Bott-Chern Secondary Characteristic Objects and Arithmetic Riemann-Roch Theorem - II -

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(Last Preliminary Version)

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Part II

Contents

Contents

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Arithmetic Riemann-Roch Theorem

Chapter II.1. Grothendieck-Riemann-Roch Theorem

In this chapter, we will give the Grothendieck-Riemann-Roch theorem in algebraic geometry and its proof. During the process, we introduce some concepts which will be used in the arithmetic Riemann-Roch theorem. Later, we state and prove the arithmetic Riemann-Roch theorem in a similar way.

For any regular algebraic variety X, there is a Chow group associated with it, denoted by $\mathrm{CH}(X)$, which is a quotient group of the free abelian group of algebraic cycles modulo rational equivalence. By the Chow moving lemma, we can introduce a unique intersection pairing on $\mathrm{CH}(X)_{\mathbf{Q}}$, and with this intersection pairing, $\mathrm{CH}(X)_{\mathbf{Q}}$ becomes a commutative ring. There are also two Grothendieck K-groups, denoted by $K_0(X)$ (resp. $K^0(X)$), which are defined as the quotient groups of the free abelian group generated by vector sheaves (resp. coherent sheaves) modulo exact sequences. Since every coherent sheaf over a regular variety has a finite vector sheaf resolution, we know that the two Grothendieck K-groups in fact are isomorphic. There is a natural multiplication on $K(X)$ and we know that there is a natural isomorphism

$$\mathrm{ch} : K(X)_{\mathbf{Q}} \to \mathrm{CH}(X)_{\mathbf{Q}}.$$

Furthermore, with respect to proper morphisms, there are functorial properties for the objects defined above. One may now state the **Grothendieck-Riemann-Roch theorem** in these terms as follows:

For any proper morphism $f : X \to Y$ of regular algebraic varieties, the diagram:

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathrm{ch}()\mathrm{td}(f)} & \mathrm{CH}(X)_{\mathbf{Q}} \\ f_K \downarrow & & \downarrow f_{\mathrm{CH}} \\ K(Y) & \xrightarrow{\mathrm{ch}()} & \mathrm{CH}(Y)_{\mathbf{Q}} \end{array}$$

is commutative.

In particular, if Y is a point, it becomes the remarkable **Hirzebruch-Riemann-Roch theorem**: For any vector sheaf $\mathcal{E}$ on a regular variety X, we have

$$\chi(X,\mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E})\mathrm{td}(X).$$

§II.1.1 Some Basic Concepts

We now introduce some basic concepts which will be used later.

II.1.1.a. Length and Order

Let R be a noetherian ring and M a finitely generated R-module. We say that M has **finite length** if there exist a positive integer l and a chain of submodules of M

$$M = M_0 \supset M_1 \supset \ldots \supset M_l = 0,$$

and maximal ideals $\mathcal{P}_i$ of R such that

$$M_{i-1}/M_i \simeq R/\mathcal{P}_i$$

for all i. Then l is the **length** of M as a R-module; more exactly, $l = l_R(M)$. It is not difficult to prove that l is well-defined. The **dimension** of R is defined as the maximum of the lengths of maximal ideals of R.

Now let R be an one-dimensional integral domain and $K := \mathrm{Frac}(R)$ the field of fractions of R. Let $f = ab^{-1} \in K^*$ with $a, b \in R$, then we define

$$\mathrm{ord}_R(f) := l_R(R/aR) - l_R(R/bR)$$

and call it the **order** of f. This is well-defined: In fact, the obvious map ord : $K^* \to \mathbf{Z}$ is a homomorphism from the multiplicative group K^* to the additive group $\mathbf{Z}$.

II.1.1.b. Torsion Modules

Let R be a Noetherian ring and M, N two R-modules. We define the **torsion modules** $\mathrm{Tor}_i^R(M,N)$ as follows: For projective resolutions $P.$ and $Q.$ of M and N respectively,

$$\mathrm{Tor}_i^R(M,N) := H_i(P. \otimes Q.).$$

II.1.1.c. Intersection Multiplicity

Let X be a noetherian and separated scheme. We say that two closed integral subschemes Y, Z of X **intersect properly**, if

$$\mathrm{codim}_X(Y \cap Z) = \mathrm{codim}_X Y + \mathrm{codim}_X Z.$$

In this case, we define the **intersection multiplicity** of Y and Z at a point $x \in Y \cap Z$ to be

$$m_x(Y,Z) := \sum_i (-1)^i l_{\mathcal{O}_{X,x}}(\mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{Y,x}, \mathcal{O}_{Z,x})).$$

It can be proved that this definition is compatible with the geometric definition in the case of compact complex manifolds.

II.1.1.d. Rational Equivalence and Chow Groups

Let X be a noetherian and separated scheme. For any integer $p \geq 0$, let $X_{(p)}$ (resp. $X^{(p)}$) be the set of points of dimension (resp. codimension) p in M, i.e. for $x \in X_{(p)}$ (resp. $X^{(p)}$), $\overline{\{x\}}$ is an irreducible closed subscheme of X of dimension (resp. codimension) p. Let $Z_p(X)$ (resp. $Z^p(X)$) be the free abelian group generated by $X_{(p)}$ (resp. $X^{(p)}$). Usually, we call the elements of $Z_p(X)$ (resp. $Z^p(X)$) dimension (resp. codimension) p-algebraic cycles; they are finite integer linear combinations of dimension (resp. codimension) p closed integral subschemes of X. We say a pair of dimension (resp. codimension) p-algebraic cycles Y, Z are **rationally equivalent** if there exist finitely many rational functions $f_i \in k(y_i)$ with $y_i \in X_{(p+1)}$ (resp. $X^{(p-1)}$), such that

$$Y - Z = \sum_i \operatorname{div}(f_i),$$

where

$$\operatorname{div}(f_i) := \sum_{x \in X_{(p)}(\text{resp.} X^{(p)}) \cap \overline{\{y_i\}}} \operatorname{ord}_{\mathcal{O}_{\overline{\{y_i\}},x}}(f_i)\overline{\{x\}}.$$

If Z is zero, we say that Y is **rationally equivalent to zero**. This equivalence relation is compatible with the addition on algebraic cycles. Hence, all elements which are rationally equivalent to zero form a subgroup $R_p(X)$ (resp. $R^p(X)$) of $Z_p(X)$ (resp. $Z^p(X)$). The p-th **Chow homology group** (resp. **Chow group**) is

$$\mathrm{CH}_p(X) := Z_p(X)/R_p(X)$$

(resp.

$$\mathrm{CH}^p(X) := Z^p(X)/R^p(X).)$$

There is also a relative theory for the above concepts. Let Y be a closed subscheme. Then we let $Z^p_Y(X)$ be the set of codimension p-algebraic cycles with support contained in Y, $R^p_Y(X)$ the subset of elements in $Z^p_Y(X)$ which are rationally equivalent to zero, and

$$\mathrm{CH}^p_Y(X) := Z^p_Y(X)/R^p_Y(X).$$

In a similar way, we have $\mathrm{CH}^Y_p(X)$.

II.1.1.e. Spectral Sequences

A **spectral sequence** consists of the following data:

(1) A family $(E^{p,q}_r)$ of objects of an abelian category, where p, q, r are integers and $p, q \geq 0, r \geq 1$.

(2) Morphisms

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

such that

$$d_r^{p+r,q-r+1} d_r^{p,q} = 0.$$

(3) The objects $E_{r+1}^{p,q}$ on the $(r+1)^{st}$ level are derived from those on the r^{th} level as follows:

$$E_{r+1}^{p,q} =: \frac{\mathrm{Ker}(d_r^{p,q})}{\mathrm{Im}(d_r^{p-r,q+r-1})}.$$

If there is a family of objects E^n, $n \geq 0$, and for each E^n a filtration:

$$0 \subset E_n^n \subset E_{n-1}^n \subset E_0^n = E^n$$

such that

$$E_p^n / E_{p+1}^n = E_\infty^{p,n-p},$$

where for each (p,q), there is an r_0 depending on (p,q) such that for all $r \geq r_0$, $d_r^{p,q} = 0 = d_r^{p-r,q+r-1}$, then we let

$$E_{r_0}^{p,q} = E_{r_0+1}^{p,q} = \ldots =: E_\infty^{p,q}.$$

Usually, we denote this situation by

$$E_1^{p,q} \Rightarrow E^n.$$

The relation between E^n and the $E_2^{p,q}$ may be made explicitly for small n. In fact, we have

$$E_2^{0,0} = E_\infty^{0,0} = E^0.$$

Then

$$E_1^1 = E_\infty^{1,0} = E_2^{1,0}, \quad E^1/E_1^1 = E_\infty^{0,1} = \mathrm{Ker}(d_2^{0,1}).$$

Hence we have an exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \to E_1^2 \to E_2^{1,1} \to E_2^{3,0},$$

where $E_1^2 = \ker(E^2 \to E_2^{0,2})$.

The most important fact for the present discussion is the following

Theorem. Let **A**,**B** and **C** be abelian categories. Assume that **A**, **B** have enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{C}$ be left exact functors. If F takes injectives to G-acyclics, then there is a spectral sequence

$$(R^pG)(R^qF)(A) \Rightarrow R^n(GF)(A)$$

for any $A \in \mathbf{A}^{\mathrm{ob}}$. In particular, we have the exact sequence

$$0 \to R^1G(FA) \to R^1(GF)A \to G(R^1F)A \to R^2G(FA) \to \ldots.$$

Furthermore, if F is exact, then

$$R^p(GF)(A) \simeq (R^pG)FA.$$

The proof of this theorem can be found in any standard book on homological algebra.

II.1.1.f Simplicial Category

Let Δ denote the category of totally ordered finite sets with monotonic maps. That is, the objects of Δ are the finite sets $[n] := \{0 < 1 < \ldots < n\}$, $n = 1, 2, 3, \ldots$, and the morphisms of Δ are generated by faces and degeneracies:

$$\partial_i : [n-1] \to [n]$$

and

$$\sigma_i : [n] \to [n-1],$$

which are defined by

$$\partial_i(j) = \begin{cases} j, & \text{if } j < i \\ j+1, & \text{otherwise;} \end{cases}$$

and

$$\sigma_i(j) = \begin{cases} j, & \text{if } j \le i \\ j-1, & \text{otherwise.} \end{cases}$$

For any category $\mathcal{C}$, the **simplicial category** $S\mathcal{C}$ of $\mathcal{C}$ is defined as follows: The objects of $S\mathcal{C}$ are contravariant functors S, $S : \Delta \to \mathcal{C}$, and the morphisms are natural transformations. If $\mathcal{A}$ is an abelian category, we denote by $C\mathcal{A}$ the category of chain complexes associated with $\mathcal{A}$. There is a natural functor N, $N : S\mathcal{A} \to C\mathcal{A}$: For any object S of $S\mathcal{A}$ let

$$(NS)_n := \begin{cases} S_0, & \text{if } n = 0; \\ \cap_{i=1}^{n} \mathrm{Ker}(d_i : S_i \to S_{i-1}), & \text{otherwise.} \end{cases}$$

Conversely, there is also a natural functor K, $K : C\mathcal{A} \to S\mathcal{A}$: For any object C in $C\mathcal{A}$, let

$$(KC)_n := \oplus_{q \le n} \oplus_\eta C_q,$$

where η runs over all surjective monotonic maps $\eta : [n] \to [q]$. It is not difficult to prove that the functors KN and NK are naturally equivalent to the corresponding identity functors. In particular, N and K are exact functors.

II.1.1.g. K-**Theory**

In this subsection, we briefly give the definition for algebraic K-theory. For more details, see [Qu 73], or Chapter 8 later.

Roughly speaking, a category $\mathcal{E}$ is an **exact category**, if there exists an abelian category $\mathcal{A}$ such that $\mathcal{E}$ is a full subcategory of $\mathcal{A}$, which is closed under extensions, i.e. for any exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

in $\mathcal{A}$, $E_1, E_3 \in \mathrm{Ob}\mathcal{E}$ implies $E_2 \in \mathrm{Ob}\mathcal{E}$.

Following Quillen, for any exact category $\mathcal{E}$, we define the K-groups $K_m(\mathcal{E})$ by using the classifying space of the Quillen construction of the category, $\mathrm{BQ}\mathcal{E}$, and let

$$K_m(\mathcal{E}) := \pi_{m+1}(\mathrm{BQ}\mathcal{E}),$$

the $(m+1)$-th homology group of $\mathrm{BQ}\mathcal{E}$. Note that $K_m(\mathcal{E})$ does not depend on the choice of $\mathcal{A}$, and that $K_0(\mathcal{E})$ is isomorphic to the **Grothendieck group** of $\mathcal{E}$: The quotient group of the free abelian group generated by the objects of $\mathcal{E}$ modulo exact sequences.

Let X be a noetherian separated scheme and Y be a closed subscheme of X, then the following are exact categories:

$\mathcal{M}(X)$: the category of coherent sheaves on X;

$\mathcal{M}_Y(X)$: the category of coherent sheaves on X, supported in Y;

$\mathcal{P}(X)$: the category of vector sheaves on X.

For $m \in Z_{\geq 0}$, we let

$$K^m(X) := K_m(\mathcal{M}(X)), \quad K^m_Y(X) := K_m(\mathcal{M}_Y(X)), \quad K_m(X) := K_m(\mathcal{P}(X)).$$

It follows from the definition that

$$K^m_Y(X) \simeq K^m(Y).$$

Also if Y is a closed subscheme of X with $U =: X - Y$, then there exists a natural exact sequence

$$K^0(Y) \to K^0(X) \to K^0(U) \to 0.$$

Examples. Let F be a field. It is known that

$$K_0(F) \simeq \mathbf{Z};$$
$$K_1(F) \simeq F^*;$$
$$K_2(F) \simeq F^* \otimes_{\mathbf{Z}} F^* / < x \otimes (1-x) : x \in F^* - \{1\} > .$$

More precisely, $K_2(F)$ may be defined by: the generators are $\{x, y\}$ for $x, y \in F^* - \{1\}$; and, the defining relations are

$$\{x_1 x_2, y\} = \{x_1, y\} + \{x_2, y\},$$
$$\{x, y_1 y_2\} = \{x, y_1\} + \{x, y_2\},$$
$$\{x, 1-x\} = 0.$$

There is also a local-global long exact sequence in algebraic K-theory. To explain, we need more terminology and notation. A category $\mathcal{S}$ is called **Serre exact** if there exists an exact category $\mathcal{E}$ such that $\mathcal{S}$ is a full subcategory of $\mathcal{E}$ and for any exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

in $\mathcal{E}$, $E_1, E_3 \in \text{ob}\mathcal{S}$ is equivalent to $E_2 \in \mathcal{S}$. For such a pair $(\mathcal{S}, \mathcal{E})$, there is a natural quotient category $\mathcal{E}/\mathcal{S}$ and there is the long exact sequence

$$\ldots \to K_{p+1}(\mathcal{E}/\mathcal{S}) \to K_p(\mathcal{S}) \to K_p(\mathcal{E}) \to K_p(\mathcal{E}/\mathcal{S}) \to K_{p-1}(\mathcal{S}) \to \ldots.$$

We will not go into greater detail, but instead, we give the following

Example. Let A be a Dedekind domain with field of fractions F, $\mathcal{E}$ the exact category of finitely generated A-modules, $\mathcal{S}$ the Serre exact (full subcategory) of torsion A-modules. Obviously,

$$\mathcal{S} \simeq \amalg_{\mathbf{p}\neq 0}\mathcal{P}(\text{Spec}(k(\mathbf{p}))), \quad \mathcal{E}/\mathcal{S} \simeq \mathcal{P}(\text{Spec}F).$$

Hence we have

$$\ldots \to K_{p+1}(F) \to \oplus_{\mathbf{p}\neq 0}K_p(k(\mathbf{p})) \to K^p(A) \to K_p(F) \to \ldots.$$

In particular, the map $K_1(F) \to \oplus_{\mathbf{p}\neq 0}K_0(k(\mathbf{p}))$ is the valuation mapping

$$F^* \to \oplus_{\mathbf{p}\neq 0}\mathbf{Z}$$

defined by $f \mapsto (\nu_{\mathbf{p}}(f))$.

In general, if X is a regular scheme, Y a closed subscheme and $\mathcal{M}_Y^p(X)$ the category of coherent sheaves $\mathcal{F}$ on X, supported on Y with $\text{codim}_X(\text{Supp}\,\mathcal{F}) \geq p$, then $\mathcal{M}_Y^p(X)$ defines a filtration of the exact category $\mathcal{M}_Y(X)$ by successive Serre exact subcategories.

Theorem. With the same notation as above, there exists a spectral sequence $E_{rY}^{p,q}(X)$ with differential

$$d_r^{p,q} : E_{rY}^{p,q}(X) \to E_{rY}^{p+r,q-r+1}(X),$$

which converges to $K_Y^{-p-q}(X)$. In particular,

$$E_{1Y}^{p,q} = \begin{cases} K_{-p-q}(\mathcal{M}_Y^p(X)/\mathcal{M}_Y^{p+1}(X)), & \text{if } p \geq 0, \text{ and } p+q \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

II.1.1.h. λ-**Ring**

A λ-**ring** is a unitary ring with operations λ^p, $\lambda^p : R \to R$, for each $p \in \mathbf{Z}_{\geq 0}$, which satisfy the following conditions:

(1) $\lambda^0 = 1$, $\lambda^1 = \mathrm{Id}_R$, and $\lambda^k(1) = 0$ for all $k > 1$.
(2) $\lambda^k(x+y) = \sum_{i=0}^k \lambda^i(x)\lambda^{k-i}(y)$.
(3) $\lambda^k(xy) = P_k(\lambda^1(x), \ldots, \lambda^k(x); \lambda^1(y), \ldots, \lambda^k(y))$, where P_r are certain universal integral coefficients polynomials.
(4) $\lambda^k(\lambda^l(x)) = P_{k,l}(\lambda^1(x), \ldots, \lambda^{kl}(x))$, where $P_{l,k}$ are certian universal integral coefficients polynomials.

If we let

$$\lambda_t(x) := \sum_k \lambda^k(x)t^k,$$

then by (2) we have

$$\lambda_t(x+y) = \lambda_t(x)\lambda_t(y).$$

Further, if $x = x_1 + \ldots + x_m$ with $\lambda^k(x_i) = 0$ for $k > 1$ and any i, we have

$$\lambda_t(x) = \prod_{i=1}^m (1 + tx_i).$$

So one may easily find what are the universal polynomials $P_{k,l}$. Similarly we can also find P_k. From this, we get the following

Verification Principle. If a universal relation among operations on a λ-ring is valid for the elements of the form $x = x_1 + \ldots + x_n$ so that $\lambda^k(x_i) = 0$ for any $k > 0$, i, then this relation holds in general.

Let

$$\phi_{-t}(x) := -t\frac{\frac{d}{dt}\lambda_t(x)}{\lambda_t(x)}.$$

Then the **Adams operations** $\phi^k(x)$ on R is defined so that

$$\sum_{k\geq 1} \phi^k(x)t^k := \phi_t(x).$$

The Adams operation is clearly a ring endmorphism of R.

§II.1.2 Algebraic Intersection Theory

There are several ways to introduce algebraic intersection theory, but basically, they have two roots: one is provided by the famous Chow moving lemma. The other is given by a natural pairing induced from higher algebraic K-theory. The Chow moving lemma has the advantage that it gives a completely geometric picture; while the K-theory approach is suitable to do everything algebraically and axiomatically. Further, the Chow moving lemma only works for a restricted numbers of categories, while the K-theory method is valid for a larger range of cases.

We consider first the algebraic intersection theory for a regular scheme X by using the K-theory approach. We then discuss the Chow moving lemma approach at the end of the chapter.

II.1.2.a. Main Theorem

In this subsection, we prove the following

Main Theorem. Let X be a regular scheme and Y be any closed subscheme. Then
(1) There is a decreasing filtration on $K_0^Y(X)$:

$$K_0^Y(X) = F^0K_0^Y(X) \supset F^1K_0^Y(X) \supset \ldots \supset F^{\dim(X)+1}K_0^Y(X) = 0;$$

a group isomorphism α,

$$\alpha : \mathrm{CH}_Y^p(X)_{\mathbf{Q}} \to \mathrm{Gr}^p K_0^Y(X),$$

such that for any morphism $f : X \to X'$,

$$f^* F^p K_0^{f(Y)}(X')_{\mathbf{Q}} \subset F^p K_0^Y(X)_{\mathbf{Q}}.$$

(2) For any closed subscheme Z, which is not contained in Y, there is a unique product

$$F^p K_0^Y(X)\, F^q K_0^Z(X) \subset F^{p+q} K_0^{Y\cap Z}(X),$$

such that under the natural map α, if Z_1, Z_2 intersect properly, then

$$[Z_1][Z_2] = [\sum_{x\in Z_1\cap Z_2} m_x(Z_1, Z_2)\overline{\{x\}}].$$

As an application, we then have the following

Theorem. Let X be a finite-dimensional regular scheme and Y, Z be closed subschemes, then there exists a pairing

$$\mathrm{CH}_Y^p(X) \otimes \mathrm{CH}_Z^q(X) \to \mathrm{CH}_{Y\cap Z}^{p+q}(X)_{\mathbf{Q}}$$

satisfying
(1) $\oplus_{Y,p}\mathrm{CH}_Y^p(X)_{\mathbf{Q}}$ is a commutative ring with a unit element $[X]$.
(2) The ring structure is compatible with the change of supports $Y' \subset Y, Z' \subset Z$.

(3) For $[Y_1] \in \mathrm{CH}^p_Y(X), [Z_1] \in \mathrm{CH}^q_Z(X)$, where Y_1, Z_1 intersect properly, we have

$$[Y_1][Z_1] = [\sum_{x \in Y_1 \cap Z_1 \cap X^{(p+q)}} m_x(Y_1, Z_1)\overline{\{x\}}].$$

Proof. By the main theorem, we know that there is a natural isomorphism α. With the help of this isomorphism, we can define a pairing as stated above and the rest is trivial.

If $Y = Z = X$, there is a multiplication on $\mathrm{CH}(X)_{\mathbf{Q}} = \oplus_p \mathrm{CH}^p(X)_{\mathbf{Q}}$ and under this multiplication, $\mathrm{CH}(X)_{\mathbf{Q}}$ is a unitary ring. Obviously, this multiplication is a direct generalization of the usual intersection pairing.

Proof Of The Main Theorem. This will be achieved in four steps:

Step 1. Putting a λ-ring structure on $\oplus_{Y \subset X} K_0^Y(X)$.
Step 2. Giving two different definitions of $F^p K_0^Y(X)$.
Step 3. Using a product from higher K-theory.
Step 4. Coincidence of two filtrations.

II.1.2.b. **The λ-Ring Structure**

Step 1. We start with the following

Lemma. Let X be a regular scheme. Then there exists a λ-ring structure on the direct sum $\oplus_{Y \subset X} K_0^Y(X)$ with the following properties:
(1) (**Naturalness**) λ^k maps $K_0^Y(X)$ to itself.
(2) (**Functoriality**) The λ-ring structure is functorial.
(3) (**Uniqueness**) If $X = \mathrm{Spec}(R), Y = \mathrm{Spec}(R/aR)$ with $a \in R$, then for the class of the Koszul complex

$$\mathrm{Kos}(a) : 0 \to R \xrightarrow{a} R \to 0,$$

we have $\phi^k([\mathrm{Kos}(a)]) = k[\mathrm{Kos}(a)]$.

Proof. First, we prove that there is a natural λ-ring structure on $K_0(X)$. Let M_n be the set of $n \times n$-matrices and let $H := \mathbf{Z}[M_n]$ be the associated Hopf algebra over $\mathbf{Z}$, where the coproduct $\mu : H \to H \otimes H$ is defined by

$$\mu(X_{ij}) := \sum_k X_{ik} \otimes X_{kj}.$$

Let

$$\lambda^1(\mathrm{Id}_n) : \mathbf{Z}^n \to H \otimes \mathbf{Z}^n$$

be defined so that, for the standard basis e_i,

$$e_i \mapsto \sum_j X_{ij} \otimes e_j.$$

This gives $\mathbf{Z}^n$ the structure of a left H-comodule. Denote the corresponding element in $R_{\mathbf{Z}}(M_n)$ by $\lambda^1(\mathrm{Id}_n)$. Here $R_{\mathbf{Z}}(M_n)$ denotes the Grothendieck group of isomorphism classes of H-left comodules, free and finitely generated over $\mathbf{Z}$. Denote $\lambda^k(\mathrm{Id}_n)$ the corresponding k-th exterior power of $\lambda^1(\mathrm{Id}_n)$. We know that

$$R_{\mathbf{Z}}(M_n) \simeq \mathbf{Z}[\lambda^1(\mathrm{Id}_n), \ldots, \lambda^n(\mathrm{Id}_n)],$$

and that there is a λ-ring structure on $R_{\mathbf{Z}}(M_n)$: Indeed, by the theory of characters,

$$R_{\mathbf{Q}}(\mathrm{GL}_n) = \mathbf{Z}[\lambda^1(\mathrm{Id}_n), \ldots, \lambda^n(\mathrm{Id}_n), \lambda^n(\mathrm{Id}_n)^{-1}].$$

Hence we have the above isomorphism over $\mathbf{Q}$. Now the result over $\mathbf{Z}$ is a consequence of a global-local discussion. On the other hand, since $R_{\mathbf{Z}}(\mathrm{GL}_n)$ is imbedded in the Grothendieck group associated with the group of diagonal matrices of size n, and all representations of T_n are direct sums of one-dimensional representations, thus there is a natural λ-ring structure on $R_{\mathbf{Z}}(M_n)$.

Now, using the natural inclusion $M_n \hookrightarrow M_{n+1}$, we take

$$R_{\mathbf{Z}}(M_\infty) := \lim_{n\to\infty} R_{\mathbf{Z}}(M_n).$$

Let A be a unitary ring. For any $m = (m_{ij}) \in M_n(A)$, the evaluating map defines a homomorphism $m : H \to A$. Hence for any representation $\rho : V \to H \otimes V$, we obtain an action $\rho(m)$ on $A \otimes V$. As an application, we get a natural action of $R_{\mathbf{Z}}(M_\infty)$ on $K_0(A)$: For any projective A-module P, there exists an A-module Q such that $P \oplus Q = A^n$ for some n. Let $p \in M_n(A)$ be the projection from A^n to P. And define

$$([\rho_\infty], [P]) \mapsto [\mathrm{Im}\, \rho_n(p)],$$

where ρ_n denotes the restriction of ρ_∞ to M_n. Obviously, this action is well-defined.

The above process may be made globally so that for any regular scheme X, there is a natural action of $R_{\mathbf{Z}}(M_\infty)$ on $K_0(X)$. And hence, we can define the λ-ring structure on $K_0(X)$ by letting $\rho_\infty = \lambda^k(\mathrm{Id})$; and for any $[\mathcal{F}] \in K_0(X)$,

$$\lambda^k([\mathcal{F}]) := [\lambda^k(\mathrm{Id})(\mathcal{F})].$$

(Written in concrete form, we see that $\lambda^k\mathcal{F}$ is the usual k-th exterior power of $\mathcal{F}$, which is also the beginning of the λ-ring theory.)

We consider the relative situation and let $\mathcal{F}.$ be a finite complex of vector sheaves over X, acyclic outside Y. Since $\lambda^k(\mathcal{F}.)$ may not be acyclic outside Y, we cannot simply give the direct definition. However we can use the discussion in the simplicial category of the last section to avoid this difficulty: For the category of vector sheaves $\mathcal{P}(X)$ and $\lambda^k : \mathcal{P}(X) \to \mathcal{P}(X)$ the k-th exterior power, we have $\lambda^k(\mathcal{F}.) := (N\lambda^k K)(\mathcal{F}.)$ induces a homotopy preserving functor from $C\mathcal{P}(X)$ to itself. So we obtain the operators λ on

$K_0^Y(X)$. Hence we get (1) and (2) of the lemma. Finally, we need to consider 3, the uniqueness. In general, by definition,

$$(KC)_n = R \oplus (\oplus_{\eta_i} E),$$

where η_i are the n surjective monotonic maps from $[n]$ to $[1]$, given by

$$\eta_i(j) := \begin{cases} 0, & \text{if } j < i \\ 1, & \text{if } j \geq i \end{cases} \quad (i = 0, \dots, n).$$

If we write $(KC)_n = F \oplus E_1 \oplus \dots \oplus E_n$, where E_i is the copy of E belonging to η_i, then by the fact that

$$\eta_i \circ \partial^j = \begin{cases} \eta_i, & \text{if } i \leq j,\ (i,j) \neq (n,n); \\ 0, & \text{if } i = 1,\ j = 0, \text{or } i = j = n; \\ \eta_{i-1}, & \text{otherwise,} \end{cases}$$

we know that

$$d_j(f, e_1, \dots, e_n) = \begin{cases} (f + u(e_1), e_2, \dots, e_n), & \text{if } j = 0; \\ (f, e_1, \dots, e_{j-1}, e_j + e_{j+1}, \dots, e_n), & \text{if } 0 < j < n; \\ (f + u(e_n), e_1, \dots, e_{n-1}), & \text{if } j = n. \end{cases}$$

Also, $\lambda^k KC$ is given by the sequence

$$(\lambda^k KC)_n = \oplus_{\substack{\alpha+\beta_1+\dots+\beta_n=k \\ (\alpha,\beta_1,\dots,\beta_n)\in\mathbb{N}^{n+1}}} \lambda^\alpha F \otimes \lambda^{\beta_1} E_1 \otimes \dots \otimes \lambda^{\beta_n} E_n,$$

where the faces d_j on $\lambda^k KC$ are induced by the faces on KC described above. Thus, we get, by a direct calculation,

$$(N\lambda^k KC)_n = \cap_{j=1}^n \mathrm{Ker}(d_j) = \oplus_{\substack{\alpha+\beta_1+\dots+\beta_n=k \\ (\alpha,\beta_1,\dots,\beta_n)\in\mathbb{N}^{n+1},\beta_i>0}} \lambda^\alpha F \otimes \lambda^{\beta_1} E_1 \otimes \dots \otimes \lambda^{\beta_n} E_n.$$

Hence, since $F = E = R$, we have $\alpha = 0, 1$ and $\beta_i = 1$. Therefore, $N\lambda^k KC = C[1-k]$, and so

$$\lambda^k[\mathrm{Kos}(a)] = \lambda^k[C] = [N\lambda^k KC] = [C[1-k]] = (-1)^{k-1}[C].$$

In particular, (3) comes from the fact that $[\mathrm{Kos}(a)]^2 = 0$.

II.1.2.c. **Two Descriptions Of Filtrations**

Step 2. In this step, we give two different methods to construct a filtration of $K_0^Y(X)$. The first one comes from the λ-ring structure for $K_0^Y(X)$, from which the natural product may be given easily; while the second one comes from the algebraic cycles with supports, which gives a natural relation with the Chow groups. Finally, as one may imagine, once we have proved that the two coincide, then we have the proof of the main theorem.

We begin with the first method. Since there is a λ-ring structure on $K_0^Y(X)$, thus for a fixed $k > 0$ and $i \geq 0$, we naturally let the **weight i-part** of $K_0^Y(X)$ be

$$K_0^Y(X)^{(i)} := \{\alpha \in K_0^Y(X)_{\mathbf{Q}} : \phi^k(\alpha) = k^i\alpha\}.$$

By the property of a λ-ring, we know that this definition does not depend on k. Now we let

$$F^p K_0^Y(X)_{\mathbf{Q}} := \oplus_{i\geq p} K_0^Y(X)^{(i)}.$$

Next we introduce a product on $F^p K_0^-(X)$: If

$$\alpha =: \sum_{i\geq p} \alpha_i \in F^p K_0^Y(X)_{\mathbf{Q}},\ \beta =: \sum_{j\geq q} \beta_j \in F^q K_0^Z(X)_{\mathbf{Q}},$$

are their corresponding decompositions, we define their product as follows:

$$\alpha\cdot\beta := \sum_{i,j} \alpha_i\,\beta_j.$$

By the fact that

$$\phi^k(\alpha_i\beta_j) = k^{i+j}\alpha_i\beta_j,$$

where $i + j \geq p + q$, we know that $\alpha\cdot\beta \in F^{p+q} K_0^{Y\cap Z}(X)_{\mathbf{Q}}$ as desired.

But on the other hand, we do not know if this definition gives a filtration. So we introduce another interpretation, which is based on the following very easy

Lemma. Let X be a regular scheme and Y a closed subscheme. There exists a short exact sequence

$$0 \to \cup_{\substack{Z\subset Y\\ \operatorname{codim}_X Z\geq \operatorname{codim}_X Y+1}} \operatorname{Im}\big(K_0^Z(X) \to K_0^Y(X)\big) \to K_0^Y(X)$$
$$\to \oplus_{x\in Y\cap X^{(m)}} K_0^{\overline{\{x\}}}(\mathcal{O}_{X,x}) \to 0.$$

Proof. Since there is the following exact sequence

$$K^0(Z) \to K^0(Y) \to K^0(Y - Z) \to 0,$$

and $K^0(Y) = K_0^Y(X)$, we get an exact sequence

$$0 \to \operatorname{Im}\big(K_0^Z(X) \to K_0^Y(X)\big) \to K_0^Y \to K_0^{Y-Z}(X - Z) \to 0.$$

We take the inductive limit over all closed subschemes $Z \subset Y$ with $\operatorname{codim}_X Z \geq \operatorname{codim}_X Y+1$, to get the exact sequence

$$0 \to \cup_{\substack{Z\subset Y\\ \operatorname{codim}_X Z\geq \operatorname{codim}_X Y+1}} \operatorname{Im}(K_0^Z(X) \to K_0^Y(X)) \to K_0^Y(X)$$
$$\to \lim_{\substack{Z\subset Y\\ \operatorname{codim}_X Z\geq \operatorname{codim}_X Y+1}} K_0^{Y-Z}(X - Z) \to 0.$$

Now the assertion comes from the fact that

$$\begin{aligned}\lim_{\substack{Z\subset Y\\ \operatorname{codim}_X Z\geq \operatorname{codim}_X Y+1}} K_0^{Y-Z}(X-Z) &\simeq \lim_{\substack{Z\subset Y\\ \operatorname{codim}_X Z\geq \operatorname{codim}_X Y+1}} K^0(Y-Z)\\ &= \oplus_{x\in Y\cap X^{(m)}} K^0(\overline{\{x\}}) \simeq \oplus_{x\in Y\cap X^{(m)}} K_0^{\overline{\{x\}}}(\mathcal{O}_{X,x}).\end{aligned}$$

We can now define a filtration of $K_0^Y(X)$ by letting

$$F'^p K_0^Y(X) := \cup_{\substack{Z\subset Y\\ \operatorname{codim}_X Z\geq p}} \operatorname{Im}(K_0^Z(X)\to K_0^Y(X)).$$

The advantage of this definition is that there is a natural map between it and the relative Chow groups: Let $Z\in Z_Y^p(X)$ be a codimension p-cycle contained in Y, then there is a finite vector sheaf resolution $\mathcal{F}.$ of the direct image of $\mathcal{O}_Z$ on X. Define a map

$$\alpha : \mathrm{CH}_Y^p(X) \to F'^p K_0^Y(X)$$

by letting

$$\alpha([Z]) = [\mathcal{F}.].$$

Furthermore, taking the quotient, we have the map

$$\alpha_{\mathbf{Q}} : \mathrm{CH}_Y^p(X)_{\mathbf{Q}} \to \mathrm{Gr}'^p K_0^Y(X)_{\mathbf{Q}}.$$

Thus the main theorem may be proved from the following

Theorem. (1) Let X be a regular scheme, Y a closed subscheme. Then

$$F'^p K_0^Y(X)_{\mathbf{Q}} = F^p K_0^Y(X)_{\mathbf{Q}}.$$

(2) $\alpha_{\mathbf{Q}}$ is an isomorphism.

Proof. We only prove (1) here, and leave (2) for later.

We begin with a proof of the assertion

$$F^p K_0^Y(X)_{\mathbf{Q}} \supseteq F'^p K_0^Y(X)_{\mathbf{Q}},$$

which is equivalent to: For $\alpha \in F'^p K_0^Y(X)_{\mathbf{Q}}$, there exists a unique decomposition $\alpha = \sum_{i\geq p}\alpha_i$ so that $\phi^k(\alpha_i) = k^i\alpha_i$. This can be proved by induction on the dimension of Y.

If the dimension is 0, Y is a point x and

$$K_0^{\{x\}}(X) \simeq K^0(\{x\}) \simeq K^0(k(x)) \simeq \mathbf{Z}.$$

So we have to show that the action of ϕ^k on a single non-zero element of $K_0^{\{x\}}(X) = F^{\dim X} K_0^Y(X)$ is multiplication by k^d. For this, let R be the regular local ring $\mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m} = (a_1, \dots, a_d)$, where $a_1, \dots, a_d$ is a regular sequence, which gives a system of parameters. We know that

$$\mathrm{Kos}(a_1, \dots, a_d) = \otimes_{i=1}^{d} \mathrm{Kos}(a_i)$$

is a resolution of $R/\mathrm{m} = k(x)$. So we get an element in $K_0^{\{x\}}(X)$. Thus the assertion is a consequence of the lemma in step 1.

Now we assume the assertion is valid for all closed subschemes of dimension less than $d-m$ and consider the case with $\dim Y = d-m$. One naturally tries to use the exact sequence in the previous lemma. In fact, if $p > m$, then $\alpha \in F'^{m+1}K_0^Y(X)$, i.e. $\alpha \in \mathrm{Im}(K_0^Z(X) \to K_0^Y(X))$ for some Z with $\mathrm{codim}_Z(X) \geq m+1$. Hence, the desired decomposition comes from the induction hypothesis. On the other hand, if $p = m$, we have the short exact sequence

$$0 \to F'^{m+1}K_0^Y(X) \to K_0^Y(X) \xrightarrow{\varepsilon} \oplus_{x \in Y \cap X^{(m)}} \overline{K_0^{\{x\}}}(\mathcal{O}_{X,x}) \to 0.$$

Hence,

$$\phi^k(\varepsilon(\alpha)) = k^m \varepsilon(\alpha).$$

That is, $\phi(\alpha) - k^m\alpha \in F'^{m+1}K_0^Y(X)$. By the induction hypothesis, we get a decomposition for

$$\phi(\alpha) - k^m\alpha = \sum_{i>m} \beta_i.$$

Thus we get a decomposition by setting

$$\alpha_i := \begin{cases} (k^i - k^m)^{-1}\beta_i, & \text{if } i > m; \\ \alpha - \sum_{i>m} \alpha_i, & \text{if } i = m. \end{cases}$$

The uniqueness is rather trivial: Suppose we have another decomposition $\sum_{i \geq m} \alpha_i'$, then apply ϕ^k to each of them and find that

$$\sum_{i>m} (k^i - k^m)\alpha_i = \sum_{i>m} (k^i - k^m)\alpha_i'.$$

By induction hypothesis, $\alpha_i = \alpha_i'$ for $i > m$, and hence $\alpha_m = \alpha_m'$.

Now we consider

$$F'^p K_0^Y(X)_{\mathbf{Q}} \supseteq F^p K_0^Y(X)_{\mathbf{Q}}.$$

If $\alpha \in F^p K_0^Y(X)_{\mathbf{Q}}$ with $\alpha \in F'^q K_0^Y(X)_{\mathbf{Q}} - F'^{q+1}K_0^Y(X)_{\mathbf{Q}}$, we have

$$\sum_{i \geq p} \alpha_i = \alpha = \sum_{j \geq q} \beta_j,$$

where $\beta_q \neq 0$. Hence $\alpha_q \neq 0$, so $p \leq q$. So we have the assertion.

In order to prove that the natural map

$$\alpha_{\mathbf{Q}} : \mathrm{CH}^p_Y(X)_{\mathbf{Q}} \to \mathrm{Gr}^p K^Y_0(X)_{\mathbf{Q}}$$

is an isomorphism, we need another description of the Chow groups.

II.1.2.d. **Completion Of The Proof**

Step 3. Another Description Of Chow Groups via $\mathcal{K}$-Theory.

By results from the algebraic K-theory in 1.g, if X is a regular scheme, Y a closed subscheme, then there exists a spectral sequence $E^{p,q}_{rY}(X)$ which converges to $K_Y^{-p-q}(X)$. On the other hand, we know that

$$K_m(\mathcal{M}^p_Y(X)/\mathcal{M}^{p+1}_Y(X)) \simeq \oplus_{x\in X^{(p)}\cap Y} K_m(k(x)).$$

Thus we can describe $E^{-p,p}_{2Y}(X)$ and $E^{p-1,-p}_{2Y}(X)$ precisely. In fact, by definition, we have

$$d_1 = d_1^{p-1,-p} : E^{p-1,-p}_{1Y}(X) \to E^{p,-p}_{1Y}(X) \to 0.$$

For a field F, $K_0(F) \simeq \mathbf{Z}$, $K_1(F) \simeq F^*$, so we have

$$d_1 : \oplus_{y\in X^{(p-1)}\cap Y} k(y)^* \to Z^p_Y(X) \to 0.$$

Then, by a local realization in the sense of Example 1.g, we have the following

Lemma 1. With the same notation as above,

$$E^{p,-p}_{2Y}(X) \simeq \mathrm{CH}^p_Y(X).$$

In the same spirit, note that since

$$E^{p-2,-p}_{1Y}(X) \simeq \oplus_{z\in X^{(p-2)}\cap Y} K_2(k(z)),$$

we have the natural morphism

$$d_1 : \oplus_{z\in X^{(p-2)}\cap Y} K_2(k(z)) \to \oplus_{y\in X^{(p-1)}\cap Y} k(y)^*.$$

We know that $d_1 : K_2(k(z)) \to k(y)^*$ is zero, unless $y \in Z := \overline{\{z\}}$. If $y \in Z$, d_1 may be described as follows: First, if the local ring $\mathcal{O}_{Z,y}$ is regular, it is a discrete valuation ring with a valuation ν, the quotient field is $k(z)$ and the residue field $k(y)$. In this case, the map d_1 is nothing but the **tame symbol** ∂_ν:

$$d_1(\{f,g\}) = \partial_\nu(\{f,g\}) := \text{the class of } (-1)^{\nu(f)\nu(g)} f^{\nu(g)} g^{-\nu(f)},\ \forall f,g \in k(z)^* - \{1\}.$$

In the non-regular case, let $W := \mathrm{Spec}(\tilde{\mathcal{O}}_{Z,y})$ with $\tilde{\mathcal{O}}_{Z,y}$ the normalization of $\mathcal{O}_{Z,y}$ inside $k(z)$ and denote by $y_1, \ldots, y_l$ the preimages of y in W. Then

$$d_1(\{f,g\}) := \prod_{i=1}^{l} N_{k(y_i)/k(y)} \partial_{\nu_i}(\{f,g\}),$$

where ν_i is the corresponding valuation of y_i. Therefore, by the fact that

$$\mathrm{CH}_Y^{p,p-1}(X) = E_{2Y}^{p-1,-p}(X) = \mathrm{Coker}\, d_1,$$

we have

Lemma 2.

$$\mathrm{CH}_Y^{p,p-1}(X) \simeq \frac{\{(f_y) \in \oplus_{y \in X^{(p-1)} \cap Y} k(y)^* : \sum_y \mathrm{div}(f_y) = 0\}}{\{(d_1(\{f_z, g_z\})) : (\{f_z, g - z\}) \in \oplus_{z \in X^{(p-2)} \cap Y} K_2(k(z))\}}.$$

Step 4. Relation Of Two K-Theory Descriptions: $\alpha_{\mathbf{Q}}$ Is An Isomorphism.

In this step, we prove the following

Lemma. There is a natural isomorphism

$$E_{2Y}^{p,-p}(X)_{\mathbf{Q}} \simeq \mathrm{Gr}^p K_0^Y(X)_{\mathbf{Q}}.$$

Hence $\alpha_{\mathbf{Q}}$ is an isomorphism.

To prove this lemma, we need still another description of $K_Y^m(X)$. Denote by $\mathcal{K}$ the Zariski simplicial sheaf associated with the presheaf

$$U \mapsto \mathbf{Z} \times \lim_{\overrightarrow{n}} \mathrm{BGL}_n(\Gamma(U, \mathcal{O}_X))^+ =: \mathbf{Z} \times \mathrm{BGL}(\Gamma(U, \mathcal{O}_X))^+.$$

By constructing flasque resolutions for Zariski simplicial sheaves, we obtain a cohomology theory for them. In particular for $\mathcal{K}$, we have

$$K_Y^m(X) \simeq H_Y^{-m}(X, \mathcal{K}).$$

There is a big advantage for this description when we introduce the λ-ring structure on $\oplus_{m,Y} K_Y^m(X)$. In fact, for all integers k, $0 \le k \le n$, there are exterior power maps of sheaves

$$\lambda^k : \mathrm{GL}_n(\mathcal{O}_X) \to \mathrm{GL}_{\binom{n}{k}}(\mathcal{O}_X).$$

Then, we have a map of sheaves

$$\lambda^k : \mathbf{Z} \times \mathrm{BGL}_n(\mathcal{O}_X)^+ \to \mathcal{K},$$

which is compatible with the inclusion $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$. Hence, there is an operation

$$\lambda^k : H_Y^{-m}(X, \mathcal{K}) \to H_Y^{-m}(X, \mathcal{K}).$$

We know that these operations λ^k induce operations on the spectral sequence

$$\lambda^k : E_{rY}^{p,q}(X) \to E_{rY}^{p,q}(X),$$

which converge to the original operation

$$\lambda^k : K_Y^{-p-q}(X) \to K_Y^{-p-q}(X).$$

In particular, for the isomorphism

$$i_* : \oplus_{x \in X^{(p)} \cap Y} K_{-p-q}(k(x)) \simeq E_{1Y}^{p,q}(X),$$

the Adams operations ϕ^k satisfy $\phi^k(i_* f)) = k^p i_*(\phi^k(f))$ for any choice of the element $f \in \oplus_{x \in X^{(p)} \cap Y} K_{-p-q}(k(x))$. Thus, we know that the operations ϕ^k act on $E_{rY}^{p,q}(X)$ as multiplication by k^{-q} for $q = -p, -p-1$.

Since the differentials $d_r^{p-1,-p}$ commute with the Adams operations ϕ^k, we have the relation

$$(k^{r-1} - 1) d_r^{p-1,-p} = 0.$$

After tensoring with $\mathbf{Q}$, this implies $d_r^{p-1,-p}$ vanishes. Therefore

$$E_{2Y}^{p,-p}(X)_{\mathbf{Q}} = E_{\infty Y}^{p,-p} \simeq \mathrm{Gr}^p K_Y^0(X)_{\mathbf{Q}}.$$

With this, the final result is a consequence of the fact that for a regular scheme X, if Y is a closed subset, then

$$K_Y^0(X) \simeq K^0(Y) \simeq K_0^Y(X).$$

This completes the proof of the lemma, and hence the main theorem.

§II.1.3 Grothendieck-Riemann-Roch Theorem

II.1.3.a. **Algebraic Theory Of Chern Character**

We have seen above that with a regular scheme X, we can associate a Chow ring and a λ-ring $K(X)$. There is also a natural filtration on $K(X)_{\mathbf{Q}}$, from which we construct the associated graded ring $\mathrm{Gr}K(X)_{\mathbf{Q}}$. After a local discussion, we know that there is a natural isomorphism $\alpha_{\mathbf{Q}}$ between $\mathrm{CH}(X)_{\mathbf{Q}}$ and $\mathrm{Gr}K(X)_{\mathbf{Q}}$, hence we have an algebraic intersection pairing on $\mathrm{CH}(X)_{\mathbf{Q}}$. This intersection pairing coincides with the classical one. In this subsection, we introduce a natural homomorphism from $K(X)$ to $\mathrm{Gr}K(X)_{\mathbf{Q}}$. The composition of this homomorphism with the inverse isomorphism of $\alpha_{\mathbf{Q}}$ is usually called the **Chern character**.

The Chern character

$$\mathrm{ch} : K(X) \to \mathrm{CH}(X)_{\mathbf{Q}}$$

is characterised by the following axioms. If X is a regular scheme:

(1) ch is a homomorphism of rings;
(2) If $f : Y \to X$ is a morphism of regular schemes, then $f^* \circ \mathrm{ch} = \mathrm{ch} \circ f^*$.
(3) If $\mathcal{L}$ is a line sheaf on X, then

$$\mathrm{ch}\,[\mathcal{L}] := \exp(\mathrm{div}(s)),$$

where s is a non-zero section of $\mathcal{L}$.

There are several ways to define the Chern character. Here we adopt some algebraic methods: Either projective bundles or Grassmannians can be used to describe them precisely. We do not discuss either here; instead, we give a construction from the λ-ring structure on $K(X)$. Let $\mathcal{E}$ be a vector sheaf on X. Define an operator γ^i on the λ-ring $K(X)$,

$$\gamma^i : K(X) \to K(X)$$

by the series

$$\gamma_t(x) := \lambda_{t/(t-1)}(x) = \sum \gamma^i(x) t^i.$$

Then the γ^i's define another λ-ring structure on $K(X)$. In particular, with this new λ-ring structure, we have another description for $F^p K(X)$. That is,

$$F^1 K(X) := \mathrm{Ker}(K(X) \xrightarrow{\mathrm{rk}} \mathbf{Z})$$

and $F^p K(X)$ = the $\mathbf{Z}$-module generated by the elements $\gamma^{r_1}(x_1) \dots \gamma^{r_k}(x_k)$ with $x_i \in F^1 K(X), \sum_i r_i \geq n$.

For any vector sheaf $\mathcal{E}$ on X, we define the i-th **Chern class** c_i by letting

$$c_i(\mathcal{E}) := [\gamma^i([\mathcal{E}] - \mathrm{rk}\,\mathcal{E})] \in \mathrm{Gr}^i K(X).$$

For example, we know that if $\mathcal{L}$ is a line sheaf on X, then $c_1(\mathcal{L}) = [[\mathcal{L}] - 1]$ and $c_i(\mathcal{L}) = 0$ for $i > 1$. (In general, by the splitting principle, we know that $c_i(\mathcal{E}) = 0$ for $i > \mathrm{rk}\,\mathcal{E}$.) Therefore, if we put

$$c_t(\mathcal{E}) := 1 + \sum_{i=1}^{\infty} c_i(\mathcal{E}) t^i,$$

the induced map

$$c_t : K(X) \to 1 + \oplus_{i=1}^{\infty} \mathrm{Gr}^i K(X) t^i$$

is a group homomorphism; since for any short exact sequence of vector sheaves

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

we have

$$c_t(\mathcal{E}_2) = c_t(\mathcal{E}_1)c_t(\mathcal{E}_3).$$

Finally, if we let

$$c_t(\mathcal{E}) := \prod_{i=1}^{\mathrm{rk}\mathcal{E}}(1 + a_i(\mathcal{E})t),$$

then

$$\mathrm{ch}(\mathcal{E}) := \sum \exp(a_i(\mathcal{E}))$$

is a ring homomorphism

$$\mathrm{ch} : K(X) \to \mathrm{Gr}K(X)_{\mathbf{Q}}.$$

We can also prove that $\mathrm{ch}_{\mathbf{Q}}$ is an isomorphism between $K(X)_{\mathbf{Q}}$ to $\mathrm{Gr}(X)_{\mathbf{Q}}$. The inverse map is defined by

$$g : \mathrm{Gr}(X)_{\mathbf{Q}} \to K(X)_{\mathbf{Q}}$$

with the relations

$$\phi^p(g(x)) = p^m g(x),$$

and

$$x = g(x) \bmod F^{m+1}K(X)_{\mathbf{Q}},$$

for any $x \in \mathrm{Gr}^m K(X)_{\mathbf{Q}}$ and $p \geq 2$. So finally, we get the Chern character. One may also show that if X is a complex manifold, then the above definition coincides with that given in Part I.

With the help of Chern classes, we can define other characteristic classes by using splitting principle. For the application to Grothendieck-Riemann-Roch theorem, we need the following

Definition-Lemma. There is a unique natural map

$$\mathrm{td} : K(X) \to \mathrm{Gr}K(X)_{\mathbf{Q}}$$

such that the following conditions are satisfied:

(1) $\mathrm{td}(x_1 + x_2) = \mathrm{td}(x_1)\mathrm{td}(x_2)$.
(2) For any morphism $f : Y \to X$,

$$\mathrm{td} \circ f^* = f^* \circ \mathrm{td}.$$

(3) For any line sheaf $\mathcal{L}$,

$$\mathrm{td}([\mathcal{L}]) := (\sum_{i=1}^{\infty}(-1)^i \frac{c_1(\mathcal{L})^{i-1}}{l!})^{-1} (= \frac{c_1(\mathcal{L})}{1 - e^{-c_1(\mathcal{L})}}).$$

We call td the **Todd characteristic class.**

II.1.3.b. Push-Out And Pull-Back Morphisms For Chow Rings And Algebraic K-Groups

A closed imbedding $i : X \hookrightarrow Y$, where X, Y are schemes, is a **regular imbedding** of codimension d if every point x in X has an affine neighborhood U in Y, such that if A is the coordinate ring of U, $\mathcal{I}$ is the ideal of A defining X, then $\mathcal{I}$ is generated by a regular sequence of length d. A morphism $f : X \to Y$ of regular schemes is called a **local complete intersection morphism**, l.c.i. morphism for short, of codimension d if f admits a factorization into a closed regular imbedding, followed by a smooth morphism. By a local discussion, we know that if $f = g \circ i$ is any factorization with i a closed imbedding and g a smooth morphism of a l.c.i. morphism f, then i is a regular imbedding. It follows that certain properties do not depend on the chosen factorization.

We now give the definitions of the push-out morphism and the pull-back morphism of the Chow rings and algebraic K-groups for algebraic regular varieties.

$$f_{\mathrm{CH}} : \mathrm{CH}(X)_{\mathbf{Q}} \to \mathrm{CH}(Y)_{\mathbf{Q}}, \quad f^{\mathrm{CH}} : \mathrm{CH}(Y)_{\mathbf{Q}} \to \mathrm{CH}(X)_{\mathbf{Q}};$$

$$f_K : K(X) \to K(Y), \qquad f^K : K(Y) \to K(X).$$

First, let us look at the push-out morphism. This is very simple. In fact, we can go further. Say, we define the push-out for a proper morphism. (The choice of proper morphisms is very natural. In fact, once we try to define the push-out morphism, the first thing we need to know is that the morphism in question maps closed subsets to closed subsets, which is the most important property of proper morphisms.) Then we make the following definition:

$$f_{\mathrm{CH}}[Z] := \deg(Z/f(Z))[f(Z)],$$

where Z is a subvariety of X, and

$$\deg(Z/f(Z)) := \begin{cases} [k(Z) : k(f(Z))], & \text{if } \dim f(Z) = \dim Z; \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to check that the definition does not depend on the representative we choose. On the other hand, if f is proper, we know that for any coherent sheaf $\mathcal{E}$ on X, its higher direct images $R^i f_*(\mathcal{E})$ are coherent sheaves on Y [Ha 77]. Thus we may use them to define f_K by

$$f_K(\mathcal{E}) := \sum_i (-1)^i R^i f_*(\mathcal{E}).$$

Now let us look at the pull-back morphism. The K-theory morphism is very simple: If $\mathcal{E}$ is a vector sheaf on Y, then

$$f^K(\mathcal{E}) := f^*(\mathcal{E}).$$

The morphisms for the algebraic Chow rings are slightly complicated. By the definition of l.c.i. morphisms, it is enough to make the definition for regular closed immersions and smooth morphisms respectively. For closed immersions, with the help of algebraic intersection theory, we naturally let $i^*([Z]) := [Z\,X]$ with $Z\,X$ the algebraic intersection of Z and X. For smooth morphisms, we can go slightly further: We may only assume that $f : X \to Z$ is a flat morphism. Then we set

$$f^*([Z]) := [f^{-1}(Z)].$$

(Since f is flat, $f^{-1}(Z)$ is a pure dimension $\dim Z + \dim f$ subvariety.) We know that all of them are well-defined. Finally, for any local complete intersection morphism $f : X \to Y$, we get the corresponding morphisms by taking the composition. Usually, if there is no risk of confusion, we let f_* and f^* denote the push-out morphism and pull-back morphism respectively. There are several very important properties for these morphisms, such as the functorial property, etc. among these, we recall the following

Projective Formula. Let $f : X \to Y$ be a l.c.i. morphism, then for $\alpha \in \mathrm{CH}(X)$, $\beta \in \mathrm{CH}(Y)$, we have

$$f_*(\alpha\, f^*\beta) = f_*(\alpha)\,\beta.$$

We end this subsection by noting the following properties for the pull back of relative Chow groups. Obviously, it is possible for us to consider slight further, i.e. those for regular schemes which are flat and of finite type over a fixed excellent regular noetherian domain A in our discussion. (Later we usually assume that our schemes have such a property.) By the definition, after a tedious discussion, we have the following

Theorem. Let $f : X \hookrightarrow Y$ be a closed immersion of regular schemes. If $T \subset Y$ is a closed subset, there is a morphism

$$i^* : \mathrm{CH}^n_T(Y) \to \mathrm{CH}^n_{X\cap T}(X)$$

such that
(1) If $\alpha \in Z^n_T(Y)$ is an algebraic cycle supported on T, then $i^*(\alpha)$ is given by Serre's multiplicity formula.
(2) If $g : Y \hookrightarrow Z$ is another regular closed immersion with a closed subset $S \subset Z$, we have

$$f^*g^* = (g\,f)^* : \mathrm{CH}^n_S(Z) \to \mathrm{CH}^n_{S\cap X}(X).$$

(3) Suppose that $g : Y \to W$ is a flat map so that $S \subset W$ is a closed subset. Then if either $h = g \circ f$ is flat or g is smooth and h is a regular closed immersion, we have

$$h^* = f^* \circ g^* : \mathrm{CH}^n_S(W) \to \mathrm{CH}^n_{h^{-1}(s)}(X).$$

(4) Suppose that $g : W \to Y$ is flat and form the Cartesian square:

$$\begin{array}{ccc} W \times_Y X & \xrightarrow{g_f} & X \\ f_g \downarrow & & \downarrow f \\ W & \xrightarrow{g} & Y. \end{array}$$

Suppose $W \times_Y X$ is of the type we want, then

$$g_f^* \circ f^* = f_g^* \circ g^* : \mathrm{CH}_T^n(Y) \to \mathrm{CH}_{(f\, g_f)^{-1}(T)}(W \times_Y X).$$

(5) Suppose that D is a Cartier divisor on Y and that the support $|D|$ meets both T and X properly. Then if $\alpha \in \mathrm{CH}_T^n(Y)$,

$$f^*([D]\,\alpha) = [f^*(D)]\,f^*\alpha$$

in $\mathrm{CH}_{|D| \cap X \cap T}(X)$.
(6) The map $f^* : \mathrm{CH}_T^n(Y) \to \mathrm{CH}_{X \cap T}^n(X)$ givess the same map

$$\mathrm{CH}_T^n(Y)_{\mathbf{Q}} \to \mathrm{CH}_{X \cap T}^n(X)_{\mathbf{Q}}$$

as induced by the isomorphism

$$\mathrm{CH}_T^{\cdot}(Y) \simeq \mathrm{Gr}^{\cdot} K_T^0(X)_{\mathbf{Q}}.$$

With the above preparation, we may go to our main result in this chapter.

II.1.3.c. **Grothendieck-Riemann-Roch Theorem**

Let $f : X \to Y$ be an l.c.i. morphism of regular varieties. Then there is a decomposition of f given by a closed immersion $i : X \hookrightarrow P$ followed by a projection $p : P \to Y$. Here P is a $\mathbf{P}^n$-bundle on Y. We define the **tangent element** of f associated with this decomposition as the element in $K(X)$

$$\mathcal{T}_f := [i^*\mathcal{T}_p] - [\mathcal{N}_i]$$

i.e. as the difference of the relative tangent vector sheaf of p and the normal vector sheaf of the closed immersion i. (This is a very natural choice at this stage, since if f itself is smooth, then we have the short exact sequence

$$0 \to \mathcal{T}_f \to i^*\mathcal{T}_p \to \mathcal{N}_i \to 0.)$$

Furthermore, by the fact that for any two decompositions of f as above, we may chose a third one, which dominates the original two, then it follows that as an element in $K(X)$, the above $\mathcal{T}_f$ is well-defined.

The Grothendieck-Riemann-Roch Theorem. Let $f : X \to Y$ be an l.c.i. morphism of regular varieties X, Y. Then we have the following commutative diagram:

$$\begin{array}{ccc} K(X) & \overset{\mathrm{ch}()\mathrm{td}(\mathcal{T}_f)}{\rightarrow} & \mathrm{CH}(X)_{\mathbf{Q}} \\ f_K \downarrow & & \downarrow f_{\mathrm{CH}} \\ K(Y) & \overset{\mathrm{ch}}{\rightarrow} & \mathrm{CH}(Y)_{\mathbf{Q}}. \end{array}$$

Proof. If we let

$$\mathrm{Err}(\mathcal{E}, f) := f_{\mathrm{CH}}(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(\mathcal{T}_f)) - \mathrm{ch}(f_K(\mathcal{E})),$$

the theorem states that, for any vector sheaf $\mathcal{E}$ on X, $\mathrm{Err}(\mathcal{E}, f) = 0$. We prove this last statement by the following steps.

Step 1. Prove Err = 0 for projective bundles.
Step 2. Prove Err = 0 for closed immersions.
Step 3. Prove Err = 0 in general.

As a remark, we first look at the properties of Err. By definition and the properties of characteristic classes, we know that Err is compatible with any flat base change. That is, if g is a flat base change, we have

$$g^* \mathrm{Err} = \mathrm{Err}\, g^*.$$

On the other hand, if $h : Y \to Z$ is any l.c.i. morphism, then one can check that

$$\mathrm{Err}(\mathcal{E}, g \circ f) = \mathrm{Err}(f_K(\mathcal{E}), g) + g_{\mathrm{CH}}(\mathrm{Err}(\mathcal{E}, f)\,\mathrm{td}(\mathcal{T}_g)).$$

(Since the algebraic K group $K(X)$ is generated by the f-acyclic vector sheaves, we only need to check those properties for f-acyclic vector sheaves.) Thus, in the proof, we may make any flat base change, and decompose the morphism f into several others.

Step 1. Projective Bundles.

Let $p : P = \mathbf{P}_Y(\mathcal{F}) \to Y$ be a projective bundle over Y with a vector sheaf $\mathcal{F}$ on Y of rank r. We prove the theorem in this situation by induction on the rank r. The basic idea is to use the precise description for a $\mathbf{P}^1$-bundle and the deformation technique for algebraic K-theory.

If $r = 2$, then P is a $\mathbf{P}^1$-bundle. So $K(P)_{\mathbf{Q}}$ as a $K(Y)_{\mathbf{Q}}$-module is generated by two elements $\mathcal{O}_P$ and $\mathcal{O}_P(-1)$. Then, the theorem follows from the calculation for those two elements by considering the following natural exact sequence

$$0 \to \mathcal{O}_P \to p^*\mathcal{E} \otimes \mathcal{O}_P(1) \to \mathcal{T}_p \to 0.$$

The details are left to the reader.

Now let $\mathcal{F}$ be a vector sheaf with $r > 2$. Since Err is compatible with a smooth base change, by the splitting principle, we may assume that $\mathcal{F}$ has a quotient vector sheaf $\mathcal{F}'$ of rank $r-1$. Consider $i : P' := \mathbf{P}(\mathcal{F}') \hookrightarrow P$, the Cartier divisor of $\mathcal{O}_P(1)$ on P. Then, as a $K(Y)_{\mathbf{Q}}$-module, $K(P)$ is generated by $K(P')$ and $\mathcal{O}_P(-1)$. If we can prove that

$$\mathrm{Err}(\mathcal{E}, p') = \mathrm{Err}(i_*\mathcal{E}, p)$$

for any coherent sheaf $\mathcal{E}$ on P', by induction, we only need to prove Grothendieck-Riemann-Roch theorem for $\mathcal{O}_p(-1)$ for p.

We first prove that

$$\mathrm{Err}(\mathcal{O}_P(-1),p)=0.$$

For this, we need some more notation. Let $\mathrm{Flag}\,\mathcal{F}$ be the flag scheme which classifies complete filtrations

$$0=\mathcal{F}_0\subset\mathcal{F}_1\subset\ldots\subset\mathcal{F}_r=\mathcal{F}.$$

There is a natural morphism

$$\pi:\mathrm{Flag}\,\mathcal{E}\to P,$$

which is a composition of the forgetting maps. Hence π is a composition of projective bundles of rank at most $r-1$. Since for any projective bundle, the direct image of the structure sheaf upstairs is the structure sheaf downstairs, so by the projective formula, it is enough to prove

$$\mathrm{Err}(\pi^*\mathcal{O}_P(-1),\pi\circ p)=0.$$

To get this assertion, we use another decomposition of $\pi\circ p$. Let $\mathrm{Flag}'\mathcal{F}$ be the flag scheme which classifies partial filtrations

$$0=\mathcal{F}_0\subset\mathcal{F}_2\subset\ldots\subset\mathcal{F}_r=\mathcal{F},$$

where $\mathcal{F}_i$ is of rank i. Then $\mathrm{Flag}\,\mathcal{F}$ is a projective line bundle over $\mathrm{Flag}'\mathcal{F}$. That is, we have the following commutative diagram:

$$\begin{array}{ccc}\mathrm{Flag}\mathcal{F} & \stackrel{\mathbf{P}^1}{\to} & \mathrm{Flag}'\mathcal{F}\\ \pi\downarrow & & \downarrow\\ P & \stackrel{p}{\to} & Y.\end{array}$$

With respect to this projective line bundle, the canonical line sheaf $\mathcal{O}_{\mathrm{Flag}\mathcal{F}}(-1)$ is nothing but $\pi^*\mathcal{O}_P(-1)$. Therefore, by the case for $r=2$,

$$\mathrm{Err}(\pi^*\mathcal{O}_P(-1),\mathrm{Flag}\,\mathcal{F}\to\mathrm{Flag}'\mathcal{F})=0.$$

On the other hand, we know that with respect to the projective line bundle, both terms in the formal difference of Err for $\mathcal{O}(-1)$ are zero, hence the Grothendieck-Riemann-Roch theorem holds for $\mathcal{O}(-1)$ in any case.

Now we come back to show that for the *codimension one* closed immersion $i:P'\hookrightarrow P$,

$$\mathrm{Err}(\mathcal{F},p')=\mathrm{Err}(i_*\mathcal{F},p).$$

This is a consequence of the following facts about the deformation to the normal cone: Let $i:X\hookrightarrow Z$ be a closed immersion over Y with smooth structure morphisms $f:X\to Y$, $g:Z\to Y$ and let $W:=B_{X\times\{\infty\}}Z\times\mathbf{P}^1$ be the blowing-up of $Z\times\mathbf{P}^1$ along $X\times\{\infty\}$. By I.9.1, the fiber W_0 of W over 0 is isomorphic to Z; the fiber W_∞ of W over ∞ is a union of two subschemes: W_∞^1, which is isomorphic to $\mathbf{P}(\mathcal{N}_i\oplus\mathcal{O}_X)$, and a scheme W_∞^2, which is isomorphic to the blowing-up B_XZ. Here $\mathcal{N}_i$ denotes the normal bundle of X in Z. Let $I:X\times\mathbf{P}^1\hookrightarrow W$ be the canonical embedding. The fiber I_0 of I over 0 is essentially

$i_0 : X \hookrightarrow W_0$, which is the original closed imbedding. The fiber at infinity, the morphism I_∞, is essentially the morphism $i_\infty : X \hookrightarrow W^1_\infty$, which is the zero section of P, by the fact that $X \times \mathbf{P}^1$ does not intersect W^2_∞.

Lemma. With the same notation as above, for the codimension one closed immersion i, we have
$$\mathrm{Err}(i_{0*}\mathcal{E}, g_0) = \mathrm{Err}(i_{\infty*}\mathcal{E}, g_\infty).$$

Proof. In fact,
$$\begin{aligned}&\mathrm{Err}(i_{0*}\mathcal{E}, g_0) - \mathrm{Err}(i_{\infty*}\mathcal{E}, g_\infty)\\ =&g_{0*}\big(\mathrm{ch}(i_{0*}\mathcal{E})\mathrm{td}(\mathcal{T}_{g_0})\big) - g_{\infty*}\big(\mathrm{ch}(i_{\infty*}\mathcal{E})\mathrm{td}(\mathcal{T}_{g_\infty})\big)\\ =&G_*\{j_{0*}\big(\mathrm{ch}(i_{0*}\mathcal{E})\mathrm{td}(\mathcal{T}_{G_0})\big) - j_{\infty*}\big(\mathrm{ch}(i_{\infty*}\mathcal{E})\mathrm{td}(\mathcal{T}_{G_\infty})\big)\},\end{aligned}$$

since $X \times \mathbf{P}^1$ does not intersect W^1_∞. We know that if we let $D\mathcal{E}$ be the pull-back of $\mathcal{E}$ to $X \times \mathbf{P}^1$, then $i_{0*}\mathcal{E} = j_0^*(I_*D\mathcal{E})$. Furthermore, $\mathrm{ch}(I_*D\mathcal{E})$ is supported on $I(X \times \mathbf{P}^1)$, and if $\mathcal{T}(-\log\infty)$ denotes the relative logarithmic tangent vector sheaf of $W/\mathbf{P}^1$ along W_∞/∞, i.e.,
$$\mathcal{T}(-\log\infty) := \text{the dual of } \Omega^1_W/Y(d\log W_\infty)/\Omega^1_{\mathbf{P}^1}(d\log\infty)),$$
then, near $i_0(X)$, $\mathcal{T}(-\log\infty)$ is isomorphic to $\mathcal{T}_Z$. Similarly, near $i_\infty(X)$, $\mathcal{T}(-\log\infty)$ is isomorphic to $\mathcal{T}_{W^1_\infty}$. Hence, the difference of two Err terms above is the G image of the product of $\mathrm{ch}(I_*D\mathcal{E})\mathrm{td}(\mathcal{T}(-\log\infty))$ with the divisor of the rational function $t : W \to \mathbf{P}^1$ induced by the second projection. Thus, the lemma is a direct consequence of the fact that the divisor of a rational function is rationally equivalent to zero.

Hence, we can always consider a codimension one closed immersion as the zero section of a projective bundle. But, for the zero section of a projective bundle, we have the following

Lemma. Let $f : X \hookrightarrow Y$ be a smooth morphism. Let $\mathcal{F}$ be a vector sheaf on X of the form $\mathcal{O}_X \oplus \mathcal{N}$. Denote by P the projective bundle $\mathbf{P}_X(\mathcal{F})$ on X. Let $i : X \to P$ be the codimension one closed imbedding corresponding to the morphism $\mathcal{F} \to \mathcal{O}_X$. Let $p : P \to X$ be the structure morphism. Then the Grothendieck-Riemann-Roch theorem for p implies that
$$\mathrm{Err}(\mathcal{E}, f) = \mathrm{Err}(i_*\mathcal{E}, f \circ p).$$

Proof. In fact, by definition, we have
$$\begin{aligned}\mathrm{Err}(i_*\mathcal{E}, f\circ p) &= (f\circ p)_*\big(\mathrm{ch}(i_*\mathcal{E})\mathrm{td}(\mathcal{T}_{f\circ p})\big) - \mathrm{ch}\big((f\circ p)_*(i_*\mathcal{E})\big)\\ &= (f_*\circ p_*)\big(\mathrm{ch}(i_*\mathcal{E})\,p^*(\mathrm{td}(\mathcal{T}_f))\mathrm{td}(\mathcal{T}_p)\big) - \mathrm{ch}(f_*\mathcal{E}).\end{aligned}$$

By the assumption that the Grothendieck-Riemann-Roch theorem is valid for p and the fact that $p_*\mathcal{O}_P = \mathcal{O}_X$, the projection formula implies that
$$\begin{aligned}\mathrm{Err}(i_*\mathcal{E}, f\circ p) =&f_*\big(\mathrm{ch}(p_*(i_*\mathcal{E}))\,\mathrm{td}(\mathcal{T}_f)\big) - \mathrm{ch}(f_*\mathcal{E})\\ =&f_*\big(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(\mathcal{T}_f)\big) - \mathrm{ch}(f_*\mathcal{E})\\ =&\mathrm{Err}(\mathcal{E}, f).\end{aligned}$$

Now we can finish the proof of the theorem for $\mathbf{P}^r$-bundles. It follows from the fact that $i : P' \hookrightarrow P$ is a codimension one closed immersion.

Step 2. **Closed Immersions.** We start with a special situation $i : X \hookrightarrow Z := \mathbf{P}_X(\mathcal{F})$: As above, $\mathcal{F}$ has the form $\mathcal{O}_X \oplus \mathcal{N}$ as a vector sheaf on X, and the closed immersion is induced from $\mathcal{F}. \to \mathcal{O}_X \to 0$. That is, X is the zero section of P over X. (The motivation for considering this special situation comes from the deformation to the normal cone.)

Lemma. For any vector sheaf $\mathcal{E}$ on Z, we have

$$\mathrm{ch}(i_*\mathcal{E}) = i_*(\mathrm{td}(\mathcal{N}_i)^{-1}\,\mathrm{ch}(\mathcal{E})).$$

Proof. Let Q be the universal quotient bundle on Y. Let s be the section of Q determined by the projection of the trivial factor in $p^*(\mathcal{N}_i \oplus \mathcal{O}_X)$ to Q, which vanishes precisely along X. By I.9.1, we know that the Koszul complex determined by s:

$$0 \to \wedge^d Q^\vee \to \ldots \to \wedge^2 Q^\vee \to Q^\vee \to \mathcal{O}_Z \to i_*\mathcal{O}_X \to 0$$

is a resolution of $i_*\mathcal{O}_X$. Therefore, for any vector sheaf $\mathcal{E}$ on X, we have a concrete vector sheaf resolution of $i_*\mathcal{E}$:

$$0 \to \wedge^d Q^\vee \otimes p^*\mathcal{E} \to \ldots \to \wedge^2 Q^\vee \otimes p^*\mathcal{E} \to Q^\vee \otimes p^*\mathcal{E} \to p^*\mathcal{E} \to i_*\mathcal{E} \to 0.$$

Therefore,

$$\mathrm{ch}(i_*\mathcal{E}) = \sum_{q=0}^{d}(-1)^q\,\mathrm{ch}(\wedge^q Q^\vee)\,\mathrm{ch}(p^*\mathcal{E}).$$

On the other hand, by definition, we know that

$$\sum(-1)^p\,\mathrm{ch}(\wedge^p Q^\vee) = c_{\mathrm{top}}(Q)\,\mathrm{td}(Q)^{-1},$$

so that

$$\mathrm{ch}(i_*\mathcal{E}) = c_{\mathrm{top}}(Q)\,\mathrm{td}(Q)^{-1}\,\mathrm{ch}(p^*\mathcal{E}).$$

But, for any $\alpha \in \mathrm{CH}(Y)$, the projective formula gives us

$$i_*(i^*\alpha) = \alpha\, i_*[X].$$

Thus, by the fact that $i^*Q = \mathcal{N}_i$ together with the definition that s is the section of Q determined by the projection of the trivial factor in $p^*(\mathcal{N}_i \oplus \mathcal{O}_X)$ to Q, which vanishes precisely along X, we get

$$\alpha\, i_*[X] = \alpha\, c_{\mathrm{top}}(Q),$$

which proves the lemma.

In general, let $\mathcal{E}$ be a vector sheaf on X, and $D\mathcal{E} = p_1^*\mathcal{E}$, where p_1 is the first projection from $X \times \mathbf{P}^1$ to X. Choose a resolution $\mathcal{G}.$ on W for $I_*(D\mathcal{E})$:

$$0 \to \mathcal{G}_n \to \mathcal{G}_{n-1} \to \ldots \to \mathcal{G}_0 \to I_*D\mathcal{E} \to 0.$$

Since $X \times \mathbf{P}^1$ and W are flat over $\mathbf{P}^1$, it follows that the restrictions of this exact sequence to the fibers W_0 and W_∞ remain exact. Therefore $j_0^*\mathcal{G}.$ is a resolution of $j_0^*(I_*(D\mathcal{E}))$ and $j_\infty^*\mathcal{G}.$ is a resolution of $j_\infty^*(I_*(D\mathcal{E}))$. Since $j_0^*(I_*(D\mathcal{E})) = I_{0*}(t_0^*(D\mathcal{E})) = I_{0*}(\mathcal{E})$, on $Z = W_0$, $j_0^*\mathcal{G}.$ resolves $I_{0*}\mathcal{E}$. Similarly, on W_∞, $j_\infty^*\mathcal{G}.$ resolves $I_{\infty*}\mathcal{E}$. But $I_\infty(X)$ is disjoint from W_∞^2, hence $k^*\mathcal{G}.$ resolves $i_{\infty*}\mathcal{E}$ on $W_\infty^1 = \mathbf{P}(\mathcal{N}_i \oplus \mathcal{O}_X)$ and $l^*\mathcal{G}.$ is acyclic, where l is the natural morphism $W_\infty^2 \hookrightarrow W$. Therefore in $\mathrm{CH}(W)_{\mathbf{Q}}$, we have

$$\begin{aligned} j_{0*}(\mathrm{ch}(i_{0*}\mathcal{E})) &= j_{0*}(\mathrm{ch}(j_0^*\mathcal{G}.)) \\ &= \mathrm{ch}(\mathcal{G}.)\, j_{0*}([Z]) \\ &= \mathrm{ch}(\mathcal{G}.)(k_*([W_\infty^1]) + l_*([W_\infty^2])), \end{aligned}$$

since in $\mathrm{CH}(M)$, $[W_0] = [W_\infty]$, were k is the natural morphism $W_\infty^1 \hookrightarrow W$. Thus by the projective formula again, we have

$$\begin{aligned} j_{0*}\big(\mathrm{ch}(i_{0*}\mathcal{E})\big) &= k_*\big(\mathrm{ch}(k^*\mathcal{G}.)\big) + l_*\big(\mathrm{ch}(l^*\mathcal{G}.)\big) \\ &= k_*\big(\mathrm{ch}(i_{\infty*}\mathcal{E})\big) + 0. \end{aligned}$$

In this way, we deduce the calculation on the section of a projective bundle. So, by the lemma above, we have

$$j_{0*}\big(\mathrm{ch}(i_{0*}\mathcal{E})\big) = k_*\big(i_{\infty*}(\mathrm{td}(\mathcal{N}_i^{-1})\,\mathrm{ch}(\mathcal{E}))\big).$$

Now let $q : W \to Z$ be the composition of the blowing-down from W to $Z \times \mathbf{P}^1$, followed by the projection to Z. We have $q \circ j_0 = \mathrm{Id}_Z$ and $q \circ k \circ i_\infty = i$. So applying q_*, we have

Theorem. In $\mathrm{CH}(Z)_{\mathbf{Q}}$,

$$\mathrm{ch}(i_*\mathcal{E}) = i_*\big(\mathrm{td}(\mathcal{N}_i)^{-1}\mathrm{ch}(\mathcal{E})\big).$$

Step 3. **L.C.I. Morphisms.**

By the result above, we know that if we let $f : X \to Y$ be the composition of a closed imbedding $i : X \hookrightarrow P$ and a projection $p : P \to X$, then

$$\begin{aligned} & f_*\big(\mathrm{ch}(\mathcal{E})\mathrm{td}(\mathcal{T}_f)\big) \\ =& p_*\big(i_*(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(\mathcal{N}_i)^{-1}\, i^*(\mathrm{td}(\mathcal{T}_p)))\big) \\ =& p_*\big(i_*(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(\mathcal{N}_i)^{-1})\,\mathrm{td}(\mathcal{T}_p)\big) \\ =& p_*\big(\mathrm{ch}(i_*\mathcal{E})\,\mathrm{td}(\mathcal{T}_p)\big) \\ =& \mathrm{ch}\big(p_*(i_*\mathcal{E})\big) \\ =& \mathrm{ch}(f_*\mathcal{E}). \end{aligned}$$

This completes the proof.

All the results above are from algebraic geometry. In what follows, we will generalize them to the arithmetic situation. Then, we can obtain an arithmetic version of the Riemann-Roch theorem by putting the results in the corresponding arithmetic notation. That is, we have the commutative diagram

$$\begin{array}{ccc} K_{\mathrm{Ar}}(X)_{\mathbf{Q}} & \xrightarrow{\mathrm{ch}_{\mathrm{Ar}}()\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)} & \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}} \\ f_K \downarrow & & \downarrow f_{\mathrm{CH}} \\ K_{\mathrm{Ar}}(Y)_{\mathbf{Q}} & \xrightarrow{\mathrm{ch}_{\mathrm{Ar}}()} & \mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}. \end{array}$$

§II.1.4. Algebraic Intersection Theory: A Geometric Description

In section 2, we gave a K-theory description of algebraic intersection theory. Here we give a more geometric definition, which is much more concrete. The price we pay for this is that now we can only deal with non-singular projective varieties. We will not give all the details, instead, we explain the basic idea behind the proof and also give the Chow moving lemma at the level of K_1.

Let X be a regular projective variety. If Y and Z are two integral subschemes of X, which intersect properly in X, i.e. $\mathrm{codim}_X(Y \cap Z) = \mathrm{codim}_X Y + \mathrm{codim}_X Z$, then there is a natural intersection

$$[Y][Z] := \sum_k \mu_k [S_k],$$

where S_k is the irreducible component of $Y \cap Z$, and μ_k the intersection multiplicity of Y and Z along S_k. This is not always true in general, since two algebraic cycles do not usually have proper intersection. In order to deal with the general situation, we first note that for any rational function f on a closed subscheme, the intersection of $\mathrm{div}(f)$ with any algebraic cycle is zero. (In fact, even through it is very simple, this is the most important principle in the intersection theory, e.g., for any meromorphic function f on $\mathbf{C}$, the number of zeros is exactly the same as the number of poles.) Therefore, for any two integral subschemes Y, Z, there always exist rational functions f, such that $Y + \mathrm{div}(f)$ intersects Z properly. Hence, we may get the algebraic intersection generally. Such an easy example give us the motivation to introduce the following

Chow Moving Lemma. Let X be a regular projective variety and α, β algebraic cycles on X. Then, there exists an algebraic cycle α', which is rationally equivalent to α, such that α' meets β properly.

We may assume that α and β are prime algebraic cycles Y and Z, then one can prove the moving lemma as follows: We need some further notation. For Y and Z as above, we define the **exceeds** $e(Y, Z)$ of Y with respect to Z by

$$e(Y, Z) := \mathrm{codim}_X Y + \mathrm{codim}_X Z - \mathrm{codim}_X(Y \cap Z).$$

The proof is achieved by induction on $e(Y, Z)$. Since X is projective, there is a closed immersion $i : X \hookrightarrow \mathbf{P}^n$. Let $L \subset \mathbf{P}^n$ be a linear subspace, and $Y \subset \mathbf{P}^n$ a subvariety for which $L \cap Y = \emptyset$. Then, there is a subvariety $C_L(Y) \subset \mathbf{P}^n$, called the **cone over** Y **with the vertex** L, viz. $C_L(Y) := \pi_L^{-1}(\pi_L(Y))$, where $\pi_L : \mathbf{P}^n - L \to \mathbf{P}^r$ is the projection and $r + 1$ is the codimension of L in P^n. We know that:

(a) For a generic L, $C_L(Y)$ meets X properly, and this intersection is generically transversal along Y, i.e.

$$i^*[C_L(Y)] = [Y] + \gamma_L,$$

where γ_L is a cycle on X which does not contain Y.

(b) If $e(Y, Z) > 0$, then for generic L,

$$\mathrm{codim}_X(T_j \cap Z) > \mathrm{codim}_X(Y \cap Z),$$

where $\gamma_L := \sum_j n_j[T_j]$.

From these two results there is a clear way to make induction on e. Then, we can deduce the Chow moving lemma from the fact that algebraic cycles on $\mathbf{P}^n$ can be moved. See Roberts [Ro 72] for more details.

The application to the arithmetic intersection theory requires a refined version of the Chow moving lemma at the level of algebraic K_1-chains.

For a regular scheme X, let

$$R_p^i(X) := \oplus_{x \in X^{(i)}} K_{p-i}(k(x)).$$

For example,

$$R_p^p(X) = Z^p(X), \quad R_p^{p-1}(X) = \oplus_{x \in X^{(p-1)}} k(x)^*.$$

The elements of $R_p^{p-1}(X)$ are K_1-chains, and we write a K_1-chain as $f = \sum_W [f_W]$, where $f_W \in k(W)^*$ and W runs through a finite set of integral codimension $p-1$ closed subschemes of X. For each $p \geq 1$, we have a natural homomorphism

$$\begin{array}{rccc} \mathrm{div}: & R_p^{p-1}(X) & \to & R_p^p(X) = Z^p(X) \\ & \sum_W [f_W] & \mapsto & \sum_W \mathrm{div}(f_W). \end{array}$$

For any K_1-chain $f = \sum_W [f_W]$, the **support** of f is the union of all W for which $f_W \neq 1$. We say that a K_1 chain f meets a collection of integral closed subschemes $\sum$ of X **almost properly** if for any element $Z \in \sum$, $\mathrm{div}(f_W)$ meets Z properly, and that a K_1-chain f meets $\sum$ **properly**, if f meets $\sum$ almost properly, while each W, for which $f_W \neq 1$, meets $\sum$ properly.

For any closed integral subscheme W of codimension $p-1$ in X, we think of $f \in k(W)^*$ as a K_1-chain. The product of K_1-chains with algebraic cycles is defined as follows.

(1) Let Z be a codimension q algebraic cycle of X, which meets W and $\mathrm{div}(f)$ properly. Define a K_1-chain $[f]Z$ as follows: Since W meets Z properly, we have

$$[W]Z = \sum_k \mu_k[S_k].$$

By the fact that $\mathrm{div}(f)$ also meets Z properly, we have rational functions $f|_{S_k} \in k(S_k)^*$, and hence can define

$$[f]Z := \sum [f^{\mu_k}|_{S_k}].$$

(2) If Z meets $\mathrm{div}(f)$ almost properly, then $[W]Z = \sum_k \mu_k[S_k] + t$, where $W \cap |Z| = S \cup T$, with S is the whole proper part. Usually, t is a class in $\mathrm{CH}_T^{p+q-1}(X)_{\mathbf{Q}}$. Since f_T is a unit, we have a class $[f]t \in \mathrm{CH}^{p+q,p+q-1}(X)_{\mathbf{Q}}$, and we can define

$$[f]Z := \sum [f^{n_k}|_{S_k}] + [f]t,$$

which is well-defined as an element of $(R_P^{p-1}(X)_{\mathbf{Q}}/d(R_p^{p-2}(X)))_{\mathbf{Q}}$. Obviously, by algebraic intersection theory, we have

$$\mathrm{div}([f]Z) = \mathrm{div}(f)Z.$$

(3) In order to introduce a product of K_1-chains with algebraic cycles in general, we need the following

Chow Moving Lemma For K_1-Chains. Let X be a non-singular quasi-projective variety. Suppose that $f \in R_p^{p-1}(X)$ is a K_1-chain such that $\mathrm{div}(f)$ meets a finite collection $\sum = \{Z_1, \ldots, Z_r\}$ of subvarieties of X properly. Then, there exists a K_1 chain g, such that
(a) $\mathrm{div}(g) = \mathrm{div}(f)$.
(b) $g - f$ represents 0 in $\mathrm{CH}^{p,p-1}(X)$.
(c) g meets $\sum$ almost properly.

Surely, by this moving lemma and the definition in (1), (2), we have a product of K_1-chains with algebraic cycles.

Proof Of The Lemma. Let L, $L \subset \mathbf{P}^n$, be a linear subspace, and Z, $Z \subset \mathbf{P}^n$, a subvariety for which $L \cap Z = \emptyset$. If $\dim Z < r$, then, there exists an open dense Zariski set in the Grassmannian of all $(n-r-1)$ planes in $\mathbf{P}^n$, such that, for any L in this subset, the map $Z \to \pi_L(Z)$ is birational, and hence, there is a canonical inclusion $k(Z) \subset k(C_L(Z))$. Therefore if $f \in R_{p+1}^p(\mathbf{P}^n)$ and $p \geq n - r + 1$, then for a generic L, we have a well-defined K_1-chain $C_L(f) \in R_{p+r-n+1}^{p+r-n}(\mathbf{P}^n)$ and $\mathrm{div}(C_L(f)) = C_L(\mathrm{div}(f))$. Also, if f is supported on a subvariety $X \subset \mathbf{P}^n$ with $L \cap X = \emptyset$, then $C_L(f)$ meets X properly. Thus, by (a) and (b) after the Chow moving lemma, there exist linear subspaces $L_1, \ldots, L_e$, such that

$$f = \sum_{j=1}^{e} (-1)^{j-1} C_{L_j}(f_j)X + (-1)^e f_e,$$

where f_e meets $\sum$ almost properly. We can also find $\tau_j \in \mathrm{Aut}(\mathbf{P}^n)$ such that $\tau_j C_{L_j}(f_j)$ meets $\sum$ properly. Joining each τ_j to Id in $\mathrm{Aut}(\mathbf{P}^n)$ by a rational curve, we obtain a family f_t of K_1-chains on X, parameterized by $t \in \mathbf{P}^1$, such that $f_0 = f$, f_∞ meets $\sum$ almost properly, and for all but finitely many values t, $\mathrm{div}(f_t)$ meets $\sum$ properly. That is, the family f_t forms a K_1-chain $Df = \sum_V [Df_V]$ on $X \times \mathbf{P}^1$ with each V flat over $\mathbf{P}^1$ and Df meeting $\mathrm{div}(t)$ properly. Since each V is flat over $\mathbf{P}^1$ and meets $\mathrm{div}(t)$ properly, we have an element

$$\sum_W \{t, f_W\} \in \oplus_{x \in (X \times \mathbf{P}^1)^{(p-1)}} K_2(k(x)).$$

Under the differential

$$d : \oplus_{x \in (X \times \mathbf{P}^1)^{(p-1)}} K_2(k(x)) \to \oplus_{x \in (X \times \mathbf{P}^1)^{(p)}} K_1(k(x)),$$

we have

$$d(\sum_W \{t, f_W\}) = \mathrm{div}(t) Df - \{t\} \mathrm{div}(Df),$$

which equals

$$f_0 \times \{0\} - f_\infty \times \{\infty\} - \{t\} \mathrm{div}(Df).$$

Hence, for each $Z \in \sum$, the K_1-chain $[t]\,\mathrm{div}(Df)$ meets $Z \times P^1$ almost properly. Push-out this element to X by the natural projection $p : X \times \mathbf{P}^1 \to X$, and we see that $p_*(\{t\}\mathrm{div}(Df))$ meets Z almost properly and

$$d(p_*(\sum_W \{t, f_W\})) = f - \big(f_\infty + p_*(\{t\}\mathrm{div}(Df))\big).$$

Therefore $g = f_\infty + p_*(\{t\}\mathrm{div}(Df))$ satisfies the conditions of the lemma above. This completes the proof of the lemma.

Chapter II.2
Arithmetic Intersection Theory

We now generalize the results in the last a few chapters to the arithmetic situation. Classically, this procedure was noticed by A. Weil and Russian mathematics school, guided by Shafarevich. For dimension 1 case, A. Weil noted a certain analogy between function fields of curves and number fields. For Russian mathematicians, they achieved this kind of analogy by studying the famous Shafarevich conjecture, which deals with the relative dimension 1 case, i.e. a curve over a function field or a number field. Since for a function field, the model is a complex surface, which is complete, there is a classical procedure to introduce intersection theory and corresponding results, such as the Riemann-Roch theorem. However, for the object over a number field, the problem is rather difficult, since this model is non-complete: We only consider the objects over the spectrum of the ring of integers, which is affine. At first sight, we do not get a satisfactory theory for this model. A natural idea is to complete this arithmetic model. One found that points in the function field case correspond to valuations. Therefore, in order to complete the arithmetic model, one needs to include also the archimedian valuations. There is then another problem: To find a local intersection theory which is valid for both finite valuations and Archimedian valuations. In order to solve this problem, mathematicians spent almost thirty years. By the work of Néron, we can define the intersection for the relative model by a purely local method. Parshin and Arakelov were the first to solve the Shafarevich conjecture for function fields at the beginning of 1970's. After that, Arakelov obtained a good analogy for the concept of the p-adic distance at infinity, the Arakelov- Green function, by choosing the so-called Arakelov metric at infinity on the corresponding Riemann surfaces. Hence one knows how to introduce the local intersection at infinity. In western countries, it was Faltings who first gave a systematic treatment of the theory for arithmetic surfaces. But this was almost ten years after Arakelov introduced his wonderful idea. With the Arakelov theory, at that time, Faltings also proved the Mordell conjecture for number fields. Soon after that, Deligne developed a more general theory for arithmetic surfaces with an arbitrary metric at infinity. Several important ideas are introduced in [De 86]. Now we follow Gillet and Soulé to define a higher dimensional arithmetic intersection theory [GS 91].

This chapter consists of seven sections. In section one, we discuss arithmetic varieties. After this, we know that there are two well-organized parts in the theory of an arithmetic variety: The finite part and the infinite part, which correspond to the model over the spectrum of the ring of integers and the model over archimedian valuations, respectively.

For the finite part, there is a natural local intersection, which comes from the algebraic intersection theory. For the infinite part, since we may associate with a complex manifold. Thus, we have to discuss certain objects in complex geometry. All this is basically the contents in section 2. In section 3, we introduce arithmetic Chow groups and their homology properties. In section 4, we use the result in section 2 and the results from algebraic intersection theory to give the arithmetic intersection theory. In section 5, we discuss the functorial properties for the arithmetic intersection theory. In section 6, we give a few concrete examples. Finally, in section 7, we give a generalization of arithmetic intersection theory, i.e. we give a cap product between arithmetic Chow cohomology and homology. The reader is advised to skip this final section in the first reading.

§II.2.1 Arithmetic Varieties

II.2.1.a **Arithmetic Rings.**

Even through we restrict ourselves generally to the varieties over number fields, we can go slightly further. Instead of number fields, we introduce a more general concept of an arithmetic ring. We say a triple (A, Σ, F_∞) is an **arithmetic ring**, if A is an excellent regular noetherian integral domain, Σ is a finite nonempty set of monomorphisms $\sigma : A \to \mathbf{C}$ and $F_\infty : \mathbf{C}^\Sigma \to \mathbf{C}^\Sigma$ is a conjugate linear involution of $\mathbf{C}$-algebras, such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\delta} & \mathbf{C}^\Sigma \\ \mathrm{Id}_A \downarrow & & \downarrow F_\infty \\ A & \xrightarrow{\delta} & \mathbf{C}^\Sigma \end{array}$$

commutes. Here δ is the natural product map induced by the family of maps $\{\sigma : A \to \mathbf{C} : \sigma \in \Sigma\}$.

Examples.

(1) Let A be a ring whose field of fractions is a number field F, let $\Sigma := \mathrm{Hom}(A, \mathbf{C})$ be the set of all embedding of A into $\mathbf{C}$ and let F_∞ be the usual Frobenius on $\mathbf{C} \simeq \mathbf{C} \otimes_{\mathbf{Q}} F$ induced by complex conjugation.

(2) Let $A = \mathbf{C}$. There is an isomorphism $\mathbf{C} \otimes_{\mathbf{R}} A \simeq \mathbf{C} \times \mathbf{C}$ sending $z \otimes w$ to $(zw, z\bar{w})$. The composition of this map with the natural map $A \to \mathbf{C} \otimes_{\mathbf{R}} A$ sending a to $1 \otimes a$ is the map $\delta : a \mapsto (a, \bar{a})$. Therefore $(\mathbf{C}, \{\mathrm{Id}_{\mathbf{C}}, c\}, F_\infty)$ is an arithmetic ring. Here $c : \mathbf{C} \to \mathbf{C}$ is complex conjugation and $F_\infty(a, b) = (\bar{b}, \bar{a})$.

We say a pair $f = (f_1, f_2)$ is a **homomorphism of arithmetic rings**,

$$f : (A, \Sigma, F_\infty) \to (A', \Sigma', F'_\infty),$$

if $f_1 : A \to A'$ is a ring homomorphism, $f_2 : \mathbf{C}^\Sigma \to \mathbf{C}^{\Sigma'}$ is a homomorphism of $\mathbf{C}$-algebras, such that $f_2\delta = \delta' f_1$ and $f_1\, F_\infty = F'_\infty\, f_2$. Obviously, $\mathbf{Z}$ as an arithmetic ring is an initial object in the category of arithmetic rings. More generally, if E/F is an extension of number fields, then the natural homomorphism

$$\mathcal{O}_F \hookrightarrow \mathcal{O}_E$$

induces an homomorphism of arithmetic rings.

II.2.1.b **Arithmetic Varieties**

Let (A, Σ, F_∞) be an arithmetic ring. We say $(X, \pi : X \to S := \operatorname{Spec} A)$ is an **arithmetic variety** over A, if π is projective, of finite type, and X_F is smooth. Here F denotes the fraction field of A and X_F is the generic fiber of X. Usually, if $s \in S$ we denote by X_s the fiber of X over s, while if $\sigma \in \Sigma$, we write $X_\sigma := X \otimes_\sigma \mathbf{C}$ and $X_\Sigma := \amalg_{\sigma \in \Sigma} X_\sigma = X \otimes_A \mathbf{C}^\Sigma$. Finally, we also denote by $X_\infty := X_\Sigma(\mathbf{C})$ the analytic space associated with the scheme X_Σ.

It follows that an arithmetic variety consists of two parts: One is the flat finite type scheme X over S, while the other is its associated infinite part $X_\Sigma(\mathbf{C})$. In the next section, we deal with the aspect of X at infinity, which is a complex manifold.

§II.2.2 Green's Currents

We consider the Green current on a complex manifold, which is needed to introduce the arithmetic intersection on an arithmetic variety.

II.2.2.a **Currents**

Let X be a complex compact manifold of dimension d. Denote by

$$A^n(X) := \oplus_{p+q=n} A^{p,q}(X)$$

the space of differential forms of degree n on X. There are natural boundary morphisms

$$\partial : A^{p,q}(X) \to A^{p+1,q}(X), \quad \bar{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$$

and $d : A^n(X) \to A^{n+1}(X)$ is the usual differential. We say that a linear function T on $A^n(X)$ is a **current**, if T is continuous in the sense of Schwartz: for any sequence $\{\omega_r\} \subset A^n(X)$ with the supports contained in certain fixed compact subset K, then $T(\omega_r) \to 0$ if all the coefficients of ω_r together with their derivatives tend uniformly to zero for $r \to \infty$. The set of currents forms a topological dual space $A(X)^*$ of $A(X)$. We denote by $D_n(X) := A^n(X)^*$. There is a natural decomposition of $D_n(X)$:

$$D_n(X) = \oplus_{p+q=n} D_{p,q}(X),$$

where $D_{p,q}(X)$ is the dual of $A^{p,q}(X)$. It is convenient for us to let

$$D^{p,q}(X) := D_{d-p,d-q}(X).$$

The morphism $\partial, \bar{\partial}$, and d induce morphisms from the space $D^{p,q}$ to the following spaces of currents $D^{p+1,q}(X), D^{p,q+1}(X), D^{p+1,q+1}(X)$ respectively. We denote them by $\partial', \bar{\partial}', d'$, e.g. we have

$$(\partial' T)(\alpha) := T(\partial \alpha).$$

Examples. (1) There is a natural inclusion

$$\begin{array}{ccc} A^{p,q}(X) & \hookrightarrow & D^{p,q}(X) \\ \omega & \mapsto & [\omega] \end{array},$$

where

$$[\omega](\alpha) := \int_X \omega \wedge \alpha$$

for any $\alpha \in A^{d-p,d-q}(X)$. We say that a current T is **smooth** if there exists a smooth form ω such that $T = [\omega]$. In particular, if $p+q=n$, it follows by Stokes' theorem that

$$\begin{aligned} [d\omega](\alpha) &= \int_X d\omega \wedge \alpha \\ &= \int_X d(\omega \wedge \alpha) - \int_X (-1)^n \omega \wedge d\alpha \\ &= (-1)^{n+1} \int_X \omega \wedge d\,\alpha = (-1)^{n+1}(d'[\omega])(\alpha). \end{aligned}$$

Therefore, if we let $\partial, \bar{\partial}, d$ on the currents be $(-1)^{n+1}\partial'$, $(-1)^{n+1}\bar{\partial}'$, $(-1)^{n+1}d'$ respectively, and let $d^c := \frac{1}{4\pi i}(\partial - \bar{\partial})$. Then

$$dd^c = -\frac{1}{2\pi i}\partial\bar{\partial}$$

is a real operator, and we have the following commutative diagram:

$$\begin{array}{ccc} A^{p,q}(X) & \hookrightarrow & D^{p,q}(X) \\ \partial \downarrow & & \downarrow \partial \\ A^{p+1,q}(X) & \hookrightarrow & D^{p+1,q}(X) \end{array}$$

etc..

(2) Let $i : Y \hookrightarrow X$ be an irreducible subvariety of codimension p. We get a current $\delta_Y \in D^{p,p}(X)$ by letting

$$\delta_Y(\alpha) := \int_{Y^{ns}} i^* \alpha$$

for any $\alpha \in A^{d-p,d-p}(X)$. Here Y^{ns} denotes the non-singular locus of Y. We call this current the **Dirac symbol** of Y. By one of Hironaka's theorem, another equivalent definition is that there exists a resolution of singularities $\tilde{Y}$ of Y with the following properties:

(a) $\tilde{Y}$ is smooth;

(b) The exceptional divisor E is a divisor of $\tilde{Y}$ with normal crossings;

(c) The natural morphism $\pi : \tilde{Y} - E \to Y^{ns}$ is an isomorphism.

Then we have

$$\delta_Y(\alpha) = \int_{\tilde{Y}} \pi^*(i^*\alpha).$$

We end this subsection by the following basic

Theorem. (1) With the boundary morphisms ∂, $\bar{\partial}$, d, there exist natural isomorphisms between the cohomologies of X in the sense of differential forms and currents.
(2) Let γ be a current on X such that $dd^c\gamma$ is smooth. Then there exist currents ω, α, β such that $\gamma = \omega + \partial\alpha + \bar{\partial}\beta$, with ω smooth.
(3) As a current, if ω is smooth and $\omega = \partial u + \bar{\partial} v$, then there exist smooth currents α, β such that $\omega = \partial\alpha + \bar{\partial}\beta$.
(4) If X is a Kähler manifold, and $\eta \in D^{p,q}(X)$, $p, q \geq 1$, is d-closed and is either d, ∂, $\bar{\partial}$ exact. Then there exists $\gamma \in D^{p-1,q-1}(X)$ such that

$$dd^c\gamma = \eta.$$

In particular, if $\eta = 0$, we may choose $\gamma = \omega + \partial\alpha + \bar{\partial}\beta$, where ω is a harmonic form.

Proof. (1) Just as in the situation for de Rham complex

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \ldots \to \mathcal{A}^{p,d} \to 0,$$

we have a complex

$$0 \to \Omega^p \to \mathcal{D}^{p,0} \ldots \to \mathcal{D}^{p,d} \to 0.$$

Here $\mathcal{D}$ denotes the sheaf of currents and we note that the sheaves $\mathcal{D}$ admit partitions of unity, so that $H^k(X, \mathcal{D}^{p,q}) = 0$ for $k > 0$. Therefore, by the proof of de Rham theorem and Dolbeault theorem, (1) is a consequence of the exactness of the above complex for currents. We have to establish the $\bar{\partial}$-Poincaré lemma for currents. Since this is a local problem, we may work over $\mathbf{C}^d$. For any two complex manifolds M, N with local holomorphic coordinates z and w, let

$$A^{(p,q)(r,s)}(M \times N)$$

be the C^∞-forms having the type (p,q) with respect to z and (r,s) with respect to w. We set

$$\Phi(\xi) := d\xi_1 \wedge \ldots \wedge d\xi_d,$$

$$\Phi_i(\xi) := (-1)^{i-1}\xi_i d\xi_1 \wedge \ldots \wedge \hat{d\xi_i} \wedge \ldots \wedge d\xi_d,$$

and define the **Bochner-Martinnelli kernel** on $\mathbf{C}^d \times \mathbf{C}^d$ by

$$k(z,w) := C_d \frac{\sum \overline{\Phi_i(z-w)} \wedge \Phi(w)}{\|z-w\|^{2d}}.$$

Thus

$$k(z,w) \in \oplus_{q=1}^d L^{(0,q)(d,d-p)}(\mathbf{C}^d \times \mathbf{C}^d, \mathrm{loc}),$$

and $k(z,w)$ is a local integrable form. Thus there is a natural morphism

$$K : A_c^{0,q}(\mathbf{C}^d) \to A^{0,q-1}(\mathbf{C}^d)$$

defined by

$$(K\phi)(z) = \int_{w\in\mathbf{C}^d} k(z,w) \wedge \phi(w).$$

With this definition, we know that

$$K\bar\partial + \bar\partial K = \mathrm{Id}.$$

The above homotopy formula now can be used to prove the $\bar\partial$-Poincaré lemma, and hence completes the proof of (1). In fact, if we only consider the situation for smooth forms, then, for a $\bar\partial$-closed form $\phi \in A^{0,q}(U)$, where $U \subset \mathbf{C}^n$ is an open set, we may find a relatively compact open subset $V \subset U$ and bump function $\rho \in C_c^\infty(U)$ with $\rho \equiv 1$ on V. Then $\rho\phi \in A_c^{0,q}(\mathbf{C}^n)$, and

$$(\rho\phi)(z) = \bar\partial(K\rho\phi)(z) + K(\bar\partial(\rho\phi))(z).$$

Restricting to V, we get

$$\phi(z) = \bar\partial(K\rho\phi)(z)$$

for all $z \in V$. With this, we easily see that the above process works also for a compactly supported current T, if we define KT by

$$KT(\phi) = T(K\phi),$$

for all $\phi \in A_c^{n,n-q-1}(\mathbf{C})$. Indeed, we see that for any test form $\varphi \in A_c^{n,n-q}(\mathbf{C}^n)$,

$$(\bar\partial(KT))(\varphi) + (K(\bar\partial T))(\varphi) = (KT(\bar\partial\varphi) + (\bar\partial T)(K\varphi)$$
$$= T(K\bar\partial\varphi + \bar\partial K\varphi) = T_\varphi.$$

So, the homotopy formula above even makes sense for compactly supported currents. Hence, we get (1).

(2) is a direct consequence of (1). In fact, as a smooth form $dd^c\gamma =: \eta$, we have $\eta = d(\bar\partial g)$ for a certain current g, i.e., η is a d-exact form. Thus, by (1), there exists a smooth form ϕ such that $\eta = d\phi$. In particular,

$$d(\bar\partial\gamma - \phi) = 0.$$

Hence, by (1) again, we know that there exists a closed smooth form α and a current S such that

$$\bar\partial\gamma - \phi = \alpha + dS.$$

That is,

$$\bar{\partial}\gamma = \alpha' + dS$$

with α' smooth. So, there exists an expression

$$\bar{\partial}\gamma = \phi' + \partial S_1 + \bar{\partial} S_2,$$

where ϕ', $\bar{\partial} S_1$, ∂S_2 are smooth. But then, from (1),

$$S_1 = \alpha_1 + \bar{\partial}\, u, \quad S_2 = \alpha_2 + \partial v$$

with α_1, α_2 smooth. With this, we get

$$\bar{\partial}\gamma = \phi'' + \partial\bar{\partial}(u - v)$$

with ϕ'' smooth. Equivalently,

$$\bar{\partial}(\gamma + \partial(u - v)) = \phi''.$$

Therefore, using (1) once more, we get

$$\gamma = x + \bar{\partial} w + \partial(u - v),$$

with x smooth, which completes the proof of (2).

(3) is a consequence of (1) and (2). Indeed, $\bar{\partial}\omega = \bar{\partial}\partial u$ is smooth. So, by (2),

$$u = \alpha + \partial x + \bar{\partial} y,$$

where α is smooth; and hence

$$\partial u = \partial\alpha + \partial\bar{\partial} y.$$

Similarly,

$$\bar{\partial} v = \bar{\partial}\beta + \bar{\partial}\partial z$$

with β smooth. Therefore

$$\omega = \partial\alpha + \bar{\partial}\beta + \partial\bar{\partial}(y - z).$$

By (2) again, $y - z = \phi + \partial s + \bar{\partial} t$ with ϕ smooth. So, we have

$$\omega = \partial(\alpha + \bar{\partial}\phi) + \bar{\partial}\beta,$$

which completes the proof of (3).

(4) may be deduced from the Hodge theorem. In fact, if η is smooth, we obtain an explicit solution of $\partial\bar{\partial}\gamma = \eta$ by $\pm\partial^*\bar{\partial}G_{\bar{\partial}}^2\eta$, where $G_{\bar{\partial}}$ is the Green's operator associated with the $\bar{\partial}$-Laplacian, which is a smooth form. Since the operators ∂^*, $\bar{\partial}^*$ and $G_{\bar{\partial}}$ extend to currents, the same expression also gives a solution of the equation when η is a current. Now the assertion comes from the Hodge decomposition theorem.

II.2.2.b. **Green's Currents**

Let Y be a codimension p analytic subvariety of X. We say that a current $g \in D^{p-1,p-1}(X)$ is a **Green's current** of Y if

$$dd^c g = [\omega] - \delta_Y$$

for some $\omega \in A^{p,p}(X)$. The main result for Green's currents is the following

Theorem. If X is a Kähler manifold, then
(1) Green's currents exist on X.
(2) If g_1 and g_2 are two Green's currents for Y, then

$$g_1 - g_2 = [\eta] + \partial S_1 + \bar{\partial} S_2,$$

where $\eta \in A^{p-1,p-1}(X)$.
(3) (The Poincaré-Lelong equation.) Let $(\mathcal{L}, \rho)$ be a hermitian line sheaf on X and s a non-zero meromorphic section of $\mathcal{L}$. Then $-\log|s|_\rho^2 \in L^1(X)$, and hence induces a distribution $[-\log|s|_\rho^2] \in D^{0,0}(X)$. We then have

$$dd^c[-\log|s|_\rho^2] = [c_1(\mathcal{L}, \rho)] - \delta_{\mathrm{div}(s)}.$$

That is, $[-\log|s|_\rho^2]$ is a Green's current of $\mathrm{div}(s)$.

Proof. Note that by Stokes' formula, we know that $d\delta_Y = 0$. So, by the fact that

$$\delta - [\omega] = da,$$

which is a consequence of (1) of the theorem in the last section, we see that (1) and (2) are consequences of (4) and (3) of the theorem in the last subsection respectively.

(3) By definition, we know that

$$-dd^c \log|s|_\rho^2 = c_1(\mathcal{L}, \rho)$$

on $X - \mathrm{div}(s)$. Thus we only need to consider the equality over $\mathrm{div}(s)$. Let $\{U, \ldots\}$ be a finite open covering of $\mathrm{div}(s)$ such that $\mathcal{L}|_U$ is trivial and on U, $\mathrm{div}(s)$ is defined by the equation $z_1 = 0$, where $z = (z_1, \ldots, z_d)$ is a local coordinate of U. By the Weierstrass preparation theorem, we may assume that $\mathrm{div}(s) \cap U$ does not contain any singular point. Thus it is sufficient to prove that for any $\omega \in A^{d-1,d-1}(X)$,

$$\int_U \log|s|_\rho^2 \, dd^c\omega = -\int_U c_1(\mathcal{L}|_U, \rho|_U) \wedge \omega + \int_{\mathrm{div}(s) \cap U} \omega.$$

Now note that since $c_1(\mathcal{L}|_U, \rho_U) = 0$ and $s = z_1 h$ for some non-vanishing holomorphic function h, it is enough to prove

$$\int_U \log|z_1|^2 dd^c\omega = \int_{U, z_1 = 0} \omega.$$

On the other hand,

$$\lim_{\epsilon\to 0}\int_{U,|z_1|\geq\epsilon}\log|z_1|^2 dd^c\omega=\int_U \log|z_1|^2 dd^c\omega.$$

So the final assertion is obtained by using Stokes' theorem for the left hand side.

From above, we see that for a given closed subvariety Y of X, there exist many Green's currents associated with Y. We also know how to measure the difference of two different Green's currents. On the other hand, we can give an explicit Green's current for a divisor. In the intersection theory, there is a general principle, which says that if certain objects can be constructed for a divisor, then we can deduce the general situation from this special situation. With this in mind, we introduce the next subsection.

II.2.2.c **Green's Currents with Logarithmic Growth**

In this subsection, we give a generalization of the Poincaré-Lelong equation for higher codimension subvarieties.

We always assume, from now on, that X is a (quasi)-projective complex manifold. For any irreducible subvariety Y, we say a smooth form α on $X-Y$ has **logarithmic growth** along Y, if there exists a proper morphism $\pi:\tilde{X}\to X$ such that $E:=\pi^{-1}(Y)$ is a divisor with normal crossings, $\pi:\tilde{X}-E\simeq X-Y$ and α is the direct image of a form β on $\tilde{X}-E$ by π with the following property:

Near each $x\in\tilde{X}$, let $z_1\ldots z_k=0$ be a local defining equation of E. Then, there exists d-closed smooth forms α_i and a smooth form γ such that

$$\beta=\sum_{i=1}^k \alpha_i\log|z_i|^2+\gamma.$$

Obviously, such an α is always locally integrable on X, and hence defines a current $[\alpha]$, which is the direct image by π of the current $[\beta]$.

By the definition, we easily have the following properties of forms with logarithmic growth.

Pull-Back Property. Let $f:X'\to X$ be a morphism of smooth projective varieties, and let α be a form on $X-Y$ of logathmic growth along Y. If $f^{-1}(Y)$ does not contain any component of X', then the form $f^*(\alpha)$ is of logarithmic growth along $f^{-1}(Y)$.

Push-Out Property. Let $f:X\to X'$ be a morphism of smooth projective varieties, and let α be a form on $X-Y$ of logathmic growth along Y. If f is smooth outside Y and $f(Y)$ does not contain any component of X', then the form $f_*(\alpha)$ is of logarithmic growth along $f(Y)$ and $f_*([\alpha])=[f_*(\alpha)]$.

As a generalization of the Poincaré-Lelong equation, we have the following

Theorem. For every irreducible subvariety $Y \subset X$, there exists a smooth form g_Y on $X - Y$ with logarithm growth along Y such that $[g_Y]$ is a Green's current for Y.

Proof. We prove this theorem by the following steps.

Step 1. Suppose Y is an irreducible codimension-1 subvariety. In this case, we may associate Y with a line sheaf $\mathcal{L}$. Since X is projective, there is a natural metric on $\mathcal{L}$, one induced by the pull-back of the Fubini-Study metric. Now the result follows from the Poincaré-Lelong equation in subsection b.

Step 2. Let $i : Y \hookrightarrow X$ be an irreducible subvariety of X of codimension p. Then, by Hironaka's theorem, there exist a smooth projective complex variety $\tilde{X}$ and a proper morphism $\pi : \tilde{X} \to X$ such that $\tilde{X} - E \simeq X - Y$, where $E = \cup_i E_i := \pi^{-1}(Y)$ is a divisor with normal crossings. Consider the cycle $[Y] \in \mathrm{CH}^p_Y(X)$: we have

$$\pi^*[Y] \in \mathrm{CH}^p_E(\tilde{X}) \simeq \mathrm{CH}^{p-1}(E) = \oplus_i \mathrm{CH}^{p-1}(E_i).$$

Hence $\pi^*[Y] = \sum_i [\eta_i]$ with $\eta_i \in \mathrm{CH}(E_i)$. Therefore, we see that the corresponding cohomology class $\pi^*\mathrm{cl}(Y) \in H^{p,p}_E(\tilde{X}, \mathbf{R})$ decomposes as $\sum_i j_{i*}(\mathrm{cl}(\alpha_i))$, where α_i is a closed form of $A^{p-1,p-1}(E_i)$, and $j_i : E_i \hookrightarrow E$ is the natural inclusion. Therefore, the result is a consequence of the following lemmas.

Lemma 1. With the same notation as above, we have

$$\delta_Y = \sum_i \pi_*(j_{i*}[\alpha_i]).$$

Lemma 2. Let X be a complex manifold and let $i : Y \hookrightarrow X$ be a closed immersion of a codimension-p smooth submanifold Y. Then, for any closed form $\alpha \in A^{n,n}(Y)$, there exist a form g on $X - Y$, which is of logarithmic growth along Y, of type $(n+p-1, n+p-1)$ and a smooth $(p+n, p+n)$-form β, such that

$$dd^c[g] = [\beta] - i_*[\alpha],$$

Proof of Lemma 1. Consider the resolution of singularities $\tilde{Z}_i$ of $Z_i := \pi(E_i) \subset X$, we have the following diagram

$$\begin{array}{ccccc} \tilde{E}_i & \xrightarrow{\tilde{\pi}_i} & \tilde{Z}_i & & \\ q_i \downarrow & & \downarrow p_i & & \\ E_i & \xrightarrow{\pi_i} & Z_i & \xhookrightarrow{j} & X, \end{array}$$

where q_i is birational and $\tilde{E}_i$ smooth. Hence

$$\pi_*(j_{i*}[\alpha_i]) = j_*(p_{i*}(\tilde{\pi}_{i*}[q_i^*\alpha_i])).$$

Note that if $\operatorname{codim}_X Z_i > \operatorname{codim}_X Y$, then $p_{i*}(\tilde{\pi}_{i*}[q_i^* \alpha_i]) = 0$, so we have $Z_i = Y$ and

$$\operatorname{codim}_X Z_i = \operatorname{codim}_X Y.$$

Therefore

$$\sum_i \pi_*(j_{i*}[\alpha_i]) = p_*(S)$$

for $p : \tilde{Y} \to Y$ a resolution of singularities of Y and $S \in D^{0,0}(\tilde{Y})$ is a closed current. In particular, we know that an S is a constant multiple of $\delta_{\tilde{Y}}$. Hence

$$\pi_*(\sum_{i=1}^{k} j_{i*}[\alpha_i]) = a\,\delta_Y$$

for some $a \in \mathbf{R}$. But $a\,\delta$ represents $\pi_*(\pi^*[Y])$ in $H_Y^{2p}(X, \mathbf{R})$. So, by the fact that π is birational, (and hence $\pi_*(\pi^*[Y]) = [Y]$,) we have $a = 1$. This gives the proof of Lemma 1.

The second lemma is a consequence of the following more general

Lemma 3. Let $f : Y \to X$ be a holomorphic map of complex manifolds of dimension d', d respectively. Then, for the graph $\Gamma(f) := \{(y, x) : x = f(y)\} \subset Y \times X$, there exists a logarithmic growth Green's current g_Γ along Γ for Γ.

Step 3. Proof of Lemma 3.

The basic idea here is to transform the situation to the divisor situation by a blowing-up process and then use the Poincaré-Lelong equation.

Let $W := B_\Gamma(Y \times X)$, then we have

$$\begin{array}{ccccc} E & \overset{j}{\hookrightarrow} & W & & \\ \pi_\Gamma \downarrow & & \downarrow \pi & & \\ \Gamma & \overset{i}{\hookrightarrow} & Y \times X & & \\ p_\Gamma \searrow & & \swarrow p_1 & \searrow p_2 & \\ & Y & & X & \end{array}$$

with E the exceptional divisor. We claim that there exists $\alpha \in A^{d-1,d-1}(W)$ such that $\pi_*(\delta_E \wedge [\alpha]) = \delta_\Gamma$.

In fact, the cohomology class $\mathrm{cl}(\Gamma)$ is an element of $H_\Gamma^{d,d}(Y \times X, \mathbf{R})$, hence

$$\pi^* \mathrm{cl}(\Gamma) \in H_E^{d,d}(W, \mathbf{R}) \simeq H^{d-1,d-1}(E, \mathbf{R}).$$

But

$$E = \mathbf{P}(\mathcal{N}_{Y \times X/\Gamma}) \simeq \mathbf{P}(i^*(p_2^* \mathcal{T}_X)).$$

Hence $\oplus_p H^{p,p}(E,\mathbf{R})$ is a free module over $\oplus_p H^{p,p}(\Gamma,\mathbf{R})$ with basis $\xi^0,\ldots,\xi^{d-1}$ and $\xi = j^*\mathrm{cl}(E)$ is the first Chern class of the tautological line bundle $\pi_\Gamma^{-1}(i^*(p_2^*\mathcal{T}_X))$. Hence we have

$$\pi^*\mathrm{cl}(\Gamma)\cap\mathrm{cl}(W) = \sum_i \pi_\Gamma^*(a_i)\xi^i,$$

where $a_i \in H^{d-1-i,d-1-i}(\Gamma,\mathbf{R})$. Thus, by the fact that $\pi^* = j^*\pi^*p^*(p_\Gamma^*)^{-1}$, if we let

$$b_i := \pi^*\circ p^*((p_\Gamma)^{-1}(a_i)) \in H^{d-1-i,d-i-1}(W,\mathbf{R}),$$

we have

$$\pi^*\mathrm{cl}(\Gamma)\cap\mathrm{cl}(W) = j^*(\sum_i b_i\mathrm{cl}(E)).$$

Now set $\alpha := [\sum_i b_i\,\mathrm{cl}(E)^i]$ and by the projective formula, we have the claim.

Since E is a divisor on W, it follows by the Poincaré-Lelong equation that

$$dd^c[\log|s|_\rho^2\alpha] = -[\beta\wedge\alpha] + \delta_E\wedge[\alpha],$$

for some section s of the line sheaf $\mathcal{O}(E)$. By the claim above, we know that $[\beta\wedge\alpha]$ represents the cohomology class $\pi^*\mathrm{cl}(\Gamma)$. Taking $\omega\in A^{d,d}(Y\times X)$ such that $\pi^*\omega$ represents $\pi^*\mathrm{cl}(\Gamma)$, we know that there exists $\phi\in A^{d-1,d-1}(W)$ such that

$$dd^c\phi = \beta\wedge\alpha - \pi^*\omega.$$

Let $\tilde{g}_\Gamma := -(\log|s|^2\alpha+\phi)$ and denote by g_Γ the form corresponding to $\tilde{g}_\Gamma$ via the isomorphism $W - E \simeq Y\times X - \Gamma$. Since

$$dd^c[(\log\|s\|^2)\alpha+\phi] = -[\beta\wedge\alpha] + \delta_E\wedge[\alpha] - [\pi^*\omega],$$

by the claim above,

$$dd^c[g_\Gamma] = -\pi_*(\delta_E\wedge[\alpha] - [\pi^*\omega]) = [\omega] - \delta_\Gamma.$$

So we only need to check the logarithmic growth condition. But then it is a direct consequence of the push-out property listed previously.

Step 4. The Proof of Lemma 2.

For the closed immersion $i: Y\hookrightarrow X$ is a closed immersion, from the proof of Lemma 3, we have a Green's current g_Γ for Γ with logarithmic growth in $Y\times X$. In particular, the form $g := p_{1*}(g_\Gamma\wedge p_2^*\alpha)$ is smooth on $X-Y$ of type $(n+p-1, n+p-1)$. Here p_i denotes the projections of $Y\times X$ to its factors. Furthermore, by the push-out property listed above, g is of logarithmic growth along Y. On the other hand, for all η of approciate degree,

$$\begin{aligned} p_{1*}(\delta_\Gamma\wedge[p_2^*\alpha])(\eta) &= (\delta_\Gamma\wedge[p_2^*\alpha])(p_{1*}\eta)\\ &= \int_\Gamma p_2^*\alpha\wedge p_1^*\eta = \int_\Gamma p_\Gamma^*\alpha\wedge p_1^*\eta\\ &= \int_Y \alpha\wedge(p_\Gamma^*)^{-1}p_1^*\eta = \int_Y \alpha\wedge i^*\eta\\ &= [\alpha](i^*\eta) = i_*[\alpha](\eta).\end{aligned}$$

Here $p_\Gamma = p_2|_\Gamma : \Gamma \simeq Y$. Therefore, we finally get

$$\begin{aligned} dd^c[g] &= dd^c[p_{1*}(g_\Gamma \wedge p_2^*\alpha)] = [p_{1*}dd^c(g_\Gamma \wedge p_2^*\alpha)] \\ &=[p_{1*}(dd^c g_\Gamma \wedge p_2^*\alpha)] = p_{1*}(dd^c[g_\Gamma] \wedge p_2^*\alpha) \\ &=p_{1*}(([\omega] - \delta_\Gamma) \wedge [p_2^*\alpha]) = [\beta] - i_*[\alpha]. \end{aligned}$$

Here $\beta := p_{1*}(\omega_\Gamma \wedge p_2^*\alpha) \in A^{p+n,p+n}(X)$. This completes of the proof of Lemma 2 and hence the theorem.

The use of forms with the logarithmic growth singularities has many advantages. For example, we have the following

Proposition. Let X be a smooth projective complex variety and Y a closed analytic subset. Suppose that α is a smooth form on $X - Y$ which has logarithmic growth along Y, then

$$d[\alpha] = [d\alpha].$$

Proof. This is a local problem and we may assume that X is the polydisc

$$\Delta^d := \{z = (z_1, \dots, z_d) \in \mathbf{C}^d : |z_i| < 1\}$$

and $Y = \cup_{j=1}^n Y_j$, with $Y_j := \{z \in \Delta^d : z_j = 0\}$. For any $\varepsilon > 0$, let

$$\begin{aligned} U_\varepsilon &:=\{z \in X : \inf_j |z_j| < \varepsilon\}, \\ W_\varepsilon^j &:=\{z \in X : |z_j| = \varepsilon, |z_k| \geq \varepsilon, k \neq j\}, \text{ and} \\ W_\varepsilon &:= \cup_j W_\varepsilon^j \end{aligned}$$

which is the boundary of U_ε. By Stokes' theorem, we have

$$(d[\alpha] - [d\alpha])(\omega) = \lim_{\varepsilon \to 0} \int_{\partial(X - U_\varepsilon)} \alpha \wedge \omega = -\lim_{\varepsilon \to 0} \int_{W_\varepsilon} \alpha \wedge \omega.$$

Hence, by the logarithmic growth condition,

$$\alpha \wedge \omega = \sum_i (a_i dz_1 + b_i d\bar{z}_i) \prod_{j \neq i} dz_j \wedge d\bar{z}_j,$$

where

$$|a_i|, |b_i| \leq C|\sum_j \log|z_j|^2|^k$$

for some positive constant C. Therefore

$$\begin{aligned} |\int_{W_\varepsilon^i} \alpha \wedge \omega| &\leq \int_{|z_i| = \varepsilon} (|a_i| dz_i + |b_i| d\bar{z}_i) \prod_{j \neq i} dz_j \wedge d\bar{z}_j \\ &\leq 2C\varepsilon \int_0^{2\pi} \int_{\Delta^{d-1}} |\sum_{j \neq i} \log|z_j|^2 + \varepsilon^2|^k d\theta_i \prod_{j \neq i} dz_i \wedge d\bar{z}_j = O(\varepsilon), \end{aligned}$$

where θ_i is $\arg(z_i)$. This completes the proof.

In general, if α is a smooth form on $X - Y$ such that α and $d\alpha$ are locally integrable on X, then we call the difference

$$d[\alpha] - [d\alpha]$$

the **residue** of α and denote it by $\mathrm{Res}_Y(\alpha)$. The above proposition may now be stated as follows:

Proposition′. If α has logarithmic growth along Y, then its residue is zero.

We need the following technical lemma, which will be used in the arithmetic intersection theory.

Lemma 4. Let $Y = \cup_j Y_j$ be a divisor of a smooth complex projective variety X with normal crossings. Suppose α is smooth over $X - Y$ and α is $O(r^{-1})$ near Y. Then

$$\mathrm{Res}_Y(\alpha) = \sum_i T_i,$$

where T_i is a **current of order** 0 supported on Y_i; i.e. if $\{U\}$ is an open covering of X such that U is isomorphic to some open subset of $\mathbf{C}^d$ and $\bar{U}$ is compact, and A is a positive constant, then there exists a positive constant B such that for any smooth form ω of X with $\omega|_U = \sum_{I,J} f_{I,J,U} dz_I \wedge d\bar{z}_J$, $|f_{I,J,U}| < A$, we have $|T(\omega)| \leq B$.

Proof. This is also a local problem. We let X and Y have the forms as in the last proposition. With the same notation, we have

$$\mathrm{Res}_Y(\alpha)(\omega) = -\lim_{\epsilon\to 0} \sum_j \int_{W_\epsilon^j} \alpha \wedge \omega.$$

Since

$$\alpha \wedge \omega = \sum_i (a_i dz_i + b_i d\bar{z}_i) \prod_{j\neq i} dz_j \wedge d\bar{z}_j,$$

where $|a_i|, |b_i| \leq C|z_1 \ldots z_n|^{-1}$ for some constant C, we have

$$\left|\int_{W_\epsilon^j} \alpha \wedge \omega\right| \leq 2C \int_0^{2\pi} \int_{\Delta^{d-1}} d\theta_i \prod_{j\neq i} \frac{dz_j \wedge d\bar{z}_j}{|z_j|} \prod_{k\geq n+1} dz_k \wedge d\bar{z}_k < \infty.$$

Now it is sufficient to prove the existence of $\lim_{\epsilon\to 0} \int_{W_\epsilon^j} \alpha \wedge \omega$. Choose a C^∞ cut-off function h_j^δ on Δ^d such that

(a) $0 \leq h_j^\delta \leq 1$;

(b) $h_j^\delta = 0$ for $|z_j| > \delta$;

(c) $h_j^\delta = 1$ for $|z_j| < \delta/2$.

We claim that $\lim_{\delta\to 0}\mathrm{Res}(\alpha)(\omega h_j^\delta)$ exists. In fact, if $\delta_1 > \delta_2 > 0$, and $\varepsilon < \delta_2/2$, then on W_ε^j,

$$\alpha \wedge \omega(h_j^{\delta_1} - h_j^{\delta_2}) = 0,$$

and for $k \neq j$,

$$\int_{W_\varepsilon^k} \alpha \wedge \omega(h_j^{\delta_1} - h_j^{\delta_2}) = O(\delta_1 - \delta_2),$$

which is independent of ε. Therefore $\lim_{\delta\to 0}\mathrm{Res}(\alpha)(\omega h_j^\delta)$ exists, say $T_j(\omega)$.

For $\varepsilon < \delta/2$, we have

$$\int_{W_\varepsilon} \alpha \wedge \omega h_j^\delta = \int_{W_\varepsilon} \alpha \wedge \omega + \sum_{k\neq j} \int_{W_\varepsilon^k} \alpha \wedge \omega h_k^\delta.$$

So

$$\int_{W_\varepsilon} \alpha \wedge \omega h_j^\delta - \int_{W_\varepsilon^j} \alpha \wedge \omega = O(\delta).$$

Thus the assertion follows by noting that

$$\begin{aligned}
&|\int_{W_\varepsilon^j} \alpha \wedge \omega - T_j(\omega)| \\
\leq &|\int_{W_\varepsilon^j} \alpha \wedge \omega - \int_{W_\varepsilon} \alpha \wedge \omega h_j^\delta| \\
&+ |\int_{W_\varepsilon} \alpha \wedge \omega h_j^\delta - \mathrm{Res}_Y(\alpha)(\omega h_j^\delta)| \\
&+ |\mathrm{Res}_Y(\alpha)(\omega h_j^\delta) - T_i(\omega)|.
\end{aligned}$$

We end this section with the following observation: Let X be a non-singular quasi-projective variety over $\mathbf{C}$ and Y a codimension-p algebraic cycle on X, then we may approximate an L^1 Green's form (C^∞ on $X - |Y|$) for Y by a C^∞ forms as follows: Choose a locally finite open covering of X by coordinate sharts and, for each $\varepsilon > 0$, let ρ_ε be a C^∞ real valued function on X such that

1. $0 \leq \rho_\varepsilon \leq 1$;
2. $\rho_\varepsilon \equiv 1$ outside the neighborhood $N_\varepsilon(Y)$ of radius ε of $|Y|$ in each coordinate charts;
3. $\rho_\varepsilon \equiv 0$ in some open neighborhood of $|Y|$.

Then we have the following

Lemma 5. For each $\varepsilon > 0$, let $g_Y^\varepsilon = \rho_\varepsilon g_Y$ with g_Y a Green's current of Y. Then
(a) g_Y^ε is a C^∞ form on X;
(b) $dd^c g_Y^\varepsilon = \omega_Y - \omega_Y^\varepsilon$ with ω_Y^ε a C^∞ form supported in the union of the closures of the $N_\varepsilon(Y)$;
(c) $\lim_{\varepsilon\to 0}[g_Y^\varepsilon] = [g_Y]$;
(d) $\lim_{\varepsilon\to 0}[\omega_Y^\varepsilon] = \delta_Y$.

The proof of this lemma which is easy is left to the reader.

§II.2.3 Arithmetic Chow Groups

II.2.3.a Arithmetic Chow Groups

We introduce now the arithmetic Chow groups and their cohomological properties.

Let X be a regular arithmetic variety over an arithmetic ring (A, Σ, F_∞). The conjugate linear automorphism F_∞ of C^Σ induces an orientation reversing continuous involution on X_∞. Since X_F is a smooth variety, X_∞ is a complex manifold. We define

$$A^{p,q}(X) := A^{p,q}(X_\infty), \quad D^{p,q}(X) := D^{p,q}(X_\infty);$$

$$A^{p,p}(X_{\mathbf{R}}) := \{\alpha \in A^{p,p}(X) : F_\infty^* \omega = (-1)^p \omega\};$$

$$D^{p,p}(X_{\mathbf{R}}) := \{\alpha \in D^{p,p}(X) : F_\infty^*(\alpha) = (-1)^p \omega\};$$

$$\tilde{A}^{p,p}(X_{\mathbf{R}}) := A^{p,p}(X_{\mathbf{R}})/(\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}); \quad \tilde{A}(X_{\mathbf{R}}) := \oplus_p \tilde{A}^{p,p}(X_{\mathbf{R}});$$

$$\tilde{D}^{p,p}(X_{\mathbf{R}}) := D^{p,p}(X_{\mathbf{R}})/(\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}); \quad \tilde{D}(X_{\mathbf{R}}) := \oplus_p \tilde{D}^{p,p}(X_{\mathbf{R}}).$$

Similarly, if X is projective, we let

$$H^{p,p}(X_{\mathbf{R}}) := \{\alpha \in H^{p,p}(X) : F_\infty^* \alpha = (-1)^p \alpha\}.$$

Since dd^c is a real operator, we know that dd^c is compatible with all of these definitions.

Let Y be a codimension-p integral subscheme of X, then Y_∞ is a F_∞-invariant analytic subspace of X_∞. Hence, integration over Y_∞ defines a current in $D^{p,p}(X_{\mathbf{R}})$ and we denote this current also by δ_Y. We say that an element $(Z, g_Z) \in Z^p(X) \oplus \tilde{D}^{p-1,p-1}(X_{\mathbf{R}})$ is an **arithmetic p-cycle** if g_Z is a Green's current of Z, i.e.

$$dd^c g_z = \omega(Z, g_Z) - \delta_Z$$

for some $\omega_Z := \omega(Z, g_Z) \in A^{p,p}(X_{\mathbf{R}})$. We denote by $Z^p_{\mathrm{Ar}}(X)$ the abelian group generated by arithmetic p-cycles.

Next, we define arithmetic rational equivalence among the arithmetic cycles. Let $i : Y \hookrightarrow X$ be an integral subscheme of codimension $p-1$. There is a resolution of singularities of Y_∞, $\pi : \tilde{Y}_\infty \to Y_\infty$ with π proper. For any rational function $f \in k(Y)^*$, define a rational function $\tilde{f}$ on $\tilde{Y}_\infty$ such that $\log|\tilde{f}|^2$ is L^1 on $\tilde{Y}_\infty$. Hence $\tilde{f}$ is contained in $D^{0,0}(\tilde{Y})$. Let $\tilde{i}_\infty : \tilde{Y}_\infty \to X_\infty$ be the natural induced morphism, then

$$\tilde{i}_{\infty *}[\log|\tilde{f}|^2] \in D^{p-1,p-1}(X),$$

and is independent of the choice of $\tilde{Y}$. We denote it by $i_*[\log|f|^2]$. Since f is F_∞-invariant, by the Poincaré-Lelong equation, we know that

$$\operatorname{div}_{\mathrm{Ar}}(f) := (\operatorname{div}(f), -i_*[\log|f|^2]) \in Z^p_{\mathrm{Ar}}(X).$$

We say that such an arithmetic cycle is **arithmetically rationally equivalent to zero.** Let $R^p_{\mathrm{Ar}}(X)$ be the subgroup of $Z^p_{\mathrm{Ar}}(X)$ generated by $\operatorname{div}_{\mathrm{Ar}}(f)$ for $f \in k(W)^*$, with W a codimension-$(p-1)$ integral subscheme. We define the **p-th arithmetic Chow group**, denoted by $\mathrm{CH}^p_{\mathrm{Ar}}(X)$, to be the quotient group $Z^p_{\mathrm{Ar}}(X)/R^p_{\mathrm{Ar}}(X)$. Let

$$\mathrm{CH}_{\mathrm{Ar}}(X) := \oplus_p \mathrm{CH}^p_{\mathrm{Ar}}(X).$$

We define in a similar way the **p-th homology group** $\mathrm{CH}_p^{\mathrm{Ar}}(X)$. In order to define the arithmetic intersection, we first need to define a product among Green's currents, which is what we will do in the next subsection.

II.2.3.b The *-product of Green's Currents

Let X be a smooth projective complex variety, $Y \subset X$ a closed irreducible subset and $f : Z \to X$ a proper morphism of irreducible projective varieties over $\mathbb{C}$ such that $f(Z) \not\subset Y$. There exists a differential form g_Y of Y such that g_Y is smooth on $X - Y$ and has logarithmic growth along Y; that is, there exists a proper birational morphism $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is smooth and projective, $E := \pi^{-1}(Y)$ is a divisor with normal crossing, $\pi : \tilde{X} - E \simeq X - Y$ and the form $\pi^* g_Y$ is $O(|\log r^2|^{2K})$ near E, while $\pi^* dg_Y$ is $O(r^{-1})$ near E. In this case, we denote the associated current by $[g_Y]$. Thus by resolving the singularities of Z, we can construct a commutative diagram

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{j} & \tilde{X} \\ p\downarrow & \searrow q & \downarrow\pi \\ Z & \xrightarrow{f} & X, \end{array}$$

such that $D = j^{-1}(E)$ is a divisor with normal crossings, $\tilde{Z}$ is projective and smooth, and p is birational. (In fact, we may choose $\tilde{Z}$ as the resolution of the singularities of the Zariski closure of $Z - Y$ embedded in $Z \times \tilde{X}$ as the graph of f.) Obviously $q^* g_Y$ on $\tilde{Z}$ has the same growth as g_Y on X, so, if Z is smooth, it makes sense to define

$$f^*[g_Y] := p_*[q^* g_Y] \in D^{n-1,n-1}(Z).$$

Also, if $f : Z \hookrightarrow X$ is a closed immersion, we define a current $[g_Y] \wedge \delta_Z = \delta_Z \wedge [g_Y]$ in $D^{m+n-1,m+n-1}(X)$ by

$$[g_Y] \wedge \delta_Z := q_*[q^* g_Y].$$

Furthermore, if g_Z is an arbitrary Green's current of Z, we define the *-product of $[g_Y]$ and g_Z by

$$[g_Y] * g_Z := [g_Y] \wedge \delta_Z + [dd^c g_Y + \delta_Y] \wedge g_Z.$$

Theorem. Let X be a smooth projective variety over $\mathbf{C}$, and let $Y = \sum_i a_i[Y_i]$ be a codimension-n cycle on X such that g_{Y_i}, a Green's form of Y_i, has logarithmic growth. Then

(1) If $Z = \sum_j b_j[Z_j]$ is a codimension-m cycle on X such that $Z_j \not\subset |Y|$ for all j, and g_Z a Green's current for Z, then

$$dd^c([g_Y] * g_Z) = [\omega_Y \wedge \omega_Z] - (\sum_k \mu_k \delta_{S_k} + t).$$

Here $|Y| \cap |Z| = S \cup T$, where S is the union of the components $S_1, \ldots, S_l$ of codimension-$(m+n)$, T is the union of the components of codimension $< (m+n)$. That is,

$$[Y][Z] = \sum_k \mu_k S_k + \tau,$$

and t is a current supported on T which represents the homology class of τ.

(2) For any two Green's currents g_Y, g'_Y of Y with logarithmic growth, as an element of $\tilde{D}(X)$, we have

$$[g_Y] * g_Z = [g'_Y] * g_Z$$

for any Green's current g_Z. Hence, we may also define the *-product among Green's currents.

(3) Let Y, Z be two algebraic cycles with Green's currents g_Y, g_Z, then

$$g_Y * g_Z = g_Z * g_Y.$$

(4) Let Y, Z and W be algebraic cycles of X with Green's currents g_Y, g_Z and g_W, respectively, we have

$$(g_Y * g_Z) * g_W = g_Y * (g_Z * g_W).$$

Proof. (1) We may assume that Y and Z are prime cycles, i.e. irreducible subvarieties of X. With the notation as above and by the result about residues, we have

$$\begin{aligned} dd^c([g_Y] * g_Z) &= dd^c(q_*[q^* g_Y]) = q_* dd^c[q^* g_Y] \\ &= q_* \mathrm{Res}_D(d^c g_Y) + \delta_Z \wedge \omega_Y = R + \delta_Z \wedge \omega_Y, \end{aligned}$$

where $R := q_* \mathrm{Res}_D(d^c g_Y)$ is a current of order zero supported on $Y \cap Z$.

On the other hand, $\mathrm{cl}(Y) \in H^{2n}_Y(X)$ is represented by $(\delta_Y, 0) \in D^{2n}(X) \oplus D^{2n-1}(X)$ and also by $(\omega_Y, (-1)^{2n+1} d^c g_Y)$. Similarly, for Z, we know that $\mathrm{cl}(Z) \in H^{2m}_Z(X)$ is represented by $(\delta_Z, 0)$. Since $\delta_Z \wedge d^c g_Y$ extends to a current $d^c(\delta_Z \wedge [g_Y])$ on X, hence $\mathrm{cl}([Y][Z]) = \mathrm{cl}(Y) \cup \mathrm{cl}(Z) \in H^{2m+2n}_{Y \cap Z}(X)$ is represented by $(\omega_Y \delta_Z, (-1)^{2m+2n+1} d^c(\delta_Z \wedge [g_Y]))$, which is also represented by $(R, 0)$, where R represents $\mathrm{cl}(Y\ Z)$ in $H^{2m+2n}_{Y \cap Z}(X)$.

If $T = \emptyset$, (1) is a consequence of the following

King's Lemma [Ki 74]. If Y and Z intersect properly, $R = \sum_k \mu_k \delta_{S_k}$.

Otherwise, we consider the restriction of R on $X - T$. Obviously, we have

$$R|_{X-T} = \sum_{i=1}^{k} \mu_i \delta_{S_i}|_{X-T}.$$

Hence $t = R + \sum_{i=1}^{k} \mu_i \delta_{S_i}$ is a current of order zero supported in T, which necessarily represents the cohomology class of $-\tau \in H_T^{2m+2n}(X)$. Hence we have the assertion with the help of King's lemma. The proof of King's lemma is not really complicated: One may use the smooth approximation to deal with it. We leave the proof to the reader.

For the proof of (2), (3), (4), we first give equivalent statements. We start with (4). By (1), we know that

$$\begin{aligned}[g_Y] * ([g_Z] * g_W) &= [g_Y] * g_{[Z][W]} \\ &= g_Y \wedge \delta_{[Z][W]} + \omega_Y \wedge g_Z \wedge \delta_W + \omega_Y \wedge \omega_Z \wedge g_W.\end{aligned}$$

On the other hand, by the C^∞ approximation at the end of the last section, we have

$$\begin{aligned}&([g_Y] * [g_Z]) * g_W \\ =&\lim_{\epsilon\to 0}([g_Y] * [g_Z]) \wedge \omega_W^\epsilon + \omega_Y \wedge \omega_Z \wedge g_W \\ =&\lim_{\epsilon\to 0} g_Y \wedge \delta_Z \wedge \omega_W^\epsilon + \lim_{\epsilon\to 0} \omega_Y \wedge g_Z \wedge \omega_W^\epsilon + \omega_Y \wedge \omega_Z \wedge g_W \\ =&\lim_{\epsilon\to 0} g_Y \wedge \delta_Z \wedge \omega_W^\epsilon + \omega_Y \wedge g_Z \wedge \delta_W + \omega_Y \wedge \omega_Z \wedge g_W \\ =&(g_Y \wedge \delta_Z \wedge \omega_W - \lim_{\epsilon\to 0} g_Y \wedge \delta_Z \wedge dd^c g_W^\epsilon) + \omega_Y \wedge g_Z \wedge \delta_W + \omega_Y \wedge \omega_Z \wedge g_W \\ =&g_Y \wedge \delta_Z \wedge \omega_W - \lim_{\epsilon\to 0} dd^c(g_Y \wedge \delta_Z) \wedge g_W^\epsilon + \omega_Y \wedge g_Z \wedge \delta_W + \omega_Y \wedge \omega_Z \wedge g_W \\ =&g_Y \wedge \delta_Z \wedge \omega_W - \lim_{\epsilon\to 0}(-\delta_{[Y][Z]} \wedge g_W^\epsilon + \omega_Y \wedge \delta_Z \wedge g_W^\epsilon) + \omega_Y \wedge g_Z \wedge \delta_W + \omega_Y \wedge \omega_Z \wedge g_W \\ =&g_Y \wedge \delta_Z \wedge \omega_W - (-\delta_{[Y][Z]} + \omega_Y \wedge \delta_Z) \wedge g_W + \omega_Y \wedge g_Z \wedge \delta_W + \omega_Y \wedge \omega_Z \wedge g_W.\end{aligned}$$

Therefore, in order to prove (4), after interchanging Z and W, it is enough to show the following

Claim.

$$[g_Y] \wedge (\sum_k \mu_k \delta_{S_k} + t) + \omega_Y \wedge [g_Z] \wedge \delta_W = \delta_{[Y][W]} \wedge g_Z + g_Y \wedge \omega_Z \wedge \delta_W.$$

Here S_k is the proper part of the intersection of Z and W.

But this formula also gives the proof of (3), since for that case, we may let $W = X$.

Proof of the claim. The details are rather formal and tedious. By Hironaka's theorem II [Hi 64], there exists a diagram

$$\begin{array}{ccc} \tilde{W} & & \\ p\downarrow & \searrow h & \\ W & \overset{j}{\hookrightarrow} & X. \end{array}$$

such that

(1) $\tilde{W}$ is non-singular and quasi-projective;
(2) p is birational;
(3) The inverse image of $Y \cup Z, Y, Z, Y \cap Z$ of h are all divisors with normal crossings;
(4) h^*g_Y, h^*g_Z have logarithmic growth near the inverse image of Y, Z, respectively;
(5) For each irreducible component E of $h^{-1}(Y \cap Z)$, there exists a commutative diagram

$$\begin{array}{ccc} & & B_T\bar{X} \\ & \bar{h}_T \nearrow & \downarrow f \\ \tilde{W} & \xrightarrow{\bar{h}} & \bar{X} \\ & h \searrow & \downarrow \\ & & X. \end{array}$$

Here $\bar{X}$ is smooth, $\bar{h}(E)$ is contained in a smooth subvariety $T \subset \bar{X}$ of codimension at least $p+q+r$ and f is the blowing-up of $\bar{X}$ along T.

Hence h^*g_Y and h^*g_Z have logarithmic singurities near $h^{-1}(Y)$ and $h^{-1}(Z)$ respectively. On the other hand,

$$dd^c(h_*[h^*g_Y]) = \omega_Y \wedge \delta_W - \delta_{[Y][W]},$$

and

$$dd^c(h_*[h^*g_Z]) = \omega_Z \wedge \delta_W - (\sum_k \mu_k \delta_{S_k} + t).$$

Also $h^*(g_Y \wedge \bar{\partial} g_Z)$ is $O(r^{-1}|\log r^2|^k)$ near $h^{-1}(Y \cup Z)$, hence it is an L^1 form on $\tilde{W}$. Now by a local calculation, we have

$$\partial(h_*[h^*(g_Y \wedge \bar{\partial} g_Z)] = h_*[h^*(\partial g_Y \wedge \bar{\partial} g_Z)] - 2\pi i(\delta_W \wedge g_Y \wedge \omega_Z - g_Y \wedge (\sum_k \mu_k \delta_{S_k} + t)).$$

Interchanging ∂ and $\bar{\partial}$, Y and Z, we have

$$\bar{\partial}(h_*[h^*(g_Z \wedge \partial g_Y)] = h_*(h^*(\bar{\partial} g_Z \wedge \partial g_Y)] - 2\pi i(\delta_W \wedge g_Z \wedge \omega_Y - g_Z \wedge \delta_{[Y][W]}).$$

Therefore in $\tilde{D}^{p+q+r-1,p+q+r-1}(X)$, we have

$$[g_Y] \wedge (\sum_k \mu_k \delta_{S_k} + t) + \omega_Y \wedge [g_Z] \wedge \delta_W = \delta_{[Y][W]} \wedge g_Z + g_Y \wedge \omega_Z \wedge_W,$$

which proves the lemma.

So we have (3) and (4). Now (2) can be proved as follows. We know that in $D(X)$, any Green's current may be represented by a Green's current with the logarithmic growth. Thus by

$$g_Y * g_Z - g'_Y * g_Z = (g_Y - g'_Y) \wedge \delta_Z.$$

But $g_Y - g'_Y = 0$ in $\tilde{D}(X)$, so the fact that δ_Z is closed implies (2).

II.2.3.c Cohomology Properties Of Arithmetic Chow Groups

We study now the cohomology properties of arithmetic Chow groups. For this purpose, we need several morphisms involving $\mathrm{CH}_{\mathrm{Ar}}(X)$, viz.

(1) $\zeta : \mathrm{CH}^p_{\mathrm{Ar}}(X) \to \mathrm{CH}(X), \quad (Z, g_Z) \mapsto Z.$
(2) $a : \tilde{A}^{p-1,p-1}(X_{\mathbf{R}}) \to \mathrm{CH}^p_{\mathrm{Ar}}(X), \quad \alpha \mapsto (0, \alpha).$
(3) $\omega : \mathrm{CH}^p_{\mathrm{Ar}}(X) \to A^{p,p}(X_{\mathbf{R}}), \quad (Z, g_Z) \mapsto dd^c g_Z + \delta_Z.$
(4) $\rho : \mathrm{CH}^{p-1,p}(X) \to \tilde{A}^{p-1,p-1}(X), \quad (f_y) \mapsto \sum_y -[\log|f_y|^2].$
(5) $c : \mathrm{CH}^p(X) \to H^{p,p}(X_{\mathbf{R}})$, the cycle class map;
(6) $h : Z^{p,p}(X) \to H^{p,p}(X_{\mathbf{R}})$, where $Z^{p,p}(X) :=$ the closed forms in $A^{p,p}(X_{\mathbf{R}})$, sends a closed form to its cohomology class.

Theorem. The above morphisms are well-defined. Furthermore, there are two exact sequences

(1) $\ldots \to \mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} H^{p-1,p-1}(X) \xrightarrow{a} \mathrm{CH}^p_{\mathrm{Ar}}(X) \xrightarrow{(\zeta,-\omega)} \mathrm{CH}^p(X) \oplus Z^{p,p}(X_{\mathbf{R}}) \xrightarrow{c-h} H^{p,p}(X_{\mathbf{R}}) \to 0.$
(2) $\ldots \to \mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} \tilde{A}^{p-1,p-1}(X) \xrightarrow{a} \mathrm{CH}^p_{\mathrm{Ar}}(X) \xrightarrow{\zeta} \mathrm{CH}^p(X) \to 0.$

Proof. (a) Clearly, the last morphisms $c - h$, ζ in the sequences (1) and (2) are surjective.

(b.)Exactness at $\mathrm{CH}^p(X) \oplus Z^{p,p}(X)$:

We have

$$(c-h)([Z], \omega) = 0 \Leftrightarrow \exists g \in D^{p-1,p-1}(X) : dd^c g = [\omega] - \delta_Z \Leftrightarrow (\zeta, -\omega)([(Z, g)]) = ([Z], \omega).$$

(c) Exactness of $\mathrm{CH}^p_{\mathrm{Ar}}(X)$. For (2), we have

$$\zeta([Z, g_Z]) = 0 \Leftrightarrow Z = \sum_y \mathrm{div}(f_y), f_y \in k(y)^*, y \in X^{(p-1)}.$$

That is,

$$[(Z, g_Z)] = [(\sum_y \mathrm{div}(f_y), g_Z)] = [(0, g_Z + \sum_y [\log|f_y|^2])].$$

Let $\tilde{g} = g_Z + \sum_y [\log|f_y|^2] \in \mathrm{CH}^p_{\mathrm{Ar}}(X)$ and we have

$$dd^c \tilde{g} = [\omega_Z],$$

Hence there exist $\eta \in A^{p-1,p-1}(X)$, $S_1 \in D^{p-2,p-1}(X)$, $S_2 \in D^{p-1,p-2}(X)$, such that

$$\tilde{g} = [\eta] + \partial S_1 + \bar{\partial} S_1.$$

Thus

$$\zeta([(Z, g_Z)]) = 0 \Leftrightarrow [(Z, g_Z)] = [(0, \tilde{g})] = [(0, [\eta])] = a([\eta]).$$

For (1), we note that in addition,

$$dd^c\tilde{g} = 0.$$

Then η is a closed form.

(d) Exactness at $\tilde{A}^{p-1,p-1}(X)$ (resp. $H^{p-1,p-1}(X)$): In fact,

$$a(\eta) = 0 \Leftrightarrow (0,[\eta]) = \sum_y (\mathrm{div}(f_y), -[\log|f_y|^2]) + (0, \partial S_1 + \bar{\partial} S_2) \in Z^p_{\mathrm{Ar}}(X)$$

$$\Leftrightarrow \sum_y \mathrm{div}(f_y) = 0, \text{ and } [\eta] = \sum_y -[\log|f_y|^2] + \partial S_1 + \bar{\partial} S_2 \Leftrightarrow \rho((f_y)) = \eta.$$

(e) Finally, we need to prove that ρ is well-defined. By the definition in 1.4, we know that

$$\mathrm{CH}^{p-1,p}(X) = E_{2X}^{p-1,p}(X) = \{(f_y) \in \oplus_{y \in X^{(p-1)}} k(y)^* : \sum_y \mathrm{div}(f_y) = 0\}/\mathrm{Im}\, d_1,$$

where

$$d_1 : \oplus_{x \in X^{(p-2)}} K_2(k(x)) \to \oplus_{y \in X^{(p-1)}} k(y)^*$$

is given by the tame symbol. Thus it is sufficient to prove that $\rho \circ d_1 = 0$. If X is a smooth projective complex variety, $Z \subset X$ an irreducible subvariety of codimension-$(p-2)$, then for $f, g \in \mathbf{C}(Z)^*$, we should have

$$\rho \circ d_1(\{f, g\}) = 0.$$

For this, we first reduce the problem to the situation in which $\mathrm{div}(f) \cup \mathrm{div}(g)$ is a divisor with normal crossings. In fact, we have the following fact:

Let $\pi : \tilde{Z} \to Z$ be a resolution of singularities of $\mathrm{div}(f) \cup \mathrm{div}(g)$ with π proper and

$$D = \pi^{-1}(\mathrm{div}(f) \cup \mathrm{div}(g))$$

a divisor with normal crossings. By the functoriality of algebraic K-theory, we have the commutative diagram:

$$\begin{array}{ccccc}
K_2(\mathbf{C}(\tilde{Z})) & \xrightarrow{\tilde{d}_1} & \oplus_{\tilde{y} \in \tilde{Z}^{(1)}} \mathbf{C}(\tilde{y})^* & \xrightarrow{\rho} & \tilde{D}^{1,1}(\tilde{Z}) \\
\| & & \downarrow \pi_* & & \downarrow \pi_* \\
K_2(\mathbf{C}(Z)) & \xrightarrow{d_1} & \oplus_{y \in Z^{(1)}} \mathbf{C}(y)^* & \xrightarrow{\rho} & \tilde{D}^{1,1}(Z).
\end{array}$$

Thus $d_1 \circ \rho = 0$ will follow from $\tilde{d}_1 \circ \rho = 0$, which is an immediate consequence of the following

Lemma. With the same notation as above, we have

$$\rho \circ \tilde{d}_1(\{f,g\}) = -\frac{i}{2\pi}(\partial[\alpha] + \bar{\partial}[\beta]),$$

where $\alpha = \log|g|^2 \wedge \bar{\partial} \log|f|^2$ and $\beta = \log|f|^2 \wedge \partial \log|g|^2$.

Proof. The problem is a local one, we may assume that $\tilde{Z} = \Delta^m$; and by the linearity of the symbols $\{f,g\}$, we are reduced to the following two cases:

(a) $f = z_1 = g$. By definition, we have

$$\tilde{d}_1(\{z_1,z_1\}) = (-1)^{v(z_1)v(z_1)} z_1^{v(z_1)} z_1^{-v(z_1)} = -1.$$

So

$$\rho \circ \tilde{d}_1(\{z_1,z_1\}) = -[\log|-1|^2] = 0.$$

On the other hand, we have

$$\begin{aligned}&(\partial[\alpha] + \bar{\partial}[\beta])(\omega)\\ =&\int_{\Delta^m} (\partial\alpha + \bar{\partial}\beta) \wedge \omega\\ =&\lim_{\epsilon\to 0} \int_{|Z_1|=\epsilon} (\alpha+\beta) \wedge \omega.\end{aligned}$$

If we write $z_1 = re^{i\theta}$, $\alpha+\beta$ becomes $2/r \log r^2\, dr$, and hence the integral vanishes when $\epsilon \to 0$.

(b) $\mathrm{div}(f)$ and $\mathrm{div}(g)$ intersect properly. By definition, we have

$$\rho \circ d_1(\{f,g\}) = +[\log|f|^2] \wedge \delta_{\mathrm{div}(g)} - [\log|g|^2] \wedge \delta_{\mathrm{div}(f)}.$$

So

$$\rho \circ d_1(\{f,g\}) = -[\log|f|^2 * [\log|g|^2] + [\log|g|^2] * [\log|f|^2].$$

Now the result follows from the more refined

Claim. Let Y and Z intersect properly, and let g_Y, g_Z be Green's forms for Y and Z with the logarithmic growth along Y and Z respectively. Then we have

$$[g_Y] * [g_Z] - [g_Z] * [g_Y] = \frac{1}{2\pi i}(\partial\,[g_Y \wedge \bar{\partial} g_Z] + \bar{\partial}\,[g_Z \wedge \partial g_Y]).$$

Proof of the claim. In fact, by definition

$$\begin{aligned}&[g_Y] * [g_Z] - [g_Z] * [g_Y]\\ =&[g_Y] \wedge \delta_Z + [\omega_Y] \wedge [g_Z] - [g_Z] \wedge \delta_Y - [\omega_Z] \wedge [g_Y]\\ =&-[g_Y] \wedge ([\omega_Z] - \delta_Z) + [g_Z] \wedge ([\omega_Y] - \delta_Y)\\ =&-[g_Y] \wedge dd^c[g_Z] + [g_Z] \wedge dd^c[g_Y]\\ =&\frac{1}{2\pi i}(\partial([g_Y] \wedge \bar{\partial}[g_Z]) + \bar{\partial}([g_Z] \wedge \partial[g_Y]))\\ =&\frac{1}{2\pi i}(\partial[g_Y \wedge \bar{\partial} g_Z] + \bar{\partial}[g_Z \wedge \partial g_Y]).\end{aligned}$$

§II.2.4. Arithmetic Intersection Theory

In this section, we obtain an arithmetic intersection theory for arithmetic varieties by using the results in the previous sections.

Let X be an arithmetic variety over an arithmetic ring (A, Σ, F_∞). Let

$$\begin{aligned} Z^p_{\text{fin}}(X) :=&\{Z \in Z^p(X) : Z \cap X_F = \emptyset\}; \\ \text{CH}^p_{\text{fin}}(X) :=&Z^p_{\text{fin}}(X)/ < \text{div}(f) : \forall f \in k(y)^*, y \in X^{(p-1)} - X_F >; \\ Z^p_{\text{Ar}}(X_F) :=&\{(Z, g_Z) : Z \in Z^p(X_F), g_Z \text{ Green's current for } Z\}. \end{aligned}$$

Then there is a natural morphism:

$$\text{div}_{\text{Ar}} : \oplus_{x \in X_F^{(p-1)}} k(x)^* \to \text{CH}^p_{\text{fin}}(X) \oplus Z^p_{\text{Ar}}(X_F)$$

which is given by

$$\text{div}_{\text{Ar}}(f) := (\text{div}(f), -[\log|f|^2]),$$

where $\text{div}(f) = Z_1 + Z_1$, $Z_1 \in Z^p_{\text{fin}}(X)$ and $Z_2 \in Z^p(X_F)$. Then we have proved

Lemma. With the notation as above, there is an exact sequence:

$$\oplus_{x \in X_F^{(p-1)}} k(x)^* \xrightarrow{\text{div}_{\text{Ar}}} \text{CH}^p_{\text{fin}}(X) \oplus Z^p_{\text{Ar}}(X_F) \to \text{CH}^p_{\text{Ar}}(X) \to 0.$$

Suppose that Y and Z are integral subschemes of X with codimensions-p, q, respectively, and that Y, Z intersect properly on X_F. Note that $[Y][Z]$ is not necessarily well-defined as a cycle on X, since Y and Z may not intersect properly on X. However $[Y][Z]$ is well-defined as a class in $\text{CH}^{p+q}_{Y \cap Z}(X)_{\mathbf{Q}}$. On the other hand, there is a canonical morphism

$$\text{CH}^p_W(X) \to \text{CH}^p_{\text{fin}}(X) \oplus Z^p_{W_F}(X_F),$$

where W is a closed subscheme of generic codimension-p. So $[Y][Z]$ may be thought as an element of

$$\text{CH}^{p+q}_{\text{fin}}(X) \oplus Z^{p+q}(X_F).$$

If g_Y, g_Z are Green's currents of Y, Z, respectively, we define

$$([Y], g_Y)([Z], g_Z) := ([Y][Z], g_Y * g_Z),$$

which is an element of $(Z^{p+q}(X_F) \oplus \text{CH}^{p+q}_{\text{fin}}(X))_{\mathbf{Q}} \oplus \tilde{D}^{p+q-1,p+q-1}(X_{\mathbf{R}})$. By the result for the $*$-product of Green's currents, we have the following

Fact. If Y and Z intersect properly on the whole of X, then

$$([Y], g_Y)([Z], g_Z) = ([Y][Z], g_Y * g_Z) \in Z^{p+q}_{\text{Ar}}(X).$$

Now we may state basic facts of arithmetic intersection theory in the following

Theorem. Let $A = (A, \Sigma, F_\infty)$ be an arithmetic ring with field of fractions F. Suppose that X is an arithmetic variety over A which is regular and has a quasi-projective generic fiber X_F. Then

(1) For each pair of natural numbers (p, q), there is a pairing

$$\begin{array}{ccc} \mathrm{CH}^p_{\mathrm{Ar}}(X) \otimes \mathrm{CH}^q_{\mathrm{Ar}}(X) & \to & \mathrm{CH}^{p+q}_{\mathrm{Ar}}(X)_{\mathbf{Q}} \\ \alpha \otimes \beta & \mapsto & \alpha\beta. \end{array}$$

The pairing is uniquely determined by the following property: If Y and Z are integral subschemes of X which intersect properly on X_F, and g_Y and g_Z are Green's currents for Y and Z, then $([Y], g_Y)([Z], g_Z)$ is given as above.

(2) The product above makes $\mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}} := \oplus_p \mathrm{CH}^p_{\mathrm{Ar}}(X)_{\mathbf{Q}}$ a commutative, associative $\mathbf{Q}$-algebra.

(3) The natural morphism

$$\zeta : \oplus_p \mathrm{CH}^p_{\mathrm{Ar}}(X)_{\mathbf{Q}} \to \oplus_p (\mathrm{CH}^p(X) \oplus Z^{p,p}(X))_{\mathbf{Q}}$$

is a $\mathbf{Q}$-algebra homomorphism.

Proof. Let $([Y], g_Y) \in \mathrm{CH}^p_{\mathrm{Ar}}(X)$ and $([Z], g_Z) \in \mathrm{CH}^q_{\mathrm{Ar}}(X)$. To define the arithmetic product, we assume that Y and Z are irreducible. If Y and Z intersect properly on X_F, we already have the definition for the intersection. Therefore, we need to deal with the situation when Y and Z do not intersect properly on X_F. By the Chow moving lemma, we know that there are rational functions $f_y \in k(y)^*, y \in X_F^{(p-1)}$, such that $(Y + \sum_y \mathrm{div}(f_y))_F$ and Z_F intersect properly so we can reduce to the generic proper intersection case. It remains to prove the following

Claim. If $g_y \in k(y)^*, y \in X_F^{(p-1)}$ is another choice of rational functions such that $(Y + \sum_y \mathrm{div}(g_y))_F$ and Z_F intersect properly, then

$$(\sum_y \mathrm{div}_{\mathrm{Ar}}(f_y) - \sum_y \mathrm{div}_{\mathrm{Ar}}(g_y))(Z, g_Z) \in < \mathrm{div}_{\mathrm{Ar}}(f); (0, \mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial))_{\mathbf{Q}} \subset Z^{p+q}_{\mathrm{Ar}}(X)_{\mathbf{Q}}.$$

By the Chow moving lemma for K_1-chains, there exists an element

$$u \in \oplus_{z \in X_F^{(p-2)}} K_2(k(z))$$

such that if $(h_y) := (f_y g_y^{-1}) + d_1(u)$, then the K_1-chain (h_y) intersects Z almost properly, i.e., $\mathrm{div}(h_y)$ meet Z properly for all Y, even through $(f_Y g_y^{-1})$ does not have this property. Note that since $\mathrm{div} \circ d_1 = d_1^2 = 0$, we have

$$\sum_y \mathrm{div}(h_y) = \sum_y \mathrm{div}(f_y) - \mathrm{div}(g_y).$$

Furthermore, since $\rho \circ d_1 = 0$, we have

$$\sum_y [\log|f_y|^2] - \sum_y [\log|g_y|^2] = \sum_y [\log|h_y|^2]$$

modulo $\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}$. Therefore, it is sufficient to prove the following

Lemma 1. With the same notation as above, we have that $\mathrm{div}_{\mathrm{Ar}}(h_y)(Z, g_Z)$ lies in

$$< \mathrm{div}_{\mathrm{Ar}}(f); (0, \mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}) >_{\mathbf{Q}} .$$

Proof. Since arithmetic intersection is a natural generalization of algebraic intersection, we may neglect the finite part. Hence, without loss of generality, we may assume that $Z_{\mathrm{fin}} = \emptyset$. Also, we will simply write h as h_y. Now if $W := \mathrm{Supp}(h)$, then $|W_F| \cap |Z_F| = S \cap T$, where $\mathrm{codim}_{X_F} S = p + q - 1 > \mathrm{codim}_{X_F} T$. In $\mathrm{CH}^{p+q-1}_{S \cup T}(X_F)_{\mathbf{Q}}$, we have

$$[W_F][Z_F] = \sum_k \mu_k [S_k] + \tau.$$

Here S_k are the irreducible components of S, μ_k the Serre intersection multiplicities, and $\tau \in \mathrm{CH}^{p+q-1}_T(X_F)_{\mathbf{Q}}$. Since $(\mathrm{div}(h))_F$ intersects Z_F properly, by the fact that $(\mathrm{div}(h))_F$ does not have a component of codimension-$(p+q-1)$, we know that $h|_{S_k} \in k(S_k)^*$; and by the fact that $(\mathrm{div}(h))_F \cap T = \emptyset$, we have $h|_T$ is a unit. With this, by the definition of the algebraic intersection of a K_1-chain and an algebraic cycle in 1.4, we know that

$$h\,Z = \prod_i (h|_{S_i})^{\mu_i} \, (h|_T\, t) \in \oplus_{y \in X_F^{(p+q-1)}} K_1(k(y)),$$

where $t \in Z^{p+q-1}_T(X_F)_{\mathbf{Q}}$ is a representative of τ and the product $(h|_T\, t)$ has to be understood in K-theoretic terms. Also by the fact that $h\,Z$ is only defined up to $\mathrm{Im}\,d_1$, so by the fact that $\mathrm{div} \circ d_1 = 0$, $\rho \circ d_1 = 0$ of the cohomological properties of arithmetic Chow groups, we know that $\mathrm{div}_{\mathrm{Ar}}(h\,Z)$ is well defined. In particular, we see that the claim is a direct consequence of the following

Lemma 2. With the same notation as above, we have

$$\mathrm{div}_{\mathrm{Ar}}(h)\,(Z, g_Z) = \mathrm{div}_{\mathrm{Ar}}(h\,Z) \mod (0, \mathrm{Im}\partial + \mathrm{Im}\bar{\partial}).$$

Proof. First consider the part for algebraic cycles. Let $H \in k(X)^*$ be such that $H|_W = h$. Then

$$\mathrm{div}(H)\,W = \mathrm{div}(H|_W) = \mathrm{div}(h).$$

Furthermore, we have

$$\begin{aligned}\mathrm{div}(h)\,Z =& (\mathrm{div}(H)\,W)\,Z_F = \mathrm{div}(H)\,(W_F\,Z_F) \\ =& \mathrm{div}(H)\,(\sum_i \mu_i S_i + t) = \sum_i \mu_i\,\mathrm{div}(H|_{S_i}) + \mathrm{div}(H|_t) \\ =& \sum_i \mu_i \mathrm{div}(h|_{S_i}) + \mathrm{div}(h|_T\,t) = \mathrm{div}(h\,Z).\end{aligned}$$

On the other hand, for Green's currents, the left hand side becomes

$$\begin{aligned}(-[\log|h|^2]) * g_Z =& g_Z * (-\log|h|^2) \\ =& - g_Z \wedge \delta_{\mathrm{div}(h)} + [\omega_Z] \wedge (-[\log|h|^2]) \mod (\mathrm{Im}\partial + \mathrm{Im}\bar\partial).\end{aligned}$$

While for the right hand side, we have

$$\begin{aligned}-[\log|h\, Z|^2] =& [\log|H^2|] \wedge \delta_{W\, Z} \\ =& - [\log|H^2|] * (g_W * g_Z) = g_Z * (-[\log|H^2|] * g_W) \\ =& - g_Z \wedge \delta_{\mathrm{div}(H)W} + [\omega_Z] \wedge (-[\log|H^2|] * g_W) \\ =& - g_Z \wedge \delta_{\mathrm{div}(h)} + [\omega_Z] \wedge ([\log|H^2|] \wedge \delta_W) \\ =& - g_Z \wedge \delta_{\mathrm{div}(h)} - [\omega_Z] \wedge (-[\log|h^2|]) \mod (\mathrm{Im}\partial + \mathrm{Im}\bar\partial).\end{aligned}$$

Here, in the last step, we use the following discussion. Since $-\log|f|^2$ is a Green's current for $\mathrm{div}(f)$,

$$\log|f|^2 \wedge \delta_Z = \log|f|^2 * g_Z,$$

for any choice of g_Z. Thus

$$\log|f|^2 \wedge \delta_Z = \log|f|^2 \wedge \omega - \delta_{\mathrm{div}(f)} \wedge g_Z.$$

If $\tilde f$ is chosen so that $\tilde f|_W = f$, then

$$f([Z]) = \tilde f([W][Z]) = \tilde f(\sum_k \mu_k[S_k] + t),$$

for $[W][Z] = \sum_k \mu_k[S_k] + t$. Hence

$$\begin{aligned}\log|f_Z|^2 =& \log|\tilde f|^2 \wedge (\delta_{\sum_k \mu_k[S_k]} + \delta_t) \\ =& \log|\tilde f|^2 \wedge \omega_Z \wedge \delta_W - \delta_{\mathrm{div}(\tilde f)} \wedge g_Z \\ =& \log|f|^2 \wedge \omega_Z - \delta_{\mathrm{div}(f)} \wedge g_Z \\ =& \log|f|^2 \wedge \delta_Z.\end{aligned}$$

With the above definition $\mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}}$ is a commutative associative $\mathbf{Q}$-algebra, since we know that the $*$-product is associative and commutative.

§II.2.5. Functorial Properties.

Once we have the definitions for arithmetic intersection theory, we can establish the properties of it with respect to morphisms of arithmetic varieties. This is what we discuss now. As usual, in order to give a good definition for the pull back morphisms, we need certain condition on the fibers, which allow us to assume that the morphism is flat since we only need the pull back morphism. On the other hand, we should also have a natural pull back with respect to closed immersions. Naturally, once we discuss the push out morphism, we need to know that the image of a closed subset should also be a closed subset. So, when we talk about push out morphisms, we shall assume that the morphism for arithmetic varieties is proper.

Theorem. Let $f : X \to Y$ be a morphism of regular arithmetic varieties over an arithmetic ring A. Then

(1) If f is flat, there is a pull-back morphism

$$f^* = f^{\mathrm{CH}} : \mathrm{CH}^p_{\mathrm{Ar}}(Y) \to \mathrm{CH}^p_{\mathrm{Ar}}(X)_{\mathbf{Q}}.$$

(2) If f is proper, $f_F : X_F \to Y_F$ is smooth and X, Y are equidimensional, then there is a push-out morphism

$$f_* = f_{\mathrm{CH}} : \mathrm{CH}^p_{\mathrm{Ar}}(X) \to \mathrm{CH}^{p-r}_{\mathrm{Ar}}(Y).$$

Here r denotes the relative dimension of f.

(3) Where the notation makes sense, we have the projective formula

$$f_*(f^*(\alpha)\beta) = \alpha f_*(\beta).$$

Proof. (1) Let $[(Z, g_Z)] \in \mathrm{CH}^p_{\mathrm{Ar}}(Y)$. We assume that Z is irreducible. If it happens that $\mathrm{codim}_{X_F}(f^{-1}(Z)_F) = p$, then we have $f^*[Z] \in \mathrm{CH}^p_{f^{-1}(Z)}(X)_{\mathbf{Q}}$, and we also denote the image of $f^*([Z])$ under the map

$$\mathrm{CH}_{f^{-1}(Z)}(X)_{\mathbf{Q}} \to \mathrm{CH}^p_{\mathrm{fin}}(X)_{\mathbf{Q}} \oplus Z^p_{f^{-1}(Z)_F}(X_F)_{\mathbf{Q}}$$

by $f^*([Z])$. This is the definition for the algebraic cycles. Since $f^* g_Z$ is defined, we may put

$$f^*[(Z, g_Z)] := [(f^*[Z], f^* g_Z)] \in \mathrm{CH}^p_{\mathrm{Ar}}(X)_{\mathbf{Q}}.$$

Obviously, this is well-defined. In general, if we do not have the exact codimension relation, we may use the moving lemma to achieve the required result; this has the same pattern as in 2.4. The details are left to the reader.

(2) We first construct a map from $Z^p_{\mathrm{Ar}}(X)$ to $Z^{p-r}_{\mathrm{Ar}}(Y)$ as follows: Let $(Z, g_Z) \in Z^p_{\mathrm{Ar}}(X)$, with Z irreducible, i.e. $Z = \overline{\{z\}}$ with z the generic point of Z. As in section 1.3, we set

$$f_*(Z) := \begin{cases} [k(z) : k(f(z))]\, \overline{\{f(z)\}}, & \text{if } \dim f(z) = \dim z; \\ 0, & \text{otherwise.} \end{cases}$$

Then, for Green's currents, we know that for any $\eta \in \tilde{A}^{\dim Y(\mathbf{C})-p, \dim Y(\mathbf{C})-p}(X(\mathbf{C}))$,

$$\begin{aligned} (f_*\delta_Z)(\eta) &= \delta_Z(f^*\eta) = \int_{Z(\mathbf{C})} f^*\eta = \int_{Z(\mathbf{C})} f^*(\eta|_{f(Z(\mathbf{C}))}) \\ &= \begin{cases} \deg(Z(\mathbf{C})/f(Z(\mathbf{C}))) \int_{f(Z(\mathbf{C}))} \eta, & \text{if } Z(\mathbf{C}) \to f(Z(\mathbf{C})) \text{ is finite;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence $f_*\delta_Z = \delta_{f_*(Z)}$, and

$$dd^c(f_*g_Z) = [f_*\omega_Z] - \delta_{f_*(Z)}.$$

That is, f_*g_Z defines a Green's current of $f_*(Z)$. Therefore we may put

$$f_*(Z, g_Z) := (f_*Z, f_*g_Z) \in Z^{p-r}_{\mathrm{Ar}}(Y).$$

Furthermore, it is not difficult to check that this definition is compatible with the arithmetic rational equivalence, and hence we get a push-out morphism f_* for arithmetic Chow rings.

(3) The proof here comes from the fact that the projective formula is valid for both algebraic cycles and Green's currents.

Next, we define the pull back morphism for regular closed immersions. This may be done formally by the arithmetic intersection theory. Let $i : X \hookrightarrow Y$ be a regular closed immersion of arithmetic varieties over an arithmetic ring A. If Z is a codimension-p cycle in Y, which meets X properly on the generic fiber, then there is a well defined cycle $i^*[Z] \in Z^p(X_F) \oplus \mathrm{CH}^p_{\mathrm{fin}}(X)$. Since Z_F meets X_F properly, let g_Z be a Green's current for Z on Y, then i^*g_Z is a Green's current for $i^*[Z]$ by a slight modification of Theorem 2.3.c. We can define

$$i^*(Z, g_Z) := (i^*[Z], i^*g_Z) \in Z^p_{\mathrm{Ar}}(X) \oplus \mathrm{CH}^p_{\mathrm{fin}}(X).$$

Suppose now that $\phi \in R^{p-1}_p(Y_F)$ is a K_1-chain such that $\mathrm{div}(\phi)$ meets X_F properly. By the Chow moving lemma for K_1-chains, there is a K_1-chain φ such that $\mathrm{div}(\phi) = \mathrm{div}(\varphi)$ and φ meets X_F almost properly. Furthermore, $\varphi - \phi$ represents zero in $\mathrm{CH}^{p-1,p}(Y_F)$. Hence, $\log|\varphi|^2 = \log|\phi|^2$, and therefore $\mathrm{div}_{\mathrm{Ar}}(\varphi) = \mathrm{div}_{\mathrm{Ar}}(\phi) \in Z^p_{\mathrm{Ar}}(Y) \oplus \mathrm{CH}^p_{\mathrm{fin}}(Y)$. If $Z := \mathrm{Supp}(\mathrm{div}(\varphi))$, $i^*(\varphi)$ is well defined in $\mathrm{CH}^{p-1,p}(X - (X \cap Z))$. By the theorem above, we have $i^*\mathrm{div}_{\mathrm{Ar}}(\varphi) = \mathrm{div}_{\mathrm{Ar}}(i^*\varphi)$. So i^* induces a map $\mathrm{CH}_{\mathrm{Ar}}(Y) \to \mathrm{CH}_{\mathrm{Ar}}(X)$.

In practice, the situation is as folows: We may first try to use the cohomological properties of the arithmetic Chow groups. Since we have the definition and properties for algebraic cycles in 1.3.b, we now consider the situation for K_1-chains. We assume that we work with regular schemes which are flat and of finite type over a fixed excellent regular noetherian domain A.

If $\phi \in R^{n-1}_n(Y)$ is a K_1-chain, we let $Z := \mathrm{Supp}(\phi)$, $T := \mathrm{Supp}(\mathrm{div}(\phi))$ and $U := Z - T$, then ϕ determines, and is determined by the class $\{\phi\} \in \mathrm{CH}^{n-1,n}_U(Y - T)$. Furthermore, the image of $\{\phi\}$ under the boundary map

$$\partial : \mathrm{CH}^{n-1,n}_U(Y - T) \to \mathrm{CH}^n_T(Y)$$

is the class of $\mathrm{div}(\phi)$, where ∂ is the boundary map induced by the exact sequence of complexes:

$$0 \to R^*_n(Y)_T \to R^*_n(Y)_Z \to R^*_n(Y - T)_U \to 0.$$

Here we use the following notation: for $V \subset W$,

$$R^*_n(W)_V := \mathrm{Ker}(R^*_n(W) \to R^*_n(W - V)),$$

and $\mathrm{CH}^{i,j}_V(W) := H^i(R^*_j(W)_V)$. Now, for the proof of the fact that the definition above is well-defined, we need to use the deformation to the normal cone. Let $\mathbf{A}^1_A = \mathrm{Spec}\, A[t]$, $\mathbf{A}^1_Y = Y \times_A \mathbf{A}^1_A$, and let $\tilde{W}$ be the blowing up along $X \times \{0\} \subset \mathbf{A}^1_Y$. If $\tilde{p} : \tilde{W} \to \mathbf{A}^1_A$ is the projection map, $\tilde{p}^{-1}(0)$ is the union of two divisors, $\mathbf{P}(\mathcal{N}_X(Y) \oplus 1)$, which is the projective completion of the normal bundle from X to Y, and to $\tilde{Y}$, which is the blowing up of Y along X. Define $W := \tilde{W} - \tilde{Y}$. We have

(1) The projection $p : W \to \mathbf{A}^1_A$ is flat;
(2) $W_0 := p^{-1}(\{0\}) \simeq N_X(Y)$;
(3) $W - W_0 \simeq Y \times \mathbf{G}_m = Y \times_A \mathrm{Spec}(A[t, t^{-1}])$;
(4) $f : X \to Y$ induces a map $\tilde{f} : X \times \mathbf{A}^1 \to W$, such that for $t = 0$, $X \times \{0\} \to W_0 = N_X(Y)$ is the zero section,while for $t \neq 0$, $X \times \mathbf{G}_m \to Y \times \mathbf{G}_m$ is the map induced by the base change from $f : X \to Y$.

Observe that t is a unit on $W - W_0$, so that t defines a class $\{t\}$ in $H^0(W - W_0, K_1(\mathcal{O}_W) = \mathcal{O}^*_W)$. For the construction of $f^*\{\phi\}$, it is convenient to consider the deformation to the normal cone construction for the inclusion of $X - (X \cap Y)$ in $Y - T$. We write $f' : X' \to Y'$ for this inclusion, and W' for the corresponding scheme flat over $\mathbf{A}^1_A$. Since $p' : Y' \times \mathbf{G}_m = W' - W'_0 \to Y'$ is flat, there is a pull back map

$$p'^* : R^*_n(Y) \to R^*_n(Y \times \mathbf{G}_m),$$

and hence a map

$$p'^* : \mathrm{CH}^{n-1,n}_U(Y') \to \mathrm{CH}^{n-1,n}_{U \times \mathbf{G}_m}(W - W_0).$$

Associated with the short exact sequence

$$R^*_n(W'_0)[1] \to R^*_{n-1}(W') \to R^*_{n+1}(W' - W'_0),$$

we have a long exact sequence

$$\ldots \to \mathrm{CH}^{i,j}_V(W'_0) \to \mathrm{CH}^{i+1,j+1}_{\bar{U}}(W') \to \mathrm{CH}^{i+1,j+1}_{U \times \mathbf{G}_m}(W - W'_0) \xrightarrow{\partial} \mathrm{CH}^{i+1,j}_V(W'_0) \to \ldots.$$

Here $\bar{U}$ is the Zariski closure of $U \times \mathbf{G}_m$ in W' and $V = \bar{U} \cap W'_0 = C_{U \cap X'}(U)$ is the normal cone of $U \cap X'$ in U. By [Gi 81], there is a natural product for any noetherian scheme S,

$$\mathcal{R}^*_{m,S} \otimes K_n(\mathcal{O}_S) \to \mathcal{R}^*_{m+n,S}[-n],$$

where $\mathcal{R}^*_{m,S}$ is the complex of sheaves $U \mapsto R^*_m(U)$ on S. Hence there are products

$$H^0(W' - W'_0, K_1(\mathcal{O}_{W'}) = \mathcal{O}^*_{W'}) \otimes \mathrm{CH}^{n-1,n}_{p^{-1}(U)}(W' - W'_0) \to \mathrm{CH}^{n-1,n+1}_{p^{-1}(U)}(W' - W'_0).$$

By the fact that $V \subset \pi^{-1}(X' \cap U)$, we may combine the boundary map and the product, and get a map

$$\begin{array}{cccccc} \sigma_t : & \mathrm{CH}^{n-1,n}_U(Y') & \to & \mathrm{CH}^{n-1,n}_V(W'_0) & \to & \mathrm{CH}^{n-1,n}_{\pi^{-1}(X' \cap U)}(W_0) \\ & & & \{\phi\} & \mapsto & \partial(\{t\} * \{\phi\}). \end{array}$$

We observe finally that

$$\pi^* : \mathrm{CH}^{n-1,n}_{X'\cap U}(X') \to \mathrm{CH}^{n-1,n}_{\pi^{-1}(X'\cap U)}(W_0)$$

is an isomorphism. Composing this with the map σ_t above, we get a map

$$f^* = (\pi^*)^{-1} \circ \sigma_t : \mathrm{CH}^{n-1,n}_U(Y') \to \mathrm{CH}^{n-1,n}_{U\cap X'}(X').$$

Theorem. Suppose that $f : X \to Y$ is a regular closed immersion. If $\phi \in R^{n-1}_n(Y)$ is a K_1-chain with support Z, $\mathrm{Supp}(\mathrm{div}(\phi)) = T$ and $U = Z - T$, then we have
(1) $\partial(f^*\{\phi\}) = f^*(\partial\{\phi\}) \in \mathrm{CH}^n_{X\cap T}(X)$ where ∂ is the boundary map

$$\mathrm{CH}^{n-1,n}_{B-C}(A - C) \to \mathrm{CH}^n_C(A)$$

for $C \subset B \subset A$.
(2) If $\phi = \sum[g_W]$ meets X almost properly, with g regular at the generic points of S, and $\mathrm{div}(g) \cap T$ is empty, then

$$f^*[g_W] = \sum_i \mu_i [g_W|_{S_i}] + \sum_j g_W\, \tau_j,$$

and $f^*(\phi) = \sum_W f^*[g_W]$. Here μ_i is the intersection multiplicity of Z and W at the generic point of the irreducible component S_i of S, and τ_j is the cycle class on the connected component T_j of T representing the component of $f^*[W]$ in $\mathrm{CH}^{n-1}_{T_j}(X) \subset \mathrm{CH}^{n-1}_{S\cup T}(X)$. The product $g_W\, \tau_j$ is defined since g_W is a regular function on T_j.

Before proving the theorem, we need the following

Lemma. Let S be a noetherian scheme, A and B closed subschemes, and $C := A \cap B$. Let $D \subset S$ be any closed subset. Then the square

$$\begin{array}{ccc}
\mathrm{CH}^{i,j}_{D-D\cap(A\cup B)}(S-(A\cup B)) & \xrightarrow{\partial} & \mathrm{CH}^{i+1,j}_{(B-C)\cap D}(S-A) \\
\partial \downarrow & & \downarrow \partial \\
\mathrm{CH}^{i+1,j}_{D\cap(A-C)}(S-B) & \xrightarrow{\partial} & \mathrm{CH}^{i+2,j}_{C\cap D}(S)
\end{array}$$

is commutative up to a factor -1.

Proof. This is a direct consequence of the following diagram

$$\begin{array}{ccccc}
R^*_j(S)_{C\cap D} & \to & R^*_j(S)_{A\cap D} & \to & R^*_j(S-B)_{A-C\cap D} \\
\downarrow & & \downarrow & & \downarrow \\
R^*_j(S)_{B\cap D} & \to & R^*_j(S)_D & \to & R^*_j(S-B)_{D\cap(S-B)} \\
\downarrow & & \downarrow & & \downarrow \\
R^*_j(S)_{(B-C)\cap D} & \to & R^*_j(S-A)_{(S-A)\cap D} & \to & R^*_j(S-B)_{D-(A\cup B)\cap D}.
\end{array}$$

Proof Of The Theorem. In the lemma above, let $S = M$, D be the Zariski closure of $Z \times \mathbf{G}_m$ in W, A the Zariski closure of $T \times \mathbf{G}_m$ in W and $B = W_0$. Then $C = C_{T\cap X}(T)$ and C is contained in $D \cap W_0 = C_{Z\cap X}(Z)$. We have the diagram

$$\begin{array}{ccc}
\mathrm{CH}_U^{n-1,n}(Y') & \xrightarrow{\partial} & \mathrm{CH}_T^n(Y) \\
p^* \downarrow & I & \downarrow p^* \\
\mathrm{CH}_{p^{-1}(U)}^{n-1,n}(Y' \times \mathbf{G}_m) & \xrightarrow{\partial} & \mathrm{CH}_{p^{-1}(T)}^n(Y \times \mathbf{G}_m) \\
\{t\} * () \downarrow & II & \downarrow \{t\} * () \\
\mathrm{CH}_{p^{-1}(U)}^{n-1,n+1}(Y' \times \mathbf{G}_m) & \xrightarrow{\partial} & \mathrm{CH}_{p^{-1}(T)}^{n,n+1}(Y \times \mathbf{G}_m) \\
\partial \downarrow & III & \downarrow \partial \\
\mathrm{CH}_{D\cap W_0'}^{n-1,n}(W_0') & \xrightarrow{\partial} & \mathrm{CH}_{C_{T\cap X}(T)}^n(W_0).
\end{array}$$

In this diagram, square I is commutative because p is flat. It can be checked at the level of complexes that square II is anti-commutative, while square III is anti-commutative by the lemma. It follows that if $\phi \in \mathrm{CH}_U^{n-1,n}(Y')$, then $\partial\sigma_t(\phi) = \sigma_t(\partial\phi) \in \mathrm{CH}_{\pi^{-1}(T\cap X)}^n(W_0)$, which gives (1). For (2), suppose $\phi = [g_W]$ for $g \in k(W)^*$, and let $\tilde{g} \in k(Y)^*$ be a rational function which is regular at the generic point of W and X, and is such that $\tilde{g}|_W = g$. Write $\tilde{D} = \mathrm{div}(\tilde{g})$, so that $\tilde{D} \cap W = D$. Then $\phi = \{\tilde{g}\}\,[W]$ under the product

$$H^0(Y - \tilde{D}, K_1(\mathcal{O}_Y)) \otimes \mathrm{CH}_W^{n-1}(Y) \to \mathrm{CH}_{W-D}^{n-1,n}(Y - D).$$

One can see from the construction of

$$f^* : \mathrm{CH}_{W-D}^{i,j}(Y - D) \to \mathrm{CH}_{X\cap(W-D)}^{n-1,n}(X \cap (Y - D))$$

that if $\alpha \in H^r(Y - \tilde{D}, K_*(\mathcal{O}_Y))$, and $\beta \in \mathrm{CH}_{W-D}^{i,j}(Y - D)$, then $f^*(\alpha\,\beta) = f^*(\alpha)\,f^*(\beta)$, where $f^*(\alpha)$ is the pull back on the sheaf cohomology induced by the pull back on the K-theory. Hence, if $X\,W = \sum_i \mu_i[S_i] + \sum_j \tau_j$,

$$\begin{aligned}
f^*[g_W] &= f^*\{\tilde{g}\}\,f^*[W] = \{\tilde{g}|_X\}(\sum_i \mu_i[S_i] + \sum_j \tau_j) \\
&= \sum_i \mu_i[\tilde{g}|_{S_i}] + \sum_j \tilde{g}\,\tau_j = \sum_i \mu_i[g|_{S_i}] + \sum_j g\,\tau_j,
\end{aligned}$$

which completes the proof of (2) of the theorem, and hence justifies the discussion above for the closed immersions.

§II.2.6. Examples

In this section, we consider some examples which explain the general discussion in a more concrete way. Let X be a regular scheme, projective and flat over $\mathbf{Z}$. Then we have

(a) $\mathrm{CH}_{\mathrm{Ar}}^0(X) = \mathrm{CH}^0(X) = \mathbf{Z}^{\pi_0(X)}$.

(b) Denote by $\mathrm{Pic}_{\mathbf{Ar}}(X)$ the group of isomorphism classes of hermitian line sheaves on X. Then we have

Proposition. There is an isomorphism

$$\begin{array}{rccc} c_{1,\mathrm{Ar}}: & \mathrm{Pic}_{\mathbf{Ar}}(X) & \to & \mathrm{CH}^1_{\mathbf{Ar}}(X) \\ & \mathrm{cl}(\mathcal{L},\rho) & \mapsto & [(\mathrm{div}(s), -[\log|s|^2_\rho])], \end{array}$$

where s is a non-zero section of $\mathcal{L}$.

Proof. We need to construct the inverse map:

$$[(Z, g_Z)] \mapsto \mathrm{cl}(\mathcal{O}_X(Z), \rho),$$

where the metric ρ is given locally by the formula

$$|\mathbf{1}|^2_\rho := e^{-g_Z}.$$

Here $\mathbf{1}$ is the section of $\mathcal{L}$ with Z as its divisor. The other properties are obvious.

(c) We may consider $X = \mathrm{Spec}(\mathcal{O}_F)$ as an 1-dimensional arithmetic variety, where $\mathcal{O}_F$ is the ring of integers of a number field F. In this special case, classically, we have

$$\begin{aligned} \mathrm{CH}^{0,1}(X) &= \mathcal{O}_F^*; \\ \tilde{A}^{0,0}(X) &= A^{0,0}(X) = \oplus_{\sigma\in\Sigma}\mathbf{R}; \\ \mathrm{CH}^1(X) &= \mathrm{Cl}(\mathcal{O}_F). \end{aligned}$$

The cohomological properties of the associated arithmetic Chow groups give the exact sequence

$$\ldots \to \mathcal{O}_F^* \xrightarrow{\rho} \mathbf{R}^{r_1+r_2} \xrightarrow{\epsilon} \mathrm{Pic}_{\mathbf{Ar}}(X) \xrightarrow{\zeta} \mathrm{Cl}(\mathcal{O}_F) \to 0.$$

Notice that ρ is, up to a factor, the classical Dirichlet regulator map, hence $\mathrm{Ker}\,\rho = \mu_F$, the roots of unity of F. Furthermore, we get a arithmetic degree morphism

$$\begin{array}{rccc} \deg_{\mathbf{Ar}}: & \mathrm{Pic}_{\mathbf{Ar}}(X) & \to & \mathrm{Pic}_{\mathbf{Ar}}(\mathbf{Z}) = \mathbf{R}, \\ & \mathrm{cl}(\mathcal{L},\rho) & \mapsto & \log\left(\frac{\prod_{\sigma\in\Sigma}|s|^2_\rho}{[L:\mathcal{O}_{F,s}]}\right), \end{array}$$

where s is a non-zero section of $\mathcal{L}$. Thus the compactness of $\mathrm{Pic}_{\mathbf{Ar}}(X)$,

$$\mathrm{Pic}^0_{\mathbf{Ar}}(X) := \mathrm{Ker}\,\deg_{\mathbf{Ar}}$$

is equivalent to the finiteness of the ideal class group $\mathrm{Cl}(\mathcal{O}_F)$ together with the Dirichlet unit theorem, i.e. $\rho(\mathcal{O}_F^*)$ is a lattice of rank $r_1 + r_2 - 1$ in $\mathbf{R}^{r_1+r_2}$. We know finally that $\mathrm{vol}(\mathrm{Pic}^0_{\mathbf{Ar}}(X)) = h_F\, R_F$, where h_F is the class number of F and R_F the regulator of F, respectively.

Dually, we have the following exact sequence in terms of cohomology theory:

$$\{1\} \to \mu(F) \to \mathcal{O}_F^* \xrightarrow{\rho} \mathbf{R}^{r_1+r_2} \xrightarrow{\epsilon} \mathrm{CH}^1_{\mathrm{Ar}}(X) \xrightarrow{\zeta} \mathrm{Cl}(\mathcal{O}_F) \to 0.$$

Here $\mu(F)$ denotes the group of the roots of unity in F. Hence, we may also have the identification

$$\mathrm{CH}^1_{\mathrm{Ar}}(X) \simeq F^*\backslash J(F)/U_F,$$

where $J(F)$ denotes the ideal group of F and U_F is the maximal compact subgroup of $J(K)$. In this content, the arithmetic degree is given by

$$\begin{array}{llll} \deg_{\mathrm{Ar}}: & \mathrm{CH}^1_{\mathrm{Ar}}(\mathrm{Spec}(\mathcal{O}_F)) & \to & \mathbf{R}, \\ & (Z = \sum_i n_i[\mathcal{P}_i], g_Z = \{g_\sigma\}_{\sigma\in\Sigma}) & \mapsto & \log\#(Z) + \frac{1}{2}\int_X g := \log\sum n_i\#(\mathcal{O}_F/\mathcal{P}_i) + \frac{1}{2}\sum_{\sigma\in\Sigma} g_\sigma. \end{array}$$

(d) We now discuss the Arakelov varieties. As we stated at the beginning of this chapter, Arakelov introduced his theory for certain admissible metrics at infinity. We call a pair (X, g_0) an **Arakelov variety** if X is a regular scheme, projective and flat over $\mathbf{Z}$, and g_0 is a Kähler metric on $X(\mathbf{C})$, invariant under F_∞. (Arakelov only considered the situation when X is an arithmetic surface, and g_0 is given by

$$\frac{i}{2g}\sum_j \omega_j \wedge \bar{\omega}_j,$$

where g is the genus of $X(\mathbf{C})$, and $\omega_1, \ldots, \omega_g$ form an orthonormal basis of the space of the holomorphic 1-forms on X, $\Gamma(X(\mathbf{C}), \Omega^1_{X(\mathbf{C})})$, whenever the symbols make sense.)

By the Hodge decomposition theorem, we have

$$A^{p,p}(X) = \mathcal{H}^{p,p}(X) \oplus \mathrm{Im}\,\partial \oplus \mathrm{Im}\,\partial^*,$$

where $\mathcal{H}^{p,p}(X) := \mathrm{Ker}(\Delta_d) \subset A^{p,p}(X)$ denotes the space of real harmonic forms on $X(\mathbf{C})$ of type (p,p), invariant under F_∞ up to the factor $(-1)^p$. Recall then that there are maps

$$\zeta: \mathrm{CH}_{\mathrm{Ar}}(X) \to \mathrm{CH}^p(X) \oplus Z^{p,p}(X)$$

given by $\zeta([(Z, g_Z)]) = ([Z], \omega_Z)$. We denote the second component by ω, i.e. $\omega(g_Z) = \omega_Z$. Now we may introduce the **Arakelov Chow group** by letting

$$\mathrm{CH}_{\mathrm{Ara}}(X, g_0) := \omega^{-1}(\mathcal{H}^{p,p}(X)) \subset \mathrm{CH}_{\mathrm{Ar}}(X).$$

There is also a Hodge decomposition for currents

$$D^{p,p}(X) = \mathcal{H}^{p,p}(X) \oplus \mathrm{Im}\,\partial \oplus \mathrm{Im}\,\partial^*,$$

and we denote by $H: D^{p,p}(X) \to \mathcal{H}^{p,p}(X)$ the orthogonal projection. We have the following

Proposition. With the notation as above,
(1) $\mathrm{CH}_{\mathrm{Ara}}(X, g_0) \simeq (Z^p(X) \oplus \mathcal{H}^{p-1,p-1}(X)) / < \mathrm{div}(f), -H[\log|f|^2] >$.
(2) $\mathrm{CH}_{\mathrm{Ara}}(X, g_0)$ is a direct summand of $\mathrm{CH}_{\mathrm{Ar}}(X)$.
(3) There is an exact sequence

$$\ldots \to \mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} \mathcal{H}^{p-1,p-1}(X) \xrightarrow{\epsilon} \mathrm{CH}^p_{\mathrm{Ara}}(X, g_0) \xrightarrow{\zeta} \mathrm{CH}^p(X) \to 0.$$

§II.2.7. Arithmetic Chow Homology Groups

We now show that the arithmetic intersection above may be extended to give a cap product between arithmetic Chow cohomology and homology. These cap products are described somewhat in the style of Fulton's operation formalism.

First notice that the real vector space $\tilde{A}(X_{\mathbf{R}})$ is a contravariant functor from arithmetic varieties to rings without unit, where we consider the $*$-product: $\phi_* \varphi := \phi \wedge dd^c(\varphi)$ on $\tilde{A}(X_{\mathbf{R}})$. Given a class $x \in \mathrm{CH}^{\mathrm{Ar}}_{\mathrm{q}}(X)$ and $\phi \in \tilde{A}(X_{\mathbf{R}})$, we define their cap product

$$\phi \cap x = \phi \cap (Z, g_Z) := (0, \phi_* g_Z) = a(\phi \cdot \omega(x)).$$

Here we have written $x = (Z, g_Z)$ and $\cdot$ for the product

$$A^{p,p}(X_{\mathbf{R}}) \otimes D_{q-r+1,q-r+1}(X_{\mathbf{R}}) \to D_{q-p-r+1,q-p-r+1}(X_{\mathbf{R}})$$

which is induced by the wedge product of forms with distribution coefficients. Naturally, we have the following

Theorem. Given a map $f : X \to Y$ of arithmetic varieties with Y regular, there is a unique cap product:

$$\begin{array}{ccc} \mathrm{CH}^{\mathrm{p}}_{\mathrm{Ar}}(Y) \otimes \mathrm{CH}^{\mathrm{Ar}}_{\mathrm{q}}(X) & \to & \mathrm{CH}^{\mathrm{Ar}}_{\mathrm{q-p}}(X)_{\mathbf{Q}} \\ y \otimes x & \mapsto & y \cdot_f x \end{array}$$

which we also denote by $y \cap x$, or more simply by $y \cdot x$ if $X = Y$, such that
(a) $\omega(y \cdot_f x) = f^* \omega(y) \wedge \omega(x)$, and, for any $\eta \in \tilde{A}(Y_{\mathbf{R}})$, $a(\phi) \cdot_f x = a(f^* \phi) \cap x$.
(b) $\mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)_{\mathbf{Q}}$ is a graded $\mathrm{CH}^{\cdot}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}$-module.
(c) If $g : Y \to Y'$ is a map of arithmetic varieties with Y' regular, $y' \in \mathrm{CH}^{\mathrm{p}}_{\mathrm{Ar}}(Y')$ and $x \in \mathrm{CH}^{\mathrm{Ar}}_{\mathrm{q}}(X)$, then $y' \cdot_{f \circ g} x = (g^*(y')) \cdot_f x$.
(d) If $h : X' \to X$ is projective, and smooth over X_F, then, after tensoring with $\mathbf{Q}$, the push-out map h_* is a map of $\mathrm{CH}^{\cdot}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}$-modules.
(e) If $h : X' \to X$ is flat and smooth over F, or an l.c.i. morphism, then, after tensoring with $\mathbf{Q}$, the pull back map h^* is a map of $\mathrm{CH}^{\cdot}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}$-modules.
(f) Let $i : D \hookrightarrow X'$ be the inclusion of a principal effective Cartier divisor, $h : X \to X'$ a morphism which meets D_F properly, and $i_X : h^{-1}(D) \hookrightarrow X$ the inclusion induced

by i. Then for any $x \in \mathrm{CH}^{\mathrm{Ar}}(X)$ and $y \in \mathrm{CH}^{\mathrm{Ar}}(Y)$, the following holds in $\mathrm{CH}^{\mathrm{Ar}}(|h^{-1}(D)|)$:

$$y \cdot_{f \circ i_X} i^*(x) = i^*(y \cdot_f x).$$

Proof. First we offer the definition for $y \cdot_f x$. Without loss of generality, we may suppose that Y is equidimensional of dimension n.

Let $x = (V, g_V) \in \mathrm{CH}_q^{\mathrm{Ar}}(X)$, with V an algebraic prime cycle on X. By the Chow moving lemma, we may assume that $y = (W = \sum_i n_i W_i, g_W) \in \mathrm{CH}^p_{\mathrm{Ar}}(Y)$, where each $f^{-1}(W_i)$ meets V properly on the generic fiber X_F. So, to define the cap product of arithmetic cycles, it will be sufficient to define an algebraic cycle $[V] \cdot_f [W] \in \tilde{Z}_{q-p}(X)$, together with a Green's current for it. Here $\tilde{Z}_p(X)$ denotes the quotient of $Z_p(X)$ by the subgroups consisting of all $\mathrm{div}(f)$ for which f is a rational function on a $(p+1)$-dimensional subvariety $W \subset X$ such that $W \cap X_F$ is empty.

First let us look at the algebraic cycle side. We may use the K-theory description discussed in the first chapter: In practice, we shall produce this cycle in the group $\mathrm{CH}_{q-p}(V \cap f^{-1}(|W|))_{\mathbf{Q}}$, which maps naturally to $\tilde{Z}_{q-p}(X)$, since each $f^{-1}(W_i)$ meets V properly on the generic fiber X_F.

Since Y is regular, $[\mathcal{O}_{W_i}] \in K_0^{W_i}(Y)$, and hence $f^*[\mathcal{O}_{W_i}] \in K_0^{V \cap f^{-1}(W_i)}(X)$. So we have $f^*[\mathcal{O}_{W_i}] \cap [\mathcal{O}_V] \in K^0(V \cap f^{-1}(W_i))$. But by Chapter 1, we know that

$$K^0(V \cap f^{-1}(W_i))_{\mathbf{Q}} \simeq \oplus_{r \geq 0} \mathrm{CH}_r(V \cap f^{-1}(W_i))_{\mathbf{Q}},$$

where

$$\mathrm{CH}_r(V \cap f^{-1}(W_i))_{\mathbf{Q}} \simeq \mathrm{Gr}_r K^0(V \cap f^{-1}(W_i))_{\mathbf{Q}}.$$

Thus it is sufficient to show that

$$f^*[\mathcal{O}_{W_i}] \cap [\mathcal{O}_V] \in F_{q-p} K^0(V \cap f^{-1}(W_i))_{\mathbf{Q}}.$$

For this, by the fact that X is quasi-projective, we can factor f as $\pi \circ i$, where $\pi : U \to Y$ is the smooth projection from a Zariski open subset U of $\mathbf{P}^n_Y$, and i is a closed immersion. By the associativity of the tensor product, $f^*[\mathcal{O}_{W_i}] \cap [\mathcal{O}_V]$ can be calculated in the K-theory with supports of U, i.e. via the isomorphism $F^{N-\cdot} K_0^V(U)_{\mathbf{Q}} \simeq F.K_0(V)_{\mathbf{Q}}$, where N is the dimension of U, and $F^{\cdot}$ is the filtration induced by the codimension support as in Chapter 1. The assertion about the cap product cycle now follows from the multiplicativity of the filtration by codimension of supports on K-theory with rational coefficients for a regular scheme stated in Chapter 1.

Next, we consider the construction of the associated Green's current. Let $\widetilde{V}(\mathbf{C})$ be a resolution of singularities of $V(\mathbf{C})$, and let $k : \widetilde{V}(\mathbf{C}) \to X(\mathbf{C})$ be the map induced by the inclusion $V \hookrightarrow X$. Then, as before, since W_i meets f properly over F, the current

$$\delta_V \wedge f^*(g_{W_i}) := k_*((f \circ k)^*(g_{W_i}))$$

is well-defined if we choose for g_{W_i} a Green's form with the logarithmic growth along $W_i(\mathbf{C})$. By section 2, such a choice always exists after adding an element of the form $\partial(u) + \bar{\partial}(v)$. We now set

$$(V, g_V)\cdot_f(W_i, g_{W_i}) := ([V]\cdot_f[W_i], \delta_V \wedge f^* g_{W_i} + g_V \wedge \omega_W),$$

which is an arithmetic cycle.

As usual, once we have a definition, we need to check that if it is well-defined. In this case, just as in section 4, we have to consider the problem at the K_1 level. Suppose that $x = (V, g_V)$ with V a subvariety of X, as above, and that (W, g_W), $(W', g_{W'})$ are arithmetic cycles on X, representing the same class $y \in \mathrm{CH}^{\cdot}_{\mathrm{Ar}}(Y)$ and both meeting V_F and f_F properly. Then, there is a K_1-chain ϕ, which meets V_F and f_F almost properly, such that

$$(W, g_W) - (W', g_{W'}) = \mathrm{div}_{\mathrm{Ar}}(\phi).$$

Furthermore, we may assume that $\phi = [Z]\cdot\{\tilde{\phi}\}$ with $\tilde{\phi}$ a rational function on Y, the divisor of which meets V_F and f_F properly, and which is a unit on any component of $f^{-1}(V_F) \cap W_F$ for which $f^{-1}(V_F) \cap W_F$ has the excess dimension. Thus $(V, g_V)\cdot_f \mathrm{div}_{\mathrm{Ar}}(\phi) = \mathrm{div}_{\mathrm{Ar}}(\varphi)$, where φ is the K_1-chain on X which is equal to $([V]\cdot_f[Z])\cdot f^*(\tilde{\phi})$.

Similarly, if (V, g_V), $(V', g_{V'})$ are representatives of x, we can write

$$(V, g_V) - (V', g_{V'}) = \mathrm{div}_{\mathrm{Ar}}(\phi),$$

where ϕ is a K_1-chain on X. By the Chow moving lemma for K_1-chains and the fact that the cap product is independent of the choice of representative for y, we can choose a representative $y = (W, g_W)$, with W_F meeting f_F and ϕ_F properly on the fiber over F. As before,

$$((V, g_V) - (V', g_{V'}))\cdot_f(W, g_W) = \mathrm{div}_{\mathrm{Ar}}(\phi\cdot_f[W]),$$

where the K_1-chain $\phi\cdot_f[W]$ is defined by the cap product

$$K^1(|\phi| - |\mathrm{div}(\phi)|) \otimes K_0^W(Y) \to K^1((|\phi| - |\mathrm{div}(\phi)|) \cap f^{-1}(|W|)).$$

Hence we see that the cap product above is well-defined.

Now we turn to the proof of the properties. Almost all of them are direct consequences of the definition. Here we give only a rough sketch, and leave the details to the reader.

First, (a) follows immediately from the definition. For (b), we assume that x, y and y' are represented by arithmetic cycles (V, g_V), (W, g_W) and $(W', g_{W'})$ respectively, such that W and W' meet properly on Y_F, meet f_F properly, and $f^{-1}(W), f^{-1}(W')$ meet V properly on X_F. Then the required associativity is a consequence of two facts: First, we have the following identities in $\tilde{D}^{\cdot}(X_{\mathbf{R}})$

$$g_V * f^*(g_W * g_{W'}) = g_V * (f^*(g_W) * f^*(g_{W'})),$$

and

$$g_V * (f^*(g_W) * f^*(g_{W'})) = (g_V * f^*(g_W)) * f^*(g_{W'}).$$

Second, the product on K-theory with support is associative, which follows from the associativity of the tensor product.

The proof of (c) for Green's current comes from the fact that if we represent the classes x and y by arithmetic cycles (V, g_V) and (W, g_W) for which the associated algebraic cycles intersect properly, then the pull-back $(f' \circ f)^* g_W$ and the product $g_V * (f' \circ f)^* g_W$ are both defined using pull-backs and wedge products of smooth forms with the logarithmetic growth, and hence are functorial and associative. For the cycles, we just appeal again to the associativity of the tensor product.

For (d), the proof for cycles uses the projection formula for K-theory, while for Green's currents, assuming proper intersection and representing Green's currents by forms with the logarithmetic growth, we are reduced to the projection formula for the integration of smooth forms over the fibers of a proper smooth map. So it is rather natural.

To prove (e), we suppose first that h is flat. Let $x = (V, g_V) \in \mathrm{CH}^{\mathrm{Ar}}_{\mathrm{q}}(X)$ with V a prime cycle, and let $y = (W, g_W)$ with W a prime cycle meeting V_F properly. It follows that W also meets $h^{-1}(V_F)$ properly. The equality $h^*(g_V) * f^*(g_W) = h^*(g_V) * (f \circ h)^*(g_W)$ follows from Theorem 5, and the fact that the pull-back does not destroy the property of being of logarithmetic growth, so that $(f \circ h)^* g_W = h^* \circ f^* g_W$ at the level of forms. Next we check that there is equality of cycles

$$h^*([V]\cdot_f [W]) = [h^{-1}(V)]\cdot_{f\circ h}[W]$$

in $\mathrm{CH}_{q-p+d}\big(h^{-1}(V) \cap (f \circ h)^{-1}(W)\big)_{\mathbf{Q}}$. This is obtained from

$$h^*([\mathcal{O}_V] \cap f^*[\mathcal{O}_W]) = [\mathcal{O}_{h^*(V)}] \cap (f \circ h)^*[\mathcal{O}_W] \in F_{q-p+d}K^0\big(h^{-1}(V) \cap (f \circ h)^{-1}(W)\big)_{\mathbf{Q}}$$

by the associativity of the tensor product and the flatness of h.

If h is an l.c.i. morphism, since a smooth map is flat, we only need to consider the case of a regular immersion $h : X' \hookrightarrow X$. Again the equation of Green's currents follows from the regular case, since X'_F and X_F are smooth. For cycles, we use the compatibility of the pull-back on cycles via deformation to the normal cone with products on K-theory, which may be verified by embedding the whole deformation of the normal cone family in a regular variety.

Finally, by definition, with a process as above, we have (f).

As expected, we also have a projective formula for this cap product.

Proposition. Let $f : X \to Y$ be a map of arithmetic varieties with Y regular, and suppose $p : P \to Y$ is a proper smooth map of arithmetic varieties of relative dimension d. Then if we write $f_p : X \times_Y P \to P$ and $p_f : X \times_Y P \to X$ for the projections, i.e.

$$\begin{array}{ccc} X \times_Y P & \xrightarrow{f_p} & P \\ p_f \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y, \end{array}$$

we have, for all $\alpha \in \mathrm{CH}^{\mathrm{Ar}}_{\mathrm{p}}(X)$, $\gamma \in \mathrm{CH}^{\mathrm{q}}_{\mathrm{Ar}}(P)$,

$$p_{f\,*}(p_f^*(\alpha)\cdot_{f_p}\gamma) = \alpha\cdot_f p_*(\gamma).$$

Proof. Suppose that α (resp. γ) is the class of the arithmetic cycle (Z, g_Z) (resp. (W, g_W)), where Z and W are prime cycles which are flat over the base. (If not, the statement is purely algebraic, see below.) We assume that g_Z and g_W have logarithmic growth along $Z(\mathbf{C})$, $W(\mathbf{C})$, respectively, and by the Chow moving lemma, that the closed sets $f_p^{-1}(W_F)$ and $p_f^{-1}(Z_F)$ meet properly in $(X \times_M P)_F$. When $\dim p(W_F) \neq \dim(W_F)$, the cycle $p_*(W)$ is zero by definition of p_* on cycles. Then any fiber of the natural map $p : W \to p(W)$ has positive dimension, and hence the same is true for any fiber of the map $p_f : p_f^{-1}(Z) \cap f_p^{-1}(W) \to Z \cap f^{-1}(W)$. It follows that in this case the cycle component of both $p_{f\,*}(p_f^*(\alpha)\cdot_{f_p}\gamma)$ and $\alpha\cdot_f p_*(\gamma)$ vanish for the choice of representatives of α and γ. On the other hand, when $\dim p(W_F) = \dim(W_F)$, by the transversality assumption we know that any component of $p_f^{-1}(Z_F) \cap f_p^{-1}(W_F)$ is generically finite over its image, and that the component of $p_f\big(p_f^{-1}(Z_F) \cap f_p^{-1}(W_F)\big)$, which is just $Z_F \cap f^{-1}(p(W_F))$, has the same dimension. Furthermore, their multiplicities are equal by the Tor formula and the projection formula, since p_f and p are smooth. Thus, for the algebraic cycle side, we need to show that the cycle classes $p_{f\,*}(p_f^*[Z]\cdot_{f_p}[W])$ and $[Z]\cdot_f p_*[W]$ are equal in the algebraic Chow group $\mathrm{CH}_.(Z \cap f^{-1}(p(W)))_{\mathbf{Q}}$. Since the cap product on Chow homology is defined using algebraic K-theory, this follows from the identity of the derived functors $Lf^*\, Rp_* = Rq_*\, Lg^*$, i.e. the base change for direct images in K-theory with supports, say Proposition III.9.3 [Ha 77].

For the current side, we only need to note that the equality of currents

$$\begin{aligned} p_{f\,*}(p_f^*(g_Z) * f_p^*(g_W)) &= p_{f\,*}(p_f^*(\delta_Z)\, f_p^*(g_W) + p_f^*(g_Z)\, f_p^*(\omega_W)) \\ &= \delta_Z\, p_{f\,*}(f_p^*(g_W)) + g_Z\, p_{f\,*}(f_p^*(\omega_W)) = g_Z * p_{f\,*}(f_p^*(g_W)), \end{aligned}$$

when tested on compactly supported forms of the appropriate degree, is an equality of indefinite integrals on the open set $X(\mathbf{C}) - \Big(Z(\mathbf{C}) \cap f^{-1}\big(p(W(\mathbf{C}))\big)\Big)$, except when $p(W(\mathbf{C})) = M(\mathbf{C})$, in which case the statement is easily checked. Thus, the projective formula stated in the proposition follows from the fact that integration of forms along fibers of p and p_f commutes with the base change by the map $Z(\mathbf{C}) \to M(\mathbf{C})$. This completes the proof.

Chapter II.3 · Arithmetic Characteristic Classes

In this chapter, we will introduce arithmetic characteristic classes by certain axioms, which are very similar to the axioms for characteristic classes in algebraic geometry. As one may imagine, the corresponding concept in arithmetic geometry for vector sheaves is that for hermitian vector sheaves. The arithmetic characteristic classes attached to an hermitian vector sheaf are the arithmetic cycles in the arithmetic Chow ring.

However, for hermitian vector sheaves, since the corresponding characteristic form associated with an exact sequence in complex geometry usually gives the classical Bott-Chern secondary characteristic form, and the Chern characteristic class gives the natural isomorphism between the algebraic K-group and the algebraic Chow group, thus it is quite natural to define the hermitian K-theory as the quotient group of the free abelian group generated by hermitian vector sheaves and smooth (p,p) forms by a subgroup generated by exact sequences and corresponding classical Bott-Chern secondary characteristic forms. We prove that the arithmetic K-group also provides a λ-ring structure which is isomorphic to the arithmetic Chow ring, but with $\mathbf{Q}$ coefficients. If we only consider complex manifolds, all of this may be thought as a refined version of the corresponding results at the level of differentials. Essentially, this chapter comes from [GS 91b].

§II.3.1 Arithmetic K-Groups

Let X be an arithmetic variety over an arithmetic ring $A = (A, \Sigma, F_\infty)$. A hermitian vector sheaf on X is a pair $(\mathcal{E}, \rho)$ where $\mathcal{E}$ is a vector sheaf on X, and ρ is an F_∞-invariant hermitian metric on the pull-back vector sheaf of $\mathcal{E}$ over $X(\mathbf{C})$. Then we define the **arithmetic K-group** $K_{\mathrm{Ar}}(X)$ as the group of the free abelian group generated by $((\mathcal{E},\rho),\eta)$, where $(\mathcal{E},\rho)$ is a hermitian vector sheaf on X and $\eta \in \tilde{A}(X_{\mathbf{R}})$ is an F_∞-invariant C^∞ form on $X(\mathbf{C})$ modulo the subgroup generated by the following relations: For any short exact sequence of vector sheaves on X,

$$\mathcal{E}.: \quad 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

let ρ_i be F_∞-invariant hermitian metrics on the pull-back of $\mathcal{E}_i$ over $X(\mathbf{C})$, then

$$((\mathcal{E}_1,\rho_1),\eta_1) + ((\mathcal{E}_3,\rho_3),\eta_3) = ((\mathcal{E}_2,\rho_2), -\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho_1,\rho_2,\rho_3) + \eta_1 + \eta_3).$$

Here $\mathrm{ch}_{BC}(\mathcal{E}, \rho_1, \rho_2, \rho_3)$ denotes the classical Bott-Chern secondary characteristic form associated with the hermitian vector sheaves complex on $X(\mathbf{C})$ corresponding to the exact sequence $\mathcal{E}.$ on X with respect to ch.

The aim of this chapter is to define an arithmetic Chern characteristic class

$$\mathrm{ch}_{\mathrm{Ar}} : K_{\mathrm{Ar}}(X) \to \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}}$$

and to introduce a λ-ring structure on $K_{\mathrm{Ar}}(X)$ such that $\mathrm{ch}_{\mathrm{Ar}}$ is a ring-isomorphism. To do so, we first need to introduce the concept of arithmetic characteristic classes for a hermitian vector sheaf on X, which will be constructed by using certain axioms and the universal property of Grassmannians.

As an example, recall that at the end of the last chapter, we introduced the first arithmetic Chern class for a hermitian line sheaf: Let $(\mathcal{L}, \rho)$ be a hermitian vector sheaf on an arithmetic variety X. Then for any non zero section s of $\mathcal{L}$, by the Poincaré-Lelong equation, we know that $(\mathrm{div}(s), -[\log|s|_\rho^2])$ is an element of $\mathrm{CH}^1_{\mathrm{Ar}}(X)$. Define $c_{\mathrm{Ar},1}(\mathcal{L}, \rho)$ as the class of this element in the arithmetic Chow ring. In general, by the universal properties of Grassmannians, we may split any vector sheaf. Thus, by functorial properties, and hence define the arithmetic characteristic classes.

II.3.2 Axioms For Arithmetic Characteristic Classes

Let B be a subring of real number field $\mathbf{R}$, and let $\phi \in B[[T_1, \ldots, T_n]]$ be a symmetric power series. The arithmetic characteristic class associated with ϕ will satisfy the following axioms:

To every hermitian vector sheaf $(\mathcal{E}, \rho)$ of rank n on X, there exists an arithmetic characteristic class

$$\phi_{\mathrm{Ar}}(\mathcal{E}, \rho) \in \mathrm{CH}_{\mathrm{Ar}}(X)_B$$

such that

(1) **Functoriality**: For any morphism $f : Y \to X$ of arithmetic varieties,

$$f^*(\phi_{\mathrm{Ar}}(\mathcal{E}, \rho)) = \phi_{\mathrm{Ar}}(f^*\mathcal{E}, f^*\rho).$$

(2) **Summation Rule**: If $(\mathcal{E}, \rho) = (\mathcal{L}_1, \rho_1) \oplus \ldots \oplus (\mathcal{L}_n, \rho_n)$ is an orthogonal direct sum of hermitian line sheaves,

$$\phi_{\mathrm{Ar}}(\mathcal{E}, \rho) = \phi(c_{\mathrm{Ar},1}(\mathcal{L}_1, \rho_1), \ldots, c_{\mathrm{Ar},1}(\mathcal{L}_n, \rho_n)).$$

(3) **Product Rule**: Let ϕ_i be defined by

$$\phi(T_1 + T, \ldots, T_n + T) = \sum_i \phi_i(T_1, \ldots, T_n) T^i.$$

Suppose that $(\mathcal{L}, \tau)$ is a hermitian line sheaf on X, then

$$\phi_{\mathrm{Ar}}(\mathcal{E} \otimes \mathcal{L}, \rho \otimes \tau) = \sum_i \phi_{\mathrm{i,Ar}}(\mathcal{E}, \rho) c_{\mathrm{Ar},1}(\mathcal{L}, \tau)^i.$$

(4) **Forgetful Rule**: In $\tilde{A}(X_{\mathbf{R}})$, we have

$$\omega(\phi_{\mathrm{Ar}}(\mathcal{E}, \rho)) = \phi(\mathcal{E}_\infty, \rho).$$

The first main result in this chapter is

Theorem. With the same notation as above, there is a unique way to define $\phi_{\mathrm{Ar}}(\mathcal{E}, \rho)$ which satisfies axioms 1-4.

There are several methods to prove this theorem, all of which have their roots in algebraic geometry. But basically, to do this, we need a kind of splitting process. Just at this stage, we may have its diversity: we may use the Grassmannians, or we may use the projective bundles. For the first, we need to take the pull back; while for the second, we need to consider the push out. Here we shall only discuss the first method, as it is more in keeping with the axioms. We will give a brief discussion later of the second method in terms of arithmetic Segre classes.

The proof will be given in the following several sections. The basic idea is as follows: Let $f : X \to Y$ be a map of complex manifolds, and $\mathcal{E}$ a vector sheaf on Y. Then the pull back $f^*\mathcal{E}$ is a vector sheaf over X. As a consequence, if the Chern classes of $\mathcal{E}$ vanish, so do those of $f^*\mathcal{E}$ by the functoriality. Thus we may think of the Chern classes as a measure of the twisting of a vector sheaf, and we show that the pull back "dilutes" a vector sheaf, i.e. makes it less twisted. One extreme example is when f is constant, in which case, $f^*\mathcal{E}$ is trivial. Another example is the flag construction Flag($\mathcal{E}$) of $\mathcal{E}$. (In this case, the pull back of $\mathcal{E}$ on Flag($\mathcal{E}$) splits as a direct sum of line sheaves.) Now naturally, one may ask if there exists a vector sheaf that is so twisted that every vector sheaf is a pull back of this universal vector sheaf. Such a vector sheaf does exist, at least for manifolds of finite type: it is the universal quotient vector sheaf on the Grassmannian.

From this later discussion, we may first define the arithmetic characteristic classes for the universal quotient bundle on the Grassmannians. Then we use the universal properties of this data to deal with the general situation by the pull back. This is quite natural following Axiom 1. Finally, we have to prove that the definition does not depend on the data we use in the construction.

§II.3.3. The Construction On Grassmannians

II.3.3.a. The Definition Of The Grassmannian In The Complex Case.

Let V be a complex vector space of dimension $m+n$. The Grassmannian $G(n,V)$ is the set of n-dimensional linear subspaces of V. We write $G_{m,n}$ for $G(n,V)$; obviously, $G_{m,n}$ is a complex manifold. By the Plücker imbedding, we know that in fact $G_{m,n}$ is a projective complex manifold.

Let $\mathbf{C}^{m+n} \times G_{m,n}$ denote the trivial vector bundle of rank $m+n$ over $G_{m,n}$. We define the **universal subbundle** $S \to G_{m,n}$, whose fiber at each point $\Lambda \in G_{m,n}$ is just the subspace $\Lambda \subset V$. Then S is clearly a holomorphic subbundle of $\mathbf{C}^{m+n} \times G_{m,n}$. The quotient $Q := (\mathbf{C}^{m+n} \times G_{m,n})/S$ is also a vector bundle, which is called the **universal quotient bundle** of $G_{m,n}$.

Proposition. Let $\mathcal{E}$ be a vector sheaf of rank n over a complex manifold X. Suppose there are $n+m$ global sections of $\mathcal{E}$ which span the fiber at each point. Then there is a map

$$f : X \to G_n(\mathbf{C}^{n+m}),$$

such that $\mathcal{E}$ is the pull back of the universal quotient bundle Q via f, i.e. $\mathcal{E} = f^*Q$.

Proof. Indeed, if $s_1, \ldots, s_{n+m}$ are n spanning global sections of $\mathcal{E}$, we let V be the complex vector space with basis $s_1, \ldots, s_{n+m}$. Then, for each point $x \in X$, the evaluation map

$$\mathrm{ev}_x : V \to \mathcal{E}_x$$

is surjective. Hence $\mathrm{Ker}\,\mathrm{ev}_x$ is a codimension-k subspace of V, and the fiber of the universal quotient bundle Q at the point $\mathrm{Ker}\,\mathrm{ev}_x$ of the Grassmannian $G_k(V)$ is $V/\mathrm{Ker}\,\mathrm{ev}_x = \mathcal{E}_x$. If the map $f : X \to G_k(V)$ is defined by

$$f : x \mapsto \mathrm{Ker}\,\mathrm{ev}_x,$$

then the quotient bundle Q is a pull back to $\mathcal{E}$.

II.3.3.b. The Algebraic Aspect Of The Grassmannians

Let A be an arithmetic ring. For any two positive integers m,n, let $G = G_{m,n} := \mathrm{Grass}_n(\mathcal{O}^{m+n}_{\mathrm{Spec}(A)})$ be the Grassmannian over $\mathrm{Spec}(A)$ representing the functor which associates with each $\mathrm{Spec}(A)$-scheme T the set of rank n locally free quotients of $\mathcal{O}_T^{m+n}$.

We consider some properties of G. Assume $m = qn$ with $q \geq 1$. Let $P := (G_{q,1})^n$ and $\mu : P \to G$ be the map given by the direct sum. There is a natural action of the symmetric group S_n on P by permuting the factors.

Lemma. 1. Assume $p \leq q$, then μ induces an isomorphism

$$\mu^* : \mathrm{CH}^p(G)_{\mathbf{Q}} \to \mathrm{CH}^p(P)^{S_n}_{\mathbf{Q}}.$$

2. If we endow the natural $U(m+n)$-invariant metric on $G_\sigma(\mathbf{C})$, then as Arakelov varieties, we have a natural isomorphism

$$\mu^* : \mathrm{CH}^p_{\mathrm{Ara}}(G)_{\mathbf{Q}} \to \mathrm{CH}^p_{\mathrm{Ara}}(P)^{S_n}_{\mathbf{Q}}.$$

Proof. We may assume that A is $\mathbf{Q}$ by the universal coefficient principle. In fact, for any Grassmannian G over $\mathbf{Z}$ with $\dim G > 0$, there exist Grassmannians G' and G'' such that $\dim G' < \dim G > \dim G''$ and there is a closed immersion $G' \subset G$ such that $G - G'$ is an affine bundle over G''. Thus we may use induction on the dimension to prove that, for any product of Grassmannians X, the morphism

$$\mathrm{CH}(X_{\mathbf{Z}}) \to \mathrm{CH}(X_{\mathbf{Q}})$$

is an isomorphism; that the canonical morphism

$$\mathrm{CH}(\mathrm{Spec}(A))_{\mathbf{Q}} \otimes \mathrm{CH}(X_{\mathbf{Z}})_{\mathbf{Q}} \to \mathrm{CH}(X_{\mathbf{Z}} \otimes_{\mathbf{Z}} A)_{\mathbf{Q}}$$

is an isomorphism, and that the canonical morphism

$$\oplus_{p\geq 1}\mathrm{CH}^{p,p-1}(\mathrm{Spec}(A))_{\mathbf{Q}} \otimes \mathrm{CH}(X_{\mathbf{Z}}) \to \oplus_{p\geq 1}\mathrm{CH}^{p,p-1}(X_{\mathbf{Z}} \times_{\mathbf{Z}} A)_{\mathbf{Q}}$$

is an epimorphism. Therefore, we may assume that A is $\mathbf{Q}$.

The first assertion is a consequence of a direct calculation. Let $\mathcal{L}_\alpha$ be the line bundle on P defined by the pull-back of the universal quotient bundle on $G_{q,1}$ by the α-th projection, $x_\alpha := c_1(\mathcal{L}_\alpha) \in \mathrm{CH}^1(P)$ and let c_i be the i-th Chern class of the universal quotient bundle of rank n on G, then

$$\mathrm{CH}(P) = \mathbf{Z}[x_1, \ldots, x_n]/(x_1^{q+1}, \ldots, x_n^{q+1})$$

and

$$\mathrm{CH}(G) = \mathbf{Z}[c_1, \ldots, c_n]/\mathcal{I},$$

where the ideal $\mathcal{I}$ is generated by elements of degree greater than q. Now by the fact that $\mu^*(c_i)$ is the i-th elementary symmetric function of the x_α's, we have the first assertion.

For the second assertion, we use the five lemma applied to the exact sequence associated with the Arakelov Chow groups.

II.3.3.c. **The Construction of $\phi_{\mathrm{Ar}}(Q_{m,n}, \rho_{m,n})$.**

Let $Q_{m,n}$ be the universal quotient bundle on $G = G_{m,n}$, endowed with a $U(m+n)$-invariant hermitian metric. Suppose that $m = q\,n$ and $q \geq \deg\phi$. Let $\mu : P = (G_{q,1})^n \to G$ be the direct sum morphism. Then the pull-back of $(Q_{m,n}, \rho_{m,n})$ by μ splits as an orthogonal direct sum

$$\mu^*(Q_{m,n}, \rho_{m,n}) = (L_1, \rho_1) \oplus \ldots \oplus (L_n, \rho_n).$$

Since $\omega(c_{\mathrm{Ar},1}(L_\alpha,\rho_\alpha)) = c_1(L_{\alpha\infty},\rho_\alpha)$ is S_n-invariant, $\phi(c_{\mathrm{Ar},1}(L_1,\rho_1),\ldots,c_{\mathrm{Ar},1}(L_n,\rho_n))$ of $\mathrm{CH}_{\mathrm{Ara}}(P)_B$ lies in $\mathrm{CH}_{\mathrm{Ara}}(P)_B^{S_n}$. By base change, we may assume that A is $\mathbf{Z}$. Note that since the degree of this element is at most q, therefore by Lemma b, there exists a unique class,

$$\phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n}) \in \mathrm{CH}_{\mathrm{Ara}}(G_{m,n})_B,$$

such that

$$\mu^*(\phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n})) = \phi(c_{\mathrm{Ar},1}(L_1,\rho_1),\ldots,c_{\mathrm{Ar},1}(L_n,\rho_n)).$$

So we have a construction of $\phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n})$.

Before going further, let us consider some properties of $\phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n})$.

Properties. (1) Let $i : G_{m,n} \hookrightarrow G_{m+n,n}$ be the canonical inclusion. Then

$$i^*(\phi_{\mathrm{Ar}}(Q_{m+n,n},\rho_{m+n,n})) = \phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n}).$$

(2) Let $\mu : G_{m_1,n_1} \times G_{m_2,n_2} \to G_{m,n}$ be the direct-sum map, where $m_1 = qn_1,\ m_2 = qn_2,\ m = m_1 + m_2$, and $n = n_1 + n_2$. Define $\phi_\alpha \varphi_\alpha$ by

$$\phi(T_1,\ldots,T_n) = \sum_\alpha \phi_\alpha(T_1,\ldots,T_{n_1})\varphi_\alpha(T_{n_1+1},\ldots,T_n).$$

We have

$$\mu^*(\phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n})) = \sum_\alpha \phi_{\alpha,\mathrm{Ar}}(Q_{m_1,n_1},\rho_{m_1,n_1})\varphi_{\alpha,\mathrm{Ar}}(Q_{m_2,n_2},\rho_{m_2,n_2}).$$

(3) Let $m'' = mm' + nm' + n$ and let $\nu : G_{m,n} \times G_{m',1} \to G_{m'',n}$ be the map induced by the tensor product. Then

$$\nu^*(\phi_{\mathrm{Ar}}(Q_{m'',n})) = \sum_{i\geq 0} \phi_{i,\mathrm{Ar}}(Q_{m,n},\rho_{m,n})c_{\mathrm{Ar},1}(Q_{m',1},\rho_{m',1})^i.$$

(4) Let $g \in \mathrm{GL}_{m+n}(A)$. Then

$$g^*\phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n}) = \phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n}) + \phi_{\mathrm{BC}}(Q_{m,n};\rho_{m,n},g^*\rho_{m,n}).$$

Proof. The first three properties come from the following commutative diagrams:

$$\begin{array}{ccc} (G_{q,1})^n & \to & (G_{q+1,1})^n \\ \mu\downarrow & & \downarrow\mu \\ G_{m,n} & \to & G_{m+n,n}; \end{array}$$

$$\begin{array}{ccc} (G_{q,1})^{n_1} \times (G_{q,1})^{n_2} & \xrightarrow{\mathrm{Id}} & (G_{q,1})^n \\ \mu\times\mu\downarrow & & \downarrow\mu \\ G_{m_1,n_1} \times G_{m_2,n_2} & \xrightarrow{\mu} & G_{m,n}; \end{array}$$

and

$$\begin{array}{ccc} (G_{q,1})^n \times G_{m',1} & \xrightarrow{\tilde{\nu}} & (G_{r,1})^n \\ \mu \times 1 \downarrow & & \downarrow \mu \\ G_{m,n} \times G_{m',1} & \xrightarrow{\nu} & G_{m'',n}. \end{array}$$

Here we let $r := qm' + n' + 1$. vskip 0.30cm For (4), let

$$c(g) := -g^* \phi_{\mathrm{Ar}}(Q_{m,n}, \rho_{m,n}) + \phi_{\mathrm{Ar}}(Q_{m,n}, \rho_{m,n}) + \phi_{\mathrm{BC}}(Q_{m,n}; \rho_{m,n}, g^* \rho_{m,n}).$$

Since g acts trivially on $\mathrm{CH}(G_{m,n})$, we see that $z(c(g)) \doteq 0$ and $\omega(c(g)) = 0$. Thus $c(g)$ lies in the image of $\oplus_p H^{p,p}((G_{m,n})_{\mathbf{R}})$. On the other hand, for any two elements g_1, g_2, we have

$$c(g_1 g_2) = c(g_1) + g_1^* c(g_2).$$

Since the action of GL on the cohomology is trivial, we have

$$c(g_1 g_2) = c(g_1) + c(g_2).$$

But c is trivial on the commutators, so we have $c = 0$.

II.3.3.d. The Construction In General.

Let X be an arithmetic variety over an arithmetic ring A. Let $(\mathcal{E}, \rho)$ be a hermitian vector sheaf on X. Since X is quasi-projective over $\mathrm{Spec}(A)$, there are ample line sheaves on X. In particular, there exists a line sheaf $\mathcal{L}$ on X such that $\mathcal{F} := \mathcal{E} \otimes \mathcal{L}$ is spanned by its global sections. Let $\{ : \mathcal{O}_X^{m+n} \to \mathcal{F}$ be an epimorphism with $m = qn, q \geq \deg(\phi)$. By Proposition a for $G_{m,n}$ and $Q_{m,n}$, we know that there exist a morphism

$$f : X \to G_{m,n}$$

over A and an isomorphism

$$f^*(Q_{m,n}) \simeq \mathcal{F}.$$

Choose an arbitrary metric τ on $\mathcal{L}$ and let ρ' be the metric coming from the isomorphism $\mathcal{E} \simeq f^*(Q_{m,n}) \otimes \mathcal{L}$. We define

$$\phi_{\mathrm{Ar},\mathcal{L},\tau,\{}(\mathcal{E}, \rho) := \sum_{i \geq 0} f^*(\phi_{i,Ar}(Q_{m,n}, \rho_{m,n})) c_{\mathrm{Ar},1}(\mathcal{L}, \tau)^i + \phi_{\mathrm{BC}}(\mathcal{E}_\infty, \rho', \rho).$$

In the following, we prove that the above construction does not depend on the choice of $(\mathcal{L}, \tau)$ and $\{$. In fact, this is a consequence of the properties of $\phi_{\mathrm{Ar}}(Q_{m,n}, \rho_{m,n})$ listed in subsection c.

§II.3.4. The Independence Of The Construction

II.3.4.a. **The Independence of** $(\mathcal{L}, \rho)$.

First note that since the Picard group of X is generated by these line sheaves which are spanned by their global sections, it is sufficient to check the assertion for them.

Let $\mathcal{L}'$ be another line sheaf on X and let $\xi' : \mathcal{O}_X^{m'+1} \to \mathcal{L} \otimes (\mathcal{L}')^{-1}$ be an epimorphism. By definition, there are the morphism $f' : X \to G_{m',1}$ and an isomorphism $\mathcal{L}' \simeq (f')^*(Q_{m',1})^{-1} \otimes \mathcal{L}$. Choose a metric on $\mathcal{L}'$ such that the above isomorphism is an isometry. Let $\mathcal{F}' = \mathcal{F} \otimes \mathcal{L} \otimes (\mathcal{L}')^{-1}$. Then $\xi \otimes \xi'$ dominates $\mathcal{F}'$ and defines an isomorphism

$$\mathcal{F}' \simeq (f \otimes f')^*(Q_{m'',n}).$$

Here $f \otimes f'$ is the composition of the morphism ν defined in the proof of Property (3) in subsection c with the direct product $f \times f'$. Choose the metric on $\mathcal{F}'$ so that the above isomorphism is an isometry. Then, by a similar process, we have the construction for $\phi_{\mathrm{Ar},\mathcal{L}',\tau',\xi\otimes\xi'}(\mathcal{E}, \rho)$. Next we prove that

$$\phi_{\mathrm{Ar},\mathcal{L}',\tau',\xi\otimes\xi'}(\mathcal{E}, \rho) = \phi_{\mathrm{Ar},\mathcal{L},\tau,\xi}(\mathcal{E}, \rho).$$

Since $\mathcal{F} \otimes \mathcal{L} \simeq \mathcal{F}' \otimes \mathcal{L}'$ is also an isometry for the associated metrics, we may assume that $\mathcal{E} = \mathcal{F} \otimes \mathcal{L}$ even with metrics. Thus, we have

$$\phi_{\mathrm{Ar},\mathcal{L}',\tau',\xi\otimes\xi'}(\mathcal{E}, \rho) = \sum_{i\geq 0}((\xi \otimes \xi')^*(\phi_{i,\mathrm{Ar}}(Q_{m'',n}, \rho_{m'',n}))(c_{\mathrm{Ar},1}(\mathcal{L}', \tau'))^i.$$

Thus, by axiom 3, which may be checked independently, we have

$$\phi_{\mathrm{Ar},\mathcal{L}',\tau',\xi\otimes\xi'}(\mathcal{E}, \rho) = \sum_{i,j\geq 0} f^*(\phi_{ij\,\mathrm{Ar}}(Q_{m,n}, \rho_{m,n}))(f'^*(c_{\mathrm{Ar},1}(Q_{m',1}, \rho_{m',1})))^j c_{\mathrm{Ar},1}(\mathcal{L}', \rho')^i.$$

Here ϕ_{ij} is defined by

$$\phi_i(T_1 + T, \ldots, T_n + T) =: \sum_{j\geq 0} \phi_{ij}(T_1, \ldots, T_n)T^j.$$

Thus we have,

$$\begin{aligned} \phi(T_1 + T + U, \ldots, T_n + T + U) &= \sum_i \phi_i(T_1, \ldots, T_n)(T + U)^i \\ &= \sum_i \phi_i(T_1 + T, \ldots, T_n + T)U^i \\ &= \sum_{i,j\geq 0} \phi_{ij}(T_1, \ldots, T_n)T^j U^i. \end{aligned}$$

But

$$c_{\mathrm{Ar},1}(\mathcal{L},\tau)=c_{\mathrm{Ar},1}(\mathcal{L}',\tau')+f'^{*}(c_{\mathrm{Ar},1}(Q_{m',1},\rho_{m',1})).$$

It follows that

$$\phi_{\mathrm{Ar},\mathcal{L}',\tau',\{\otimes\{'}(\mathcal{E},\rho)=\sum_i f^{*}\phi_{i\,\mathrm{Ar}}(Q_{m,n},\rho_{m,n})(c_{\mathrm{Ar},1}(\mathcal{L},\tau))^i=\phi_{\mathrm{Ar},\mathcal{L},\tau,\{}(\mathcal{E},\rho).$$

Finally the dependence of the metric τ may be checked directly.

II.3.4.b. **The Independence of $\{$.**

With the same notation as above, let $\{' : \mathcal{O}_X^{m'+n}\to\mathcal{F}$ be an epimorphism with $m'=q'n, q'\geq \deg\phi$. Then we have a morphism $f' : X\to G_{m',n}$ and an isomorphism $\mathcal{F}\simeq (f')^{*}(Q_{m',n})$. We show that

$$\phi_{\mathrm{Ar},\mathcal{L},\{}(\mathcal{E},\rho)=\phi_{\mathrm{Ar},\mathcal{L},\{'}(\mathcal{E},\rho).$$

First, we have

$$\begin{aligned}&\sum_{i\geq 0}f^{*}(\phi_{i,\mathrm{Ar}}(Q_{m,n},\rho_{m,n}))c_{\mathrm{Ar},1}(\mathcal{L},\tau)^i-\phi_{\mathrm{BC}}((\mathcal{E},\rho)\to f^{*}(Q_{m,n},\rho_{m,n})\otimes(\mathcal{L},\tau))\\&=\sum_{i\geq 0}f'^{*}(\phi_{i,\mathrm{Ar}}(Q_{m',n},\rho_{m',n}))c_{\mathrm{Ar},1}(\mathcal{L},\tau)^i-\phi_{\mathrm{BC}}((\mathcal{E},\rho)\to f'^{*}(Q_{m',n},\rho_{m',n})\otimes(\mathcal{L},\tau)).\end{aligned}$$

By the result about classical Bott-Chern secondary characteristic forms in I.1.4, we know that

$$\begin{aligned}&\phi_{\mathrm{BC}}((\mathcal{E},\rho)\to f^{*}(Q_{m,n},\rho_{m,n})\otimes(\mathcal{L},\tau))-\phi_{\mathrm{BC}}((\mathcal{E},\rho)\to f^{*}(Q_{m,n},\rho_{m,n})\otimes(\mathcal{L},\tau))\\&=\phi_{\mathrm{BC}}(f'^{*}(Q_{m',n},\rho_{m',n})\otimes(\mathcal{L},\tau)\to f^{*}(Q_{m,n},\rho_{m,n})\otimes(\mathcal{L},\tau))\\&=\sum_{i\geq 0}\phi_{\mathrm{BC}}(f'^{*}(Q_{m',n},\rho_{m',n})\to f^{*}(Q_{m,n},\rho_{m,n}))c_1(\mathcal{L},\tau)^i.\end{aligned}$$

Hence it is enough to prove

$$f^{*}(\phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n}))=f'^{*}(\phi_{\mathrm{Ar}}(Q_{m',n},\rho_{m',n}))+\phi_{\mathrm{BC}}(f^{*}(Q_{m,n},\rho_{m,n})\to f'^{*}(Q_{m',n},\rho_{m',n})).$$

To do this, we use properties 1 and 4 for $\phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n})$ in subsection c: Since $\mathcal{O}_X^{m+n}$ and $\mathcal{O}_X^{m'+n}$ are free, we may choose morphisms

$$\alpha:\mathcal{O}_X^{m+n}\to\mathcal{O}_X^{m'+n},\quad \beta:\mathcal{O}_X^{m'+n}\to\mathcal{O}_X^{m+n}$$

such that $\{=\{'\circ\alpha$ and $\{'=\{\circ\beta$. Thus on $\mathcal{O}_X^{m+n}\oplus\mathcal{O}_X^{m'+n}$, the automorphism

$$g=\begin{pmatrix}1-\beta\alpha & \beta\\ -\alpha & 1\end{pmatrix}$$

and the composition of the projective morphisms

$$\zeta : \mathcal{O}_X^{m+n} \oplus \mathcal{O}_X^{m'+n} \to \mathcal{O}_X^{m+n} \xrightarrow{\zeta} \mathcal{F}$$

$$\zeta' : \mathcal{O}_X^{m+n} \oplus \mathcal{O}_X^{m'+n} \to \mathcal{O}_X^{m'+n} \xrightarrow{\zeta'} \mathcal{F}$$

satisfy the equality

$$\zeta \circ g = \zeta'.$$

Let $m'' = m + m' + n$, then we have

$$\begin{array}{ccc} G_{m,n} & \to & G_{m'',n} \\ f\uparrow & h\nearrow & | \\ X & & g \\ f'\downarrow & h'\searrow & \downarrow \\ G_{m',n} & \to & G_{m'',n}. \end{array}$$

Therefore, we have

$$\begin{aligned} & f^*(\phi_{\mathrm{Ar}}(Q_{m,n}, \rho_{m,n})) - f'^*(\phi_{\mathrm{Ar}}(Q_{m',n}, \rho_{m',n})) \\ &= h^* f^*(\phi_{\mathrm{Ar}}(Q_{m'',n}, \rho_{m'',n})) - h'^* f'^*(\phi_{\mathrm{Ar}}(Q_{m'',n}, \rho_{m'',n})) \\ &= h^*(\phi_{\mathrm{BC}}(g)) \\ &= \phi_{\mathrm{BC}}(f^*(Q_{m,n}, \rho_{m,n}) \to f'^*(Q_{m',n}, \rho_{m',n})). \end{aligned}$$

So far, we have already shown that the arithmetic characteristic class associated with ϕ defined in subsection 3.d does not depend on the various data used in the definition. Hence, we may denote it by $\phi_{\mathrm{Ar}}(\mathcal{E}, \rho)$.

We next check the axioms step by step.

§II.3.5. Checking The Axioms

II.3.5.a. The Functorial Property.

Let $\varphi : Y \to X$ be a morphism of arithmetic varieties and let $(\mathcal{E}, \rho)$ be a hermitian vector sheaf on X. With the same notation as in the definition, we have

$$\begin{aligned} & \phi_{\mathrm{Ar}}(\varphi^*(\mathcal{E}, \rho)) \\ &= \sum_{i \geq 0} (f \circ \varphi)^*(\phi_{i,\mathrm{Ar}}(Q_{m,n}, \rho_{m,n})) c_{\mathrm{Ar},1}(\varphi^*(\mathcal{L}, \tau))^i \\ &\quad - \phi_{\mathrm{BC}}(\varphi^*(\mathcal{E}, \rho) \to \varphi^*(\mathcal{F}, f^*\rho_{m,n}) \otimes \varphi^*(\mathcal{L}, \tau)). \end{aligned}$$

Thus the result now follows from the functorial properties of $c_{\mathrm{Ar},1}$ and ϕ_{BC}.

II.3.5.b. **The Additive Rule.**

Let $(\mathcal{E},\rho) = (\mathcal{L}_1,\tau_1) \oplus \ldots \oplus (\mathcal{L}_n,\tau_n)$. Choose a hermitian line sheaf $(\mathcal{L},\tau)$ such that for every $j = 1,\ldots,n$, $\mathcal{L}_j \otimes \mathcal{L}^{-1}$ is spanned by its global sections. Choose epimorphisms $f_j : \mathcal{O}_X^{q+1} \to \mathcal{L}_j \otimes \mathcal{L}$. Then

$$\mathcal{L}_j \otimes \mathcal{L}^{-1} \simeq f_j^*(Q_{q,1}).$$

So $\mathcal{F} := \oplus_j \mathcal{L}_j \otimes \mathcal{L}$ is classified by $f = \mu \circ (f_j)$, where

$$\mu : (G_{q,1})^n \to G_{m,n}$$

is the direct sum morphism. Taking the associated metrics, we have

$$\begin{aligned}&\phi_{\mathrm{Ar}}(\mathcal{E},\rho)\\ &\quad= \sum_{i\geq 0} f^*(\phi_{i,\mathrm{Ar}}(Q_{m,n},\rho_{m,n}))c_{\mathrm{Ar},1}(\mathcal{L},\tau)^i - \phi_{\mathrm{BC}}((\mathcal{E},\rho) \to (\mathcal{F}, f^*\rho_{m,n}) \otimes (\mathcal{L},\tau)).\end{aligned}$$

But

$$\begin{aligned}f^*\phi_{i,\mathrm{Ar}}(Q_{m,n},\rho_{m,n}) &= (f_j^*)\mu^*\phi_{i,\mathrm{Ar}}(Q_{m,n},\rho_{m,n})\\ &= (f_j^*)\phi_i(c_{\mathrm{Ar},1}(Q_{q,1},\rho_{q,1}),\ldots,c_{\mathrm{Ar},1}(Q_{q,1},\rho_{q,1}))\\ &= \phi_i(c_{\mathrm{Ar},1}(\mathcal{L}_1 \otimes \mathcal{L}^{-1}, f_1^*\rho_{q,1}),\ldots,c_{\mathrm{Ar},1}(\mathcal{L}_n \otimes \mathcal{L}^{-1}, f_n^*\rho_{q,1})).\end{aligned}$$

By definition,

$$\begin{aligned}&\sum_{i\geq 0}\phi_i\big(c_{\mathrm{Ar},1}(\mathcal{L}_1 \otimes \mathcal{L}^{-1}, f_1^*\rho_{q,1}),\ldots,c_{\mathrm{Ar},1}(\mathcal{L}_n \otimes \mathcal{L}^{-1}, f_n^*\rho_{q,1})\big)c_{\mathrm{Ar},1}(\mathcal{L},\tau)^i\\ &\quad= \phi\big(c_{\mathrm{Ar},1}(\mathcal{L}_1 \otimes \mathcal{L}^{-1}, f_1^*\rho_{q,1}) + c_{\mathrm{Ar},1}(\mathcal{L},\tau),\ldots,c_{\mathrm{Ar},1}(\mathcal{L}_n \otimes \mathcal{L}^{-1}, f_n^*\rho_{q,1}) + c_{\mathrm{Ar},1}(\mathcal{L},\tau)\big).\end{aligned}$$

Note that since

$$\begin{aligned}&c_{\mathrm{Ar},1}(\mathcal{L}_j \otimes \mathcal{L}^{-1}, f_j^*\rho_{q,1}) + c_{\mathrm{Ar},1}(\mathcal{L},\tau)\\ &\quad= c_{\mathrm{Ar},1}\big((\mathcal{L}_j \otimes \mathcal{L}^{-1}) \otimes \mathcal{L}, f_j^*\rho_{q,1} \otimes \tau\big)\\ &\quad= c_{\mathrm{Ar},1}(\mathcal{L}_j,\tau_j) + c_{1,\mathrm{BC}}\big((\mathcal{L}_j,\tau_j) \to ((\mathcal{L}_j \otimes \mathcal{L}^{-1}) \otimes \mathcal{L}, f_j^*\rho_{q,1} \otimes \tau)\big),\end{aligned}$$

we have

$$\begin{aligned}&\phi(c_{\mathrm{Ar},1}\big(\mathcal{L}_1 \otimes \mathcal{L}^{-1}, f_1^*\rho_{q,1}),\ldots,c_{\mathrm{Ar},1}(\mathcal{L}_n \otimes \mathcal{L}^{-1}, f_n^*\rho_{q,1})\big)\\ &= \phi\big(c_{\mathrm{Ar},1}(\mathcal{L}_1,\tau_1),\ldots,c_{\mathrm{Ar},1}(\mathcal{L}_n,\tau_n)\big)\\ &\quad + \sum_j \phi\Big(c_1(\mathcal{L}_1,\tau_1),\ldots,c_1(\mathcal{L}_{j-1},\tau_{j-1}), c_{1,\mathrm{BC}}\big((\mathcal{L}_j,\tau_j) \to ((\mathcal{L}_j \otimes \mathcal{L}^{-1}) \otimes \mathcal{L}, f_j^*\rho_{q,1} \otimes \tau)\big),\\ &c_1\big((\mathcal{L}_{j+1} \otimes \mathcal{L}^{-1}, f_{j+1}^*\rho_{q,1}) \otimes (\mathcal{L},\tau)\big),\ldots,c_1\big((\mathcal{L}_n \otimes \mathcal{L}^{-1}, f_n^*\rho_{q,1}) \otimes (\mathcal{L},\tau)\big)\Big).\end{aligned}$$

The properties of the classical Bott-Chern secondary characteristic classes in I.1.4 again show that

$$\begin{aligned}&\phi\big(c_{\mathrm{Ar},1}(\mathcal{L}_1\otimes\mathcal{L}^{-1},f_1^*\rho_{q,1}),\dots,c_{\mathrm{Ar},1}(\mathcal{L}_n\otimes\mathcal{L}^{-1},f_n^*\rho_{q,1})\big)\\&=\phi\big(c_{\mathrm{Ar},1}(\mathcal{L}_1,\tau_1),\dots,c_{\mathrm{Ar},1}(\mathcal{L}_n,\tau_n)\big)\\&\quad+\phi_{\mathrm{BC}}\big((\mathcal{E},\rho)\to(\mathcal{F}\otimes\mathcal{L},\oplus_j f_j^*\rho_{q,1}\otimes\tau)\big).\end{aligned}$$

Therefore, we have

$$\phi_{\mathrm{Ar}}(\mathcal{E},\rho)=\phi\big(c_{\mathrm{Ar},1}(\mathcal{L}_1,\tau_1),\dots,c_{\mathrm{Ar},1}(\mathcal{L}_n,\tau_n)\big).$$

II.3.5.c. **The Product Rule.**

Let $\mathcal{E}$ and $\mathcal{L}$ be the same as in Axiom 3. We assume first that $\mathcal{F}=\mathcal{E}\otimes\mathcal{L}^{-1}$ is spanned by its global sections. We have

$$\phi_{\mathrm{Ar}}(\mathcal{E},\rho)=\sum_i\phi_{i,\mathrm{Ar}}(\mathcal{F},f^*\rho_{m,n})c_{\mathrm{Ar},1}(\mathcal{L},\tau)^i-\phi_{\mathrm{BC}}\big((\mathcal{E},\rho)\to(\mathcal{F},f^*\rho_{m,n})\otimes(\mathcal{L},\tau)\big).$$

But

$$\phi_{\mathrm{BC}}\big((\mathcal{E},\rho)\to(\mathcal{F},f^*\rho_{m,n})\otimes(\mathcal{L},\tau)\big)=\sum_i\phi_{i,\mathrm{BC}}\big((\mathcal{E},\rho)\otimes(\mathcal{L},\tau)^{-1}\to(\mathcal{F},f^*\rho_{m,n})\big)c_1(\mathcal{L},\tau)^i.$$

Hence, we have

$$\phi_{\mathrm{Ar}}(\mathcal{E},\rho)=\sum_i\phi_{i,\mathrm{Ar}}\big((\mathcal{E},\rho)\otimes(\mathcal{L},\tau)^{-1}\big)c_{\mathrm{Ar},1}(\mathcal{L},\tau)^i$$

as required.

In the general case, we choose a hermitian line sheaf $(\mathcal{L}',\tau')$ such that $\mathcal{E}\otimes\mathcal{L}\otimes\mathcal{L}'^{-1}$ is spanned by its global sections. Now applying the result above to $\mathcal{E}\otimes\mathcal{L}$ and $\mathcal{L}'$, $\mathcal{E}$ and $\mathcal{L}\otimes\mathcal{L}'^{-1}$, respectively, we have

$$\phi_{\mathrm{Ar}}\big((\mathcal{E},\rho)\otimes(\mathcal{L},\tau)\big)=\sum_i\phi_{i,\mathrm{Ar}}\big((\mathcal{E},\rho)\otimes(\mathcal{L},\tau)\otimes(\mathcal{L}',\tau')^{-1}\big)c_{\mathrm{Ar},1}(\mathcal{L}',\tau')^i$$

and

$$\phi_{i,\mathrm{Ar}}\big((\mathcal{E},\rho)\otimes(\mathcal{L},\tau)\otimes(\mathcal{L}',\tau')^{-1}\big)=\sum_j\phi_{ij,\mathrm{Ar}}(\mathcal{E},\rho)\,c_{\mathrm{Ar},1}\big((\mathcal{L},\tau)\otimes(\mathcal{L}',\tau')^{-1}\big)^j.$$

Thus, as in 3.4.a, we have

$$\begin{aligned}&\phi_{\mathrm{Ar}}\big((\mathcal{E},\rho)\otimes(\mathcal{L},\tau)\big)\\&=\sum_i\phi_{i,\mathrm{Ar}}(\mathcal{E},\rho)\Big(c_{\mathrm{Ar},1}\big((\mathcal{L},\tau)\otimes(\mathcal{L}',\tau')^{-1}\big)+c_{\mathrm{Ar},1}(\mathcal{L}',\tau')\Big)^i\\&=\sum_i\phi_{i,\mathrm{Ar}}(\mathcal{E},\rho)c_{\mathrm{Ar},1}(\mathcal{L},\tau)^i.\end{aligned}$$

II.3.5.d. **The Forgetful Rule.**

Since
$$\omega\big(\phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n})\big) = \phi(Q_{m,n},\rho_{m,n}),$$
we have
$$\begin{aligned}\omega(\phi_{\mathrm{Ar}}(\mathcal{E},\rho)) =& \sum_i f^*\phi_i(Q_{m,n},\rho_{m,n})c_1(\mathcal{L},\tau)^i + dd^c\phi_{\mathrm{BC}}(\mathcal{E};\rho,f^*\rho_{m,n}\otimes\tau)\\ =&\phi(f^*(Q_{m,n},\rho_{m,n})\otimes(\mathcal{L},\tau)) + dd^c\phi_{\mathrm{BC}}(\mathcal{E};\rho,f^*\rho_{m,n}\otimes\tau)\\ =&\phi(\mathcal{E},\rho).\end{aligned}$$

II.3.5.e. **Uniqueness.**

Since $\phi_{\mathrm{Ar}}(Q_{m,n},\rho_{m,n})$ is unique, hence the uniqueness is a direct consequence of the following
$$\phi_{\mathrm{Ar}}(\mathcal{E},\rho) - \phi_{\mathrm{Ar}}(\mathcal{E},\rho') = \phi_{\mathrm{BC}}(\mathcal{E},\rho,\rho').$$
This will be proved in establishing some further properties of arithmetic characteristic classes in the next subsection.

II.3.5.f **Properties of Arithmetic Characteristic Classes.**

The most important properties of arithmetic characteristic classes are found in the following

Theorem. (1) $z(\phi_{\mathrm{Ar}}(\mathcal{E},\rho)) = \phi(\mathcal{E}) \in \mathrm{CH}(X)_B$.

(2) Let
$$\mathcal{E}.:\quad 0\to\mathcal{E}_1\to\mathcal{E}_2\to\mathcal{E}_3\to 0$$
be an exact sequence of vector sheaves on X. Put F_∞-invariant hermitian metrics ρ_i on $\mathcal{E}_i$ for $i=1,2,3$. Then
$$\phi_{\mathrm{Ar}}(\mathcal{E}_2,\rho_2) = \phi_{\mathrm{Ar}}(\mathcal{E}_1\oplus\mathcal{E}_3,\rho_1\oplus\rho_3) + a(\phi_{\mathrm{BC}}(\mathcal{E}.,\rho_1,\rho_2,\rho_3)).$$

(3) Let $n=n_1+n_2$ and define ϕ_α, φ_α by
$$\phi(T_1,\dots,T_{n_1},U_1,\dots,U_{n_2}) = \sum_\alpha \phi_\alpha(T_1,\dots,T_{n_1})\varphi_\alpha(U_1,\dots,U_{n_2}).$$

Then
$$\phi_{\mathrm{Ar}}(\mathcal{E}_1\oplus\mathcal{F}_3,\rho_1\oplus\rho_3) = \sum_\alpha \phi_{\alpha,\mathrm{Ar}}(\mathcal{E}_1,\rho_1)\varphi_{\alpha,\mathrm{Ar}}(\mathcal{E}_3,\rho_3).$$

(4) Let $n = n_1 n_2$ and define ϕ_β, φ_β by

$$\phi(T_1 + U_1, T_1 + U_2, \dots, T_{n_1} + U_{n_2}) = \sum_\beta \phi_\beta(T_1, \dots, T_{n_1})\varphi_\beta(U_1, \dots, U_{n_2}).$$

Then

$$\phi_{\mathrm{Ar}}(\mathcal{E}_1 \otimes \mathcal{E}_2, \rho_1 \otimes \rho_2) = \sum_\beta \phi_\beta(\mathcal{E}_1, \rho_1)\varphi_\beta(\mathcal{E}_2, \rho_2).$$

The proofs of (1), (3) and (4) can be deduced from the definitions. For (2), we use the $\mathbf{P}^1$-deformation technique. By the construction of the classical Bott-Chern secondary characteristic form associated with $\mathcal{E}.$ in I.1.2.b, with the same notation, we have

$$i_0^*\phi_{\mathrm{Ar}}(D\mathcal{E}_2, D\rho_2) - i_\infty^*\phi_{\mathrm{Ar}}(D\mathcal{E}_2, D\rho_2) = a(-\int_{\mathbf{P}^1} \omega(\phi_{\mathrm{Ar}}(\tilde{\mathcal{E}}_2, \tilde{\rho}_2))\log|z|^2).$$

Therefore, we have our result by the functorial property .

§II.3.6. Arithmetic Chern Characteristic Classes.

II.3.6.a. Main Results

We prove in this section the fact that $\mathrm{ch}_{\mathrm{Ar}}$ induces an isomorphism between the arithmetic K-group and the arithmetic Chow group, but with $\mathbf{Q}$-coefficients. The result is the following

Main Theorem. Let X be an arithmetic variety over an arithmetic ring (A, Σ, F_∞). Then there is a natural λ-ring structure on $K_{\mathrm{Ar}}(X)$ such that if we denote by $K_{\mathrm{Ar}}^{(p)}(X)$ the eigenspace of the associated Adams operator φ^k with eigenvalues k^p, then,

$$\mathrm{ch}_{\mathrm{Ar}} : K_{\mathrm{Ar}}^{(p)}(X) \to \mathrm{CH}_{\mathrm{Ar}}^p(X)_{\mathbf{Q}}$$

is an isomorphism for all $p \geq 0$.

The basic idea to prove this theorem is to use the five lemma. We know by Chapter 1 that there exists an exact sequence

$$\dots \to \oplus_{p\geq 1}\mathrm{CH}^{p,p-1}(X)_{\mathbf{Q}} \to \tilde{A}(X_{\mathbf{R}}) \to \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}} \to \mathrm{CH}(X)_{\mathbf{Q}} \to 0.$$

Therefore it is natural for us to prove the following

Theorem. For any arithmetic variety X over an arithmetic ring (A, Σ, F_∞), there is a natural exact sequence

$$\dots \to K_1(X)_{\mathbf{Q}} \to \tilde{A}(X_{\mathbf{R}}) \to K_{\mathrm{Ar}}(X)_{\mathbf{Q}} \to K(X)_{\mathbf{Q}} \to 0,$$

and a natural local Chern character

$$\mathrm{ch} : K_1(X) \to \oplus_{p\geq 1}\mathrm{CH}^{p,p-1}(X)_{\mathbf{Q}},$$

such that the following diagram is commutative:

$$\begin{array}{ccccccccc} \ldots \to & K_1(X)_{\mathbf{Q}} & \to & \tilde{A}(X_{\mathbf{R}}) & \to & K_{\mathrm{Ar}}(X)_{\mathbf{Q}} & \to & K(X)_{\mathbf{Q}} & \to 0 \\ & \mathrm{ch}\downarrow & & \mathrm{Id}\downarrow & & \mathrm{ch}_{\mathrm{Ar}}\downarrow & & \mathrm{ch}\downarrow & \\ \ldots \to & \oplus_{p\geq 1}\mathrm{CH}^{p,p-1}(X)_{\mathbf{Q}} & \to & \tilde{A}(X_{\mathbf{R}}) & \to & \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}} & \to & \mathrm{CH}(X)_{\mathbf{Q}} & \to 0. \end{array}$$

In particular, we see that $\mathrm{ch}_{\mathrm{Ar}}$ is an isomorphism.

We prove this in the following several subsections: It is a direct consequence of the exact sequence of the higher K-theory associated with a fiber space.

II.3.6.b. The Construction of A Fiber Space

We consider an exact sequence for the arithmetic K-group. Let $\mathcal{P}(X)$ be the category of vector sheaves on X. Define a simplicial set $G(X)$ as follows:

Let $[n]$ be the standard ordered set with $n+1$ elements,

$$[n] := \{0 < 1 < \ldots < n\}.$$

We view $[n]$ as a category. Let $\mathrm{Ar}[n]$ be the category of maps in $[n]$. Denote by $F(X)_n$ the set of functors

$$P : \mathrm{Ar}[n] \to \mathcal{P}(X)$$

such that:

(1) For every $i \leq j \leq k$, the sequence

$$0 \to P(i,j) \to P(i,k) \to P(j,k) \to 0$$

is exact;

(2) If $i > 0$, $P(i,i) = 0$.

We may think of $P \in F(X)_n$ as a sequence of inclusions

$$P_0 \to P_1 \to \ldots \to P_n,$$

where $P_i = P(i,0)$ and $P(i,j) = P_i/P_j$ in $\mathcal{P}(X)$.

Now define $G(X)_n$ as the set of pairs

$$(P,Q) = ((P_0,Q_0) \to (P_1,Q_1) \to \ldots \to (P_n,Q_n))$$

in $F(X)_n$, together with isomorphisms

$$P(i,j) \simeq Q(i,j), \forall j > 0$$

which are compatible with the maps

$$P(i,j) \to P(k,l), \quad Q(i,j) \to Q(k,l),$$

for $i \leq k$ and $0 < j \leq l$. There are natural face maps

$$d_k : G(X)_n \to G(X)_{n-1},$$

and degenerate maps

$$s_l : G(X)_{n-1} \to G(X)_n,$$

for $0 \leq k \leq n, 1 \leq l \leq n-1$. The direct sum gives a composition law on $G(X)$. Thus one can check that $\pi_m(G(X))$ gives another description of $K_m(X)$ in the sense of Quillen [GG].

We now formulate an arithmetic analogue $G_{\mathrm{Ar}}(X)$ of $G(X)$. Put an F_∞ invariant hermitian metric on every vector sheaf in $P(X)$. Let $G_{\mathrm{Ar}}(X)_n$ be the set of triples (P,Q,η) with $(P,Q) \in G(X)_n$ and

$$\eta = (\eta_0, \ldots, \eta_n) \in \tilde{A}(X_{\mathbf{R}})^{n+1},$$

such that, for all $i = 1, \ldots, n$,

$$\begin{aligned} \eta_{i-1} - \eta_i =& \mathrm{ch}_{\mathrm{BC}}(0 \to (P_{i-1}, \rho_{i-1}) \to (P_i, \rho_i) \to (P_{i-1,i}, \rho_{i-1,i}) \to 0) \\ &- \mathrm{ch}_{\mathrm{BC}}(0 \to (Q_{i-1}, \tau_{i-1}) \to (Q_i, \tau_i) \to (Q_{i-1,i}, \tau_{i-1,i}) \to 0) \\ &- \mathrm{ch}_{\mathrm{BC}}((P_{i-1,i}, \rho_{i-1,i}) \to (Q_{i-1,i}, \tau_{i-1,i})). \end{aligned}$$

The face map

$$d_k : G_{\mathrm{Ar}}(X)_n \to G_{\mathrm{Ar}}(X)_{n-1}$$

sends (P,Q,η) to $(d_k(P,Q), d_k(\eta))$ with

$$d_k(\eta_0, \ldots, \eta_n) := \begin{cases} (\eta_1, \ldots, \eta_n), & \text{if } k = 0; \\ (\eta_0, \ldots, \eta_k + \eta_{k+1}, \ldots, \eta_n), & \text{if } 0 < k < n; \\ (\eta_0, \ldots, \eta_{n-1}), & \text{if } k = n. \end{cases}$$

Similarly, the degenerate maps $s_l : G_{\mathrm{Ar}}(X)_{n-1} \to G_{\mathrm{Ar}}(X)_n$ are defined by

$$s_l(\eta_0, \ldots, \eta_{n-1}) = (\eta_0, \ldots, \eta_l, \eta_l, \ldots, \eta_{n-1}).$$

It is not difficult to show that these maps are well-defined. The simplicial set $G_{\mathrm{Ar}}(X)$ has a composition law defined by

$$(P,Q,\eta) + (P',Q',\eta') = (P \oplus P', Q \oplus Q', \eta + \eta').$$

Furthermore, the natural forgetful map $G_{\mathrm{Ar}}(X) \to G(X)$, defined by sending (P, Q, η) to (P, Q), is a covering space with the group $\tilde{A}(X_{\mathbf{R}})$. Indeed, we may define a action of $\tilde{A}(X_{\mathbf{R}})$ on $G_{\mathrm{Ar}}(X)$ by

$$\alpha + (P, Q, (\eta_0, \ldots, \eta_n)) := (P, Q, (\eta_0 + \alpha, \ldots, \eta_n + \alpha)),$$

which is free on each $G_{\mathrm{Ar}}(X)_n$. Now the assertion follows from the fact that for a given $(P, Q) \in G(X)_n$, an element (P, Q, η) is determined by the choice of $\eta_0 \in \tilde{A}(X_{\mathbf{R}})$.

II.3.6.c. **The Exact Sequence for Arithmetic K-Theory**

With the same notation as above and by a general fact from algebraic K-theory, we have the exact sequence

$$\ldots \to K_1(X) \to \tilde{A}(X_{\mathbf{R}}) \to \pi_0(G_{\mathrm{Ar}}(X)) \to K(X) \to 0.$$

To go further, we need the following

Lemma. In the exact sequence above, $\pi_0(G_{\mathrm{Ar}}(X)) \simeq K_{\mathrm{Ar}}(X)$ and the morphism $K_{\mathrm{Ar}}(X) \to K(X)$ is defined by sending $[((\mathcal{E}, \rho), \eta)]$ to $[\mathcal{E}]$.

Proof. The group $\pi_0(G_{\mathrm{Ar}}(X))$ is generated by triples (P_0, Q_0, η_0) with P_0, Q_0 in $P(X)$ and η_0 in $\tilde{A}(X_{\mathbf{R}})$, and satisfies the following relations: For every

$$(((P_0, Q_0) \to (P_1, Q_1)), (\eta_0, \eta_1)) \in G_{\mathrm{Ar}}(X)_1,$$

(P_0, Q_0, η_0) is equivalent to (P_1, Q_1, η_1). Furthermore, in $\pi_0(G_{\mathrm{Ar}}(X))$,

$$(P_0, Q_0, \eta_0) + (P_0', Q_0', \eta_0') = (P_1 \oplus P_0', Q_0 \oplus Q_0', \eta_0 + \eta_0').$$

Define a map

$$\pi_0(G_{\mathrm{Ar}}(X)) \to K_{\mathrm{Ar}}(X)$$

by sending (P_0, Q_0, η_0) to $[(P_0, \rho_0)] - [(Q_0, \tau_0)] + \eta_0$. Conversely, we can define a map

$$K_{\mathrm{Ar}}(X) \to \pi_0(G_{\mathrm{Ar}}(X))$$

by sending $((P, \rho'), \eta)$ to $(P, 0, \eta + \mathrm{ch}_{\mathrm{BC}}(P, \rho, \rho'))$. It is easy to check that these two maps are well-defined and that they are isomorphisms. The naturality follows easily from the definitions above. In this way, we have a natural exact sequence for arithmetic K-groups.

II.3.6.d. λ-Structure on $K_{\mathrm{Ar}}(X)_{\mathbf{Q}}$

For regular arithmetic varieties, there also exists a natural λ-ring structure on the arithmetic K-group. We give the definition and leave the simple proof to the reader.

We start with the fact that any λ ring structure on any group is uniquely determined by its Adams operators, and for any graded algebra $A = \oplus A_i$, there is a canonical λ-ring structure defined by: The Adams operator φ^k which acts on A^i as k^i. In this sense, there exist canonical λ-ring structures on $Z(X_{\mathbf{R}})$ and $Z(X_{\mathbf{R}})\oplus\tilde{A}(X_{\mathbf{R}})$, for any arithmetic variety X over an arithmetic ring (A, Σ, F_∞). By the definition, we also know that in $Z(X_{\mathbf{R}})$, we have

$$\mathrm{ch}(\lambda^k(\mathcal{E},\rho)) = \lambda^k(\mathrm{ch}(\mathcal{E},\rho)).$$

The first λ-ring structure comes from the wedge product. In fact, this is an easy consequence of the splitting principle.

We can now define a λ-ring structure on K_{Ar} by the following:

$$\lambda^k((\mathcal{E},\rho),\eta) = (\lambda^k(\mathcal{E},\rho),[\lambda^k(\mathrm{ch}(\mathcal{E},\rho),\eta)]).$$

The only nontrivial part is to show that this is well-defined. For this, let

$$\mathcal{E}.: 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

be a short exact sequence of vector sheaves on X with F_∞-invariant hermitian metrics ρ_i on $\mathcal{E}_i$ for $i = 1,2,3$. Then the reader has to complete the proof of the following equality:

$$\lambda^k((\mathcal{E}_2,\rho_2), -\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)) = (\lambda^k(\mathcal{E}_1\oplus\mathcal{E}_3,\rho_1\oplus\rho_3),0).$$

II.3.6.e. The Proof of The Theorem.

In this subsection, we show how to 'complete' the proof of the main theorem.

First, by above discussion, we may have the following diagram:

$$\begin{array}{ccccccccc} K_1(X)_{\mathbf{Q}} & \to & \tilde{A}(X_{\mathbf{R}}) & \to & K_{\mathrm{Ar}}(X)_{\mathbf{Q}} & \to & K(X)_{\mathbf{Q}} & \to & 0 \\ \mathrm{ch}\downarrow & & \mathrm{Id}\downarrow & & \mathrm{ch}_{\mathrm{Ar}}\downarrow & & \mathrm{ch}\downarrow & & \\ \oplus_{p\geq 1}\mathrm{CH}^{p,p-1}(X)_{\mathbf{Q}} & \to & \tilde{A}(X_{\mathbf{R}}) & \to & \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}} & \to & \mathrm{CH}(X)_{\mathbf{Q}} & \to & 0, \end{array}$$

except for the morphism ch on K_1. But this is an algebraic morphism, which may be naturally defined in algebraic K-theory. (See Chapter. Usually, we call it the local Chern character.) By the naturality of this theory, we expect that all squares of this diagram are commutative. Even through this may be done, it is rather complicated and tedious, so we do not give the proof here. Instead, we shall assume that there is this natural local Chern character, which makes the above diagram commute. Hence by the five lemma, we know that $\mathrm{ch}_{\mathrm{Ar}}$ is a group isomorphism of $K_{\mathrm{Ar}}(X)_{\mathbf{Q}}$ and $\mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}}$. This completes the proof of the theorem.

We can also introduce a multiplication on $K_{\mathrm{Ar}}(X)$:

$$((\mathcal{E},\rho),\eta)\otimes((\mathcal{E}',\rho'),\eta') := ((\mathcal{E}\otimes\mathcal{E}',\rho\otimes\rho'),[(\mathrm{ch}(\mathcal{E},\rho),\eta)*(\mathrm{ch}(\mathcal{E}',\rho'),\eta')]).$$

Moreover, on the real vector space $Z(X_{\mathbf{R}})\oplus\tilde{A}(X_{\mathbf{R}})$, with Z denoting the closed forms, we can define a pairing by letting

$$(\omega,\eta)*(\omega',\eta') := (\omega\wedge\omega',\omega\wedge\eta'+\eta\wedge\omega'+dd^c\eta\wedge\eta').$$

It is not difficult to show that the above definitions make $K_{\mathrm{Ar}}(X)$ a commutative, associative and unitary ring. (As an illustration, if

$$\mathcal{E}.:0\to\mathcal{E}_1\to\mathcal{E}_2\to\mathcal{E}_3\to 0$$

is an exact sequence of vector sheaves on X with F_∞-invariant hermitian metrics ρ_i on $\mathcal{E}_i$ for $i=1,2,3$, then, we have

$$\begin{aligned}
&((\mathcal{E}_1,\rho_1),0)\otimes((\mathcal{E}',\rho'),\eta')+((\mathcal{E}_3,\rho_3),0)\otimes((\mathcal{E}',\rho'),\eta')\\
&\quad-((\mathcal{E}_2,\rho_2),-\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.))\otimes((\mathcal{E}',\rho'),\eta')\\
=&((\mathcal{E}_1\otimes\mathcal{E}',\rho_1\otimes\rho'),\mathrm{ch}(\mathcal{E}_1,\rho_1)\wedge\eta')+((\mathcal{E}_3\otimes\mathcal{E}',\rho_3\otimes\rho'),\mathrm{ch}(\mathcal{E}_3,\rho_3)\wedge\eta')\\
&\quad-((\mathcal{E}_2\otimes\mathcal{E}',\rho_2\otimes\rho'),\mathrm{ch}(\mathcal{E}_2,\rho_2)\wedge\eta')+\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)\wedge\mathrm{ch}(\mathcal{E}',\rho')-dd^c(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)\wedge\eta')\\
=&((\mathcal{E}_1\otimes\mathcal{E}',\rho_1\otimes\rho'),0)+((\mathcal{E}_3\otimes\mathcal{E}',\rho_3\otimes\rho'),0)\\
&\quad-((\mathcal{E}_2\otimes\mathcal{E}',\rho_2\otimes\rho'),-\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)\wedge\mathrm{ch}(\mathcal{E}',\rho'))\\
=&((\mathcal{E}_1\otimes\mathcal{E}',\rho_1\otimes\rho'),0)+((\mathcal{E}_3\otimes\mathcal{E}',\rho_3\otimes\rho'),0)\\
&\quad-((\mathcal{E}_2\otimes\mathcal{E}',\rho_2\otimes\rho'),-\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.\otimes\mathcal{E}',\rho.\otimes\rho'))\\
=&0.
\end{aligned}$$

Therefore, the definition makes sense. The checking for others is very similar.)

We then have the following

Theorem. The arithmetic Chern characteristic class

$$\mathrm{ch}_{\mathrm{Ar}}:K_{\mathrm{Ar}}(X)_{\mathbf{Q}}\to\mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}}$$

is a ring isomorphism.

Proof. It is enough to prove that $\mathrm{ch}_{\mathrm{Ar}}$ preserves multiplication. In fact, by definition and Theorem 5.f, we know that

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{Ar}}\Big(((\mathcal{E},\rho),\eta)\otimes((\mathcal{E}',\rho'),\eta')\Big)\\
=&\mathrm{ch}_{\mathrm{Ar}}((\mathcal{E},\eta)\otimes(\mathcal{E}',\rho'))+\Big((\mathrm{ch}(\mathcal{E},\rho),\eta)*(\mathrm{ch}(\mathcal{E}',\rho'),\eta')\Big)\\
=&\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}',\rho')+\mathrm{ch}(\mathcal{E},\rho)\wedge\eta'+\eta\wedge\mathrm{ch}(\mathcal{E}',\rho')+dd^c\eta\wedge\eta'\\
=&\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}',\rho')+\omega(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho))\wedge\eta'+\eta\wedge\omega(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}',\rho'))+\omega(\eta)\wedge\eta'\\
=&(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)+\eta)(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}',\rho')+\eta').
\end{aligned}$$

§II.3.7. Cap Product: A Dual Version

In this section, we give a dual version of the result in section 2.7. More precisely, we have the following

Theorem. There is a biadditive pairing

$$\begin{array}{ccc} K^0_{\mathrm{Ar}}(X)\otimes \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X) & \to & \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)_{\mathbf{Q}} \\ \alpha\otimes x & \mapsto & \mathrm{ch}_{\mathrm{Ar}}(\alpha)\cap x \end{array}$$

with the following properties:

(a). If $f: X \to Y$ is a morphism of arithmetic varieties, with Y regular, $\alpha \in K_{\mathrm{Ar}}(Y)$ and $x \in \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)$, then

$$\mathrm{ch}_{\mathrm{Ar}}(f^*\alpha)\cap x = \mathrm{ch}_{\mathrm{Ar}}(\alpha)._f x.$$

(b). If $(0,\eta) \in K_{\mathrm{Ar}}(X)$ and $x \in \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)$, then

$$\mathrm{ch}_{\mathrm{Ar}}((0,\eta))\cap x = a(\eta\,\omega(x)).$$

(c). If $\alpha \in K_{\mathrm{Ar}}(X)$ and $x \in \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)$, then

$$\omega(\mathrm{ch}_{\mathrm{Ar}}(\alpha)\cap x) = \mathrm{ch}(\alpha)\cap \omega(x).$$

(d). The pairing makes $\mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)_{\mathbf{Q}}$ into a $K^0_{\mathrm{Ar}}(X)$-module.

(e). If $f: X \to Y$ is proper, and smooth over Y_F, let $\alpha \in K_{\mathrm{Ar}}(Y)$ and $x \in \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)$, then

$$f_*(\mathrm{ch}_{\mathrm{Ar}}(f^*\alpha)\cap x) = \mathrm{ch}_{\mathrm{Ar}}(\alpha)\cap f_*(x).$$

(f). If $f: Y \to X$ is flat and smooth over F, or a l.c.i. morphism, let $\alpha \in K_{\mathrm{Ar}}(X)$ and $x \in \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)$. Then

$$f^*(\mathrm{ch}_{\mathrm{Ar}}(\alpha)\cap x) = \mathrm{ch}_{\mathrm{Ar}}(f^*\alpha)\cap f^*(x).$$

(g). Let $i: D \hookrightarrow X$ be the inclusion of a principal effective Cartier divisor, $f: Y \to X$ a morphism which meets D_F properly, $i_Y: f^{-1}(D) \hookrightarrow Y$ the inclusion induced by i, and $(\mathcal{E},\rho)$ a hermitian vector sheaf on Y. Then for any $x \in \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)$, we have

$$\mathrm{ch}_{\mathrm{Ar}}(i_Y^*(\mathcal{E},\rho)\cap i^*(x)) = i^*\left(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\cap x\right).$$

Proof. We only make the definition, and do not give the proof of the properties, since all of them can be deduced in a standard way. Take a generator $\alpha = ((\mathcal{E},\tau),\eta)$ of $K^0_{\mathrm{Ar}}(X)$. Since we assume that all the varieties are quasi-projective over A, there exists a vector sheaf $\mathcal{U}$ over an arithmetic variety G, with G smooth over A, a map $u: X \to G$, and an isomorphism $\theta: \mathcal{E} \to u^*\mathcal{U}$. Fix an arbitrary F_∞-invariant hermitian metric ρ on $\mathcal{U}$. Then we have an arithmetic Chern character $\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U},\tau) \in \mathrm{CH}^{\cdot}_{\mathrm{Ar}}(G)_{\mathbf{Q}}$. Given $x \in \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)$, consider the class

$$\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U},\rho)._{u}x + a(\mathrm{ch}_{\mathrm{BC}}(\theta,\rho_\theta)) \cap x + a(\eta) \cap x.$$

We claim that this is independent of the choice the triple $(u,(\mathcal{U},\rho),\theta)$: In fact, if another choice is $(u',(\mathcal{U}',\rho'),\theta')$, let $\mathrm{Iso}(\mathcal{U}',\mathcal{U})$ be the variety, smooth over A, which parameterizes isomorphisms $\mathcal{U}' \to \mathcal{U}$. There are projections $p : \mathrm{Iso}(\mathcal{U}',\mathcal{U}) \to G$ and $p' : \mathrm{Iso}(\mathcal{U}',\mathcal{U}) \to G'$, and an isomorphism $\varphi : p^*\mathcal{U} \to p'^*\mathcal{U}'$. By the definition of $\mathrm{Iso}(\mathcal{U}',\mathcal{U})$, there is unique map $\delta : X \to \mathrm{Iso}(\mathcal{U}',\mathcal{U})$ such that $p\circ\delta = u$, $p'\circ\delta = u'$ and $\delta^*(\varphi)$ is the isomorphism $\theta'(\theta)^{-1}$. So given $x \in \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)$, we get

$$\begin{aligned}
&x._{u}\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U},\rho) - x._{u'}\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U}',\rho')\\
&= x._{p\circ\delta}\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U},\rho) - x._{p'\circ\delta}\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U}',\rho')\\
&= x._{\delta}\left(p^*\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U},\rho) - p'^*\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U},\rho)\right)\\
&= x._{\delta}\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(p^*(\mathcal{U},\rho),p'^*(\mathcal{U}',\rho'),\varphi)\big)\Big)\\
&= x \cap \delta^*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(p^*(\mathcal{U},\rho),p'^*(\mathcal{U}',\rho'),\varphi)\big)\Big)\\
&= x \cap \Big(a\big(\mathrm{ch}_{\mathrm{BC}}(u^*(\mathcal{U},\rho),u'^*(\mathcal{U}',\rho'),\theta'(\theta)^{-1})\big)\Big)\\
&= x \cap \Big(a\big(\mathrm{ch}_{\mathrm{BC}}((\mathcal{E},\tau),p'^*(\mathcal{U}',\rho'),\theta)\big) - a\big(\mathrm{ch}_{\mathrm{BC}}((\mathcal{E},\tau),p^*(\mathcal{U},\rho),\theta')\big)\Big)
\end{aligned}$$

as desired.

Since the cap product is biadditive, this pairing is additive in x. In order to show that we get a map $K_{\mathrm{Ar}}(X) \otimes \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X) \to \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)_{\mathbf{Q}}$, it suffices to show that for any exact sequence

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

of vector sheaves on X with F_∞-invariant hermitian metrics ρ_i on $\mathcal{E}_i$, we have

$$\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_2,\rho_2) - \mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_1,\rho_1) - \mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_3,\rho_3)\big) \cap x = a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)\big) \cap x.$$

This can be deduced as follows: Choose triples $(u_i,(\mathcal{U}_i,\tau_i),\theta_i)$ representing the corresponding terms $(\mathcal{E}_i,\rho_i)$ in the exact sequence. There is a variety P, smooth over A, which parameterizes exact sequences

$$0 \to \mathcal{U}_1 \to \mathcal{U}_2 \to \mathcal{U}_3 \to 0.$$

That is, there are projections $q_i : P \to G_i$ and a universal exact sequence

$$q^*_{\cdot}\mathcal{U}. : 0 \to q_1^*\mathcal{U}_1 \to q_2^*\mathcal{U}_2 \to q_3^*\mathcal{U}_3 \to 0$$

with the obvious universal property. In particular, there are a map $f : X \to P$ such that $q_i \circ f = u_i$ and an isomorphism of exact sequence $f^*(q^*_{\cdot}\mathcal{U}.) \simeq \mathcal{E}.$. Since P is smooth over A, by I.4, (the axiom for classical Bott-Chern secondary characteristic forms,) or better by II.3.5.f, (the property of arithmetic characteristic classes,) we know that

$$q_2^*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U}_2,\tau_2)) - q_1^*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U}_1,\tau_1)) - q_3^*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{U}_3,\tau_3)) = a(\mathrm{ch}_{\mathrm{BC}}(q.^*\mathcal{U}.,q^*.\tau.)).$$

From the biadditivity of the cap product $\mathrm{CH}^{\cdot}_{\mathrm{Ar}}(M) \otimes \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X) \to \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)_{\mathbf{Q}}$, we see that

$$\begin{aligned}
&(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_2,\rho_2) - \mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_1,\rho_1) - \mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_3,\rho_3)) \cap x \\
&= \mathrm{ch}_{\mathrm{Ar}}(\mathcal{U}_2,\tau_2)\cdot_{u_2} x - \mathrm{ch}_{\mathrm{Ar}}(\mathcal{U}_1,\tau_1)\cdot_{u_1} x - \mathrm{ch}_{\mathrm{Ar}}(\mathcal{U}_3,\tau_3)\cdot_{u_3} x \\
&\quad + a(\mathrm{ch}_{\mathrm{BC}}((\mathcal{E}_2,\rho_2), u_2^*(\mathcal{U}_2,\tau_2);\theta_2)) \cap x \\
&\quad - a(\mathrm{ch}_{\mathrm{BC}}((\mathcal{E}_1,\rho_1), u_1^*(\mathcal{U}_1,\tau_1);\theta_1)) \cap x - a(\mathrm{ch}_{\mathrm{BC}}((\mathcal{E}_3,\rho_3), u_3^*(\mathcal{U}_3,\tau_3);\theta_3)) \cap x \\
&= \mathrm{ch}_{\mathrm{Ar}}(q_2^*(\mathcal{U}_2,\tau_2))\cdot_f x - \mathrm{ch}_{\mathrm{Ar}}(q_1^*(\mathcal{U}_1,\tau_1))\cdot_f x - \mathrm{ch}_{\mathrm{Ar}}(q_3^*(\mathcal{U}_3,\tau_3))\cdot_f x \\
&\quad + a(\mathrm{ch}_{\mathrm{BC}}((\mathcal{E}_2,\rho_2), u_2^*(\mathcal{U}_2,\tau_2);\theta_2)) \cap x \\
&\quad - a(\mathrm{ch}_{\mathrm{BC}}((\mathcal{E}_1,\rho_1), u_1^*(\mathcal{U}_1,\tau_1);\theta_1)) \cap x - a(\mathrm{ch}_{\mathrm{BC}}((\mathcal{E}_3,\rho_3), u_3^*(\mathcal{U}_3,\tau_3);\theta_3)) \cap x \\
&= f^*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(q^*\mathcal{U}_.,q^*.\tau_.)\big)\Big) \cap x \\
&\quad + a\Big(\mathrm{ch}_{\mathrm{BC}}\big((\mathcal{E}_2,\rho_2), u_2^*(\mathcal{U}_2,\tau_2);\theta_2\big)\Big) \cap x \\
&\quad - a\Big(\mathrm{ch}_{\mathrm{BC}}\big((\mathcal{E}_1,\rho_1), u_1^*(\mathcal{U}_1,\tau_1);\theta_1\big)\Big) \cap x - a\Big(\mathrm{ch}_{\mathrm{BC}}\big((\mathcal{E}_3,\rho_3), u_3^*(\mathcal{U}_3,\tau_3);\theta_3\big)\Big) \cap x \\
&= a(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_.,\rho_.)) \cap x.
\end{aligned}$$

So we have the assertion.

Remark: The same method may be used to define other arithmetic characteristic classes, for example, the arithmetic Todd characteristic class $\mathrm{td}_{\mathrm{Ar}}(\mathcal{E},\rho)\cap x$ for any hermitian vector sheaves from the regular case. For more details, see section 8.

§II.3.8. Arithmetic Todd Classes

II.3.8.a. A Technical Lemma

From the Grothendieck-Riemann-Roch theorem in algebraic geometry, we know that the Todd character and its inverse should be very useful in the theory of arithmetic Riemann-Roch theorem for l.c.i. morphisms. In this section, we list the properties without proof. As a hint, we observe that the Todd character is a multiplicative character, and hence, the reader may use the techniques developed above, such as the $\mathbf{P}^1$-deformation technique, to prove the assertions made here.

First notice that for a given exact sequence

$$\mathcal{E}_. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

of holomorphic vector sheaves on a complex manifold X, endowed with arbitrary hermitian metrics, the following equality holds in $\mathrm{CH}_{\mathrm{Ar}}(X)$:

$$\mathrm{td}_{\mathrm{BC}}^{-1}(\mathcal{E}_.,\rho_.) = -\mathrm{td}_{\mathrm{BC}}(\mathcal{E}_.,\rho_.)\,\mathrm{td}^{-1}(\mathcal{E}_1,\rho_1)\,\mathrm{td}^{-1}(\mathcal{E}_2,\rho_2)\,\mathrm{td}^{-1}(\mathcal{E}_3,\rho_3).$$

To check this, note that both sides have the same image by dd^c, then depend functorially on $\mathcal{E}_.$, and vanish when $(\mathcal{E}_.,\rho_.)$ is split. Therefore they coincide by I.1.

Lemma. (1). Let

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & S & = & S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A & \to & B & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & C & \to & D & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & & ,
\end{array}$$

be a commutative diagram of vector sheaves on X, with exact lines and columns. Call these respectively C_1, C_2, C_3 and L_1, L_2, L_3 (from the left to the right and from the top to the bottom). Choose arbitrary metrics on all vector sheaves. Then the following identity holds in $\tilde{A}(X)$:

$$\begin{aligned}
&\mathrm{td}_{\mathrm{BC}}(L_3,\rho_{L_3})\,\mathrm{td}(A,\rho_A)\,\mathrm{td}^{-1}(C,\rho_C)\,\mathrm{td}^{-1}(D,\rho_D)+\mathrm{td}_{\mathrm{BC}}(C_2,\rho_{C_2})\,\mathrm{td}^{-1}(D,\rho_D)\\
&\quad=\mathrm{td}_{\mathrm{BC}}(L_2,\rho_{L_2})\,\mathrm{td}^{-1}(D,\rho_D)+\mathrm{td}_{\mathrm{BC}}(C_1,\rho_{C_1})\,\mathrm{td}^{-1}(C,\rho_C).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathrm{td}_{\mathrm{BC}}(L_3,\rho_{L_3})\,\mathrm{td}(B,\rho_B)+\mathrm{td}_{\mathrm{BC}}(C_2,\rho_{C_2})\,\mathrm{td}^{-1}(D,\rho_D)\\
&\quad=\mathrm{td}_{\mathrm{BC}}(L_2,\rho_{L_2})\,\mathrm{td}^{-1}(C,\rho_C)\,\mathrm{td}^{-1}(Q,\rho_Q)+\mathrm{td}_{\mathrm{BC}}(C_1,\rho_{C_1})\,\mathrm{td}^{-1}(C,\rho_C).
\end{aligned}$$

(2) Let

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S & = & S & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & S' & \to & A & \to & B & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & S' & \to & C & \to & D & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & ,
\end{array}$$

be a commutative diagram of vector sheaves on X, with exact lines and columns. Call these respectively C_1, C_2, C_3, L_1, L_2, L_3 (from the left to the right and from the top to the bottom). Choose arbitrary metrics on all vector sheaves. Then the following identity holds in $\tilde{A}(X)$:

$$\begin{aligned}
&\mathrm{td}_{\mathrm{BC}}(L_3,\rho_{L_3})\,\mathrm{td}(A,\rho_A)\,\mathrm{td}^{-1}(C,\rho_C)\mathrm{td}^{-1}(D,\rho_D)\mathrm{td}^{-1}(S',\rho_{S'})+\mathrm{td}_{\mathrm{BC}}(C_2,\rho_{C_2})\mathrm{td}^{-1}(C,\rho_C)\\
&\quad=\mathrm{td}_{\mathrm{BC}}(L_2,\rho_{L_2})\mathrm{td}^{-1}(D,\rho_D)\mathrm{td}^{-1}(S',\rho_{S'})+\mathrm{td}_{\mathrm{BC}}(C_3,\rho_{C_3})\mathrm{td}^{-1}(D,\rho_D).
\end{aligned}$$ ∎

(3) Let

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_1 & \to & E_1 & \to & Q_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_2 & \to & E_2 & \to & Q_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_3 & \to & E_3 & \to & Q_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}$$

be a commutative diagram of vector sheaves on X, with exact lines and columns. Call these respectively C_1, C_2, C_3 and L_1, L_2, L_3 (from the left to the right and from the top to the bottom). Choose arbitrary metrics on all vector sheaves. Then the following identity holds in $\tilde{A}(X)$:

$$\begin{aligned}
&\mathrm{td}_{\mathrm{BC}}(L_3,\rho_{L_3})\,\mathrm{td}(Q_1,\rho_{Q_1})\,\mathrm{td}^{-1}(Q_3,\rho_{Q_3})\,\mathrm{td}(E_1,\rho_{E_1})\\
&\ +\mathrm{td}_{\mathrm{BC}}(L_1,\rho_{L_1})\,\mathrm{td}^{-1}(Q_1,\rho_{Q_1})\,\mathrm{td}^{-1}(Q_3,\rho_{Q_3})\,\mathrm{td}(E_3,\rho_{E_3})\\
&\ +(\mathrm{td}_{\mathrm{BC}}(L_1,\rho_{L_1})*\mathrm{td}_{\mathrm{BC}}(L_3,\rho_{L_3}))\,\mathrm{td}^{-1}(Q_1,\rho_{Q_1})\,\mathrm{td}^{-1}(Q_3,\rho_{Q_3})\\
&\ -\mathrm{td}_{\mathrm{BC}}(L_2,\rho_{L_2})\,\mathrm{td}^{-1}(Q_2,\rho_{Q_2})\\
=&\mathrm{td}_{\mathrm{BC}}(C_1,\rho_{C_1})-\mathrm{td}_{\mathrm{BC}}(C_2,\rho_{C_2})\,\mathrm{td}^{-1}(Q_1,\rho_{Q_1})\,\mathrm{td}^{-1}(Q_3,\rho_{Q_3})\\
&\ +\mathrm{td}_{\mathrm{BC}}(C_2,\rho_{C_2})*\mathrm{td}_{\mathrm{BC}}^{-1}(C_3,\rho_{C_3}))-\mathrm{td}_{\mathrm{BC}}^{-1}(C_3,\rho_{C_3})\,\mathrm{td}(E_1,\rho_{E_1})\,\mathrm{td}(E_3,\rho_{E_3}).
\end{aligned}$$

II.3.8.b. **The Arithmetic Tangent Elements For L.C.I. Morphisms**

We now make the definition of arithmetic tangent elements for certain l.c.i. morphisms and their associated arithmetic Todd characteristic classes. This definition is motivated by the Grothendieck-Riemann-Roch theorem in algebraic geometry.

Recall that, in defining arithmetic characteristic classes, we always assumed that the arithmetic K-group is generated by hermitian vector sheaves among others. In algebraic geometry, we know that if the variety is regular, then the algebraic K-group has two different expressions: One is given by the vector sheaves, while the other is given by coherent sheaves. So naturally, we can consider the parallel situation in arithmetic geometry. There is no problem for algebraic cycles. But for Green's currents, we do meet an essential problem: How can one attach a metric to a coherent sheaf? So far nobody knows how to deal with this ambiguous object. One possible way is to take a vector sheaf resolution of the coherent sheaf in question and then put hermitian metrics on the vector sheaves, and finally to define the arithmetic element associated with this coherent sheaf as the alternating sum of the hermitian vector sheaves but with a modification by a 'Bott-Chern secondary characteristic current' associated with the above resolution. This sounds good, but how to get a very interesting Bott-Chern secondary characteristic current is far from being understood. Now we know only a few examples; the situation for closed immersions for instance. But this idea suggests us the following approach: First, we should extend the arithmetic K-group to include certain kinds of currents, which properly includes the smooth (p,p) forms, the relative Bott-Chern secondary characteristic currents with respect to closed immersions; secondly, one needs to re-examine the classical logarithmic short exact sequence in the sense of hypercohomology. With this, now we can define $K_0^{\mathrm{Ar}}(X)$ as the quotient group of the free abelian group generated by the elements $((\mathcal{F},\tau),\eta)$ but under the same relations as before. Here $\mathcal{F}$ is a coherent sheaf on X such that the pull back of $\mathcal{F}$ at infinity is a vector sheaf on $X(\mathbf{C})$, and ρ an F_∞-invariant hermitian metric on this pull back, while $\eta \in \tilde{D}(X_{\mathbf{R}})$. So in this sense, it is natural to make the following restriction on morphisms between arithmetic varieties: The corresponding infinite part of the morphism is smooth.

Next we use the results in the previous section about arithmetic Chow homology groups and cup products to give the the arithmetic tangent elements for l.c.i. morphisms.

Let $f : X \to Y$ be an l.c.i. morphism of arithmetic varieties, which is smooth over the generic fiber Y_F over an arithmetic ring A. Choose an F_∞-invariant hermitian metric on the complex relative tangent vector sheaf $\mathcal{T}_{f_{\mathbf{C}}}$. In the following, we attach this data to an arithmetic Todd character $\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)$.

Since X is quasi-projective, we can imbed X in a projective space $\mathbf{P}_A^N$ and let $i : X \hookrightarrow X \times_A \mathbf{P}_A^N = P$ be the product of this imbedding with the map f: We get a factorization of f as $g \circ i$, where $g : P \to Y$ is the first projection. Furthermore, i is regular by the fact that f is an l.c.i. morphism. Denote by $N = N_{X/P}$ the normal bundle of X in P and by T_g the relative tangent bundle of g. Choose F_∞-invariant hermitian metrics on N and T_g. Then there is an exact sequence of vector sheaves on $X(\mathbf{C})$:

$$\mathcal{T}_{f,i} : 0 \to \mathcal{T}_{f_{\mathbf{C}}} \to i^*\mathcal{T}_{g_{\mathbf{C}}} \to \mathcal{N}_{\mathbf{C}} \to 0.$$

Hence there is an element $\mathrm{td}_{\mathrm{BC}}(\mathcal{T}_{f,i},\rho_{\mathcal{T}_{f,i}}) \in \tilde{A}(X_{\mathbf{R}})$, the classical Bott-Chern secondary characteristic forms associated with the above exact sequence. Now let

$$\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g}) := \mathrm{td}_{\mathrm{BC}}(\mathcal{T}_{f,i},\rho_{\mathcal{T}_{f,i}})\,\mathrm{td}(\mathcal{N},\rho_{\mathcal{N}})^{-1} \in \tilde{A}(X_{\mathbf{R}}),$$

and we have

$$dd^c\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g}) = \mathrm{td}(\mathcal{T}_{f_{\mathbf{C}}},\rho_{\mathcal{T}_{f_{\mathbf{C}}}}) - \mathrm{td}(i^*\mathcal{T}_{g_{\mathbf{C}}},\rho_{i^*\mathcal{T}_{g_{\mathbf{C}}}})\,\mathrm{td}(\mathcal{N}_{\mathbf{C}},\rho_{\mathcal{N}_{\mathbf{C}}})^{-1}.$$

So, for any $\alpha \in \mathrm{CH}^{\mathrm{Ar}}.(X)$, we can make a natural definition of $\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)$ by the following formula:

$$\begin{aligned}&\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)\cap\alpha\\ &:=\mathrm{td}_{\mathrm{Ar}}(i^*\mathcal{T}_g,\rho_{i^*\mathcal{T}_g})\cap(\mathrm{td}_{\mathrm{Ar}}^{-1}(\mathcal{N},\rho_{\mathcal{N}})^{-1}\cap\alpha)+\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\cap\alpha \in \mathrm{CH}^{\mathrm{Ar}}.(X)_{\mathbf{Q}}.\end{aligned}$$

If $Y = \mathrm{Spec}(A)$, we also write $\mathrm{td}_{\mathrm{Ar}}(X)$ instead of $\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)$, and $\mathrm{td}_{\mathrm{BC}}(X/P,\rho_{X/P})$ instead of $\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})$. Furthermore, if X and Y are regular, we have the following

$$\mathrm{td}_{\mathrm{Ar}}(f,\rho_f) := \mathrm{td}_{\mathrm{Ar}}(i^*\mathcal{T}_g,\rho_{i^*\mathcal{T}_g})\mathrm{td}_{\mathrm{Ar}}^{-1}(\mathcal{N},\rho_{\mathcal{N}}) + \mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g}) \in \mathrm{CH}^{\mathrm{Ar}}.(X)_{\mathbf{Q}}.$$

Proposition. (1) The class $\mathrm{td_{Ar}}(f,\rho_f)$ depends only on the choice of the metric on $\mathcal{T}_{f_\mathbf{C}}$, and not on the choice of i, g, nor on the choice of metrics on $\mathcal{N}$ and $\mathcal{T}_g$.
(2) Let $f : X \to Y$ and $g : Y \to Z$ be two maps of regular arithmetic varieties X, Y. Assume that f and g are smooth over F and choose F_∞-invariant hermitian metrics on $\mathcal{T}_{f_\mathbf{C}}$, $\mathcal{T}_{g_\mathbf{C}}$ and $\mathcal{T}_{(g\circ f)_\mathbf{C}}$ respectively. Then the following identity holds in $\mathrm{CH}^{Ar}.(X)$:

$$\mathrm{td_{Ar}}(g\circ f,\rho_{g\circ f}) = \mathrm{td_{Ar}}(f,\rho_f)\, f^*(\mathrm{td_{Ar}}(g,\rho_g)) - a(\mathrm{td_{BC}}(\mathcal{T}.,\rho.)),$$

where $\mathcal{T}.$ is the exact sequence

$$0 \to \mathcal{T}_{f_\mathbf{C}} \to \mathcal{T}_{(g\circ f)_\mathbf{C}} \to f^*\mathcal{T}_{g_\mathbf{C}} \to 0.$$

Proof. (1) For any two factorizations $f = g_1 \circ i_1 = g_2 \circ i_2$ as above, we may consider the fiber product $P_1 \times_Y P_2$ and the diagonal imbedding. So we are led to consider a diagram

$$\begin{array}{ccccc} X & \overset{i}{\hookrightarrow} & P & \overset{j}{\hookrightarrow} & P' \\ & f\searrow & \downarrow g & \swarrow h & \\ & & Y & & \end{array},$$

where g and h are smooth. We need to show that, for arbitrary choice of metrics,

$$\begin{aligned}&\mathrm{td_{Ar}}(i^*\mathcal{T}_g,\rho_{i^*\mathcal{T}_g})\cap(\mathrm{td}^{-1}_{\mathrm{Ar}}(\mathcal{N}_{X/P},\rho_{\mathcal{N}_{X/P}})^{-1}\cap\alpha)+\mathrm{td_{BC}}(f/g,\rho_{f/g})\cap\alpha\\ &=\mathrm{td_{Ar}}((j\circ i)^*\mathcal{T}_h,\rho_{(j\circ i)^*\mathcal{T}_h})\cap(\mathrm{td}^{-1}_{\mathrm{Ar}}(\mathcal{N}_{X/P'},\rho_{\mathcal{N}_{X/P'}})^{-1}\cap\alpha)+\mathrm{td_{BC}}(f/h,\rho_{f/h})\cap\alpha.\end{aligned}$$

Since there is an exact sequence on P:

$$0 \to T_g \to j^*T_h \to N_{P/P'} \to 0,$$

therefore by subsection 5.f, we have

$$\mathrm{td_{Ar}}(\mathcal{T}_g,\rho_{\mathcal{T}_g})\cap\alpha = \mathrm{td_{Ar}}(j^*\mathcal{T}_h,\rho_{j^*\mathcal{T}_h})\cap(\mathrm{td}^{-1}_{\mathrm{Ar}}(\mathcal{N}_{P/P'},\rho_{\mathcal{N}_{P/P'}})\cap\alpha)+\mathrm{td_{BC}}(g/h,\rho_{g/h})\cap\alpha.$$

On $X(\mathbf{C})$, we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \mathcal{T}_{f_\mathbf{C}} & = & \mathcal{T}_{f_\mathbf{C}} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & i^*\mathcal{T}_{g_\mathbf{C}} & \to & (j\circ i)^*\mathcal{T}_{h_\mathbf{C}} & \to & \mathcal{N}_{P(\mathbf{C})/P'(\mathbf{C})} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathcal{N}_{X(\mathbf{C})/P(\mathbf{C})} & \to & \mathcal{N}_{X(\mathbf{C})/P'(\mathbf{C})} & \to & i^*\mathcal{N}_{P(\mathbf{C})/P'(\mathbf{C})} & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array},$$

Thus by Lemma a.(1), we have the assertion.

(2) is a consequence of Lemma a.(3). In fact, since f and g are l.c.i. morphisms, by a standard argument, there exists a commutative diagram

$$\begin{array}{ccccc} X & \overset{i}{\hookrightarrow} & M' & \overset{j}{\hookrightarrow} & M \\ & f\searrow & \downarrow p & & \downarrow q \\ & & Y & \overset{k}{\hookrightarrow} & M'' \\ & & & g\searrow & \downarrow r \\ & & & & Z, \end{array}$$

in which p, q and r are smooth, i, j and k are regular immersions, and the square is Cartesian. Let us write $\mathcal{T}_p$, $\mathcal{T}_q$, $\mathcal{T}_r$, $\mathcal{T}_{r\circ q}$, $\mathcal{N}_i$, $\mathcal{N}_j$, $\mathcal{N}_k$ and $\mathcal{N}_{j\circ i}$ for the relative tangent bundles and normal bundles respectively of the maps p, q, r, ,$r\circ q$, i, j, k and $j\circ i$. Note that $\mathcal{N}_j \simeq p^*\mathcal{N}_k$, while $\mathcal{T}_p \simeq j^*\mathcal{T}_p$. We choose arbitrary metrics on these bundles, except for the condition that the two isomorphisms we have just mentioned are isometries. On $X(\mathbf{C})$, we get the following commutative diagram with exact lines and exact columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{T}_f & \rightarrow & i^*\mathcal{T}_p & \rightarrow & \mathcal{N}_i & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{T}_{gf} & \rightarrow & i^*j^*\mathcal{T}_{rq} & \rightarrow & \mathcal{N}_{ji} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & f^*\mathcal{T}_g & \rightarrow & f^*k^*\mathcal{T}_r & \rightarrow & f^*\mathcal{N}_k & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & . \end{array}$$

We have the assertion by Lemma a.(3).

There are similar results for $\mathrm{td}_{\mathrm{BC}}(f/g, \rho_{f/g})$. We leave them to the reader.

§II.3.9. Arithmetic Segre's Classes

We end this chapter by introducing the arithmetic Segre classes, which give an alternative way to define the arithmetic characteristic classes in general.

Let X be an arithmetic variety and $(\mathcal{E}, \rho)$ a hermitian vector sheaf of rank r on X. Denote by $P := \mathbf{P}(\mathcal{E})$ the projective bundle of $\mathcal{E}$ in the sense of Grothendieck, i.e. the bundle which represents all rank one quotients of $\mathcal{E}$, and $p : P \to X$ the projection. There is a canonical universal exact sequence on P

$$\mathrm{UN} : 0 \to \mathcal{S} \to p^*\mathcal{E} \to \mathcal{O}(1) \to 0,$$

where $\mathcal{O}(1)$ is the tautological line sheaf. We equip $p^*\mathcal{E}$ with the metric $p^*\rho$, and $\mathcal{S}$ (resp. $\mathcal{O}(1)$) with the induced (resp. quotient) metric. For every integer $k \geq 0$, define

$$t_{\mathrm{Ar,k}}(\mathcal{E}, \rho) := p_*\left(c_{1,\mathrm{Ar}}^{k+r-1}(\mathcal{O}(1), \rho_{\mathcal{O}(1)})\right) \in \mathrm{CH}^{\mathrm{k}}_{\mathrm{Ar}}(X).$$

Also, for each $k > 0$, define an element $R_k \in \tilde{A}^{k-1,k-1}(X)$ as follows: Let $\mathrm{UN}^\vee.$ be the dual of UN. and $\mathrm{UN}^\vee(1). := \mathrm{UN}^\vee. \otimes \mathcal{O}(1)$. For the obvious choice of metrics, let $c_{r,\mathrm{BC}}(\mathrm{UN}^\vee(1).,\rho_{\mathrm{UN}^\vee(1).}) \in \tilde{A}^{r-1,r-1}(P)$ be the classical Bott-Chern secondary characteristic class of this exact sequence with respect to the r-th Chern class c_r. Then R_k is the k-th coefficient of the formal power series

$$\sum_{k>0} R_k x^k = \Big(\sum_{k>0} p_*\big(c_1^{k-1}(\mathcal{O}(1),\rho_{\mathcal{O}(1)})\, c_{r,\mathrm{BC}}(\mathrm{UN}^\vee(1).,\rho_{\mathrm{UN}^\vee(1).})\big)x^k\Big)\Big(\sum_{j\geq 0} c_j(\mathcal{E},\rho)(-x)^j\Big)^{-1},$$

where we have used the module structure of $\tilde{A}^\cdot(X)$ on the ring $A^\cdot(X)$.

Now we define the **arithmetic Segre's class** $s_{k\,\mathrm{Ar}}(\mathcal{E},\rho) \in \mathrm{CH}^k_{\mathrm{Ar}}(X),\ k \geq 0$ by

$$s_{k\,\mathrm{Ar}}(\mathcal{E},\rho) = \begin{cases} 1, & \text{if } k = 0; \\ t_{k\mathrm{Ar}}(\mathcal{E},\rho) + a(R_k), & \text{if } k > 0. \end{cases}$$

The relation between the arithmetic Serge classes and the arithmetic Chern classes is expressed in terms of this definition by the following

Theorem. With the same notation as above,

$$\sum_{j\geq 0} c_{j\mathrm{Ar}}(\mathcal{E},\rho)(-t)^j = \Big(\sum_{k\geq 0} s_{k\mathrm{Ar}}(\mathcal{E},\rho)t^k\Big)^{-1}.$$

Proof. From the exact sequence

$$UN^\vee(1) : 0 \to \mathcal{O}_P \to p^*(\mathcal{E}^\vee)(1) \to S^\vee(1) \to 0,$$

by the additivity of Chern classes and the behavior under tensoring by a line sheaf, note that the rank of $\mathcal{E}$ is r, we get

$$\begin{aligned} a\big(c_{r\,\mathrm{BC}}(UN^\vee(1),\rho_{UN^\vee(1)})\big) &= c_{r\,\mathrm{Ar}}(\mathcal{O}_P \oplus S^\vee(1),\rho_{\mathcal{O}_P\oplus S^\vee(1)}) - c_{r\mathrm{Ar}}(p^*(\mathcal{E},\rho)^\vee(1)) \\ &= -\,c_{r\mathrm{Ar}}(p^*(\mathcal{E},\rho)^\vee(1)) \\ &= -\sum_{j\geq 0} p^*\big(c_{j\mathrm{Ar}}((\mathcal{E},\rho)^\vee)\big)c_{1\mathrm{Ar}}^{r-j}(\mathcal{O}(1),\rho_{\mathcal{O}(1)}) \\ &= -\sum_{j\geq 0} p^*\big((-1)^j c_{j\mathrm{Ar}}((\mathcal{E},\rho))\big)c_{1\mathrm{Ar}}^{r-j}(\mathcal{O}(1),\rho_{\mathcal{O})(1)}). \end{aligned}$$

On the other hand, if we apply the morphism a to the defining equation of R_k above, then, by the projective formula, we have

$$\begin{aligned}
&\sum_{k>0} a(R_k)x^k \\
&= -\Big(\sum_{\substack{k>0\\ j\geq 0}} p_*\big(c_{1,\mathrm{Ar}}^{k-1}(\mathcal{O}(1), \rho_{\mathcal{O}(1)})\, p^*((-1)^j c_{j\mathrm{Ar}}((\mathcal{E},\rho)))c_{1\mathrm{Ar}}^{r-j}(\mathcal{O}(1), \rho_{\mathcal{O})(1)})\big)x^k\Big) \\
&\quad \big(\sum_{j\geq 0} c_{j\,\mathrm{Ar}}(\mathcal{E},\rho)^\vee(-x)^j\big)^{-1} \\
&= -\Big(\sum_{j\geq 0} c_{j\mathrm{Ar}}(\mathcal{E},\rho)(-x)^j\big(\sum_{k>0} p_*(c_{1\,\mathrm{Ar}}^{k-j+r-1}(\mathcal{O}(1), \rho_{\mathcal{O}(1)})x^{k-j}\big)\Big)\big(\sum_{j\geq 0} c_{j\,\mathrm{Ar}}(\mathcal{E},\rho)(-x)^j\big)^{-1}.
\end{aligned}$$

Therefore, by the fact that $f_*\big(c_{1\,\mathrm{Ar}}^n(\mathcal{O}(1), \rho_{\mathcal{O}(1)})\big) = 0$ unless $n \geq r-1$, we have

$$\sum_{k>0} a(R_k)x^k = -\sum_{k\geq 0} t_{k\,\mathrm{Ar}}(\mathcal{E},\rho)x^k + \big(\sum_{j\geq 0} c_{j\,\mathrm{Ar}}(\mathcal{E},\rho)(-x)^j\big)^{-1}.$$

This completes the proof.

Remark. It follows from this theorem that arithmetic Segre's classes provide an alternative way to define arithmetic Chern classes. Hence we may define alternatively all arithmetic classes of hermitian vector sheaves, rather than using the splitting principle as before.

Chapter II.4. Arithmetic Riemann-Roch Theorem For Smooth Morphisms

In this chapter, we prove an arithmetic Riemann-Roch theorem for smooth morphisms of regular arithmetic varieties $f : X \to Y$, which was first given in [Fa 92]. To do so, the first problem we meet is how to define a push-out morphism for arithmetic K-groups. It is at this stage that we have to use the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms at infinity induced by f developed in Part I. We are going to give this definition in section 1. In section 2 we state the arithmetic Riemann-Roch theorem for smooth morphisms. Finally, in section 3, we use the axioms for the relative Bott-Chern secondary characteristic forms with respect to a smooth morphism to prove the arithmetic Riemann-Roch theorem stated in section 2.

§II.4.1. The Push-Out Morphism of Arithmetic K-Groups.

We now make a natural definition of the push-out morphism of arithmetic K-groups with respect to a smooth morphism $f : X \to Y$ of regular arithmetic varieties:

$$f_K : K_{\mathrm{Ar}}(X) \to K_{\mathrm{Ar}}(Y).$$

Since $K_{\mathrm{Ar}}(X)$ is generated by f-acyclic hermitian vector sheaves and elements in $\tilde{A}(X_{\mathbf{R}})$, it is sufficient to give a definition for both of these elements and to check the compatibility with the equivalence relations among these objects.

II.4.1.a. For f-Acyclic Hermitian Vector Sheaves

Let $f : X \to Y$ be a smooth morphism of regular arithmetic varieties X, Y over an arithmetic ring (A, Σ, F_∞). Let $(\mathcal{E}, \rho)$ be an f-acyclic hermitian vector sheaf on X. There is a natural element $(f_*\mathcal{E}, f_*\rho)$ in $K_{\mathrm{Ar}}(Y)$. On the other hand, by the Riemann-Roch theorem in the algebraic sense, it is quite natural to consider another element in $K_{\mathrm{Ar}}(Y)$, which corresponds to the element $f_{\mathrm{CH}}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}, \rho)\mathrm{td}_{\mathrm{Ar}}(f, \rho_f))$ via the isomorphism $\mathrm{ch}_{\mathrm{Ar}}$ between $K_{\mathrm{Ar}}(X)$ and $\mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}}$. In Part I, we introduced the relative Bott-Chern secondary characteristic form with respect to the above data, which is supposed to be the object measuring the difference in $K_{\mathrm{Ar}}(Y)$ between $(f_*\mathcal{E}, f_*\rho)$ and the element corresponding to $f_{\mathrm{CH}}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\mathrm{td}_{\mathrm{Ar}}(f,\rho_f))$ (see 3.5.f). Therefore, by the definition of Green's currents, we make the following definition:

$$f_K(\mathcal{E},\rho) := (f_*\mathcal{E}, f_*\rho) + \mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho; f,\rho_f).$$

II.4.1.b. For Differential Forms

Since the arithmetic Riemann-Roch theorem is supposed to be a generalization of the classical algebraic Riemann-Roch theorem, we make the following definition: For any element $\omega \in \tilde{A}(X_{\mathbf{R}})$,

$$f_K(\omega) := f_*(a(\omega)\,\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)).$$

II.4.1.c. General Situation.

We are now ready to give the definition of f_K for general elements. By linearity, it is enough to give the definition of f_K for a hermitian vector sheaf $(\mathcal{F},\tau)$.

There exists an f-acyclic vector sheaf resolution for $\mathcal{F}$: In fact since f is proper, there exists a relative sufficient ample line sheaf $\mathcal{L}$ on X. Therefore for any free resolution of $\mathcal{F}\otimes\mathcal{L}^{-1}$, we can get an f-acyclic vector sheaf resolution

$$\mathcal{E}.: 0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

where $\mathcal{E}_i$ are f-acyclic vector sheaves. Now put hermitian metrics ρ_i on $\mathcal{E}_i$, and let

$$f_K(\mathcal{F},\tau) := \sum_{i=0}^{n}(-1)^i f_K(\mathcal{E}_i,\rho_i) - f_K(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho_i,\tau)).$$

We need the following

Proposition. Let $f : X \to Y$ be a smooth morphism of regular arithmetic varieties over an arithmetic ring (A,Σ,F_∞). Then
(1) f_K is well-defined;
(2) f_K is a group morphism.

II.4.1.d. Proof of The Proposition.

It is sufficient to show that for any short exact sequence of f-acyclic vector sheaves

$$\mathcal{E}.: 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

on X and any F_∞-invariant hermitian metrics ρ_i for $i = 1, 2, 3$, we have

$$f_K(\mathcal{E}_1, \rho_1) + f_K(\mathcal{E}_3, \rho_3) = f_K(\mathcal{E}_2, \rho_2) - f_K(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.)). \qquad (*)$$

In fact, suppose we have this assertion, then f_K is well-defined, since for any hermitian vector sheaf $(\mathcal{F}, \tau)$, any two f-acyclic vector sheaf resolutions are dominated by a common third one, and hence the proposition comes from a very simple relation between the associated classical Bott-Chern secondary characteristic forms, say, Theorem I.1.2.c.

The proof of (*) is also very simple. In fact, in $K_{\mathrm{Ar}}(Y)$, we have

$$(f_*\mathcal{E}_1, f_*\rho_1) + (f_*\mathcal{E}_3, \rho_3) = (f_*\mathcal{E}_2, \rho_2) - \mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{E}., f_*\rho.).$$

Here, we let

$$f_*\mathcal{E}. : 0 \to f_*\mathcal{E}_1 \to f_*\mathcal{E}_2 \to f_*\mathcal{E}_3 \to 0$$

be the image of the exact sequence $\mathcal{E}.$. Thus, we have to show that

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_2, \rho_2; f, \rho_f) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_1, \rho_1; f, \rho_f) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_3, \rho_3; f, \rho_f)\\ =& f_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.)\mathrm{td}(f, \rho_f)) - \mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{E}., f_*\rho.),\end{aligned}$$

which is nothing but Axiom 3 of relative Bott-Chern secondary characteristic forms with respect to smooth morphisms in I.2.

II.4.2. Arithmetic Riemann-Roch Theorem for Smooth Morphisms

In this section, we state and explain the arithmetic Riemann-Roch theorem for smooth morphisms.

However, before stating the theorem, we need to introduce some more notation: Let B be a subring of $\mathbf{R}$, and let $P(x) \in B[[x]]$ be any power series. Then for any hermitian vector sheaf $(\mathcal{E}, \rho)$, by the splitting principle, there exists a unique additive characteristic class $P(\mathcal{E}) \in H^{\mathrm{ev}}(X)$, where $H^{\mathrm{ev}}(X)$ is the even homology of X. We define the **modified arithmetic Todd characteristic class with respect to** P by letting

$$\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(\mathcal{E}, \rho) := \mathrm{td}_{\mathrm{Ar}}(\mathcal{E}, \rho)(1 - a(P(\mathcal{E}))).$$

With this, we have the following

Arithmetic Riemann-Roch theorem For Smooth Morphisms. (Faltings [F 92]) There exists a unique power series $R(x)$ such that for any smooth morphism of regular arithmetic varieties $f : X \to Y$ over an arithmetic ring (A, Σ, F_∞) with an F_∞-invariant hermitian metric ρ_f on the relative tangent sheaf of f, the following diagram is commutative:

$$\begin{array}{ccc} K_{\mathrm{Ar}}(X) & \xrightarrow{\mathrm{ch}_{\mathrm{Ar}}()\,\mathrm{Td}^{\mathrm{R}}_{\mathrm{Ar}}(f,\rho_f)} & \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}} \\ f_K \downarrow & & \downarrow f_{\mathrm{CH}} \\ K_{\mathrm{Ar}}(Y) & \xrightarrow{\mathrm{ch}_{\mathrm{Ar}}()} & \mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}. \end{array}$$

Here $\mathrm{Td}^{\mathrm{R}}_{\mathrm{Ar}}$ denotes the modified arithmetic Todd characteristic class with respect to R.

Next, we prove this theorem.

II.4.2.a. Several Intermediary Results.

With the same notation as above, for any power series P, for any smooth morphism $f : X \to Y$ of regular arithmetic varieties over an arithmetic ring (A, Σ, F_∞), any f-acyclic hermitian vector sheaf $(\mathcal{E}, \rho)$ on X, let

$$\mathrm{Err}(\mathcal{E}, \rho; f, \rho_f; P) := \mathrm{ch}_{\mathrm{Ar}}(f_K(\mathcal{E}, \rho)) - f_{\mathrm{CH}}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}, \rho)\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(f, \rho_f)).$$

To prove the theorem, it is sufficient to show that there exists a unique power series $R(x)$ so that

$$\mathrm{Err}(\mathcal{E}, \rho; f, \rho_f; R) = 0.$$

For this we need some intermediary results.

Proposition 1. Let $f : X \to Y$ be a smooth morphism of regular arithmetic varieties with an F_∞-invariant hermitian metric ρ_f on the relative tangent vector sheaf of f. Then for any short exact sequence of f-acyclic hermitian vector sheaves

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

with F_∞-invariant hermitian metrics ρ_i on $\mathcal{E}_i$ for $i = 1, 2, 3$, we have

$$\mathrm{Err}(\mathcal{E}_1, \rho_1; f, \rho_f; P) + \mathrm{Err}(\mathcal{E}_3, \rho_3; f, \rho_f; P) = \mathrm{Err}(\mathcal{E}_2, \rho_2; f, \rho_f; P).$$

In particular, $\mathrm{Err}(\mathcal{E}, \rho; f, \rho_f; P)$ does not depend on the metric ρ. Furthermore, we have that $\mathrm{Err}(\mathcal{E}, \rho; f, \rho_f; P)$ lies in the a-image of harmonic forms.

Proposition 2. Let $f : X \to Y$ and $g : Y \to Z$ be two smooth morphisms of regular arithmetic varieties which have F_∞-invariant hermitian metrics ρ_f, ρ_g and $\rho_{g\circ f}$ on the relative tangent vector sheaves of f, g and $g \circ f$ respectively. Let $(\mathcal{E}, \rho)$ be an f-acyclic hermitian vector sheaf on X such that $f_*\mathcal{E}$ is g-acyclic. Then

$$\mathrm{Err}(\mathcal{E}, \rho; g \circ f, \rho_{g\circ f}; P) = \mathrm{Err}(f_*\mathcal{E}, f_*\rho; g, \rho_g; P) + g_*(\mathrm{Err}(\mathcal{E}, \rho; f, \rho_f; P)\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(g, \rho_g)).$$

In particular, $\mathrm{Err}(\mathcal{E}, \rho; f, \rho_f; P)$ does not depend on the metric ρ_f.

Remark. Because of these two propositions, we denote $\mathrm{Err}(\mathcal{E}, \rho; f, \rho_f; P)$ simply by $\mathrm{Err}(\mathcal{E}; f; P)$.

Proposition 3. There is a natural morphism

$$\mathrm{Err} : K(X_F) \to H(Y_{\mathbf{R}})/\rho(\mathrm{CH}^{(1,0)}(Y))_{\mathbf{Q}},$$

such that $\mathrm{Err}(\mathcal{E};P)=\mathrm{Err}(E;f;P)$.

Proposition 4. Let $f: X \to Y$ be a smooth morphism of regular arithmetic varieties with an F_∞-invariant hermitian metric ρ_f on the relative tangent vector sheaf of f. Then, for any flat base change $g: Z \to Y$, we have

$$g^*\mathrm{Err}(\mathcal{E};f;P)=\mathrm{Err}(g_f^*\mathcal{E};f_g;P).$$

Here we use the following diagram

$$\begin{array}{ccc} Z\times_Y X & \xrightarrow{g_f} & X \\ f_g\downarrow & & \downarrow f \\ Z & \xrightarrow{g} & Y. \end{array}$$

Proposition 5. There is a unique power series $R(x)$ such that for any $\mathbf{P}^1$-bundle

$$p: X=\mathbf{P}_Y(\mathcal{F})\to Y,$$

$\mathrm{Err}(\mathcal{E};p;R)=0$.

Finally, we consider Err for closed immersions. In this case, we have to introduce a new Err term. That is, let $i: X\hookrightarrow Z$ be a closed immersion with the smooth structure morphisms $f: X\to Y$ and $g: Z\to Y$ of regular arithmetic varieties, then we define

$$\mathrm{Err}(\mathcal{E};i;P):=\mathrm{Err}(\mathcal{E};f;P)-\mathrm{Err}(i_*\mathcal{E};g;P).$$

By Proposition 3, this definition makes sense, even through $i_*\mathcal{E}$ is usually only a coherent sheaf.

Proposition 6. Let $i: X\hookrightarrow Z$ be a *codimension-one* regular closed immersion of regular arithmetic varieties over an arithmetic variety Y with smooth structure morphisms $f: X\to Y$ and $g: Z\to Y$. Let $(\mathcal{E},\rho)$ be an f-acyclic hermitian vector sheaf on X such that $i_*\mathcal{E}$ is g-acyclic, then

$$\mathrm{Err}(\mathcal{E};i;P)=0.$$

II.4.2.b. **The Proof of The Propositions.**

Proof of Proposition 1. Since the independence of Err with respect to the metric is a consequence of the first assertion, it is sufficient to prove that Err is additive and is in the a-image of harmonic forms.

We go back to the definition, and have

$$\begin{aligned}&\mathrm{Err}(\mathcal{E}_1,\rho_1;f,\rho_f;P)+\mathrm{Err}(\mathcal{E}_3,\rho_3;f,\rho_f;P)\\ =&\mathrm{ch}_{\mathrm{Ar}}(f_K(\mathcal{E}_1,\rho_1))-f_{\mathrm{CH}}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_1,\rho_1)\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(f,\rho_f))\\ &+\mathrm{ch}_{\mathrm{Ar}}(f_K(\mathcal{E}_3,\rho_3))-f_{\mathrm{CH}}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_3,\rho_3)\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(f,\rho_f)).\end{aligned}$$

On the other hand, by Proposition 4.1.2, we know that

$$f_K(\mathcal{E}_1,\rho_1)+f_K(\mathcal{E}_3,\rho_3)=f_K(\mathcal{E}_2,\rho_2)-f_K(\mathrm{ch_{BC}}(\mathcal{E}.,\rho.)).$$

Hence by the fact that

$$\begin{aligned}&\mathrm{ch_{B\check{C}}}(\mathcal{E}.,\rho.)\\&=\mathrm{ch_{Ar}}(\mathcal{E}_2,\rho_2)-\mathrm{ch_{Ar}}(\mathcal{E}_1,\rho_1)-\mathrm{ch_{Ar}}(\mathcal{E}_3,\rho_3),\end{aligned}$$

and that the image of a is a square zero ideal, we have

$$\begin{aligned}&\mathrm{Err}(\mathcal{E}_1,\rho_1;f,\rho_f;P)+\mathrm{Err}(\mathcal{E}_3,\rho_3;f,\rho_f;P)\\&=\mathrm{ch_{Ar}}(f_K(\mathcal{E}_2,\rho_2))-f_K(\mathrm{ch_{BC}}(\mathcal{E}.,\rho.))\\&\quad-f_{\mathrm{CH}}(\mathrm{ch_{Ar}}(\mathcal{E}_1,\rho_1)\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(f,\rho_f))-f_{\mathrm{CH}}(\mathrm{ch_{Ar}}(\mathcal{E}_3,\rho_3)\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(f,\rho_f))\\&=\mathrm{ch_{Ar}}(f_K(\mathcal{E}_2,\rho_2))-f_{\mathrm{CH}}(\mathrm{ch_{Ar}}(\mathcal{E}_2,\rho_2)\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(f,\rho_f))\\&=\mathrm{Err}(\mathcal{E}_2,\rho_2;f,\rho_f;P).\end{aligned}$$

Now we prove the third assertion. That is, Err lies in the image of harmonic forms. This assertion has two aspects. First, for algebraic cycles, note that the arithmetic definition is a natural generalization of the algebraic one, and the arithmetic Riemann-Roch theorem is a natural generalization of the classical theorem. We know that the image of Err, via the forgetting map on algebraic cycles, is zero. Secondly, for currents, under the natural morphism

$$\omega:\mathrm{CH_{Ar}}(Y)_{\mathbf{Q}}\to\tilde{A}(Y_{\mathbf{R}}),$$

which sends $((\mathcal{F},\tau),\alpha)$ to $\mathrm{ch}(\mathcal{F},\tau)+dd^c\alpha$, we know that

$$\begin{aligned}&\omega(\mathrm{Err}(\mathcal{E},\rho;f,\rho_f;P))\\&=\mathrm{ch}(f_*\mathcal{E},f_*\rho)-f_*(\mathrm{ch}(\mathcal{E},\rho)\mathrm{td}(f,\rho_f))+dd^c\mathrm{ch_{BC}}(\mathcal{E},\rho;f,\rho_f).\end{aligned}$$

Thus by Axiom 1 of relative Bott-Chern secondary characteristic forms with respect to smooth morphisms in I.2, we have

$$\omega(\mathrm{Err}(\mathcal{E},\rho;f,\rho_f;P))=0.$$

So we may complete the proof by considering the structure of the homology exact sequence associated with the arithmetic Chow groups in 2.3.c .

Proof of Proposition 2. By putting $g=\mathrm{Id}_Y$, we find that the second statement of this proposition is a consequence of the first one.

Now we prove the first statement. The result for Err is equivalent to

$$\begin{aligned}&\mathrm{ch_{Ar}}((g\circ f)_*\mathcal{E},(g\circ f)_*\rho)+\mathrm{ch_{BC}}(\mathcal{E},\rho;g\circ f,\rho_{g\circ f})-(g\circ f)_{\mathrm{CH}}(\mathrm{ch_{Ar}}(\mathcal{E},\rho)\,\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(g,\circ f,\rho_{g\circ f}))\\&=g_{\mathrm{CH}}(\mathrm{ch_{Ar}}(f_*\mathcal{E},f_*\rho)\,\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(g,\rho_g))+g_{\mathrm{CH}}(\mathrm{ch_{BC}}(\mathcal{E},\rho;f,\rho_f)\,\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(g,\rho_g))\\&\quad-(g\circ f)_{\mathrm{CH}}(\mathrm{ch_{Ar}}(\mathcal{E},\rho)\,\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(f;\rho_f)\,f^*(\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(g,\rho_g)))\\&\quad+\mathrm{ch_{Ar}}((g\circ f)_*\mathcal{E},(g\circ f)_*\rho)+\mathrm{ch_{BC}}(f_*\mathcal{E},f_*\rho;g,\rho_g)\\&\quad-g_{\mathrm{CH}}(\mathrm{ch_{Ar}}(f_*\mathcal{E},f_*\rho)\,\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(g,\rho_g)).\end{aligned}$$

By definition, this equation is nothing but

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;g\circ f,\rho_{g\circ f})-\mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{E},f_*\rho;g,\rho_g)-g_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f,\rho_f)\,\mathrm{td}(g,\rho_g))\\&=(g\circ f)_{\mathrm{CH}}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)(\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(g\circ f,\rho_{g\circ f})-\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(f,\rho_f)\,f^*(\mathrm{Td}^{\mathrm{P}}_{\mathrm{Ar}}(g,\rho_g)))).\end{aligned}$$

We see that the image of a is a square zero ideal, and

$$\mathrm{td}_{\mathrm{BC}}(f,g,g\circ f)=\mathrm{td}_{\mathrm{Ar}}(g\circ f,\rho_{g\circ f})-\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)\,f^*(\mathrm{td}_{\mathrm{Ar}}(g,\rho_g)).$$

So the previuos assertion is a direct consequence of Axiom 4 of relative Bott-Chern secondary characteristic forms with respect to smooth morphisms in I.2.

Proof of Proposition 3. By Proposition 1, 2, it is sufficient to prove that Err may be descended to $K(X_F)$. Consider the relation:

$$\mathrm{Ker}(K(X)\to K(X(F)))=\mathrm{Im}(K(X)_{\mathrm{fin}}\to K(X)),$$

where $K(X)_{\mathrm{fin}}=\oplus_p K^{X_p}(X)$ is the Grothendieck group of X with support in finite fibres. We also have the commutative diagram

$$\begin{array}{ccc}K(X)_{\mathrm{fin}} & \to & K(X)\\ \mathrm{ch}()\downarrow & & \downarrow \mathrm{ch}_{\mathrm{Ar}}\\ \mathrm{CH}(X)_{\mathrm{fin}} & \xrightarrow{i} & \mathrm{CH}_{\mathrm{Ar}}(X),\end{array}$$

where $i(z)=(z,0)$. So the assertion is a consequence of the classical Grothendieck-Riemann-Roch theorem with supports.

Proof of Proposition 4. Note that everything in the expression is compatible with a flat base change, so we have the assertion.

Proof of Proposition 5. This is a consequence of Proposition 3 and a direct calculation. As a $K(Y)$-module, $K(X)$ is generated by $\mathcal{O}_X$, and $\mathcal{O}_X(-1)$, so by Proposition 3, it is sufficient to show that Err is zero for both of these two elements: We prove this by using the functorial property of Err. Since Err does not depend on the metrics, we can normalize the metrics as follows: Take an F_∞-invariant hermitian metric ρ on $\mathcal{F}$ and put the induced metric on $\mathcal{O}(1)$. (Since we have the universal exact sequence

$$p^*\mathcal{F}^\vee\to\mathcal{O}_X(1)\to 0.)$$

Then we have an induced metric on $\mathcal{O}(n)$ for any integer n. By the exact sequence

$$0\to\mathcal{O}_X\to p^*\mathcal{F}\otimes\mathcal{O}_X(1)\to\mathcal{T}_{X/Y}\to 0,$$

we have $\mathcal{T}_{X/Y}\simeq(\det\mathcal{F})\otimes\mathcal{O}_X(2)$. In particular, we can take the metric on $\mathcal{T}_{X/Y}$ such that the algebraic isomorphism is an isometry.

Let $Y_n := G_{2,n}$ denote the Grassmannian of rank 2 subgroups of $\mathbf{Z}^{n+2}$, and let

$$X_n := \mathbf{P}_{Y_n}(\mathcal{S}_n) \to Y_n$$

be the projective bundle associated with the universal subbundle $\mathcal{S}_n$ (which is a $\mathbf{P}^1$-bundle). By the universal property of Grassmannians, the natural map a on $H^*(Y_n)$ is an injection, and by a standard argument, say Prop. 23.2 [BT 82], we know that

$$\lim_{\leftarrow} H(Y_n) = \mathbf{R}[[c_1(\mathcal{F},\rho), c_2(\mathcal{F},\rho)]].$$

So we may prove the proposition by the following argument. First, the arithmetic Riemann-Roch theorem for projective morphisms holds up to an error term in $H^*(Y_n)$. Secondly, these error terms correspond to each other with respect to n under the natural injection $Y_n \hookrightarrow Y_{n+1}$ (by the definition) and remain unchanged if we tensor $\mathcal{S}_n$ with a line sheaf. (See also 3.3.) Thus, we obtain universal classes in $\lim_{\leftarrow} H(Y_n)$ so that for any rank two vector bundle $\mathcal{F}$, which is generated by its global sections, these classes define the errors in arithmetic Riemann-Roch theorem for $\mathbf{P}_Y(\mathcal{F}) \to Y$ once and for all.

Now we determine these errors. Note that the direct image of the structure sheaf is the structure sheaf below. Let $A = c_1(\mathcal{O}(1), \rho_{\mathcal{O}(1)}) + \frac{1}{2}c_1(\mathcal{F},\rho)$. Then

$$c_1(\mathcal{T}_{X/Y}, \rho_{\mathcal{T}}) = 2A,\ A^2 = -c_2(\mathcal{F},\rho) + \frac{1}{4}c_1(\mathcal{F},\rho)^2$$

and

$$f_*A^{2m} = 0,\ \ f_*A^{2m+1} = (A^2)^m f_*A = (A^2)^m$$

for all positive integer m. If we tensor $\mathcal{S}_n$ with a sufficient very ample line sheaf, we will see that the error term for $\mathcal{O}_X$ is given by a power series in $c_1^2(\mathcal{F},\rho) - 4c_2(\mathcal{F},\rho)$. The same is true for $\mathcal{O}_X(-1)$ if we multiply the error term by $\exp(\frac{1}{2}c_1(\mathcal{F}^\vee, \rho^\vee))$.

Finally, in order to complete the proof of the proposition, we have to show that one may adjust P uniquely such that both errors become zero. Obviously, if we change P by δ, then the error of arithmetic Riemann-Roch theorem expression for $\mathcal{O}_X$ changes by

$$f_*(\frac{2A}{1-e^{-2A}}\delta(2A))$$

and similarly for $\mathcal{O}_X(-1)$, (up to $\exp(\frac{1}{2}c_1(\mathcal{F}^\vee,\rho^\vee)$,) the error changes by

$$f_*(\frac{2Ae^{-A}}{1-e^{-2A}}\delta(2A)).$$

In the second expression, the factor before δ has a series in even powers. So, we may choose the odd part, δ^{odd}, such that the error vanishes for $\mathcal{O}_X(-1)$ uniquely. Hence, we may assume that δ is even, and try to cancel the error for $\mathcal{O}_X$. But, for $\mathcal{O}_X$, only the odd part in the power series matters, that is, we have to consider

$$f_*(A\delta(2A)).$$

Again this is unique. This completes the proof.

Proof of Proposition 6. We prove this proposition as follows:

(a) Consider the special situation with codimension-one closed immersions as the zero sections of a $\mathbf{P}^1$-bundle;
(b) Reduce the general codimension-one closed immersion to the zero section of a $\mathbf{P}^1$-bundle by the deformation to the normal cone technique.

(a) Suppose $i_1 : X \hookrightarrow Z$ is a zero section of a $\mathbf{P}^1$-bundle. Then, we have the following diagram

$$\begin{array}{ccc} Y & \overset{i_1}{\hookrightarrow} & \mathbf{P}^1_Y(\mathcal{F}) \\ & \mathrm{Id}_Y \searrow \quad \swarrow p_1 & \\ & Y. & \end{array}$$

Hence, for any vector sheaf $\mathcal{E}$ on X,

$$\mathrm{Err}(\mathcal{E}, i_1) = \mathrm{Err}(\mathcal{E}, \mathrm{Id}) - \mathrm{Err}(i_{1*}\mathcal{E}, p_1).$$

The arithmetic Riemann-Roch theorem is clearly valid for the identity morphisms. So we have

$$\mathrm{Err}(\mathcal{E}, \mathrm{Id}) = 0.$$

On the other hand, since p_1 is a natural projection from a $\mathbf{P}^1$-bundle, by Proposition 5, we know that

$$\mathrm{Err}(i_{1*}\mathcal{E}, p_1) = 0.$$

Therefore, we get

$$\mathrm{Err}(\mathcal{E}, i_1) = 0.$$

Hence, if we can reduce any codimension-one closed immersion to the zero section of certain $\mathbf{P}^1$-bundle, we have proved the proposition.

(b) Now we make the reduction required by the last statement in (a). Throughout this part, we must know that now we are working with a special closed immersion, i.e. a closed immersion of codimension one.

Recall the diagram from the deformation to the normal cone from I.9.2.a. We have

$$\begin{array}{ccccccc} X & \leftarrow & X \times \{\infty\} & \overset{i_\infty}{\hookrightarrow} & \mathrm{P}(\mathcal{N} \oplus \mathcal{O}_X) + B_X Z = W_\infty & \rightarrow & \{\infty\} \\ \downarrow & & \downarrow & & j_\infty \downarrow & & \downarrow \\ X & \overset{p}{\leftarrow} & X \times \mathbf{P}^1 & \overset{I}{\hookrightarrow} & B_{X \times \infty} Z \times \mathbf{P}^1 = W & \overset{p}{\rightarrow} & \mathbf{P}^1 \\ \uparrow & & \uparrow & & j_0 \uparrow & & \uparrow \\ X & \leftarrow & X \times \{0\} & \overset{i_0}{\hookrightarrow} & Z \times \{0\} = W_0 & \rightarrow & \{0\}. \end{array}$$

Lemma. With the same notation as above, we have

$$\mathrm{Err}(\mathcal{E}, i_0) = \mathrm{Err}(\mathcal{E}, i_\infty).$$

Obviously, the proposition is a direct consequence of the lemma.

Proof of the lemma. First, by definition, the equation may be read as follows:

$$\mathrm{Err}(\mathcal{E}, f_0) - \mathrm{Err}(i_{0*}\mathcal{E}, g_0) = \mathrm{Err}(\mathcal{E}, f_\infty) - \mathrm{Err}(i_{\infty*}\mathcal{E}, g_\infty).$$

Now the difficult is that $i_*\mathcal{E}$ is usually only a coherent sheaf. One may get rid of this difficult by the natural exact sequence

$$0 \to \mathcal{I}\mathcal{E} \to \mathcal{E} \to i_*\mathcal{E} \to 0.$$

Since i is a codimension-one closed immersion, so in fact $\mathcal{I}\,\mathcal{E}$ is a vector sheaf. Formally we may consider in the arithmetic K-group that $i_*\mathcal{E}$ as the summation of the hermitian vector sheaves associated with $\mathcal{I}\,\mathcal{E}$, $\mathcal{E}$, and the relative Bott-Chern secondary characteristic current with respect to i. Thus, by the facts that the arithmetic Riemann-Roch theorem is valid for forms, or better currents (just by the definition), and that $X \times \mathbf{P}^1$ does not intersect W^2_∞ in W, we know that, to prove the above lemma, it is sufficient to show that

$$\mathrm{Err}(i_{0*}\mathcal{E}, g_0) = \mathrm{Err}(i_{\infty*}\mathcal{E}, g_\infty).$$

Indeed, we know that the Err has nothing to do with the associated metrics, and could be descent to the algebraic K group, so the above formal process makes sense. In particular, we see that the latest assertion is equivalent to the following

Lemma$'$. Let $\mathcal{E}.: 0 \to \mathcal{I}_{X\times\mathbf{P}^1}\mathcal{E}(W^2_\infty) \to \mathcal{E}(W^2_\infty)$ be an exact sequence on W. Then

$$\mathrm{Err}(\mathcal{E}.|_0, g_0) = \mathrm{Err}(\mathcal{E}.|_\infty, g_\infty).$$

Proof of the lemma$'$. By definition, the equality in the lemma$'$ may be written as

$$\begin{aligned} &g_{0*}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.|_0, \rho.|_0)\,\mathrm{Td}^R_{\mathrm{Ar}}(g_0, \rho_{g_0})) - g_{\infty*}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.|_\infty, \rho.|_\infty)\,\mathrm{Td}^R_{\mathrm{Ar}}(g_\infty, \rho_{g_\infty}))\\ &\quad=\mathrm{ch}_{\mathrm{Ar}}(g_{0*}(\mathcal{E}.|_0, \rho.|_0) + \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.|_0, \rho.|_0; g_0, \rho_{g_0})\\ &\qquad - \mathrm{ch}_{\mathrm{Ar}}(g_{\infty*}(\mathcal{E}.|_\infty, \rho.|_\infty)) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.|_\infty, \rho.|_\infty; g_\infty, \rho_{g_\infty})). \end{aligned}$$

We first study the difference on the right hand side by using the deformation theory for the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms in I.9.2: On one hand, from the proof of Proposition 3.5.f.2 (via the $\mathbf{P}^1$-deformation), by the fact that

$$\mathrm{div}_{\mathrm{Ar}}(z) = ((0) - (\infty), -[\log|z|^2]),$$

where z is the usual coordinate of $\mathbf{P}^1$, we know that

$$\mathrm{ch}_{\mathrm{Ar}}(g_{0*}(\mathcal{E}.|_0, \rho.|_0)) - \mathrm{ch}_{\mathrm{Ar}}(g_{\infty*}(\mathcal{E}.|_\infty, \rho.|_\infty))$$

offers the classical Bott-Chern secondary characteristic forms with respect to the change of the metrics from 0 to ∞ for $\mathcal{F}.$, which is defined by the direct image of the exact sequence of

$$0 \to \mathcal{I}_X\mathcal{E} \to \mathcal{E}$$

on Y. (Remember that $X \times \mathbf{P}^1$ does not intersect W^2_∞.) That is,

$$\begin{aligned}&\mathrm{ch}_{\mathrm{Ar}}(g_{0*}(\mathcal{E}.|_0,\rho.|_0)) - \mathrm{ch}_{\mathrm{Ar}}(g_{\infty*}(\mathcal{E}.|_\infty,\rho.|_\infty))\\ &=(0,\int_{\mathbf{P}^1}\mathrm{ch}(\mathcal{F}.,\rho_{\mathcal{F}}.))\,[\log|z|^2].\end{aligned}$$

Hence, by the axiom for the Bott-Chern ternary charactertistic forms with respect to the smooth morphisms in I.9.2.a, we see that the right hand side is

$$\int_{\mathbf{P}^1}\omega(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.;G,\rho_G))\,[\log|z|^2].$$

We claim that this is just the element given by the difference of the left hand side above.

To prove the latest claim, we need first note the fact that $\omega(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.;G,\rho_G))$ gives the difference

$$G_*(\mathrm{ch}(\mathcal{E}.,\rho)\,\mathrm{td}(\mathcal{T}_G(-\log\infty),\rho_G)) - \mathrm{ch}(G_*\mathcal{E}.,G_*\rho.).$$

Therefore, by the fact that $[\log|z|^2]$ has total mass zero on $\mathbf{P}^1$, we see that the right hand of the Err relation above only offers the G-direct image of the element

$$(0,\int_{\mathbf{P}^1}\mathrm{ch}(\mathcal{E}.,\rho)\,\mathrm{td}(\mathcal{T}_G(-\log\infty),\rho_G)\log|z|^2).$$

Thus by the Axiom 1 for the relative Bott-Chern secondary characteristic forms with respect to the smooth forms in I.2, or better, the proof of Proposition 3.5.f.2, we see that lemma′ is implied from the following

Sublemma. With the same notation as above, for all $t \in \mathbf{P}^1$

$$\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.|_t,\rho.|_t)))\,\mathrm{Td}^R_{\mathrm{Ar}}(g_t,\rho_{g_t}) = i_t^*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.,\rho.)\,\mathrm{Td}^R_{\mathrm{Ar}}(\mathcal{T}_G(-\log\infty),\rho_G)),$$

where g_∞ denotes the restriction of G_∞ on W^1_∞.

In fact, suppose the sublemma is proved, then we have

$$\begin{aligned}&g_{0*}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.|_0,\rho.|_0)\,\mathrm{Td}^R_{\mathrm{Ar}}(g_0,\rho_{g_0})) - g_{\infty*}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.|_\infty,\rho.|_\infty)\,\mathrm{Td}^R_{\mathrm{Ar}}(g_\infty,\rho_{g_\infty}))\\ &=G_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.,\rho.)\,\mathrm{Td}^R_{\mathrm{Ar}}(\mathcal{T}_G(-\log\infty),\rho_G))((0)-(\infty),0)\\ &=\big(0,G_*(\int_{\mathbf{P}^1}[\log|z|^2]\mathrm{ch}(\mathcal{E}.,\rho.)\,\mathrm{Td}^R(\mathcal{T}_G(-\log\infty),\rho_G))\big).\end{aligned}$$

So we have the lemma′, hence complete the proof of Proposition 6.

Proof of the sublemma. The proof is rather formal but standard. The key point is that now $X \times \mathbf{P}^1$ does not intersect W^2_∞. So, in the discussion, essentially, we may pay no attention on that part. Also we may let the tangent bundle T_{W_0} (resp. $T_{W^1_\infty}$) is the restriction of the logarithmic tangent bundle, and put a metric on the logarithmic ralative tangent sheaf, such that outside a neighborhood U of ∂W_∞, the restriction for the relative tangent bundle offers an isometry over each point $t \in \mathbf{P}^1$. More precisely, in practice, we may go as follows:

First, introduce a relative arithmetic Chow group $\mathrm{CH}_{\mathrm{Ar}}^{X,W-\bar{U}}(Z)$ as the quotient of of group generated by arithmetic cycles cycles (Z, g_Z) with $Z \subset X$, $\mathrm{Supp}(g_Z) \subset W - \bar{U}$, modulo the arithmetic divisors of rational functions on cycles in X, together with the forms $\partial\alpha + \bar{\partial}\beta$, where α and β are currents with support in X. Then for any hermitian line sheaf $(\mathcal{L}, \tau)$ of X we can introduce the action of $c_{1,\mathrm{Ar}}(\mathcal{L}, \tau)$ on $\mathrm{CH}_{\mathrm{Ar}}^{X,W-\bar{U}}(X)$ in a natural way. Any two $c_{1,\mathrm{Ar}}(\mathcal{L}, \tau)$'s actions commute. Furthermore, if we have two embeddings $X \hookrightarrow Z_1$ and $X \hookrightarrow Z_2$, such that there are Zariski open neighbourhoods V_1 respectively V_2 of X and an isomorphism $V_1 \simeq V_2$, which fixes X so that if $U_1 \subset V_{1,\mathbf{C}}$ and $U_2 \subset V_{2,\mathbf{C}}$ are the corresponding open neighborhoods, then naturally,

$$\mathrm{CH}_{\mathrm{Ar}}^{Y,W_1-\bar{U}_1}(Z_1) \simeq \mathrm{CH}_{\mathrm{Ar}}^{X,W_2-\bar{U}_2}(Z_2).$$

Moreover, this isomorphism is compatible with the action of $c_{1,\mathrm{Ar}}(\mathcal{L}, \tau)$'s, if there is an isomorphism $\mathcal{L}_1|_{V_1} \simeq \mathcal{L}_2|_{V_2}$ which is an isometry over $W_1 - \bar{U}_1 \simeq W_2 - \bar{U}_2$.

We now discuss the above action for any hermitian vector sheaf. Naturally, by the splitting principle, we may have an action. That is, we may have the following situation: Suppose that we have two hermitian vector sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ which are isomorphic over a Zariski open set $V \subset X$ and isometric over $W - \bar{U} \subset V_{\mathbf{C}}$. Let (A, g_A) be an arithmetic cycle in $\mathrm{CH}_{\mathrm{Ar}}^{X,W-\bar{U}}(X)$. Denote by Z_1 (resp. Z_2) the complete flag varieties of $\mathcal{E}_1$ (resp. $\mathcal{E}_2$), by $\pi_i : Z_i \to Z$ the natural projections and $V_i = \pi_i^{-1}(V)$, etc.. Then we have the corresponding arithmetic cycles $\pi_i^*(A, g_A)$ in $\mathrm{CH}_{\mathrm{Ar}}^{X_i,W_i-\bar{U}_i}(X_i)$ for $i = 1, 2$. In particular, $\pi_i^*(\mathcal{E}_i)$ splits on Z_i, so there is a complete filtration by line sheaves $\mathcal{L}_{j,i}$ for this splitting. Any polynomial in the $c_{1,\mathrm{Ar}}(\mathcal{L}_{j,i})$'s operates on the arithmetic cycles, and obviously the two actions for $i = 1, 2$ correspond to each other via the natural isomorphism on the relative arithmetic Chow groups introduced above. So, in the later discussion, we may neglect the above difference. In particular, we can use the splitting principle to introduce a multiplication by $\mathrm{ch}_{\mathrm{Ar}}$-class, and hence have similar results. In fact, if $P(c_{i,\mathrm{Ar}}(\mathcal{E}, \rho))$ is a polynomial in the arithmetic Chern classes of $(\mathcal{E}, \rho)$, we can find an operator $Q(c_{1,\mathrm{Ar}}(\mathcal{L}_j)) + R$ on $\mathrm{CH}_{\mathrm{Ar}}^{X_i,W_i-\bar{U}_i}(Z_i)$, with Q a polynomial in arithmetic Chern classes, and R multiplication by the classical Bott-Chern secondary characteristic form, such that in $\mathrm{CH}_{\mathrm{Ar}}(Z)$,

$$P(c_{j,\mathrm{Ar}}(\mathcal{E}_i))\,(A, g_A) = \pi_{i*}\Big(\big(Q(c_{1,\mathrm{Ar}}(\mathcal{L}_k))\big) + R\Big)\,\pi_i^*(A, g_A).$$

It follows that both for $\mathcal{E}_1$ and $\mathcal{E}_2$, the left hand sides can be obtained from the same class in $\mathrm{CH}_{\mathrm{Ar}}^{X_i,W_i-\bar{U}_i}(Z_i)$, by first projecting them to $\mathrm{CH}_{\mathrm{Ar}}^{X,W-\bar{U}}(Z)$ and then mapping to $\mathrm{CH}_{\mathrm{Ar}}(Z)$. Thus the operations on cycles with support in $(X, W - \bar{U})$ of $P(c_{j,\mathrm{Ar}}(\mathcal{E}_i))$ coincide.

As a consequence of the above general discussion, by the fact that $\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_\cdot, \rho_\cdot)$ is of the form (M, g_M), where M is an algebraic cycle supported in $X \times \mathbf{P}^1$ and g_M a Green's current supported in $W - \bar{U}$, we get the assertion in the sublemma.

II.4.3. The Proof Of Arithmetic Riemann-Roch Theorem For Smooth Morphisms

Now we use the propositions stated and proved in the last two sections to show the arithmetic Riemann-Roch theorem for smooth morphisms.

For simplicity, we let p_n denote the projection from the $\mathbf{P}^n$-bundle, and let i_1 denote the codimension-one closed immersion.

It is enough to prove Err's are zero for both closed immersions and projections from $\mathbf{P}^n$-bundles. For this, we use the trick of Faltings to reduce the problems for Err to those for just p_1 and i_1. If it is so, then the main theorem is a direct consequence of Proposition 2.5 and Proposition 2.6. In practice, we use the induction step for p_n to reduce the problem to that for p_1 and i_1, while for any closed immersion, we use the deformation to the normal cone technique to reduce the problem to p_n and i_1, using Proposition 3 above.

II.4.3.a. Projection Cases

We start the induction on n. If $n = 1$, by Proposition 2.5, we know that

$$\mathrm{Err}(\mathcal{E}; p_1) \equiv 0.$$

Suppose that for any $m < n$, we have

$$\mathrm{Err}(\mathcal{E}, p_m) \equiv 0.$$

We shall prove that

$$\mathrm{Err}(\mathcal{E}, p_n) \equiv 0.$$

In order to prove this, consider the generator of $K(X)$ for $X = \mathbf{P}_Y(\mathcal{F})$, where $\mathcal{F}$ is a rank $n+1$ vector sheaf on Y. Note that by Proposition 2.4, everything above is compatible with flat base-change, so by the splitting principle, we may assume that $\mathcal{F}$ has a rank 1 sub-line sheaf $\mathcal{L}$ such that $\mathcal{F}/\mathcal{L}$ is also a vector sheaf. Now, it is an easy observation that we may have the following simple but very important

Fact. As a $K(Y)$-module, $K(X)$ is generated by $\mathcal{O}_X(-1)$ and the direct image of $i_*(K(\mathbf{P}_Y(\mathcal{F}/\mathcal{L})))$. Here

$$i : \mathbf{P}_Y(\mathcal{F}/\mathcal{L}) \hookrightarrow \mathbf{P}_Y(\mathcal{F})$$

is the natural codimension-one closed imbedding.

In order to prove the main theorem, we need only to show that, for $\mathcal{O}_X(-1)$ and the elements in $i_*(K(\mathbf{P}_Y(\mathcal{F}/\mathcal{L})))$, Err is zero. We deal first with the element in $i_*(K(\mathbf{P}_Y(\mathcal{F}/\mathcal{L})))$. For this purpose, we need to use Proposition 2.6. In fact, since

$$i_1 : \mathbf{P}_Y(\mathcal{F}/\mathcal{L}) \hookrightarrow \mathbf{P}_Y(\mathcal{F})$$

is a codimension-one closed imbedding, we have $\mathrm{Err}(\alpha, i_1) = 0$ for $\alpha \in K(\mathbf{P}_Y(\mathcal{F}/\mathcal{L}))$. But by definition,

$$\mathrm{Err}(\alpha, i) = \mathrm{Err}(\alpha, p_{n-1}) - \mathrm{Err}(i_* \alpha, p_n).$$

Here p_{n-1} (resp. p_n) denotes the natural projection from $\mathbf{P}_Y(\mathcal{F})$ (resp. $\mathbf{P}_Y(\mathcal{F}/\mathcal{L})$) to Y. Moreover, by the induction step,

$$\mathrm{Err}(\alpha, p_{n-1}) \equiv 0,$$

hence we have

$$\mathrm{Err}(i_* \alpha, p_n) \equiv 0,$$

which exactly means that the arithmetic Riemann-Roch formula is valid for the elements in the direct image of $K(\mathbf{P}_Y(\mathcal{F}/\mathcal{L}))$.

Now let us consider the term $\mathrm{Err}(\mathcal{O}_X(-1), p_n)$. For this special purpose, we need some more notation.

Let $\mathrm{Flag}_Y(\mathcal{F})$ be the Flag variety of $\mathcal{F}$ on Y. That is, the variety which classifies complete filtrations of $\mathcal{F}$:

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{n+1} = \mathcal{F},$$

where the succesive vector sheaf quotients are of rank 1. There is a natural morphism from $\mathrm{Flag}_Y(\mathcal{F})$ to X which is just the composition of the forgetting maps. Hence the morphism from $\mathrm{Flag}_Y(\mathcal{F})$ to X is a composition of $\mathbf{P}^m$-bundles with $m < n$. Therefore, by Proposition 2.2 and the induction hypothesis, the arithmetic Riemann-Roch theorem holds for the morphism $\mathrm{Flag}_Y(\mathcal{F}) \to X$. On the other hand, we can consider the pull-back of the line sheaf $\mathcal{O}_X(-1)$ over $\mathrm{Flag}_Y(\mathcal{F})$. It is well-known that the push-out to X of this pull-back line sheaf on $\mathrm{Flag}_Y(\mathcal{F})$ is just $\mathcal{O}_X(-1)$ itself. Thus, if we can prove that, for the natural morphism $\mathrm{Flag}_Y(\mathcal{F}) \to Y$ that Err of the pull-back of $\mathcal{O}_X(-1)$ is zero, then Err of $\mathcal{O}_X(-1)$ with respect to $X \to Y$ is also zero by Proposition 2.2.

In order to deal with the morphism $\mathrm{Flag}_Y(\mathcal{F}) \to Y$, we introduce another decomposition: Let $\mathrm{Flag}'_Y \mathcal{F}$ be another flag variety which classifies the following partial filtrations of $\mathcal{F}$:

$$0 = \mathcal{F}_0 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_{n+1} = \mathcal{F},$$

where the rank of $\mathcal{F}_k$ is k. Then the natural morphism from $\mathrm{Flag}_Y \mathcal{F}$ to $\mathrm{Flag}'_Y \mathcal{F}$, followed by the natural morphism from $\mathrm{Flag}'_Y \mathcal{F}$ to Y, is just $\mathrm{Flag}_Y(\mathcal{F}) \to Y$. But, the morphism from $\mathrm{Flag}_Y \mathcal{F}$ to $\mathrm{Flag}'_Y \mathcal{F}$ is a $\mathbf{P}^1$-bundle. Therefore, the arithmetic Riemann-Roch formula is valid for the pull-back of $\mathcal{O}_X(-1)$ on $\mathrm{Flag}_Y \mathcal{F}$, with respect to the morphism from $\mathrm{Flag}_Y \mathcal{F}$ to $\mathrm{Flag}'_X \mathcal{F}$, by our Proposition 2.5 for $\mathbf{P}^1$-bundles. On the other hand, note that both sides of the arithmetic Riemann-Roch formula for the $\mathbf{P}^1$-bundle case, with respect to the pull back of the line sheaf $\mathcal{O}(-1)$, consist only of forms over $\mathrm{Flag}'_Y \mathcal{F}$. So, by the fact that the arithmetic Riemann-Roch formula for any smooth form with respect to any morphism holds, we know that Err for the pull-back of $\mathcal{O}_X(-1)$ on $\mathrm{Flag}_Y \mathcal{F}$ with respect to $\mathrm{Flag}_Y(\mathcal{F}) \to Y$ is zero. In this way, we know that $\mathrm{Err}(i_1) = 0$ and $\mathrm{Err}(p_1) = 0$ implies $\mathrm{Err}(p_n) = 0$ for all n.

II.4.3.b. **Closed Immersions**

Next we deal with a regular closed immersion via the deformation to the normal cone technique. Here the aim is to reduce the general immersion case to the following situation:

(1) The zero section of a projectivised affine bundle;
(2) Codimension-one closed immersions.

Let us now suppose that we can reduce the problem with respect to a general closed immersion to the case (1) and (2) above, then by Proposition 2.6, we solve the Err problem i.e. Err=0, for (2). Also by the above result about projections, we solve the Err problem for (1), since the composition of the zero section immbedding and the projection is the identity map, and the arithmetic Riemann-Roch theorem is of couse valid for the identity map. (See part a of the proof of Proposition 1.6.) Hence we have a proof for the arithmetic Riemann-Roch theorem with respect to smooth morphisms.

To reduce the case of an arbitrary closed immersion to (1) and (2) above, we use deformation to the normal cone theory as usual. For this, we recall the following basic fact concerning the theory of deformation to the normal cone

Fact. With the same notation as in (*) of I.9.2.a, the following two morphisms

$$X \stackrel{i_\infty}{\hookrightarrow} \mathbf{P}_X(\mathcal{N} \oplus \mathcal{O}_X) \stackrel{j_\infty}{\hookrightarrow} W$$

and

$$X \stackrel{i_0}{\hookrightarrow} W_0 \stackrel{j_0}{\hookrightarrow} W$$

induce the same morphism for K-groups.

In this way, by Proposition 2.3, we see that

$$\mathrm{Err}(\mathcal{E}, j_\infty \circ i_\infty) = \mathrm{Err}(\mathcal{E}, j_0 \circ i_0).$$

But by definition, we know that

$$\mathrm{Err}(\mathcal{E}, j_\infty \circ i_\infty) = \mathrm{Err}(\mathcal{E}, i_\infty) + \mathrm{Err}(i_{\infty *}\mathcal{E}, j_\infty)$$

and

$$\mathrm{Err}(\mathcal{E}, j_0 \circ i_0) = \mathrm{Err}(\mathcal{E}, i_0) + \mathrm{Err}(i_{0 *}\mathcal{E}, j_0).$$

Thus to complete the proof, it is sufficient to prove that

$$\mathrm{Err}(\mathcal{E}, i_\infty) \equiv 0,$$

$$\mathrm{Err}(i_{\infty *}\mathcal{E}, j_\infty) \equiv 0,$$

and

$$\mathrm{Err}(i_{0*}\mathcal{E}, j_0) \equiv 0.$$

Note that each of the three closed immersions, i_∞, j_0 and j_∞, is either codimension-one closed immersion or the zero section of a $\mathbf{P}^n$-bundle, it follows that it is sufficient to deal with the zero sections of projectivized affine bundles and codimension-one closed immersions. These are just the situations in (1) and (2) above. It is in this way that we complete the proof of the arithmetic Riemann-Roch theorem for smooth morphisms.

We end this chapter with the following remark. As we stated in Chapter II.1, we hope that the arithmetic Riemann-Roch theorem can be obtained by replacing the concepts in the Grothendieck-Riemann-Roch theorem by the corresponding arithmetic concepts. However, it seems to be the case that our final arithmetic Riemann-Roch theorem does not have this form, since we have to modify the arithmetic Todd characteristic class with a power series of R. But, this may be easily removed by changing the push-out morphism for arithmetic K-theory: Instead of defining $f_K(\mathcal{E}, \rho)$ by

$$f_K(\mathcal{E}, \rho) := (f_*\mathcal{E}, f_*\rho) + \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; f, \rho_f)$$

for an f-acyclic vector sheaf $\mathcal{E}$, we define $f_K^R(\mathcal{E}, \rho)$ by

$$\begin{aligned} f_K^R(\mathcal{E}, \rho) :=& (f_*\mathcal{E}, f_*\rho) + \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; f, \rho_f) \\ &+ f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}, \rho)\, \mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_f, \rho_f)\, a(R(\mathcal{T}_f, \rho_f))). \end{aligned}$$

Since $\mathrm{ch}_{\mathrm{Ar}}$ is an isomorphism which maps forms to forms, we see that the above definition makes sense. Also, since R is unique, if there is no risk of confusion, we can also denote f_K^R by f_K. Thus we get a perfect situation:

The Arithmetic Riemann-Roch theorem For Smooth Morphisms′.
Let $f : X \to Y$ be a smooth morphism of regular arithmetic varieties X, Y over an arithmetic ring (A, Σ, F_∞), with an F_∞-invariant hermitian metric ρ_f on the relative tangent sheaf of f. Then we have the following commutative diagram

$$\begin{array}{ccc} K_{\mathrm{Ar}}(X) & \xrightarrow{\mathrm{ch}_{\mathrm{Ar}}()\,\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)} & \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}} \\ f_K \downarrow & & \downarrow f_{\mathrm{CH}} \\ K_{\mathrm{Ar}}(Y) & \xrightarrow{\mathrm{ch}_{\mathrm{Ar}}()} & \mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}. \end{array}$$

Now one may say that historically, for Hirzebruch to find his famous Riemann-Roch theorem, he actually did show that the Todd genus is the only one which makes the Riemann-Roch theorem hold, by checking certain concrete examples. But here, at the last minute, we have changed our direction. Well, we may argue that this last change makes the arithmetic Riemann-Roch theorem totally similar to the Grothendieck-Riemann-Roch theorem. So, even philosophically, this change makes sense.

Chapter II.5 Arithmetic Riemann-Roch Theorem For Closed Immersions

We now use the relative Bott-Chern secondary characteristic currents with respect to closed immersions to prove the arithmetic Riemann-Roch theorem for a special kind of closed immersion, which was first given in [BGS 91].

§II.5.1 An Arithmetic Riemann-Roch Theorem For Closed Immersions

Let $A = (A, \Sigma, F_\infty)$ be an arithmetic ring and let

$$\begin{array}{ccccc} X & & \overset{i}{\hookrightarrow} & & Z \\ & f\searrow & & \swarrow g & \\ & & Y & & \end{array}.$$

be a closed immersion of regular arithmetic varieties over Y with *smooth* structure morphisms f and g. Put F_∞-invariant hermitian metrics on the relative tangent vector sheaves and on the normal vector sheaf of i.

Let $(\mathcal{E}, \rho)$ be a hermitian vector sheaf on X and $\mathcal{F}. \to i_*\mathcal{E} \to 0$ a vector sheaf resolution of $i_*\mathcal{E}$ on Z. By the results in I.7, we know that there exist F_∞-invariant hermitian metrics $\tau.$ on $\mathcal{F}.$ such that Bismut condition (A) is satisfied with respect to ρ_N and ρ. As before, we let $a : \tilde{A}(X_\mathbf{R}) \to \mathrm{CH}_{\mathrm{Ar}}(X)$ and $\omega : \mathrm{CH}_{\mathrm{Ar}}(X) \to A(X_\mathbf{R})$ be the morphisms defined by $a(\alpha) = (0, \alpha)$ and $\omega(A, g_A) := dd^c g_A + \delta_A$, respectively.

In order to obtain the arithmetic Riemann-Roch theorem for closed immersions, we first recall the Grothendieck-Riemann-Roch theorem in this situation. By Lemma 1.3.c, we know that in $\mathrm{CH}(Z)_\mathbf{Q}$,

$$\mathrm{ch}(i_*\mathcal{E}) = i_*(\mathrm{td}^{-1}(N)\,\mathrm{ch}(\mathcal{E})).$$

As this is only valid at the level of cohomology classes, we need to be more careful when we deal with the arithmetic Riemann-Roch theorem for closed immersions, since this is supposed to be a refined version of the classical Grothendieck-Riemann-Roch theorem at the level of differential forms (say, for complex manifolds). Generally speaking, to find such an arithmetic Riemann-Roch theorem with respect to closed immersions, the first difficulty is that $i_*\mathcal{E}$ is usually only a coherent sheaf on Z. Nevertheless, this difficulty can be avoided by introducing the combination of a hermitian super-vector sheaf $(\mathcal{F}., \tau.)$ and $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; i, \rho_i; \mathcal{F}., \tau.)$, the relative Bott-Chern secondary characteristic current with respect to the closed immersion $i_{\mathbf{C}}$. On the other hand, if we do things in this way, we know that this element is not in $K_{\mathrm{Ar}}(Z)$, which is assumed to be generated by hermitian vector sheaves and smooth forms. Indeed, we may in principle handle this by the introduction of the relative theory of arithmetic Chow groups, arithmetic K-theory, etc. for closed immersions. However, this is tedious and we will not do it here. We hope that we may come back to this point. (In fact, basically, we may do it as follows: First discuss the situation for codimension one case to get a good feeling, then use the deformation theory via MacPherson's Grassmannian graph construction to give the theory in general.) For our present purpose, we can go around, i.e. to consider the product of these elements with any element in $\Psi \in \mathrm{CH}_{\mathrm{Ar}}(Z)$, and push them out to the base arithmetic variety Y. In this sense, the left hand side of the arithmetic Riemann-Roch theorem for closed immersions would become

$$g_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}.,\tau.)\Psi) + g_*(a(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;i,\rho_i;\mathcal{F}.,\tau.)\omega(\Psi)).$$

So we may state the following

An Arithmetic Riemann-Roch Theorem For Closed Immersions. ([BGS 91]) For any $\Psi \in \mathrm{CH}_{\mathrm{Ar}}(Z)$, the following identity holds in $\mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}$

$$\begin{aligned} &g_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}.,\tau.)\Psi\big) + g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;i,\rho_i;\mathcal{F}.,\tau.)\omega(\Psi)\big)\Big)\\ =&f_*\big(\mathrm{td}_{\mathrm{Ar}}^{-1}(\mathcal{N},\rho_{\mathcal{N}})\,\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)i^*(\Psi)\big).\end{aligned}$$

We prove this theorem in the following sections.

§II.5.2 Several Intermediate Results

The basic idea to prove the above arithmetic Riemann-Roch theorem with respect to closed immersions is as follows: First we give an explanation for the element $g_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}.,\tau.)\Psi)$ via the deformation to the normal cone technique. Then note that at infinity, the new closed immersion is just the zero section of $\mathbf{P}(N\oplus 1)$, therefore we may use the Koszul complex to calculate it precisely. Surely, during this process, we have to use the deformation theory for the relative Bott-Chern secondary characteristic current with respect to closed immersions, which was developed in Chapter I.9.

II.5.2.a Deformation to the Normal Cone

We use the deformation to the normal cone technique to give another expression for the term $g_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}.,\tau.)\Psi)$. In order to do so, we recall the following diagram for the deformation to the normal cone with respect to $i : X \hookrightarrow Z$:

$$\begin{array}{ccccccc} X & \leftarrow & X\times\{\infty\} & \overset{i_\infty}{\hookrightarrow} & \mathbf{P}(\mathcal{N}\oplus\mathcal{O}_X)+B_XZ=W_\infty & \rightarrow & \{\infty\}\\ \downarrow & & \downarrow & & j_\infty\downarrow & & \downarrow\\ X & \overset{p_X}{\leftarrow} & X\times\mathbf{P}^1 & \overset{I}{\hookrightarrow} & B_{X\times\infty}Z\times\mathbf{P}^1=W & \overset{\pi_2}{\rightarrow} & \mathbf{P}^1\\ \uparrow & & \uparrow & & j_0\uparrow & & \uparrow\\ X & \leftarrow & X\times\{0\} & \overset{i_0}{\hookrightarrow} & Z\times\{0\}=W_0 & \rightarrow & \{0\}.\end{array}$$

Let

$$\pi : W \to Z, \quad \pi_2 : W \to \mathbf{P}^1$$

and

$$p_X : X \times \mathbf{P}^1 \to X, \quad p : P = \mathbf{P}_X(\mathcal{N} \oplus 1) \to X$$

be the natural projections. Then $W_\infty = \pi_2^{-1}(\infty)$ has two components: W_∞^1, the projective space $P = \mathbf{P}_X(N \oplus 1)$ over X, and W_∞^2, the blowing-up $B_X Z$ of Z along X. Let $D\mathcal{E} := p_X^*(\mathcal{E})$ and let $D\mathcal{F}. \to I_*D\mathcal{E} \to 0$ be a metrized vector sheaf resolution of $I_*D\mathcal{E}$ such that, on $W_0 = \pi_2^{-1}(0)$, $D\mathcal{F}.$ coincides with $\mathcal{F}.$. By the results in I.9.1, we can assume that the metrics on $D\mathcal{F}.$ are compatible with the metrics on the normal bundle of $X \times \mathbf{P}^1$ in W, which is nothing but $\mathcal{N}(-1) := \mathcal{N} \otimes \mathcal{O}(-1)$. Furthermore, the restriction of $D\mathcal{F}.$ to $B_X Z$ is split acyclic even as a complex of hermitian vector sheaves.

We then have the element

$$\mathrm{ch}_{\mathrm{Ar}}(D\mathcal{F}., D\tau.) := \sum_{i \geq 0} (-1)^i \mathrm{ch}_{\mathrm{Ar}}(D\mathcal{F}_i, D\tau_i) \in \mathrm{CH}_{\mathrm{Ar}}(W)_{\mathbf{Q}}.$$

Moreover, we can measure the difference of the restrictions of this element to the fibers of π_2 over $\{0\}$ and $\{\infty\}$ by the classical Bott-Chern secondary characteristic forms in the sense of Theorem 3.5.e.2. More precisely, the difference

$$g_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}., \tau.)\Psi) - f_*(p_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}_\infty., \tau_\infty.))i^*\Psi)$$

is measured by an associated classical Bott-Chern secondary characteristic form. The construction of the classical Bott-Chern secondary characteristic forms in I.1.4 shows that it is defined as an integrate of Chern characteristic forms with respect to $\log|z|^2$ over $\mathbf{P}^1$. With this in mind, we need the following

Proposition. For any element $\Psi \in \mathrm{CH}_{\mathrm{Ar}}(Z)$, in $\mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}$

$$\begin{aligned} & g_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}., \tau.)\Psi\big) - f_*\big(p_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}_\infty., \tau_\infty.))i^*\Psi\big) \\ =& a\Big(g_*\big(\pi_*(\mathrm{ch}(D\mathcal{F}., D\tau.)\mathrm{Log}|z|^2)\omega(\Psi)\big)\Big). \end{aligned}$$

Proof. The proof below shows that

$$dd^c g_*\Big(\pi_*\big(\mathrm{ch}(D\mathcal{F}., D\tau.)\mathrm{Log}|z|^2\big)\omega(\Psi)\Big)$$

is smooth and hence the current

$$g_*\Big(\pi_*\big(\mathrm{ch}(D\mathcal{F}., D\tau.)\mathrm{Log}|z|^2\big)\omega(\Psi)\Big)$$

is in $\tilde{A}(Y_{\mathbf{R}})$. So the equality in the proposition makes sense.

Now let $\mathrm{ch}_{\mathrm{Ar}}(D\mathcal{F}., D\tau.)$ be an arithmetic cycle (A, g_A). Since $D\mathcal{F}.$ is acyclic outside $X \times \mathbf{P}^1$, we may assume that, A is supported on $X \times \mathbf{P}^1$ up to the rational equivalence, and that $A = p_X^*(T) + S \times \{1\}$, where S, T are two cycles on X.

Since

$$0 = \mathrm{div}_{\mathrm{Ar}}(z) = (\mathrm{div}(z), -\mathrm{Log}|z|^2),$$

we have

$$g_*\Big(\pi_*\big(\mathrm{ch}_{\mathrm{Ar}}(D\mathcal{F}., D\tau.)\,\mathrm{div}_{\mathrm{Ar}}(z)\big)\Psi\Big) = 0.$$

But by definition and the fact that $S \times \{1\}$ does not intersect $\mathrm{div}(z) = W_0 - W_\infty$, we have

$$\begin{aligned}&\mathrm{ch}_{\mathrm{Ar}}(D\mathcal{F}., D\tau.)\,\mathrm{div}_{\mathrm{Ar}}(z)\\ &=(T \times \{0\} - T \times \{\infty\}, g_A|_{W_0} - g_A|_{W_\infty} - \mathrm{ch}(D\mathcal{F}., D\tau.)\mathrm{Log}|z|^2).\end{aligned}$$

The fact that the restriction of $D\mathcal{F}.$ to $B_X Z$ is split acyclic (even as a hermitian complex) shows that $g_A|_{B_X Z} = 0$. Hence

$$\begin{aligned}0 =&g_*\Big(\pi_*\big(\mathrm{ch}_{\mathrm{Ar}}(D\mathcal{F}., D\tau.)\,\mathrm{div}_{\mathrm{Ar}}(z)\big)\Psi\Big)\\ =&g_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}., \tau.)\Psi\big) - f_*\Big(p_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}_\infty., \tau_\infty.)\big)i^*(\Psi)\Big)\\ &- g_*\Big(\pi_*\big((0, \mathrm{ch}(D\mathcal{F}., D\tau.)\mathrm{Log}|z|^2)\omega(\Psi)\big)\Big).\end{aligned}$$

This completes the proof.

II.5.2.b **A Calculation Via The Koszul Complex**

For the proof of the arithmetic Riemann-Roch theorem for closed immersions, we need to use the Koszul complex. Then, by I.9.3, we need only know the arithmetic Euler class for a hermitian vector sheaf.

Let X be an arithmetic variety over an arithmetic ring A. Let (E,ρ) be a hermitian vector bundle of rank r on X. Then there exists a section s of E over X such that, at infinity, s is chosen as in chapter I.9.3.e. Namely, if $x \in X(\mathbf{C})$ is such that $s_{\mathbf{C}}(x) = 0$, and $d(s_{\mathbf{C}})$ is the differential of $s_{\mathbf{C}}$ at x, then $\mathrm{Im}[ds_{\mathbf{C}}(x)] = E$. Denote by $A_F \in Z^r(X_F)$ the zero set of s at a generic fiber X_F of X. Hence by the result in I.9.3.e, we know that $g_A := -s_{\mathbf{C}}^*(e_{\mathrm{BC}}(E,\rho))$ is a Green's current for A_F. On the other hand, viewing s as an imbedding of X in the total space X^E, we see $e_{\mathrm{fin}}(E) := s^*([X]) \in \mathrm{CH}^r_{\mathrm{fin}}(X)$. Here $[X] \in \mathrm{CH}^r_{\mathrm{fin}}(X^E)$ is the part of the zero section supported only on the special fibres, i.e. those fibers over the non-Archimedean places. Since $\mathrm{CH}^p_{\mathrm{Ar}}(X)$ is a quotient of the group

$$\mathrm{CH}^p_{\mathrm{fin}}(X) \oplus Z^p(X_F) \oplus \tilde{D}^{p-1,p-1}(X_{\mathbf{R}}),$$

by the result in Section 2.4, it follows that the triple $(e_{\mathrm{fin}}, A_F, g_A)$ defines a class $e_{\mathrm{Ar}}(E,\rho;s)$ in $\mathrm{CH}^r_{\mathrm{Ar}}(X)$. We call it the **arithmetic Euler class** of $(\mathcal{E},\rho)$. (Note that in algebraic geometry, the Euler class may be realized as the top Chern class.) We have

Proposition 1. With the same notation as above,
(a) $e_{\mathrm{Ar}}(E,\rho;s)$ does not depend on the choice of s and we denote it simply as $e_{\mathrm{Ar}}(E,\rho)$;
(b) In $\mathrm{CH}^r_{\mathrm{Ar}}(X)$, we have

$$c_{r,\mathrm{Ar}}(E,\rho) = e_{\mathrm{Ar}}(E,\rho).$$

Proof. One can show directly that if $q : X^E \to X$ is the natural projection and s_0 is the zero section of q, then

$$e_{\mathrm{Ar}}(E,\rho;s) = s_0^*\big(e_{\mathrm{Ar}}(q^*E, q^*\rho; s_0)\big).$$

On the other hand, as an element in $\mathrm{CH}_{\mathrm{Ar}}(X^E)$,

$$e_{\mathrm{Ar}}(q^*E, q^*\rho; s_0) = \big(s_0(X), -e_{\mathrm{BC}}(q^*E, q^*\rho)\big).$$

So $e_{\mathrm{Ar}}(E,\rho;s)$ does not depend on s.

Indeed, just as for $c_{r,\mathrm{Ar}}(E,g^E)$, (by definition,) it is easily shown that $e_{\mathrm{Ar}}(E,g^E;s)$ has the following properties:

Functorial Rule. Let $f : M \to X$ be a morphism of arithmetic varieties over A, then

$$f^*e_{\mathrm{Ar}}(E,\rho;s) = e_{\mathrm{Ar}}(f^*E, f^*\rho; s_f).$$

Product Rule. $e_{\mathrm{Ar}}(E \oplus E', \rho \oplus \rho'; (s,s')) = e_{\mathrm{Ar}}(E,\rho;s)\, e_{\mathrm{Ar}}(E',\rho';s')$.
Forgetful Rule For Morphism a. The a-image of $e_{\mathrm{Ar}}(E,\rho;s)$ in $\mathrm{CH}^r(X)$ is the Euler class of E.
Forgetful Rule For Morphism ω. $\omega\big(e_{\mathrm{Ar}}(E,\rho;s)\big) = e(E,\rho)$.
Uniqueness Rule. Let (L,τ) be a hermitian line bundle, then

$$e_{\mathrm{Ar}}(L,\rho;s) = s_0^*\big(\mathrm{div}(y), -\mathrm{Log}|y|^2\big),$$

where y is the tautological section of p^*L on X^L and s_0 is the zero section.

By a similar process to that in the proof of the arithmetic Riemann-Roch theorem for smooth morphisms, i.e. using the $\mathbf{P}^1$-deformation technique, etc., we can show that the difference

$$\mathrm{Err}(E,\rho;s) := e_{\mathrm{Ar}}(E,\rho;s) - c_{r,\mathrm{Ar}}(E,\rho)$$

is in the image of a. It may be first decreased into the ordinary K-theory, since there are similar properties; the downstairs rule, the functorial rule, the uniqueness rule, for example.

In order to prove that Err is zero, we use an induction on r. When $r = 1$, it follows by the uniqueness rule that Err is zero. In general, we consider the problem on $\mathbf{P}(E)$. By the functorial rule, we know that it is sufficient to show that $\mathrm{Err}(p^*E; p^*s) = 0$. But this is a consequence of the fact that on $\mathbf{P}(E)$, there is a canonical exact sequence of vector sheaves

$$0 \to S \to p^*E \to \mathcal{O}(1) \to 0.$$

Hence (b) follows easily and so does (a).

We next discuss the Koszul complex. For the notation and the results, we ask the reader to consult I.9.4.

Let

$$\mathcal{H} := \mathrm{Ker}\big(p^*(\mathcal{N}_i^* \oplus \mathcal{N}_{\infty/\mathbf{P}^1}^*) \to \mathcal{O}_P(1)\big),$$

associated with $i^*\mathcal{F}.$, then we have two short exact sequences of vector sheaves:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{Z}_j & \to & i^*\mathcal{F}_j & \to & \mathcal{B}_j & \to & 0; \\ 0 & \to & \mathcal{B}_{j+1} & \to & \mathcal{Z}_j & \to & \mathcal{H}_j(i^*\mathcal{F}) & \to & 0. \end{array}$$

Let H be the associated vector bundle of $\mathcal{H}$ on P. Then by the natural associated inclusion $\mathcal{H} \hookrightarrow p^*(\mathcal{N}^*_{X/Z} \oplus \mathcal{N}^*_{\infty/\mathbf{P}^1})$, there is an induced metric ρ_H on H. On $K_j := \wedge^j H$, we put the metric induced from ρ_H. A standard argument from the definition varifies

Proposition 2. With the same notation as above,

$$\mathrm{ch}_{\mathrm{Ar}}(K, \rho_K) = c_{r,\mathrm{Ar}}(H^*, \rho_{H^*})\mathrm{td}^{-1}_{\mathrm{Ar}}(H^*, \rho_{H^*}) = e_{r,\mathrm{Ar}}(H^*, \rho_{H^*})\mathrm{td}^{-1}_{\mathrm{Ar}}(H^*, \rho_{H^*}),$$

where r is the rank of H.

(In fact, the equality is a direct generalization of the fact that at the cohomology level,

$$\mathrm{ch}(K) = e_r(H^*)\mathrm{td}^{-1}(H^*).)$$

With this, let σ be the canonical section of H^* and let $\sigma^*(e_{\mathrm{BC}}(H^*, g^{H^*}))$ be the pull-back of the corresponding Euler-Green current on P by I.9.3.e. For each $j \geq 0$, we have the exact sequence of complexes of vector sheaves:

$$A_j : 0 \to p^*\mathcal{L}. \to k^*D\mathcal{F}. \to K. \otimes p^*\mathcal{E} \to 0,$$

where

$$\mathcal{L}_j = (\mathcal{B}_{j+1} \otimes \mathcal{N}^j_{\infty/\mathbf{P}^1}) \oplus (\mathcal{B}_j \otimes \mathcal{N}^{j-1}_{\infty/\mathbf{P}^1}),$$

and k is the natural inclusion of P in W. Let $L.$ be the associated vector bundle of $\mathcal{L}.$. Then we may put the orthogonal direct sum of the induced metrics, so that the complex L attached to $\mathcal{L}.$ becomes split acyclic as a complex of hermitian holomorphic vector bundles. In this way, we can metrize the complex A_j: We get a classical Bott-Chern secondary characteristic form $\mathrm{ch}_{\mathrm{BC}}(A_j, \rho_{A_j})$ on P. In particular, we see that

$$\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}_{\infty}., \tau_{\infty}.) = \mathrm{ch}_{\mathrm{Ar}}(K. \otimes p^*\mathcal{E}, \rho_{K.} \otimes p^*\rho) + \sum_{j\geq 0}(-1)^j a(\mathrm{ch}_{\mathrm{BC}}(A_j, \rho_{A_j})).$$

Here, we use the fact that $\mathrm{ch}_{\mathrm{Ar}}$ for $\mathcal{L}$ contributes nothing. Hence, by the projection formula, we have the following

Proposition 3. With the same notation as above,

$$\begin{aligned} &p_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}_{\infty}., \tau_{\infty}.)\big) \\ &\quad= p_*\big(\mathrm{ch}_{\mathrm{Ar}}(K. \otimes p^*\mathcal{E}, \rho_{K.} \otimes p^*\rho)\big) + \sum_{j\geq 0}(-1)^j a\big(p_*(\mathrm{ch}_{\mathrm{BC}}(A_j, \rho_{A_j}))\big) \\ &\quad= p_*\big(\mathrm{ch}_{\mathrm{Ar}}(K., \rho_{K.})\big)\, \mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}, \rho) + \sum_{j\geq 0}(-1)^j a\Big(p_*\big(\mathrm{ch}_{\mathrm{BC}}(A_j, \rho_{A_j})\big)\Big). \end{aligned}$$

§II.5.3. The Proof Of The Theorem.

We can now complete the proof of the arithmetic Riemann-Roch theorem for closed immersions. We have to show that

$$g_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}_\cdot,\tau_\cdot)\Psi)+g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;i,\rho^i;\mathcal{F}_\cdot,\tau_\cdot)\omega(\Psi)\big)\Big)=f_*\big(\mathrm{td}_{\mathrm{Ar}}^{-1}(N,\rho_N)\,\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)i^*(\Psi)\big).$$

But by Proposition 2.a, we know that

$$g_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}_\cdot,\tau_\cdot)\Psi\big)-f_*\Big(p_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}_{\infty\cdot},\tau_{\infty\cdot})\big)i^*\Psi\Big)=a\Big(g_*\big(\pi_*(\mathrm{ch}(D\mathcal{F}_\cdot,D\tau_\cdot)\mathrm{Log}|z|^2)\omega(\Psi)\big)\Big).$$

So by the expression for $\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}_{\infty\cdot},\tau_{\infty\cdot})$ in terms of the Koszul complex (K,ρ_K) at the end of the last section, it remains to prove that

$$\begin{aligned}
&f_*\big(\mathrm{td}_{\mathrm{Ar}}^{-1}(N,\rho_N)\,\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)i^*(\Psi)\big)\\
=&f_*\Big(p_*\big(\mathrm{ch}_{\mathrm{Ar}}(K_\cdot,\rho_{K_\cdot})\big)\,\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)i^*(\Psi)\Big)\\
&+\sum_{j\geq 0}(-1)^j a\Big(p_*\big(\mathrm{ch}_{\mathrm{BC}}(A_j,\rho_{A_j})\big)\Big)i^*(\Psi)\\
&+\Big(g_*\big((0,\pi_*(\mathrm{ch}(D\mathcal{F}_\cdot,D\tau_\cdot)\mathrm{Log}|z|^2))\omega(\Psi)\big)\Big)\\
&+g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;i,\rho^i;\mathcal{F}_\cdot,\tau_\cdot)\omega(\Psi)\big)\Big).
\end{aligned}$$

On the other hand, by Theorem I.9.4,

$$\begin{aligned}
0=&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;i,g^i;\mathcal{F}_\cdot,\tau_\cdot)+\pi_*\big(\mathrm{Log}|z|^2\mathrm{ch}(D\mathcal{F}_\cdot,D\tau_\cdot)\big)\\
&-\Big(p_*\big(\mathrm{td}^{-1}(H^*,g^{H^*})\sigma^*(e_{\mathrm{BC}}(H^*,g^{H^*}))\big)\mathrm{ch}(\mathcal{E},\rho)+p_*\big(\sum_{j=0}^{m}(-1)^j\mathrm{ch}_{\mathrm{BC}}(A_j,\rho_{A_j})\big)\Big)\delta_X.
\end{aligned}$$

Hence, it is enough to prove that

$$\begin{aligned}
&\mathrm{td}_{\mathrm{Ar}}^{-1}(N,\rho_N)\,\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)-p_*\big(\mathrm{ch}_{\mathrm{Ar}}(K_\cdot,\rho_{K_\cdot})\big)\,\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\\
&\quad=a\Big(p_*\big(\mathrm{td}^{-1}(H^*,\rho_{H^*})\,\sigma^*(e_{\mathrm{BC}}(H^*,\rho_{H^*}))\big)\,\mathrm{ch}(\mathcal{E},\rho)\Big).
\end{aligned}$$

From Proposition 2.b.2, we know that

$$\mathrm{ch}_{\mathrm{Ar}}(K_\cdot,\rho_{K_\cdot})=e_{\mathrm{r},\mathrm{Ar}}(H^*,\rho_{H^*})\mathrm{td}_{\mathrm{Ar}}^{-1}(H^*,\rho_{H^*}),$$

so it is sufficient to show that

$$\begin{aligned}
&\mathrm{td}_{\mathrm{Ar}}^{-1}(N,\rho_N)\,\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)-p_*\big(e_{\mathrm{r},\mathrm{Ar}}(H^*,\rho_{H^*})\,\mathrm{td}_{\mathrm{Ar}}^{-1}(H^*,\rho_{H^*})\big)\,\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\\
&\quad=a\Big(p_*\big(\mathrm{td}^{-1}(H^*,\rho_{H^*})\,\sigma^*(e_{\mathrm{BC}}(H^*,\rho_{H^*}))\big)\,\mathrm{ch}(\mathcal{E},\rho)\Big):
\end{aligned}$$

By Theorem 3.5.f.2, this last assertion is a direct consequence of the facts that $e_{\mathrm{Ar}}(H^*,\rho_{H^*})$ is the class

$$(X\times\{\infty\},-\sigma^*(e_{\mathrm{BC}}(H^*,\rho_{H^*}))$$

in $\mathrm{CH}_{\mathrm{Ar}}(P)$, and that the restriction of H^* to $X\times\{\infty\}$ coincides with N.

Chapter II.6. Arithmetic Riemann-Roch Theorem For L.C.I. Morphisms

In this chapter, we will give the main result for this part; that is, the arithmetic Riemann-Roch theorem for l.c.i. morphisms of regular arithmetic varieties. By chapters 4 and 5 of this part, it seems to be the case that we already have had such a theorem, as we have given the arithmetic Riemann-Roch theorems for both smooth morphisms and closed immersions. However, the situation here is much more complicated: E.g., for closed immersions, we have assumed that the structure morphisms are smooth even at the finite part, which is in general not true. The effort to remove the assumption of the smoothness has so far only reached its first stage: We only have the result for a l.c.i. morphism of arithmetic varieties, which is smooth at infinity. As we stated as earlier as in 3.8.b, the main difficult is that we do not know what a hermitian K-theory for coherent sheaves should be. In algebraic geometry, which usually deals with everything at the level of cohomology classes, the coherent sheaves are essentially the vector sheaves (both of them correspond to algebraic cycles). This latest statement is no longer true in arithmetic geometry, which treats everything at the level of differential forms, or better, at the level of currents. To avoid this difficulty up to certain degree, we now assume that our object in the finite part concerns coherent sheaves, while at infinity, the object is only concerned with vector sheaves. So we may introduce a concept for hermitian coherent sheaves: These are coherent sheaves in the finite part, while their pull backs at the infinity are hermitian vector sheaves. Hence we may attach them with arithmetic cycles. In the same spirit, in the sequel, we shall only deal with l.c.i. morphisms of arithmetic varieties which are smooth at infinity.

This chapter consists of two sections. In section one, we give a natural definition for the push-out morphisms of arithmetic K-groups with respect to l.c.i. morphisms, and also the arithmetic Riemann-Roch theorem for l.c.i. morphisms. In section two, we reduce the proof of the arithmetic Riemann-Roch theorem for l.c.i. morphisms to an arithmetic Riemann-Roch theorem for closed immersions, which is similar to the one proved in the last chapter, but without the assumption that the structure morphism are smooth. We leave the complete proof of this theorem to the next chapter, after making certain natural generalizations of the theory.

§II.6.1. The Arithmetic Riemann-Theorem For L.C.I. Morphisms

We give a definition of the push-out morphism of arithmetic K-groups for l.c.i. morphisms of regular arithmetic varieties, which are smooth at infinity. As was stated in the introduction, the definition of a push-out morphism of arithmetic K-groups for l.c.i. morphisms of regular arithmetic varieties will include the classical Bott-Chern secondary characteristic forms, the relative Bott-Chern secondary characteristic forms for smooth morphisms and the relative Bott-Chern secondary characteristic currents for closed immersions.

Let $f : X \to Y$ be an l.c.i. morphism of regular arithmetic varieties. We then have a decomposition: A closed immersion $i : X \hookrightarrow P$ followed by a projection $g : P \to Y$. That is, we have the following commutative diagram:

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & P \\ & f\searrow \quad \swarrow g & \\ & Y & \end{array}.$$

To define a morphism

$$f_K : \mathrm{K}_{\mathrm{Ar}}(X)_{\mathbf{Q}} \to \mathrm{K}_{\mathrm{Ar}}(Y)_{\mathbf{Q}},$$

we need introduce the push-out morphism of arithmetic K-groups for the closed immersion i, since we have already had a good definition for the smooth morphism g in chapter 3. Even through this can be done in principle: We need a theory for a relative arithmetic intersection theory and a relative arithmetic K-theory for closed immersions. Since we do not want to develop the relative theory here, we give a direct definition.

Note that since the arithmetic K-group $\mathrm{K}_{\mathrm{Ar}}(X)$ is generated by f-acyclic hermitian vector sheaves and smooth forms, we need only to make the definition of f_K for each of them, and then prove the compatibility.

For the decomposition of f, since $f_{\mathbf{C}}$ is smooth, we have the following short exact sequence:

$$N : 0 \to \mathcal{T}_{f_{\mathbf{C}}} \to i^*\mathcal{T}_{g_{\mathbf{C}}} \to \mathcal{N}_{i,\mathbf{C}} \to 0.$$

With this, in subsection 3.7.b and for any $\Upsilon \in \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}}$, we define

$$\begin{aligned} &\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)\Upsilon := \\ &\quad = \mathrm{td}_{\mathrm{Ar}}(i^*\mathcal{T}_g, i^*\rho_g)\,(\mathrm{td}_{\mathrm{Ar}}^{-1}(\mathcal{N}_i,\rho_{\mathcal{N}_i})\Upsilon) + \mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\Upsilon \in \mathrm{CH}.^{\mathrm{Ar}}(X)_{\mathbf{Q}}. \end{aligned}$$

Here $\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})$ denotes the intersection of the classical Bott-Chern secondary characteristic forms associated with the short exact sequence N above and $\mathrm{td}^{-1}(\mathcal{N}_i,\rho_i)$. (See Section 3.7.)

If $\alpha \in \tilde{A}(X_{\mathbf{R}})$, it is natural to let

$$f_K(\alpha) := f_*(\alpha\ \mathrm{td}_{\mathrm{Ar}}(f,\rho_f)).$$

Next we give the definition of f_K for f-acyclic hermitian vector sheaves. Let $(\mathcal{E},\rho)$ be an f-acyclic hermitian vector sheaf on X. From I.7, we know that there is a resolution of vector sheaves on P for $i_*\mathcal{E}$:

$$0 \to \mathcal{F}_n \to \dots \to \mathcal{F}_1 \to \mathcal{F}_0 \to i_*\mathcal{E} \to 0.$$

Let $\mathcal{F}_j$ be equipped with F_∞-invariant hermitian metrics τ_j, which satisfy Bismut condition (A), then we make the following definition:

$$\begin{aligned} f_K(\mathcal{E},\rho) =: & g_K(\mathcal{F}_\cdot,\tau_\cdot) \\ & + g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;i,\rho_i;\mathcal{F}_\cdot,\tau_\cdot)\big)\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\Big) \\ & + f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\,a\big(\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\big)\Big). \end{aligned}$$

As in Chapter II.3, for any g-acyclic hermitian vector sheaf $(\mathcal{F},\tau)$, we have

$$\begin{aligned} g_K(\mathcal{F},\tau) = & (g_*\mathcal{F},g_*\tau) + \mathrm{ch}_{\mathrm{BC}}(\mathcal{F},\tau;g,\rho_g) \\ & + g_*\Big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F},\rho)\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\,a\big(R(\mathcal{T}_g)\big)\Big), \end{aligned}$$

where R is the unique power series defined in the arithmetic Riemann-Roch theorem for smooth morphisms. We also use the isomorphism $\mathrm{ch}_{\mathrm{Ar}}$ between K_{Ar} and $\mathrm{CH}_{\mathrm{ArQ}}$ and think of the element

$$g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho,i,\rho_i)\big)\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\Big) + f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\,a\big(\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\big)\Big)$$

in $\mathrm{CH}_{\mathrm{ArQ}}$ as being in K_{ArQ}: Since they are in the image of a, the meaning of this element in K_{Ar} is clear.

As usual, once we make a definition, we need to show that it is well-defined. So we need the following

Proposition. With the above definition for smooth forms and f-acyclic hermitian sheaves, we have a well-defined group morphism

$$f_K : K_{\mathrm{Ar}}(X) \to K_{\mathrm{Ar}}(Y)_{\mathbf{Q}}.$$

Here one may ask how f_K depends on the with various data. This may be deduced (by 3.7.b) from the following

Arithmetic Riemann-Roch Theorem For l.c.i. Morphisms. With the notation above, for any element $\Upsilon \in K_{\mathrm{Ar}}(X)_{\mathbf{Q}}$, we have

$$\mathrm{ch}_{\mathrm{Ar}}(f_K(\Upsilon)) = f_{\mathrm{CH}}(\mathrm{ch}_{\mathrm{Ar}}(\Upsilon)\ \mathrm{td}_{\mathrm{Ar}}(f,\rho_f)).$$

Remark: Actually, in this definition, we need the cup product for the arithmetic intersection in the sense of arithmetic Chow homology. But since it is rather formal, and one can understand the above formula without it, so we will not give more attention to this formalism, but leave consideration of it to the next chapter. The serious reader may first wish to look at that chapter, and then go on with this chapter. We treat matters in this way so that the reader may get a feeling for why a certain formalism is necessary for "the" general theory.

Now we return to the dependence of f_K on various data. By Proposition 3.7.b, we know that $\mathrm{td}_{\mathrm{Ar}}(f, \rho_f)$ depends only on the choices of metric on $\mathcal{T}_{f_{\mathbf{C}}}$, and not on the choice of i, g, nor on the metrics on $\mathcal{N}_i$ and $\mathcal{T}_g$. Therefore, we know that f_K also depends only on the choice of the metric on $\mathcal{T}_{f_{\mathbf{C}}}$, and not on the rest.

§II.6.2. The Proof Of The Results

In this section, we give the proofs of the results stated in the last section using a direct generalization of the arithmetic Riemann-Roch theorem for closed immersions in Chapter 5. The proof of this generalization is not to be given until the next chapter.

The Arithmetic Riemann-Roch Theorem For Closed Immersions.
Let $A = (A, \Sigma, F_\infty)$ be an arithmetic ring and let

$$\begin{array}{ccccc} X & & \stackrel{i}{\hookrightarrow} & & Z \\ & f\searrow & & \swarrow g & \\ & & Y & & \end{array}$$

be closed immersions of regular arithmetic varieties over Y with f proper, $f_{\mathbf{C}}$ smooth, and g smooth. Put F_∞-invariant hermitian metrics on the relative tangent sheaves and the normal sheaf of i. Let $(\mathcal{E}, \rho)$ be a hermitian vector sheaf on X and $\mathcal{F}. \to i_*\mathcal{E} \to 0$ a vector sheaf resolution of $i_*\mathcal{E}$ on Z. Put F_∞-invariant hermitian metrics $\tau.$ on $\mathcal{F}.$ such that Bismut condition (A) is satisfied with respect to ρ_N and ρ. Then, for any $\Psi \in \mathrm{CH}_{\mathrm{Ar}}(Z)$, the following identity holds on $\mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}$

$$\begin{aligned} &f_*\big(\mathrm{td}_{\mathrm{Ar}}^{-1}(N, \rho_N)\,\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}, \rho) i^*(\Psi)\big) \\ &\quad= g_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}., \tau.)\Psi\big) + g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; i, \rho^i; \mathcal{F}., \tau.)\omega(\Psi)\big)\Big). \end{aligned}$$

Now we give the proof of the proposition stated in section 1.

Proof of the proposition: By the proof of Proposition 4.1.d, it is sufficient to prove that for any short exact sequence of f-acyclic hermitian vector sheaves

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

with F_∞-invariant hermitian metrics ρ_k on $\mathcal{E}_k$,

$$f_K(\mathcal{E}_2,\rho_2)-f_K(\mathcal{E}_1,\rho_1)-f_K(\mathcal{E}_3,\rho_3)=f_K(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)).$$

This equality can be proved by using Axiom 3 of relative Bott-Chern secondary characteristic current for the closed immersion case and the arithmetic Riemann-Roch theorem for closed immersions. First take a vector sheaf resolution of $i_*(\mathcal{E}.)$

$$\mathcal{F}_k.\to i_*(\mathcal{E}_k)\to 0,$$

so that we have a short exact sequence

$$\mathcal{F}_{\cdot j}:0\to\mathcal{F}_{1j}\to\mathcal{F}_{2j}\to\mathcal{F}_{3j}\to 0.$$

Equip the corresponding terms with F_∞-invariant hermitian metrics such that Bismut condition (A) is satisfied. Then, by definition, we know that

$$\begin{aligned}
&f_K(\mathcal{E}_2,\rho_2)-f_K(\mathcal{E}_1,\rho_1)-f_K(\mathcal{E}_3,\rho_3)\\
&=g_K(\mathcal{F}_2.,\tau.)+g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_2,\rho_2;i,\rho_i;\mathcal{F}_2.,\tau_2.)\big)\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\Big)\\
&\quad+f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_2,\rho_2)\,a\big(\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\big)\Big)\\
&\quad-g_K(\mathcal{F}_1.,\tau_1.)-g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_1,\rho_1;i,\rho_i;\mathcal{F}_1.,\tau_1.)\big)\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\Big)\\
&\quad-f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_1,\rho_1)\,a\big(\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\big)\Big)\\
&\quad-g_K(\mathcal{F}_3.,\tau_3.)-g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_3,\rho_3;i,\rho_i;\mathcal{F}_3.,\tau_3.)\big)\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\Big)\\
&\quad-f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_3,\rho_3)\,a\big(\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\big)\Big).
\end{aligned}$$

By the arithmetic Riemann-Roch theorem for closed immersions stated above, and Axiom 3 for relative Bott-Chern secondary characteristic currents for closed immersions, we know that

$$\begin{aligned}
&f_K(\mathcal{E}_2,\rho_2)-f_K(\mathcal{E}_1,\rho_1)-f_K(\mathcal{E}_3,\rho_3)\\
&=f_{\mathrm{CH}}\big(\sum_k(-1)^k\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_k,\rho_k)\,\mathrm{td}_{\mathrm{Ar}}^{-1}(N,\rho_N)\,i^*\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\big)\\
&\quad+f_*\Big(\sum_k(-1)^k\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}_k,\rho_k)\,a\big(\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\big)\Big)\\
&=f_{\mathrm{CH}}\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)\big)\,\mathrm{td}_{\mathrm{Ar}}^{-1}(N,\rho_N)\,i^*\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\Big)\\
&\quad+f_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)\big)\,a\big(\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\big)\Big)\\
&=f_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)\big)\,\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)\Big)\\
&=f_K\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)\big)
\end{aligned}$$

as required.

We end the chapter by proving the arithmetic Riemann-Roch theorem for l.c.i. morphisms.

Proof of the arithmetic Riemann-Roch theorem for l.c.i. morphisms. The proof comes from the arithmetic Riemann-Roch theorems for both smooth morphisms and closed immersions. It is clearly sufficient to prove the formula for f-acyclic hermitian vector sheaves. Let $(\mathcal{E},\rho)$ be such an element. We have

$$\begin{aligned}\mathrm{ch}_{\mathrm{Ar}}\big(f_K(\mathcal{E},\rho)\big) &- f_{\mathrm{CH}}\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\ \mathrm{td}_{\mathrm{Ar}}(f,\rho_f)\big)\\ &=\sum_{j=0}^{n}(-1)^j\mathrm{ch}_{\mathrm{Ar}}\big(g_K(\mathcal{F}_j,\tau_j)\big)+g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho,i,\rho_i)\big)\ \mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\Big)\\ &+f_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\big)-f_{\mathrm{CH}}\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\ \mathrm{td}_{\mathrm{Ar}}(f,\rho_f)\big).\end{aligned}$$

By the arithmetic Riemann-Roch theorem for the smooth morphism g, the above combination of terms is equal to

$$\begin{aligned}&\sum_{j=0}^{m}(-1)^j g_{\mathrm{CH}}\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}_j,\tau_j)\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\big)+g_*\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;i,\rho_i;\mathcal{F}_\cdot,\tau_\cdot)\mathrm{td}_{\mathrm{Ar}}(g,\rho_g)\big)\\ &+f_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\big)-f_{\mathrm{CH}}\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\ \mathrm{td}_{\mathrm{Ar}}(f,\rho_f)\big).\end{aligned}$$

On the other hand, by the arithmetic Riemann-Roch theorem for the closed immersion i, the last quantity is equal to

$$\begin{aligned}f_*\big(&\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})+\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\mathrm{td}_{\mathrm{Ar}}{}^{-1}(\mathcal{N}_i,\rho_{\mathcal{N}_i})\ i^*\mathrm{td}_{\mathrm{Ar}}(g,\rho_g)\\ &-\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\mathrm{td}_{\mathrm{Ar}}(f,\rho_f)\big),\end{aligned}$$

and this is 0 by definition. So we have proved the assertion.

We end of this chapter with the following remark: Since we assume that f at infinity is smooth, it makes sense to talk about the associated relative Bott-Chern secondary characteristic forms with respect to f_∞. As a corollary, we may give a direct definition for the push-out morphism of arithmetic K-groups, without using any decomposition of f as above. More precisely, if $f: X \to Y$ is an l.c.i. morphism of regular arithmetic varieties over an arithmetic ring (A,σ,F_∞), which is smooth at infinity. Directly define the push-out morphism $f_K^{\mathrm{Ar}}: K_0^{\mathrm{Ar}}(X) \to K_0^{\mathrm{Ar}}(Y)$ by letting

$$\begin{aligned}f_K^{\mathrm{Ar}}(\mathcal{E},\rho)=&(f_*\mathcal{E},f_*\rho)+\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f,\rho_f)\\ &+f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\ \mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_f,\rho_f)\,a\big(R(\mathcal{T}_f)\big)\Big)\end{aligned}$$

for any f-acyclic hermitian vector sheaf $(\mathcal{E},\rho)$ as in Chapter II.4, where R is the unique power series defined in the arithmetic Riemann-Roch theorem for smooth morphisms. Then, we have the following

Arithmetic Riemann-Roch Theorem For l.c.i. Morphisms′. With the same notation above, for any element $\Upsilon \in K_{\mathrm{Ar}}(X)_{\mathbf{Q}}$, we have

$$\mathrm{ch}_{\mathrm{Ar}}(f_K(\Upsilon)) = f_{\mathrm{CH}}(\mathrm{ch}_{\mathrm{Ar}}(\Upsilon)\ \mathrm{td}_{\mathrm{Ar}}(f,\rho_f)).$$

We leave the proof of this theorem in Chapter II.7 too. As a consequence of this theorem, we can give the relation between various Bott-Chern secondary characteristic objects. Indeed, from the equalities in the above two arithmetic Riemann-Roch theorems for l.c.i. morphisms, we find that the right hand sides are just the same. Thus, by the fact that $\mathrm{ch}_{\mathrm{Ar}}$ is an isomorphism between $K_{\mathrm{Ar}\mathbf{Q}}$ and $\mathrm{CH}_{\mathrm{Ar}\mathbf{Q}}$, we see that the two definitions about f_K should be the same. So, with the same notation as above, we find that for f-acyclic hermitian vector sheaf $(\mathcal{E},\rho)$,

$$\begin{aligned}(f_*\mathcal{E}, f_*\rho) + \mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho; f,\rho_f)&\\ &+ f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_f,\rho_f)\,a\big(R(\mathcal{T}_f)\big)\Big)\\ &= g_K(\mathcal{F}.,\tau.)\\ &+ g_*\Big(a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho; i,\rho_i;\mathcal{F}.,\tau.)\big)\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_g,\rho_g)\Big)\\ &+ f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\,a\big(\mathrm{td}_{\mathrm{BC}}(f/g,\rho_{f/g})\big)\Big).\end{aligned}$$

In particular, we now have the uniqueness of the relative Bott-Chern secondary characteristic currents with respect to closed immersions, since in the latest relation, all others are unique for a fixed decomposition of $f = g \circ i$.

Chapter II.7
Grassmannian Graph Construction In Arithmetic Geometry

We now prove the arithmetic Riemann-Roch theorem for closed immersions stated in 6.1. During this process, (in fact, even at the very beginning, in order to understand the theorem), we need to introduce the arithmetic Chern character with supports in the language of arithmetic Chow homology groups. As one may imagine, the Grassmannian graph construction of MacPherson is very useful. Here we will follow [GS 92] to expose the whole theory.

This chapter consists of the following four sections. In section one, we introduce the Grassmannian graph construction. In section two, we introduce the arithmetic Chern characteristic class with supports. In section three, we complete the proof of the arithmetic Riemann-Roch theorem for l.c.i. morphisms, and give a more direct definition for the push-out morphism for arithmetic K-groups. Finally, in section four, we give a more general discussion.

§II.7.1. The Grassmannian Graph Construction

In this section, we will give MacPherson's Grassmannian graph construction. This is a generalization of the deformation to the normal cone theory for a regular closed immersion and has its root in homotopy theory: How to find an easily handled object in a homotopy class?

II.7.1.a. Deformation To The Normal Cone

We recall the deformation to the normal cone theory for a regular closed immersion and consider what are the most important facts concerning it. Let $i : X \hookrightarrow Z$ be a closed immersion of regular varieties. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
X & \leftarrow & X \times \{\infty\} & \overset{i_\infty}{\hookrightarrow} & \mathbf{P}(\mathcal{N} \oplus \mathcal{O}_X) + B_X Z = W_\infty & \rightarrow & \{\infty\} \\
\downarrow & & \downarrow & & j_\infty \downarrow & & \downarrow \\
X & \overset{p_X}{\leftarrow} & X \times \mathbf{P}^1 & \overset{f}{\hookrightarrow} & B_{X \times \infty} Z \times \mathbf{P}^1 = W & \overset{\pi_2}{\rightarrow} & \mathbf{P}^1 \\
\uparrow & & \uparrow & & j_0 \uparrow & & \uparrow \\
X & \leftarrow & X \times \{0\} & \overset{i_0}{\hookrightarrow} & Z \times \{0\} = W_0 & \rightarrow & \{0\}
\end{array}$$

with the following properties:

(1) The fiber of I over 0 is i.
(2) The fiber of I over ∞ is the union of $B_X Z$ and $\mathbf{P}_X(N \oplus 1)$ and they intersect transversally along the exceptional divisor of $B_X Z$; the imbedding of $X \times \{\infty\}$ in W_∞ is just the zero section of $\mathbf{P}(N \oplus 1)$.
(3) The I image of $X \times \mathbf{P}^1$ does not meet $B_X Z$ in W.
(4) The projection $W \to \mathbf{P}^1$ is flat.

Since we have the above properties, if $\mathcal{F}.$ is a vector sheaf resolution of the i direct image of a vector sheaf $\mathcal{E}$ on Z, then with this deformation, we can use the natural associated Koszul complex to make a certain precise calculation. For more details, see I.9, II.1 and II.5. Next, we generalize the above basic properties to a more general context by using the Grassmannian graph construction, which comes from the following vivid observation for the deformation to the normal cone:

Assume E is a vector bundle on Z, and s is a section of E whose zero-scheme is X. Then for each scalar λ, the graph of λs is a line in $E \oplus 1$. So we get an imbedding

$$Z \times \mathbf{A}^1 \hookrightarrow \mathbf{P}(E \oplus 1) \times \mathbf{P}^1,$$

by the natural map $(z, \lambda) \mapsto (\text{graph of } \lambda s(z), [1 : \lambda])$. In this way, we may find that the deformation space W is in fact the closure of $Z \times \mathbf{A}^1$ in this imbedding.

II.7.1.b. The Grassmannian Graph Construction

Let X be an integral scheme. Let $\mathcal{E}.$ be a chain complex of vector sheaves on X. Denote by $C. := C(\mathcal{E}.)$ the split acyclic complex with $C_i = \mathcal{E}_i \oplus \mathcal{E}_{i-1}$ and differential $d_i : C_i \to C_{i-1}$ being $d_i(x, y) = (y, 0)$. Obviously, $C(\mathcal{E}.)$ is an additive functor of graded vector sheaves $\mathcal{E}.$. Furthermore, there is a natural morphism of complexes

$$\begin{array}{rccc} \gamma : & \mathcal{E}. & \to & C(\mathcal{E}.) \\ & x & \mapsto & (x, d(x)), \end{array}$$

which is the inclusion of a subvector sheaf in each degree. This construction is also compatible with the morphisms between complexes: If $\phi : \mathcal{E}. \to \mathcal{F}.$ is a morphism of complexes, then $C(\phi) \circ \gamma_{\mathcal{E}.} = \gamma_{\mathcal{F}.} \circ \phi$. If ϕ is quasi-isomorphic to zero, or equivalently, null-homotopic, i.e. if there exists h such that $\phi = d\,h + h\,d$, then $C(\phi)$ is also null-homotopic. In fact, there is a map

$$\begin{array}{rccc} C(h) : & \mathcal{E}_i \oplus \mathcal{E}_{i-1} & \to & \mathcal{F}_i \oplus \mathcal{F}_{i-1} \\ & (x, y) & \mapsto & (h(x), -h(y) + \phi(x)). \end{array}$$

So, on $C(\mathcal{E}.)$, $C(\phi) = d \circ C(h) + C(h) \circ d$ and this homotopy is compatible with the natural transformation γ, i.e. $C(h) \circ \gamma_{\mathcal{E}.} = \gamma_{\mathcal{F}.} \circ h$.

We suppose now that $\mathcal{E}_i = 0$ for $i < 0$. Let $\mathbf{P}^1$ be the projective line over $\mathbf{Z}$, and $\mathcal{O}_{\mathbf{P}^1}(i\infty)$ the line sheaf of meromorphic functions on $\mathbf{P}^1$ which have poles of at most order i along the divisor ∞ and are regular on the affine line $\mathbf{A}^1 = \mathbf{P}^1 - \{\infty\}$. Naturally, $\mathcal{O}_{\mathbf{P}^1}(i\infty)$

is contained in $\mathcal{O}_{\mathbf{P}^1}((i+1)\infty)$. By a pull-back along the projection $X \times \mathbf{P}^1 \to X$, we may view $\mathcal{E}.$ as a complex of vector sheaves on $X \times \mathbf{P}^1$. Let $DC. := DC(\mathcal{E}.)$ be the C-construction applied to the graded vector sheaf $\oplus_i \mathcal{E}_i(i)$, where $\mathcal{E}(i) := \mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^1}(i\infty)$ is the twisted vector sheaf. We know that the sheaf $\mathcal{E}_i$ is a subsheaf of $\mathcal{E}_i(i)$ and that they are equal on $X \times \mathbf{A}^1$. Hence, via the map $\gamma_{\mathcal{E}.}$, $\mathcal{E}_i|_{X\times\mathbf{A}^1}$ is a subvector sheaf of $DC(\mathcal{E}.)|_{X\times\mathbf{A}^1}$. Let $\pi : \mathbf{G} \to X \times \mathbf{P}^1$ be the product of Grassmannian bundle $\mathbf{G}(n_i, DC_i)$ parameterizing the rank $n_i = \mathrm{rank}(C_i)$ subvector sheaves of DC_i over $X \times \mathbf{P}^1$ for all i. Over $X \times \mathbf{A}^1$, the map $\gamma_{\mathcal{E}.}$ defines a section s of π. Thus, following Baum, Fulton and MacPherson [BFM 75], we define the **Grassmannian graph** of $\mathcal{E}.$ as the Zariski closure $W := W(\mathcal{E}.)$ of $s(X \times \mathbf{A}^1)$ in $\mathbf{G}$. Next we look at properties of this construction.

First, since π is proper so is its restriction to W, which we shall also denote by π. Since $X \times \mathbf{A}^1$ is integral, so is W. By the construction, π is an isomorphism on $X \times \mathbf{A}^1$. However, the effective Cartier divisor $W_\infty := \pi^{-1}(X \times \{\infty\})$, cut out by W at infinity, may not be isomorphic to X. There is also a subvector sheaf $D\mathcal{E}_i \subset \pi^*(DC_i)$ which coincides with $\mathcal{E}_i$ over $X \times \mathbf{A}^1$. Indeed, this later property characterizes $D\mathcal{E}_i$ as a subvector sheaf of $\pi^*(DC_i)$, since W is integral.

Second, if the restriction of $\mathcal{E}.$ to a nonempty open subset $U \subset X$ is acyclic. Then there is a canonical splitting of π over $U \times \mathbf{P}^1$. Denote by $\tilde{X}$ the closure in W_∞ of the image of $U \times \{\infty\}$ by the section s. Then the cycle $Z = [W_\infty] - [\tilde{X}]$ is supported in the inverse image by π of $X - U$, and the restriction of $D\mathcal{E}.$ to $\tilde{X}$ is split acyclic.

Finally, we look at how this construction generalizes the above deformation to the normal cone theory for closed immersions. Let $i : X \hookrightarrow P$ be a regular closed immersion. Let $\mathcal{F}$ be a vector sheaf on X, and let $\mathcal{E}. \to i_*\mathcal{F} \to 0$ be a finite vector sheaf resolution. Then, by a direct calculation, we see that $W(\mathcal{E}.)$ is isomorphic to the total space W of the deformation to the normal cone construction stated in subsection 7.1.a. Hence,

(1) $W_\infty \simeq \mathbf{P}(\mathcal{N}_{X/P} \oplus 1) \cup \tilde{P}$. Here, $\tilde{P}$ is the blow-up of P along X.
(2) The immersion $X \times \mathbf{P}^1 \hookrightarrow P \times \mathbf{P}^1$ induces a closed immersion $I : X \times \mathbf{P}^1 \hookrightarrow W$, such that $D\mathcal{E}.$ is a resolution of $I_*\mathcal{F}$.
(3) $|Z|$ is the projective completion $\mathbf{P}(\mathcal{N}_{X/P} \oplus 1)$ of the normal bundle of X in P.
(4) There is an exact sequence on $|Z|$

$$0 \to \mathcal{G}. \to D\mathcal{E}.|_{\mathbf{P}(\mathcal{N}_{X/P}\oplus 1)} \to K.(\mathcal{H}) \otimes \pi^*(\mathcal{F}) \to 0,$$

where $\mathcal{G}.$ is acyclic and $K.(\mathcal{H})$ is the tautological Koszul complex on $\mathbf{P}(\mathcal{N}_{X/P} \oplus 1)$, which is a resolution of $\mathcal{O}_X$ when X is imbedded in $\mathbf{P}(\mathcal{N}_{X/P} \oplus 1)$ by the zero section.

II.7.1.c. Additional Properties

We consider how $W(\mathcal{E}.)$ depends upon $\mathcal{E}.$. For this, we need the following

Lemma 1. $D\mathcal{E}.$ is a subcomplex of $\pi^*(DC.)$.

Proof. Since $DC.$ is a complex, it is sufficient to show that $d_{DC.}(D\mathcal{E}_i) \subset D\mathcal{E}_{i-1}$. By definition, we know that this is true on the dense open subset $X \times \mathbf{A}^1 \subset W$. But

W is integral and, by the definition of the Grassmannian, $D\mathcal{E}_{i-1}$ is a subvector sheaf in $\pi^*(DC_{i-1})$. So we deduce the assertion from the following easy

Sublemma. Let V be an integral scheme, and suppose that $\mathcal{A} \subset \mathcal{E}$ and $\mathcal{B} \subset \mathcal{F}$ are vector subsheaves of vector sheaves. Then we have
(1) If $\phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism which vanishes on a Zariski open dense subset of V, then ϕ vanishes on the whole of V;
(2) If $\mathcal{B} \subset \mathcal{F}$ is a subvector sheaf, and $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism such that, over a Zariski dense open subset of V, $\phi(\mathcal{A}) \subset \mathcal{B}$, then $\phi(\mathcal{A}) \subset \mathcal{B}$ over the whole of V.

Proof of the sublemma. We have the first assertion since $\mathcal{B}$ is a torsion free module. Applying (1) to the induced map $\mathcal{A} \to \mathcal{F}/\mathcal{B}$, we have (2).

Before we go to the most general case, we look at a very special Grassmannian graph construction.

Proposition 1. If $\mathcal{E}.$ is a complex as above, but with the homology sheaves being vector sheaves, then $W(\mathcal{E}.) \simeq X \times \mathbf{P}^1$.

Proof. We know that the complex $\mathcal{E}.$ breaks up into two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{Z}_i & \to & \mathcal{E}_i & \xrightarrow{d} & \mathcal{B}_{i-1} & \to & 0; \\ 0 & \to & \mathcal{B}_i & \to & \mathcal{Z}_i & \to & H_i(\mathcal{E}.) & \to & 0, \end{array}$$

where $\mathcal{Z}$ and $\mathcal{B}$ denote the subsheaves of $\mathcal{E}$ consisting of cycles and boundaries respectively. Since we assume that the H_i's are vector sheaves, it follows by induction on i that all the sheaves in the above exact sequences are also vector sheaves. From this, we introduce a natural morphism

$$\eta_i : \mathcal{E}_i(i) \oplus \mathcal{E}_{i-1}(i-1) \to \frac{\mathcal{B}_{i-1}(i) \oplus \mathcal{E}_{i-1}(i-1)}{\mathcal{B}_{i-1}(i-1)}$$

as follows: η_i maps (u, v) to the class of (du, v). Here $\mathcal{B}_{i-1}(i-1)$ is mapped diagonally into $\mathcal{B}_{i-1}(i) \oplus \mathcal{E}_{i-1}(i-1)$ by the inclusions $\mathcal{B}_{i-1}(i-1) \subset \mathcal{B}_{i-1}(i)$ and $\mathcal{B}_{i-1}(i-1) \subset \mathcal{Z}_{i-1}(i-1) \subset \mathcal{E}_{i-1}(i-1)$. So by the fact that the sheaves in the above exact sequences are all vector sheaves, we know that the image of η_i is also a vector sheaf, and hence so is its kernel. On the other hand, over $X \times \mathbf{A}^1$, the homomorphism η_i is equivalent to the map

$$\mathcal{E}_i(i) \oplus \mathcal{E}_{i-1}(i-1) \to \mathcal{E}_{i-1}(i-1),$$

which sends (u, v) to $v - du$. So the restriction of Ker (η_i) to $X \times \mathbf{A}^1$ is isomorphic to the inclusion of $\mathcal{E}_i$ into $\mathcal{E}_i \oplus \mathcal{E}_{i-1}$ via $x \mapsto (x, dx)$. Thus the subvector sheaf Ker$(\eta_i) \subset DC_i$ defines an extension of the section $s : X \times \mathbf{A}^1 \to \mathbf{G}(n_i, C_i)$ over $X \times \mathbf{P}^1$. Therefore, $W(\mathcal{E}.) = X \times \mathbf{P}^1$.

We now consider the functorial property of the Grassmannian graph construction.

Proposition 2. Let $\mathcal{E}.$ be a complex of vector sheaves on X, and let $f : Y \to X$ be a flat map. Then $W(f^*\mathcal{E}.) = Y \times_X W(\mathcal{E}.)$.

Proof. By definition, obviously, we have $D(f^*\mathcal{E}.) = (f \times \mathrm{Id}_{\mathbf{P}^1})^*(DC(\mathcal{E}.))$. Hence $W(f^*\mathcal{E}.)$ is the Zariski closure of $Y \times \mathbf{A}^1$ in $Y \times \mathbf{G}$, which is equal to the Zariski closure of $f^{-1}(s(X \times \mathbf{A}^1))$. On the other hand, by the fact that f is flat, we know that f is open and f^{-1} preserves the operation of taking Zariski closure. So $W(f^*\mathcal{E}.) = f^{-1}(W(\mathcal{E}.)) = Y \times_X W(\mathcal{E}.)$.

For the dependence of $W(\mathcal{E}.)$ on $\mathcal{E}.$ in general, we have the following

Proposition 3. Let $\phi : \mathcal{E}. \to \mathcal{F}.$ be a morphism of complexes of vector sheaves on X, such that ϕ is a monomorphism and Coker (ϕ) is an acyclic complex of vector sheaves. Then $W(\mathcal{E}.) = W(\mathcal{F}.)$ and ϕ extends, from $X \times \mathbf{A}^1$, to a uniuqe morphism of complexes $D\phi : D\mathcal{E}. \to D\mathcal{F}.$. Furthermore, Coker$(D\phi)$ is acyclic, and split acyclic over $\{\infty\}$.

Proof. By definition, we know that $X \times \mathbf{A}^1$ is dense in W, so we can get at most one isomorphism $W(\mathcal{E}.) \simeq W(\mathcal{F}.)$ such that its restriction to $X \times \mathbf{A}^1$ is the identity. Now, we define one. Since we may work locally on X, we assume that X is affine and therefore that the complex $\mathcal{F}.$ is the direct sum of $\mathcal{E}.$ with an acyclic complex $\mathcal{G}.$. (This can be shown by induction on the degree.) Therefore $DC(\mathcal{F}.) \simeq DC(\mathcal{E}.) \oplus DC(\mathcal{G}.)$. Let m_i, n_i and p_i be the ranks of $\mathcal{E}_i$, $\mathcal{F}_i$ and $\mathcal{G}_i$, respectively. Then we have a closed embedding

$$\mathbf{G}(m_i, DC_i(\mathcal{E}.)) \times_{X \times \mathbf{P}^1} \mathbf{G}(p_i, DC_i(\mathcal{G}.)) \to \mathbf{G}(n_i, DC_i(\mathcal{F}.)),$$

which is compatible with the section on $X \times \mathbf{A}^1$. By Proposition 1, $W(\mathcal{G}.) = X \times \mathbf{P}^1$, hence under the embedding above, $W(\mathcal{E}.) \simeq W(\mathcal{F}.)$.

Other assertions come from the following facts: On $W(\mathcal{E}.) \simeq W(\mathcal{F}.)$, there is an exact sequence

$$0 \to DC(\mathcal{E}.) \to DC(\mathcal{F}.) \to DC(\mathcal{G}.) \to 0.$$

Hence the induced sequence

$$0 \to D\mathcal{E}. \to D\mathcal{F}. \to D\mathcal{G}. \to 0$$

is exact, since this is true locally.

As an immediate consequence of this proposition, we have

Corollary 1. Let $\phi : \mathcal{E}. \to \mathcal{F}.$ be a quasi-isomorphism between bounded complexes of vector sheaves on X. Then $W(\mathcal{E}.) = W(\mathcal{F}.)$, and the complexes $D\mathcal{E}.$ and $D\mathcal{F}.$ are quasi-isomorphic as complexes on $W(\mathcal{E}.)$.

If we apply the result to a regular closed immersion, we get

Corollary 2. Let P be an integral regular scheme, and $i : X \hookrightarrow P$ a closed subscheme. Let $W(X/P)$ denote the Grassmannian graph construction for any vector sheaf resolution of $i_*\mathcal{O}_X$ on P. Then, given a vector sheaf $\mathcal{F}$ on X, any resolution $\mathcal{E}. \to i_*\mathcal{F} \to 0$ by vector sheaves on P may extend to a complex of vector sheaves on $W(X/P)$ which is a complex of subvector sheaves of $DC(\mathcal{E}.)$.

Proof. By Corollary 1, we know that W does not depend on the choice of the resolution. On the other hand, locally on X, $\mathcal{E}.$ is quasi-isomorphic to a direct sum of copies of such a resolution $\mathcal{F}.$. By Proposition 3, $\mathcal{F}.$ and any finite direct sum $\mathcal{F}.^{\oplus N}$ extends to a subcomplex of $DC(\mathcal{F}.^{\oplus n}) \simeq (DC(\mathcal{F}.)^{\oplus n})$ on W. Now $W = \cup W(X \cap U/U)$ as U runs through any open cover of P. If we choose the open cover so that, on each U, $\mathcal{E}.$ is quasi-isomorphic to a sum of copies of $\mathcal{F}.$, then $\mathcal{E}.$ extends as a subcomplex of $DC(\mathcal{E}.)$ on each $W(X \cap U/U)$, and hence, by the uniqueness of such extensions, it extends as a subcomplex on the whole of W.

As a matter of fact, we also have the following

Proposition 4. If $\mathcal{E}.$ and $\mathcal{F}.$ are chain complexes of vector sheaves on X, the identity map on $X \times \mathbf{A}^1$ extends to a unique map from the Zariski closure $W(\mathcal{E}., \mathcal{F}.)$ of $X \times \mathbf{A}^1$ in $W(\mathcal{E}.) \times_{X \times \mathbf{P}^1} W(\mathcal{F}.)$ to $W(\mathcal{E}. \oplus \mathcal{F}.)$.

Proof. On the variety $W(\mathcal{E}.) \times_{X \times \mathbf{P}^1} W(\mathcal{F}.)$ by a pull-back from the two factors, we obtain subvector sheaves $D\mathcal{E}.$ of $DC(\mathcal{E}.)$ and $D\mathcal{F}$ of $DC(\mathcal{F}.)$ extending $\mathcal{E}.$ and $\mathcal{F}.$ from $X \times \mathbf{A}^1$ respectively. The direct sum $D\mathcal{E}. \oplus D\mathcal{F}.$ is a subvector sheaf of $DC(\mathcal{E}.) \oplus DC(\mathcal{F}.) \simeq DC(\mathcal{E}. \oplus \mathcal{F})$ and hence is classified by a map from the fiber product $W(\mathcal{E}.) \times_{X \times \mathbf{P}^1} W(\mathcal{F}.)$ to the Grassmannian of subvector sheaves of $DC(\mathcal{E}. \oplus \mathcal{F})$. This map agrees with the standard section s over $X \times \mathbf{A}^1$, and hence maps $W(\mathcal{E}., \mathcal{F}.)$ to $W(\mathcal{E}. \oplus \mathcal{F}.)$.

So, together with sublemma 1, we have the following

Corollary 3. (a) Any map $\phi : \mathcal{E}. \to \mathcal{F}.$ of chain complexes of vector sheaves on X extends to a map $D\phi : D\mathcal{E}. \to D\mathcal{F}.$ on $W(\mathcal{E}., \mathcal{F}.)$. The operation $\phi \mapsto D\phi$ is additive.
(b) If $\phi : \mathcal{E}. \to \mathcal{F}.$ and $\varphi : \mathcal{F}. \to \mathcal{G}.$ are maps of chain complexes of vector sheaves on X, then on $W(\mathcal{E}., \mathcal{F}., \mathcal{G}.)$ (which is defined analogously to $W(\mathcal{E}., \mathcal{F}.)$), we have $D(\varphi \circ \phi) = D\varphi \circ D\phi$.

Next, we simplify temporarily our notation, and write W for $W(\mathcal{E}., \mathcal{F}.)$. Let W_∞ be the inverse image of $X \times \{\infty\}$ under the projection $\pi : W \to X \times \mathbf{P}^1$. It follows from Proposition 1 that $[W_\infty] = Z + [\tilde{X}]$. The map $\pi|_{\tilde{X}} : \tilde{X} \to X$ is birational, and the support $|Z|$ of Z is contained in the inverse image of the proper closed subset of X, where the homology sheaves $H(\mathcal{E}. \oplus \mathcal{F}.)$ are not vector sheaves. By definition, if U is the complement of this closed subset in X, $\tilde{X}$ is the closure in W_∞ of the image of $U \times \{\infty\}$ by the section of π over $U \times \mathbf{P}^1$ obtained in Proposition 1.

Proposition 5. Let $\phi : \mathcal{E}. \to \mathcal{F}.$ be a morphism of complexes of vector sheaves on X and let h be a null homotopy of ϕ. Then ϕ extends uniquely from $X \times \mathbf{A}^1$ to a null homotopy Dh of $D\phi$ on W. Furthermore, the restriction of Dh to W_∞ depends only on ϕ and not on the choice of h; it is additive in ϕ; And the restriction of Dh to $|Z|$ depends only on the restriction of ϕ to $\pi(|Z|)$ and is additive.

Proof. We can define a map on $X \times \mathbf{P}^1$

$$DC_i(h) : \mathcal{E}_i(i) \oplus \mathcal{E}_{i-1}(i-1) \to \mathcal{F}_{i+1}(i+1) \oplus \mathcal{F}_i(i)$$

by $(x, y) \mapsto (h(x), -h(y) + \phi(x))$, where we use the embedding of $\mathcal{O}_{\mathbf{P}^1}(i\infty)$ naturally into $\mathcal{O}_{\mathbf{P}^1}((i+1)\infty)$. By the sublemma, we see the existence and the uniqueness of Dh, since

$DC(h)$ restricted to the dense open subset $X \times \mathbf{A}^1 \subset W$ is a null-homotopy of $C(\phi)$. On the other hand, on W, we have the commutative diagram

$$\begin{array}{ccc} D\mathcal{E}_i & \rightarrow & \mathcal{E}_i(i) \oplus \mathcal{E}_{i-1}(i-1) \\ Dh \downarrow & & \downarrow h_C \\ D\mathcal{F}_{i+1} & \rightarrow & \mathcal{F}_{i+1}(i+1) \oplus \mathcal{F}_i(i), \end{array}$$

where

$$h_C := \begin{pmatrix} h(1) & 0 \\ \phi & -h(1) \end{pmatrix}$$

and $h(1) : \mathcal{E}_i(i) \rightarrow \mathcal{F}_{i+1}(i+1)$ is the composition of h with the natural inclusion $\mathcal{F}_{i+1}(i) \hookrightarrow \mathcal{F}_{i+1}(i+1)$. Since the restriction of this inclusion to $X \times \{\infty\}$ vanishes, so does the restriction of $h(1)$. Hence the restriction of Dh to infinity does not depend on the choice of h, it depends on the restriction to $\mathcal{E}.$ of the map

$$\begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}$$

from $DC(\mathcal{E}.)$ to $DC(\mathcal{F}.)$. Therefore, at a point $w \in W$, it depends linearly on the map ϕ at the image of w in X. This completes the proof.

II.7.1.d. **A Technical Result**

We finish this section by considering the relations that hold between various complexes associated with the Grassmannian graph construction, and will be used later.

Let X be an integral scheme, quasi-projective over a regular noetherian integral domain A. Suppose that $j : X \hookrightarrow M$ and $i : X \hookrightarrow P$ are two immersions of X into regular varieties M and P, and that there is a smooth map $p : P \rightarrow M$, such that $p \circ i = j$. Then we have the following diagram:

$$\begin{array}{ccccc} X & \overset{f}{\hookrightarrow} & X \times_M P & \overset{j_p}{\rightarrow} & P \\ & \mathrm{Id} \searrow & p_j \downarrow & & \downarrow p \\ & & X & \overset{j}{\hookrightarrow} & M. \end{array}$$

Here $f = (\mathrm{Id}, i)$ is the induced map by the Cartesian product. So f is a regular immersion. In particular, the direct image by f_* of any vector sheaves on X has a finite global resolution by vector sheaves on $X \times_M P$. (This is a standard fact: Being regular, f is perfect and since $X \times_M P$ is quasi-projective, we can apply II. Prop. 2.2.9.b [SGA 6].)

Let $\mathcal{V}. \rightarrow f_*\mathcal{O}_X \rightarrow 0$ be a vector sheaf resolution on $X \times_M P$. If $\mathcal{F}$ is a coherent sheaf on X, $f_*\mathcal{O}_X$ and $p_j^*\mathcal{F}$ are Tor-independent (p_j is smooth), and so $\mathcal{V}. \otimes p_j^*\mathcal{F}$ is a resolution of $f_*\mathcal{F}$ by coherent sheaves. Now let $\mathcal{E}. \rightarrow j_*\mathcal{F} \rightarrow 0$ be a vector sheaf resolution of $j_*\mathcal{F}$ on M, and let $W := W(\mathcal{E}.)$ be the corresponding Grassmannian graph construction.

Choose a resolution of $j_{p*}(\mathcal{V}. \otimes p_j^*\mathcal{F})$ on P by a double complex $\mathcal{G}..$ of vector sheaves so that, for each i, $\mathcal{G}_i. \rightarrow j_{p*}(\mathcal{V}_i \otimes p_j^*\mathcal{F}) \rightarrow 0$ is a resolution: We proceed by induction on i, using the fact that this is true when $\mathcal{V}.$ has length one. By Corollary c.2, since each $\mathcal{G}_i.$ is a resolution of a vector sheaf on X, it extends as a complex $D\mathcal{G}_i.$ to $W((X \times_M P)/P)$.

Now observe that the horizontal component d' of the differential on $\mathcal{G}_{..}$ can be viewed as determining for each i a map of complexes $d' : \mathcal{G}_{i.} \to \mathcal{G}_{i-1.}$, where the differential on $\mathcal{G}_{i.}$ is $(-1)^i d$. Hence by Proposition 1.c.4, we get a map $Dd' : D\mathcal{G}_{i.} \to D\mathcal{G}_{i-1.}$, such that $Dd'^2 = 0$. So we have a double complex $D\mathcal{G}_{..}$ on $W(X/M) \times_M P$.

Therefore, by the construction, we see that the double complex $D\mathcal{G}_{..}$ has the following properties: If $\pi : W \times_M P \to \mathbf{P}^1 \times P$ is the projection, then for each l, $D\mathcal{G}_{l.}$ is acyclic on $\pi^{-1}(\mathbf{P}^1 \times (P - p^{-1}(X)))$. Over $\{0\} \in \mathbf{P}^1$, $D\mathcal{G}_{..} \simeq \mathcal{G}_{...}$ Over $\{\infty\} \in \mathbf{P}^1$, if we define $\tilde{M} \subset W_\infty$ as before, $W_\infty \times_M P \simeq (Z \cup \tilde{M}) \times_M P$ with $\pi(Z) \subset X \times_M P \subset P$. So the restriction of the double complex $D\mathcal{G}_{..}$ to $\tilde{M} \times_M P$ has split acyclic columns $D\mathcal{G}_{l.}$ and $\operatorname{Tot} D\mathcal{G}|_{\tilde{M} \times_M P}$, the associated total complex, is therefore acyclic.

Let $|Z|$ be the support in W of the algebraic cycle $Z = [W_\infty] - [\tilde{M}]$, denote by $v : |Z| \times_M P \to X \times_M P$ and $p_\infty : |Z| \times_M P \to Z$ the projections induced by π and p respectively, and by $\mathcal{E}_.^Z$ the restriction to $|Z|$ of the canonical extension of $\mathcal{E}_.$ to W, we have $\pi(|Z|) = X$ and the following

Proposition. There is an isomorphism in the derived category of bounded complexes of vector sheaves on $|Z| \times_M P$:

$$\operatorname{Tot}(D\mathcal{G}_{..}|_{|Z| \times_M P}) \simeq v^*(\mathcal{V}_.) \times p_\infty^*(\mathcal{E}_.^Z).$$

Proof. The proof of this proposition follows from two lemmas in homological algebra.

Lemma 1. Let $\mathcal{E}_.$ and $\mathcal{F}_.$ be bounded complexes of sheaves of abelian groups on a topological space X. Suppose that there is a finite open cover $\{U_\alpha\}$ of X and quasi-isomorphisms

$$\varphi_\alpha : \mathcal{E}_.|_{U_\alpha} \to \mathcal{F}_.|_{U_\alpha},$$

such that on each intersection $U_\alpha \cap U_\beta$, φ_α and φ_β are homotopic, i.e. there exists a map

$$\varphi_{\beta\alpha} : \mathcal{E}_.|_{U_\alpha \cap U_\beta} \to \mathcal{F}_.|_{U_\alpha \cap U_\beta}$$

such that,

$$\varphi_\beta - \varphi_\alpha = d \circ \varphi_{\beta\alpha} + \varphi_{\beta\alpha} \circ d,$$

and such that, on each triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$,

$$\varphi_{\beta\alpha} - \varphi_{\gamma\alpha} + \varphi_{\gamma\beta} = 0.$$

Then $\mathcal{E}_.$ and $\mathcal{F}_.$ are isomorphic in the derived category of bounded complexes of sheaves.

Proof. Consider the complex $\check{C}^*(\{U_\alpha\}, \mathcal{F}_.)$ of sheaves on X with sections over each open subset U which consist of the total complex of the Čech bi-complex $\check{C}^*(\{U_\alpha\} \cap U, \mathcal{F}_.)$. Then the natural augmentation $\eta : \mathcal{F}_. \to \check{C}^*(\{U_\alpha\}, \mathcal{F}_.)$ is a quasi-isomorphism. Thus it suffices to show that there exists a quasi-isomorphism $\mathcal{E}. \to \check{C}^*(\{U_\alpha\}, \mathcal{F}.)$. We define such a map $\varphi^\cdot$ as follows: $\varphi^\cdot := \oplus \varphi^k$, where $\varphi^k : \mathcal{E}_i \to \oplus_{l-k=i} \check{C}^k(\{U_\alpha\}, \mathcal{F}_l)$ is given by

$$\varphi^k = \begin{cases} \oplus_\alpha \varphi_\alpha, & \text{if } k = 0; \\ \oplus_{\alpha\beta}\varphi_{\alpha\beta}, & \text{if } k = 1; \\ 0, & \text{if } k > 1. \end{cases}$$

It is straightforward to check that $\varphi^\cdot$ is a map of complexes. To see that $\varphi^\cdot$ is a quasi-isomorphism, we may work locally and assume that $X = U_\alpha$ for some α. Then $\eta\varphi_\alpha$ and $\varphi^\cdot$ are homotopic. So by the fact that φ_α and η are quasi-isomorphisms, it follows that $\varphi^\cdot$ is too.

Lemma 2. Suppose that $\mathcal{E}..$ and $\mathcal{F}..$ are double complexes in an abelian category with d' and d'' the first and second differentials, respectively. Assume that $\varphi..$ is a map of bigraded objects such that

(a) For each k, $\varphi_{k\cdot} : \mathcal{E}_{k\cdot} \to \mathcal{F}_{k\cdot}$ is a quasi-isomorphism of complexes.

(b) For each k, l, there exists a map $\chi_{kl} : \mathcal{E}_{kl} \to \mathcal{F}_{k-1\, l+1}$ such that

$$d' \circ \varphi_{k\cdot} - \varphi_{k-1\cdot} \circ d' = (-1)^{k-1} d'' \circ \chi_{k\cdot} + (-1)^k \chi_{k\cdot} \circ d''.$$

(c) $d' \circ \chi + \chi \circ d'$ vanishes.

Then the map $\varphi + (-1)^k \chi$ on $\mathcal{E}_{k\cdot}$ induces a quasi-isomorphism between the total complexes of $\mathcal{E}..$ and $\mathcal{F}..$.

Proof. We check that $\phi := \varphi + (-1)^k\chi$ on $\mathcal{E}_{k\cdot}$ induces a chain map on the total complex $\mathrm{Tot}(\mathcal{E}..)$. Let $D = d' + d''$ be the total differential. Then, $D \circ \phi = \phi \circ D$: In fact, by definition, we see that

$$D\phi - \phi D = (d'\varphi - \varphi d') + (d''\varphi - \varphi d'') + (-1)^k(d'\chi + \chi d') + (-1)^k(d''\chi - \chi d'').$$

But the second and the third terms on the right hand side of this equation vanish by (a) and (c), while the other two terms have sum zero by (b).

On the other hand, for any given double complex $X..$, we can introduce its associated filtration by letting $F_i X.. := \oplus_{k \le i} X_{k\cdot}$. Thus ϕ preserves this filtration on $\mathcal{E}..$ and $\mathcal{F}..$. Thus the induced map on the associated graded objects is the sum of the quasi-isomorphisms $\varphi_{k\cdot}$ and hence ϕ itself is a quasi-isomorphism.

With these two lemmas in mind, we can prove the proposition by, first, constructing the morphisms as in the lemmas above and then checking the conditions there.

For every integer k, since each $\mathcal{V}_k$ is a vector sheaf on $X \times_M P$, there is a small open set where $\mathcal{V}_k$ is trivial of rank r_k and we get that

$$j_{p*}(\mathcal{V}_k \otimes p_j^*\mathcal{F}) = j_{p*}(p_j^*\mathcal{F})^{r_k} = p^*(j_*\mathcal{F})^{r_k}$$

has a resolution by $p^*\mathcal{E}.^{r_k}$. It also has a global resolution by $\mathcal{G}_{k\cdot}$. Hence there exists a locally finite affine covering $\{U_\alpha\}$ of P, and for each U_α, an isomorphism $\bar{\theta}_\alpha : \mathcal{V}_k \to \mathcal{O}^{r_k}_{X \times_M P}$ on $U_\alpha \cap (X \times_M P)$, and a chain equivalence $\phi_\alpha : \mathcal{E}.^{r_k} \to \mathcal{G}_{k\cdot}$, on U_α, resolving the isomorphism

$$j_{p*}(\bar{\theta}_\alpha \otimes 1) : j_{p*}(\mathcal{V}_k \otimes p_j^*\mathcal{F}) \to j_{p*}(p_j^*\mathcal{F})^{r_k}.$$

In particular, on $U_\alpha \cap U_\beta \cap (X \times_M P)$, there is a transition matrix $\bar\theta_{\beta\alpha} := \bar\theta_\beta \bar\theta_\alpha^{-1}$ for $\mathcal{V}_k$, which we may lift as an $r_k \times r_k$ matrix of functions $\theta_{\beta\alpha}$ on $U_\alpha \cap U_\beta$.

Now consider the two maps ϕ_α and $\phi_\beta(\theta_{\beta\alpha} \otimes 1)$ from $\mathcal{E}_\cdot^{r_k}$ to $\mathcal{G}_{k\cdot}$: The decompositions of these maps with the quasi-isomorphism from $\mathcal{G}_{k\cdot}$ to $j_{p*}(\mathcal{V}_k \otimes p_j^*\mathcal{F})$ are the same, and hence, since $\mathcal{E}^{r_k}_\cdot$ is a complex of vector sheaves, these maps are homotopic over the affine open subset $U_\alpha \cap U_\beta$. So we may choose homotopy morphisms $\phi_{\beta\alpha}$ such that

$$d(\phi_{\beta\alpha}) = \phi_\beta(\theta_{\beta\alpha} \otimes 1) - \phi_\alpha.$$

By Proposition c.4, the map ϕ_α, ϕ_β, $\theta_{\beta\alpha}$ and $\phi_{\beta\alpha}$ extend to the inverse images of the open subsets U_α, U_β, and $U_\alpha \cap U_\beta$ in $W \times_M P$, respectively. We denote these extensions by $D\phi_\alpha$, $D\phi_\beta$, $D\theta_{\beta\alpha}$ and $D\phi_{\beta\alpha}$. Notice that $D\theta_{\beta\alpha}$ is the inverse image by π of $\theta_{\beta\alpha}$, we see that its restriction to $|Z| \times_M P$ coincides with $v^*(\bar\theta_{\beta\alpha}) = v^*(\bar\theta_\beta)\, v^*(\bar\theta_\alpha^{-1})$. On the covering $\{|Z| \times_M U_\alpha\}$, we consider the trivialization $D\theta_\alpha = v^*(\bar\theta_\alpha)$ of $v^*(\mathcal{V}_k)$ and the maps

$$\varphi_\alpha := D\phi_\alpha(D\theta_\alpha \otimes 1) : v^*(\mathcal{V}_k \otimes p_\infty^*(\mathcal{E}_\cdot^Z))|_{|Z| \times_M (P \cap U_\alpha)} \to D\mathcal{G}_{k\cdot}|_{|Z| \times_M (P \cap U_\alpha)}.$$

On the intersection $|Z| \times_M (U_\alpha \cap U_\beta)$ the map

$$\varphi_{\beta\alpha} := D\phi_{\beta\alpha}(D\theta_\alpha \otimes 1)$$

is a homotopy between φ_β and φ_α, since

$$\begin{aligned} d\varphi_{\beta\alpha} =& d(D\phi_{\beta\alpha})|_{|Z| \times_M P}(D\theta_\alpha \otimes 1) = (D\phi_\beta D\theta_{\beta\alpha} - D\phi_\alpha)|_{|Z| \times_M P}(D\theta_\alpha \otimes 1) \\ =& D\phi_\beta|_{|Z| \times_M P}((v^*(\bar\theta_{\beta\alpha})v^*(\bar\theta_\alpha)) \otimes 1) - D\phi_\alpha|_{|Z| \times_M P}(v^*(\bar\theta_\alpha) \otimes 1) \\ =& D\phi_\beta(D\theta_\beta \otimes 1) - D\phi_\alpha(D\theta_\alpha \otimes 1) = \varphi_\beta - \varphi_\alpha. \end{aligned}$$

Furthermore, by Proposition c.5, $D\phi_{\beta\alpha}|_{|Z| \times_M P}$ depends only on the restriction of $\phi_\beta\theta_{\beta\alpha} - \phi_\alpha$ to $X \times_M P$. Therefore, on $|Z| \times_M (U_\alpha \cap U_\beta \cap U_\gamma)$,

$$\varphi_{\beta\alpha} - \varphi_{\gamma\alpha} + \varphi_{\gamma\beta} = 0.$$

Indeed, $(\varphi_{\beta\alpha} - \varphi_{\gamma\alpha} + \varphi_{\gamma\beta})(D\theta_\alpha \otimes 1)^{-1}$ is the restriction to $|Z|$ of the extension to W of a null homotopy of

$$(\phi_\beta\theta_{\beta\alpha} - \phi_\alpha) - (\phi_\gamma\theta_{\gamma\alpha} - \phi_\alpha) + (\phi_\gamma\theta_{\gamma\beta} - \phi_\beta),$$

whose restriction to $X \times_M P$ is zero since

$$\bar\theta_{\gamma\beta}\bar\theta_{\beta\alpha} = \bar\theta_\gamma\bar\theta_\beta^{-1}\bar\theta_\beta\bar\theta_\alpha^{-1} = \bar\theta_{\gamma\alpha}.$$

Hence, by Lemma 1 above, the family $\{\varphi_\alpha, \varphi_{\beta\alpha}\}$ defines a quasi-isomorphism of complexes of sheaves on $|Z| \times_M P$:

$$v^*(\mathcal{V}_k) \otimes p_\infty^*(\mathcal{E}_\cdot^Z) \to \mathrm{Tot}(\check{C}^\cdot(\{U_\alpha\}, D\mathcal{G}_{k\cdot})).$$

On the other hand, the differentials $d_k : \mathcal{V}_k \to \mathcal{V}_{k-1}$ can also be lifted locally to maps of complexes $d_{k,\alpha} : \mathcal{E}_k^{r_k} \to \mathcal{E}_{k-1}^{r_{k-1}}$, over each U_α. After composition with the augmentation from $\mathcal{G}_{k-1}$. to $j_{p*}(\mathcal{V}_{k-1} \otimes p_j^* \mathcal{F})$, the two maps $d_k^{\mathcal{G}} \circ \phi_\alpha$ and $\phi_\alpha \circ d_{k-1}$ coincide. Hence these maps are homotopic.

Finally let ϕ'_α be the homotopy and let χ_α be the restriction to $|Z| \times_M P$ of the canonical extension of ϕ'_α to $W \times_M P$. Restricting to $|Z| \times_M P$, there is over each U_α a diagram

$$\begin{array}{ccc} \mathcal{V}_k \otimes D\mathcal{E}. & \stackrel{\varphi_\alpha}{\to} & D\mathcal{G}_k. \\ d' \downarrow & & \downarrow d' \\ \mathcal{V}_{k-1} \otimes D\mathcal{E}. & \stackrel{\varphi_\alpha}{\to} & D\mathcal{G}_{k-1}. \end{array}$$

Here the first d' is just $d_k^{\mathcal{V}} \otimes 1$ and $d'\varphi_\alpha - \varphi_\alpha d' = (-1)^{k-1} d'' \chi_\alpha + (-1)^k \chi_\alpha (1 \otimes d_{D\mathcal{E}})$. From the equality

$$d'(d'\phi_\alpha - \phi_\alpha d') + (d'\phi_\alpha - \phi_\alpha d')d' = 0$$

on $X \times_M P$, we get, by Proposition c.5, that $d'\chi_\alpha + \chi_\alpha d' = 0$. Hence, by Lemma 2 above, $\varphi_\alpha + (-1)^k \chi_\alpha$ defines a quasi-isomorphism

$$\mathrm{Tot}(v^*(\mathcal{V} \otimes p_\infty^*(D\mathcal{E}.)) \to \mathrm{Tot}(D\mathcal{G}..)$$

on U_α. Thus we are led to consider

$$\Phi := \{\varphi_\alpha + (-1)^k \chi_\alpha, \varphi_{\beta\alpha}\} : \mathrm{Tot}(v^*(\mathcal{V} \otimes p_\infty^*(\mathcal{E}^Z.)) \to \mathrm{Tot}(\check{C}^\cdot(\{U_\alpha\}, D\mathcal{G}..)).$$

Obviously, the proposition is equivalent to saying that Φ is a quasi-isomorphism, which can be proved by using Lemma 2 once more: For this the only identity left to be shown is $d'\varphi_{\beta\alpha} - \varphi_{\beta\alpha} d' = (-1)^k (\chi_\beta - \chi_\alpha)$. This follows from Proposition c.5 by an argument as above, where we compose both sides of this equality with $(D\theta_\alpha \otimes 1)^{-1}$ and notice that d' commutes with $\bar{\theta}_{\beta\alpha} \otimes 1$ on $X \times_M P$.

§II.7.2. The Arithmetic Chern Character With Supports

In this section, we use the results in the previous section to define the arithmetic Chern character with supports, which will be used in the next section to prove an arithmetic Riemann-Roch theorem for general closed immersions stated in 6.2.

Let P be a regular arithmetic variety (over an arithmetic ring A), and let $i : X \hookrightarrow P$ be a closed arithmetic proper subvariety, with a choice of F_∞-invariant hermitian metric on the normal bundle to $X(\mathbf{C})$ in $P(\mathbf{C})$. Suppose that $\mathcal{E}.$ is a bounded complex of vector sheaves on P, acyclic off X over the generic fiber P_F, which is a resolution of a hermitian vector sheaf $(\mathcal{F}, \tau)$ on X_F. Then we can define an arithmetic Chern character with supports, $\mathrm{ch}_{\mathrm{Ar}P}^X(\mathcal{E}.) \in \mathrm{CH}^{\mathrm{Ar}}_\cdot(X)_{\mathbf{Q}}$, as follows:

Let $\pi : W \hookrightarrow P \times \mathbf{P}^1$ be the Grassmannian graph construction associated with the complex $\mathcal{E}..$ As in section 1, we consider the cycle $Z = [W_\infty] - [\tilde{P}]$ and write $|Z|$ for its support. Let $D\mathcal{E}.$ be the extension of $\mathcal{E}.$ to W, and $\mathcal{E}.^Z$ its restriction to $|Z|$. Since $\mathcal{E}.^Z$ is a resolution over $|Z|_F$ of the direct image of $\mathcal{F}$, we can equip the bundles $\mathcal{E}_i^Z$ with F_∞-invariant hermitian metrics which satisfy Bismut condition (A). By I.8, there is the relative Bott-Chern secondary characteristic current $\mathrm{ch_{BC}}(\mathcal{E}.^Z, \rho_{\mathcal{E}.^z})$ of this complex with respect to the corresponding closed immersion. Now we change the notation a little bit. We write $\pi^Z : |Z| \to X$ for the projection induced by the map $\pi : W \to P$. Hence $dd^c(\pi_*^Z(\mathrm{ch_{BC}}(\mathcal{E}.^Z, \rho_{\mathcal{E}.^z}))$ is smooth on X and hence $a(\pi_*^Z(\mathrm{ch_{BC}}(\mathcal{E}.^Z, \rho_{\mathcal{E}.^z})))$ is an element in $\mathrm{CH^{Ar}}(X)$. Furthermore, by the dimension reason, we know that the cycle Z can be viewed as giving a class in $\mathrm{CH^{Ar}_{dim(Z)}}(|Z|) = \mathrm{CH_{dim(Z)}}(|Z|)$.

With this, we may define the **arithmetic Chern character of** $(\mathcal{E}., \rho.)$ **with supports in** X by

$$\mathrm{ch_{Ar}}{}_P^X(\mathcal{E}., \rho.) := \pi_*^Z\left(\mathrm{ch_{Ar}}(\mathcal{E}.^Z, \rho_{\mathcal{E}.^z}) \cap Z\right) + a\left(\pi_*(\mathrm{ch_{BC}}(\mathcal{E}.^Z, \rho_{\mathcal{E}.^z})\right).$$

We shall sometimes write $\mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{E}., \rho.)$ rather than $\mathrm{ch_{Ar}}{}_P^X(\mathcal{E}., \rho.)$. As usual, we now need to check that the definition makes sense: We claim that the class depends on the choice of metrics on $\mathcal{F}$ and on the normal bundles to $X(\mathbf{C})$ in $P(\mathbf{C})$, but it is independent of the choice of metrics on $\mathcal{E}.^Z$. In fact, this is a special case of the following proposition with $\mathcal{E}_1. = \mathcal{E}_2. = \mathcal{E}..$

Proposition 1. If $k : \mathcal{E}_1. \to \mathcal{E}_2.$ is a quasi-isomorphism which, induces a morphism of resolutions of $\mathcal{F}$ over F, then

$$\mathrm{ch_{Ar}}{}_P^X(\mathcal{E}_1., \rho_1.) = \mathrm{ch_{Ar}}{}_P^X(\mathcal{E}_2., \rho_2.).$$

Proof. Replacing the quasi-isomorphism by its mapping cone, if necessary, we may assume that k is injective in each degree. By Proposition c.3, we know that $W(\mathcal{E}_1.) = W(\mathcal{E}_2.)$ and that the map k induces a map $k^Z : \mathcal{E}_1^Z. \to \mathcal{E}_2^Z.$ which is a monomorphism with cokernel a split acyclic complex of vector sheaves. We now choose arbitrary F_∞-invariant hermitian metrics on the complexes $\mathcal{E}_1^Z.$, $\mathcal{E}_2^Z.$, which satisfy Bismut condition (A), and make a choice of F_∞-invariant hermitian metrics on the quotient complex compatible with the splitting. By Theorem 3.7,

$$\begin{aligned}&\left(\mathrm{ch_{Ar}}(\mathcal{E}_2^Z., \rho_{\mathcal{E}_2^z.}) - \mathrm{ch_{Ar}}(\mathcal{E}_1^Z., \rho_{\mathcal{E}_1^z.}) - \mathrm{ch_{Ar}}(\mathcal{E}_2^Z/\mathcal{E}_1^Z., \rho_{\mathcal{E}_2^z./\mathcal{E}_1^z})\right) \cap Z \\ &= a\Big(\sum_m (-1)^m \mathrm{ch_{BC}}(0 \to \mathcal{E}_{1m}^Z \to \mathcal{E}_{2m}^Z \to \mathcal{E}_{2m}^Z/\mathcal{E}_{1m}^Z \to 0, \rho._m^Z)\Big).\end{aligned}$$

However, by axioms of the Bott-Chern secondary characteristic objects, the last expression is nothing but

$$a\left(\mathrm{ch_{BC}}(\mathcal{E}_{2.}^Z, \rho_{\mathcal{E}._{2.}^z}) - \mathrm{ch_{BC}}(\mathcal{E}_{1.}^Z, \rho_{\mathcal{E}._{1.}^z}) - \mathrm{ch_{BC}}(\mathcal{E}_{2.}^Z/\mathcal{E}_{1.}^Z, \rho_{\mathcal{E}._{2.}^z/\mathcal{E}._{1.}^z})\right).$$

With this, the assertion comes from the fact that $\mathcal{E}._{2.}^Z/\mathcal{E}._{1.}^Z$ is split acyclic.

Corollary 1. Let $(\mathcal{F},\tau)$ be a hermitian vector sheaf on X and suppose that P is regular. For any vector sheaf resolution $\mathcal{E}.$ of $\mathcal{F}$ by a complex of vector sheaves on P, the class $\mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{E}.,\rho.) \in \mathrm{CH}^{\mathrm{Ar}}(X)$ is independent of the choice of the resolution. We shall denote this class by $\mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{F},\tau)$.

Next, we give a few down-to earth properties of $\mathrm{ch}_{\mathrm{Ar}}{}^X_P(\mathcal{E}.,\rho.)$.

Proposition 2. With the same notation as above,
(a). $\omega(\mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{E}.,\rho.)) = \mathrm{ch}(\mathcal{F},\tau)\,\mathrm{td}^{-1}(\mathcal{N},h_{\mathcal{N}})$.
(b). If two metrics h_1 and h_2 are given on the normal bundle of $X(\mathbf{C})$ in $P(\mathbf{C})$, then the difference of the associated arithmetic Chern characters with supports is given by the formula

$$\mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{E}.,\rho.)_1 - \mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{E}.,\rho.)_2 = a\big(\mathrm{ch}(\mathcal{F},\tau)\,\mathrm{td_{BC}}^{-1}(\mathcal{N};h_1,h_2)\big).$$

(c). Let

$$\mathcal{A}: 0 \to \mathcal{E}_1. \to \mathcal{E}_2. \to \mathcal{E}_3. \to 0$$

be a short exact sequence of complexes of vector sheaves, which on the generic fiber is a resolution of a exact sequence of hermitian vector sheaves:

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0.$$

Then

$$\mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{E}_2.,\rho_{\mathcal{E}_2.}) - \mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{E}_1.,\rho_{\mathcal{E}_1.}) - \mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{E}_3.,\rho_{\mathcal{E}_3.}) = a(\mathrm{ch_{BC}}(\mathcal{A},\rho_{\mathcal{A}})\,\mathrm{td}^{-1}(\mathcal{N},h)).$$

Applying this construction to $K_{\mathrm{Ar}}(X)$, we immediately get the following

Corollary 2. Assume P is regular, then the map

$$((\mathcal{F},\tau),\gamma) \mapsto \mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{F},\tau) + a(\gamma)$$

defines a homomorphism of abelian groups

$$K_0^{\mathrm{Ar}}(X) \to \mathrm{CH}^{\mathrm{Ar}}(X)_{\mathbf{Q}}.$$

Here $K_0^{\mathrm{Ar}}(X)$ is generated by hermitian coherent vector sheaves, i.e. the coherent sheaf on X, but with a vector sheaf pull back at infinite places, and so on.

We end this section with some more properties, which are quite natural.

Property 1. (Module Property) If $\mathcal{G}$ is a vector sheaf with an F_∞-invariant hermitian metric on P, then

$$\mathrm{ch}_{\mathrm{Ar}}{}^X_P(\mathcal{E}.\otimes i^*\mathcal{G},\rho_{\mathcal{E}.\otimes i^*\mathcal{G}}) = \mathrm{ch}_{\mathrm{Ar}}{}^X_P(\mathcal{E}.,\rho.)\,;\mathrm{ch}_{\mathrm{Ar}}(\mathcal{G},\tau).$$

Proof. By the fact that $W(\mathcal{E}.\otimes\mathcal{G}) = W(\mathcal{E}.)$, and that on this scheme, we have $D(\mathcal{E}.\otimes\mathcal{G}) \simeq D\mathcal{E}.\otimes\mathcal{G}$. So by the restriction to $|Z|$, we have

$$(\mathcal{E}.\otimes\mathcal{G})^Z = \mathcal{E}^Z.\otimes\pi^{Z*}i^*\mathcal{G}.$$

Now observe that

$$\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}^Z.\otimes\pi^{Z*}i^*\mathcal{G},\rho_{\mathcal{E}^Z.\otimes\pi^{Z*}i^*\mathcal{G}}) = \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}^Z.,\rho_{\mathcal{E}^Z.})\,\pi^{Z*}i^*\mathrm{ch}(\mathcal{G},\tau).$$

The assertion follows by the projective formula for currents.

Property 2. Let $i : X \hookrightarrow P$ be a closed immersion of arithmetic varieties which is proper over an arithmetic variety Y and with the structure morphisms f and g that are smooth at infinity. Suppose that $(\mathcal{E}.,\rho.)$ is a complex of hermitian vector sheaves on P, acyclic off X, which on the generic fiber P_F is a resolution of hermitian coherent vector sheaf $(\mathcal{F},\tau)$ on X_F satisfying Bismut condition (A). Then in $\mathrm{CH}^{\mathrm{Ar}}_{\cdot}(Y)_{\mathbf{Q}}$

$$f_*\big(\mathrm{ch}_{\mathrm{Ar}}{}_P^X(\mathcal{E}.,\rho.)\cdot_i\alpha\big) = g_*\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.,\rho.)\cap\alpha\big) + a\big(g_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.))\,\omega(\alpha)\big),$$

where $\alpha = \mathrm{ch}_{\mathrm{Ar}}(x)$ for some $x \in K_{\mathrm{Ar}}(P)_{\mathbf{Q}}$, or P is regular and $\alpha \in \mathrm{CH}^{\cdot}_{\mathrm{Ar}}(P)$.

Proof. We fix an F_∞-invariant hermitian metric on the normal bundle of $X\times\mathbf{P}^1$ in the Grassmannian graph construction $W = W(\mathcal{E}.)$ via the natural isomorphism

$$\mathcal{N}_{X\times\mathbf{P}^1/W} \simeq p^*\mathcal{N}_{X/P}(-\infty),$$

where p is the projection from $Y\times\mathbf{P}^1$ to Y.

Note that at infinity, the map $\tilde{\pi} = g\circ\pi$ from W to $Y\times\mathbf{P}^1$ induces a proper map of complex manifolds. Choose F_∞-invariant hermitian metrics on $D\mathcal{E}.$ such that the restriction to 0 coincides with the one on $\mathcal{E}.$, the restriction to $\tilde{P}$ is split acyclic, and the restriction to $|Z|$ satisfies Bismut condition (A).

Then, we consider the class

$$\tilde{\pi}_*\big(\mathrm{ch}_{\mathrm{Ar}}(D\mathcal{E}.,\rho_{D\mathcal{E}.})\cdot_q\alpha\big) + a\big(\tilde{\pi}_*\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}.,\rho_{D\mathcal{E}.})\,q^*\omega(\alpha)\big)$$

in $\mathrm{CH}^{\mathrm{Ar}}_{\cdot}(Y\times\mathbf{P}^1)_{\mathbf{Q}}$, where $q : W \to P$ is the projection. The restriction of this class to $\{\infty\}$ is the left hand side of the equation of this property, while the restriction to $\{0\}$ is the right hand side. Thus by Axiom 1 for the classical Bott-Chern secondary characteristic forms, or better, by Theorem 3.5.f, the difference of these two elements is

$$\int_{\mathbf{P}^1}\tilde{\pi}_*\Big(\omega\big(\mathrm{ch}_{\mathrm{Ar}}(D\mathcal{E}.,\rho_{D\mathcal{E}.}) + a(\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}.,\rho_{D\mathcal{E}.}))\big)\,q^*\omega(\alpha)\Big)\,dd^c[-\log|z|^2],$$

where z is the parameter on $\mathbf{P}^1$. However, by Axiom 1 for the relative Bott-Chern secondary characteristic currents with respect to closed immersions,

$$\omega\big(\mathrm{ch}_{\mathrm{Ar}}(D\mathcal{E}.,\rho_{D\mathcal{E}.}) + a(\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}.,\rho_{D\mathcal{E}.}))\big)$$

is equal to $j_*(\mathrm{ch}(\mathcal{F},\tau)\,\mathrm{td}^{-1}(\mathcal{N}_{X\times\mathbf{P}^1/W}))$, where $j : X \times \mathbf{P}^1 \hookrightarrow W$ is the natural inclusion. Applying the projection formula for the integration over the fiber, the assertion comes from the fact that

$$\int_{\mathbf{P}^1} \log|z|^2 \mathrm{ch}(p^* \mathcal{N}_{Y/P}(-\infty), \rho_{p^*\mathcal{N}_{Y/P}(-\infty)})) = 0.$$

Property 3 (Uniqueness Rule.) (a). Suppose that $i : X \hookrightarrow P$ is a regular closed immersion and $(\mathcal{F},\tau)$ is a hermitian vector sheaf on X. For any vector sheaf resolution $\mathcal{E}. \to i_*\mathcal{F} \to 0$ on P, we have

$$\mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{E}.,\rho.) = \mathrm{ch}_{\mathrm{Ar}}(\mathcal{F},\tau) \cap \left(\mathrm{td}_{\mathrm{Ar}}{}^{-1}(\mathcal{N}_i,\rho_i) \cap [X]\right).$$

(b). Suppose that both X and P are regular. Then for any hermitian coherent vector sheaf $(\mathcal{F},\tau)$ on X,

$$\mathrm{ch}^X_{\mathrm{Ar}}(\mathcal{E}.,\rho.) = \mathrm{ch}_{\mathrm{Ar}}(\mathcal{F},\tau) \cap \mathrm{td}_{\mathrm{Ar}}{}^{-1}(\mathcal{N}_i,\rho_i).$$

Proof. We only need to prove (a), since then (b) is a direct consequence.

The inclusion $i : X \hookrightarrow P$ is a regular immersion, the cycle Z at infinity is irreducible and is equal to $\mathbf{P}(\mathcal{N}\oplus 1)$, with $\mathcal{N} = \mathcal{N}_{X/P}$. Let $\mathcal{S}$ be the tautological codimension-one sub-vector sheaf of $\mathcal{N}\oplus 1$ on $\mathbf{P}(\mathcal{N}\oplus 1)$. Then the Koszul complex $K.(\mathcal{S})$ is a resolution of $s_*\mathcal{O}_X$, where s is the zero section. Hence $K.(\mathcal{S})\otimes\pi^*(\mathcal{F})$ is a resolution of $s_*\mathcal{F}$. Furthermore, we know from the consturction and Theorem I.9.1.c, that there is a quasi-isomorphism

$$\phi : \mathcal{E}.^Z \simeq K.(\mathcal{S}) \oplus \pi^*(\mathcal{F})$$

for which ϕ is an epimorphism with a split acyclic kernel. We equip $K.(\mathcal{S})$ with the F_∞-invariant hermitian metric induced by thinking of $\mathcal{S}$ as a sub-vector sheaf of $\mathcal{N}\oplus 1$, then with the induced metrics, $K.(\mathcal{S})\otimes\pi^*(\mathcal{F})$ satisfies Bismut condition (A) as a vector sheaf resolution of $s_*\mathcal{F}$. Hence, as in Property 1,

$$\begin{aligned} &a\big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.^Z,\rho_{\mathcal{E}.^Z}) - \mathrm{ch}_{\mathrm{BC}}(K.(\mathcal{S})\oplus\pi^*(\mathcal{F}),\rho_{K.(\mathcal{S})\oplus\pi^*(\mathcal{F})})\big)\\ &\quad=\big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.^Z,\rho_{\mathcal{E}.^Z}) - \mathrm{ch}_{\mathrm{Ar}}(K.(\mathcal{S})\oplus\pi^*(\mathcal{F}),\rho_{K.(\mathcal{S})\oplus\pi^*(\mathcal{F})})\big)\cap[Z]. \end{aligned}$$

So we have the assertions by the proof of Theorem 5.1.

§II.7.3. Completion Of The Proof Of Theorems In Section 6.2

In this section, we are going to prove two theorems in Section 6.2.

II.7.3.a. **The Proof Of The Arithmetic Riemann-Roch Theorem For Closed Immersions.**

In this subsection, we will give a proof of the following

Arithmetic Riemann-Roch Theorem For Closed Immersions.
Let $A = (A, \Sigma, F_\infty)$ be an arithmetic ring and let

$$\begin{array}{ccccc} X & & \overset{i}{\hookrightarrow} & & Z \\ & f\searrow & & \swarrow g & \\ & & Y & & \end{array}$$

be closed immersions of regular arithmetic varieties over Y with f proper, $f_{\mathbf{C}}$ smooth, and g smooth. Put F_∞-invariant hermitian metrics on the relative tangent sheaves and the normal sheaf of i. Let $(\mathcal{E}, \rho)$ be a hermitian vector sheaf on X and $\mathcal{F}. \to i_*\mathcal{E} \to 0$ a vector sheaf resolution of $i_*\mathcal{E}$ on Z. Put F_∞-invariant hermitian metrics $\rho.$ on $\mathcal{F}.$ such that Bismut condition (A) is satisfied with respect to ρ_N and ρ. Then, for any $\Psi \in \mathrm{CH}_{\mathrm{Ar}}(Z)$, the following identity holds on $\mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}$

$$\begin{aligned} &f_*(\mathrm{td}_{\mathrm{Ar}}^{-1}(N, \rho_N)\,\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}, \rho) i^*(\Psi)) \\ &= g_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}., \rho.)\Psi) + g_*\Big(a(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; i, \rho^i; \mathcal{F}., \rho.)\omega(\Psi))\Big). \end{aligned}$$

First, from the fact that the closed immersion $i : X \hookrightarrow P$ is regular, the Grassmannian graph $W = W(\mathcal{F}.)$ is isomorphic to the deformation to the normal cone, and $|Z| = \mathbf{P}_X(N \oplus 1)$ (see 1.b). Let $D\mathcal{F}.$ be the extension of $\mathcal{F}.$ to W, and $\mathcal{F}.^Z$ its restriction to $|Z|$. Choose F_∞-invariant hermitian metrics on $D\mathcal{F}.$ whose restriction to $\tilde{P}(\mathbf{C})$ is split acyclic, and which satisfy Bismut condition (A). Define

$$\beta := \int_{W(\mathbf{C})/P(\mathbf{C})} [\log|z|^2]\mathrm{ch}(D\mathcal{F}., \rho_{D\mathcal{F}.})$$

in $\tilde{A}^{\cdot}(P_{\mathbf{R}})$. By the same proof as in Proposition 5.2.a, we have

$$g_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}., \rho.) \cap \Psi) = f_* \circ p_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{F}.^Z, \rho_{\mathcal{F}.^Z}) \cap p^* \circ i^*(\Psi)) + a(g_*(\beta\,\omega(\Psi))),$$

where $p : \mathbf{P}_X(N \oplus 1) \to X$ is the smooth projection. Moreover, the right hand side of the equality may be computed as in the proof of Theorem 5.1, i.e. by comparing $\mathcal{F}.^Z$ with the Koszul complex $K.(\mathcal{H}) \otimes p^*\mathcal{E}$ considered in 5.2.

More precisely, we need to have

$$\begin{aligned} &p_*(\mathrm{ch}_{\mathrm{Ar}}(K.(\mathcal{H}) \otimes p^*\mathcal{E}, \rho_{K.(\mathcal{H})\otimes p^*\mathcal{E}}) \cap p^* \circ i^*(\Psi)) \\ &= p_*(\mathrm{ch}_{\mathrm{Ar}}(K.(\mathcal{H}), \rho_{K.(\mathcal{H})}) \cap p^* \circ i^*(\Psi)) \cap \mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}, \rho), \end{aligned}$$

which follows from Theorem 3.7.e, the projection formula. With this, next we use the the cap product formalism to make the corresponding changes.

In fact, as in the proof of Theorem 3.8, we can choose a map $h : X \to M$, where M is regular, and a vector bundle N' on M such that there exists a metric on N' with $(N,\rho) = h^*(N', \rho_{N'})$. Note that for the proof of the arithmetic Riemann-Roch theorem for closed immersions above, it is enough to have a proof for just one metric on N. Hence, we assume that there exists an F_∞-invariant hermitian metric such that

$$(\mathcal{N}, \rho_{\mathcal{N}}) = h^*(\mathcal{N}', \rho_{\mathcal{N}'}).$$

We consider the canonical hyperplane bundle $H' \subset p'^*(N') \oplus 1$ on $P_M(N' \oplus 1)$, where $p' : \mathbf{P}(N' \oplus 1) \to M$ is the projection, and the Koszul complex $K.(H')$ has the metric induced by $\rho_{\mathcal{N}'}$. Using the projection formula in Proposition 2.8, we may get

$$p_*(\mathrm{ch}_{\mathrm{Ar}}(K.(H), \rho_{K.(H)}) \cap p^* \circ i^*(\Psi)) = i^*(\Psi)._h p'_*(\mathrm{ch}_{\mathrm{Ar}}(K.(H), \rho_{K.(H)})).$$

With above, we then complete the proof by replacing the corresponding relations in the proof of Theorem 5.1.

II.7.3.b. The Proof Of The Arithmetic Riemann-Roch Theorem For L.C.I. Morphisms'

First, we recall the arithmetic Riemann-Roch theorem for l.c.i. morphisms'.

Let $f : X \to Y$ be a l.c.i. morphism of regular arithmetic varieties over an arithmetic ring (A, σ, F_∞), which is smooth at infinity. Define the push-out morphism $f_K^{\mathrm{Ar}} : K_0^{\mathrm{Ar}}(X) \to K_0^{\mathrm{Ar}}(Y)$ by letting

$$\begin{aligned} f_K^{\mathrm{Ar}}(\mathcal{E}, \rho) =& (f_*\mathcal{E}, g_*\rho) + \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; f, \rho_f) \\ &+ f_*\Big(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}, \rho)\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_f, \rho_f)\, a\big(R(\mathcal{T}_f)\big)\Big) \end{aligned}$$

for any f-acyclic hermitian vector sheaf $(\mathcal{E}, \rho)$ as in Chapter II.4, where R is the unique power series defined in the arithmetic Riemann-Roch theorem for smooth morphisms. In particular, we have the following

Arithmetic Riemann-Roch Theorem For l.c.i. Morphisms'. With the same notation above, for any element $\Upsilon \in K_{\mathrm{Ar}}(X)_{\mathbf{Q}}$, we have

$$\mathrm{ch}_{\mathrm{Ar}}(f_K(\Upsilon)) = f_{\mathrm{CH}}(\mathrm{ch}_{\mathrm{Ar}}(\Upsilon)\ \mathrm{td}_{\mathrm{Ar}}(f, \rho_f)).$$

This theorem is very similar to the one in Chapter 4. The difference is that now we do not assume that f is smooth: We only have the condition that f is proper and f at infinity is smooth. So, we may imitate the proof of the main theorem in Chapter 4 to offer a proof here.

More precisely, we consider the difference of two sides of the relation in the theorem. As before, we call it as Err. Then, by the fact that the part for algebraic cycles is just the Grothendieck-Riemann-Roch theorem for l.c.i. morphism, so we only need to consider the part for the associated Green currents. Then, we may also know that Err is compatible with the composition of morphisms, is compatible with flat base change, and has nothing to do with the associated metrics, etc.. All in all, we see that the only difficulty is associated with the deformation to the normal cone theory, as now f at the finite place is not smooth: We use it to show that for any codimension one closed immersion, all could be deformed from 0 to these on just one component at ∞. (One might also worry about the arithmetic tangent elements. But this could not create any more problems, since we have already taken care of them in the corresponding results of 3.8.b.)

For the deformation to the normal cone, essentially, there should have no problem neither, since now only the infinite part matters: By our assumption, $f_{\mathbb{C}}$ is smooth as before. That is, if we replace the deformation to the normal cone theory by the generalized theory about the Grassmannian graph construction associated with the vector sheaf resolution of $i_*\mathcal{E}$, then, by the corresponding result from algebraic geometry (see [BFM 75]), the algebraic cycle part could be deformed from the original (codimension one) closed immersion to the corresponding part over a section of a projective vector bundle. But then, once we shift to the infinity part of our arithmetic varieties, by 1.b, we see that at infinity, the Grassmannian graph construction is isomorphic to the deformation to the normal cone construction at infinity. Furthermore, we see that at infinite, the arithmetic tangent element defined by using the decomposition of f as $g \circ i$ in 3.8.b is just the same as the relative tangent vector bundle for the smooth morphism $f_{\mathbb{C}}$. Thus, if we replace the proof of Sublemma 4.2.b by the discussion in Section 2, we know that, in the proof of Theorem 4.1, everything works well here. This completes the proof.

We end this subsection by the following remark: Actually, one may still go slightly further. Instead of assuming that f at infinity is smooth, we may only assume that the logarithmic relative tangent vector sheaf at infinity exists, e.g. [De 70]. But then we need to use Melrose's b-calculus to build up the foundation for Part I.

§II.7.4. The Construction Of τ_{Ar}

So far, we have already proved the arithmetic Riemann-Roch theorem for l.c.i. morphisms of arithmetic varieties under the additional assumption that the induced morphisms are smooth at infinity. In order to remove this assumption, as one may imagine, we need to give a hermitian theory for coherent sheaves, together with an arithmetic analogue of the Riemann-Roch transform used by Baum, Fulton and MacPherson in proving the singular Riemann-Roch theorem in algebraic geometry. Though we do not know how to obtain a hermitian theory for coherent sheaves, we can provide the arithmetic Riemann-Roch transform.

More precisely, if $K_0^{\mathrm{Ar}}(X)$ denotes the arithmetic K group generated by hermitian coherent vector sheaves, (i.e. coherent sheaves such that the pull-backs at infinity are hermitian vector sheaves, and so on,) then we have a map

$$\tau_{\mathrm{Ar}} : K_0^{\mathrm{Ar}}(X) \to \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)_{\mathbf{Q}},$$

which depends only on the choice of an F_∞-invariant hermitian metric on the tangent bundle to $X(\mathbf{C})$.

Motivated by the situation in algebraic geometry, we are led to the following

Theorem. Let X be an arithmetic variety over an arithmetic ring A. Put an F_∞-invariant hermitian metric on the tangent coherent vector sheaf of X at infinity. Then there is a unique natural morphism, the **arithmetic Riemann-Roch transform**,

$$\tau_{\mathrm{Ar}} : K_0^{\mathrm{Ar}}(X) \to \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)_{\mathbf{Q}},$$

which satisfies the following properties:
(1) For any hermitian coherent vector sheaf $(\mathcal{F}, \tau)$ on X,

$$\omega\big(\tau_{P,\mathrm{Ar}}(\mathcal{F},\tau)\big) = \mathrm{ch}(\mathcal{F},\tau)\,\mathrm{td}(X(\mathbf{C}), \rho_{X(\mathbf{C})}).$$

(2) τ_{Ar} only depends on the F_∞-invariant hermitian metric on the tangent coherent vector sheaf of X at infinity.
(3) τ_{Ar} induces a canonical isomorphism of $\mathbf{Q}$-vector spaces

$$\tau_{\mathrm{Ar}} : K_0^{\mathrm{Ar}}(X)_{\mathbf{Q}} \to \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)_{\mathbf{Q}},$$

which maps the class of $((\mathcal{F},\tau),\eta)$ to $\tau_{\mathrm{Ar}}(\mathcal{F},\tau) + a\big(\eta\,\mathrm{td}(X,\rho_X)\big)$.
(4) For any $x \in K_0^{\mathrm{Ar}}(X)$ and $y \in K_{\mathrm{Ar}}(X)$,

$$\tau_{\mathrm{Ar}}(x \cap y) = \tau_{\mathrm{Ar}}(x) \cap \mathrm{ch}_{\mathrm{Ar}}(y).$$

(5) If X is regular, for any $x \in K_{\mathrm{Ar}}(X) \simeq K_0^{\mathrm{Ar}}(X)$,

$$\tau_{\mathrm{Ar}}(x) = \mathrm{ch}_{\mathrm{Ar}}(x)\,\mathrm{td}_{\mathrm{Ar}}(X)$$

in $\mathrm{CH}^{\cdot}_{\mathrm{Ar}}(X)_{\mathbf{Q}} = \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)_{\mathbf{Q}}$.

The rest of this section is devoted to a proof of this theorem. First, we give a definition for τ_{Ar}. It is sufficient to give the definition for a hermitian coherent vector sheaf $(\mathcal{F},\tau)$ on X. Let $i : X \hookrightarrow P$ be a closed immersion of X into a regular irreducible arithmetic variety P. Fix F_∞-invariant hermitian metrics on the normal bundle of $X(\mathbf{C})/P(\mathbf{C})$ and related tangent bundles. Then by Corollary 2.1, we get from these data a class $\mathrm{ch}_{\mathrm{Ar}}{}^X_P(\mathcal{F},\tau) \in \mathrm{CH}^{\mathrm{Ar}}_{\cdot}(X)_{\mathbf{Q}}$. In addition, we fix F_∞-invariant hermitian metrics on the tangent bundles of $X(\mathbf{C})$ and $P(\mathbf{C})$. We define

$$\tau_{P,\mathrm{Ar}}(\mathcal{F},\tau) := \mathrm{ch}_{\mathrm{Ar}}{}^X_P(\mathcal{F},\tau) \cdot_i \mathrm{td}_{\mathrm{Ar}}(P,\rho_P) + a\big(\mathrm{ch}(\mathcal{F},\tau)\,\mathrm{td}_{\mathrm{BC}}(X/P,\rho_{X/P})\big).$$

We clearly have (1) above.

Next, we check (2): For a given $(\mathcal{F}, \tau)$, the arithmetic class defined above only depends on the choice of the metric on the tangent bundle to X. Hence we must show the independence of all other choices we made. To prove this, we proceed as follows:

From the definition, we easily see that

Lemma 1. The class $\tau_{P,\mathrm{Ar}}(\mathcal{F}, \tau)$ does not depend on the choice of the metrics on $\mathcal{T}_P$ and $\mathcal{N}_{X/P}$.

It now only remains to show the following

Proposition. The class $\tau_{P,\mathrm{Ar}}(\mathcal{F}, \tau)$ does not depend on the choice of the embedding of X into the regular integral variety P and we denote it simply $\tau_{\mathrm{Ar}}(\mathcal{F}, \tau)$.

Proof of the proposition. We divide the proof into several steps.

Step 1. We start with a special situation.

Lemma 2. Let $j : X \hookrightarrow M$ and $k : X \hookrightarrow P$ be two closed immersions of X into regular integral varieties M and P, and suppose that there is a smooth map $q : P \to M$ such that $q \circ k = j$. Then

$$\tau_{P,\mathrm{Ar}}(\mathcal{F}, \tau) = \tau_{M,\mathrm{Ar}}(\mathcal{F}, \tau).$$

Step 2. From Lemma 2, we see that $\tau_{\mathrm{Ar}}(\mathcal{F}, \tau)$ is independent of the embedding $k : X \hookrightarrow P$ for P smooth and integral. Indeed, given $k : X \hookrightarrow P$ and $j : X \to M$ two closed embedding of X into smooth varieties, we can consider the product embedding $i : X \hookrightarrow P \times M$ and apply Lemma 2 to the two projections from the product. This leads to

Step 3. The study for the situation in general. Given a closed immersion $j : X \hookrightarrow M$ with M regular and integral, we choose a closed immersion $f : M \hookrightarrow P$ with P smooth and integral. Let $N = N_{M/P}$ be the normal bundle of M in P and let $s : M \hookrightarrow \mathbf{P}(N \oplus 1)$ be the zero section. Note that $\mathbf{P}(N \oplus 1)$ is regular, and that the projection $q : \mathbf{P}(N \oplus 1) \to M$ is smooth. Then by Lemma 2

$$\tau_{P,\mathrm{Ar}}(\mathcal{F}, \tau) = \tau_{\mathbf{P}(N\oplus 1),\mathrm{Ar}}(\mathcal{F}, \tau),$$

where we embed X into $\mathbf{P}(N \oplus 1)$ via $s \circ j$. So to complete the proof of the Proposition and verify (2) in the theorem, it suffices to prove the following

Lemma 3. With the same notation as above,

$$\tau_{P,\mathrm{Ar}}(\mathcal{F}, \tau) = \tau_{\mathbf{P}(N\oplus 1),\mathrm{Ar}}(\mathcal{F}, \tau).$$

Obviously, once we have these lemmas, we have the Proposition. The other assertions in the theorem may be deduced similarly from the construction in a quite standard way.

Proof of Lemma 2. We are in the situation of Proposition 7.1.d. So, there is a complex $D\mathcal{G}. = \mathrm{Tot}(\mathcal{G}..)$ over $W \times_M P$ and the restriction $\mathcal{G}.^Z$ to $|Z| \times_M P$ is quasi-isomorphic to $v^*(\mathcal{V}.) \otimes p_\infty^*(\mathcal{E}.^Z)$ by that proposition.

Choose hermitian metrics on vector bundles in question. The normal vector sheaf of X in $X \times_M P$ coincides with $k^*\mathcal{T}_{P/M}$, and

$$|Z|(\mathbf{C}) = \mathbf{P}(\mathcal{N}_{X(\mathbf{C})/P(\mathbf{C})} \oplus 1).$$

The normal vector sheaf of $X(\mathbf{C})$ in $|Z|(\mathbf{C}) \times_M P(\mathbf{C})$ is isomorphic to

$$\mathcal{N}_{X(\mathbf{C})/M(\mathbf{C})} \oplus k^*\mathcal{T}_{P(\mathbf{C})/M(\mathbf{C})}.$$

We endow it with the original direct sum of the two corresponding metrics and write $\mathrm{td}_{\mathrm{BC}}^{-1}(X, P, M)$ for the classical Bott-Chern secondary characteristic forms associated with the characteristic class td^{-1} with respect to the exact sequence of vector sheaves on $X(\mathbf{C})$:

$$0 \to k^*\mathcal{T}_{P(\mathbf{C})/M(\mathbf{C})} \to \mathcal{N}_{X(\mathbf{C})/P(\mathbf{C})} \to \mathcal{N}_{X(\mathbf{C})/M(\mathbf{C})} \to 0.$$

We now have

Sublemma 1. If $Y \subset |Z| \times_M P$ is the support of the homology of $\mathcal{G}.^Z$, and $h : Y \to X$ is the projection, then, for the above choice of metrics on normal vector sheaves, we have

$$\mathrm{ch}_{\mathrm{Ar}P}^{X}(\mathcal{F}, \tau) = \sum_\beta n_\beta h_*^\beta\big(\mathrm{ch}_{\mathrm{Ar}Z_\beta \times_M P}^{Y_\beta}(\mathcal{G}.^Z|_{Z_\beta}, \rho_{\mathcal{G}.^Z|_{Z_\beta}})\big) - a\big(\mathrm{td}_{\mathrm{BC}}^{-1}(X, P, M)\mathrm{ch}(\mathcal{F}, \tau)\big).$$

Here the Z_β are the irreducible components of $|Z|$, $Z = \sum_\beta n_\beta Z_\beta$, $Y^\beta := Z_\beta \times_M P \subset Y$, and $h^\beta : Y^\beta \to X$ is the induced projection.

We leave the proof of this sublemma later. With it, noting that for each irreducible component Z_β of Z, $Z_\beta \times_M P = Z_\beta \times_X (X \times_M P)$, and we have a Cartesian diagram:

$$\begin{array}{ccc} Z_\beta \times_M P & \overset{p_{f\beta}}{\to} & Z_\beta \\ v_\beta \downarrow & & \downarrow b_\beta \\ X \times_M P & \overset{p}{\to} & X. \end{array}$$

Choose F_∞-invariant hermitian metrics on $\mathcal{G}.^Z$, $\mathcal{V}.$ and $\mathcal{E}.^Z$, which satisfy Bismut condition (A) with respect to the choice of metrics on $\mathcal{F}$ and on the normal bundles involved. Let $f := (\mathrm{Id}, k) : X \hookrightarrow X \times_M P$ be the natural inclusion induced by Id and k. Then, for all β, the inverse image by v_β of $f(X(\mathbf{C}))$ is transverse in the complex points of $Z_\beta \times_M P$ to the inverse image by p_f of the zero section $j^Z : X(\mathbf{C}) \to \mathbf{P}(\mathcal{N}_\mathbf{C} \oplus 1)$. Therefore the complex $v_\beta^*\mathcal{V}. \otimes p_{f\beta}^*\mathcal{E}.^Z$, with its natural induced metrics, is a resolution of $(j^Z \times f)_*\mathcal{F}$, and satisfies Bismut condition (A) for the one component Z_β of Z with multiplicity 1 which is non-empty.

Since both the complexes $\mathcal{G}.^Z|_{Z_\beta\times_M P}$ and $v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z$ are resolutions of the hermitian vector sheaf $\mathcal{F}$, if $Z_\beta(\mathbf{C})\neq\emptyset$, we have that

$$h_*^\beta(\mathrm{ch}_{\mathrm{Ar}}{}_{Z_\beta\times_M P}^{Y^\beta}(\mathcal{G}^Z.|_{Z_\beta},\rho_{\mathcal{G}^Z.|_{Z_\beta}}))=h_*^\beta(\mathrm{ch}_{\mathrm{Ar}}{}_{Z_\beta\times_M P}^{Y^\beta}(v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z|_{Z_\beta},\rho_{v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z}|_{Z_\beta}))).$$

Moreover, if $Z_\beta(\mathbf{C})=\emptyset$, by the fact that the complexes are quasi-isomorphic, the same formula is true as an identity in $\mathrm{CH}^{\mathrm{Ar}}_.(b_\beta(Z_\beta))=\mathrm{CH}.(b_\beta(Z_\beta))$.

The projection map $\pi=p\circ v=b\circ f$ from $|Z|\times_M P$ to X is smooth at infinity, and maps $Y(\mathbf{C})$ isomorphically onto $X(\mathbf{C})$ via h. Hence, applying Property 2.2, to the maps $\pi^\beta: Z_\beta\times_M P\to X$ and $h^\beta: Y^\beta=Y\cap(Z_\beta\times_M P)\to X$ and noting that $Z_\beta(F)$ is empty for all but one β, we find that

$$\begin{aligned}\sum_\beta n_\beta h_*^\beta(\mathrm{ch}_{\mathrm{Ar}}{}_{Z_\beta\times_M P}^{Y^\beta}(v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z|_{Z_\beta},\rho_{v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z|_{Z_\beta}}))\\ =\sum_\beta n_\beta\pi_*^\beta(\mathrm{ch}_{\mathrm{Ar}}(v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z|_{Z_\beta},\rho_{v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z|_{Z_\beta}})\cap[Z_\beta\times_M P])\\ +a\Big(\pi_*(\mathrm{ch}_{\mathrm{BC}}(v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z,\rho_{v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z}))\Big).\end{aligned}$$

We now compute the first term in the above expression, treating each term in the sum separately. Noting that $\pi^\beta=b_\beta\circ p_{f\beta}$, we start with the direct image by $p_{f\beta}$. In $\mathrm{CH}^{\mathrm{Ar}}_.(Z_\beta)$, we have, by Theorem 3.7 (d) and (e), that

$$\begin{aligned}&p_{f\beta *}(\mathrm{ch}_{\mathrm{Ar}}(v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z|_{Z_\beta},\rho_{v_\beta^*\mathcal{V}.\otimes p_{f\beta}^*\mathcal{E}.^Z|_{Z_\beta}})\cap[Z_\beta\times_M P])\\ &=p_{f\beta *}(\mathrm{ch}_{\mathrm{Ar}}(p_{f\beta}^*\mathcal{E}.^Z|_{Z_\beta},\rho_{p_{f\beta}^*\mathcal{E}.^Z|_{Z_\beta}})\cap(\mathrm{ch}_{\mathrm{Ar}}(v_\beta^*\mathcal{V}.,\rho_{v_\beta^*\mathcal{V}.})\cap[Z_\beta\times_M P]))\\ &=\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.^Z|_{Z_\beta},\rho_{\mathcal{E}.^Z|_{Z_\beta}})\cap p_{f\beta *}(\mathrm{ch}_{\mathrm{Ar}}(v_\beta^*\mathcal{V}.,\rho_{v_\beta^*\mathcal{V}.})\cap[Z_\beta\times_M P]).\end{aligned}$$

Now $v_\beta^*(\mathcal{V}.)$ is a resolution of $s_*(\mathcal{O}_{Z_\beta})$, where $s: Z_\beta\to Z_\beta\times_M P$ is the section of the smooth morphism $p_{f\beta}: Z_\beta\times_M P\to Z_\beta$ induced by the section $f: X\to X\times_M P$. Over the complex manifold $(Z_\beta\times_M P)(\mathbf{C})$, the associated metrics on $v_\beta^*(\mathcal{V}.)$ satisfy Bismut condition (A), since it is the pull-back by a submersion between complex manifolds of a complex which satisfies Bismut condition (A). Applying Property 2.2 with $\alpha=1$ to the diagram

$$\begin{array}{ccc} Z_\beta & \xrightarrow{s} & Z_\beta\times_M P\\ & & \downarrow p_{f\beta}\\ & & Z_\beta,\end{array}$$

we find that

$$p_{f\beta *}(\mathrm{ch}_{\mathrm{Ar}}(v_\beta^*\mathcal{V}.,\rho_{v_\beta^*\mathcal{V}.})\cap[Z_\beta\times_M P])+a(p_{f\beta *}\mathrm{ch}_{\mathrm{BC}}(\mathcal{V}.,\rho_{\mathcal{V}.}))=\mathrm{ch}_{\mathrm{Ar}}{}_{Z_\beta\times_M P}^{Z_\beta}(\mathcal{O}_{Z_\beta},\rho_{\mathcal{O}_{Z_\beta}})\cap[Z_\beta],$$

which, by Property 2.3, is equal to

$$\mathrm{td}_{\mathrm{Ar}}^{-1}(\mathcal{N}_{Z_\beta/Z_\beta\times_M P},\rho_{\mathcal{N}_{Z_\beta/Z_\beta\times_M P}})\cap[Z_\beta]=\mathrm{td}_{\mathrm{Ar}}^{-1}(b_\beta^*\mathcal{N}_{X/X\times_M P},\rho_{b_\beta^*\mathcal{N}_{X/X\times_M P}})\cap[Z_\beta].$$

But

$$\mathrm{ch_{Ar}}{}_P^X(\mathcal{F},\tau) = \sum_\beta n_\beta b_*^\beta \mathrm{ch_{Ar}}(\mathcal{E}.^Z|_{Z_\beta}, \rho_{\mathcal{E}.^Z|_{Z_\beta}}) + a\big(b_*\mathrm{ch_{BC}}(\mathcal{E}.^Z, \rho_{\mathcal{E}.^Z})\big),$$

so by Sublemma 1 we find that

$$\begin{aligned}
\mathrm{ch_{Ar}}{}_P^X(\mathcal{F},\tau) =& \Big(\mathrm{ch_{Ar}}{}_M^X(\mathcal{F},\tau) - a\big(b_*\mathrm{ch_{BC}}(\mathcal{E}.^Z,\rho_{\mathcal{E}.^Z})\big)\Big)\,\mathrm{td}_{\mathrm{Ar}}^{-1}(\mathcal{N}_{X/X\times_M P}, \rho_{\mathcal{N}_{X/X\times_M P}})\\
&- a\Big(b_*\big(\mathrm{ch}(\mathcal{E}.^Z,\rho_{\mathcal{E}.^Z})\,p_{f\infty *}\mathrm{ch_{BC}}(v^*\mathcal{V}.,\rho_{v^*\mathcal{V}.})\big)\Big)\\
&+ a\big(\pi_*\mathrm{ch_{BC}}(v^*\mathcal{V}.\otimes p_{f\infty}^*\mathcal{E}.^Z, \rho_{v^*\mathcal{V}.\otimes p_{f\infty}^*\mathcal{E}.^Z})\big)\\
&- a\big(\mathrm{td}_{\mathrm{BC}}^{-1}(X,P,M)\,\mathrm{ch}(\mathcal{F},\tau)\big)\\
=& \big(\mathrm{ch_{Ar}}{}_M^X(\mathcal{F},\tau)\big)\,\mathrm{td}_{\mathrm{Ar}}^{-1}(\mathcal{N}_{X/X\times_M P}, \rho_{\mathcal{N}_{X/X\times_M P}})\\
&- a\big(b_*\mathrm{ch_{BC}}(\mathcal{E}.^Z,\rho_{\mathcal{E}.^Z})\,\mathrm{td}^{-1}(\mathcal{N}_{X(\mathbf{C})/X(\mathbf{C})\times_{M(\mathbf{C})}P(\mathbf{C})}, \rho_{\mathcal{N}_{X(\mathbf{C})/X(\mathbf{C})\times_{M(\mathbf{C})}P(\mathbf{C})}})\big)\\
&- a\big(b_*\mathrm{ch}(\mathcal{E}.^Z,\rho_{\mathcal{E}.^Z})\,p_{f\infty *}\mathrm{ch_{BC}}(v^*\mathcal{V}.,\rho_{v^*\mathcal{V}.})\big)\\
&+ a\big(\pi_*\mathrm{ch_{BC}}(v^*\mathcal{V}.\otimes p_{f\infty}^*\mathcal{E}.^Z, \rho_{v^*\mathcal{V}.\otimes p_{f\infty}^*\mathcal{E}.^Z})\big)\\
&- a\big(\mathrm{td}_{\mathrm{BC}}^{-1}(X,P,M)\,\mathrm{ch}(\mathcal{F},\tau)\big).
\end{aligned}$$

On the other hand, by Axiom 4 of relative Bott-Chern secondary characteristic currents with respect to closed immersions, together with the projection formula for direct image currents,

$$\begin{aligned}
\pi_*\big(&\mathrm{ch_{BC}}(v^*\mathcal{V}.\otimes p_{f\infty}^*\mathcal{E}.^Z, \rho_{v^*\mathcal{V}.\otimes p_{f\infty}^*\mathcal{E}.^Z})\big)\\
=& b_*\mathrm{ch}(\mathcal{E}.^Z,\rho_{\mathcal{E}.^Z})\,p_*\mathrm{ch_{BC}}(\mathcal{V}.,\rho_{\mathcal{V}.})\\
&+ \mathrm{td}^{-1}(\mathcal{N}_{X(\mathbf{C})/X(\mathbf{C})\times_{M(\mathbf{C})}P(\mathbf{C})}, \rho_{\mathcal{N}_{X(\mathbf{C})/X(\mathbf{C})\times_{M(\mathbf{C})}P(\mathbf{C})}})\,b_*\mathrm{ch_{BC}}(\mathcal{E}.^Z,\rho_{\mathcal{E}.^Z}),
\end{aligned}$$

so

$$\mathrm{ch_{Ar}}{}_P^X(\mathcal{F},\tau) = \big(\mathrm{ch_{Ar}}{}_M^X(\mathcal{F},\tau)\big)\,\mathrm{td}_{\mathrm{Ar}}^{-1}(\mathcal{N}_{X/X\times_M P}, \rho_{\mathcal{N}_{X/X\times_M P}}) - a\big(\mathrm{td}_{BC}^{-1}(X,P,M)\,\mathrm{ch}(\mathcal{F},\tau)\big).$$

From this equality and the definition of τ_{Ar}, we deduce that

$$\begin{aligned}
\tau_{P,\mathrm{Ar}}(\mathcal{F},\tau) =& \mathrm{ch_{Ar}}{}_P^X(\mathcal{F},\tau)\cdot_k\mathrm{td}_{\mathrm{Ar}}(P,\rho_P) + a\big(\mathrm{td_{BC}}(X/P,\rho_{X/P})\,\mathrm{ch}(\mathcal{F},\tau)\big)\\
=& \big(\mathrm{ch_{Ar}}{}_M^X(\mathcal{F},\tau)\cap\mathrm{td}_{\mathrm{Ar}}^{-1}(\mathcal{N}_{X/X\times_M P}, \rho_{\mathcal{N}_{X/X\times_M P}})\big)\cdot_k\mathrm{td}_{\mathrm{Ar}}(P,\rho_P)\\
&- a\Big(\mathrm{ch}(\mathcal{F},\tau)\mathrm{td}_{\mathrm{BC}}^{-1}(X,P,M)k^*\big(\mathrm{td}(\mathcal{T}_{P(\mathbf{C})},\rho_{\mathcal{T}_{P(\mathbf{C})}})\big)\\
&+ \mathrm{ch}(\mathcal{F},\tau)\,\mathrm{td_{BC}}(X/P,\rho_{X/P})\Big).
\end{aligned}$$

But, the normal bundle of X in $X\times_M P$ coincides with $k^*\mathcal{T}_{P/M} = k^*\mathcal{T}_p$. Therefore, by applying Proposition 3.8.b.(2) to the map p and the structure map of Y, and Theorem 2.7.(c) for $j = p\circ k$, we get, from the equation above, that

$$\tau_{P,\mathrm{Ar}}(\mathcal{F},\tau) = \mathrm{ch_{Ar}}{}_M^X(\mathcal{F},\tau)\cdot_j\mathrm{td}_{\mathrm{Ar}}(M,\rho_M) + a\big(\mathrm{ch}(\mathcal{F},\tau)\,x\big),$$

where

$$
\begin{aligned}
x =& \mathrm{td}_{\mathrm{BC}}(X/P, \rho_{X/P}) - \mathrm{td}_{\mathrm{BC}}^{-1}(X, P, M) \cap k^* \mathrm{td}(\mathcal{T}_{P(\mathbf{C})}, \rho_{\mathcal{T}_{P(\mathbf{C})}}) \\
& - \mathrm{td}_{\mathrm{BC}}(0 \to k^*\mathcal{T}_{P(\mathbf{C})/M(\mathbf{C})} \to k^*\mathcal{T}_{P(\mathbf{C})} \to j^*\mathcal{T}_{M(\mathbf{C})} \to 0, \rho.) \\
& \mathrm{td}^{-1}(k^*\mathcal{T}_{P(\mathbf{C})/M(\mathbf{C})}, \rho_{k^*\mathcal{T}_{P(\mathbf{C})/M(\mathbf{C})}}) \mathrm{td}^{-1}(\mathcal{N}_{X(\mathbf{C})/M(\mathbf{C})}, \rho_{X(\mathbf{C})/M(\mathbf{C})}),
\end{aligned}
$$

which is nothing but $\mathrm{td}_{\mathrm{BC}}(X/M, \rho_{X/M})$, and therefore

$$\tau_{P,\mathrm{Ar}}(\mathcal{F}, \tau) = \tau_{M,\mathrm{Ar}}(\mathcal{F}, \tau).$$

Proof of Lemma 3. Choose metrics on the normal bundles of $X(\mathbf{C})$ in $P(\mathbf{C})$ and $M(\mathbf{C})$, and on $N = N_{M/P}$. The normal bundle of $X(\mathbf{C})$ in $\mathbf{P}(N \oplus 1)(\mathbf{C})$ is the direct sum of $j^*(N_{\mathbf{C}})$ and $N_{X(\mathbf{C})/M(\mathbf{C})}$. We endow it with the orthogonal sum of the corresponding metrics. Given the formula for $\tau_{P,\mathrm{Ar}}$ in terms of the arithmetic Chern character with supports, it then suffices to compare $\mathrm{ch}_{\mathrm{Ar}P}^{X}(\mathcal{F}, \tau)$ with $\mathrm{ch}_{\mathrm{Ar}\mathbf{P}(N\oplus 1)}^{X}(\mathcal{F}, \tau)$. If we can show that the arithmetic Chern character with supports is compatible with restriction to principal divisors, (see Sublemma 2 below,) then by $\mathbf{P}^1$-deformation theory in I.9, we have that the difference of these two classes is the integral over $\mathbf{P}^1$ of

$$\mathrm{ch}(\mathcal{F}, \tau)\mathrm{td}^{-1}(\mathcal{N}_{(X\times\mathbf{P}^1)(\mathbf{C})/W(\mathbf{C})}, \rho_{\mathcal{N}})[\log|z|^2],$$

where W is the deformation to the normal cone for the inclusion of M into P, and the map $X \times \mathbf{P}^1 \to W$ is the natural inclusion. But the vector sheaf $\mathcal{N}_{(X\times\mathbf{P}^1)(\mathbf{C})/W(\mathbf{C})}$ is an extension of $\mathcal{N}_{(M\times\mathbf{P}^1)(\mathbf{C})/W(\mathbf{C})}$ by $\mathcal{N}_{X(\mathbf{C})/M(\mathbf{C})}$ which coincides with $\mathcal{N}_{X(\mathbf{C})/P(\mathbf{C})}$ over $X(\mathbf{C}) \times \{0\}$, and the normal bundle of $X(\mathbf{C})$ in $\mathbf{P}(N \oplus 1)(\mathbf{C})$ over $X(\mathbf{C}) \times \{\infty\}$. Hence, we get

$$
\begin{aligned}
&\mathrm{ch}_{\mathrm{Ar}P}^{X}(\mathcal{F}, \tau) - \mathrm{ch}_{\mathrm{Ar}\mathbf{P}(N\oplus 1)}^{X}(\mathcal{F}, \tau) \\
&\quad = -a\big(\mathrm{ch}(\mathcal{F}, \tau)\, \mathrm{td}_{\mathrm{BC}}^{-1}(0 \to \mathcal{N}_{X(\mathbf{C})/M(\mathbf{C})} \to \mathcal{N}_{X(\mathbf{C})/P(\mathbf{C})} \to j^*(\mathcal{N}_{\mathbf{C}}) \to 0, \rho.)\big).
\end{aligned}
$$

Similarly, since $\mathcal{T}_{\mathbf{P}(N\oplus 1)(\mathbf{C})}$ as a hermitian vector sheaf is the orthogonal direct sum of $\mathcal{T}_{M(\mathbf{C})}$ and $\mathcal{N}_{\mathbf{C}}$, we get, on M,

$$
\begin{aligned}
&p^*\big(\mathrm{td}_{\mathrm{Ar}}(P, \rho_P)\big) \\
&\quad = s^*\big(\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_{\mathbf{P}(N\oplus 1)}, \rho_{\mathcal{T}_{\mathbf{P}(N\oplus 1)}}) - a(\mathrm{td}_{\mathrm{BC}}(0 \to \mathcal{T}_{M(\mathbf{C})} \to p^*\mathcal{T}_{P(\mathbf{C})} \to \mathcal{N}_{\mathbf{C}} \to 0, \rho.)\big).
\end{aligned}
$$

From these two equations and the definition with $i = p \circ j$, we have

$$\tau_{P,\mathrm{Ar}}(\mathcal{F}, \tau) = \tau_{\mathbf{P}(N\oplus 1),\mathrm{Ar}}(\mathcal{F}, \tau) + a\big(\mathrm{ch}(\mathcal{F}, \tau)\, y\big),$$

where

$$
\begin{aligned}
y =& \mathrm{td}_{\mathrm{BC}}(X/P, \rho_{X/P}) \\
& - \mathrm{td}_{\mathrm{BC}}^{-1}(0 \to \mathcal{N}_{X(\mathbf{C})/M(\mathbf{C})} \to \mathcal{N}_{X(\mathbf{C})/P(\mathbf{C})} \to j^*(\mathcal{N}_{\mathbf{C}}) \to 0, \rho.)\, k^*\mathrm{td}(\mathcal{T}_{P(\mathbf{C})}, \rho) \\
& - \mathrm{td}_{\mathrm{BC}}(X/\mathbf{P}(N \oplus 1), \rho_{X/\mathbf{P}(N\oplus 1)}) \\
& - j^*\big(\mathrm{td}_{\mathrm{BC}}(0 \to \mathcal{T}_{M(\mathbf{C})} \to p^*\mathcal{T}_{P(\mathbf{C})} \to \mathcal{N}_{\mathbf{C}} \to 0, \rho.)\, \mathrm{td}^{-1}(\mathcal{N}_{X(\mathbf{C})/\mathbf{P}(N\oplus 1)(\mathbf{C})}, \rho)\big).
\end{aligned}
$$

So the assertion comes from the fact that $y = 0$, which follows from the fact that

$$\mathrm{td}_{\mathrm{BC}}(X/\mathbf{P}(N \oplus 1), \rho_{X/\mathbf{P}(N\oplus 1)}) = \mathrm{td}_{\mathrm{BC}}(X/M, \rho_{X/M})$$

and Proposition 3.8.b. This completes the proof of Lemma 3, provided we have the following

Sublemma 2. Let $\mathcal{E}.$ be a complex of vector sheaves on an arithmetic variety P and let $X \subset P$ be the support of the homology of $\mathcal{E}.$. Suppose that X_F, viewed as a reduced subscheme of P_F, is smooth over F, and that on P_F, $\mathcal{E}.$ is a resolution of a hermitian vector sheaf on X_F. Let $i : D \hookrightarrow P$ be an arithmetic subvariety which is a principal effective Cartier divisor that meets X transversally over F. Write $\{D_\alpha\}$ for the irreducible components of D, n_α for the multiplicity of D_α in the Weil divisor $[D] = \sum_\alpha n_\alpha[D_\alpha]$ associated with D, and let $\eta^\alpha : X \cap D_\alpha \hookrightarrow X \cap D$ be the natural inclusion. Then

$$i_X^* \mathrm{ch}_{\mathrm{Ar}}{}_P^X(\mathcal{E}., \rho.) = \sum_\alpha n_\alpha \eta_*^\alpha \mathrm{ch}_{\mathrm{Ar}}{}_{D_\alpha}^{X\cap D_\alpha}(\mathcal{E}.|_{D_\bullet}, \rho_{\mathcal{E}.|_{D_\alpha}}) \in \mathrm{CH}_\cdot^{\mathrm{Ar}}(X \cap D),$$

where $i_X : X \cap D \hookrightarrow X$ is the inclusion. Moreover, the above relation is also true more generally. For example, if D is a divisor on P, and is contained in a Zariski open subset $U \subset P$, as a principal divisor on U.

Proof of Sublemma 1. Let T be the support of $W \times_M P$ of the homology of $D\mathcal{G}.$. There is a natural projection from T to $X \times \mathbf{P}^1$, which is an isomorphism over $X \times \mathbf{A}^1$ and such that the inverse image of $X \times \{\infty\}$ is Y. Given any $t \in \mathbf{P}^1$, we write T_t for the inverse image in T of $X \times \{t\}$. Notice that the generic fiber of T is isomorphic to that of $X \times \mathbf{P}^1$. Let $j_0 : P \hookrightarrow W \times_M P$ and $j_\infty : W_\infty \times_M P \hookrightarrow W \times_M P$ be the inclusions corresponding to $\{0\}$ and $\{\infty\}$ in $\mathbf{P}^1$.

The normal vector sheaf of $X(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$ in $W(\mathbf{C})$ is naturally isomorphic to the pull-back of the normal sheaf of $X(\mathbf{C})$ in $M(\mathbf{C})$, i.e., $h^*(\mathcal{N}_{X(\mathbf{C})/M(\mathbf{C})}(-1))$, where $h : X \times \mathbf{P}^1 \to X$ is the projection. We metrize it by choosing a metric on $h^*(\mathcal{N}_{X(\mathbf{C})/M(\mathbf{C})})$ and tensoring with the standard metric on the tautological line sheaf over $\mathbf{P}^1(\mathbf{C})$. By Sublemma 2 above, since $T \cap (W_\infty \times_M P) = Y$, $Y \cap (\tilde{M} \times_M P) = \emptyset$, we know that

$$j_0^*(\mathrm{ch}_{\mathrm{Ar}}{}_{W\times_M P}^T(D\mathcal{G}., \rho_{D\mathcal{G}.})) = \mathrm{ch}_{\mathrm{Ar}}{}_P^X(\mathcal{G}., \rho_{\mathcal{G}.}) = \mathrm{ch}_{\mathrm{Ar}}{}_P^X(\mathcal{F}, \tau),$$

and that

$$j_\infty^*(\mathrm{ch}_{\mathrm{Ar}}{}_{W\times_M P}^T(D\mathcal{G}., \rho_{D\mathcal{G}.})) = \sum_\beta n_\beta h_*^\beta(\mathrm{ch}_{\mathrm{Ar}}{}_{Z_\beta\times_M P}^{Y^\beta}(\mathcal{G}^Z.|_{Z_\beta}, \rho_{\mathcal{G}^Z.|_{Z_\beta}})).$$

By the method in I.9 for the deformation to the normal cone, we see that

$$\mathrm{ch}_{\mathrm{Ar}}{}_P^X(\mathcal{F}, \tau) - \sum_\beta n_\beta h_*^\beta(\mathrm{ch}_{\mathrm{Ar}}{}_{Z_\beta\times_M P}^{Y^\beta}(\mathcal{G}^Z.|_{Z_\beta}, \rho_{\mathcal{G}^Z.|_{Z_\beta}}))$$

is equal to the integral over $\mathbf{P}^1$ of $\mathrm{ch}(\mathcal{F},\tau)\mathrm{td}^{-1}(N_{T(\mathbf{C})/W(\mathbf{C})},\rho_{N_{T(\mathbf{C})/W(\mathbf{C})}})[\log|z|^2]$, and hence

$$\mathrm{ch}_{\mathrm{Ar}P}^{X}(\mathcal{F},\tau)-\sum_{\beta} n_\beta h_*^\beta\left(\mathrm{ch}_{\mathrm{Ar}Z_\beta\times_M P}^{Y^\beta}(\mathcal{G}^Z_\cdot|_{Z_\beta},\rho_{\mathcal{G}^Z_\cdot|_{Z_\beta}})\right)=-a\left(\mathrm{td}_{\mathrm{BC}}^{-1}(X,P,M)\mathrm{ch}(\mathcal{F},\tau)\right).$$

This completes the proof of the sublemma.

Finally, we complete the proof by the following

Proof of Sublemma 2. We fix certain notation. Write G_P for the product of Grassmannian bundle $\prod_m G(n_m, DC_m(\mathcal{E}.))$ over $P\times\mathbf{P}^1$, G_D for the restriction of G_P to $D\times\mathbf{P}^1$ (which may be identified with the corresponding product of Grassmannians for the restriction of $\mathcal{E}.$ to D). Let $i_G: G_D\hookrightarrow G_P$ be the inclusion of the divisor G_D in G_P. Let G_{P_∞} be the fiber of G_P over $P\times\{\infty\}$, which is a divisor in G_P; and let $i_\infty: G_{D_\infty}\hookrightarrow G_{P_\infty}$ be the inclusion for the corresponding divisor in G_D. Write $j: G_{P_\infty}\hookrightarrow G_P$ for the inclusion, and for each α, let j_{D_α} be the corresponding inclusion over $D_\alpha\times\mathbf{P}^1$. Obviously, $G_{P_\infty}\cap G_D=G_{D_\infty}$ and G_{D_∞} is a principal divisor in the pull-back G_D^0 of G_D over $D\times\mathbf{A}^1-\{0\}$. (We can see G_P^0 in a similar way). Write $t=0$ for the equation of this divisor and we have pull-back maps $j_D^*: Z_k(G_D)\to Z_{k-1}(G_{D_\infty})$ on cycles as well as on $\mathrm{CH}^{\mathrm{Ar}}$, together with similar pull-back maps j^*, i_∞^* and $i_{G_D}^*$.

Let $W\subset G_P$ be the Grassmannian graph of $\mathcal{E}.$ and $Z=j^*([W])-[\tilde{P}]$ on G_{P_∞}. On the complement of the divisor G_{P_∞}, the variety W is the image of the section of G_P over $P\times\mathbf{A}^1$ corresponding to the graphs of the differentials in $\mathcal{E}.$. On the open subset $G_P-G_{P_\infty}$, we have an equality of cycles

$$i^*([W])=\sum_\alpha n_\alpha[W_{D_\alpha}].$$

Also

$$j_{D_\alpha}^*([W_{D_\alpha}])=Z_{D_\alpha}+[\tilde{D}_\alpha].$$

However, we know that the maps $j^*\circ i^*$ and $i^*\circ j^*$ agree modulo rational equivalence. Thus there is a K_1-chain ϕ on G_{D_∞} such that

$$\mathrm{div}(\phi)=j^*\circ i^*[W]-i^*\circ j^*[W].$$

We claim that this K_1-chain can be chosen so that its support does not intersect the generic fiber. Since the varieties G_D, W and G_{P_∞} all meet transversally over F, hence the cycle $j^*\circ i^*[W]-i^*\circ j^*[W]$ is supported over the special fibers. Thus, using K-theory and the Gersten complex, one may show that the K_1-chain ϕ can be constructed by blowing up the components of the intersection which have the excess intersection which in this case are all supported over special fibers: Let f be an equation for D, then the symbol $\{f,t\}$ defines an element in $K_2(W\cap(G_P-(G_{|D|}\cup G_{P_\infty})))$, and hence in K_2 of the function field of W. The differential of this element in the Gersten complex is compatible with the natural product. We know that, on the components of $\mathrm{div}(f)$ where t does not vanish, φ is equal to $\varphi_t=\mathrm{div}(t)*\{f\}$. Now observe that $\mathrm{div}(\varphi_t)=j_D^*\circ i_G^*[W]$, while $\mathrm{div}(\varphi_t)=i_{G_D}^*\circ j^*[W]$. Since the composition of two differentials in the Gersten complex is zero, we have $\mathrm{div}(\varphi) = 0$, and hence

$$j^* \circ i^*[W] - i^* \circ j^*[W] = \mathrm{div}(\phi),$$

or

$$\sum_\alpha n_\alpha (Z_{D_\alpha} + [\tilde{D}_\alpha]) - i^*(Z + [\tilde{P}]) = \mathrm{div}(\phi),$$

where $\phi = \varphi - \varphi_t + \varphi_f$ is a K_1-chain supported on $\mathrm{div}(f) \cap \mathrm{div}(t) \cap W$. Since D and X are smooth and meet transversally over F, with the identification of the generic fiber of the Grassmannian graph construction for the deformation to the normal cone of that over F, the two cycles $Z_{D_F} + [\tilde{D}_F]$ and $i^*(Z_F + [\tilde{P}_F])$ coincide. Hence the cycle ϕ constructed above is supported only over the special fibers.) Furthermore, since away from X, W and W_{D_α}, W and W_{D_α} are isomorphic to $P \times \mathbf{P}^1$ and $D_\alpha \times \mathbf{P}^1$ respectively, we see that the support of ϕ lies over $X \cap |D|$.

Now choose metrics on $D\mathcal{E}.$ as in the proof of Property 2.2, and consider the class

$$i^* \mathrm{ch_{Ar}}^X(\mathcal{E}., \rho.) = i^* \Big(\pi_*^Z (\mathrm{ch_{Ar}}^X(\mathcal{E}.^Z, \rho_{\mathcal{E}.^Z}) \cap Z) + a(\pi_*^Z \mathrm{ch_{BC}}^X(\mathcal{E}.^Z, \rho_{\mathcal{E}.^Z})) \Big).$$

Since $X(\mathbf{C})$ and $D(\mathbf{C})$ are smooth and intersect transversely in $P(\mathbf{C})$,

$$i^* \Big(a(\pi_*^Z \mathrm{ch_{BC}}^X(\mathcal{E}.^Z, \rho_{\mathcal{E}.^Z})) \Big) = a(\pi_*^{Z_D} \mathrm{ch_{BC}}^X(\mathcal{E}.^{Z_D}, \rho_{\mathcal{E}.^{Z_D}})).$$

Therefore, it suffices to show that, in $\mathrm{CH}_\cdot^{\mathrm{Ar}}(X \cap |D|)$,

$$i^* \pi_* (\mathrm{ch_{Ar}}(\mathcal{E}.^Z, \rho_{\mathcal{E}.^Z}) \cap Z) = \sum_\alpha n_\alpha \eta_*^\alpha \Big(\pi_*^{Z_{D_\alpha}} (\mathrm{ch_{Ar}}^X(\mathcal{E}.^{Z_{D_\alpha}}, \rho_{\mathcal{E}.^{Z_{D_\alpha}}})) \cap Z_{D_\alpha}) \Big),$$

where $\eta^\alpha : X \cap D_\alpha \hookrightarrow X \cap |D|$ is the natural inclusion. But $\eta^\alpha \circ \pi^{Z_{D_\alpha}}$ factors through the inclusion of Z_{D_α} into $|Z| \cap |X| := (\pi^Z)^{-1}(|D|)$ followed by the projection from $|Z| \cap |D|$ to $X \cap |D|$, we know, using Theorem 3.8.e for this inclusion, that the right hand side of this formula is equal to

$$\pi_*^{|Z| \cap |D|} (\mathrm{ch_{Ar}}(\mathcal{E}.^Z|_{|Z| \cap |D|}, \rho_{\mathcal{E}.^Z|_{|Z| \cap |D|}}) \cap (\sum_\alpha n_\alpha Z_{D_\alpha})).$$

Also, by Theorem 3.8.b, applied to i, the left hand side is equal to

$$\pi_*^{|Z| \cap |D|} (\mathrm{ch_{Ar}}(\mathcal{E}.^Z|_{|Z| \cap |D|}, \rho_{\mathcal{E}.^Z|_{|Z| \cap |D|}}) \cap i^* Z).$$

On the other hand,

$$i^* Z = \sum_\alpha n_\alpha Z_{D_\alpha} - \mathrm{div}(\phi) + \tau$$

in $\mathrm{CH}_\cdot^{\mathrm{Ar}}(|Z| \cap |D|)$, where $\tau := \sum_\alpha [\tilde{D}_\alpha] - i^*[\tilde{P}]$. Thus by the fact that the support of τ is contained in $\tilde{P} \cap |Z|$, we have $\mathcal{E}.^Z|_{|\tau|}$ is metrically split. So $\mathrm{ch_{Ar}}(\mathcal{E}.^Z|_{|\tau|}, \rho_{\mathcal{E}.^Z|_{|\tau|}}) \cap \tau = 0$ in

$\mathrm{CH}^{\mathrm{Ar}}(|\tau|)$. Moreover, the fact that the support of the K_1-chain ϕ does not meet X_F implies $\mathrm{div}_{\mathrm{Ar}}(\phi) = (\mathrm{div}(\phi), 0)$, and therefore

$$\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.^{Z}|_{|\phi|}, \rho_{\mathcal{E}.^{Z}|_{|\phi|}}) \cap \mathrm{div}_{\mathrm{Ar}}(\phi) = 0$$

in $\mathrm{CH}^{\mathrm{Ar}}.(|\phi|)$. So we have

$$\begin{aligned}&\pi_*^{|Z|\cap|D|}(\mathrm{ch}_{\mathrm{Ar}}{}^{X}(\mathcal{E}.^{Z}|_{|Z|\cap|D|}, \rho_{\mathcal{E}.^{Z}|_{|Z|\cap|D|}}) \cap i^* Z)\\ &\quad = \pi_*^{|Z|\cap|D|}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E}.^{Z}|_{|Z|\cap|D|}, \rho_{\mathcal{E}.^{Z}|_{|Z|\cap|D|}}) \cap (\sum_\alpha n_\alpha Z_{D_\alpha})),\end{aligned}$$

which completes the proof of the theorem.

With above, we may also obtain a version of arithmetic Riemann-Roch theorem for morphisms $f : X \to Y$ of arithmetic varieties only with the condition that, at infinity, $f_{\mathbf{C}} : X(\mathbf{C}) \to Y(\mathbf{C})$ is a smooth morphism of Kähler manifolds. We leave this formulation to the reader.

Chapter II.8.
Arithmetic K-Theory
I: A Definition Of Higher Arithmetic K-Groups

The associated Dedekind zeta function for a number field has a simple pole at $s = 1$. Moreover, the analytic class number formula in classical algebraic number theory gives the corresponding residue in terms of the class number and the Dirichlet units regulator of the field. This result may be thought of as a higher, non-trivial, local-global principle. On one hand, for $\mathrm{Re}(s) > 1$, the Dedekind zeta function is defined as a convergent Euler product, given completely in terms of the local arithmetic of the field, which has a meromorphic continuation to the whole complex plane. On the other hand, the ideal class group and the group of units are global arithmetic invariants of the field. Nevertheless, it appears that the existence of such a local-global principle is not an isolated phenomenon. The special values, or better, the leading coefficients at integer points of the L-functions of arithmetic varieties, seem to be closely related to the global arithmetic properties of these varieties. In this direction, the conjecture of Birch and Swinnerton-Dyer shows an extraordinary example, which deals with the arithmetic of abelian varieties. In general, Beilinson has developed a completely general conjectural formalism which connects the transcendental parts of the leading coefficients to the so-called regulators [Be 85]. Many mathematicians have tried hard to verify these conjectures. Motivated by these examples, we introduce another mathematical object, an arithmetic K-theory, to give regulators in a more general sense. That is, regulators should give relations between the algebraic properties and the analytic properties of arithmetic varieties.

More precisely, we first give a definition of higher arithmetic K-groups, following Quillen's definition of higher algebraic K-groups. Then we use this arithmetic K-theory to connect the algebraic K-theory, which is a purely algebraic object, with a homotopy theory, which is a purely analytic object. We then have a global triangle diagram, with the boundary morphisms from algebraic objects to analytic objects being generalized regulators. (See below.) Comparing this picture with the classical one, we have one advantage: we now look at an arithmetic object globally, while the classical picture only gives two aspects of the object. (If we look at the triangle mentioned above formally, the classical picture gives two vertexes and one side, but the picture given in this paper has three sides and three vertexes, even through we only add one more point.)

The strategy for giving a definition of higher arithmetic K-groups is to imitate what Quillen did for higher algebraic K-groups. I asked A. J. Berrick about this possibility, he told me that one should start with the exact category theory used by Quillen. This has proved to be a very important suggestion.

It is well-known that when Quillen introduced his definition for higher algebraic K-groups, with the fact that the K-group defined by Grothendieck is the same as the one introduced by himself using the classifying space, he starts with a special kind of category: exact categories. Beginning with any exact category, Quillen could apply his Q-construction to define higher algebraic K-groups [Qu 73]. Unfortunately, this does not work for the arithmetic situation, simply because we cannot construct an exact category for which the Grothendieck group is exactly the arithmetic K-group $K_0^{\rm Ar}(X)$ of an arithmetic variety X, which was introduced by Gillet and Soulé [GS 91b]. Indeed, among all axioms for an exact category, the most difficult one to deal with is the one that concerns the composition of morphisms. As a consequence, we cannot find any bi-product construction for arithmetic objects in the classical sense. (For more details, see section 3 and section 4 of this chapter.)

Yet, in category language for our purpose, the bi-product only has its meaning for the construction of a special kind of pull-backs. So in this account, we decided not to start with an exact category, but rather to consider the essential properties of exact categories, which are used in the definition of Quillen's algebraic K-theory. From this point of view, by noting that many properties of exact categories are just consequences of the fact that every abelian category can be realized as a full subcategory of the R-module category over a ring R, we finally get a category, which is called a taips category, to which Quillen's construction can be applied, and hence make the definition of higher arithmetic K-groups. Roughly speaking, the taips category is a right category to apply Quillen's construction.

Now we introduce the main results of this chapter. For notation, see the later part of this paper.

Main Theorem 1. Let X be a regular arithmetic variety over an arithmetic ring A. Then there is a taips category $\mathcal{T}_{\rm Ar}(X)$ so that the Quillen construction can be applied. In particular,

$$\pi_1(BQ\mathcal{T}_{\rm Ar}(X),(0,0;0)) \simeq K_0^{\rm Ar}(X).$$

Thus we may define an arithmetic K-theory by letting

$$K_i^{\rm Ar}(X) := \pi_{i+1}(BQ\mathcal{T}_{\rm Ar}(X),(0,0;0)),$$

for all $i \geq 0$.

Main Theorem 2. (a) Let $QF : Q\mathcal{T}_{\rm Ar}(X) \to Q\mathcal{P}(X)$ be the natural functor induced by the forgetful functor

$$\begin{array}{rcl} F:\mathcal{T}_{\rm Ar}(X) & \to & \mathcal{P}(X) \\ (\mathcal{E},\rho;\eta) & \mapsto & \mathcal{E}. \end{array}$$

(Here, as usual, we denote $\mathcal{P}(X)$ as the category of vector bundles on X.) Then for any object $\mathcal{E}$ of $\mathcal{P}(X)$, and object $(\mathcal{E},\rho;\eta)$ of $\mathcal{T}_{\rm Ar}(X)$, we have a natural induced long exact sequence of abelian groups

$$\begin{array}{ccccc} K_{i+1}(X) & \leftarrow & \dots & & \\ \downarrow R & & & & \\ \pi_{i+1}(\mathcal{E}\backslash QF,((\mathcal{E},\rho;\eta),\mathrm{Id}_{\mathcal{E}})) & \xrightarrow{N} & K_i^{\mathrm{Ar}}(X) & \xrightarrow{F} & K_i(X) \\ & & & & R\downarrow \\ & & \dots & \leftarrow & \pi_i(\mathcal{E}\backslash QF,((\mathcal{E},\rho;\eta),\mathrm{Id}_{\mathcal{E}})). \end{array}$$

(b) At its lower level, after tensoring with $\mathbf{Q}$, the above long exact sequence becomes the natural exact sequence

$$K_1(X)_{\mathbf{Q}} \to \tilde{A}(X_{\mathbf{R}}) \to K_0^{\mathrm{Ar}}(X)_{\mathbf{Q}} \to K_0(X)_{\mathbf{Q}} \to 0$$

This chapter is organized as follows: In section 1, we recall the classicl definition of algebraic K-groups $K_0(R)$, $K_1(R)$ and $K_2(R)$ for a ring R, in order to get a good feeling; in section 2, we recall the Quillen construction for an exact category and give Quillen's definition for higher algebraic K-groups; in section 3, we describe the essential properties of exact categories that are used in Quillen's construction; in section 4, we give the definition of a taips category associated with an arithmetic veriety and hence make the definition of an arithmetic K-theory; finally, in section 5, we formulate the global triangle relation.

II.8.1. The Classical K-Theory

There are plenty of references for classical K-theory. By classical here we mean the K-theory before Quillen's historical paper "Higher Agebraic K-Theory I". The reader should consult the books of Bass or Milnor when necessary.

Let R be a ring and denote by $\mathcal{P}(R)$ the category of finitely generated projective R-modules. The **Grothendieck group** $K_0(R)$ is the quotient group

$$K_0(R) = \mathcal{F}/\mathcal{R},$$

where $\mathcal{F}$ is the free abelian group on the isomorphism classes of projective modules in $\mathcal{P}(R)$ and $\mathcal{R}$ is the subgroup generated by the elements

$$[P \oplus Q] - [P] - [Q],$$

for all $P, Q \in \mathcal{P}(R)$. For P, $Q \in \mathcal{P}(R)$, $[P] = [Q]$ in $K_0(R)$ is equivalent to if $P \oplus P' \simeq Q \oplus P'$ for some $P' \in \mathcal{P}(R)$ (if and only if $P \oplus R^n \simeq Q \oplus R^n$ for some $n \geq 0$).

We now give the definition of $K_1(R)$ in the classical sense. Let $\mathrm{Gl}_n(R)$ be the group of invertible matrices of size n over R. Denote by $E_n(R)$ the subgroup of **elementary matrices**, defined to be the group generated by the matrices $e_{ij}^{(n)}(\lambda)$, with $1 \leq i \neq j \leq n$, $\lambda \in R$, where $e_{ij}^{(n)}(\lambda)$ is the unipotent matrix whose diagonal entries are all 1 and whose only non-zero off-diagonal entry is λ in the $(i,j)^{\text{th}}$-position. Let $\text{Gl}_n(R) \hookrightarrow \text{Gl}_{n+1}(R)$ be the natural map given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

and let $\text{Gl}(R) := \lim_{\rightarrow} \text{Gl}_n(R)$. Similarly, we let $E(R) := \lim_{\rightarrow} E_n(R)$. Since $e_{ij}^{(n)}(\lambda) \mapsto e_{ij}^{(n+1)}(\lambda)$ under $E_n(R) \hookrightarrow E_{n+1}(R)$, we obtain elements $e_{ij}(\lambda) \in E(R)$ as the common image of all $e_{ij}^{(n)}(\lambda)$ for $1 \leq i,j \leq n$. Hence $E(R)$ is the subgroup of $\text{Gl}(R)$ generated by the $e_{ij}(\lambda)$. An easy calculation shows that $E_n(R)$ is perfect for $n \geq 3$ and so is $E(R)$. In greater detail,

$$[E_n(R), E_n(R)] = E_n(R) \text{ and } [E(R), E(R)] = E(R).$$

In particular, by the fact that for any $A \in \text{Gl}_n(R)$,

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in E_{2n}(R),$$

we see that

$$[E(R), E(R)] = [\text{Gl}(R), \text{Gl}(R)].$$

We define

$$\begin{aligned} K_1(R) :=& \text{Gl}(R)/E(R) = \text{Gl}(R)/[\text{Gl}(R), \text{Gl}(R)] \\ =& \text{Gl}(R)^{\text{ab}} \simeq H_1(\text{Gl}(R), \mathbf{Z}). \end{aligned}$$

Finally, we give the classical definition of $K_2(R)$.

The n^{th} **Steinberg group** $\text{St}_n(R)$ is defined to be the quotient group of the free abelian group on symbols $x_{ij}^{(n)}(\lambda)$ for $1 \leq i \neq j \leq n$ and $\lambda \in R$, modulo the normal subgroup corresponding to the relations

1 $x_{ij}^{(n)}(\lambda)\, x_{ij}^{(n)}(\mu) = x_{ij}^{(n)}(\lambda + \mu)$ for any i, j.
2 $[x_{ij}^{(n)}(\lambda), x_{kl}^{(n)}(\mu)] = 1$ for $i \neq l, k \neq j$.
3 $[x_{ij}^{(n)}(\lambda), x_{jk}^{(n)}(\mu)] = x_{ik}^{(n)}(\lambda\, \mu)$ for $i \neq k$.

There is a natural surjection

$$\phi_n : \text{St}_n(R) \rightarrow E_n(R),$$

given by $\phi_n(x_{kl}^{(n)}(\mu)) = e_{kl}^{(n)}(\mu)$. We also have a natural homomorphism

$$\text{St}_n(R) \rightarrow \text{St}_{n+1}(R).$$

The **infinite Steinberg group** is

$$\text{St}(R) := \lim_{\rightarrow} \text{St}_n(R)$$

and there is the surjection

$$\phi : \mathrm{St}(R) \to E(R).$$

Then we define

$$K_2(R) := \mathrm{Ker}\,\phi.$$

The most important properties of $K_2(R)$ are the following

Fact. (1). $\mathrm{St}_n(R)$ $n \geq 3$ and $\mathrm{St}(R)$ are perfect.
(2). $K_2(R) = H_2(E(R), \mathbf{Z})$.

So far, we have given the classical definitions for $K_i(R)$ with $i = 0, 1, 2$, and this suggests that we can use the homotopy groups of certain spaces to define a general algebraic K-theory; this is just what Quillen did.

II.8.2. The Q-Construction From Exact Categories

In this section, we review the Q-construction of Quillen for an exact category. In order to do this, it is also necessary to describe the construction of the classifying space for a small category. The reference here is [Q 73] and [Sr 91].

II.8.2.a. The Classifying Space Of A Small Category

We start with some topological preparation. Let Δ be the following category: for each non-negative integer n, let $\underline{n} := \{0 < 1 < \ldots < n\}$ be the ordered set consisting of $0, 1, \ldots, n$; the objects of Δ are the ordered sets $\underline{n}$, and the morphisms are the monotonic maps.

For each positive integer n, there are $n+1$ maps in Δ

$$\partial_i^n : \underline{n-1} \to \underline{n},$$

which are injective and are given by

$$\partial_i^n(j) := \begin{cases} i, & \text{if } j < i, \\ j+1, & \text{if } j \geq i. \end{cases}$$

These are the **face maps**. Dually, there are n maps

$$s_i^{n-1} : \underline{n} \to \underline{n-1},$$

which are surjective and are given by

$$s_i^{n-1}(j) := \begin{cases} j, & \text{if } j \leq i, \\ j-1, & \text{if } j > i. \end{cases}$$

These are the **degeneracy maps**. The compositions of face maps and degeneracy maps give all morphisms in Δ. Then we make the following definition:

A **simplicial object** of a category C is a contravariant functor $\Delta \to C$. A morphism of simplicial objects in C is a natural transformation. A **simplicial set** is a functor $\Delta^{\text{op}} \to \underline{\text{Set}}$, where $\underline{\text{Set}}$ denotes the category of sets. Similarly, a **simplicial space** is a functor $\Delta^{\text{op}} \to \underline{\text{Top}}$, where $\underline{\text{Top}}$ denotes the category of topological spaces.

Suppose $F : \Delta^{\text{op}} \to \underline{\text{Set}}$ is a simplicial set. Then for each non-negative integer n, $F(\underline{n})$ is a set, called the **set of n-simplices** of F. The maps ∂_i^n give rise to $n+1$ maps of sets $F(\underline{n}) \to F(\underline{n-1})$, called the **face maps**, which associate with each n-simplex in $F(\underline{n})$ a collection of $n+1$ $(n-1)$-simplices in $F(\underline{n-1})$, called its **faces**. Dually, the n maps s_i^{n-1} give maps $F(\underline{n-1}) \to F(\underline{n})$, which associate with each $(n-1)$-simplex a collection of n degenerate n-simplices. These maps $F(\underline{n-1}) \to F(\underline{n})$ are called **degeneracies**. For $\delta \in F(\underline{n})$, we call $F(\partial_i^n)(\delta)$ the i^{th} **face** of δ, and $F(s_i^n)(\delta) \in F(\underline{n+1})$ the i^{th} **degenerate simplex** of δ.

A natural example of the above concepts is constructed as follows (it is also the motivation for the notion). Let X be a topological space. Let $S(X)$ denote the **total singular complex** of X, so that $S_n(X)$, the set of all n-simplices of $S(X)$, is just the set of singular n-simplices in X, i.e. $S_n(X)$ is the set of all continuous maps $\Delta_n \to X$, where Δ_n is the standard n-simplex

$$\Delta_n := \{(t_0, \dots, t_n) \in \mathbf{R}^{n+1} : t_i \geq 0, \sum_i t_i = 1\}.$$

If $f : \underline{m} \to \underline{n}$ is a morphism in Δ, then we introduce the natural map

$$\tilde{f} : \Delta_m \to \Delta_n$$

such that $\underline{n} \mapsto \Delta_n$, $f \mapsto \tilde{f}$ is a functor $\Delta \to \underline{\text{Top}}$ as follows:

$$\tilde{f}((s_0, \dots, s_m)) = (t_0, \dots, t_n)$$

where $t_i = \sum_{f(j)=i} s_j$, with $t_i = 0$ if $\{j : f(j) = i\} = \emptyset$. It is easily checked that, $S(X) := \{S_n(X)\}_{n \geq 0}$ becomes a simplicial set.

With each simplicial set $F : \Delta^{\text{op}} \to \underline{\text{Set}}$, we can associate a topological space $|F|$, called the **geometric realization** of F; $|F|$ is defined as the quotient space

$$(\coprod_{n \geq 0} F(\underline{n}) \times \Delta_n)/\sim,$$

where for each $n \geq 0$, $F(\underline{n})$ is regarded as a discrete topological space. The equivalence relation $\sim$ is defined as follows: given $f : \underline{m} \to \underline{n}$ in Δ, let $\tilde{f} : \Delta_m \to \Delta_n$ be the map described above. Then for any $\delta \in F(\underline{n})$, we set

$$(\delta, \tilde{f}(y)) \sim (F(f)(\delta), y)$$

for all $y \in \Delta_m$. Clearly, the construction of the geometric realization is functorial.

For any simplicial set F, a simplex $\delta \in F(\underline{n})$ is **non-degenerate** if it is not the degenerate simplex assigned to any $(n-1)$-simplex by one of the degeneracies. The very important property of $|F|$ is that $|F|$ is homeomorphic to a CW-complex, which has one n-cell corresponding to each non-degenerate n-simplex of F. We have the following general facts about this construction.

Facts. (1) If F, G are simplicial sets, such that $|F|$ and $|G|$ are locally compact, then $|F \times G|$ is homeomorphic to $|F| \times |G|$.
(2) Let $\Delta(n) := \mathrm{Hom}_\Delta(-, \underline{n})$, then $|\Delta(n)| \simeq \Delta_n$.
(3) The homotopy of $|F|$ may be computed as follows: Let $C_n(F)$ be the free abelian group on $F(\underline{n})$, and let $\bar{\partial}_i^n : C_n(F) \to C_{n-1}(F)$ be the map induced by $F(\partial_i^n)$. Then

$$C(F) := (C_n(F),\ d_n := \sum (-1)^i \bar{\partial}_i^n)_{n \geq 1}$$

is a chain complex such that for any abelian group A,

$$H_*(|F|, A) \simeq H_*(C(F) \otimes_{\mathbf{Z}} A).$$

(4) Let X be a topological space, and let $S(X)$ denote the total singular complex. There is a continuous surjective map $f : |S(X)| \to X$. For any base point $x \in X$, denote by $S(x)$ the subcomplex of $S(X)$ such that $|S(x)|$ is a point whose image under f is x. Then $f : (|S(X)|, |S(x)|) \to (X, x)$ induces isomorphisms on the homotopy groups. Hence if X is a CW-complex, f is also a homotopy equivalence.

Now we have the idea of the classifying space of a small category.

A category $\mathcal{C}$ is a **small category** if its objects form a set. The **nerve** of $\mathcal{C}$, denoted by $N\mathcal{C}$, is defined in the following way: An n-simplex of $N\mathcal{C}$ is a diagram

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \dots \xrightarrow{f_n} A_n,$$

where $A_i \in \mathrm{Ob}\,\mathcal{C}$, $f_i \in \mathrm{Mor}\,\mathcal{C}$. Given a map $f : \underline{m} \to \underline{n}$ in Δ, the corresponding map $N\mathcal{C}(\underline{m}) \to N\mathcal{C}(\underline{n})$ maps the above n-simplex to the m-simplex

$$B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2 \xrightarrow{g_3} \dots \xrightarrow{g_m} B_m,$$

where $B_j := A_{f(j)}$, and $g_j : B_{j-1} \to B_j$ is the composite map $A_{f(j-1)} \to A_{f(j)}$. (Here if $f(j-1) = f(j)$, we let $A_{f(j-1)} \to A_{f(j)}$ be the identity map.) In particular, the i^{th} face of the above n-simplex is the $(n-1)$-simplex

$$A_0 \xrightarrow{f_1} A_1 \to \dots \to A_{i-1} \xrightarrow{f_{i+1} \circ f_i} A_{i+1} \to \dots \to A_n,$$

while the i^{th} degenerate simplex of the above n-simplex is the $(n+1)$-simplex

$$A_0 \xrightarrow{f_1} A_1 \to \dots \to A_i \xrightarrow{\mathrm{Id}} A_i \xrightarrow{f_{i+1}} A_{i+1} \to \dots \to A_n.$$

The **classifying space** of $\mathcal{C}$ is the geometric realization of $N\mathcal{C}$, and is denoted by $B\mathcal{C}$; in other words,

$$B\mathcal{C} := |N\mathcal{C}|.$$

As an example, let $\mathcal{C}$ be the category $\{0 < 1\}$, consisting of two objects 0, 1 and a unique (non-identity) morphism $0 \to 1$. Then $B(\{0 < 1\}) = I := [0,1]$, the unit interval.

The above construction is functorial, the verification is left to the reader.

II.8.2.b. Exact Categories

In this subsection, we review certain axioms in category theory.

A **category** $\mathcal{C}$ is a pair $(\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C})$, where $\mathrm{Ob}\,\mathcal{C}$ consists of the objects of $\mathcal{C}$, the collection $\mathrm{Mor}\,\mathcal{C}$ consists of morphisms between pairs of objects of $\mathcal{C}$. That is, for any pair of objects X, Y of $\mathcal{C}$, there is a collection $\mathrm{Hom}_{\mathcal{C}}(X,Y)$; An element $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is denoted as $f : X \to Y$. If $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, then there is a unique element $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X,Z)$ and the following properties hold.

(1) For any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, $h \in \mathrm{Hom}_{\mathcal{C}}(Z,W)$, then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

(2) For any $X \in \mathrm{Ob}\,\mathcal{C}$, there is a (unique) element $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that, for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$, we have

$$f \circ \mathrm{Id}_X = f, \quad \mathrm{Id}_X \circ g = g.$$

(3) If (X,Y), (X',Y') are not the same, then $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(X',Y')$ are disjoint.

We call $\mathcal{D}$ a **subcategory** of a category $\mathcal{C}$, if all objects of $\mathcal{D}$ are objects of $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{D}}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$, and the composition of morphisms in $\mathcal{D}$ is the same as the one in $\mathcal{C}$. A subcategory is called a **full subcategory** if for any pair of objects X, Y of $\mathcal{D}$, we have

$$\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

An object e is called a **final object**, if for any $X \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,e)$ consists of only one element. Dually, an object e' is called an **initial object** if for any $X \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(e,X)$ consists of only one element.

If for any pair of objects X, Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an abelian group so that the following conditions are satisfied, we call $\mathcal{C}$ an **Ab-category**:

(1) $h \circ (f+g) = h \circ f + h \circ g$ whenever they make sense.

(2) $(f+g)\circ h = f\circ h + g\circ h$ whenever they make sense.
(3) There is a unique zero object 0, i.e. 0 is an initial object and is also a final object.

If, furthermore, for any two objects X, Y in an Ab-category, there is a unique bi product $X \times Y$, then we call $\mathcal{C}$ an **additive category**. Here the bi-product means that for any two objects X, Y, there is an object W and four morphisms $p \in \mathrm{Hom}_{\mathcal{C}}(W,X)$, $q \in \mathrm{Hom}_{\mathcal{C}}(W,Y)$, $i \in \mathrm{Hom}_{\mathcal{C}}(X,W)$, $j \in \mathrm{Hom}_{\mathcal{C}}(Y,W)$, such that

$$p\circ i = \mathrm{Id}_X, \quad i\circ p + j\circ q = \mathrm{Id}_W, \quad q\circ j = \mathrm{Id}_Y.$$

If for any object X in $\mathcal{C}$, there is a unique object $F(X)$ in $\mathcal{C}'$ and, for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, there is a unique morphism $F(f) \in \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y))$ so that

$$F(g\circ f) = F(g)\circ F(f), \quad F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)},$$

then we call F a **covariant functor**.

For a certain additive category $\mathcal{C}$, we can also introduce the kernel, cokernel, image and coimage of a morphism $f : X \to Y$. Moreover, an additive category is called an **abelian category** if every morphism has its kernel and cokernel and, for any morphism f, the natural induced morphism $\mathrm{Coker}(f) \to \mathrm{Im}(f)$ is an isomorphism. Surely in this case, we may introduce the concept about exact. Hence we also have the exact functor, etc.. A very important fact about abelian categories is the following

Theorem. Every abelian category can be imbedded in an R-module category by an exact covariant functor.

This theorem has many corollaries. For instance, if $g\circ f$ is a monomorphism of an abelian category, then f is a monomorphism. Dually, there is a similar statement for epimorphisms.

Next we introduce the concepts and definitions for exact categories, following [Qu 73].

Let $\mathcal{C}$ be an additive category which is embedded as a full subcategory of an abelian category $\mathcal{A}$. Suppose that $\mathcal{C}$ is closed under the extension in $\mathcal{A}$ in the sense that if an object A of $\mathcal{A}$ has a subobject A' such that A' and A/A' are isomorphic to objects of $\mathcal{C}$, then A itself is isomorphic to an object in $\mathcal{C}$. Let $\underline{\mathcal{E}}$ be the class of sequences

$$(*) \qquad 0 \to X' \xrightarrow{i} X \xrightarrow{j} X'' \to 0$$

in $\mathcal{C}$ which are exact in $\mathcal{A}$. A map in $\mathcal{C}$ is an admissible monomorphism (resp. admissible epimorphism) if it occurs as the map i (resp. j) of some member of $\underline{\mathcal{E}}$. We will also use $\rightarrowtail$ (resp. $\twoheadrightarrow$) to denote a monomorphism (resp. an epimorphism).

Obviously, we have the following facts:

(1) Any sequence in $\mathcal{C}$ which is isomorphic to a sequence in $\underline{\mathcal{E}}$ is in $\underline{\mathcal{E}}$. For any X', X'' in $\mathcal{C}$, the sequence

$$0 \to X' \xrightarrow{(\mathrm{Id}_{X'},0)} X' \oplus X'' \xrightarrow{\mathrm{pr}_2} X'' \to 0$$

is in $\mathcal{C}$. For any sequence in $\underline{\mathcal{E}}$, i is a kernel for j and j is a cokernel for i in $\mathcal{C}$.
(2) The classes of admissible epimorphisms is closed under composition and under the base change by arbitary maps in $\mathcal{C}$. The dual version still holds.
(3) Let $X \to X''$ be a map with a kernel in $\mathcal{C}$. If there exists a map $Y \to X$ in $\mathcal{C}$ such that $Y \to X \to X''$ is an admissible epimorphism, then $X \to X''$ is an admissible epimorphism. A similar statement holds for admissible monomorphisms.

An **exact category** is an additive category $\mathcal{C}$ equipped with a family of sequences of the form $(*)$, called the **short exact sequences** of $\mathcal{C}$, such that the properties 1, 2 and 3 hold. An **exact functor** $F : \mathcal{C} \to \mathcal{C}'$ between exact categories is an additive functor carrying exact sequences in $\mathcal{C}$ into exact sequences in $\mathcal{C}'$.

II.8.2.c. The Quillen Construction

Let $\mathcal{C}$ be a small exact category. Recall that if $M, N \in \operatorname{Ob}\mathcal{C}$, an arrow $i : M \to N$ is called to be an **admissible monomorphism** if there is an exact sequence in $\mathcal{C}$

$$0 \to M \xrightarrow{i} N \to P \to 0$$

for some object P in $\mathcal{C}$. Similarly, $q : M \to N$ is an **admissible epimorphism** if there is an exact sequence

$$0 \to P \to M \xrightarrow{q} N \to 0$$

for some object P in $\mathcal{C}$.

Following Quillen, we can form a new category $Q\mathcal{C}$ having the same objects as $\mathcal{C}$, but with morphisms defined in the following way. Let M and M' be objects in $\mathcal{C}$ and consider all diagrams

$$M \xleftarrow{j} N \xrightarrow{i} M'$$

where j is an addmissible epimorphism and i is an admissible monomorphism. A morphism $M \to N$ in $Q\mathcal{C}$ is an equivalence class of diagram $M \longleftarrow M' \to N$ above, where

$$M \longleftarrow M'' \to N$$

is an equivalent diagram if and only if there is an isomorphism $u : M' \to M''$ making the folowing diagram commute:

$$\begin{array}{ccccc} M & \longleftarrow & M' & \to & N \\ \mathrm{Id}\downarrow & & u\downarrow & & \downarrow\mathrm{Id} \\ M & \longleftarrow & M'' & \to & N. \end{array}$$

The composition of morphisms is defined as follows. Given diagrams $M \longleftarrow M' \to N$, $N \longleftarrow N' \to P$, the composite morphism $M \to P$ in $Q\mathcal{C}$ is represented by the diagram

$M \longleftarrow M' \times_N N' \to P$. That is, the diagram, with the square being a special pull back over an admissible monomorphism, as follows

$$\begin{array}{ccccc} M' \times_N N' & \overset{\mathrm{pr}_2}{\to} & N' & \overset{i'}{\to} P \\ \mathrm{pr}_1 \downarrow & & \downarrow j' & \\ M' & \overset{i}{\to} & N & \\ j \downarrow & & & \\ M & & & \end{array}$$

In particular, if $i : M \to N$ is an admissible monomorphism, we have an associated arrow $i_! : M \to N$ in QC, given by $M \overset{\mathrm{Id}}{\twoheadleftarrow} M \overset{i}{\rightarrowtail} N$. Similarly, if $q : M \to N$ is an admissible epimorphism, we have an associated arrow $q^! : M \to N$ in QC, given by $N \overset{q}{\twoheadleftarrow} M \overset{\mathrm{Id}}{\rightarrowtail} M$. Thus in general, if $f : M \to N$ is an arbitrary arrow in QC, given by the diagram

$$M \overset{q}{\twoheadleftarrow} M' \overset{i}{\rightarrowtail} N,$$

then $f = i_! \circ q^!$, which comes immediately from the definition of the composition of morphisms in QC. We can also form the pushout square in C

$$\begin{array}{ccc} M' & \overset{i}{\to} & N \\ q \downarrow & & \downarrow q' \\ M & \overset{i'}{\to} & N', \end{array}$$

where the horizontal arrows are admissible monomorphisms and vertical arrows are admissible epimorphisms. Such a square is called to be bi-cartesian. In particular, $f = q'^! \circ i'_!$ in QC, from the following diagram

$$\begin{array}{cccc} M' & \overset{i}{\to} & N & \overset{\mathrm{Id}}{\to} N \\ q \downarrow & & \downarrow q' & \\ M & \overset{i'}{\to} & N' & \\ \mathrm{Id} \downarrow & & & \\ M & & & \end{array}$$

Thus, the assignments $i \mapsto i_!, q \mapsto q^!$ have the following properties:

(1). If i, i' are composable admissible monomorphisms, then $(i \circ i')_! = i_! \circ i'_!$; Similarly for composable admissible epimorphisms q, q', we have $(q \circ q')^! = q^! \circ q'^!$.

(2). If

$$\begin{array}{ccc} M' & \overset{i}{\to} & N \\ q \downarrow & & \downarrow q' \\ M & \overset{i'}{\to} & N' \end{array}$$

is a bi-cartesian square, then $i_! \circ q^! = q'^! \circ i'_!$.

In fact, (1) and (2) characterize QC in the following sense.

Lemma. Let $\mathcal{C}$ be an exact category, $\mathcal{D}$ a category. Assume that

(1) For each object M in $\mathcal{C}$, there is an object $F(M)$ in $\mathcal{D}$.
(2) For each admissible monomorphism $i : M' \to M$, there is an arrow $F_1(i) : F(M') \to F(M)$ such that $F_1(i \circ i') = F_1(i) \circ F_1(i')$, if i, i' are composable; Dually, for each admissible epimorphism $q : M \to N$, there is an arrow $F_2(q) : F(N) \to F(M)$ such that $F_2(q \circ q') = F_2(q') \circ F_2(q)$, if q, q' are composable;
(3) If

$$\begin{array}{ccc} M' & \xrightarrow{i} & N \\ q\downarrow & & \downarrow q' \\ M & \xrightarrow{i'} & N' \end{array}$$

is bi-cartesian, then

$$F_1(i) \circ F_2(q) = F_2(q') \circ F_1(i').$$

Then there is a well-defined functor $F : Q\mathcal{C} \to \mathcal{D}$ given by

$$M \mapsto F(M),\ (M \xleftarrow{q} M' \xrightarrow{i} N) \mapsto F_1(i) \circ F_2(q).$$

Proof. If $M \xleftarrow{q} M' \xrightarrow{i} N$ and $M \xleftarrow{q_1} M'' \xrightarrow{i_1} N$ are two equivalent diagrams giving a morphism $M \to N$ in $Q\mathcal{C}$, we have an isomorphism $u : M' \to M''$ such that $q = q_1 \circ u$, $i = i_1 \circ u$. Regarding u as an admissible monomorphism, we get $F_1(i) = F_1(i_1) \circ F_1(u)$, while regarding u as admissible epimorphism, we get $F_2(q) = F_2(u) \circ F_2(q_1)$.

On the other hand, from the bi-cartesian square

$$\begin{array}{ccc} M' & \xrightarrow{u} & M'' \\ u\downarrow & & \downarrow \mathrm{Id} \\ M'' & \xrightarrow{\mathrm{Id}} & M'' \end{array}$$

we have $F_1(u) \circ F_2(u) = F_2(\mathrm{Id}) \circ F_1(\mathrm{Id}) = \mathrm{Id}$. Hence

$$F_1(i) \circ F_2(q) = F_1(i_1) \circ F_1(u) \circ F_2(u) \circ F_2(q_1) = F_1(i_1) \circ F_2(q_1).$$

Thus $F_1(i) \circ F_2(q)$ depends only on the arrows in $Q\mathcal{C}$, and not on the particular diagram which represents it.

Next, if $M \xleftarrow{q_1} M' \xrightarrow{i_1} N$ and $N \xleftarrow{q_2} N' \xrightarrow{i_2} P$ are given and $M \xleftarrow{q} M' \times_N N' \xrightarrow{i} P$ represents the composite arrow in $Q\mathcal{C}$, we have a diagram with a bi-cartesian square,

$$\begin{array}{ccccc} M' \times_N N' & \xrightarrow{i'} & N' & \xrightarrow{i_2} & P \\ q'\downarrow & & \downarrow q_2 & & \\ M' & \xrightarrow{i_1} & N & & \\ q_1\downarrow & & & & \\ M & & & & \end{array}$$

and $q = q_1 \circ q', i = i_2 \circ i'$. Then

$$\begin{aligned} F_1(i) \circ F_2(q) &= F_1(i_2) \circ F_1(i') \circ F_2(q') \circ F_2(q_1) \\ &= F_1(i_2) \circ F_2(q_2) \circ F_1(i_1) \circ F_2(q_1). \end{aligned}$$

This proves that $(M \longleftarrow M' \to N) \mapsto F_1(i) \circ F_2(q)$ is compatible with composition in QC and so yield a well defined functor $QC \to \mathcal{D}$.

Suppose now that C is a small exact category, so that the classifying space BQC is defined. Let 0 be the zero object of C. Then the fundamental group $\pi_1(BQC, 0)$ is canonically isomorphic to the Grothendieck group $K_0(C)$. (For the proof, see the next chapter.) Motivated by this fact, Quillen was able to give the following remarkable definition of algebraic K-theory: the i^{th} **K-groups** for a small exact category C, denoted as $K_i(C)$, is defined as the $(i+1)^{\text{st}}$ homotopy group of the classifying space QC, i.e.

$$K_i(C) := \pi_{i+1}(BQC, 0).$$

In particular, if we let C be the category of finitely generated projective R-modules, then the above definition for $K_i(X)$ with $i =$ 0, 1, 2 is the same as those in the previous chapter.

II.8.3. The Essential Properties For The Q-Construction

In this section, we point out the essential properties of exact categories which make the Quillen construction work. Later, we will use similar properties to characterize taips categories of arithmetic varieties. For this purpose, we recall the following theorem and its proof.

Theorem. (Quillen [73]) There is a natural isomorphism $K_0(C) \simeq \pi_1(BQC, \{0\})$ for any small exact category C and the null object $0 \in C$, where K_0 denotes the Grothendieck group.

Lemma. The category of covering spaces of the classifying space BC of a small category C is naturally equivalent to the category of functors $F : C \to \underline{\text{Set}}$ such that $F(u)$ is a bijection for each morphism u of C.

Proof. Let $p : E \to BC$ be a covering space. For any object X of C, let $E(X)$ be the fiber over $X \in \text{Ob}\, BC$, where X is regarded as a 0-simplex in NC, and hence determines a 0-cell in BC. Given a morphism $u : X_1 \to X_2$, we may regard u as a 1-simplex in NC, which determines a path Bu in BC joining X_1 to X_2. Since p is a covering, it has the unique path lifting property, which gives a bijection $(Bu)_* : E(X_1) \to E(X_2)$, by associating to a point $y \in E(X_1)$ the second end-point of the unique path in E which lifts Bu and begins at y. Hence $X \mapsto E(X), u \mapsto (Bu)_*$ determines a functor $C \to \underline{\text{Set}}$ carrying all arrows of C into bijections of $\underline{\text{Set}}$.

Conversely, if $F : \mathcal{C} \to \underline{\text{Set}}$ is a morphism inverting functor, i.e. $F(u)$ is a bijection for each morphism u. Let $F\backslash\mathcal{C}$ be the category of pairs (X, x) with $X \in \text{Ob}\,\mathcal{C}, x \in F(X)$, where a morphism $(X, x) \to (X', x')$ is a morphism $u : X \to X'$ such that $F(u)(x) = x'$. The forgetful functor $F\backslash\mathcal{C} \to \mathcal{C}$ gives a map on classifying spaces $p_F : B(F\backslash\mathcal{C}) \to B\mathcal{C}$ with fibers $p_F^{-1}(X) = F(X)$ for any object $X \in \text{Ob}\,\mathcal{C}$.

Claim. p_F is a covering space.

Suppose the claim is true, then for any morphism $u : X \to X'$ in $\mathcal{C}$ and any $x \in F(X)$, if $x' = F(u)(x)$, u determines a morphism $(X, x) \to (X', x')$ in $F\backslash\mathcal{C}$, which gives the unique path in $B(F\backslash\mathcal{C})$ lifting Bu and beginning at $x \in p_F^{-1}(X)$. Thus the above constructions are inverse to each other, and give the desired equivalence of categories.

The proof of the claim. By a standard result from topology, we know that it is enough to show that the map of simplicial sets $N(F\backslash\mathcal{C}) \to N\mathcal{C}$ is a simplicial covering, i.e. if $\Delta(n)$ is the simplicial set $\Delta(n)(\underline{p}) = \text{Hom}_\Delta(\underline{p}, \underline{n})$, so that $|\Delta(n)| = \Delta_n$, the standard n-simplex, then for any given diagram of maps of simplicial sets

$$\begin{array}{ccc} \Delta(0) & \to & N(F\backslash\mathcal{C}) \\ \downarrow & & \downarrow \\ \Delta(n) & \to & N\mathcal{C}, \end{array}$$

we must show that there is a unique map $\Delta(n) \to N(F\backslash\mathcal{C})$ of simplicial sets making the diagram commute. Of course, a map $\sigma : \Delta(n) \to N\mathcal{C}$ determines an n-simplex $\sigma \in N_n\mathcal{C}$, so we must show that if $\sigma \in N_n\mathcal{C}$ is an n-simplex of $N\mathcal{C}$, $\sigma_0 \in N_0(F\backslash\mathcal{C})$ a 0-simplex lying over the i^{th}-vertex of σ, then there exists a unique n-simplex $\tau \in N_n(F\backslash\mathcal{C})$ which maps to σ and is such that σ_0 is the i^{th}-vertex of τ. Assume that σ is given by the diagram in $\mathcal{C}$

$$M_0 \xrightarrow{u_1} M_1 \xrightarrow{u_2} \dots \xrightarrow{u_n} M_n,$$

where the i^{th}-vertex of σ is given by the object M_i, so that σ_0 is given by an object $(M_i, x_i) \in F\backslash\mathcal{C}$ for $x_i \in F(M_i)$. Hence we have bijections

$$F(M_0) \xrightarrow{F(u_1)} F(M_1) \xrightarrow{F(u_2)} \dots \xrightarrow{F(u_n)} F(M_n),$$

which for each i give a composite bijection $f_j : F(M_j) \to F(M_i)$ for each j. In particular, for $j = i$, f_j is the identity. We have

$$f_j = f_{j+1} \circ F(u_{j+1}) : F(M_j) \xrightarrow{F(u_j)} F(M_{j+1}) \xrightarrow{f_{j+1}} F(M_i).$$

Let $x_j \in F(M_j)$ be the unique element satisfying $f_j(x_j) = x_i$. Then,

$$x_i = f_j(x_j) = f_{j+1}(F(u_{j+1})(x_j)),$$

and $x_{j+1} = F(u_{j+1})(x_j)$. Thus we have a diagram in $F\backslash\mathcal{C}$, giving an element $\tau \in N_n(F\backslash\mathcal{C})$,

$$(M_0, x_0) \xrightarrow{\bar{u}_1} (M_1, x_1) \xrightarrow{\bar{u}_2} \dots \xrightarrow{\bar{u}_n} (M_n, x_n),$$

where $\tilde{u}_j$ is the morphism induced by u_j. One sees at once that τ is the unique n-simplex lifting σ whose i^{th}-vertex is (M_i, x_i). This proves the claim, and finishes the proof of the lemma.

The proof of the theorem. Let $\mathcal{C}$ be a small exact category, $0 \in \operatorname{Ob}\mathcal{C}$ a null object. The category of covering spaces of $BQ\mathcal{C}$ is equivalent to the category $\underline{\mathcal{F}}$ of functors $F : Q\mathcal{C} \to \underline{\text{Set}}$ such that $F(u)$ is a bijection for every arrow u of $Q\mathcal{C}$.

Let $\underline{\mathcal{F}}' \subset \underline{\mathcal{F}}$ be the full subcategory consisting of functors $F : Q\mathcal{C} \to \underline{\text{Set}}$ with

$$F(M) = F(0),\ F(i_!) = \mathrm{Id}_{F(0)},$$

for any admissible monomorphism $i : M' \to M$ in $\mathcal{C}$.

We claim that $\underline{\mathcal{F}}'$ and $\underline{\mathcal{F}}$ are equivalent categories. In fact, if $F \in \operatorname{Ob}\underline{\mathcal{F}}$ is an arbitary functor, let $\bar{F} \in \operatorname{Ob}\underline{\mathcal{F}}'$ be the functor given by $\bar{F}(M) = F(0)$. If $M \xleftarrow{q} M' \xrightarrow{i} N$ represents an arrow $u : M \to N$ in $Q\mathcal{C}$, let

$$\bar{F}(u) = F(i_{M''!})^{-1} \circ F(q^!) \circ F(i_{M!}) : F(0) \overset{F(i_{M!})}{\to} F(M) \overset{F(q^!)}{\to} F(M') \overset{F(i_{M'!})^{-1}}{\to} F(0),$$

where for any $M \in \operatorname{Ob}\mathcal{C}$, we have $i_M : 0 \to M$, $q_M : M \longrightarrow 0$. Since $F(i_{M!})$ is an isomorphism in the category $\underline{\text{Set}}$, it is clear that $M \mapsto F(i_{M!})$ gives a natural transformation $\bar{F} \to F$ which is an isomorphism of functors. Thus every object of $\underline{\mathcal{F}}$ is isomorphic to an object of $\underline{\mathcal{F}}'$, and $\underline{\mathcal{F}}$ is equivalent to $\underline{\mathcal{F}}'$.

Now, to prove the theorem, it suffices to show that $\underline{\mathcal{F}}'$ is equivalent to the category of $K_0(\mathcal{C})$-sets. (By definition, a $K_0(\mathcal{C})$-set is a set on which $K_0(\mathcal{C})$ acts through permutations.)

Step 1. By the lemma above, we know that the category $BK_0(\mathcal{C})$ of covering spaces of the classifying space of the group $K_0(\mathcal{C})$ is equivalent to the category of $K_0(\mathcal{C})$-sets.

On the other hand, the universal cover $\widetilde{BK_0(\mathcal{C})}$ is an initial object in the category of covering spaces of $BK_0(\mathcal{C})$, and the automorphism group of $\widetilde{BK_0(\mathcal{C})}$ in the category of covering spaces is just $K_0(\mathcal{C})$, the fundamental group of $BK_0(\mathcal{C})$. Hence the category of covering spaces of $BQ\mathcal{C}$ also has an initial object whose automorphism group is $K_0(\mathcal{C})$.

Step 2. Define a functor $\underline{K_0(\mathcal{C}) - \text{Sets}} \to \underline{\mathcal{F}}'$.

Let S be a $K_0(\mathcal{C})$-set with $\phi : K_0(\mathcal{C}) \to \operatorname{Aut}(S)$ the permutation representation. Then we can define a functor $F_S : Q\mathcal{C} \to \underline{\text{Set}}$ by means of Lemma II.3 with the following assignments: $F_S(M) = S$ for any $M \in \operatorname{Ob}\mathcal{C}$, $(F_S)_1(i_!) = \mathrm{Id}_S$; $(F_S)_2(q^!) = \phi([\operatorname{Ker} q]) \in \operatorname{Aut}(S)$. We must show that

$$\phi([\operatorname{Ker}(q' \circ q)]) = \phi([\operatorname{Ker} q])\,\phi([\operatorname{Ker} q']).$$

But it is rather obvious. Finally, if

$$\begin{array}{ccc} M' & \xrightarrow{i} & N \\ q\downarrow & & \downarrow q' \\ M & \xrightarrow{i'} & N' \end{array}$$

is a bi-cartesian square, then $\operatorname{Ker} q \simeq \operatorname{Ker} q'$, so that $\phi([\operatorname{Ker} q]) = \phi([\operatorname{Ker} q'])$, and the conditions of Lemma II.3 hold. Therefore we do have a functor

$$\begin{array}{ccc} \underline{K_0(\mathcal{C}) - \text{Set}} & \to & \underline{\mathcal{E}}' \\ S & \mapsto & F_S. \end{array}$$

Step 3. Define a functor $\underline{\mathcal{E}}' \to \underline{K_0(\mathcal{C}) - \text{Set}}$.

For $F \in \operatorname{Ob} \underline{\mathcal{E}}'$, let $\phi_F : K_0(\mathcal{C}) \to \operatorname{Aut}(F(0))$ be given by $\phi_F([M]) = F(q^!_M)$. We can check that this gives a well-defined homomorphism on $K_0(\mathcal{C})$: for a given exact sequence in $\mathcal{C}$

$$0 \to M' \xrightarrow{i} M \xrightarrow{q} M'' \to 0,$$

there is the bicartesian square

$$\begin{array}{ccc} M' & \xrightarrow{i} & M \\ q_{M'} \downarrow & & \downarrow q \\ 0 & \xrightarrow{i_{M''}} & M'' \end{array}$$

for which $q^! \circ i_{M''!} = i_! \circ q^!_M$. Hence $F(q^!_{M'}) = F(q^!)$. Further, $q^!_M = q^! \circ q^!_{M''}$ implies

$$F(q^!_M) = F(q^! \circ q^!_{M''}) = F(q^!_{M'}) \circ F(q^!_{M''}).$$

So, by considering the split exact sequences

$$0 \to M' \to M' \oplus M'' \to M'' \to 0$$

$$0 \to M'' \to M' \oplus M'' \to M' \to 0$$

we see that $F(q^!_{M'}), F(q^!_{M''}) \in \operatorname{Aut}(F(0))$ commute. Hence ϕ_F is well defined.

Clearly, $(S, \phi) \mapsto F_S$ and $F \mapsto (F(0), \phi_F)$ give the desired equivalence of categories. This proves the Theorem.

II.8.4. A Definition Of Higher Arithmetic K-Groups

In this section, we make a definition of arithmetic K-theory, and obtain some of its elementary but most important properties. The discussion is based on that of the last section.

II.8.4.a. Arithmetic K-Groups

We begin by recalling the definition of the arithmetic K-group, $K_0^{\text{Ar}}(X)$, for an arithmetic variety X over an arithmetic ring A, following [GS 91]. Later, we use the Quillen construction to construct this group in another way.

Let X be an arithmetic variety over an arithmetic ring $A = (A, \Sigma, F_\infty)$. An hermitian vector bundle on X is a pair $(\mathcal{E}, \rho)$, where $\mathcal{E}$ a vector bundle on X, and ρ is an F_∞-invariant hermitian metric on the pull-back vector bundle of $\mathcal{E}$ over $X(\mathbf{C})$. The **arithmetic K-group** $K_0^{\mathrm{Ar}}(X)$ is defined to be the quotient group of the abelian group generated by $((\mathcal{E}, \rho); \eta)$, where $(\mathcal{E}, \rho)$ is a hermitian vector bundle on X, and $\eta \in \tilde{A}(X_{\mathbf{R}})$ an F_∞-invariant C^∞ form on $X(\mathbf{C})$, with the subgroup generated by the following relations: For any short exact sequence of vector bundles on X,

$$\mathcal{E}.: \quad 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

if ρ_i is F_∞-invariant hermitian metric on the pull-back of $\mathcal{E}_i$ over $X(\mathbf{C})$ for each i, then

$$((\mathcal{E}_1, \rho_1), \eta_1) + ((\mathcal{E}_3, \rho_3), \eta_3) = ((\mathcal{E}_2, \rho_2), \eta_1 + \eta_3 - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.)).$$

Here $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.)$ denotes the classical Bott-Chern secondary characteristic form associated with the hermitian vector bundle complex on $X(\mathbf{C})$ corresponding to the exact sequence $\mathcal{E}.$ on X, with respect to the Chern characteristic form ch.

There are several properties for $K_0^{\mathrm{Ar}}(X)$: From Chapter 2 and Chapter 3, we have the following

Theorem. Let X be an arithmetic variety over an arithmetic ring (A, Σ, F_∞). Then there is a natural λ-ring structure on $K_0^{\mathrm{Ar}}(X)$ such that if $K_0^{\mathrm{Ar},(p)}(X)$ is the eigen-space of the associated Adams operator φ^k with eigen-values k^p, then for each $p \geq 0$,

$$\mathrm{ch}_{\mathrm{Ar}} : K_0^{\mathrm{Ar},(p)}(X) \to \mathrm{CH}^{\mathrm{p}}_{\mathrm{Ar}}(X)_{\mathbf{Q}}$$

is an isomorphism.

The basic idea to prove this theorem is to use the five lemma. There exists an exact sequence

$$\oplus_{p \geq 1} \mathrm{CH}^{p,p-1}(X)_{\mathbf{Q}} \to \tilde{A}(X_{\mathbf{R}}) \to \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}} \to \mathrm{CH}(X)_{\mathbf{Q}} \to 0.$$

Therefore, it is natural to have the following

Theorem. For any arithmetic variety X over an arithmetic ring (A, Σ, F_∞), there is a natural exact sequence

$$K_1(X)_{\mathbf{Q}} \to \tilde{A}(X_{\mathbf{R}}) \to K_{\mathrm{Ar}}(X)_{\mathbf{Q}} \to K(X)_{\mathbf{Q}} \to 0,$$

and a natural local Chern character

$$\mathrm{ch} : K_1(X) \to \oplus_{p \geq 1} \mathrm{CH}^{p,p-1}(X)_{\mathbf{Q}},$$

such that the following diagram commutes:

$$\begin{array}{ccccccccc}
K_1(X)_{\mathbf{Q}} & \to & \tilde{A}(X_{\mathbf{R}}) & \to & K_{\mathrm{Ar}}(X)_{\mathbf{Q}} & \to & K(X)_{\mathbf{Q}} & \to & 0 \\
\mathrm{ch} \downarrow & & \mathrm{Id} \downarrow & & \mathrm{ch}_{\mathrm{Ar}} \downarrow & & \mathrm{ch} \downarrow & & \\
\oplus_{p \geq 1} \mathrm{CH}^{p,p-1}(X)_{\mathbf{Q}} & \to & \tilde{A}(X_{\mathbf{R}}) & \to & \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}} & \to & \mathrm{CH}(X)_{\mathbf{Q}} & \to & 0.
\end{array}$$

In particular, we see that $\mathrm{ch}_{\mathrm{Ar}}$ is an isomorphism.

Putting these theorems together, we see that

(1). The arithmetic K-group works like the algebraic K-group.
(2). At the lowest level, there is a triangle relation as follows:

$$\begin{array}{ccc} H_\bullet^{\mathrm{An}}(*) & \overset{N}{\to} & K_\bullet^{\mathrm{Ar}}(X) \\ & R \nwarrow \quad \swarrow F & \\ & K_*(X) & \end{array}.$$

Here $H_\bullet^{\mathrm{An}}(*)$ denotes certain analytic homology groups, N is the natural induced map, F denotes the forgetful map, and R means certain regulators, which are generalizations of the classical regulator maps.

II.8.4.b. The Taips Category Of An Arithmetic Variety

Let X be an arithmetic variety over an arithmetic ring (A, F_∞, Σ). There are two possible ways to define the objects of the taips category to be constructed. The first is that the objects consist of prime elements, i.e. hermitian vector bundles $(\mathcal{E}, \rho)$ on X, together with the classes of F_∞-invariant differential forms, modulo the exact forms about ∂, $\bar{\partial}$. If so, it is difficult to study the classical Bott-Chern secondary characteristic forms. Hence, it is natural to make the other choice: we let the objects be triples $(\mathcal{E}, \rho; \eta)$, where $(\mathcal{E}, \rho)$ is a hermitian vector bundle over X, and η is a differential form in $\tilde{A}(X_{\mathbf{R}})$. One may think naively that we can define a "sequence"

$$0 \to (\mathcal{E}_1, \rho_1; \eta_1) \to (\mathcal{E}_2, \rho_2; \eta_2) \to (\mathcal{E}_3, \rho_3; \eta_3) \to 0$$

to be exact if

(1). The induced maps on the vector bundles give the short exact sequence

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0;$$

(2). $\eta_2 - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.) = \eta_1 + \eta_3$.

Then, one may try to go further and say that this definition gives us an exact category and make the Quillen construction possible and hence we could reconstruct the arithmetic K-group as the Grothendieck group of this exact category. Unfortunately, this does not work well because there is no exact category at this level. (The reason will be seen later.)

From now on, we assume that the objects of the taips category associated with an arithmetic variety X are triples $(\mathcal{E}, \rho; \eta)$ as above. We next define morphisms between objects and, for this purpose, we find out that the terminology 'essential' is quite useful. (Here 'essential' means that, basically and theoretically, the whole story works just because we have them, but they do not work in practice.) Following the suggestion of A. J. Berrick, we call the sequence

$$0 \to (\mathcal{E}_1, \rho_1; \eta_1) \to (\mathcal{E}_2, \rho_2; \eta_2) \to (\mathcal{E}_3, \rho_3; \eta_3) \to 0$$

an **essential short exact sequence** if

(1). The induced map on vector bundles gives a short exact sequence

$$\mathcal{E}.: 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0;$$

(2). $\eta_2 - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.) = \eta_1 + \eta_3$.

As a special case,

$$(\mathcal{E}, \rho; \eta) \to (\mathcal{F}, \tau; \omega)$$

is said to be an **essential isomorphism** if

(1). The induced map on vector bundles is an isomorphism $\mathcal{E} \simeq \mathcal{F}$.
(2). $\omega - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \mathcal{F}; \rho, \tau) = \eta$.

It is clear that

$$(\mathcal{E}, \rho; \eta) \to (\mathcal{F}, \tau; \omega)$$

is an essential isomorphism if and only if there is an essential short exact sequence

$$0 \to (0, 0; 0) \to (\mathcal{E}, \rho; \eta) \to (\mathcal{F}, \tau; \omega) \to 0.$$

We now make a detailed analysis of the possible morphisms among objects. To explain this, we consider the situation for abelian groups: suppose $\alpha : E \to F$ is a morphism of abelian groups E, F. Then we have the following diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \uparrow \\
\mathrm{Ker}(\alpha) & & \mathrm{Coker}(\alpha) \\
\downarrow & & \uparrow \\
E & \xrightarrow{\alpha} & F \\
\downarrow & & \uparrow \\
E/\mathrm{Ker}(\alpha) & \simeq & \mathrm{Im}(\alpha) \\
\downarrow & & \uparrow \\
0 & & 0,
\end{array}$$

where the columns are exact and the row at the bottom is an isomorphism. For any two objects $(\mathcal{E}, \rho; \eta)$, $(\mathcal{F}, \tau; \omega)$, we shall call the diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \uparrow \\
(\mathrm{Ker}(\alpha), \rho|; \eta_3(\alpha)) & & (\mathrm{Coker}(\alpha), \bar{\tau}; \omega^3(\alpha)) \\
\downarrow & & \uparrow \\
(\mathcal{E}, \rho; \eta) & \xrightarrow{(\alpha, \eta_3(\alpha))} & (\mathcal{F}, \tau; \omega) \\
\downarrow & & \uparrow \\
(\mathcal{E}/\mathrm{Ker}(\alpha), \bar{\rho}; \eta_1(\alpha)) & \simeq & (\mathrm{Im}(\alpha), \tau|; \omega^1(\alpha)) \\
\downarrow & & \uparrow \\
0 & & 0
\end{array}$$

a **general morphism**, where the two columns are essential short exact sequences, the row at the bottom is an essential isomorphism, $|$ means the restriction, and $^{-}$ means the quotient.

In particular, for general morphisms, we have the following very important relation between $\eta_3(\alpha)$ and $\omega^3(\alpha)$:

$$\eta_3(\alpha) - \omega^3(\alpha) = \eta - \omega + \mathrm{ch}_{\mathrm{BC}}(\alpha : \mathcal{E} \to \mathcal{F}; \eta, \tau),$$

where $\mathrm{ch}_{\mathrm{BC}}(\alpha : \mathcal{E} \to \mathcal{F}; \eta, \tau)$ denotes the classical Bott-Chern secondary characteristic form associated with the natural exact sequence

$$0 \to (\mathrm{Ker}(\alpha), \rho|) \to (\mathcal{E}, \rho) \xrightarrow{\alpha} (\mathcal{F}, \tau) \to (\mathrm{Coker}(\alpha), \bar{\tau}) \to 0.$$

Remark 1. We will see later that morphisms, in the taips category of an arithmetic variety X, are not the general morphisms: Roughly speaking, a morphism in the associated taips category is a special general morphism.

Remark 2. For any general morphism as above, from the definition of an essential exact sequence, we can easily determine $\eta_1(\alpha)$, $\omega^1(\alpha)$, and $\omega^3(\alpha)$ in terms of the corresponding classical Bott-Chern secondary characteristic forms and $\eta_3(\alpha)$. In particular, if the objects and the morphism for vector bundles are given, then $\eta_1(\alpha)$, $\omega^1(\alpha)$, and $\omega^3(\alpha)$ are uniquely determined by $\eta_3(\alpha)$.

We define the **kernel** of a general morphism $(\alpha, \eta_3(\alpha))$ as $(\mathrm{Ker}(\alpha), \rho|; \eta_3(\alpha))$. Similarly, we define the **cokernel** of a general morphism $(\alpha, \eta_3(\alpha))$ as $(\mathrm{Coker}(\alpha), \bar{\tau}; \omega^3(\alpha))$. For convenience, we call $(\mathcal{E}/\mathrm{Ker}(\alpha), \bar{\rho}; \eta_1(\alpha))$ the **quotient** of $(\alpha, \eta_3(\alpha))$, and $(\mathrm{Im}(\alpha), \tau|; \omega^1(\alpha))$ the **image** of $(\alpha, \eta_3(\alpha))$. Also, a general morphism $(\alpha, \eta_3(\alpha))$ is a **general monomorphism** if its kernel is $(0,0;0)$. Dually, a general morphism $(\alpha, \eta_3(\alpha))$ is a **general epimorphism** if its cokernel is $(0,0;0)$. With these definitions, we see that an essential isomorphism is a general morphism which is both a general monomorphism and a general epimorphism. Further, there is an essential short exact sequence, which comes from a general monomorphism α followed by a general epimorphism β, for which the kernel of β is essentially isomorphic to the quotient of α.

Before we make the definition for morphisms, we give the properties of the category to be constructed, which are needed to apply Quillen's construction. For brevity, we denote this category by $\mathcal{T}_{\mathrm{Ar}}(X)$, and call it the **taips category** determined by the arithmetic variety X, even though, at this moment, we do not have the definition of morphisms in $\mathcal{T}_{\mathrm{Ar}}(X)$.

A summary of the last section shows that in order to use the Quillen construction for $\mathcal{T}_{\mathrm{Ar}}(X)$, among others, we need the following items:

(i). A zero element 0 for the category. That is, an object 0 such that for any object $(\mathcal{E}, \rho; \eta)$, there is a unique morphism from 0 to $(\mathcal{E}, \rho; \eta)$, and a unique morphism from $(\mathcal{E}, \rho; \eta)$ to 0.

(ii). For any two objects $(\mathcal{E},\rho;\eta)$, $(\mathcal{F},\tau;\omega)$, the set $\mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}}((\mathcal{E},\rho;\eta),(\mathcal{F},\tau;\omega))$ must be an abelian group.

(iii). There is a composition for morphisms

$$\mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}^{\cdot}}((\mathcal{E},\rho;\eta),(\mathcal{F},\tau;\omega)) \times \mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}}((\mathcal{F},\tau;\omega),(\mathcal{G},\nu;\chi)) \\ \to \mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}}((\mathcal{E},\rho;\eta),(\mathcal{G},\nu;\chi)),$$

denoted by

$$(f,g) \mapsto g \circ f,$$

such that the composition is bi-linear with respect to the (group) addition of Hom. That is, whenever it makes sense,

$$g \circ (f + f') = g \circ f + g \circ f', \quad (g + g') \circ f = g \circ f + g' \circ f.$$

(iv). Whenever it makes sense,

$$\mathrm{Id} \circ f = f, \quad g \circ \mathrm{Id} = g.$$

(v). There are definitions of monomorphisms and epimorphisms so that

$$\mathrm{Mono} \circ \mathrm{Mono} \in \mathrm{Mono}, \quad \mathrm{Epi} \circ \mathrm{Epi} \in \mathrm{Epi}.$$

That is, the composition of two monomorphisms is a monomorphism, and the composition of two epimorphisms is an epimorphism.

(vi). There are certain special pull-back constructions in $\mathcal{T}_{\mathrm{Ar}}(X)$. More precisely, we need the pull-back of monomorphisms with respect to epimorphisms.

Remark 3. In (vi), we drop out the condition that the bi-product should exist, which is apparently needed for Quillen's construction. In order to explain (vi), we look at the situation in set theory: there the pull-back is defined by the diagram

$$\begin{array}{ccc} M \times_Q N & \to & N \\ \downarrow & & \downarrow p \\ M & \xrightarrow{q} & Q. \end{array}$$

Furthermore, the set $M \times_N Q$ is constructed as the set $\{(x,y) \in M \times N : p(x) = q(y)\}$. Therefore the essential point for the bi-product in the Quillen construction is that the bi-product has its very important consequence to create the pull-back. In this sense, we may omit the axiom for the bi-product. Also by considering the Quillen construction, we know that we do not need to have a general pull-back construction. What we need is the special pull-back of monomorphisms with respect to epimorphisms.

Once we make all these items clear, we can discuss how to introduce morphisms for $\mathcal{T}_{\mathrm{Ar}}(X)$.

First, for any object $(\mathcal{E}, \rho; \eta)$, quite naturally, we can introduce the **general identity morphism** for $(\mathcal{E}, \rho; \eta)$ by letting it be $(\mathrm{Id}, 0)$. That is, we have the following diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \uparrow \\ (0,0;0) & & (0,0;0) \\ \downarrow & & \uparrow \\ (\mathcal{E},\rho;\eta) & \xrightarrow{(\mathrm{Id},0)} & (\mathcal{E},\rho;\eta) \\ \downarrow & & \uparrow \\ (\mathcal{E},\rho;\eta) & \overset{(\mathrm{Id},0)}{\simeq} & (\mathcal{E},\rho;\eta) \\ \downarrow & & \uparrow \\ 0 & & 0. \end{array}$$

There is an obvious choice for the zero object of $\mathcal{T}_{\mathrm{Ar}}(X)$ viz. the object $(0,0;0)$, where the first 0 is the zero bundle, the second 0 is the zero metric, while the third 0 is the zero form. We need to show that $(0,0;0)$ is an initial object and is also a finial object. Since we have not yet made a definition for morphisms, we cannot consider this now. However, we shall still think of $(0,0;0)$ as the zero object, and call $(0,0;0)$ the **general zero element**. Hence we also call $(0;\eta)$ the **general zero morphism** for any two objects $(\mathcal{E}, \rho; \eta)$, $(\mathcal{F}, \tau; \omega)$. That is, we have the following diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \uparrow \\ (\mathcal{E},\rho;\eta) & & (\mathcal{F},\tau;\omega) \\ \downarrow & & \uparrow \\ (\mathcal{E},\rho;\eta) & \xrightarrow{(0,\eta)} & (\mathcal{F},\tau;\omega) \\ \downarrow & & \uparrow \\ (0,0;0) & \simeq & (0,0;0) \\ \downarrow & & \uparrow \\ 0 & & 0. \end{array}$$

Now note that only four types of morphisms are needed in the items above, i.e. the zero morphism, the identity morphism, monomorphisms, and epimorphisms, so we can begin with a detailed discussion for them in the sense of general morphisms. In particular, we pay special attention to the structure of $\eta_3(\alpha)$. So it is convenient to recall the following:

For a general morphism

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \uparrow \\ (\mathrm{Ker}(\alpha),\rho|;\eta_3(\alpha)) & & (\mathrm{Coker}(\alpha),\bar{\tau};\omega^3(\alpha)) \\ \downarrow & & \uparrow \\ (\mathcal{E},\rho;\eta) & \xrightarrow{(\alpha,\eta_3(\alpha))} & (\mathcal{F},\tau;\omega) \\ \downarrow & & \uparrow \\ (\mathcal{E}/\mathrm{Ker}(\alpha),\bar{\rho};\eta_1(\alpha)) & \simeq & (\mathrm{Im}(\alpha),\tau|;\omega^1(\alpha)) \\ \downarrow & & \uparrow \\ 0 & & 0, \end{array}$$

we have

$$\eta_3(\alpha) - \omega^3(\alpha) = \eta - \omega + \mathrm{ch}_{\mathrm{BC}}(\alpha : \mathcal{E} \to \mathcal{F}; \eta, \tau).$$

In the following, we denote $\mathrm{ch}_{\mathrm{BC}}(\alpha : \mathcal{E} \to \mathcal{F}; \eta, \tau)$ by $\mathrm{ch}_{\mathrm{BC}}(\alpha)$.

We discuss the above situations case by case.

a. The general zero morphism is given by

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \uparrow \\ (\mathcal{E}, \rho; \eta) & & (\mathcal{F}, \tau; \omega) \\ \downarrow & & \uparrow \\ (\mathcal{E}, \rho; \eta) & \xrightarrow{0} & (\mathcal{F}, \tau; \omega) \\ \downarrow & & \uparrow \\ (0,0;0) & \simeq & (0,0;0) \\ \downarrow & & \uparrow \\ 0 & & 0. \end{array}$$

So $\eta_3(0) = \eta$.

b. The general identity morphism is given by

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \uparrow \\ (0,0;0) & & (0,0;0) \\ \downarrow & & \uparrow \\ (\mathcal{E}, \rho; \eta) & \xrightarrow{(\mathrm{Id},0)} & (\mathcal{E}, \rho; \eta) \\ \downarrow & & \uparrow \\ (\mathcal{E}, \rho; \eta) & \simeq & (\mathcal{E}, \rho; \eta) \\ \downarrow & & \uparrow \\ 0 & & 0. \end{array}$$

So $\eta_3(\mathrm{Id}) = 0$.

c. A general monomorphism is given by

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \uparrow \\ (0,0;0) & & (\mathrm{Coker}(\alpha), \bar{\tau}; \omega^3(\alpha)) \\ \downarrow & & \uparrow \\ (\mathcal{E}, \rho; \eta) & \xrightarrow{(\alpha, \eta_3(\alpha))} & (\mathcal{F}, \tau; \omega) \\ \downarrow & & \uparrow \\ (\mathcal{E}, \rho; \eta) & \simeq & (\mathrm{Im}(\alpha), \tau|; \omega^1(\alpha)) \\ \downarrow & & \uparrow \\ 0 & & 0. \end{array}$$

So $\eta_3(\alpha) = 0$.

d. A general epimorphism is given by

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \uparrow \\ (\mathrm{Ker}(\alpha), \rho|; \eta - \omega + \mathrm{ch}_{\mathrm{BC}}(\alpha)) & & (0,0;0) \\ \downarrow & & \uparrow \\ (\mathcal{E}, \rho; \eta) & \stackrel{(\alpha;\eta_3(\alpha))}{\longrightarrow} & (\mathcal{F}, \tau; \omega) \\ \downarrow & & \uparrow \\ (\mathcal{E}/\mathrm{Ker}(\alpha), \bar{\rho}; \eta_1(\alpha)) & \simeq & (\mathcal{F}, \tau; \omega) \\ \downarrow & & \uparrow \\ 0 & & 0. \end{array}$$

So $\eta_3(\alpha) = \eta - \omega + \mathrm{ch}_{\mathrm{BC}}(\alpha)$.

Therefore, we find that $\eta_3(\alpha)$ in these four cases has the form $a\,\eta + b\,(\eta - \omega) + c\,\mathrm{ch}_{\mathrm{BC}}(\alpha)$ with a, b, $c \in \{0,1\}$. More precisely, we have the following observation.

a. The general zero morphism gives

$$(a,b,c) = (1,0,0).$$

b. The general identity morphism gives

$$(a,b,c) = (0,1,1).$$

c. A general monomorphism gives

$$(a,b,c) = (0,0,0).$$

d. A general epimorphism gives

$$(a,b,c) = (0,1,1).$$

In these terms, we can describe the morphism as $(\alpha; a, b, c)$, where α denotes the morphism for vector bundles in the usual sense, a, $b \in \{0,\ 1\}$ and $c \in \mathbf{Z}$. In this way, we may use an addition and a multiplication for α, $\mathbf{Z}_2$ and $\mathbf{Z}$ to define addition and composition for morphisms of the form $(\alpha; a, b, c)$. But this does not work well. For example, (a,b,c) for the zero morphism is not $(0,0,0)$, while (a,b,c) for the identity morphism is not $(1,1,1)$. So we still cannot get the corresponding items (i), ..., (vi) listed above. In order to overcome this problem, we introduce two maps from the field $\mathbf{Z}_2 = \{0, e\}$ to the set $\{0,1\} \subset \mathbf{Z}$:

$$\begin{array}{rcl} J : \{0,e\} & \to & \{0,1\}, \\ 0 & \mapsto & 0; \\ e & \mapsto & 1; \end{array}$$

and

$$\begin{array}{rcl} J' : \{0,e\} & \to & \{0,1\}, \\ 0 & \mapsto & 1; \\ e & \mapsto & 0. \end{array}$$

Main Definition A. (1). For any two objects $(\mathcal{E},\rho;\eta)$, $(\mathcal{F},\tau;\omega)$ of $\mathcal{T}_{\mathrm{Ar}}(X)$, neither of which is $(0,0;0)$, we let

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}(X)}((\mathcal{E},\rho;\eta),\ (\mathcal{F},\tau;\omega))\\ &\quad = \{(\alpha;a,b,c) : \alpha\in \mathrm{Hom}(\mathcal{E},\mathcal{F}),\ a,\, b\in \mathbf{Z}_2,\ c\in\mathbf{Z}\}.\end{aligned}$$

If either one of $(\mathcal{E},\rho;\eta)$, $(\mathcal{F},\tau;\omega)$ is $(0,0;0)$, then

$$\mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}(X)}((\mathcal{E},\rho;\eta),\ (\mathcal{F},\tau;\omega)) = \{(0;0,0,0)\},$$

where the first 0 means the zero morphism of vector bundles.

(2). Suppose

$$(\alpha;a,b,c),\ (\alpha';a',b',c')\in \mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}(X)}((\mathcal{E},\rho;\eta),\ (\mathcal{F},\tau;\omega)),$$

then

$$(\alpha;a,b,c) := (\alpha';a',b',c')$$

if and only if

$$\alpha=\alpha',\ a=a',\ b=b',\ c=c'.$$

(3). Suppose

$$(\alpha;a,b,c),\ (\alpha';a',b',c')\in \mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}(X)}((\mathcal{E},\rho;\eta),\ (\mathcal{F},\tau;\omega)),$$

then

$$(\alpha;a,b,c)+(\alpha';a',b',c') := (\alpha+\alpha';a+a',b+b',c+c'),$$

where for the right hand side, the first addition means the usual addition of morphisms of vector bundles, the second and the third additions mean the addition in the group $\mathbf{Z}_2$, while the last addition means the addition in $\mathbf{Z}$.

(4). Suppose

$$(\alpha;a,b,c),\ (\alpha';a',b',c')\in \mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}(X)}((\mathcal{E},\rho;\eta),\ (\mathcal{F},\tau;\omega)),$$

then

$$(\alpha;a,b,c)\circ(\alpha';a',b',c') := (\alpha\circ\alpha';a\circ a',b\circ b',c\circ c'),$$

where for the right hand side, the first $\circ$ means the usual composition of morphisms of vector bundles, the second and the third mean the multiplication in the field $\mathbf{Z}_2$, while the last $\circ$ means the multiplication in the integer ring $\mathbf{Z}$.

(5). Suppose

$$(\alpha;a,b,c)\in \mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}(X)}((\mathcal{E},\rho;\eta),\ (\mathcal{F},\tau;\omega))\neq\{0\},$$

then the **kernel** of the morphism $(\alpha;a,b,c)$ is defined to be

$$(\mathrm{Ker}(\alpha),\rho|;J'(a)\eta+J(b)(\eta-\omega)+c\,\mathrm{ch}_{\mathrm{BC}}(\alpha)).$$

That is, the kernel of the general morphism

$$(\alpha,J'(a)\eta+J(b)(\eta-\omega)+c\,\mathrm{ch}_{\mathrm{BC}}(\alpha)),$$

which is usually called the **associated general morphism** of $(\alpha; a, b, c)$. The **cokernel** of the morphism $(\alpha; a, b, c)$ is defined to be the cokernel of the associated general morphism

$$(\alpha, J'(a)\eta + J(b)(\eta - \omega) + c\,\mathrm{ch}_{\mathrm{BC}}(\alpha)).$$

Similarly, we define the **image** of the morphism $(\alpha; a, b, c)$ to be the image of the associated general morphism

$$(\alpha, J'(a)\eta + J(b)(\eta - \omega) + c\,\mathrm{ch}_{\mathrm{BC}}(\alpha)),$$

and the **quotient** of the morphism $(\alpha; a, b, c)$ as the quotient of the associated general morphism

$$(\alpha, J'(a)\eta + J(b)(\eta - \omega) + c\,\mathrm{ch}_{\mathrm{BC}}(\alpha)).$$

For the morphism connected with $(0,0;0)$, we define its kernel, image and quotient to be that associated with $(0;0,0,0)$.

(6). For any object $(\mathcal{E}, \rho; \eta)$, which is not $(0,0;0)$, we define the **identity morphism** of $(\mathcal{E}, \rho; \eta)$ to be $(\mathrm{Id}_{\mathcal{E}}; e, 1, 1)$.

(7). In $\mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}(X)}((\mathcal{E}, \rho; \eta),\ (\mathcal{F}, \tau; \omega))$, we define the **zero morphism** to be $(0;0,0,0)$.

(8). Suppose

$$(\alpha; a, b, c) \in \mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}(X)}((\mathcal{E}, \rho; \eta),\ (\mathcal{F}, \tau; \omega)) \neq \{0\}.$$

Then $(\alpha; a, b, c)$ is an **isomorphism** if

(a). $(a, b, c) = (e, 1, 1)$.

(b). α is an isomorphism of vector bundles.

(c). The kernel of $(\alpha; a, b, c)$ is $(0,0;0)$.

In particular, an identity morphism is an isomorphism.

(9). Suppose

$$(\alpha; a, b, c) \in \mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}(X)}((\mathcal{E}, \rho; \eta),\ (\mathcal{F}, \tau; \omega)) \neq \{0\}.$$

Then $(\alpha; a, b, c)$ is a **monomorphism** if it is the identity morphism, or it is an isomorphism, or it satisfies the following condition

(a). $\mathcal{E}$ is a subvector sheaf of $\mathcal{F}$ and α is the natural inclusion.

(b). $(a, b, c) = (e, 0, 0)$.

For convenience, all morphisms starting from $(0,0;0)$ are also called monomorphisms of $\mathcal{T}_{\mathrm{Ar}}(X)$.

(10). Suppose

$$(\alpha; a, b, c) \in \mathrm{Hom}_{\mathcal{T}_{\mathrm{Ar}}(X)}((\mathcal{E}, \rho; \eta),\ (\mathcal{F}, \tau; \omega)) \neq \{0\}.$$

Then $(\alpha; a, b, c)$ is an **epimorphism** if

(a). α is a surjective morphism of vector bundles.

(b). $(a, b, c) = (e, 1, 1)$.

For convienence, all morphisms which end with $(0,0;0)$ are also called epimorphisms of $\mathcal{T}_{\mathrm{Ar}}(X)$. In particular, an isomorphism is an epimorphism.

(11). The sequence from $\mathcal{T}_{\mathrm{Ar}}(X)$

$$(0,0;0) \to (\mathcal{E}_1, \rho_1; \eta_1) \overset{(\alpha_1; a_1, b_1, c_1)}{\longrightarrow} (\mathcal{E}_2, \rho_2; \eta_2) \overset{(\alpha_2; a_2, b_2, c_2)}{\longrightarrow} (\mathcal{E}_3, \rho_3; \eta_3) \to (0,0;0)$$

is a **short exact sequence**, if

(a). $(\alpha_1; a_1, b_1, c_1)$ is a monomorphism.

(b). $(\alpha_2; a_2, b_2, c_2)$ is an epimorphism.

(c). The associated sequence for general morphisms is an essential short exact sequence.

(12). With the above definition, we call $\mathcal{T}_{\mathrm{Ar}}(X)$ the **taips category** associated with the arithmetic variety X.

Proposition. (1). The category $\mathcal{T}_{\mathrm{Ar}}(X)$, satisfies the items (i), (ii), (iii), (iv), and (v), which are need for the Quillen construction.

(2). If $(\alpha; a, b, c)$ is a monomorphism, then its associated general morphism is a general monomorphism; In particular, the kernel is $(0, 0; 0)$.

(3). If $(\alpha; a, b, c)$ is an epimorphism, then its associated general morphism is a general epimorphism; in particular, the cokernel is $(0, 0; 0)$.

(4). If $(\alpha; a, b, c)$ is the identity morphism, then its associated general morphism is the general identity morphism.

(5). If $(\alpha; a, b, c)$ is an isomorphism, then its associated general isomorphism is an essential isomorphism.

This proposition is easily proved from the definition. We leave it to the reader. The proposition tells us that the definitions above give the essential properties which are needed for an arithmetic K-theory. In particular, the definition does include the action of the classical Bott-Chern secondary characteristic forms for exact sequences.

Thus, to apply the Quillen construction, what we need now is the construction of a special pull-back for monomorphisms with respect to epimorphisms. Before we give this pull-back construction, we have to find the necessary conditions for it. Start with the picture

$$\begin{array}{ccc} & & (\mathcal{E}_1, \rho_1; \eta_1) \\ & & (\alpha_1; a_1, \downarrow b_1, c_1) \\ (\mathcal{E}_3, \rho_3; \eta_3) & \xrightarrow{(\alpha_3; a_3, b_3, c_3)} & (\mathcal{E}_2, \rho_2; \eta_2) \end{array} ,$$

where $(\alpha_1; a_1, b_1, c_1)$ is an epimorphism and $(\alpha_3; a_3, b_3, c_3)$ is a monomorphism, we need to find an object $(\mathcal{E}_4, \rho_4; \eta_4)$, an epimorphism $(\alpha_2; a_2, b_2, c_2)$, and a monomorphism $(\alpha_4; a_4, b_4, c_4)$ so that we can complete the diagram as follows:

$$\begin{array}{ccc} (\mathcal{E}_4, \rho_4; \eta_4) & \xrightarrow{(\alpha_4; a_4, b_4, c_4)} & (\mathcal{E}_1, \rho_1; \eta_1) \\ (\alpha_2; a_2, \downarrow b_2, c_2) & & (\alpha_1; a_1, \downarrow b_1, c_1) \\ (\mathcal{E}_3, \rho_3; \eta_3) & \xrightarrow{(\alpha_3; a_3, b_3, c_3)} & (\mathcal{E}_2, \rho_2; \eta_2). \end{array}$$

Furthermore, we need the following additional very strong condition: there is a natural essential isomorphism

$$\mathrm{Ker}(\alpha_2; a_2, b_2, c_2) \simeq \mathrm{Ker}(\alpha_1; a_1, b_1, c_1),$$

which will be needed in the proof of the main theorem. There are basically three different situations. The first is that in which

$$(\mathcal{E}_2, \rho_2; \eta_2) = (0, 0; 0).$$

In this situation, by the additional condition above, we let $(\mathcal{E}_4, \rho_4; \eta_4)$ be the kernel of the epimorphism $(\alpha_1; a_1, b_1, c_1)$. The associated morphisms are easily determined and we have the following picture.

$$\begin{array}{ccc} (\mathrm{Ker}(\alpha_1), \rho_1|; \eta_1 - \eta_2 + \mathrm{ch}_{\mathrm{BC}}(\alpha_1)) & \xrightarrow{(\alpha_4; a_4, b_4, c_4)} & (\mathcal{E}_1, \rho_1; \eta_1) \\ (\alpha_2; a_2, \downarrow b_2, c_2) & & (\alpha_1; a_1, \downarrow b_1, c_1) \\ (0,0;0) & \xrightarrow{(\alpha_3; a_3, b_3, c_3)} & (\mathcal{E}_2, \rho_2; \eta_2), \end{array}$$

where α_4 is the natural inclusion. Note that once we know the objects and the morphisms for corresponding vector bundles, then the associated epimorphism or the associated monomorphism is easily determined by definition.

Now we come to the second situation, in which

$$(\mathcal{E}_2, \rho_2; \eta_2) \neq (0,0;0),$$

and the corresponding monomorphism is not an isomorphism. In this case, for vector bundles, the answer is natural: we take $\mathcal{E}_4$ as the pull-back of $\mathcal{E}_3$ by the map α_1. Then $\mathcal{E}_4$ is a subvector bundle of $\mathcal{E}_1$. The associated morphisms for vector bundles are also easily determined: α_4 is just the natural inclusion, while α_2 is the surjective induced by α_1. Next, we have to choose a metric on $\mathcal{E}_4$, but this is also quite obvious, since we may choose the restriction metric from $\mathcal{E}_1$. Thus, for $(\mathcal{E}_4, \rho_4; \eta_4)$, the only term we need to determine is the differential form η_4. But this is also not very difficult. In fact, from the additional condition about the kernel, which now has a definite meaning, the isomorphism of vector sheaves between the kernel of α_2 and the kernel of α_1 is determined. So, by the definition of the essential short exact sequence, we have a unique choice for the form η_4.

Finally, we discuss the third case, in which the corresponding monomorphism is an isomorphism. So we have $\mathcal{E}_4 = \mathcal{E}_1 \times_{\mathcal{E}_2} \mathcal{E}_3$ with the corresponding hermitian metric $\rho_1 \times_{\rho_2} \rho_3$. Note that now α_3 is an isomorphism, we see that all these make sense. Let α_4 be the first projection, which is an isomorphism, and α_2 be the second projection. Put $(a_2, b_2, c_2) = (a_4, b_4, c_4) = (e, 1, 1)$. Thus for the definition, the final problem is to find out η_4. For doing so, we only need to put the condition that $(\alpha_2; a_4, b_4, c_4)$ is an isomorphism in $\mathcal{T}_{\mathrm{Ar}}(X)$. With this definition, we also need to show that the relation for the kernels holds: Recall that the classical Bott-Chern secondary characteristic forms associated to a short exact sequence of hermitian vector bundles may be constructed by using the $\mathbf{P}^1$-deformation, so twisting by an isometry does not change this secondary form.

We now have

Proposition-Definition. For any picture

$$\begin{array}{ccc} & & (\mathcal{E}_1, \rho_1; \eta_1) \\ & & (\alpha_1; a_1, \downarrow b_1, c_1) \\ (\mathcal{E}_3, \rho_3; \eta_3) & \xrightarrow{(\alpha_3; a_3, b_3, c_3)} & (\mathcal{E}_2, \rho_2; \eta_2) \ , \end{array}$$

where $(\alpha_1; a_1, b_1, c_1)$ is an epimorphism and $(\alpha_3; a_3, b_3, c_3)$ is a monomorphism, there is a unique pull-back construction in $\mathcal{T}_{\mathrm{Ar}}(X)$:

$$\begin{array}{ccc} (\mathcal{E}_4, \rho_4; \eta_4) & \xrightarrow{(\alpha_4; a_4, b_4, c_4)} & (\mathcal{E}_1, \rho_1; \eta_1) \\ (\alpha_2; a_2, \downarrow b_2, c_2) & & (\alpha_1; a_1, \downarrow b_1, c_1) \\ (\mathcal{E}_3, \rho_3; \eta_3) & \xrightarrow{(\alpha_3; a_3, b_3, c_3)} & (\mathcal{E}_2, \rho_2; \eta_2), \end{array}$$

where $(\alpha_2; a_2, b_2, c_2)$ is an epimorphism, and $(\alpha_4; a_4, b_4, c_4)$ is a monomorphism. We call this square a **bi-cartesian square** in $\mathcal{T}_{\mathrm{Ar}}(X)$, and denote $(\mathcal{E}_4, \rho_4; \eta_4)$ by

$$(\mathcal{E}_1, \rho_1; \eta_1) \times_{(\mathcal{E}_2, \rho_2; \eta_2)} (\mathcal{E}_3, \rho_3; \eta_3).$$

Furthermore, we have

$$\mathrm{Ker}\,(\alpha_2; a_2, b_2, c_2) \simeq \mathrm{Ker}\,(\alpha_1; a_1, b_1, c_1).$$

Now we come to first main result.

Main Theorem I. For any regular arithmetic variety X, there is a natural isomorphism

$$K_0^{\mathrm{Ar}}(X) \simeq \pi_1(BQ\mathcal{T}_{\mathrm{Ar}}(X), (0, 0; 0)).$$

The proof of the main theorem is similar to that of Quillen's theorem in Chapter 3, but since the category $\mathcal{T}_{\mathrm{Ar}}(X)$ has its own special properties, we give a detailed proof below.

Proof. We begin with two lemmas.

Lemma 1. The category of covering spaces of the classifying space $BQ\mathcal{T}_{\mathrm{Ar}}(X)$ is naturally equivalent to the category of functors $F : Q\mathcal{T}_{\mathrm{Ar}}(X) \to \underline{\mathrm{Set}}$ such that $F(u)$ is a bijection for any morphism u of $Q\mathcal{T}_{\mathrm{Ar}}(X)$.

Lemma 2. Let $\mathcal{D}$ be any category. Assume that

(1). For each object M in $\mathcal{T}_{\mathrm{Ar}}(X)$, we are given only one object $F(M)$ in $\mathcal{D}$.
(2). For each monomorphism $i : M' \to M$, we are given an arrow $F_1(i) : F(M') \to F(M)$ such that if i, i' are composable $F_1(i \circ i') = F_1(i) \circ F_1(i')$.
Dually, for each epimorphism $q : M \to N$, we are given an arrow $F_2(q) : F(N) \to F(M)$ such that $F_2(q \circ q') = F_2(q') \circ F_2(q)$ if q, q' are composable;
(3). If

$$\begin{array}{ccc} M' & \xrightarrow{i} & N \\ q \downarrow & & \downarrow q' \\ M & \xrightarrow{i'} & N' \end{array}$$

is bi-cartesian, then

$$F_1(i) \circ F_2(q) = F_2(q') \circ F_1(i').$$

Then there is a well-defined functor $F : Q\mathcal{T}_{\mathrm{Ar}}(X) \to \mathcal{D}$ given by

$$M \mapsto F(M),\ (M \overset{q}{\leftarrow} M' \overset{i}{\rightarrow} N) \mapsto F_1(i) \circ F_2(q).$$

The proof of Lemma 1. Let $p : E \to BQ\mathcal{T}_{\mathrm{Ar}}(X)$ be a covering space. For any object M of $\mathcal{T}_{\mathrm{Ar}}(X)$, let $E(M)$ be the fiber over $M \in \mathrm{Ob}\, BQ\mathcal{T}_{\mathrm{Ar}}(X)$, where M is regarded as a 0-simplex in $NQ\mathcal{T}_{\mathrm{Ar}}(X)$, which determines a 0-cell in $BQ\mathcal{T}_{\mathrm{Ar}}(X)$. Given a morphism $u : M_1 \to M_2$, we regard u as a 1-simplex in $NQ\mathcal{T}_{\mathrm{Ar}}(X)$, which determines a path Bu in $BQ\mathcal{T}_{\mathrm{Ar}}(X)$ joining M_1 to M_2. Since p is a covering, it has the unique path lifting property, which gives a bijection $(Bu)_* : E(M_1) \to E(M_2)$, by associating to a point $y \in E(M_1)$ the second end-point of the unique path in E which lifts Bu and begins at y. Hence $M \mapsto E(M)$, $u \mapsto (Bu)_*$ determines a functor $Q\mathcal{T}_{\mathrm{Ar}}(X) \to \underline{\mathrm{Set}}$ carrying all arrows of $Q\mathcal{T}_{\mathrm{Ar}}$ to bijections of $\underline{\mathrm{Set}}$.

Conversely, if $F : Q\mathcal{T}_{\mathrm{Ar}}(X) \to \underline{\mathrm{Set}}$ is a morphism inverting functor, i.e. $f(u)$ is a bijection for each morphism u. Let $F\backslash Q\mathcal{T}_{\mathrm{Ar}}(X)$ be the category of pairs (M, m) with $M \in \mathrm{Ob}\, Q\mathcal{T}_{\mathrm{Ar}}(X), m \in F(M)$, where a morphism $(M, m) \to (M', m')$ is a morphism $u : M \to M'$ such that $F(u)(m) = m'$. The forgetful functor $F\backslash Q\mathcal{T}_{\mathrm{Ar}}(X) \to Q\mathcal{T}_{\mathrm{Ar}}(X)$ gives a map on classifying spaces $p_F : B(F\backslash Q\mathcal{T}_{\mathrm{Ar}}(X)) \to BQ\mathcal{T}_{\mathrm{Ar}}(X)$ with fibers $p_F^{-1}(M) = F(M)$ for any object $M \in \mathrm{Ob}\, \mathcal{T}_{\mathrm{Ar}}(X)$.

Claim. p_F is a covering space.

Suppose the claim is true. Let $u : M \to M'$ be a morphism in $Q\mathcal{T}_{\mathrm{Ar}}(X)$ and let $m \in F(M)$. If $m' = F(u)(m)$, then u determines a morphism $(M, m) \to (M', m')$ in $F\backslash Q\mathcal{T}_{\mathrm{Ar}}(X)$, which gives the unique path in $B(F\backslash Q\mathcal{T}_{\mathrm{Ar}}(X))$ lifting Bu and beginning at $m \in p_F^{-1}(M)$.

Thus the above constructions are inverse to each other, and give the desired equivalence of categories.

The proof of the claim. By a standard result from topology, we know that it is enough to show that the map of simplicial sets $N(F\backslash Q\mathcal{T}_{\mathrm{Ar}}(X)) \to NQ\mathcal{T}_{\mathrm{Ar}}(X)$ is a simplicial covering, i.e. if $\Delta(n)$ is the simplicial set $\Delta(n)(\underline{p}) = \mathrm{Hom}_{\Delta}(\underline{p}, \underline{n})$, then given any diagram of maps of simplicial sets

$$\begin{array}{ccc} \Delta(0) & \to & N(F\backslash Q\mathcal{T}_{\mathrm{Ar}}(X)) \\ \downarrow & & \downarrow \\ \Delta(n) & \to & NQ\mathcal{T}_{\mathrm{Ar}}(X), \end{array}$$

we must show that there is a unique map $\Delta(n) \to N(F\backslash Q\mathcal{T}_{\mathrm{Ar}}(X))$ of simplicial sets which makes the diagram commute. Of course, a map $\sigma : \Delta(n) \to NQ\mathcal{T}_{\mathrm{Ar}}(X)$ is just an n-simplex $\sigma \in N_nQ\mathcal{T}_{\mathrm{Ar}}(X)$, so we must show that if $\sigma \in N_nQ\mathcal{T}_{\mathrm{Ar}}(X)$ is an n-simplex of $NQ\mathcal{T}_{\mathrm{Ar}}(X)$, and $\sigma_0 \in N_0(F\backslash Q\mathcal{T}_{\mathrm{Ar}}(X))$ is a 0-simplex lying over the i^{th}-vertex of σ, then there exists a unique n-simplex $\tau \in N_n(F\backslash Q\mathcal{T}_{\mathrm{Ar}}(X))$ which maps to σ, such that σ_0 is the i^{th}-vertex of τ. Assume that σ is given by the diagram in $Q\mathcal{T}_{\mathrm{Ar}}(X)$

$$M_0 \overset{u_1}{\rightarrow} M_1 \overset{u_2}{\rightarrow} \dots \overset{u_n}{\rightarrow} M_n,$$

and the i^{th} vertex of σ is given by the object M_i, so that σ_0 is given by an object (M_i, x_i) of $F\backslash Q\mathcal{T}_{\text{Ar}}(X)$, where $x_i \in F(M_i)$. Then we have bijections

$$F(M_0) \overset{F(u_1)}{\rightarrow} F(M_1) \overset{F(u_2)}{\rightarrow} \dots \overset{F(u_n)}{\rightarrow} F(M_n),$$

which gives a composite bijection $f_j : F(M_j) \rightarrow F(M_i)$ for each j (with f_i being the identity) such that

$$f_j = f_{j+1} \circ F(u_{j+1}) : F(M_j) \overset{F(u_j)}{\rightarrow} F(M_{j+1}) \overset{f_{j+1}}{\rightarrow} F(M_i).$$

Let $m_j \in F(M_j)$ be the unique element satisfying $f_j(m_j) = m_i$. Then,

$$m_i = f_j(m_j) = f_{j+1}(F(u_{j+1})(m_j)),$$

and $m_{j+1} = F(u_{j+1})(m_j)$. Thus there is a diagram in $F\backslash Q\mathcal{T}_{\text{Ar}}(X)$,

$$(M_0, m_0) \overset{\tilde{u}_1}{\rightarrow} (M_1, m_1) \overset{\tilde{u}_2}{\rightarrow} \dots \overset{\tilde{u}_n}{\rightarrow} (M_n, m_n),$$

where $\tilde{u}_j$ is the morphism induced by u_j. This determines an element $\tau \in N_n(F\backslash Q\mathcal{T}_{\text{Ar}}(X))$. One sees at once that τ is the unique n-simplex lifting σ, whose i^{th}-vertex is (M_i, m_i). This proves that $p_F : B(F\backslash Q\mathcal{T}_{\text{Ar}}(X)) \rightarrow BQ\mathcal{T}_{\text{Ar}}(X)$ is a covering, and completes the proof of Lemma 1.

The proof of Lemma 2. If

$$\begin{array}{ccccc} (\mathcal{E}_1, \rho_1; \eta_1) & \overset{(\alpha_1; e,1,1)}{\longleftarrow} & (\mathcal{E}_0, \rho_0; \eta_0) & \overset{(\alpha_2; e,0,0)}{\longrightarrow} & (\mathcal{E}_2, \rho_2; \eta_2), \\ (\mathcal{E}_1, \rho_1; \eta_1) & \overset{(\alpha_1'; e,1,1)}{\longleftarrow} & (\mathcal{E}_0', \rho_0'; \eta_0') & \overset{(\alpha_2'; e,0,0)}{\longrightarrow} & (\mathcal{E}_2, \rho_2; \eta_2), \end{array}$$

are equivalent diagrams which give a morphism $(\mathcal{E}_1, \rho_1; \eta_1) \rightarrow (\mathcal{E}_2, \rho_2; \eta_2)$ in $Q\mathcal{T}_{\text{Ar}}(X)$, then, (by definition,) there is an isomorphism $u : \mathcal{E}_0 \rightarrow \mathcal{E}_0'$ such that

$$(\alpha_1; e, 1, 1) = (\alpha_1'; e, 1, 1) \circ (u; e, 1, 1), \ (\alpha_2; e, 0, 0) = (\alpha_2'; e, 0, 0) \circ (u; e, 1, 1).$$

So for admissible monomorphisms, we get

$$F_1(\alpha_2; e, 0, 0) = F_1(\alpha_2'; e, 0, 0) \circ F_1(u; e, 1, 1);$$

and for admissible epimorphisms, we get

$$F_2(\alpha_1; e, 1, 1) = F_2(u; e, 1, 1) \circ F_2(\alpha_1'; e, 1, 1).$$

On the other hand, from the bi-cartesian square

$$\begin{array}{ccc} (\mathcal{E}_0, \rho_0; \eta_0) & \overset{(u;e,1,1)}{\longrightarrow} & (\mathcal{E}_0', \rho_0'; \eta_0') \\ (u; e, \downarrow 1, 1) & & (\text{Id}; e, \downarrow 1, 1) \\ (\mathcal{E}_0', \rho_0'; \eta_0') & \overset{(\text{Id};e,1,1)}{\longrightarrow} & (\mathcal{E}_0', \rho_0'; \eta_0'), \end{array}$$

we have

$$F_1(u;e,1,1)\circ F_2(u;e,1,1) = F_2(\mathrm{Id};e,1,1)\circ F_1(\mathrm{Id};e,1,1) = \mathrm{Id},$$

since

$$(\mathrm{Id};e,1,1)\circ(\beta;e,0,0) = (\beta;e,0,0),$$

and

$$(\mathrm{Id};e,1,1)\circ(\beta;e,1,1) = (\beta;e,1,1).$$

Hence

$$\begin{aligned}F_1(\alpha_2;e,0,0)\circ F_2(\alpha_1;e,1,1)&\\ &=F_1(\alpha_2';e,0,0)\circ F_1(u;e,1,1)\circ F_2(u;e,1,1)\circ F_2(\alpha_1';e,1,1)\\ &=F_1(\alpha_2';e,0,0)\circ F_2(\alpha_1';e,1,1).\end{aligned}$$

Thus $F_1(\alpha_2;e,0,0)\circ F_2(\alpha_1;e,1,1)$ depends only on the arrows in $Q\mathcal{T}_{\mathrm{Ar}}(X)$, and does not depend on the particular diagram which represents it.

Next, if $M \xleftarrow{q_1} M' \xrightarrow{i_1} N$ and $N \xleftarrow{q_2} N' \xrightarrow{i_2} P$ are given, and $M \xleftarrow{q} M'\times_N N' \xrightarrow{i} P$ represents the composite arrow in $Q\mathcal{T}_{\mathrm{Ar}}(X)$, we have a diagram

$$\begin{array}{ccccc} M'\times_N N' & \xrightarrow{i'} & N' & \xrightarrow{i_2} P \\ q'\downarrow & & \downarrow q_2 & \\ M' & \xrightarrow{i_1} & N & \\ q_1\downarrow & & & \\ M, & & & \end{array}$$

where the square is bi-cartesian, $q = q_1\circ q$ and $i = i_2\circ i'$. Then

$$\begin{aligned}F_1(i)\circ F_2(q) &= F_1(i_2)\circ F_1(i')\circ F_2(q')\circ F_2(q_1)\\ &= F_1(i_2)\circ F_2(q_2)\circ F_1(i_1)\circ F_2(q_1).\end{aligned}$$

This proves that $(M \xleftarrow{q} M' \xrightarrow{i} N)\mapsto F_1(i)\circ F_2(q)$ is compatible with composition in $Q\mathcal{T}_{\mathrm{Ar}}(X)$ and so yields a well-defined functor $F : Q\mathcal{T}_{\mathrm{Ar}}(X)\to\mathcal{D}$.

Finally, we come to the proof of the main theorem.

The proof of the main theorem. By Lemma 1, the category of covering spaces of $BQ\mathcal{T}_{\mathrm{Ar}}(X)$ is equivalent to the category $\underline{\mathcal{F}}$ of functors $F : Q\mathcal{T}_{\mathrm{Ar}}(X)\to\underline{\mathrm{Set}}$ such that $F(u)$ is a bijection for every arrow u of $Q\mathcal{T}_{\mathrm{Ar}}(X)$.

Let $\underline{\mathcal{F}}'\subset\underline{\mathcal{F}}$ be the full subcategory consisting of functors $F : Q\mathcal{T}_{\mathrm{Ar}}(X)\to\underline{\mathrm{Set}}$ with $F(M) = F(0)$, $F(i_!) = \mathrm{Id}_{F(0)}$ for each admissible monomorphism $i : M'\to M$ in $\mathcal{T}_{\mathrm{Ar}}(X)$. We claim that $\underline{\mathcal{F}}'$ and $\underline{\mathcal{F}}$ are equivalent categories.

In fact, if $F\in\mathrm{Ob}\,\underline{\mathcal{F}}$ is an arbitary functor, let $\bar{F}\in\mathrm{Ob}\,\underline{\mathcal{F}}'$ be the functor given by $\bar{F}(M) = F(0)$; and if $M \xleftarrow{q} M' \xrightarrow{i} N$ represents an arrow $u : M\to N$ in $Q\mathcal{T}_{\mathrm{Ar}}(X)$, let

$$\bar{F}(u) = F(i_{M'!})^{-1}\circ F(q^!)\circ F(i_{M!}) : F(0) \xrightarrow{F(i_{M!})} F(M) \xrightarrow{F(q^!)} F(M') \xrightarrow{F(i_{M'!})^{-1}} F(0),$$

where for any $M \in \mathcal{T}_{\mathrm{Ar}}(X)$, we have that

$$i_M : 0 \to M,\ q_M : M \longrightarrow 0,$$

while for any monomorphism $(\alpha; e, 0, 0)$ and any epimorphism $(\beta; e, 1, 1)$ from $(\mathcal{E}, \rho; \eta)$ to $(\mathcal{F}, \tau; \omega)$, we denote by $(\alpha; e, 0, 0)_!$ and $(\beta; e, 1, 1)^!$ the following pictures:

$$\begin{array}{ccc} (\mathcal{E}, \rho; \eta) & \xrightarrow{(\alpha;e,0,0)} & (\mathcal{F}, \tau; \omega) \\ (\mathrm{Id}; e, \downarrow 1, 1) & & \\ (\mathcal{E}, \rho; \eta). & & \end{array}$$

and

$$\begin{array}{ccc} (\mathcal{E}, \rho; \eta) & \xrightarrow{(\mathrm{Id};e,1,1)} & (\mathcal{E}, \rho; \eta) \\ (\beta; e, \downarrow 1, 1) & & \\ (\mathcal{F}, \tau; \omega). & & \end{array}$$

One can also prove that for any two composable arrows u, v in $Q\mathcal{T}_{\mathrm{Ar}}(X)$, we have

$$F(v \circ u) = F(v) \circ (u).$$

Clearly $M \mapsto F(i_{M!})$ determines a natural transformation $\bar{F} \to F$ which is an isomorphism of functors, since $F(i_{M!})$ is an isomorphism in the category Set by the picture

$$\begin{array}{ccc} \bar{F}(M) & \xrightarrow{F(i_{M!})} & F(M) \\ \| & & \\ F(0). & & \end{array}$$

Thus every object of $\underline{\mathcal{F}}$ is isomorphic to an object of $\underline{\mathcal{F}}'$, and $\underline{\mathcal{F}}$ is equivalent to $\underline{\mathcal{F}}'$.

Claim. To prove the theorem, it suffices to show that $\underline{\mathcal{F}}'$ is equivalent to the category of $K_0^{\mathrm{Ar}}(X)$-sets.

We divide the proof of this theorem into the following 3 steps.

Step 1. The proof of the last claim.

By Lemma 1, we know that the category of covering spaces of the classifying space of the group $K_0^{\mathrm{Ar}}(X)$, $BK_0^{\mathrm{Ar}}(X)$, is equivalent to the category of $K_0^{\mathrm{Ar}}(X)$-sets.

On the other hand, the universal covering $\widetilde{BK_0^{\mathrm{Ar}}}(X)$ of $BK_0^{\mathrm{Ar}}(X)$ is an initial object in the category of covering spaces of $BK_0^{\mathrm{Ar}}(X)$, and the automorphism group of $\widetilde{BK_0^{\mathrm{Ar}}(X)}$ in the category of covering spaces is just $K_0^{\mathrm{Ar}}(X)$, the fundamental group of $BK_0^{\mathrm{Ar}}(X)$. Hence the category of covering spaces of $BQ\mathcal{T}_{\mathrm{Ar}}(X)$ also has an initial object whose automorphism group is $K_0^{\mathrm{Ar}}(X)$. Therefore, if $\underline{\mathcal{F}}'$ is equivalent to the category of $K_0^{\mathrm{Ar}}(X)$-sets, we have

$$K_0^{\mathrm{Ar}}(X) \simeq \pi_1(BQ\mathcal{T}_{\mathrm{Ar}}(X), (0, 0; 0)).$$

Step 2. Define a functor $\underline{K_0^{\mathrm{Ar}}(X)-\mathrm{Set}} \to \mathcal{F}'$.

Let S be a $K_0^{\mathrm{Ar}}(X)$-set and let $\phi : K_0^{\mathrm{Ar}}(X) \to \mathrm{Aut}(S)$ be the permutation representation. We define a functor $F_S : Q\mathcal{T}_{\mathrm{Ar}}(X) \to \underline{\mathrm{Set}}$ by means of Lemma 2 with the following assignments: $F_S(M) = S$ for any $M \in \mathrm{Ob}\,\mathcal{T}_{\mathrm{Ar}}(X)$, and

$$(F_S)_1(i_!) = \mathrm{Id}_S, \quad (F_S)_2(q^!) = \phi([\mathrm{Ker} q]) \in \mathrm{Aut}(S).$$

First, we need to show that if

$$\begin{array}{ccc} M' & \xrightarrow{i} & N \\ q\downarrow & & \downarrow q' \\ M & \xrightarrow{i'} & N' \end{array}$$

is a bi-cartesian square, then $\mathrm{Ker}\, q \simeq \mathrm{Ker}\, q'$. But this relation follows from the definition of the bi-cartesian square in $\mathcal{T}_{\mathrm{Ar}}(X)$. Therefore,

$$\phi([\mathrm{Ker}\, q]) = \phi([\mathrm{Ker}\, q']).$$

Then we need to show that, for any two composable epimorphisms q and q',

$$\phi([\mathrm{Ker}\,(q' \circ q)]) = \phi([\mathrm{Ker}\, q])\, \phi([\mathrm{Ker}\, q']).$$

But this is a direct consequence of the definitions of epimorphisms and their composition, the bi-cartesin square, the kernel, the exact sequence, and the property of the classical Bott-Chern secondary characteristic form associated with the following special 3×3 picture [2] or [5],

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Ker}(q) & \to & \mathrm{Ker}(q' \circ q) & \to & \mathrm{Ker}(q') & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Ker}(q) & \to & \mathcal{E}_1 & \xrightarrow{q} & \mathcal{E}_2 & \to & 0 \\ & & & & q' \circ q \downarrow & & \downarrow q & & \\ & & & & \mathcal{E}_3 & = & \mathcal{E}_3 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & . \end{array}$$

All the conditions of Lemma 2 are now satisfied. Hence, we have a functor

$$\begin{array}{ccc} \underline{K_0^{\mathrm{Ar}}(X)-\mathrm{Set}} & \to & \mathcal{F}' \\ S & \mapsto & F_S. \end{array}$$

Step 3. Define a functor $\mathcal{F}' \to \underline{K_0^{\mathrm{Ar}}(X)-\mathrm{Set}}$.

Now for $F \in \mathrm{Ob}\,\underline{\mathcal{F}}'$, let $\phi_F : K_0^{\mathrm{Ar}}(X) \to \mathrm{Aut}(F(0))$ be given by

$$\phi_F([M]) = F(q^!_M).$$

We need to check that this is a well-defined homomorphism on $K_0^{\mathrm{Ar}}(X)$: if we have an exact sequence in $\mathcal{T}_{\mathrm{Ar}}(X)$

$$0 \to M' \xrightarrow{i} M \xrightarrow{q} M'' \to 0$$

by definition, we can get a bi-cartesian square

$$\begin{array}{ccc} M' & \xrightarrow{i} & M \\ q_{M'} \downarrow & & \downarrow q \\ 0 & \xrightarrow{i_{M''}} & M'' \end{array}$$

for which $q^! \circ i_{M''^!} = i_! \circ q^!_M$. Hence $F(q^!_{M'}) = F(q^!)$; and $q^!_M = q^! \circ q^!_{M''}$ implies

$$F(q^!_M) = F(q^! \circ q^!_{M''}) = F(q^!_{M'}) \circ F(q^!_{M''}).$$

Therefore, if we prove that

$$F(q^!_{M''} \circ q^!_{M'}) = F(q^!_{M'}) \circ F(q^!_{M''}),$$

then ϕ_F is well-defined. For this, we only need to consider two 'split' exact sequences

$$0 \to (\mathcal{E}, \rho; \eta) \xrightarrow{(i_1;e,0,0)} (\mathcal{E} \oplus \mathcal{F}, \rho \oplus \tau; \eta + \omega) \xrightarrow{(p_2;e,1,1)} (\mathcal{F}, \tau; \omega) \to 0,$$

and

$$0 \to (\mathcal{F}, \tau; \omega) \xrightarrow{(i_2;e,0,0)} (\mathcal{E} \oplus \mathcal{F}, \rho \oplus \tau; \eta + \omega) \xrightarrow{(p_1;e,1,1)} (\mathcal{E}, \rho; \eta) \to 0.$$

So $F(q^!_{(\mathcal{E},\rho;\eta)})$, $F(q^!_{(\mathcal{F},\tau;\omega)}) \in \mathrm{Aut}(F(0))$ commute.

It follows that $(S, \phi) \mapsto F_S$, $F \mapsto (F(0), \phi_F)$ give the desired equivalence of categories. This proves Main Theorem I.

Motivated by the above theorem, we make

Main Definition B. Let X be a regular arithemetic variety over an arithmetic ring A. The $(i+1)^{\mathrm{st}}$-homotopy group of the classifying space of the Quillen construction of the taips category $\mathcal{T}_{\mathrm{Ar}}(X)$ is called the i^{th}**-arithmetic** K**-group**, denoted by $K_i^{\mathrm{Ar}}(X)$. That is, for all $i \geq 0$, we have

$$K_i^{\mathrm{Ar}}(X) := \pi_{i+1}(\hat{B}Q\mathcal{T}_{\mathrm{Ar}}(X), (0,0;0)).$$

II.8.5. The Global Triangle

From the previous section, we have arithmetic K-groups $K_i^{\mathrm{Ar}}(X)$ for $i \geq 0$ and any regular arithmetic variety X. In this chapter, we give the following "global triangle", which gives the relation between the analytic properties, the arithmetic properties and the algebraic properties of arithmetic varieties:

$$\begin{array}{ccccc} H_\bullet^{\mathrm{An}}(*) & & \xrightarrow{N} & & K_\bullet^{\mathrm{Ar}}(X) \\ & R\nwarrow & & \swarrow F & \\ & & K_\bullet(X) & & \end{array}$$

We first recall some results from category theory.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. Then for any object N of $\mathcal{C}'$, we let $N\backslash F$ denote the category consisting of pairs (M, v), where $v : N \to F(M)$, and a morphism from (M, v) to (M', v') is a map $w : M \to M'$ such that $F(w) \circ v = v'$. In particular, when F is an identity functor of $\mathcal{C}'$, we obtain the category $N\backslash\mathcal{C}'$ of objects under N.

Main Theorem II. (1). Let $QF : Q\mathcal{T}_{\mathrm{Ar}}(X) \to Q\mathcal{P}(X)$ be the natural functor induced by the forgetful functor

$$\begin{array}{rcl} F : \mathcal{T}_{\mathrm{Ar}}(X) & \to & \mathcal{P}(X) \\ (\mathcal{E}, \rho; \eta) & \mapsto & \mathcal{E}. \end{array}$$

Here, $\mathcal{P}(X)$ is the category of vector bundles on X. For any object $\mathcal{E}$ of $\mathcal{P}(X)$, and object $(\mathcal{E}, \rho; \eta)$ of $\mathcal{T}_{\mathrm{Ar}}(X)$, we have a natural induced long exact sequence of abelian groups

$$\begin{array}{ccccc} K_{i+1}(X) & \leftarrow & \dots & & \\ \downarrow R & & & & \\ \pi_{i+1}(\mathcal{E}\backslash QF, ((\mathcal{E}, \rho; \eta), \mathrm{Id}_\mathcal{E})) & \xrightarrow{N} & K_i^{\mathrm{Ar}}(X) & \xrightarrow{F} & K_i(X) \\ & & & & R\downarrow \\ & & \dots & \leftarrow & \pi_i(\mathcal{E}\backslash QF, ((\mathcal{E}, \rho; \eta), \mathrm{Id}_\mathcal{E})). \end{array}$$

(2). At its lowest level, after tensoring with $\mathbf{Q}$, the long exact sequence above becomes the natural exact sequence

$$K_1(X)_\mathbf{Q} \to \tilde{A}(X_\mathbf{R}) \to K_{\mathrm{Ar}}(X)_\mathbf{Q} \to K(X)_\mathbf{Q} \to 0,$$

- stated in Chapter IV.

The first part of the main theorem is a direct consequence of Quillen's Theorem A and Theorem B. For convenience, we recall them below.

Theorem A. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. If the category $N\backslash F$ is contractible for every object N of $\mathcal{C}'$, then the functor F is a homotopy equivalence.

Theorem B. Let $F : \underline{C} \to \underline{C}'$ be a functor such that for every arrow $N \to N'$ in $\underline{C}'$, the induced functor $N'\backslash F \to N\backslash F$ is a homotopy equivalence, then for any $F^{-1}(N)$, there is a natural induced exact sequence

$$\ldots \to \pi_{i+1}(\underline{C}', N) \to \pi_i(N\backslash F, (M, \mathrm{Id}_M)) \to \pi_i(\underline{C}, M) \to \pi_i(\underline{C}', N) \to \ldots.$$

In order to use these two theorems, we need the following result from [Qu 73].

Proposition. A category having either an initial or a final object is contractible.

Proof of the Theorem. We only need to show that for any object $\mathcal{E}$ in $Q\mathcal{P}(X)$,

$$\mathcal{E}\backslash QF = \{((\mathcal{F}, \tau; \omega), v) : (\mathcal{F}, \tau; \omega) \in \mathrm{Ob}QT_{\mathrm{Ar}}(X), v : \mathcal{E} \to \mathcal{F}\};$$

and for any arrow $u : \mathcal{E} \to \mathcal{E}$ in $Q\mathcal{P}(X)$,

$$\begin{array}{rcl} u\backslash QF : \mathcal{E}'\backslash QF & \to & \mathcal{E}\backslash QF \\ ((\mathcal{F}, \tau; \omega), v) & \mapsto & ((\mathcal{F}, \tau; \omega), v \circ u). \end{array}$$

Then for any $((\mathcal{F}, \tau; \omega), v : \mathcal{E} \to \mathcal{F})$, $u : \mathcal{E} \to \mathcal{E}'$, we have

$$\begin{array}{l} ((\mathcal{F}, \tau; \omega), v)\backslash(u\backslash QF) \\ =\{((((\mathcal{G}, \nu; \chi), w), s) : w : \mathcal{E}' \to \mathcal{G},\ s : ((\mathcal{G}, \nu; \chi), w \circ u) \to ((\mathcal{F}, \tau; \omega), v)\}. \end{array}$$

In particular, we see that $(((0,0;0),0),0)$ is an object of the category $((\mathcal{F}, \tau; \omega), v)\backslash(u\backslash QF)$. Therefore, by Quillen's results above, we have the long exact sequence

$$\begin{array}{ccccc} \pi_{i+1}(BQ\mathcal{P}(X), \mathcal{E}) & \leftarrow & \ldots & & \\ \downarrow R & & & & \\ \pi_i(\mathcal{E}\backslash QF, ((\mathcal{E}, \rho; \eta), \mathrm{Id}_{\mathcal{E}})) & \xrightarrow{N} & \pi_i(BQT_{\mathrm{Ar}}(X), (\mathcal{E}, \rho; \eta)) & \xrightarrow{F} & \pi_i(BQ\mathcal{P}(X), \mathcal{E}) \\ & & & & R \downarrow \\ & & \ldots & \leftarrow & \pi_i(\mathcal{E}\backslash QF, ((\mathcal{E}, \rho; \eta), \mathrm{Id}_{\mathcal{E}})). \end{array}$$

Thus, by the main definition B, we have Part 1.

For Part 2, the only difficult part is to show that

$$\pi_1(\mathcal{E}\backslash QF, ((\mathcal{E}, \rho; \eta), \mathrm{Id}_{\mathcal{E}})) \simeq \tilde{A}(X_{\mathbf{R}}).$$

But this can be deduced directly from the definition, or by the five lemma.

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