manifolds are found, which answers in negative a well-known question of S.S.Chern.

In this paper we are addressed to the following question:

*Which groups can be realized as the isometry group of a positively curved 4-manifold?*

Due to the work of [18] we may restrict our attention to *finite groups* acting on 4-manifold by isometries. The question may have a perfect answer once the order of the group is odd and larger than a certain constant, depending on the Gromov's Betti number bound [11].

For the sake of simplicity, in the paper we use $\mathrm{Isom}(M)$ to denote the group of orientation preserving isometries of $M$.

**Theorem 1.1.** *Let $M$ be an oriented 4-manifold with positive sectional curvature. Let $G \subset Isom(M)$ be a finite subgroup of odd order. Then there is a universal constant $C$*

Supported partially by NSF Grant 19925104 of China, 973 project of Foundation Science of China and the Max-Planck Institut für Mathematik.

*such that, if the order $|G| \geq C$, then $G$ is either abelian of rank at most 2, or non-abelian and isomorphic to a subgroup of $PU(3)$ with a presentation $\{A, B | A^m = B^n = 1, BAB^{-1} = A^r, (n(r-1), m) = 1, r \neq r^3 = 1 (mod\ m)\}$.*

We refer to Corollary 4.3 for a rough estimate of the constant value $C$. The following result characterizes the topology of the manifold $M$ in Theorem 1.1 when $G$ is abelian of rank 2 or non-abelian.

**Theorem 1.2.** *Let $M$ be an oriented 4-manifold with positive sectional curvature. Let $G \subset Isom(M)$ be a finite subgroup of odd order. If $|G| \geq C$, then*

*(1.2.1) $M$ is either homeomorphic to $S^4$ or $\mathbb{C}P^2$, provided $G$ is abelian of rank 2.*

*(1.2.2) $M$ is homeomorphic to $\mathbb{C}P^2$, provided $G$ is non-abelian.*

The case of even order is more involved, in this case we can almost characterize the group as a subgroup of $SO(5)$.

**Theorem 1.3.** *Let $M$ be a compact 4-manifold with positive sectional curvature. Let $G \subset Isom(M)$ be a finite subgroup. If $M$ is neither homeomorphic to $S^4$ nor $\pm\mathbb{C}P^2$, then there is a universal constant $C$ such that, if the order $|G| \geq C$, then $G$ contains an index 2 subgroup isomorphic to a subgroup of $SO(5)$.*

We refer to §8 and §9 for a complete list of possible finite isometry groups of positively curved 4-manifolds. The following result estimate the size of the finite isometry groups.

**Theorem 1.4.** *Let $M$ be an oriented compact 4-manifold with positive sectional curvature. Let $G \subset Isom(M)$ be a finite subgroup of order $|G| \geq C$. If $M$ is not homeomorphic to $S^4$, then $G$ contains a normal cyclic subgroup of index at most 120.*

The assumption of $M$ is not homeomorphic to $S^4$ is necessary, since clearly, $\mathbb{Z}_p \oplus \mathbb{Z}_p$ acts on $S^4$ by isometries for any $p$.

By Cheeger the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ admits a metric with non-negative sectional curvature. It is interesting to ask how many copies connected sums of $\mathbb{C}P^2$ can have a metric with positive sectional curvature. The following result gives an estimate the size of elementary abelian 2-group acting on 4-manifolds with positive sectional curvature and definite intersection forms.

**Theorem 1.5.** *Let $M$ be a compact oriented 4-manifold with positive sectional curvature and definite intersection form. If $\mathbb{Z}_2^k$ acts on $M$ effectively by isometry, then $k \leq 4$.*

We believe that one can improve the above estimate to $k \leq 3$ if $\chi(M) \geq 3$. Note that $\mathbb{Z}_2^3 = \langle T_1, T_2, T_3 \rangle$ acts on $\mathbb{C}P^2$ by isometries with respect to the Fubini-Study metric, where $T_1, T_2$ acts holomorphically, and $T_3$ acts by complex conjugation action.

**Acknowledgement:** The work was done during the author's visit to the Max-Planck Institut für Mathematik at Bonn. The author would like to thank the institute for its financial support and wonderful research atmospheres. The author is very grateful to Ian Hambleton for invaluable discussions concerning finite group actions on 4-manifolds.

## 2. Preparations

A key fact used in the paper is the following classical result due to Frankel.

**Theorem 2.1** ([10]). *Let $M$ be a compact manifold of dimension $m$ with positive sectional curvature, and let $N_1, N_2$ be two totally geodesic submanifolds of dimensions $n_1$ and $n_2$ respectively. If $n_1 + n_2 \geq m$, then $N_1 \cap N_2$ is not empty.*

Theorem 2.1 implies readily that the fixed point set of any isometry of a 4-manifold with positive sectional curvature contains at most one 2-dimensional component. For recent development much generalizing Theorem 2.1, we refer to [27][7].

**Theorem 2.2** ([11]). *Let $M$ be a compact manifold of dimension $m$ with non-negative sectional curvature. Then the total Betti number $\sum b_i(M) \leq C(n)$, where $C(n)$ is a constant depending only on $n$.*

The constant $C(n)$ is roughly $10^{10n^4}$.

**Theorem 2.3** ([21]). *Let $M$ be a simply connected compact 4-manifold. Let $G$ be a finite group acting effectively on $M$ and trivially on homology groups. If the second Betti number $b_2(M) \geq 3$, then $G$ is abelian of rank at most 2, and the fixed point set $Fix(G; M)$ is not empty.*

**Theorem 2.4** ([25][16]). *Any pseudo-free locally linear action by a finite group on a closed 4-manifold homeomorphic to a complex projective space is conjugate to the linear action of a subgroup of $PSU(3)$ on $\mathbb{C}P^2$.*

## 3. $q$-extent estimates

The $q$-extent $xt_q(X)$, $q \geq 2$, of a compact metric space $(X, d)$ is, by definition, given by the following formula:

$$xt_q(X) = \binom{q}{2}^{-1} \max\Big\{ \sum_{1 \leq i < j \leq q} d(x_i, x_j) : \{x_i\}_{i=1}^q \subset X \Big\}$$

Given a positive integer $n$ and integers $k, l \in \mathbb{Z}$ coprime to $n$, let $L(n; k, l)$ be the 3-dimensional lens space, the quotient space of a free isometric $\mathbb{Z}_n$-action on $S^3$ defined by

$$\psi_{k,l} : \mathbb{Z}_n \times S^3 \to S^3;\ g(z_1, z_2) = (\omega^k z_1, \omega^l z_2)$$

with $g \in \mathbb{Z}_n$ a generator, $\omega = e^{i\frac{2\pi}{n}}$ and $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$.

Note that $L(n; k, l)$ and $L(n; -k, l)$ (resp. $L(n; l, k)$) are isometric (cf. [29] p.536). Obviously $L(n; -k, l)$ and $L(n; n-k, l)$ are isometric. Therefore, up to isometry we may *always* assume $k, l \in (0, n/2)$ without loss of generality. The proof of Lemma 7.4 in [29] works identically for $L(n; k, l)$ with $0 < k, l < n/2$ to prove

**Lemma 3.1** ([29]). *Let $L(n;k,l)$ be a 3-dimensional lens space of constant sectional curvature one. Then*

$$xt_q(L(n;k,l)) \leq arccos\Big\{ cos(\alpha_q) cos\ \pi n^{-\frac{1}{2}} - \frac{1}{2}\{(cos\ \pi n^{-\frac{1}{2}} - cos\ \pi/n)^2 + sin^2(\alpha_q)(n^{\frac{1}{2}} sin\ \pi/n - sin\ \pi n^{-\frac{1}{2}})^2\}^{\frac{1}{2}} \Big\}$$

*where $\alpha_q = \pi/(2(2-[(q+1)/2]^{-1}))$.*

**Corollary 3.2.** *Let $L(n;k,l)$ be a 3-dimensional lens space of constant sectional curvature one. If $n \geq 61$, then $xt_5(L(n;k,l)) < \pi/3$.*

Let $X$ be a compact 4-manifold with positive sectional curvature with an effective $G$-action. We call a fixed point of $G$ is *isolated* if $G$ acts freely on the normal unit sphere of the tangent sphere at the fixed point. The following estimate is central in the paper.

**Proposition 3.3.** *Let $X$ be a compact oriented 4-manifold with positive sectional curvature. If $\mathbb{Z}_n$ acts on $X$ by isometries. Then the action has at most 5 isolated fixed points.*

*Proof.* We argue by contradiction, assuming $x_1, ..., x_6$ are six isolated $\mathbb{Z}_n$-fixed points. Let $\bar{X} = X/\mathbb{Z}_n$. Connecting each pair of points by a minimal geodesic in $\bar{X}$, we obtain a configuration consisting of twenty geodesic triangles. Because $\bar{X}$ has positive curvature in the comparison sense, the sum of the interior angles of each triangle is $> \pi$ and thus the sum of total angles of the twenty triangles, $\sum \theta_i > 20\pi$. We then estimate the sum of the total angles in the following way, first estimate from above of the ten angles around each $\bar{x}_i$ and then sum up over the six points. We claim that the sum of angles at $\bar{x}_i$ is bounded above by $10 \cdot xt_5(\bar{x}_i) \leq 10\frac{\pi}{3}$ and thus $\sum \theta_i \leq 6(10 \cdot \frac{\pi}{3}) = 20\pi$, a contradiction.

Let $S^\perp_{x_i}$ denote the unit 3-sphere in the tangent space $T_{x_i}(X)$. Since $\mathbb{Z}_n$ acts freely on every $S^\perp_{x_i}$, the open neighborhood of $\bar{x}_i$ in $\bar{X}$ is isometric to the metric cone $C(S^3/\mathbb{Z}_n)$. By Corollary 3.2 we arrive at a contradiction. □ □

## 4. Lemmas

**Lemma 4.1.** *Let $M$ be a compact 4-manifold with non-negative sectional curvature. Assume that the second Betti number $b_2(M) \geq 3$ or $b_2(M) = 2$, and $M \neq S^2 \times S^2$. If a finite group $G$ acts effectively on $M$ by isometries, then the normal subgroup $G_0$ of $G$ acting trivially on $H_2(M)$ trivially satisfies*

*(i) the index $[G_0 : G] \leq C$.*

*(ii) $G_0$ is an abelian group of rank at most 2.*

*Proof.* Consider the natural homomorphism $\rho : G \to \mathrm{Aut}(H_2(M))$. By Gromov's theorem, $b_2(M) \leq C(4)$. Therefore, the image $\rho(G)$ is a finite subgroup of $\mathrm{GL}(\mathbb{Z}, C(4))$.

It is well-known that its torsion subgroup is isomorphic into $GL(\mathbb{Z}_3, C(4))$ (cf. [6]) which clearly has order bounded, a constant depending only on $C(4)$.

Since $G_0 = \ker(\rho)$, by Theorem 2.3 we see that $G_0$ has to be abelian of rank at most 2. The desired result follows. □

It is a standard fact of topology, if $g$ is a self-diffeomorphism of $M$, the Lefschetz formula reads

$$\chi(M, g) = \chi(Fix(g)) \qquad (4.1)$$

where $\chi(M, g)$ is the Lefschetz number of $g$. In particular, if $g$ acts trivially on homology, i.e. $\rho(g) = 1$, then $\chi(M) = \chi(Fix(g))$.

Given a cyclic group $\mathbb{Z}_m$ acting effectively on $M^4$ with an isolated fixed point $p \in M$, consider the isotropy representation of $\mathbb{Z}_m$ at the tangent space $T_pM$. This representation is uniquely determined by a pair of integers $p, q$, such that the representation is given by $\alpha(u, v) = (e^{\frac{2p\pi i}{m}} u, e^{\frac{2q\pi i}{m}} v)$, where $\alpha = e^{\frac{2\pi i}{m}} \in \mathbb{Z}_m$ is a generator. Since the action is effective, the common factor $(p, q)$ must be coprime to $m$.

**Lemma 4.2.** *Let $M$ be a compact 4-manifold with positive sectional curvature. If $G$ acts effectively on $M$ by isometries. If $G_0 \subset G$ is the normal subgroup in the above lemma, and it contains an element of exponent at least $61^4$. Then $\chi(M) \leq 7$.*

*Proof.* By [29] and the above identity (4.1) we may assume that all prime factors of $|G_0|$ is smaller than 61. Let $g \in G_0$ be an element of composite order greater than 61. Consider the cyclic group $\langle g \rangle$.

Case (i) The fixed point set of Fix($g$) contains a surface $\Sigma$.

By Theorem 2.1, it contains at most a 2-dimensional component. Consider the fixed points of $g$ outside $\Sigma$, saying, $p_1, \cdots, p_n$. Note that $\Sigma$ is either a 2-sphere or a real projective plane.

If all $p_i$ are isolated fixed points, i.e., $g$ acts freely on the unit 3-sphere of the tangent space $T_{p_i}M$ for all $p_i$, by Proposition 3.3 we know that $n \leq 5$. Hence $\chi(M) = \chi(M^g) \leq 7$.

If not, let $p$ be a non-isolated fixed point of $g$ outside $\Sigma$. Let $g^k$ be an element of $\langle g \rangle$ whose fixed point set contains a surface $F$ passing through $p$. By Theorem 2.1 $\Sigma \cap F \neq \emptyset$. Thus, at an intersection point $q \in \Sigma \cap F$, $g^k$ acts trivially on both the tangent space $T_q\Sigma$ and $T_qF$. Clearly, $F \neq \Sigma$ implies the tangent spaces $T_qF \neq T_q\Sigma$. Since $g \in G_0$, $T_qF + T_q\Sigma$ is the tangent space $T_qM$. Thus $g^k$ acts trivially on $T_qM$. A contradiction since the action is effective.

Case (ii). The fixed point set Fix($g$) is zero dimensional.

As in the above, if all fixed points of $g$ are isolated, by Proposition 3.3 Fix($g$) contains at most 5 points. Therefore by (4.1) we get $\chi(M) \leq 5$.

If not, let $p_1, \cdots, p_m$ be the non-isolated fixed points of $g$. Therefore, there are surfaces $F_1, \cdots, F_m$ passing through $p_1, \cdots, p_m$ with corresponding isotropy groups of order $k_1, \cdots, k_m$. We may assume that $k_i$ are all smaller than 61, otherwise, we may use the isotropy group instead of $\langle g \rangle$ to reduce to Case (i). The $F_i$'s are all orientable surface with positive sectional curvature since $g \in G_0$, therefore $F_i = S^2$. By Theorem 2.1, $F_i \cap F_j \neq \emptyset$, for all $1 \leq i, j \leq m$. Observe that the intersections are zero dimensional if $i \neq j$. Because that $g$ acts on $F_i \cap F_j$, and the action of $g$ on a sphere has only two isolated points, $F_i \cap F_j$ is contained in $\mathrm{Fix}(g)$.

Secondly, it is easy to see that no three surfaces, say, $F_1, F_2, F_3$, can intersect at some point $p_i \in \mathrm{Fix}(g)$. Therefore, by Theorem 2.1 again, we see that $m \leq 3$. Otherwise, there are two non-intersecting totally geodesic surfaces in $M$. If $m \leq 2$, the desired result follows by Prop. 3.3 and the formula (4.1). If $m = 3$, first observe that the integers $k_1, k_2, k_3$ (at most 60) are pairwisely coprime. Since the order of $g$ is at least $61^4$, one gets readily a subgroup $G_1 \subset \langle g \rangle$ of order $|G_1| \geq 61$ with only isolated fixed points. By Proposition 3.3 we get $\chi(M) \leq 5$. □

By Lemmas 4.1 and 4.2 we get immediately that

**Corollary 4.3.** *Let $M$ be a compact 4-manifold with positive sectional curvature. Let $G \subset Isom(M)$. If $|G| \geq C$, then $\chi(M) \leq 7$, where $C = 61^8 \times |GL(\mathbb{Z}_3, C(4))|$, and $C(4) \approx 10^{10 \times 4^4}$ the Gromov constant.*

In the following we will always use $C$ to indicate this universal constant.

**Lemma 4.4.** *Let $M$ be a compact 4-manifold with positive sectional curvature, and let $G_0$ be as above. If $b_2(M) \geq 3$ or $b_2(M) = 2$ but $M \neq S^2 \times S^2$, then $G_0$ is cyclic.*

*Proof.* If not, there is a subgroup $\mathbb{Z}_p^2$ in $G_0$. By [22] we know that $\mathbb{Z}_p^2$ has exactly $b_2(M) + 2$ fixed points. It is easy to see that, at each fixed points, there are exactly two submanifolds of dimension 2 with isotropy groups isomorphic to $\mathbb{Z}_p$. If $b_2(M) \geq 2$, there are at least two totally geodesic surfaces in $M$ not intersecting to each other. A contradiction to Theorem 2.1. □

By Lemmas 4.1, 4.2, 4.3, Theorem 2.4 and [23] for $M = S^2 \times S^2$, it is straightforward to verify that

**Lemma 4.5.** *Let $M$ be a compact 4-manifold with positive sectional curvature. Let $G \subset Isom(M)$ be a finite group so that $|G| \geq C$. If $M \neq S^4$, then $G$ contains a normal cyclic subgroup of index less than $|SL(\mathbb{Z}_3, 5)|$.*

The assumption of $M \neq S^4$ is necessary, since clearly, $\mathbb{Z}_p \oplus \mathbb{Z}_p$ acts on $S^4$ by isometries for any $p$. The estimate above is not sharp. Indeed, if $b_2(M) = 4, 5$, the fixed point of some element must contain a surface $\Sigma$, and we can show much sharper estimate. If $b_2(M) = 2, 3$, since the manifold has only few homeomorphism types, and the automorphisms preserve the intersection forms, we may have a better estimate.

Recall that a polyhedral group is a subgroup of $SO(3)$. By Theorems 2.1, 2.4, Lemma 4.4 and [23] it is easy to see that

**Proposition 4.6.** *Let $M$ be a compact 4-manifold with positive sectional curvature. If $G$ acts on $M$ effectively by isometries, but trivially on homology, then there exists a constant $C$, such that if $|G| \geq C$, then either $G$ is cyclic and $3 \leq b_2(M) \leq 5$, or $b_2(M) \leq 2$ and $G$ is a polyhedral group, or a non-cyclic subgroup of $PU(3)$.*

## 5. Odd order isometries

Let $M$ be an oriented 4-manifold with positive sectional curvature. Let $G$ be a group of odd order acting on $M$ by isometries.

*Proof of Theorem 1.1.* Let $G_0$ be the normal subgroup of $G$ acting homologically trivial on $H_2(M)$. By the first section we know that $b_2(M) \leq 5$, and if $b_2(M) \geq 3$, then $G_0$ is cyclic. We will prove $G$ is always abelian of rank at most 2, in all cases below except case (3).

Case (1) $b_2(M) \geq 3$.

If the fixed point set of $G_0$ contains a surface, $\Sigma$. By the argument before, since $G$ acts on $\Sigma$, and the order of $G$ is odd, so $\Sigma$ is orientable, and $G$ acts preserving the orientation of $\Sigma$. Thus $\Sigma = S^2$, and every element of $G/G_0$ acts on $\Sigma$ with (and only) isolated fixed points. Therefore, by local isotropy representation at a fixed point in $\Sigma$ it is easy to see that $G_0$ is in the center of $G$. If $G/G_0$ acts non-effectively on $\Sigma$, it contains a normal subgroup acting trivially on $\Sigma$, which generates a cyclic subgroup with $G_0$, since both of them acts on the normal two plane of $\Sigma$ freely, and the action of $G$ is effective on $M$. For the same reasoning above, this enlarged cyclic subgroup is again in the center of $G$. We keep to use $G_0$ to denote this group. By [19] $G/G_0$ is a subgroup of $SO(3)$. Thus it must be cyclic since the order is odd. This clearly implies that $G$ is an abelian group of rank at most 2.

The same argument applies equally to the case when $G_0$ has non-trivial subgroup with 2-dimensional fixed point set. Therefore, we may assume that $G_0$ acts *pseudofreely* on $M$. From the proof of Corollary 4.3 this case we may upgrade the estimate to $\chi(M) \leq 5$. Therefore $b_2(M) = 3$, and $G_0$ is cyclic by Lemma 4.4. Observe that the fixed point set $\mathrm{Fix}(G_0)$ consists of exactly five points. Since $G_0$ is normal subgroup, $G$ acts on the five fixed points, this defines a homomorphism $\rho : G \to S_5$, the full permutation group of 5-letters. On the other hand, the manifold $M$ is homeomorphic to $\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2$ or $\mathbb{C}P^2\#S^2 \times S^2$, up to possibly an orientation reversing (cf. [10]). For these manifolds, the automorphism groups $\mathrm{Aut}(H_2(M))$ has no order 5 element, indeed, up to 2-torsion element, it has order $(2^3-1)(2^3-2)(2^3-4)$. Therefore, the odd order group $\rho(G) \subset S_5$ is isomorphic to $\mathbb{Z}_3$, or trivial. In either case, this shows that the action $G$ on the five points has at least two fixed points. For any such a fixed point, the isotropy representation gives an embedding of $G$ into $SO(4)$. Because all odd order subgroup of $SO(4)$ is abelian of rank at most 2, the desired result follows.

Case (2) $b_2(M) = 2$.

In this case, $M$ is homeomorphic to $S^2 \times S^2$, $\mathbb{C}P^2\#\mathbb{C}P^2$, or $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, up to possibly an orientation reversing (cf. [10]). It is easy to see that the automorphisms induced by $G$-action on $M$ on $H_2(M)$ has no nontrivial odd order element. Therefore, $G = G_0$.

If $G$ contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for some odd prime $p$, by [22], if $M$ is not $S^2 \times S^2$, then $G$ has four isolated fixed points. Therefore, by isotropy representation at these points we find two non-intersecting totally geodesic surfaces in $M$. A contradiction to Theorem 2.1. By [23] $G$ is cyclic, if $M = S^2 \times S^2$ and the action is pseudofree. If $M = S^2 \times S^2$ and the action is not pseudofree, we may have two disjoint totally geodesic surfaces among the isotropy representations of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ at the four fixed points. A contradiction to Theorem 2.1 again. Therefore, we may assume in the following that, $G$ **has no abelian subgroup of rank** 2 and $M \neq S^2 \times S^2$.

For any $g \in G$, by (4.1) the fixed point set has Euler charactericitic 4. If its fixed point set contains a surface $\Sigma$, then the rest must be two isolated points (by Theorem 2.1). Therefore, the normalizer $N(g)$ must fix the two points since its order is odd. This implies that $N(g)$ is a subgroup of $SO(4)$, and by the assumption we further conclude that $N(g)$ cyclic. The same argument applies also to the case the fixed points of $g$ all isolated. Therefore, for any $g \in G$, the normalizer $N(g)$ is cyclic. In particular, this implies that any $p$-Sylow subgroup of $G$ is cyclic. By the Burnside theorem (cf. [28] page 163) we know that $G$ must contain a cyclic normal subgroup of finite index, and so $G$ itself is cyclic.

Case (3) $b_2(M) = 1$.

Since $H_2(M) \cong \mathbb{Z}$, the odd order assumption implies that $G$ acts on $H_2(M)$ trivially. By Theorem 2.4 and [26] the desired result follows.

Case (4) $b_2(M) = 0$.

By [10] $M \cong S^4$. If the action of $G$ on $M$ is pseudofree, i.e. the singular set consists of isolated points, by [19] we know that $G$ is a polyhedral group or dihedral group. Consequently, $G$ must be cyclic since $|G|$ is odd. Otherwise, let $g \in G$ is an element with a 2-dimensional fixed point set $\Sigma$. For any $h \in G$, the totally geodesic surface $h(\Sigma)$ intersects with $\Sigma$ at some point $p$. Thus the isotropy group of $p$ contains the subgroup generated by $g, h$. This subgroup clearly has $p$ as a fixed point, so it must be a subgroup of $SO(4)$. In particular, $\langle g, h\rangle$ is abelian of rank at most 2. This proves that $g$ is in the center of $G$. Therefore, $G$ acts on $\Sigma$. Obviously the effective part of $G/\langle g\rangle$ action on $\Sigma$ is cyclic (note that $\Sigma = S^2$). Thus $G$ is an abelian group of rank at most 2. This completes the proof. □

*Proof of Theorem 1.2.* (1.2.2) follows readily from the proof of Theorem 1.1. It remains to prove (1.2.1).

Let $G = \mathbb{Z}_{pq} \oplus \mathbb{Z}_p$, where $p$, $q$ are odd integers. By Corollary 4.3 the second Betti number $b_2(M) \leq 5$. The fixed point set $F$ of $\mathbb{Z}_{pq}$ consists of the union of a connected surface $\Sigma$ with at most 5 points, where $\Sigma$ may be empty, $S^2$.

If $M = S^2 \times S^2$, it is easy to see that $G = G_0$, where $G_0$ is as in Lemma 4.1. The $G$ action has four fixed points. Therefore, there is a subgroup whose fixed point set is the union of two surfaces, a contradiction by Theorem 2.1.

Assume now $b_2(M) \geq 3$ or $b_2(M) = 2$ and $M$ has odd type intersection form. By Theorem 2.3 $G_0$ is cyclic. We may assume that $G_0 = \mathbb{Z}_{pq}$, and $\mathbb{Z}_p$ is a subgroup of $\mathrm{Aut}(H_2(M))$. Since $p$ is odd, by analyzing the automorphism group it remains only to consider the following cases:

(i) $b_2(M) = 3$, and $M = 3\mathbb{C}P^2$ (or $3\overline{CP}^2$);

(ii) $b_2(M) = 5$, and $M = r\mathbb{C}P^2 \# (5-r)\overline{\mathbb{C}P}^2$, $0 \leq r \leq 5$;

In all these cases, the automorphism group may have order 3 or 5 subgroups, correspondingly $p = 3$ or 5.

In case (i), note that $p = 3$ and an order 3 subgroup of $\mathrm{Aut}(H_2(M))$ has trace zero. We may assume that the second factor, $\mathbb{Z}_3$ of $G$ has trace zero on $H_2(M)$. By the formula (4.1) the fixed point set of $\mathbb{Z}_3$ has Euler characteristic 2. If $\mathbb{Z}_{3q}$ has only isolated fixed points, by (4.1) once again it has exactly 5 fixed points. Obviously, $\mathbb{Z}_3$ keeps at least two points fixed on the five points. If $\mathbb{Z}_{3q}$ has a surface $\Sigma = S^2$ in its fixed point set, $\mathbb{Z}_3$ acts on the sphere with two isolated fixed points. By Theorem 2.1 it is easy to see that $G$ can not have more than three fixed points. Therefore, we may assume in either cases that $G$ has exactly two fixed points, say $x_1$ and $x_2$. By the isotropy representations at $x_1$ and $x_2$ there is an order 3 element $g \in G$ with a 2-dimensional fixed point set so that $tr(g|_{H_2(M)}) = 0$. By (4.1) again the fixed point set of $g$ has Euler characteristic 2 and so $\mathrm{Fix}(g) = S^2$. On the other hand, by the Atiyah-Singer $G$-signature formula (cf. [20] page 266, Corollary 14.9) the $g$-signature $\mathrm{sig}(g, M) > 0$. Since the intersection form of $M$ is positive definite (or negative definite), by definition $tr(g|_{H_2(M)}) = \mathrm{sig}(g, M) > 0$ (or $< 0$). A contradiction.

In case (ii), by the proof of Corollary 4.3 we may assume that $\mathbb{Z}_{pq}$ has a surface $\Sigma = S^2$ in its fixed point set. By the Lefschetz fixed point formula (4.1) $\mathbb{Z}_{pq}$ has exactly five isolated fixed points outside $\Sigma$. Since $G$ acts on $\Sigma$ with two fixed points (note that $p$ is odd), and $G$ also acts on the five points, by Theorem 2.1 it remains only to consider the case of $p = 5$ and the second factor of $G$, $\mathbb{Z}_5$, acts freely on the five points. Since the automorphism group $\mathrm{Aut}(H_2(M))$ does not contain 5-torsion unless $r = 0$ or 5 (the author thanks Ian Hambleton for pointing out this fact), and such an order 5 automorphism has trace zero, the second factor $\mathbb{Z}_5$ has trace zero. The same argument in case (i) applies equally to arrive at a contradiction. □

## 6. Proof of Theorem 1.5

For an involution on compact 4-manifold $M$, the Atiyah-Singer $G$-signature theorem tells us

$$\mathrm{sig}(M, T) = \#(F \cap F) \qquad (6.1)$$

where $F$ is the fixed point set of $T$, and $\#(F \cap F)$ is the self-intersection number of $F$. In particular, if $T$ has no 2-dimensional fixed point set, then $\mathrm{Sig}(M, T) = 0$.

Let $M$ be a compact 4-manifold with positive definite intersection form. For an involution as above, by definition we get $\mathrm{sig}(M, T) = \mathrm{tr}(T|_{H^2(M;\mathbb{Q})})$. This together with the formula (4.1) implies that

$$\chi(\mathrm{Fix}(T)) = 2 + \mathrm{sig}(M, T) \qquad (6.2)$$

*Proof of Theorem 1.4.* Since $M$ has definite intersection form, it is easy to see that $\mathbb{Z}_2^k$ acts on the homology group $H_2(M; \mathbb{Z}_2)$ by permutation group $S_n$ with respect to the standard basis of $H_2(M; \mathbb{Z}_2)$, where $n = b_2(M)$.

For an involution $T_1 \in \mathbb{Z}_2^k$, we claim that $\mathrm{Fix}(T_1)$ is not empty. If not, by [5] VII 7.4 we know that $(T_1)_* \in S_n$ is an alternation without fixed letter, otherwise there is a surface in the fixed point set of $T_1$. Therefore, the trace $\mathrm{tr}T_1|_{H^2(M;\mathbb{Q})} = 0$. By (4.1) $\chi(\mathrm{Fix}(T_1)) = 2$. A contradiction.

If $T_1$ is an involution with only isolated fixed points, the number of its fixed points is equal to 2, otherwise, by (6.2) $\mathrm{sig}(M, T_1) \neq 0$ and then by (6.1) we get a contradiction. If $T_1$ has only two isolated fixed points, since $\mathbb{Z}_2^k$ acts on the set of the two points, there exists a point whose isotropy group contains subgroup isomorphic to $\mathbb{Z}_2^{k-1}$. This clearly implies that $T_1$ and $T_2$ has a common fixed point, for some $T_2 \in \mathbb{Z}_2^k$. Therefore, $T_2$ has a 2-dimensional fixed point set, $F$, by the local isotropy representation. Note that $F$ represents a non-trivial homology class in $H_2(M;\mathbb{Z}_2)$ and the self-intersection number is non-zero, by the definiteness of the intersection form. Consider $\mathbb{Z}_2^k$-action on $F$. Let $G_0$ denote the principal isotropy group. We first claim that $G_0$ is generated by $T_2$, otherwise, there is an element with three dimensional fixed point set. A contradiction since the action preservs the orientation. Now $\mathbb{Z}_2^k/G_0$ acts on $F$, where $F = S^2$ or $\mathbb{R}P^2$ since the curvature is positive. For the latter it is easy to see that $\mathbb{Z}_2^k/G_0$ has $\mathbb{Z}_2$ rank at most 2 (cf. [19]), and for the former, $\mathbb{Z}_2^k/G_0$ has $\mathbb{Z}_2$-rank at most 3, and so $k \leq 4$. □

## 7. An algebraic theorem

In this section we consider central extensions of cyclic groups by polyhedral groups. We will prove that all of these groups can be embedded in $SO(5)$.

**Theorem 7.1.** *Let $G$ be a finite group which is a central extension of a cyclic group $\mathbb{Z}_k$ by a group $H$, where $H$ is a finite subgroup of $SO(3)$. Then $G$ is isomorphic to a subgroup of $SO(5)$.*

Recall that the finite subgroup of $SO(3)$ (i.e. polyhedral group) is cyclic, a dihedral group, a tetrahedral group $T$, an octahedral group $O$, or an icosahedral group $I$. It is well-known that $T \cong A_4$, the alternating group of 4 letters, $O \cong S_4$, the full permutation group of 4-letters, and $I \cong A_5$, the alternating group of 5 letters.

By group extension theory, a central extension

$$1 \to \mathbb{Z}_m \to G \to \bar{G} \to 1$$

is uniquely characterized by the $k$-invariant $k \in H^2(\bar{G}, \mathbb{Z}_m)$, and the trivial extension corresponding to the product $\bar{G} \times \mathbb{Z}_m$, has trivial $k$-invariant.

**Lemma 7.2.** *The cohomology groups*

$H^2(A_4, \mathbb{Z}_m) \cong \mathbb{Z}_m \otimes \mathbb{Z}_6$;
$H^2(S_4, \mathbb{Z}_m) \cong \mathbb{Z}_m \otimes \mathbb{Z}_2$;
$H^2(A_5, \mathbb{Z}_m) \cong \mathbb{Z}_m \otimes \mathbb{Z}_2$;
$H^2(D_{2k}, \mathbb{Z}_m) = 0$, *if $m$ is odd, and*
$H^2(D_{2k}, \mathbb{Z}_m) \cong \mathbb{Z}_2 \otimes \mathbb{Z}_m$, *if $k$ is odd, and*
$H^2(D_{2k}, \mathbb{Z}_m) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \otimes \mathbb{Z}_m$, *if $k$ is even*

The lemma can be verified via spectral sequence (cf. [1]).

**Lemma 7.3.** *Let $G$ be a nontrivial central extension of $\mathbb{Z}_m$ by $S_4$ or $A_5$. Then $G$ is isomorphic to $\mathbb{Z}_m \otimes_{\mathbb{Z}_2} O^*$ or $\mathbb{Z}_m \otimes_{\mathbb{Z}_2} I^*$, where $O^*$ (resp. $I^*$) is the binary octahedral group (resp. binary icosahedral group).*

*Proof.* It is clear that the group $\mathbb{Z}_m \otimes_{\mathbb{Z}_2} O^*$ (resp. $\mathbb{Z}_m \otimes_{\mathbb{Z}_2} I^*$) is a nontrivial central extension of $\mathbb{Z}_m$ by $O$ (resp. $I$). By Lemma 7.2 this is the only nontrivial central extension, where $m$ is even. □

By Lemma 7.2 it is straightforward to verify the following three lemmas.

**Lemma 7.4.** *Let $G$ be a nontrivial central extension of $\mathbb{Z}_m$ by $A_4$.*

*(7.4.1) If $m = 3^r m_+$, $(m_+, 6) = 1$, then $G \cong \mathbb{Z}_{m_+} \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes_\alpha \mathbb{Z}_{3^r}$ where $\alpha : \mathbb{Z}_{3^r} \to \mathbb{Z}_3 \to Aut(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ is the unique nontrivial homomorphism;*

*(7.4.2) If $m = 2^r m_+$, $(m_+, 6) = 1$, then $G \cong \mathbb{Z}_{m_+} \times \mathbb{Z}_{2^r} \times_{\mathbb{Z}_2} (Q_8 \rtimes_\alpha \mathbb{Z}_3)$, where $(Q_8 \rtimes_\alpha \mathbb{Z}_3)$ is the binary tetrahedral group $T^*$.*

If $m = 2^r \times 3^s m_+$ and the $k$-invariant of the extension contains both 2 and 3-torsions,

**Lemma 7.5.** *Let $G$ be a nontrivial central extension of $\mathbb{Z}_{2^r m_+ \times 3^s}$ by $A_4$ whose $k$-invariant contains both 2 and 3-torsions, where $(m_+, 6) = 1$. Then $G \cong \mathbb{Z}_{2^r m_+} \times_{\mathbb{Z}_2} (Q_8 \rtimes_\alpha \mathbb{Z}_{3^{s+1}})$.*

**Lemma 7.6.** *Let $G$ be a nontrivial central extension of $\mathbb{Z}_m$ by $D_{2k}$, where $D_{2k}$ is the dihedral group of order $2k$ and $k$ is odd. Then $G \cong \mathbb{Z}_m \times_{\mathbb{Z}_2} D_{4k}$.*

*Proof of Theorem 7.1.* Of course $SO(4)$ contains every rank 2 abelian subgroup. If $G$ is a central extension by a cyclic subgroup of $SO(3)$, it is a rank $\leq 2$ abelian group, and so isomorphic to a subgroup of $SO(4)$.

Both groups in Lemma 7.3 can be embedded into $SO(4)$ in a standard way, since $SO(4) = S^3 \times_{\mathbb{Z}_2} S^3$, and $I^*, O^*$ are subgroups of $S^3$. If the extensions are trivial, both groups are contained in $SO(3) \times SO(2) \subset SO(5)$.

For a group of type (7.4.1), it is isomorphic to a subgroup of $SO(3) \times SO(2) \subset SO(5)$. Indeed, it is generated by the subgroup $A_4 \subset SO(3)$ and the cyclic subgroup of order $3^r m_+$ of $SO(3) \times SO(2)$ given by a block diagonal $5 \times 5$-matrix whose minor $3 \times 3$ block is the order 3 permutation matrix, and the minor $2 \times 2$ block is the order $3^r m_+$ rotation matrix.

For a group in Lemma 7.5 (and also of type (7.4.2)), note that the group $Q_8 \rtimes_\alpha \mathbb{Z}_{3^{s+1}}$ is a subgroup of $U(2)$ acting freely on $S^3$ (cf. [28] page 224). Take the cyclic group $\mathbb{Z}_{2^r m_+}$ in the center of $U(2)$, it has a common order 2 subgroup with $Q_8$ and so it together with $Q_8 \rtimes_\alpha \mathbb{Z}_{3^{s+1}}$ generates a subgroup of $U(2)$ of the type in Lemma 7.5.

It remains to consider central extensions by a dihedral group.

By Lemma 7.6 the group in that lemma is clearly a subgroup of $O(4) \subset SO(5)$. It suffices to prove the theorem by considering the extension by a dihedral group of order divisible by 4. By Lemma 7.2 we may assume further that $m = 2^r$. Consider the 2-Sylow group of $G$, saying $G_0$, which is a central extension of $\mathbb{Z}_m$ by $D_{2^s}$. By Lemma 7.2, the group $H^2(D_{2^s}; \mathbb{Z}_{2^r}) \cong \mathbb{Z}_2^3$, with generators, $\beta(x), \beta(y), w$, where $\beta$ is the Bockstein homomorphism associated to the extension $1 \to \mathbb{Z}_2 \to \mathbb{Z}_{2^{r+1}} \to \mathbb{Z}_{2^r} \to 1$.

If the extension is nontrivial over $\mathbb{Z}_{2^{s-1}} \subset D_{2^s}$, then $G$ contains an element generating a subgroup of index 2. By the classification of 2-groups with an index 2 cyclic subgroup the desired result follows (cf. Wolf's book [28] page 173). Otherwise, by [1] page 130 the proof of Lemma 2.11 we may determine the group $G$. It is easily seen that the group $G$ is a subgroup of $SO(5)$. This proves the desired result. $\square$

## 8. Even order isometries, I

In this section we consider group $G$ of isometries of 4-manifold $M$ of even order with the assumption, $G_0$ *has no element with* 2*-dimensional fixed point set*, where $G_0$ is the normal subgroup of $G$ acting trivially on the homology group $H_2(M)$. By Lemma 4.4, $G_0$ is cyclic, unless $b_2(M) \leq 1$ or $M = S^2 \times S^2$. If the order $|G| \geq C$, by Corollary 4.3 $b_2(M) \leq 5$, and this may be improved to $b_2(M) \leq 3$ under the above assumption, by Lemma 4.1 and the proof of Lemma 4.2.

**Lemma 8.1.** *Let $M$ be a compact oriented* 4*-manifold with positive sectional curvature. Let $G \subset Isom(M)$ be a finite group. Then $G$ is an extension of a cyclic group $G_0$ by a subgroup of $Aut(H_2(M))$. If additionally the intersection form of $M$ is odd type and $G_0$ has no element with a fixed point set of dimension* 2*, then $G_0$ has odd order.*

*Proof.* The former one is by Theorem 2.3. The latter fact is because any involution must have a 2-dimensional fixed point, by [5] VII Lemma 7.6. □

Let $M$ be a compact simply connected 4-manifold with $2 \leq b_2(M) \leq 3$. If the intersection form of $M$ is odd type, by [10], $M$ is homeomorphic to $3\mathbb{C}P^2$, $2\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, $2\mathbb{C}P^2$, and $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, by possibly reversing the orientation.

Let $G \subset \mathrm{Isom}(M)$ be a finite group. Consider the induced homomorphism $\rho : G \to \mathrm{Aut}(H_2(M;\mathbb{Z}))$. Let $h : \mathrm{Aut}(H_2(M;\mathbb{Z})) \to \mathrm{Aut}(H_2(M;\mathbb{Z}_2))$ be the forgetful homomorphism.

**Lemma 8.2.** *Let $M$ be a compact oriented* 4*-manifold with positive sectional curvature. Let $G \subset Isom(M)$ be a finite group with induced homomorphism $\rho : G \to Aut(H_2(M;\mathbb{Z}))$. If $2 \leq b_2(M) \leq 3$ and the intersection form is odd type, then $\rho(G) \cap ker(h)$ can not contain $\mathbb{Z}_2^2$.*

*Proof.* First note that the kernel of the forgetful homomorphism $h$ is $\mathbb{Z}_2^k$, generated by the reflections on the three factors of $\mathbb{Z}^k$, where $k = b_2(M)$. If $\rho(G) \cap \ker(h)$ contains $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, let $F$ denote the fixed point set of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. By the same spectral sequence calculation in [22] with $\mathbb{Z}_2$-coefficients the homology $H^*(F;\mathbb{Z}_2)$ has rank $b_2(M) + 2$, where $F$ has dimension at most 1.

If $b_2(M) = 2$, and $\rho(G) \cap \ker(h)$ contains $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, there is an involution $T$ whose trace on $H_2(M;\mathbb{Z})$ is $-2$. By the trace formula (6.1) again the Euler characteristic $\chi(F(M;T)) = 0$. A contradiction, since the fixed point set has positive sectional curvature, and so positive Euler characteristic.

If $b_2(M) = 3$, there is an involution $T_1 \in \mathbb{Z}_2^2$ whose trace on $H_2(M;\mathbb{Z})$ is $-1$ (it can not be $-2$ by (6.1) since the fixed point set has positive Euler characteristic). By the trace formula (6.1) the Euler characteristic $\chi(F(M;T_1)) = 1$. This together with Lemma 9.1 implies that $F(M;T_1) = \mathbb{R}P^2$ (since the positive sectional curvature). For another involution $T_2 \neq T_1 \in \mathbb{Z}_2^2$, since $F$ is equal to the fixed point set of $T_2$ on $F(M;T_1)$, which is either a point or $\mathbb{R}P^1 \cup \{pt\}$. Therefore, the homology $H^*(F;\mathbb{Z}_2)$ has rank at most 3. A contradiction. □

**Lemma 8.3.** *Let $M$ be a compact oriented* 4*-manifold with positive sectional curvature. Let $G \subset Isom(M)$. Assume that $2 \leq b_2(M) \leq 3$, and the intersection form of $M$ is* *odd type. If $G_0$ has no element with a fixed point set of dimension 2, then $G$ is one of the following type*

$$\mathbb{Z}_n,\ \mathbb{Z}_n \rtimes \mathbb{Z}_2,\ \mathbb{Z}_n \rtimes \mathbb{Z}_4,\ \mathbb{Z}_n \rtimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2);$$

*where $n$ is odd.*

*Proof.* There are four cases to consider:

Case (i) If $M = 3\mathbb{C}P^2$;

By Lemma 8.2, $G/G_0$ is a subgroup of a central extension of $\mathbb{Z}_2$ by $S_3$, the permutation group of three letters. Therefore, the 2-Sylow group of $G$ is $\mathbb{Z}_2$, $\mathbb{Z}_2^2$ or $\mathbb{Z}_4$. By Theorems 1.1 and 1.2 the odd order subgroup of $G$ is cyclic. Therefore, $G$ is isomorphic to one of the following:

$$\mathbb{Z}_n,\ \mathbb{Z}_n \rtimes \mathbb{Z}_2,\ \mathbb{Z}_n \rtimes \mathbb{Z}_4,\ \mathbb{Z}_n \rtimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$$

where $n$ is odd.

Case (ii) If $M = 2\mathbb{C}P^2$;

By Lemma 8.2 again $\rho(G) = G/G_0$ is a subgroup of an extension of $\mathbb{Z}_2$ by $S_2 = \mathbb{Z}_2$. It is easy to see that $G/G_0 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2$. Hence $G$ is either cyclic, $G = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_n \rtimes \mathbb{Z}_4$, where $n$ is odd.

Case (iii) If $M = 2\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$;

As above, by Lemma 8.2 one may show that $G/G_0$ is $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$. Thus, $G$ is cyclic, or one of the extensions $\mathbb{Z}_n \rtimes \mathbb{Z}_2$, $\mathbb{Z}_n \rtimes \mathbb{Z}_4$, $\mathbb{Z}_n \rtimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, where $n$ is odd.

Case (iv) If $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$;

By Lemma 8.2 it is easy to see that $\rho(G) \cong \mathbb{Z}_2$. Thus $G$ is cyclic or $G \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$. $\square$

By [22] one sees that $G_0$ is a polyhedral group if $M = S^2 \times S^2$. The following lemma is essentially due to McCooey.

**Lemma 8.4.** *Let $M$ be a 4-manifold homeomorphic to $S^2 \times S^2$. Let $G$ be a finite group acting effectively on $M$ preserving the orientation. Then, $G$ is an extension of a polyhedral group by a subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, if $G$ is an extension of a polyhedral group by $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $G$ is isomorphic to either of*

*(8.4.1)* $D^*_{2^n m_-} \times \mathbb{Z}_{m_+}$

*(8.4.2)* $((\mathbb{Z}_{m_-} \rtimes \mathbb{Z}_{2^{n+1}}) \times \mathbb{Z}_{m_+}) \rtimes \mathbb{Z}_2$ *with $n > 1$;*

*where $m_+, m_-, n$ are all odd integers, $(m_+, m_-) = 1$.*

A complete list of the group $G$ in Lemma 8.4 may be found in [22].

## 9. Even order isometries, II

Let $M$ be a compact oriented 4-manifold with positive sectional curvature. Let $G \subset \mathrm{Isom}(M)$ be a finite group. Let $G_0$ be the subgroup of $G$ acting trivially on homology. In this section we consider the case that some element of $G_0$ has 2-dimensional fixed point set, saying, $\Sigma$. Observe that $\Sigma = S^2$ or $\mathbb{R}P^2$, since it is totally geodesic with positive sectional curvature. Since $G_0$ is normal, $G$ acts on the surface $\Sigma$. For simplifying the notions, we may assume that $\Sigma$ is a fixed point component of $G_0$ itself (one should keep in mind it might be a cyclic subgroup of $G_0$ which does the job). This implies immediately that the effective part of $G/G_0$ action on $\Sigma$ is a subgroup of $O(3)$ (cf. [19]).

**Lemma 9.1.** *Let $M$, $G$ and $G_0$ be as above. Assume that $G_0$ is cyclic with a fixed point surface $\Sigma$. If every element of $G$ has at least an isolated fixed point in $\Sigma$. Then $G_0$ is in the center of $G$.*

*Proof.* For any $g \in G$ with a fixed point $p \in \Sigma$, since $g$ acts on $\Sigma$, the isotropy representation of $g$ at $T_pM$ splits into $T_p\Sigma \oplus \nu_p(\Sigma)$, the tangent and normal 2-planes of $\Sigma$ at $p$. Therefore, the group $\langle g, G_0\rangle$ has a representation of dimension 4 which splits into the sum of two dimensional representations. Since $p$ is isolated, $g$ must preserve the orientation of $T_p\Sigma$ and so also the orientation of $\nu_p(\Sigma)$. Similarly, $G_0$ preserves the orientation of $\nu_p(\Sigma)$. This shows that $G_0$ commutes with $g$. The desired result follows. $\square$

Case (i) $\Sigma = \mathbb{R}P^2$.

**Lemma 9.2.** *Let $M$, $G$ and $G_0$ be as above. Assume that $b_2(M) \geq 2$, and $M \neq S^2 \times S^2$. If $\mathbb{R}P^2 \subset Fix(G_0)$, then $G$ is a center extension of a cyclic group by either a cyclic group or a dihedral group.*

*Proof.* By Lemma 4.4 $G_0$ is cyclic. Since any isometry of $\mathbb{R}P^2$ has an isolated fixed point, by Lemma 9.1 $G_0$ is in the center of $G$. We may assume that $G/G_0$ acts effectively on $\mathbb{R}P^2$, otherwise, we may replace enlarge $G_0$ by the extension by the principal isotropy group of $G/G_0$, and the same argument in Lemma 9.1 applies equally. Therefore, by appealing to [19] $G/G_0$ is a dihedral group or a cyclic group. The desired result follows. $\square$

Case (ii) $\Sigma = S^2$.

Since any orientation preserving isometry of $S^2$ has only isolated fixed points, $G/G_0$ acts pseudofreely on $S^2$, i.e. the singular data consists of isolated points. Let $G_1 \subset G/G_0$ be the subgroup of isometries preserving the orientation of $S^2$. Clearly $G_1$ has index at most 2 in $G/G_0$. By [19] $G_1$ surjects onto a subgroup of $SO(3)$. The kernel of the surjection acts trivially on $S^2$. Therefore, the same argument of Lemma 9.1 implies its kernel together with $G_0$ generates a cyclic group in the center of $G$. This proves that

**Lemma 9.3.** *Let $M$, $G$ and $G_0$ be as above. Assume that $b_2(M) \geq 2$, and $M \neq S^2 \times S^2$. If $S^2 \subset Fix(G_0)$, then $G$ contains an index $\leq 2$ normal subgroup, $H$, which is a central extension of a cyclic group by a polyhedral group.*

Results in §7 gives a complete classification of the index at most 2 subgroup. However, in order to get more precise information on how the group $G$ might be in Lemma 9.3, we need to analyze how an orientation reversing element in $G$ conjugates on $G_0$. There are two types of involution acting on $S^2$ reversing the orientation, either a reflection with fixed point set a circle, or free action on $S^2$ with quotient $\mathbb{R}P^2$. In the former case we have

**Lemma 9.4.** *Let the assumptions be as in Lemma 9.3. If there is an $\alpha \in G$ so that its fixed point set on $S^2$ has dimension 1, then $G$ satisfies an extension*

$$1 \to H \to G \to \mathbb{Z}_2 \to 1$$

*where $H$ is as in Lemma 9.3, the conjugation $\alpha g_0 \alpha^{-1} = g_0^{-1}$, for any $g_0 \in G_0$.*

*Proof.* Let $S = \mathrm{Fix}(\alpha|_{S^2})$. Note that $\alpha|_{S^2}$ is a reflection on $S^2$ along $S$. Since $\alpha$ preserves the orientation of $M$, therefore $\alpha$ is also a reflection with respect to a line in the normal plane $\nu_p(S^2)$ (so $\alpha$ has order 2). Thus, $\alpha$ and $G_0$ generates a dihedral group. Therefore, the conjugation $\alpha g_0 \alpha^{-1} = g_0^{-1}$, for any $g_0 \in G_0$. □

Next let us consider the case that there is an element $\beta \in G/G_0$ acting freely on $S^2$ (reversing the orientation of $S^2$). By [19] we know that $G/G_0$ is a subgroup of $O(3)$. The element $\beta$ must be the center of $O(3)$, acting on $S^2$ by antipodal map.

**Lemma 9.5.** *Let the assumptions be as in Lemma 9.3. If the Euler characteristic $\chi(M) > 2$, and there is an involution $\beta \in G$ acting freely on $S^2$, then $|G_0|$ is odd.*

*Proof.* By Corollary 4.3 $\chi(M) \leq 7$. Therefore, if $M$ is of even intersection type, it is homeomorphic to $S^2 \times S^2$, or $S^2 \times S^2 \# S^2 \times S^2$, by [10]. If $|G_0|$ is even, by a result of Atiyah-Bott [2] (compare [20]), an involution in $G_0$ must have all components of the same dimension, since the Spin structure of $M$ is unique. A contradiction to Theorem 2.1 since $S^2$ is fixed by $G_0$, this implies all fixed point components (at least two by the trace formula (6.1)) of $G_0$ are of dimension 2.

It remains to consider the case when $M$ is non-Spin. If $\chi(M)$ is odd, by [5] VII Lemma 7.6 $\beta$ has 2-dimensional fixed point set. By Theorem 2.1 this fixed point set intersects $S^2$, a contradiction to the freeness of $\beta$ on $S^2$. If $\chi(M)$ is even, then $M$ is homeomorphic to $r\mathbb{C}P^2 \# s\overline{\mathbb{C}P}^2$, where $r + s = 2, 4$. Since $\beta([S^2]) = -[S^2] \neq 0$ as a homology class, one may verify case by case that there is always a class $x \in H^2(M; \mathbb{Z}_2)$, so that $\beta^*(x) = x$ and $x^2 \neq 0$. By [5] VII Lemma 7.4 $\beta$ has 2-dimensional fixed point set again. A contradiction. □

**Lemma 9.6.** *Let the assumptions be as in Lemma 9.5. Then $G$ satisfies an extension*

$$1 \to H \to G \to G/H \to 1$$

*where $H$ is an extension of the cyclic group $G_0$ by $\mathbb{Z}_2$, which is non-splitting when $|G_0|$ is even, and $G/H$ is either a cyclic or a dihedral group.*

*Proof.* Let $p : G \to G/G_0$ be the projection. Let $H = p^{-1}(\langle \beta \rangle)$. By Lemma 9.5, $H$ is a non-splitting extension of $G_0$ by $\mathbb{Z}_2 = \langle \beta \rangle$ if $|G_0|$ is even. By [19] $G/G_0$ is a subgroup of $O(3)$. The element $\beta$ is given by the antipodal map, which belongs to the center of $G/G_0$. Therefore, the quotient group, $G/H$, acts on $S^2/\langle \beta \rangle = \mathbb{R}P^2$. As we noticed in Case (i), this implies that $G/H$ is either a cyclic or a dihedral group. □

By Lemma 9.3 we know that the extension in the above lemma satisfies certain additional commutativity. We leaves it to the reader.

## 10. Proofs of Theorems 1.3 and 1.4

*Proof of Theorem 1.3.* Let $G_0$ be the subgroup of $G$ acting trivially on $H_*(M;\mathbb{Z})$. By Corollary 4.3 we may assume that $b_2(M) \leq 5$, and moreover $b_2(M) \leq 3$ if $G_0$ contains no element with 2-dimensional fixed point set. By Lemma 4.4 we may assume that $G_0$ is cyclic, or $M = S^2 \times S^2$. In this case, by Corollary 4.3, Theorems 8.3 and 8.4 we know that $G$ contains an index $\leq 2$ subgroup which is a subgroup of $SO(4)$, since $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ with $n$ odd is always a subgroup of $SO(4)$. On the other hand, if the action of $G_0$ contains 2-dimensional strata, by Lemmas 9.2, 9.3 and Theorem 7.1 we know that $G$ contains an index $\leq 2$ subgroup which is a subgroup of $SO(5)$, provided $M \neq S^2 \times S^2$. Finally, if $M = S^2 \times S^2$, and $G_0$-action is not pseudofree, as in the proof of Lemma 9.5, $|G_0|$ has to be odd. Therefore, by Theorem 1.2 $G_0$ is cyclic of odd order. Now the same proof of Lemmas 9.2 and 9.3 applies to conclude that $G$ contains an index $\leq 2$ subgroup embedded in $SO(5)$. The desired result follows. □

*Proof of Theorem 1.4.* By Lemmas 8.3, 8.4 and Theorem 2.4 we may assume that $G_0$ has 2-dimensional strata. By [26] we may assume that $b_2(M) \geq 2$. If $M = S^2 \times S^2$, as in the proof of Theorem 1.3 $G_0$ is cyclic of odd order. The desired result follows since $\mathrm{Aut}(H_2(M,\mathbb{Z})) = \mathbb{Z}_2^2$. Therefore, it remains to consider the case where $5 \geq b_2(M) \geq 2$, and $M \neq S^2 \times S^2$. By Lemmas 9.2 and 9.3, the index $\leq 2$ subgroup $H$ is a central extension of a cyclic group by a polyhedral group. Since a polyhedral group other than cyclic or dihedral has order at most 60, it suffices to consider when $H$ is a central extension of a cyclic group by a cyclic group or a dihedral group. Therefore, $H$ contains an abelian subgroup of index $\leq 2$ and rank at most 2. Therefore, it suffices to prove that this rank 2 abelian subgroup contains a cyclic component of index at most 30.

Assume an abelian group $A$ of rank 2 with $G_0 \subset A$. By Theorem 1.2 the odd order subgroup of $A$ is cyclic. Clearly $G_0$ has even order. By the proof of Lemma 9.5 we may assume that the intersection form of $M$ is odd type. Assume that $M = r\mathbb{C}P^2 \# s\overline{\mathbb{C}P^2}$, where $r + s \leq 5$. It is easy to see that the automorphism group $\mathrm{Aut}(H_2(M,\mathbb{Z}))$ is a normal extension of $\mathbb{Z}_2^{r+s}$ by a subgroup of $S_{r+s}$, the permutation group of $(r+s)$ letters. Therefore, the 2-Sylow subgroup of $A/G_0$ has order $\leq 16$. The desired result follows. □

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