On constructions making Dirac Operators invertible at Infinity

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Abstract

We construct index pairings of Dirac operators with **K**-cycles coming from the Higson corona of a complete Riemannian manifold. We produce operators of Callias type and compute their index as far as possible. A real construction using Clifford-indices leads to obstructions against the existence of metrics with certain positivity properties of the scalar curvature in a given quasi-isometry class.

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1 Introduction

There are several methods known to produce a Fredholm operator starting with a Dirac operator on a complete Riemannian manifold M acting on sections of a Clifford bundle $S \to M$. A very easy one is to add a suitable bundle endomorphism $\Phi \in \Gamma(M, End(S))$. Adding such a Φ should have the effect of shifting away the essential spectrum such that $B := D + \Phi$ has a gap at zero. We call such endomorphisms admissible. If S has a $\mathbf{Z}_2$-grading by z and $deg\,\Phi = 1$ then B has a well defined index

$$ind\,B = tr_{ker\,B} z.$$

It takes into account the geometry of D as well as the behaviour of Φ at infinity. Especially in the odd-dimensional case the index of B vanishes if D is itself Fredholm. Thus $ind\,B$ depends on the properties of the essential spectrum of D near zero. There should be a relation to scattering indices as considered by Borisov/Müller/Schrader in [4]. Φ can be deformed on compact subsets of M without changing the index of B. If Φ is boundedly invertible then $ind\,B = 0$.

Operators of the form $B + \Phi$ are often called operators of Callias type because of the special case considered by Callias in [9]. Besides the problem how to find an admissible Φ one is confronted with the question of computing the index of B. At a first glance it seems hopeless to have a general effective method to do this if one takes into accout all the difficulties of index theory on open manifolds. But it turns out that there is in fact a rather effective and surprisingly easy method for computing $ind\, B$. All the information on the index of B is contained in a tubular neighbourhood of a compact hypersurface $N \subset M$ cutting off the part where Φ is invertible. Thus the computation can be reduced to an index computation on a compact manifold. In the example of Callias there is $M = \mathbf{R}^n$ and $ind\, B$ is expressed as the index of a Dirac operator D_V on a large sphere $S^{n-1} \subset \mathbf{R}^n$ twisted with a bundle of the positive eigenvectors of Φ on S^{n-1}. Callias result has been generalized by Anghel [1][2] to manifolds with cylinder like-ends where of course N becomes the cylinder base. Another example is Roe's index theorem for partitioned manifolds [23] which has a reformulation in terms of operators of Callias type by Higson [15].

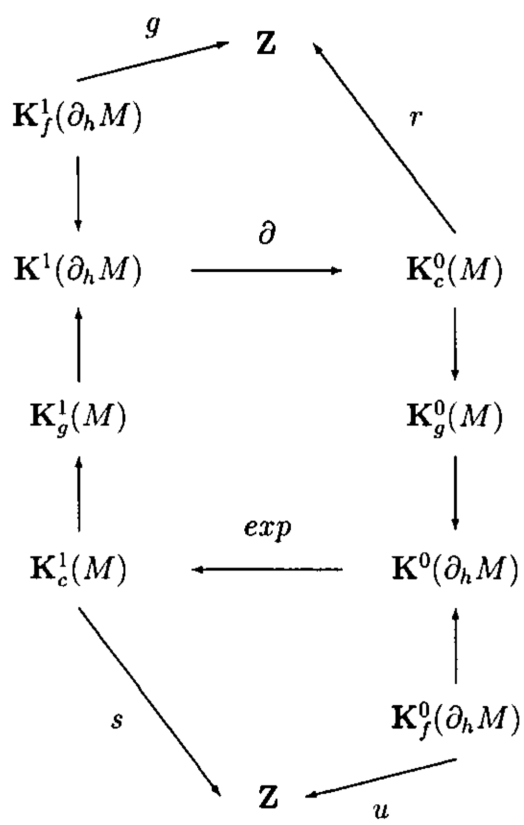
All the examples mentioned above start initially with a Dirac operator on an ungraded Clifford bundle $E \to M$. The grading enters later by forming matrices. They give non-trivial results on odd-dimensional manifolds. It turns out that there is an even-dimensional counterpart of the theory which in fact starts with $\mathbf{Z}_2$-graded Dirac operators.

In the present paper we deal with the problem of finding admissible endomorphisms Φ and with the computation of $ind\, B$. It will turn out the somehow most general Φ can be constructed out of finite $\mathbf{K}$-classes of the boundary $\partial_h M$ of a compactification $\bar{M}$ of M. This compactification has been introduced by Higson [14] and has been described in [24]. $\partial_h M$ is called the Higson corona. It is the spectrum of the C^*-algebra $C_g(M)/C_0(M)$ where $C_g(M)$ is the C^*-subalgebra of $C(M)$ generated by the smooth bounded functions with vanishing gradient at infinity. Finite $\mathbf{K}$-theory $K_f^*(\partial_h M)$ is generated by unitary matrices $(* = 1)$ or projections $(* = 0)$ in $C(\partial_h M) \otimes Mat(N)$ for $N > 0$. It comes out that constructing Φ and taking the index of $B := D + \Phi$ gives a homomorphism $u : K_f^0(\partial_h M) \to \mathbf{Z}$ when D is ungraded and $g : Kk_f^1(\partial_h M) \to \mathbf{Z}$ if D is graded. We will also give a more general interpretation of these maps taking the $\mathbf{K}$-homology content of D into account.

The maps u and g are closely related to two other constructions; namely the spectral flow pairing $s : K_c^1(M) \to \mathbf{Z}$ (constructed with an ungraded operator) and the relative index pairing $r : K_c^0(M) \to \mathbf{Z}$ (constructed with a graded Dirac operator). All these maps fit into the following commutative diagram which consists of the the long exact sequence of complex $\mathbf{K}$-theory associated to the short exact sequence of C^*-algebras

$$0 \to C_0(M) \to C_g(M) \to C(\partial_h M) \to 0$$

completed with the finite $\mathbf{K}$-theory and the index maps.



We will compute u and g by a cutting and pasting procedure simplifying M to a suitable compact manifold and employ the result to verify commutativity of the above diagram.

There should be a close relation between Roe's exotic index and the index of Callias type operators contructed above such that the theory of Roe and the theory developed here are two sides of the same thing. We will try to exhibit this elsewhere.

Having in mind several generalizations in future we will formulate our results in terms of **KK**-theory. Thus we associate to $B := D + \Phi$ a Kasparov module which represents a class $[M] \in \mathbf{KK}(C_g(M), \mathbf{C})$ and the restriction $\{M\} \in \mathbf{KK}(\mathbf{C}, \mathbf{C}) = \mathbf{Z}$ (here we use the convention that the symbol M stands for all structures over M). Our main computational tools are deformation invariance of $\{M\}$, vanishing theorems for $\{M\}$ (e.g. if Φ is invertible or if D is itself Fredholm) and the relative index theorem that states that cutting and pasting along compact hypersurfaces does not change the class $\{M\} \in \mathbf{Z}$. The idea of computation is to use cutting and pasting and the vanishing theorems in order to reduce the problem to a cylinder over some compact hypersurface N. There one can compute explicity by seperation of variables. This method is similar to that used by Roe [23] and Anghel [1]. We will describe the technical machinery in the first few sections of the present paper.

It turns out that a large part of our methods work also in the case of real operators and of operators beeing equivariant with respect to the action of some C^*-algebra A. The index of B is then of course an element of $\mathbf{K}_*(A)$ (or of $\mathbf{KO}_*(A)$ in the real case). The difficulties come when A is infinite dimensional. The are two critical points which have to be resolved. The first is that one has to verify that D^2+1 is invertible. This is not obvious (at least to the author). This is equivalent to the density of the image of D^2+1. and is true if $E \to M$ is a usual Clifford bundle twisted with a flat bundle of projective A-modules under some finiteness conditions at infinity. The second problem arrises in our computations on the cylinder. Here we explicitly identify kernels of two operators living on different spaces and want to conclude that the indices coincide. This is true if the kernels are projective modules. But in general they are not and one has to interpret the index as in [22]. We do not know any result assuring that the two kernels after reinterpretation give the same $\mathbf{K}$-class. Nevertheless we hope to solve these problems (at least partially) in a forthcoming paper. Then there would be immedeate generalizations of the results of the present paper to this more general case.

For the purpose of the present paper we consider the real case and $A = C^{0,k}$ some Clifford algebra. There should be a similar diagram as in the complex case where one has to replace the two-periodic by the eight-periodic long exact sequence of real $\mathbf{K}$-theory. The relative index pairing works in this case too. Finding the right generalization of g depends on an explicit understanding of the boundary maps in real $\mathbf{K}$-theory in terms of Clifford bundles. Up to now we have not succeded to do that. Thus we start with a real $C^{0,n}$-equivariant Dirac operator on $E \to M$ and an suitable Φ with values in a matrix algebra over $C^{k,0}$ and construct a new bundle $S \to M$ and and operator $B := D + \Phi$ beeing equivariant with respect to $C^{0,n-k}$. This operator is then Fredholm and represents an element $\{M\} \in \mathbf{KO}_{n-k}(\mathbf{R})$. Again we are able to simplify the manifold by cutting off the part where Φ is invertible and replace it by a half-cylinder $\mathbf{R}_+ \times N$. We we are not able to reduce to a tubular neighbourhood of the cylinder base in general.

In order to push through the calculation we have to make special assumptions on Φ. We use a map $F :\to \mathbf{R}^k$ with $|F| \to 1$ and $|grad\, F| \to 0$ when $|x| \to \infty$. We let $\Phi := zF$ where z is the $\mathbf{Z}_2$-graduation of E and form $B := D + \Phi$. In order to define the action of Φ on E we understand the inclusions

$$\begin{array}{ccc} \mathbf{R}^k & \subset & C^{0,k} \\ \cap & & \cap \\ \mathbf{R}^n & \subset & C^{0,n} \end{array}.$$

Let $C^{0,n-k}$ be generated by $(\mathbf{R}^k)^\perp \subset \mathbf{R}^n$. Then B is $C^{0,n-k}$-eqivariant and $ind\, B = \{M\} \in \mathbf{KO}_{n-k}(\mathbf{R})$. Let 0 be a regular value of F and $N = F^{-1}(0)$. Then we find a Dirac operator D_N on N beeing $C^{0,n-k}$-equivariant and show that

$$ind\, B = ind\, D_N.$$

If M^n is a Riemannian spin manifold and E is the real Clifford bundle associated to the spin structure then

$$ind\, B = ind\, D_N = \alpha(N)$$

where $\alpha(N) \in \mathbf{KO}_{n-k}(\mathbf{R})$ is the α-invariant defined by Hitchin [16] (see also [25] and [20]. We apply this result in order to produce obstructions against the existence of metrics with certain positivity requirements. Our results are similar to those of Roe [23] (section 6.1) but we can also cover $\mathbf{Z}_2$-valued obstructions. Moreover we can answer questions of Rosenberg and Stolz [26]. As special cases we prove:

Corollary 1.1 *If $\alpha(N) \neq 0$ there is no metric on $\mathbf{R}^k \times N$ with uniform positive scalar curvature at infinity in the quasi-isometry class of the product.*

Corollary 1.2 *If $\alpha(N) \neq 0$ then there is no metric of non-negative scalar curvature τ on $\mathbf{R}^k \times N$ with lower bound $\tau(x, n) > \frac{c}{|x|+1}$ for some $c > 0$ in the quasi-isometry class of the product.*

Corollary 1.3 *If $\alpha(N) \neq 0$ then there is no complete metric of non-egative scalar curvature on $\mathbf{R} \times N$ which is positive on some section $\{a\} \times N$.*

Corollary 1.3 has also been shown by Lesch [21].

The author was strongly influenced by discussions with S.Stolz and talks given by J. Rosenberg at a DMV-Seminar in June 1992. Moreover it was profitable reading the paper of S.Hurder [17]. The present paper came out as a by-product of the work on obstructions against positive scalar curvature metrics on open manifolds for which the maschinery of the **KK**-theoretic relative index theorem [7] was developed. It has been turned out that it is applicable also in the situations described in this paper. The author thanks M. Lesch for the suggestion how to defer Corollary 1.2 from our index theorem.

2 The Higson corona and finite K-theory

Let M be a complete Riemannian manifold with Dirac operator D acting in sections of a Clifford bundle $E \to M$. We want to construct admissible $\Phi \in \Gamma(M, End(E))$ out of **K**-classes living on some compactification $\bar{M}$ of M. An inspection of the constructions below shows that the largest compactification we can work with is the compactification by the *Higson corona* $\partial_h M$. It has been introduced by Higson [14] and is also described in [24]. **K**-classes living on smaller compactifications can be pulled back to $\partial_h M$. Thus we can apply our constructions to them too.

Let us now describe $\partial_h M$. Let $C(M)$ be the C^*-algebra of bounded continuous functions on M with the sup-norm (we consider real or complex functions depending on the context). Let $C_0(M)$ be the closure of $C_c^\infty(M)$ in $C(M)$. Define

$$C_g^\infty(M) := \{f \in C^\infty(M) \cap C(M) \mid |df| \in C_0(M, T^*M)\}$$

and let $C_g(M)$ be the closure of $C_g^\infty(M)$ in $C(M)$. There is an exact sequence of C^*-algebras

$$0 \to C_0(M) \to C_g(M) \to C(\partial_h M) \to 0$$

where $C(\partial_h M) := C_g(M)/C_0(M)$. In fact $\partial_h M := spec\, C(\partial_h M)$ is defined by duality. Then $\bar{M} := specC_g(M) = M \cup \partial_h M$ is the smallest compactification to which all functions of $C_g(M)$ extend continuously.

The Higson corona is rather large. It maps to any smaller compactification defined by a subalgebra of $C_g(M)$, especially the one-point compactification and the compactification given by the closure of the algebra of functions beeing constant outside of compact sets.

Since the Higson corona is a very large space it is not a priori clear that every element of $\mathbf{K}^*(\partial_h M) = \mathbf{K}_*(C(\partial_h M))$ can be represented by a finite matrix. That is the reason for introducing finite **K**-theory as the group of stable homotopy classes of projections (or unitaries) which can be realized by finite matrices. Here only homotopies via finite matrices are allowed as equivalence relation. Due to this restriction finite **K**-theory is not simply a subset of the usual **K**-theory.

3 The relative index theorem

3.1 Operators of Dirac type and invertibility at infinity

Let M be a complete Riemannian manifold and $S \to M$ be a Clifford bundle over M (real or complex) with associated Dirac operator $D : C_c^\infty(M, S) \to C_c^\infty(M, S)$. We consider the Sobolev spaces $H^k(M, S)$ beeing the closures of $C_c^\infty(M, S)$ with respect to the norm given by the scalar product

$$< \psi, \phi >_k := \sum_{l=0}^{k} \int_M < D^l\psi, D^l\phi > dvol.$$

for $k \geq 0$. For negative k we define H^k by duality. Then D extends to bounded operators in $B(H^k, H^{k-1})$.

Let $\Phi \in C^1(M, End(S))$ and form $B := D + \Phi$. Then $B \in B(H^k, H^{k-1})$ for $k = 2, 1$. We call B an operator of *Dirac type*. If B is an operator of Dirac type so is B^* where we take the adjoint with respect to the scalar product in $H^0 := L^2(M, S)$.

Definition 3.1 *An operator of Dirac type is invertible at infinity if there is a positive function $s \in C_c^\infty(M)$ such that $BB^* + s$ and $B^*B + s$ are invertible as bounded operators in $B(H^2, H^0)$.*

One well known example where D itself is invertible at infinity is the following. Let M be spin and assume that M has uniform positive scalar curvature at infinity. Then the Dirac operator on M associated to the spinor bundle is invertible at infinity. This follows easely from the Weizenboeck formula.

Invertibility at infinity of B implies the Fredholm property (as operator in $B(H^1, H^0)$) and the applicability of relative index theorems (see below, Gromov/Lawson [13] and [6]). In many cases invertibility at infinity is easy to verify.

3.2 Kasparov modules

Since we want to consider also real indices of Fredholm operators, i.e. indices of Fredholm operators acting on Hilbert-C^*-modules over a real Clifford algebra $C^{0,n}$ we formulate our constructions in the language of Kasparov modules. This is also most suitable for deformation arguments and the statement of the relative index theorem. The generalizations to operators beeing equivariant with respect to more complicated C^*-algebras become (at least formally, disregarding technical difficulties as described in the introduction) rather obvious.

In order to simplify notation we try to avoid Hilbert-C^*-modules where this is possible. In fact it is the same to consider a Hilbert space H with a right action of $C^{0,n}$ or a Hilbert-C^*-module H over $C^{0,n}$. There is a trace $\epsilon : C^{0,n} \to \mathbf{R}$ given by $\epsilon(1) := 1$ and $\epsilon(e_I) := 0$ where $I := \{0 < i_1 < i_2 < \ldots < i_k \leq n\}$ is some multiindex with $0 < k \leq n$ and $e_I := e_{i_1} \ldots e_{i_k}$ for an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbf{R}^n$ generating $C^{0,n}$ with relations $e_ie_j + e_je_i = -2\delta_{ij}$. One can reconstruct the $C^{0,n}$-valued scalar product $<<,>>$ of H from the Hilbert space structure by

$$<< \psi, \phi >> := \sum_I < \psi, \phi e_I > e_I^*$$

an vice versa the Hilbert space structure of H from $<<,>>$ by

$$< \psi, \phi > := \epsilon(<< \psi, \phi >>).$$

Moreover $B(H)$ is the space of $C^{0,n}$-equivariant bounded operators on H while $K(H)$ is the space of compact elements in $B(H)$. This is not true for general Hilbert-C^*-modules over infinite dimensional algebras (see [22],[3] and [18]).

A *Kasparov module* over the pair $(\mathbf{C}, \mathbf{C})$ is a $\mathbf{Z}_2$-graded Hilbert space H together with an bounded Operator $F \in B(H)$ such that

- deg(F)=1
- $F - F^* \in K(H)$,
- $F^2 - 1 \in K(H)$.

If A is a C^*-algebra acting on H from the left then for Kasparov module over the pair $(A, \mathbf{C})$ we require in addition $b[a, F] \in K(H)$ for any $a, b \in A$.

Note that $C^{0,n}$ is $\mathbf{Z}_2$-graded. A Kasparov module over the pair of real C^*-algebras $(\mathbf{R}, C^{0,n})$ is a real $\mathbf{Z}_2$-graded Hilbert space H with a graded right action of $C^{0,n}$ together with $F \in B(H)$ satisfying the same conditions as above (but with different meaning of $B(H), K(H)$). Analogously we define a Kasparov module over $(A, C^{0,n})$ where A is a real C^*- algebra.

A Kasparov module (H, F) over the pair of C^*-algebras (A, B) represents a class in $\mathbf{KK}(A, B)$. Note that we do not distinguish the real case from the complex case in our notation. Some authors use **KKR** for the real **KK**-theory. In the present paper it is always clear from the context which **KK**-groups are ment.

We will use the equivalence relations generated by compact perturbation, unitary equivalence, addition of trivial elements and homotopy (see Kasparov [19], Blackadar [3] and Jensen/Thomsen [18] for more information on **KK**-theory). We will also employ the identifications

$$\begin{aligned} \mathbf{KK}(\mathbf{C},\mathbf{C}) &= \mathbf{K}_0(\mathbf{C}) = \mathbf{Z} \\ \mathbf{KK}(\mathbf{R},C^{0,n}) &= \mathbf{KO}_n(\mathbf{R}) \end{aligned}$$

which are given by taking the index of F in the first case and by taking the class of the kernel of F as a graded $C^{0,n}$-module in the second.

We prefer to work with bounded Kasparov modules. But some constructions are simpler described with unbounded ones. Thus unbounded modules will occur in the places where we decribe in a more general framework using the intersection product what we did down to earth. We do not use the intersection product explicitly in our proofs. Implicitly it is contained in the proof that the different equivalence relations on the Kasparov modules give in fact the same **KK**-groups (see [3]).

3.3 The relative index theorem

Let M be a complete Riemannian manifold and $S \to M$ be a $\mathbf{Z}_2$-graded Clifford bundle (complex or admitting a $C^{0,n}$-right action in the real case). Let D be the associated Dirac operator and $B := D + \Phi$ be a selfadjoint operator of Dirac type of degree one which is invertible at infinity (in the real case we assume B to be $C^{0,n}$-equivariant).

As we have shown in [6] there is a $S \in K(H^0, H^1)$ such that $B + S$ is invertible. We set $A := B + S$ and $F := [B(AA^*)^{-1/2}]^{odd} \in B(H^0)$. It has been proven in [6] that (H^0, F) is a Kasparov module over the pair $(C_g(M), \mathbf{C})$ (or $(C_g(M), C^{0,n})$ in the real case). Let $\{M\}$ be the class in $\mathbf{KK}(\mathbf{C},\mathbf{C})$ (or $\mathbf{KK}(\mathbf{R}, C^{0,n})$ in the real case) represented by (H^0, F). This class does not depend on the choice of S. Another choice of S results in a compact perturbation of F.

We formulate now the relative index theorem (see [6]). Let $N \subset M$ be a compact hypersurface cutting a normal neighbourhood $U(N)$ into two pieces $U(N)_\pm$. Assume that there is a diagram

$$\begin{array}{cccccc} \Gamma & : & S_{|U(N)_-} & \longrightarrow & S_{|U(N)_-} \\ & & \downarrow & & \downarrow \\ \gamma & : & U(N)_- & \longrightarrow & U(N)_- \end{array}$$

such that γ is an isometry and Γ is an isomorphism of Clifford bundles commuting with Φ. We form a new manifold $\tilde{M}$ cutting at N and glueing the pieces together using γ. We glue a new Clifford bundle bundle $\tilde{S}$ using Γ. Then Φ extends nicely to $\tilde{\Phi}$ and we obtain an operator of Dirac type $\tilde{B}$. Suppose that $\tilde{B}$ is also invertible at infinity. Let $\{\tilde{M}\} \in \mathbf{KK}(\mathbf{C},\mathbf{C})$ (or $\mathbf{KK}(\mathbf{R}, C^{0,n})$) be the class defined with $\tilde{B}$ as described above.

Theorem 3.2 (K-theoretic relative index theorem) $\{\tilde{M}\} = \{M\}$

Special cases have been proven by Gromov/Lawson [13], Donnelly [10] and other authors. The present formulation in terms of cutting and pasting was suggested to me by S. Stolz.

4 Construction and deformation of Dirac-type operators

4.1 Admissible endomorphisms

Let M be a complete Riemannian manifold and $S \to M$ be a $\mathbf{Z}_2$-graded Clifford bundle with associated Dirac operator D (admitting a $C^{0,n}$-right action in the real case).

Definition 4.1 *An endomorphism $\Phi \in C^1(M, End(S))$ is called admissible if*

- *$deg\, \Phi = 1$ and $\Phi^* = \Phi$,*
- *$\Phi D + D\Phi$ is bounded of zero order and*
- *there is a compact set $K \subset M$ such that $\Phi D + D\Phi + \Phi^2 \geq c > 0$ on $M \setminus K$ for some constant c.*

In the real case we will require in addition equivariance with respect to certain Clifford algebra $C^{0,k} \subset C^{0,n}$ such that $B := D + \Phi$ becomes $C^{0,k}$-equivariant.

A compact set $K \subset M$ with $\Phi^2 \geq c > 0$ on $M \setminus K$ is called an essential support of Φ. Later we will show that all the information on the index of B is contained in an essential support of Φ. Note that replacing Φ by $\Phi_t := t\Phi$ one can have also $D\Phi_t + \Phi_t D + \Phi_t^2 = t(D\Phi + \Phi D) + t^2\Phi^2 \geq c > 0$ outside of a given essential support for large t. Note that B is selfadjoint for admissible Φ.

Lemma 4.2 *If $\Phi \in C^1(M, End(S))$ is admissible then $B := D + \Phi$ is invertible at infinity.*

Proof: Let $K \subset M$ such that $\Phi D + D\Phi + \Phi^2 \geq c > 0$ on $M \setminus K$ for some constant c. Let $s \in C_c^\infty(M)$ be a positive function such that $s \geq c$ on K. Then $B^2 + \lambda s$ is selfadjoint and positive for large λ. □

4.2 Deformations

Let M be a complete Riemannian manifold and $S \to M$ be a $\mathbf{Z}_2$-graded Clifford bundle with Dirac operator D. Let $\Phi \in C^1(M, End(S))$ be an admissible endomorphism and $B := D + \Phi$. Consider a perturbation B_t of B continuous in $B(H^1, H^0)$. Let $\{M_t\}$ be the class in **KK**-theory constructed with B_t. Then we have deformation invariance.

Lemma 4.3 $\{M_t\} = \{M_0\}$ *for small t.*

Proof: Let $S \in K(H^0, H^1)$ such that $B + S$ is invertible. Then for t small $B_t + S$ is invertible. Thus we can form $A_t := B_t + S$ and $F_t := [B_t(A_tA_t^*)^{-1/2}]^{odd}$. The cycle (H^0, F_t) represents $\{M_t\}$. To prove the Lemma it is enough to show that F_t induces a homotopy (do not confuse with operator homotopy). In fact it is enough to show that F_t is strongly continuous and that $F_t^* - F_t$ and $F_t^2 - 1$ are continuous in the norm. This is easely seen from the integral representations used in [6]. □

Corollary 4.4 *Let $\{B_t\}_t \in I$ be a continuous family in $B(H^1, H^0)$ such that B_t is invertible at infinity for any $t \in I$. Then $\{M_t\} = \{M\}$.*

Proof: One finds a finite partition of the interval I and applies the Lemma 4.3 finitely often. □

A continuous path $\Phi_t \in C(M, End(S))$ leads to a continuous family $B_t \in B(H^1, H^0)$. More general continuous deformation of the Clifford bundle structure or Riemannian metric in the C^0-topology (see Eichhorn [11] and [12]) (with appropriate identification of the Hilbertspaces) lead to continuous families in $B(H^1, H^0)$ and thus do not change the class $\{M\}$ constructed with B. Of course if the connection of the Clifford bundle comes from the Levi-Civita connection on has to consider C^1-continuous families of metrics in order to obtain C^0-continuous families of Clifford bundle structures.

Corollary 4.5 *If Φ is admissible and has an empty essential support (i.e. $\Phi^2 \geq c > 0$ for some constant $c > 0$) then $\{M\} = 0$.*

Proof: Consider the family $B_t := D + t\Phi$. If Φ is admissible so is $t\Phi$ for $t \geq 1$. Moreover B_t is invertible for large $t > 0$. Hence $\{M\} = \{M_t\} = 0$. □

5 The map u

5.1 The construction of u

Let M be a complete Riemannian manifold and $E \to M$ be a (complex) Clifford bundle over M with Dirac operator $D_E : C_c^\infty(M, E) \to C_c^\infty(M, E)$ which is ungraded. Let $[p] \in \mathbf{K}_f^0(\partial_h M)$ be represented by a projection $p \in C(\partial_h M) \otimes Mat(N)$ for some large N. We can find a 'lift' $P \in C_g(M) \otimes Mat(N)$ of p such that

- $P = P^*$
- $P^2 - P \in C_0^\infty(M) \otimes Mat(N)$
- $P\ (mod\, C_0(M) \otimes Mat(N))$ represents $[p]$.

Stabilization and homotopies of p can be lifted to stabilization and homotopies of P. Form the $\mathbf{Z}_2$-graded Clifford bundle $S := E\hat{\otimes}(\mathbf{C}^N \oplus (\mathbf{C}^N)^{op})$ with associated Dirac operator D given by the matrix

$$D := \begin{pmatrix} 0 & D_E \otimes 1 \\ D_E \otimes 1 & 0 \end{pmatrix}.$$

Here $\mathbf{C}^N$ is trivially graded (i.e. all elements have degree 0) and $(\mathbf{C}^N)^{op}$ consists of elements of degree 1. Let

$$\Phi := \imath \otimes \begin{pmatrix} 0 & 1 - 2P \\ 2P - 1 & 0 \end{pmatrix} \in C^1(M, End(S)).$$

Lemma 5.1 *Φ is admissible.*

Proof: Obviously $deg\Phi = 1$ and $\Phi = \Phi^*$. Moreover

$$D\Phi + \Phi D = 2\imath \begin{pmatrix} gradP & 0 \\ 0 & -gradP \end{pmatrix}.$$

But $gradP$ tends to zero at infinity while Φ^2 tends to 1. Thus there is a compact set $K \subset M$ such that $D\Phi + \Phi D + \Phi^2 \geq c > 0$ on $M \setminus K$ for some constant c. □

Let $\{M\} \in \mathbf{K}_0(\mathbf{C})$ be the class defined by the operator of Dirac type $B := D + \Phi$. It is easy to see that $\{M\}$ does only depend on the stable homotopy class of p in finite **K**-theory. In fact two different choices of the lift P can be deformed into each other such that we get a C^0-deformation of the corresponding admissible Φ's. This does not change the **K**-class of $\{M\}$ by Corollary 4.4. Stabilization results in forming a direct sum of B with some invertible operator of Dirac type. Homotopies of p inside finite matrices can be lifted to C^0-continuous homotopies of the P and thus of the Φ inside the admissible endomorphisms. Again this does not change the class $\{M\}$.

Definition 5.2 *The map $u : \mathbf{K}^0_f(\partial_h M) \to \mathbf{Z}$ is given by $p \to \{M\}$.*

Obviously u is a group homomorphism.

Since D commutes with the action of $C^{1,0}$ given by

$$\epsilon \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where ϵ generates $C^{1,0}$ with $\epsilon^2 = 1$ we have the following fact.

Proposition 5.3 *If there is a gap in the spectrum of D_E then then $u = 0$.*

Proof: Let D and Φ be as in the construction above. The important property of Φ is that it anticommutes with ϵ. Let $\lambda \in res\ D$. Consider the two-parameter family

$$B_{t,s} := D - s\lambda\epsilon + t\Phi$$

We can deform $B_{1,0}$ to $B_{0,1}$ inside the operators of Dirac type beeing invertible at infinity. In fact let first s go from zero to one and then t go from one to zero. But $B_{0,1}$ commutes with ϵ thus its index vanishes. □

This fits well into the results of Roe and others stating that the odd **K**-homology class constructed out of an ungraded Dirac operator vanishes if there is a gap in the spectrum.

5.2 Computation of u

Let $N \subset M$ be a compact hypersurface cutting $M = M_- \cup_N M_+$ such that M_- is compact and contains an essential support of Φ. Deform the metric, the Clifford bundle structure and Φ in a neighbourhood of N such that there is a product collar near $N$and Φ is constant in normal direction. This makes glueing easier. Let $M_1 := M_- \cup_N [0,\infty) \times N$. There is a Clifford bundle $S_1 \to M_1$ respecting the product structure over the cylindrical end and restricting to $S_{|M_-}$ on M_-. In fact one restricts S to S_N on N. Then $S_{1|[0,\infty)\times N} = pr_N^* S_{|N}$ with induced connection and Clifford bundle structure. One can extend $-\Phi_{|M_-}$ constantly along the cylinder obtaining Φ_1. We form $M_2 := M \cup -M_1^{op}$ where *op* means that we choose the opposite grading on the Clifford bundle over M_1 and $-$ stands for redefining the Clifford bundle structure such that $X \in TM_1$ acts as $-X$. (note that we compress all structures over the manifolds into the symbols $M, M_1, \ldots$). Then $\{M_2\} = \{M\} - \{M_1\}$.

Now we apply the relative index theorem. We cut the manifold M_2 at the two copies of N, i.e. $N \cup \{0\} \times N$ and glue together interchanging the boundary components obtaining $M_3 := M_- \cup_N -M_-^{op} \cup M_+ \cup_N -(\mathbf{R}_+ \times N)^{op}$. In order to glue the Clifford bundles we have to use an isomorphism Γ of degree one such that $\Gamma X = -X\Gamma$ for $X \in TN$ and $\Gamma n = n\Gamma$ for the unit normal vector field at N (note that the diffeomorphism used for glueing transports n to $-n$). We take Γ simply as multiplication with n. In fact we have also $n\Phi = -\Phi n$. Thus the admissible endomorphisms glue nicely. By the relative index theorem $\{M_2\} = \{M_3\}$. But $\{M_3\} = 0$. On one hand $\{M_- \cup_N -M_-^{op}\} = 0$ since this manifold is compact and hence the Dirac operator has gap in the spectrum such that we can apply Proposition 5.3. On the other hand an essential support of the admissible endomorphism over $M_+ \cup_N -(\mathbf{R}_+ \times N)^{op}$ is empty. Thus $\{M_+ \cup_N -(\mathbf{R}_+ \times N)^{op}\} = 0$ by Corollary 4.5. Hence $\{M\} = \{M_1\}$. Thus we have replaced one half of the manifold by a cylinder. In the next step we do the same with the other half.

Consider $M_4 := M_1 \cup -M_1$ where we chose the admissible endomorphism Φ_4 as $-\Phi_1$ on the first component and Φ_1 on the second. Then $\{M_4\} = 2\{M_1\}$. Now we deform Φ_4 on a compact set such that it is zero near the two copies $\{0\} \times N$ of N in M_4. Cut M_4 at these two copies and glue together again interchanging the boundary components obtaining $M_5 := M_- \cup_N -M_- \cup M_6$ where $M_6 = \mathbf{R} \times N$. For glueing the Clifford bundle we use the even isomorphism of $S_{|_N}$ given by multiplication with $\Gamma := diag(n,n)$. By the relative index theorem $2\{M_1\} = \{M_5\}$. But $\{M_- \cup_N -M_-\} = 0$ since this manifold is compact and therefore we can apply Proposition 5.3. Hence $2\{M\} = \{\mathbf{R} \times N\}$. Thus we have reduced the index computation to a cylinder.

But there we can calculate explicitly. Let Φ_c be the admissible endomorphism over the cylinder. By a further deformation over compact sets we can assume that $\Phi_c = \psi\Phi_{|N}$ where ψ is a smooth function on $\mathbf{R} \times N$ depending only on the first coordinate such that $\psi(r,n) = sign(r)$ for $|r| \geq 1$. Moreover we can deform $\Phi_{|N}$ to an involution or, what is the same, $P_{|N}$ to a projection. This amounts to an index-preserving deformation of Φ_c by Lemma 4.4. Then

$$\begin{aligned}
\{\mathbf{R} \times N\} &= index\left(n\frac{\partial}{\partial r} - \psi(r)\imath(1-2P) + D_N\right) \\
&= index\left(\frac{\partial}{\partial r} + \psi(r)\imath n(1-2P) - nD_N\right) \\
&= index\left(\frac{\partial}{\partial r} + \psi(r)n\imath(1-2P) - n(1-2P)D_N(1-2P)\right)
\end{aligned}$$

where D_N is the Dirac operator on $S_{|N}$. Note that $(1-2P)D_N(1-2P) = (1-P)D_N(1-P) + PD_NP - PD_N(1-P) - (1-P)D_NP$. But $(1-P)D_NP$ and $PD_N(1-P)$ are of zero order. Hence by an index preserving deformation

$$\{\mathbf{R} \times N\} = index\, T$$

with

$$T := \left(\frac{\partial}{\partial r} + \psi(r)n\imath(1-2P) - [n(1-P)D_N(1-P) + nPD_NP]\right).$$

By explicit calculation using separation of variables and employing the L^2-condition (similar to the caluculation of Higson [15]) one obtains that the kernel of T and T^* consists of elements of the form

$$\begin{aligned} f_v^-(r) &= = exp\left(-\int_0^r \psi(s)\imath n(1-2P)\right) v \\ f_v^+(r) &= exp\left(+\int_0^r \psi(s)\imath n(1-2P)\right) v \end{aligned}$$

with $v \in ker\, n[(1-P)D_N(1-P) + PD_NP]$. Note that $\imath n(1-2P)$ is an involution anticommuting with $n[(1-P)D_N(1-P) + PD_NP]$. Thus we can split

$$ker\, n[(1-P)D_N(1-P) + PD_NP] = W^+ \oplus W^-$$

into the eigenspaces of $\imath n(1-2P)$ to the eigenvalues ± 1. f_v^+ is in L^2 exactly when $v \in W^-$ and f_v^- is in L^2 iff $v \in W^+$. Thus $index\, T = dim\, W^+ - dim\, W^-$. Let $V_- \to N$ be the vector bundle over N given by P and V_+ be the bundle given by $(1-P)$. Then $n((1-P)D_N(1-P) + PD_NP)$ is the sum of the two twisted Dirac operators D_{N,V_+} and D_{N,V_-} and $dim\, W^- + dim\, W^- = ind_{\imath n}\, D_{N,V_+} - ind_{\imath n}\, D_{N,V_-}$ where the grading is given by the involution $\imath n$. By cobordism invariance $ind_{\imath n}\, D_{N,V_+} = -ind_{\imath n}\, D_{N,V_-}$. Thus

$$u([p]) = \{M\} = \frac{1}{2}\{\mathbf{R} \times N\} = \frac{1}{2} index\, T = ind_{\imath n} D_{N,V_+}$$

Let us formulate this result as a theorem.

Theorem 5.4 *Let $[p] \in \mathbf{K}_f^0(\partial_h M)$ and $P \in C_g(M) \otimes Mat(N)$ be a selfadjoint representer of p. Let $N \subset M$ be a compact hypersurface cutting $M = M_- \cup_N M_+$ where M_- is compact and $(1-2P)^2 \geq c > 0$ on M_+ for some constant c. Let V_+ be the subbundle of $N \times \mathbf{C}^N$ given by the eigenvalues $< \frac{1}{2}$ of $P_{|N}$. Then $u([p]) = ind D_{N,V_+}$ where D_{N,V_+} is is the Dirac operator of $E_{|N} \otimes V_+$ and the grading is given by $\imath n$ with the unit normal vector field n at N pointing into the direction of M_+.*

This reproves and generalizes the theorems of Callias [9] and Anghel [1]. It containes also the index theorem for partioned manifolds of Roe [23] in the formulation of Higson [15] as a special case. In fact a partioned manifold M is given by a complete Riemannian manifold M together with a function $\psi \in C^\infty(M)$ such that $|\psi| = 1$ outside of a compact set and $\psi^{-1}(0)$ is a smooth compact hypersurface. Then ψ induces an element $[(1+\psi)/2] \in$

$K_f^0(\partial_h M)$. Let $N_+ \cup N_-$ be the boundary of a small tubular neighbourhood M_- of N such that $\psi_{|N_-} < 0$. Hence $u([\psi]) = ind_{in} D_{N_-} = ind_{in} D_N$ where n is the unit normal vector field pointing into the region of M where ψ is negative.

Since the index of a Dirac operator on an odd-dimensional manifold vanishes we have $u = 0$ if M is even-dimensional.

5.3 Interpretation as KK-intersection product

The operator

$$\begin{pmatrix} 0 & D_E \\ D_E & 0 \end{pmatrix}$$

represents an unbounded cycle $[D] \in \mathbf{KK}(C_0(M), C^{1,0})$ where the action of the generator ϵ of $C^{1,0}$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The endomorphism

$$\Phi = \imath \begin{pmatrix} 0 & 1-2P \\ 2P-1 & 0 \end{pmatrix}$$

represents a cycle $[\Phi] \in \mathbf{KK}(C_g(M), C_0(M) \otimes C^{0,1})$ where the action of the generator of $C^{0,1}$ is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The construction of $\{M\}$ above is nothing else than forming the intersection product

$$[M] = [\Phi] \otimes_{C_0(M)} [D] \in \mathbf{KK}(C_g(M) \otimes C^{1,0}, C^{0,1}) = \mathbf{KK}(C_g(M), \mathbf{C}).$$

restricted to $\mathbf{KK}(\mathbf{C}, \mathbf{C})$. One can show this in the framework of bounded Kasparov modules using the definition given in [18] in terms of connections (This becomes easy since we are tensoring with a trivial bundle and thus $1 \otimes D$ makes sense).

Formulating things in terms of $\mathbf{K}$-homology and $\mathbf{K}$-cohomology we have $[D] \in \mathbf{K}^c_{-1}(M)$ and $[\Phi] \in \mathbf{K}_g^{-1}(M)$ and we have employed the pairing $\mathbf{K}_g^i(M) \otimes \mathbf{K}_j^c(M) \to \mathbf{K}_{j-i}^g(M)$.

6 The spectral flow pairing s

6.1 Construction

Let M be a complete Riemannian manifold and $E \to M$ be an ungraded Clifford bundle over M.

A cycle $[W] \in \mathbf{K}_c^1(M)$ is represented by an unitary matrix $W \in C^\infty(M) \otimes Mat(N)$ with $W = 1$ outside of a compact set $K \subset M$. Let $S := E \otimes \mathbf{C}^N$ and D be the associated Dirac operator. It is selfadjoint and defines a spectral projection $Q \in B(H^0)$ onto the positive spectral subspace of D. Consider $A := (1 - Q) + W^* Q$ where W acts on H^0 as unitary operator in the obvious way. It was shown in [8] that A is a Fredholm operator.

Definition 6.1 *The spectral flow pairing is given by* $s([W]) := indexA$.

In fact it is easy to see that $indexA$ depends only on the stable homotopy class of W and is additive under direct sums. One can compute this index by reducing to a compact manifold containing K. If M would be compact then $s([W])$ is given by the spectral flow of a family between D and WDW^*.

6.2 The relation between s and u

We show the commutativity of the lower part of the diagram in the introduction. Let $exp : \mathbf{K}^0(\partial_h M) \to \mathbf{K}_c(M)$ be the exponential map and $i : \mathbf{K}^0_f(\partial_h M) \to \mathbf{K}^0(\partial_h M)$.

Theorem 6.2 $u = s \circ exp \circ i$

Proof: Let $[p] \in K^0_f(\partial_h M)$ be represented by a selfadjoint $P \in C_g(M) \otimes Mat(N)$. We can choose P such that P is a projection outside of a compact set $K \subset M$. Then $exp(i([p]))$ is represented by the unitary $W := exp(2\pi \imath P)$. In fact $W = 1$ on $M \setminus K$. Choose a compact hypersurface $N \subset M$ which splits $M = M_- \cup_N M_+$ such that $K \subset M_-$. Let $I \times N$ be a tubular neighbourhood of N in M. We can choose another P_1 representing p such that $P_1 = P$ on the plus-side of $I \times N$ and $P_1 = 0$ on the minus-side. Let $W_1 = exp(2\pi \imath P_1)$. Then $W_1 = 1$ outside of the tubular neigbourhood of N and W_1 represents the same class as W. Thus consider P_1, W_1 instead of P, W. As it has been shown in [8] one can compute $s([W])$ by reduction to $M_1 := S^1 \times N$, a manifold which containes naturally our tubular neigbourhood $I \times N$. First one deforms the metric and the Clifford bundle structure near N such that it respects the product structure of the tubular neighbourhood. Then one can glue $S_{|I \times N}$ at the boundaries of of $I \times N$ obtaining a Clifford bundle $S_1 \to M_1$. Since $W = 1$ on these boundaries it extends to W_1 over M_1. Now one can form $A_1 := (1 - Q_1) + W_1^* Q_1$ where Q_1 is the positive spectral projection of the Dirac operator on S_1. By the results of [8] we have $index\, A_1 = index\, A$ and $index\, A_1$ is the spectral flow of a family between D_1 and $W_1 D_1 W_1^*$. Since W_1 is the image of a projection $P_1 := P_{|1 \times N}$ under the suspension map it is rather well known that this spectral flow equals $ind_{in} P_1 D_N P_1$. Comparison with Theorem 5.4 finishes the proof. □

Note that the spectral flow pairing must not vanish even when D has a gap in the spectrum. It gives then the spectral flow of the familiy between D and WDW^* in this gap [5].

7 The map g

7.1 Construction of g

Let M be a complete Riemannian spin manifold and $E \to M$ be a $\mathbf{Z}_2$-graded Clifford bundle over M with grading z. An element $[u] \in \mathbf{K}^1_f(\partial_h M)$ is represented by an unitary matrix $u \in C(\partial_h M) \otimes Mat(N)$ for some large N. Let $U \in C^\infty_g(M) \otimes Mat(N)$ be a lift of u (up to homotopy in order to obtain smoothness). Stabilization and homotopies of u can be lifted to stabilization and homotopies of U. Form the $\mathbf{Z}_2$-graded bundle $V := M \times (\mathbf{C}^N \oplus \mathbf{C}^N)$ and the graded tensor product $S := E \otimes V$. S is a Clifford bundle with Dirac operator D. Let

$$\Phi := z \otimes \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

and $B := D + \Phi$.

Lemma 7.1 *Φ is admissible.*

Proof: In fact $deg\,\Phi = 1$ and $\Phi^* = \Phi$. Moreover

$$D\Phi + \Phi D = z \otimes \begin{pmatrix} 0 & -grad\,U^* \\ -grad\,U & 0 \end{pmatrix}$$

vanishes at infinity while Φ^2 tends to 1 there. Hence there is some compact set $K \subset M$ such that $D\Phi + \Phi D + \Phi^2 \geq c > 0$ on $M \setminus K$ for some constant c. □

Let $\{M\} \in \mathbf{K}_0(\mathbf{C})$ be the class constructed with B. It is easy to see that it depends only on the stable homotopy class of U and therefore of u. Moreover it is additive under direct sum. Thus we can define the homomorphism g by

Definition 7.2 *The map g is given by $g([u]) := \{M\}$.*

7.2 Computation of g

Let $N \subset M$ be a compact hypersurface cutting $M = M_- \cup_N M_+$ such that M_- is compact and contains an essential support of Φ. We can deform U to an unitary matrix on M_+ using polar decomposition. Thus let us assume this. Further we assume that (after deformation near N) the metric and the Clifford bundle structure near N respect the product structure of a tubular neighbourhood $I \times N \subset M$ of N. Consider $M_1 := M \cup \mathbf{R} \times N$ where the Clifford bundle on $\mathbf{R} \times N$ comes from restriction $S_{|N}$. We extend $\Phi_{|N}$ constantly along the $\mathbf{R}$-direction obtaining Φ_1 over M_1. Then $\{M\} = \{M_1\}$ since an essential support of Φ_1 over $\mathbf{R} \times N$ is empty. We apply now the relative index theorem. We cut at the two copies $N \cup \{0\} \times N$ and glue interchanging the boundary components obtaining $M_2 = M_- \cup_N \mathbf{R}_+ \times N \cup \mathbf{R}_- \times N \cup_N M_+$. The Clifford bundles are glued using the natural isomorphism. The admissible endomorphism glues nicely to Φ_2. Then $\{M\} = \{M_1\} = \{M_2\}$. But over $\mathbf{R}_- \times N \cup M_+$ an essential support is empty and thus $\{M\} = \{M_3\}$ where $M_3 := M_- \cup_N \mathbf{R}_+ \times N$. Again we have replaced half of the manifold by a cylinder.

Consider the matrix

$$W := \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \in C^\infty(M, End(S).$$

W is even, unitary and constant in the $\mathbf{R}$-direction over the cylindrical end. Deform the connection ∇ on S_3 near $\{1\} \times N$ such that it is $W\nabla W^*$ in a neighbourhood of $\{1\} \times N$ and deform Φ_3 there such that it is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = W\Phi_{|N}W^*.$$

Now we cut $M_- \cup_N \mathbf{R}_+ \times N$ at $\{1\} \times N \cup \{2\} \times N$ and glue together obtaining $M_4 := M_- \cup_N [0,1] \times N \cup_N [2,\infty] \times N \cup S^1 \times N$. We use W to identify

$$W^* : S_{|\{1\}\times N} \rightarrow S_{|\{2\}\times N}.$$

Then the admissible endomorphisms glue nicely to Φ_4. By the relative index theorem $\{M\} = \{M_4\}$. Since $W\Phi_{|N}W^*$ can be extended as invertible over $M_- \cup_N [0,1] \times N$ we have $\{M\} = \{S^1 \times N\}$. Since this manifold is compact we can deform Φ_4 to zero there. Let us formulate this result as a theorem.

Theorem 7.3 *Let $[u] \in K^1_f(\partial_h M)$ be represented by $U \in C^\infty_g(M) \otimes Mat(N)$. Let $N \subset M$ be a compact hypersurface of M splitting $M = M_- \cup_N M_+$ such that M_- is compact and U is invertible over M_+. Then $g([u])$ is given by the index of the twisted Dirac operator D_L over $S^1 \times N$ where $L \rightarrow S^1 \times N$ is constructed from the trivial bundle $I \times N \times \mathbf{C}^N$ by glueing with*

$$U_{|N} : \{0\} \times N \times \mathbf{C}^N \rightarrow \{1\} \times N \times \mathbf{C}^N.$$

$g([u])$ is also the spectral flow of a family between D_N and $U^*_{|N} D_N U_{|N}$ where D_N is half of the Dirac operator in $S_{|N}$.

7.3 Interpretation as KK-intersection product

The operator D represents an unbounded cycle $[D] \in \mathbf{KK}(C_0(M), \mathbf{C})$ while

$$\Phi = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

represents an cycle $[\Phi] \in \mathbf{KK}(C_g(M), C_0(M))$. The construction of $\{M\}$ is nothing else than forming the intersection product $[M] = [\Phi] \otimes_{C_0(M)} [D] \in \mathbf{KK}(C_G(M), \mathbf{C})$ restricted to element $\mathbf{KK}(\mathbf{C}, \mathbf{C})$. Formulating things in terms of $\mathbf{K}$-homology and $\mathbf{K}$-cohomology we have $[D] \in K^c_0(M)$ and $[\Phi] \in K^0_g(M)$ and we have again employed the pairing $K^i_g(M) \otimes K^c_j(M) \rightarrow K^g_{j-i}(M)$.

8 The relative index pairing r

8.1 Construction

Let M be a complete Riemannian manifold and $E \rightarrow M$ be a $\mathbf{Z}_2$-graded Clifford bundle over M. An element $\tau \in K_c(M)$ is represented by a tuple (V_1, V_2, W) where V_1, V_2 are vector bundles over M and $W \in C^\infty(M, Hom(V_1, V_2))$ is invertible outside of a compact subset $K \subset M$. Let $N \subset M$ be a hypersurface cutting $M = M_- \cup_N M_+$ such that M_- is compact and contains K. Take $\tilde{M} = M_- \cup_N -M_-^{op}$ where we glue the Clifford bundle E using the odd morphism given by the multiplication with the unit normal vector field n pointing into M_+. We glue $V_{1|M_-}$ with $V_{2|-M_-^{op}}$ using the identification

$$W_{|N} : V_{1|M_-} \to V_{2|-M_-^{op}}$$

obtaining $V \to \tilde{M}$. Let $\tilde{D}_V$ be the twisted Dirac operator on $\tilde{E} \otimes V$.

Definition 8.1 *The relative index pairing is given by* $r(\tau) := ind(\tilde{D}_V)$.

It has been shown that this definition does not depend on the choice of the representer of τ and the hypersurface N (for the latter see [13])

8.2 The relation between r and g

We show the commutativity of the upper part of the diagram in the introduction. Let $i : \mathbf{K}^1_f(\partial_h M) \to \mathbf{K}^1(\partial_h M)$.

Theorem 8.2 $r \circ \delta \circ i = g$.

Proof: Let $[u] \in \mathbf{K}^1_f(\partial_h M)$ be represented by the unitary matrix $u \in C(\partial_h M) \otimes Mat(N)$ and let $U \in C_g(M) \otimes Mat(N)$ be a lift of u. We can assume that U is unitary outside of a compact set $K \subset M$. Then $\delta \circ i([u])$ is given by $\tau = [(M \times \mathbf{C}^N, M \times \mathbf{C}^N, U)]$. Let $N \subset M$ be a compact hypersurface cutting $M = M_- \cup_N M_+$ such that $K \subset M_-$. Let $I \times N$ be a tubular neighbourhood of N. We can deform U on M_- such that $U_{M_- \setminus I \times N} = 1$. Hence we can construct $r(\tau)$ as follows. We glue

$$\tilde{M} = S^1 \times N = I \times M \cup -(I \times N)^{op}.$$

We have to glue the trivial bundle $I \times N \times \mathbf{C}^N$ with the trivial bundle $-(I \times N)^{op} \times \mathbf{C}^N$ using the identifications

$$1 : \{0\} \times N \times \mathbf{C}^N \to -(\{0\} \times N)^{op} \times \mathbf{C}^N$$

and

$$U_{|N} : \{1\} \times N \times \mathbf{C}^N \to -(\{1\} \times N)^{op} \times \mathbf{C}^N$$

obtaining $V \to S^1 \times N$: Then $r(\tau) = ind\tilde{D}_V$. Comparison with Theorem 7.3 gives the desired result. □

9 A real construction

9.1 The pairing

There are geometric constructions where real Dirac operators arrise naturally [20],[25],[16]. These Dirac operators commute with a natural right action of some real Clifford algebra $C^{0,n}$ The real index is an equivalence class of the kernel as a graded module over this Clifford algebra modulo restrictions of graded modules of $C^{0,n+1}$. Let, for example, M^n be spin with spin structure $P_{Spin} \to M$. Then the bundle $P_{Spin} \times_{Spin(n)} C^{0,n}$ admits a right action of $C^{0,n}$. Here $Spin(n) \subset C^{0,n}$ acts by left multiplication on $C^{0,n}$.

The index constructions are most compactly formulated in terms of **KK**-theory. In this section we consider **KK**-theory for real C^*-algebras. Also $C_g(M), C_0(M)$, etc. consist of real functions.

Let M be a complete Riemannian manifold and $E \to M$ be a real $\mathbf{Z}_2$-graded Clifford bundle over M admitting a right action of the real Clifford algebra $C^{0,n}$. Let z be the grading operator. The Dirac operator D is $C^{0,n}$-equivariant and represents an element $[D] \in \mathbf{KK}(C_0(M), C^{0,n})$. For later purpose it is better to understand $[D] \in \mathbf{KK}(C_0(M) \otimes C^{n,0}, \mathbf{R})$. The action of an element $f \otimes e \in C_0(M) \otimes \mathbf{R}^n$ is given by $-fze$ i.e. first multiplication with $-e$ from the right followed by the action of z and left multiplication with f. Here $\mathbf{R}^n$ generates $C^{0,n}$ as well as $C^{n,0}$. This construction realizes an isomorphism

$$\mathbf{KK}(C_0(M) \otimes C^{n,0}, \mathbf{R}) = \mathbf{KK}(C_0(M), C^{0,n}).$$

In terms of K-homology we have $[D] \in \mathbf{K}_n^0(M)$. We want to pair $[D]$ with certain elements $[\Phi] \in \mathbf{KK}(C_g(M), C_0(M) \otimes C^{k,0})$. For simplicity we assume $k < n$, this can easely be obtained by tensoring $[D]$ with $C^{8,0}$ several times and using Bott periodicity. We consider $[\Phi]$ represented by a matrix $\Phi \in Mat(C_g(M) \otimes C^{k,0}, N)$ for some $N > 0$ with the properties $\Phi^* = \Phi$, $deg\, \Phi = 1$ and that $\Phi^2 - 1$ vanishes at infinity. We construct now an operator of Dirac type representing

$$[M] = [\Phi] \otimes_{C_0(M) \otimes C^{k,0}} [D] \in K^g_{n-k}(M).$$

Form the graded tensor product $S := (C^{k,0})^{\oplus N} \otimes_{C^{k,0}} E$. This is a real $\mathbf{Z}_2$-graded Clifford bundle with right action of $C^{0,n-k}$. In fact let $\mathbf{R}^k \subset \mathbf{R}^n$ generate $C^{0,k}$ and $C^{k,0}$ and $\mathbf{R}^{n-k} = (\mathbf{R}^k)^\perp$ generate $C^{0,n-k}$ then $e \in R^k$ acts by $-ze$ on E and this commutes with the action of $f \in \mathbf{R}^{n-k}$. Let D_S be the Dirac operator of S and form $B := D_S + \Phi$. Note that Φ acts naturally on S.

Lemma 9.1 *Φ is admissible.*

Then $[\Phi] \otimes_{C_0(M) \otimes C^{0,k}} [D] = [M] \in \mathbf{KK}(C_g(M), C^{0,n-k}) = \mathbf{K}^g_{n-k}(M)$ is the class constructed with B. Let $\{M\} \in \mathbf{KK}(\mathbf{R}, C^{0,n-k})$ be the restriction of $[M]$.

9.2 Computation of the real index

We want to compute $\{M\}$ in analogy to the complex case as far as possible. Let $N \subset M$ be a hypersurface cutting $M = M_- \cup_N M_+$ such that M_- is compact and contains an essential support of Φ. Again we assume (after deformation) that there is a tubular neighbourhood $I \times N \subset M$ of N where the metric and the Clifford bundle structure respect the product structure and where Φ is constant in the normal direction. Consider $M_1 := \mathbf{R} \times N$ with the induced Clifford bundle and with Φ_1 induced by $\Phi_{|N}$ constantly extended over the $\mathbf{R}$-direction. An essential support of Φ_1 is empty. Thus $\{M\} = \{M \cup M_1\}$. Now we cut at the two copies $N \cup \{0\} \times N$ and glue together again interchanging the boundary components obtaining $M_- \cup_N \mathbf{R}_+ \times N \cup \mathbf{R}_- \times N \cup_N M_+$. We apply the relative index theorem and obtain $\{M\} = \{M_2\}$ where $M_2 = M_- \cup_N \mathbf{R}_+ \times N$. The second component does not contribute since an essential support of the admissible endomorphism there is empty. So far the computation is similar to the one of g. We have replaced the nonessential part by a cylinder. It is not obvious how to go further. In the complex cases the special form of Φ or D are used. It is a question wether the index information is contained in a tubular neighbourhood of N. This is the case in the example below.

10 An example

Let M be a complete Riemannian manifold and $S \to M$ be a real $\mathbf{Z}_2$-graded Clifford bundle over M admitting a right $C^{0,n}$-action with associated Dirac operator D. Let $0 < k \leq n$ and consider $\mathbf{R}^k \subset \mathbf{R}^n$ where $\mathbf{R}^k$ generates $C^{0,k} \subset C^{0,n}$ and $(\mathbf{R}^k)^\perp$ generates $C^{0,n-k} \subset C^{0,n}$. Let $f \in C(\partial_h M) \otimes \mathbf{R}^k$ be given such that $f^2 = -1$ i.e. f has values in the unit sphere. Let $F \in C_g(M) \otimes \mathbf{R}^k$ be a lift of f. Then $\Phi := zF$ represents a cycle in $\mathbf{KK}(C_g(M), C_0(M) \otimes C^{k,0})$.

Lemma 10.1 *Φ is admissible.*

The operator of Dirac type $B := D + \Phi$ is $C^{0,n-k}$-equivariant and represents the class $[M] = [\Phi] \otimes_{C_0(M)\otimes C^{k,0}} [D]$. Let $\{M\} \in \mathbf{KK}(\mathbf{R}, C^{0,n-k})$ be the restriction of $[M]$.

10.1 Reduction to a product

We want to compute $\{M\}$. We can assume that 0 is a regular value of F. Then $N := F^{-1}(0)$ is a compact manifold of dimension $n - k$. We use F to identify a small neighbourhood of N with $D^k \times N$ where D^k is the unit disk in $\mathbf{R}^k$. After deformation of the metric we can assume that F is an isometry on this neighbourhood. Let $\tilde{F}$ be a deformation of F in a neighbourhood of N such that $\tilde{F}(x,n) = x/\|x\|$ in a neighbourhood of $S^{k-1} \times N = \partial(D^k \times N)$. Let $M_1 := \mathbf{R}^k \times N$ with $F_1(x,n) = \tilde{F}(x,n)$ on $D^k \times N$ and $x/\|x\|$ for $\|x\| \geq 1$. M_1 has a Clifford bundle S_1 induced from M respecting the product structure. Let $\Phi_1 := -zF_1$. Note that Φ_1 is admissible. Let $M_2 := M \cup -M_1^{op}$. We split M_2 at the two copies of $S^{k-1} \times N$ and glue together interchanging the boundary components obtaining $M_3 = M_4 \cup M_5$ with

$$M_4 := M \setminus D^k \times N \cup_{S^{k-1}\times N} -(R^k \setminus D^k \times N)^{op}$$

and

$$M_5 := D^k \times N \cup_{S^{k-1}\times N} -(D^k \times N)^{op}.$$

We glue the Clifford bundles using the odd morphism given by left multiplication with the unit normal vector field at the boundary $S^{k-1} \times N$. By the relative index theorem $\{M_2\} = \{M_3\}$. But $\{M_3\} = 0$ since M_5 is compact and thus Φ_5 can be deformed to zero and the Dirac operator itself without perturbation is equivariant with respect to a larger Clifford algebra. On M_4 an essential support of the admissible endomorphism is empty. Hence $\{M\} = \{M_1\}$. Thus we have reduced the problem to a product $M_1 = \mathbf{R}^k \times N$ where $\Phi_1(x, n) = zx/\|x\|$ for $\|x\| \geq 1$.

10.2 The product case

Let $M := M_1$ and $\Phi := \Phi_1$ for a moment. The Clifford bundle $S \to \mathbf{R}^k \times N$ is the pull back of a bundle S^k over N admitting a right action of $C^{0,n}$ and a left action of $C^{0,k}$ (which commutes with the action of the Clifford algebra bundle of N in a graded way). This left action is used to define the Clifford multiplication with $T\mathbf{R}^k$ on $pr_N^* S^k$ in order to obtain the Clifford bundle S over $\mathbf{R}^k \times N$. We want to use induction in k. In order to employ the product structure $\mathbf{R}^k = \mathbf{R} \times \mathbf{R}^{k-1}$ we have to deform Φ such that it respects this product structure. For $1/2 > \epsilon > 0$ let $\psi \in C^\infty(\mathbf{R})$ be such that $\psi(r) = sign(r)$ for $|r| \geq 2/\epsilon$, $\psi(r) \in [-1, 1]$ for $|r| \leq 2/\epsilon$ and $|\partial_r \psi| \leq 2\epsilon$. Form $\Phi_1(x, n) := z(\psi(x^1), \dots, \psi(x^k))$.

Lemma 10.2 *Φ_1 is admissible.*

Proof: In fact $deg\,\Phi_1 = 1$ and $\Phi_1^* = \Phi_1$. Moreover $\Phi_1 D + D\Phi_1 + \Phi_1^2 = -z\,grad(z\Phi_1) + \Phi_1^2$. But $\|grad(z\Phi_1)\| \leq 2\epsilon$ and for $\|x\| \geq 2/\epsilon$ we have $\Phi_1^2 \geq 1$. □

Let $\{M_1\}$ be the class defined by $B_1 := D + \Phi_1$.

Lemma 10.3 $\{M\} = \{M_1\}$

Proof: We can deform Φ to Φ_1 inside the admissible endomorphisms. Let $\Phi_t = t\Phi + (1-t)\Phi_1$. Then for $\|x\| \geq 2/\epsilon$

$$
\begin{aligned}
& \Phi_t D + D\Phi_t + \Phi_t^2 \\
= \; & -tz\,grad(z\Phi) - (1-t)z\,grad(z\Phi_1) + t^2\Phi^2 + (1-t)^2 \sum_{i=1}^k \psi(x^i)^2 \\
& +2t(1-t)\frac{1}{\|x\|}\sum_{i=1}^k x^i \psi(x^i) \\
\geq \; & t^2 + (1-t)^2 - t|grad(z\Phi)| - 2(1-t)k\epsilon \\
\geq \; & 1/8
\end{aligned}
$$

for ϵ small enough. □

Let now $\mathbf{R}^k = \mathbf{R} \times \mathbf{R}^{k-1}$ and n be the unit normal vector in $\mathbf{R}$-direction. n generates $C^{0,1}$ and we can split $S^k = S^{k-1} \otimes C^{0,1}$ by the following procedure. Let $e := zlr$ where l is the action of n from the left and r is the action of n from the right. e is an even involution of S^k. Let S^{k-1} be the positive eigenspace of e. Then S^{k-1} admits an $C^{0,n-1}$-action from the right where $C^{0,n-1}$ is generated by $\mathbf{R}^\perp \subset \mathbf{R}^n$, is a Clifford bundle over N and admits a $C^{0,k-1}$ action from the left commuting in a graded sense with the action of the Clifford algebra bundle of N. In fact the generators of $C^{0,k-1}$ acting from the left and of $C^{0,n-1}$ acting from the right commute with the involution e.

For $0 \le i \le k$ let $M^{(i)} := \mathbf{R}^i \times N$ with Clifford bundle induced from S^i. Construct the admissible endomorphism $\Phi_1^{(i)}$ as above and form $\{M^{(i)}\}$.

Proposition 10.4 $\{M^{(i)}\} = \{M^{(i-1)}\}$

Proof: We identify $ker(D^{(i)} + \Phi^{(i)})$ with $ker(D^{(i-1)} + \Phi^{(i-1)})$ as a $C^{0,n-k}$-module. With respect to $S^i = S^{i-1} \otimes C^{0,1}$ we have

$$D^{(i)} + \Phi^{(i)} = l\frac{\partial}{\partial x^k} + zr\psi(x^k) + [D^{(i-1)} + \Phi^{(i-1)}] \otimes 1.$$

Multiplying with $-l$ we obtain

$$A := \frac{\partial}{\partial x^k} + zlr\psi(x^k) - l[D^{(i-1)} + \Phi^{(i-1)}] \otimes 1.$$

A respects the splitting

$$L^2(\mathbf{R}^i \times N, S^i) = L^2(\mathbf{R}) \otimes ker([D^{(i-1)} + \Phi^{(i-1)}] \otimes 1) \oplus L^2(\mathbf{R}) \otimes ker([D^{(i-1)} + \Phi^{(i-1)}] \otimes 1)^{\perp}$$

and is invertible on the second component (compare Higson [15] for a similar argument). Thus the elements of $ker\ A$ are of the form

$$f_v(x^k) = exp(-zlr\int_0^{x^k} \psi(t)dt)v.$$

From the L^2-condition we see that $ker A$ is the 1-eigenspace of zlr in $ker([D^{(i-1)} + \Phi^{(i-1)}] \otimes 1)$. But this is exactly $ker(D^{(i-1)} + \Phi^{(i-1)})$. All identifications we have made respect the structure of graded $C^{0,n-k}$-modules. This proves the Proposition. □

10.3 A real index theorem and applications

We come now back to the general situation. Let M be a complete Riemannian manifold and $S \to M$ be a real $\mathbf{Z}_2$-graded Clifford bundle over M admitting a right $C^{0,n}$-action with associated Dirac operator D. Let $F \in C_g(M) \otimes \mathbf{R}^k$ for $0 < k \le n$ as above with $F^2 \to -1$ at infinity. Assume that 0 is a regular value of F and let $N := F^{-1}(0)$. We have constructed a Clifford bundle $S^0 \to N$ with Dirac operator $D_N^{(0)}$ admitting a right $C^{0,n-k}$-action such that $S_{|N} = S^0 \otimes C^{0,k}$. Let $B := D + zF$ and $\{M\} \in \mathbf{KK}(\mathbf{R}, C^{0,n-k})$ be the class constructed with B. Putting the above results together we have

Theorem 10.5 $\{M\} = ind\, D_N^{(0)}$.

10.4 Obstructions against positive scalar curvature

If M is a Riemannian spin manifold of dimension n there is a natural real Clifford bundle $S \to M$ admitting a $C^{0,n}$-action. In fact $S := P_{Spin} \times_{Spin(n)} C^{0,n}$. Let $F \in C_g(M) \otimes \mathbf{R}^k$, $n \geq k \geq 1$ and N as above. Then N is spin and S^0 is the natural Clifford bundle associated to the spin structure of N. In this case $ind\, D_N^{(0)} = \alpha(N)$ where $\alpha(N)$ is the alpha invariant defined by Hitchin [16],[20]. Let D be the Dirac operator on M. We have the Weizenboeck formula $D^2 = \Delta + \tau/4$ where τ is the scalar curvature of M.

If τ is uniformly positive at infinity then D is invertible at infinity and hence $\{M\} = 0$ for any F. In fact we can then deform F to zero without changing the index. But D is equivariant with respect to a larger Clifford algebra $C^{0,n}$ and hence the class of its kernel vanishes. It follows $\alpha(N) = 0$. Since the conditions on F depend only on the quasi-isometry class we obtain

Corollary 10.6 *Let M^n be a complete Riemannian spin manifold and $F \in C_g(M) \otimes \mathbf{R}^k$ be such that $|F| \to 1$ at infinity and 0 is a regular value. If $\alpha(F^{-1}(0)) \neq 0$ then there is no metric in the given quasi-isometry class which has uniform positive scalar curvature at infinity.*

A special case is:

Corollary 10.7 *If $\alpha(N) \neq 0$ then there is no metric with uniform positive scalar curvature at infinity on the product $\mathbf{R}^k \times N$ for any k in the quasi-isometry class of the product.*

Another application is

Corollary 10.8 *If $\alpha(N) \neq 0$ then there is no metric of non-negative scalar curvature τ on $\mathbf{R}^k \times N$ with lower bound $\tau(x,n) > \frac{c}{|x|+1}$ for some $c > 0$ in the quasi-isometry class of the product.*

Proof: Assume that $M := \mathbf{R}^k \times N$ has a metric with non-negative scalar curvature in the quasi-isometry class of the product satisfying $\tau(x,n) \geq \frac{c}{|x|+1}$ for some constant $c > 0$. Consider $B_t := D + tzF$ with $F \in C^\infty(M, \mathbf{R}^k)$ depending only on the first coordinate and satisfying $F^{-1}(0) = \{0\} \times N$ and $F(x,n) = x/|x|$ for $|x| > 1$. Then $B_t^2 = \Delta + \tau/4 - zt\, grad\, F + t^2F^2$. But $\|grad\, F(x,n)\| \leq \frac{c_1}{|x|+1}$ for some $c_1 > 0$. Thus

$$\tau/4 - zt\, grad\, F + t^2F^2 \geq \frac{c/4 - c_1 t}{|x|+1} + t^2F^2.$$

If we chose $t < \frac{c}{4c_1}$ then B_t is positive and hence invertible. It follows $\{M\} = \{M_t\} = 0$ and $\alpha(N) = 0$. □

Let M again be a complete Riemannian spin manifold with non-negative scalar curvature. Let $F \in C^\infty(M) \otimes \mathbf{R}^k$ with 0 as a regular value and which is constant of unit length outside of a tubular neighbourhood U of $N = F^{-1}(0)$. Assume that the scalar curvature is positive on U. Then

Proposition 10.9 $\alpha(N) = 0$

Proof: Consider $B_t = D + tzF$ for small $t > 0$. B_t is invertible at infinity. Thus by Theorem 10.5 and deformation invariance $\alpha(N) = \{M\} = \{M_t\}$. Then $B_t^2 = \Delta + \tau/4 - zt\, grad(F) + t^2|F|^2$. If t is small enough then $\tau/4 - zt\, grad(F) \geq 0$ and hence B_t is invertible. But then $\{M_t\} = 0$. □

(This sort of argument the author has learned from M. Lesch [21]).

Corollary 10.10 *If $\alpha(N) \neq 0$ then there is no complete Riemannian metric on $\mathbf{R} \times N$ which is positive in some section $\{a\} \times N \subset \mathbf{R} \times N$.*

All these obstructions have refinements to the case of finite fundamental groups where one employs Clifford bundles twisted with flat bundles with fibre $C_r^*(\pi_1(M))$. In order to cover the case of general fundamental groups one has to find a proof of Proposition 10.4 circumventing the problem of non-projective modules which may occur as a kernel. We hope to do this in a forthcoming paper.

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