SIXTH PAINLEVÉ EQUATION, UNIVERSAL ELLIPTIC CURVE, AND MIRROR OF P^2

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0. Introduction

0.1. Three approaches to the Painlevé equations. The differential equations studied in this paper form a family $\mathrm{PVI}_{\alpha,\beta,\gamma,\delta}$ depending on four parameters $\alpha, \beta, \gamma, \delta$, and classically written as:

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+ \frac{X(X-1)(X-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right]. \qquad (0.1)$$

They were discovered around 1906 and have been approached from at least three different directions.

a. Study of non-linear ordinary differential equations of the second order whose solutions have no movable critical points.

Their classification program was initiated by Painlevé, but he inadvertently omitted (0.1) due to an error in calculations. It was B. Gambier [G] who completed Painlevé's list and found (0.1).

b. Study of the isomonodromic deformations of linear differential equations.

c. Theory of abelian integrals depending on parameters and taken over chains with boundary (not necessarily cycles.)

These two approaches are due to R. Fuchs [F].

In the subsequent development of the theory, relationship with isomonodromic deformations proved to be most fruitful. For some recent research and bibliography the reader may consult [JM], [O1], [H1], [H2].

In this paper I take up the somewhat neglected approach via abelian integrals and algebraic geometry.

My principal motivation was the desire to understand the quantum cohomology of $\mathbf{P}^2$ and to find an algebraic-geometric object which could be reasonably called the mirror of $\mathbf{P}^2$, thus tentatively extending the scope of the mirror duality discovered for Calabi–Yau manifolds.

As was explained in a preprint version of [D] (cf. also [DFI] and [H3]), the potential of the quantum cohomology of $\mathbf{P}^2$ can be reduced by a change of variables to a particular solution to the Painlevé equation with parameters $(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, 0, \frac{1}{2})$. The Painlevé transcendents are generally "new" functions, but for certain values of parameters all or some solutions can be expressed through more classical special functions. Whether this is true for the $\mathbf{P}^2$–solution referred to above, seems an open problem. N. Hitchin completely solved the equation $(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$ in elliptic functions: cf. [H2].

Trying to understand all this, I arrived to the basically algebraic–geometric picture of all Painlevé VI equations, which in particular suggests that the mirror of $\mathbf{P}^2$ can be thought of as a pencil of elliptic curves with labelled sections of order two and an additional, possibly transcendental, multisection. More precisely, the Picard–Fuchs equation for the periods of the mirror dual Calabi–Yau family is replaced in our framework by a "non–homogeneous Picard–Fuchs equation" satisfied by the Abelian integral from zero to this additional multisection (cf. formula (1.5) below.) It would be important to understand whether this pattern persists for quantum cohomology of other Fano manifolds.

I will now briefly describe this picture stressing its geometric aspects. A reader with more analytic background may prefer to skip the following section. In the main body of the paper, a prominent role is given to the uniformization picture using elliptic and modular functions. In particular, it allows us to reduce the $\mathbf{P}^2$–equation to the beautiful form

$$\frac{d^2 z}{d\tau^2} = -\frac{1}{8\pi^2}\wp_z(z,\tau), \tag{0.2}$$

where $\wp$ is the Weierstrass function. The special solution to (0.2) corresponding to $\mathbf{P}^2$ at a point with complex multiplication by the cubic root of unity passes through a point of order three. For details, see [M2], Chapter II, 5.6.1.

0.2. Algebraic geometry of Painlevé VI: a review. We will describe a series of constructions which starts with a pencil of elliptic curves. We work in the category of complex analytic manifolds, although the most natural category for this part seems to be that of schemes over Spec $\mathbf{Z}[\frac{1}{2}]$.

a. Let $(\pi : E \to B; D_0, \dots, D_3)$ be a pencil of compact smooth curves of genus one, with variable absolute invariant, endowed with four labelled sections D_i such that if any one of them is taken as zero, the others will be of order two.

We will call E *a configuration space* of PVI (common for all values of parameters.) Solutions to all equations will be represented by some multisections of π.

b. Let $\mathcal{F}$ be the subsheaf of the sheaf of vertical 1-forms $\Omega^1_{E/B}(D_3)$ on E with pole at D_3 and residue 1 at this pole. It is an affine twisted version of $\Omega^1_{E/B}$ which is the sheaf of sections of the relative cotangent bundle $T^*_{E/B}$. Similarly, $\mathcal{F}$ itself "is" the sheaf of sections of an affine line bundle $F = F_{E/B}$ on E. More precisely, we construct such a bundle $\lambda : F \to E$ and a form $\nu_F \in \Gamma(F, \Omega^1_{F/B}(\lambda^{-1}(D_3))$ such that the map

$$\{\text{local section } s \text{ of } F\} \mapsto s^*(\nu_F)$$

identifies the sheaf of sections of F/E with $\mathcal{F}$.

We will call F *a phase space* for PVI (again, common for all parameter values.)

c. E carries a distinguished family of algebraic curves transversal to the fibers of E: considered as multisections of E/B they are of finite order (if any of D_i is chosen as zero.) It is important that each curve of this family has a canonical lifting to F (for its description, see the main text, formulas (2.12) and (2.29).)

d. F carries a closed 2-form $\omega^{(0)}$ which can be characterized by the following two properties:

i). The vertical part of $\omega^{(0)}$, i. e. its restriction to the fibers of $\pi \circ \lambda : F \to B$, coincides with $d_{F/B}(\nu_F)$.

ii). Any canonical lift to F of a connected multisection of finite order of $E \to B$, referred to above, is a leaf of the null-foliation of $\omega^{(0)}$.

e. E also carries four distinguished closed two-forms $\omega_0, \dots, \omega_3$. They are determined, up to multiplication by a constant, by the following properties.

iii). The divisor of ω_i is $\dfrac{D_j D_k D_l}{D_i^3}$ where $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

iv). Identify the sheaves Ω_E^2 and $\pi^*(\Omega_{E/B}^1)^{\otimes 3}$ on E using the Kodaira–Spencer isomorphism $\pi^*(\Omega_B^1) \cong (\Omega_{E/B}^1)^{\otimes 2}$ and the exact sequence $0 \to \pi^*(\Omega_B^1) \to \Omega_E^1 \to \Omega_{E/B}^1 \to 0$. Then the image of ω_i in $\pi^*(\Omega_{E/B}^1)^{\otimes 3}$ considered in the formal neighborhood of D_i is the cube of a vertical 1-form with a constant residue along D_i.

The affine space $P_0 := \omega^{(0)} + \sum_{i=0}^{3} \mathbf{C}\lambda^*(\omega_i)$ of closed two-forms on F is our version of the *moduli space of the PVI equations* replacing the classical $(\alpha, \beta, \gamma, \delta)$-space.

We can now summarize our definition of PVI equations and their solutions.

0.2.1. Definition. *a). A Painlevé two-form on F is a point $\omega \in P_0$.*
b). The Painlevé foliation corresponding to ω is the null-foliation of ω.
c). The solutions to the respective Painlevé equation are the leaves of this foliation (in the Hamiltonian description), or their projections on E.
The form $\omega^{(0)}$ corresponds to $(\alpha, \beta, \gamma, \delta) = (0, 0, 0, \frac{1}{2})$.

To obtain (0.1), we must specialize this description to the (projectivized) family

$$E_t : \; Y^2 = X(X-1)(X-t) \mapsto t \in B := \mathbf{P}^1 \setminus \{0, 1, \infty\} \tag{0.3}$$

and look at the variation of X along solutions.

0.3. Plan of the paper. In §1 we reproduce R. Fuchs's description of (0.1) in terms of elliptic integrals and deduce from it an analytic form of the Painlevé equations involving Weierstrass $\wp$-punction. As an application, we derive the elementary symmetries of PVI and introduce the Landin transform.

The key §2 is devoted to the Hamiltonian structure of PVI and contains proofs of all claims made in 0.2 above.

In §3 we establish the relationship of our Hamiltonian picture with that of Okamoto [O2] and sketch Okamoto's treatment of the hidden $W(D_4)$-symmetry of PVI. We also review some known solutions.

This symmetry nicely explains, why Hitchin was able to solve his $(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$-equation.

0.4. Further plans. The geometric setting advocated in this paper furnishes a convenient framework for the treatment of the following subject matters:

a. Three-dimensional Frobenius manifolds, including the quantum cohomology of $\mathbf{P}^2$.

b. Geometry of the degenerations of PVI to PV, ... , PI.

c. Generalizations to higher genus and isomonodromic deformations with many singular points.

I hope to return to these problems in future publications.

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§1. PVI and elliptic functions

1.1. Theorem (R. Fuchs, 1907). *The equation (0.1) can be written in the form*

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4}\right]\int_{\infty}^{(X,Y)}\frac{dx}{\sqrt{x(x-1)(x-t)}}=$$

$$=\alpha Y+\beta\frac{tY}{X^2}+\gamma\frac{(t-1)Y}{(X-1)^2}+(\delta-\frac{1}{2})\frac{t(t-1)Y}{(X-t)^2} \tag{1.1}$$

where $Y^2 = X(X-1)(X-t)$.

Proof. First, let us clarify the meaning of (1.1). Consider the family of elliptic curves $E \to B$ parametrized by $t \in \mathbf{P}^1 \setminus \{0,1,\infty\} := B$: the curve E_t is the projective closure of $Y^2 = X(X-1)(X-t)$. Points at infinity of $\{E_t\}$ form a section D_0 of this family which is the zero section for the standard group law on fibers. Choose in $E_t(\mathbf{C})$ a path from $D_0(t)$ to the point $(X(t), Y(t))$ of a local section. The operator

$$L_t := t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4} \tag{1.2}$$

annihilates the periods $\int \frac{dx}{y}$ along closed paths in $E_t(\mathbf{C})$ because

$$\left[t(1-t)\frac{\partial^2}{\partial t^2}+(1-2t)\frac{\partial}{\partial t}-\frac{1}{4}\right]\frac{d_{E/B}x}{y}=\frac{1}{2}d_{E/B}\frac{y}{(x-t)^2} \tag{1.3}$$

where we put $\frac{\partial}{\partial t}(x) = 0$ and $d_{E/B}t = 0$. Applying L_t to $\int_{\infty}^{(X,Y)}\frac{dx}{y}$ we get $\frac{1}{2}\frac{y}{(x-t)^2}\Big|_{\infty}^{(X,Y)}$ plus the contribution of the boundary sections which together with the right hand side of (1.1) amounts to (0.1).

1.2. μ–equations. The equation (1.1) is an instance of a general construction which was used in [M1] to prove the functional Mordell conjecture. We will recall it now.

A *μ–equation* is a system of non–linear PDE in which independent variables are (local) coordinates on a manifold B and unknown functions are represented by a section s of a family of abelian varieties (or complex tori) $\pi : A \to B$. To write this system explicitly, assume B small enough so that $\pi_*(\Omega^1_{A/B})$ and $\mathcal{D}_B$ (sheaf of differential operators on B) are $\mathcal{O}_B$–free, and make the following choices:

a. An $\mathcal{O}_B$–basis of vertical 1–forms $\omega_1, \dots, \omega_n \in \Gamma(B, \pi_*(\Omega^1_{A/B}))$.

b. A system of generators of the $\mathcal{D}_B$–module of the Picard–Fuchs equations

$$\sum_{i=1}^{n} L_i^{(j)} \int_\gamma \omega_i = 0, \quad j = 1, \dots, N, \tag{1.4}$$

where γ runs over families of closed paths in the fibers spanning $H_1(B_t)$.

c. A family of meromorphic functions $\Phi^{(j)}$, $j = 1, \dots, N$ on A.

The respective μ–equation for a local (multi)–section $s : B \to A$ reads then

$$\sum_{i=1}^{n} L_i^{(j)} \int_0^s \omega_i = s^*(\Phi^{(j)}), \quad j = 1, \dots, N, \tag{1.5}$$

where 0 denotes the zero section.

One drawback of (1.5) is its dependence on arbitrary choices. Clearly, this can be reduced by taking account of the transformation rules with respect to the changes of various generators. For elliptic pencils, the result takes a very neat form.

1.3. Elliptic μ–equations. Let again $E \to B$ be a non-constant one-dimensional family of elliptic curves. We temporarily keep the assumption that $\pi_*(\Omega^1_{E/B})$ and the tangent sheaf $\mathcal{T}_B$ are free. For any symbol of order two $\sigma \in S^2(\mathcal{T}_B)$ and any generator ω of $\pi_*(\Omega^1_{E/B})$ denote by $L_{\sigma,\omega}$ the Picard–Fuchs operator on B with the symbol σ annihilating all periods of ω.

1.3.1. Lemma. *For any local section s, the expression $L_{\sigma,\omega} \int_0^s \omega$ is $\mathcal{O}_B$–bilinear in σ and ω.*

Proof. Obviously,

$$L_{f\sigma,\omega} = f L_{\sigma,\omega}, \quad L_{\sigma,g\omega} = g L_{\sigma,\omega} \circ g^{-1},$$

where f, g are functions on B. The lemma follows.

Thus the expression

$$\mu(s) := \left(L_{\sigma,\omega} \int_0^s \omega \right) \otimes \sigma^{-1} \otimes \omega^{-1} \in S^2(\Omega^1_B) \otimes (\pi_* \Omega^1_{E/B})^{-1} \tag{1.6}$$

depends only on s and is compatible with restrictions to open subsets of B. This means that the natural domain of the right hand sides for elliptic μ–equations is the set of meromorphic sections Φ of the sheaf $\pi^* \left[S^2(\Omega^1_B) \otimes (\pi_* \Omega^1_{E/B})^{-1} \right]$.

Notice that the Kodaira–Spencer isomorphism (and eventually a choice of the theta–characteristic of B) allows us to identify Φ with a meromorphic section of $(\Omega^1_{E/B})^3$ or $\pi^*(\Omega^1_B)^{3/2}$ as well.

We will now lift the Fuchs–Painlevé equation (1.1) to the classical covering space, which in particular will make transparent the nature of its right hand side.

1.3.2. Uniformization. Consider the family of elliptic curves parametrized by the upper half-plane H: $E_\tau := \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau) \mapsto \tau \in H$. Recall that

$$\wp(z,\tau) := \frac{1}{z^2} + \sum{}' \left(\frac{1}{(z+m\tau+n)^2} - \frac{1}{(m\tau+n)^2} \right), \tag{1.7}$$

$$\wp_z(z,\tau) = -2 \sum \frac{1}{(z+m\tau+n)^3}. \tag{1.8}$$

We have

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau)-e_1(\tau))(\wp(z,\tau)-e_2(\tau))(\wp(z,\tau)-e_3(\tau)) \tag{1.9}$$

where

$$e_i(\tau) = \wp(\frac{T_i}{2},\tau),\ (T_0,\ldots,T_3) = (0,1,\tau,1+\tau) \tag{1.10}$$

and $e_1+e_2+e_3=0$. Functions $\wp$ and $\wp_z$ are invariant with respect to the shifts $\mathbf{Z}^2 : (z,\tau) \mapsto (z+m\tau+n,\tau)$ and behave in the following way under the full modular group Γ :

$$\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right) = (ct+d)^2 \wp(z,\tau), \tag{1.11}$$

$$\wp_z\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right) = (ct+d)^3 \wp_z(z,\tau). \tag{1.12}$$

Consider now the morphism of families $\varphi :\ \{E_\tau\} \to \{E_t\}$ induced by

$$(z,\tau) \mapsto \left(X = \frac{\wp(z,\tau)-e_1}{e_2-e_1}, Y = \frac{\wp_z(z,\tau)}{2(e_2-e_1)^{3/2}}, t = \frac{e_3-e_1}{e_2-e_1} \right). \tag{1.13}$$

This is a Galois covering with the group $\Gamma(2) \ltimes \mathbf{Z}^2$. We have

$$\varphi^* \left(\frac{d_{E/B}X}{Y} \right) = 2(e_2-e_1)^{1/2} d_{E/H} z. \tag{1.14}$$

In the future formulas of this type we will omit φ^* and denote differentials over a base B by $d_\downarrow$. For instance, $d_\downarrow \left(\frac{z}{c\tau+d} \right) = \frac{d_\downarrow z}{c\tau+d}$, whereas $d\left(\frac{z}{c\tau+d} \right) = \frac{dz}{c\tau+d} - \frac{czd\tau}{(c\tau+d)^2}$.

It follows from (1.14) that if we denote by γ_1 (resp. γ_2) the image of [0,1] (resp. [0,1]τ) in $\{E_t\}$, then

$$\int_{\gamma_1} \frac{d_\downarrow X}{Y} = 2(e_2-e_1)^{1/2}, \quad \int_{\gamma_2} \frac{d_\downarrow X}{Y} = 2\tau(e_2-e_1)^{1/2} \tag{1.15}$$

so that the operator L_t from (1.2) annihilates periods (1.15) as functions of τ.

1.4. Theorem. *A lift of (1.1) to the (z,τ)–space $\mathbf{C}\times H$ reads:*

$$\frac{d^2z}{d\tau^2}=\frac{1}{(2\pi i)^2}\sum_{j=0}^{3}\alpha_j\wp_z(z+\frac{T_j}{2},\tau) \tag{1.16}$$

where

$$(\alpha_0,\dots,\alpha_3):=(\alpha,-\beta,\gamma,\frac{1}{2}-\delta). \tag{1.17}$$

Proof. Following the lead of no. 1.3, we will directly calculate the μ–equation for $\{E_\tau\}$, choosing $\omega=d_\downarrow z$ (instead of $d_\downarrow X/Y$) and $\sigma=\frac{d^2}{d\tau^2}$ (instead of $t^2(1-t)^2\frac{d^2}{dt^2}$.) Since periods of $d_\downarrow z$ are generated by 1 and τ, the relevant Picard–Fuchs operator is simply $\frac{d^2}{d\tau^2}$. From the Lemma 1.3.1 and (1.15) it follows that

$$t(1-t)L_t\circ 2(e_2-e_1)^{1/2}=Z(\tau)\frac{d^2}{d\tau^2}.$$

Using (1.13) and comparing symbols, we see that

$$Z(\tau)=2\left(\frac{e_3-e_1}{e_2-e_1}\right)^2\left(\frac{e_3-e_2}{e_2-e_1}\right)^2\frac{(e_2-e_1)^4}{9(e_1e_2'-e_2e_1')^2}(e_2-e_1)^{1/2}=$$

$$=\frac{2}{9}\frac{\prod_{i>j}(e_i-e_j)^2}{(e_1e_2'-e_2e_1')^2}(e_2-e_1)^{-3/2}. \tag{1.18}$$

Since $e_1+e_2+e_3=0$, we can replace $(e_1e_2'-e_2e_1')^2$ by $(e_ie_j'-e_je_i')^2$ for any $i\neq j$. It follows that

$$C:=\frac{\prod_{i>j}(e_i-e_j)^2}{(e_1e_2'-e_2e_1')^2}$$

is a modular function for the full modular group without zeroes and poles, hence a constant. A calculation with theta–functions, here omitted, for which I am grateful to A. Levin, shows that $C=-9\pi^2$, so that finally

$$t(1-t)L_t\int_\infty^{(X(t),Y(t))}\frac{d_\downarrow x}{y}=-2\pi^2(e_2-e_1)^{-3/2}\frac{d^2}{d\tau^2}\int_0^{z(\tau)}d_\downarrow z \tag{1.19}$$

for the respective sections. We can now consecutively compare the summands in the right hand side of (1.1) with those in (1.16). The first summand gives

$$\alpha Y=\frac{\alpha}{2}(e_2-e_1)^{-3/2}\wp_z(z,\tau).$$

For the remaining ones we have to use the addition formulas

$$\wp_z(z+\frac{T_i}{2},\tau)=-\frac{(e_i-e_j)(e_i-e_k)}{(\wp(z,\tau)-e_i)^2}\wp_z(z,\tau),\quad \{i,j,k\}=\{1,2,3\},$$

so that, say, for $i=3$ we get

$$(\delta-\frac{1}{2})\frac{t(t-1)Y}{(X-t)^2}=(\delta-\frac{1}{2})\frac{(e_3-e_1)(e_3-e_2)}{(e_2-e_1)^2}\cdot\frac{\wp_z(z,\tau)}{2(e_2-e_1)^{3/2}}\cdot\frac{(e_2-e_1)^2}{(\wp(z,\tau)-e_3)^2}=$$

$$=-\frac{1}{2}(\delta-\frac{1}{2})(e_2-e_1)^{-3/2}\cdot\frac{-(e_3-e_1)(e_3-e_2)}{(\wp(z,\tau)-e_3)^2}\wp_z(z,\tau)=$$

$$=-\frac{1}{2}(\delta-\frac{1}{2})(e_2-e_1)^{-3/2}\wp_z(z+\frac{1+\tau}{2},\tau).$$

The remaining two summands are treated similarly. This finishes the proof.

As the first application, we can now describe the space of the right hand sides of Painlevé–Fuchs equations in a model–independent way (compare Introduction, 0.2e, iv)).

1.5. PVI on an arbitrary elliptic pencil. We put ourselves in the setting of 0.2a. As was explained in 1.3.1, for an invariant μ–equation (as (1.6)) the right hand side can be considered as a meromorphic section of $\pi^*(S^3(\Omega^1_{E/B}))$. The space of the right hand sides of (1.16) written in the invariant form is generated by four cubic differentials $\wp_z(z+\frac{T_i}{2},\tau)(d_\downarrow z)^3$. Looking at their Laurent series near $z+\dfrac{T_i}{2}=0$ one easily sees that they are cubes of a formal differential with a constant residue along $D_i:=\dfrac{T_i}{2}\bmod(1,\tau)$, and that this property together with identification of their divisors as $\dfrac{D_jD_kD_l}{D_i^3}$ characterizes them up to a multiplicative constant.

In the Theorem 2.5 below we will give a Hamiltonian interpretation of this space.

1.6. S_4–symmetry and the Landin transform. As the first application of 1.5 and (1.16) we will construct some natural transformations of PVI. For much deeper hidden symmetries, see §3.

a. *The classical S_4–symmetry.* Isomorphisms of (E,D_i) which do not conserve the labelling of D_i induce transformations of PVI permuting α_i. In the form (1.16), they act on solutions as compositions of the transformations of two types: $(z,\tau)\mapsto\left(\dfrac{z}{cz+\tau},\dfrac{a\tau+b}{c\tau+d}\right)$ indexed by cosets $\Gamma/\Gamma(2)$, and $(z,\tau)\mapsto(z+\dfrac{T_i}{2},\tau)$ shifting the zero section.

b. *The Landin transform.* From (1.8) one easily deduces Landin's identity

$$\wp_z(z,\frac{\tau}{2})=-2\left[\sum\frac{1}{(z+2m\frac{\tau}{2}+n)^3}+\sum\frac{1}{(z+\frac{\tau}{2}+2m\frac{\tau}{2}+n)^3}\right]=$$

$$=\wp_z(z,\tau)+\wp_z(z+\frac{\tau}{2},\tau).$$

Hence if $z(\tau)$ is a solution to PVI with parameters $(\alpha_0,\alpha_1,\alpha_0,\alpha_1)$, we have

$$\frac{d^2z(\tau)}{d\tau^2}=\alpha_0[\wp_z(z,\tau)+\wp_z(z+\frac{\tau}{2},\tau)]+\alpha_1[\wp_z(z+\frac{1}{2},\tau)+\wp_z(z+\frac{1+\tau}{2},\tau)]=$$

$$= \frac{1}{4}\frac{d^2 z(\tau)}{d(\tau/2)^2} = \alpha_0 \wp_z(z, \frac{\tau}{2}) + \alpha_1 \wp_z(z+\frac{1}{2}, \frac{\tau}{2}),$$

that is, $z(2\tau)$ is a solution to PVI with parameters $(4\alpha_0, 4\alpha_1, 0, 0)$. The converse statement is true as well. In this way we get the following bijections between the sets of solutions to (1.16):

$$(\alpha_0, \alpha_1, \alpha_0, \alpha_1) \leftrightarrow (4\alpha_0, 4\alpha_1, 0, 0) \tag{1.19}$$

and in particular

$$(\alpha_0, 0, \alpha_0, 0) \leftrightarrow (4\alpha_0, 0, 0, 0). \tag{1.20}$$

Of course, we can combine these correspondences with permutations. In this way, Hitchin's equation reduces in two steps to

$$\frac{d^2 z}{d\tau^2} = -\frac{1}{2\pi^2}\wp_z(z, \tau), \tag{1.21}$$

whereas the $\mathbf{P}^2$–equation reduces to (0.2).

1.7. Remark. A straightforward generalization of (1.16) is the following infinite–dimensional family of μ–equations:

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2}\sum_{\zeta} \alpha_\zeta \wp_z(z+\zeta, \tau), \tag{1.22}$$

where ζ runs over representatives of $(\mathbf{Q}+\mathbf{Q}\tau)/(\mathbf{Z}+\mathbf{Z}\tau)$, and $\alpha_\zeta = 0$ for almost all ζ. Most of the results of this paper readily extend to (1.22)

§2. Hamiltonian structure

2.1. The time–dependent Hamiltonian. The PVI–equation written as in (1.16) has an obvious time-dependent Hamiltonian form:

$$\frac{dz}{d\tau} = \frac{\partial \mathcal{H}}{\partial y}, \ \frac{dy}{d\tau} = -\frac{\partial \mathcal{H}}{\partial z}, \tag{2.1}$$

where

$$\mathcal{H} := \frac{y^2}{2} - \frac{1}{(2\pi i)^2}\sum_{j=0}^{3} \alpha_j \wp(z + \frac{T_j}{2}, \tau). \tag{2.2}$$

To understand the geometric meaning of these equations, we will extend the action of $\Gamma(2) \ltimes \mathbf{Z}^2$ to the (y, z, τ)–space in a way compatible with (2.1), (2.2). We start with recalling the general Hamiltonian formalism.

2.2. Hamiltonian formalism. *a. Non–degenerate case.* Let X be a manifold, $\pi \in \Gamma(X, \wedge^2 \mathcal{T}_X)$, $\omega \in \Gamma(X, \wedge^2 \mathcal{T}_X^*)$. The natural integrability conditions for such tensors are

for π: $\{f,g\}_\pi := \pi(df,dg)$ satisfies the Jacobi identity;
for ω: $d\omega = 0$.

If both π and ω are nowhere degenerate, we can write the *compatibility condition* for them meaning that they define mutually inverse isomorphisms $\mathcal{T}_x \underset{\tilde\pi}{\overset{\tilde\omega}{\rightleftarrows}} \mathcal{T}^*_X$. This relation is a bijection compatible with the two integrability conditions, which establishes the equivalence between the non–degenerate Poisson structures π on X and the symplectic structures ω on X, so that we can write the relevant Poisson bracket as $\{f,g\}_\omega$ as well. Any function $\mathcal{H}$ on X (time–independent Hamiltonian) defines a flow on X endowed with π or ω. This flow has respectively two equivalent descriptions:

Poisson: $\frac{df}{dt} = \{\mathcal{H}, f\}_\omega$, f being any function on X;
symplectic: graphs of the flow lines in the extended phase space $X \times A^1_t$ are leaves of the null–foliation of the closed form $\mathrm{pr}^*_X\omega - d\mathcal{H}\wedge dt$.

b. Degenerate case. Here the two structures diverge, and the natural compatibility relation ceases to be a bijection.

A tensor $\pi \in \Gamma(X, \wedge^2\mathcal{T}_X)$ of constant rank defines the subbundle $\mathrm{Ker}\,\tilde\pi \subset \mathcal{T}^*_X$ and the orthogonal distribution $(\mathrm{Ker}\,\tilde\pi)^\perp \subset \mathcal{T}_X$. If in addition π is Poisson, then

i). $(\mathrm{Ker}\,\tilde\pi)^\perp$ is integrable, i. e. it defines a foliation called the symplectic foliation of π.

ii). On the leaves of this foliation, π induces a nondegenerate Poisson, or equivalently, symplectic structure.

On the other hand, a tensor $\omega \in \Gamma(X, \wedge^2\mathcal{T}^*_X)$ of constant rank directly defines the distribution $\mathrm{Ker}\,\tilde\omega \subset \mathcal{T}_X$, and if ω is closed, then

i$'$). $\mathrm{Ker}\,\tilde\omega$ is integrable; its leaves form the null–foliation of ω.

ii$'$). ω induces a symplectic structure on the leaves of any foliation transversal to the null–foliation of ω and having the complementary dimension.

We will now call π and ω *compatible*, if $\mathcal{T}_X = (\mathrm{Ker}\,\tilde\pi)^\perp \oplus \mathrm{Ker}\,\tilde\omega$, and if in addition, π and ω induce the same symplectic structure on the symplectic leaves of π.

In the remaining part of this paper, we will be interested only in the (degenerate) symplectic picture (X,ω) considered as a generalization of the extended phase space. The leaves of the null–foliation will be for us solutions to a Hamiltonian system. The following simple Proposition shows that a particular case of this picture encodes the classical formalism of Hamiltonian equations with many times and time–dependent Hamiltonians.

2.2.1. Proposition. *Let $X = X_0 \times B$, $(p_i, q_i), i = 1,\dots,n$, be coordinates on X_0, $(t_1,\dots,t_m)$ coordinates on B. Let $\omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$ be a non–degenerate symplectic form on X_0, and $\omega = \sum_{i=1}^n dp_i\wedge dq_i - \sum_{j=1}^m d\mathcal{H}_j \wedge dt_j$ be a closed form of the constant rank $2n$, where $\mathcal{H}_j = \mathcal{H}_j(p,q,t)$ are functions on X. Then we have:*

a). Leaves of the null–foliation of ω form an étale covering of B iff the Hamiltonians $\mathcal{H}_j$ satisfy the integrability condition

$$\forall j,k, \quad \{\mathcal{H}_j, \mathcal{H}_k\}_{\omega_0} = \partial_{t_j}\mathcal{H}_k - \partial_{t_k}\mathcal{H}_j$$

(empty for $m = 1$), where the Poisson bracket is taken at constant times.

b). The equations of motion expressing variation of p_i, q_i along the leaves are

$$\frac{\partial p_i}{\partial t_j} = -\frac{\partial \mathcal{H}_j}{\partial q_i}, \ \frac{\partial q_i}{\partial t_j} = \frac{\partial \mathcal{H}_j}{\partial p_i}.$$

Proof. Leaves of the null-foliation form an étale covering of B, iff the null–distribution is spanned by lifts of the basic vector fields ∂_{t_j}:

$$\mathcal{D}_j = \sum_{i=1}^{n} A_i^{(j)} \partial_{p_i} + \sum_{i=1}^{n} B_i^{(j)} \partial_{q_i} + \partial_{t_j}, \ j = 1, \dots, m.$$

We have then:

$\omega(\mathcal{D}_j, \partial_{p_i}) = 0 \ \Leftrightarrow \ B_i^{(j)} = \dfrac{\partial \mathcal{H}_j}{\partial p_i},$

$\omega(\mathcal{D}_j, \partial_{q_i}) = 0 \ \Leftrightarrow \ A_i^{(j)} = -\dfrac{\partial \mathcal{H}_j}{\partial q_i},$

$\omega(\mathcal{D}_j, \partial_{t_k}) = 0 \ \Leftrightarrow \ -\sum_{i=1}^{n} A_i^{(j)} \partial_{p_i} \mathcal{H}_k - \sum_{i=1}^{n} B_i^{(j)} \partial_{q_i} \mathcal{H}_k - \partial_{t_j} \mathcal{H}_k + \partial_{t_k} \mathcal{H}_j = 0.$

Finally, for any observable f, the equations of motion are

$$\frac{\partial}{\partial t_j}(f|_L) = (\mathcal{D}_j f)|_L$$

where L is any leaf of the foliation.

Proposition 2.2.1 generally furnishes a too simplified picture. Not only Hamiltonians but the constant time slices, together with their symplectic structure, might become time–dependent (especially in the analytic context). Still worse, projection onto a time manifold B may not be a part of the data. Even a specific transversal foliation need not be present.

2.3. PVI revisited. Looking at (2.1) and (2.2), we see that in the (y, z, τ)–space solutions of a particular PVI form the null–foliation of

$$\omega = \omega(\alpha_0, \dots, \alpha_3) := 2\pi i(dy \wedge dz - d\mathcal{H} \wedge d\tau) = \ .$$

$$= 2\pi i(dy \wedge dz - y dy \wedge d\tau) + \frac{1}{2\pi i} \sum_{j=0}^{3} \alpha_j \wp_z(z + \frac{T_j}{2}, \tau) dz \wedge d\tau. \qquad (2.3)$$

(The extra factor $2\pi i$ makes ω defined over $\mathbf{Q}$, cf. below.)

2.3.1. Proposition. *The standard action of $\Gamma(2)$ (resp. $\mathbf{Z}^2$) on $\mathbf{C} \times H$ has a unique extension to $\mathbf{C} \times \mathbf{C} \times H$ leaving (2.3) invariant:*

$$(y, z, \tau) \mapsto \left(y(c\tau + d) - cz, \frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right), \qquad (2.4)$$

$$(y, z, \tau) \mapsto (y + m, z + m\tau + n, \tau). \qquad (2.5)$$

Proof. Let $(y,z,\tau)\mapsto\left(\tilde{y},\tilde{z}=\dfrac{z}{c\tau+d},\tilde{\tau}=\dfrac{a\tau+b}{c\tau+d}\right)$, $d\tilde{y}=Ady+Bdz+Cd\tau$, be a transformation from $\Gamma(2)$ preserving the form of (2.3):

$$\omega\equiv 2\pi i(d\tilde{y}\wedge d\tilde{z}-\tilde{y}d\tilde{y}\wedge d\tilde{\tau})+\frac{1}{2\pi i}\sum_{j=0}^{3}\alpha_j\wp_z(\tilde{z}+\frac{T_j}{2},\tilde{\tau})d\tilde{z}\wedge d\tilde{\tau}. \tag{2.6}$$

From (1.12) it follows that the terms in (2.6) involving the Weierstrass function are automatically invariant. Comparing coefficients of $dy\wedge dz$ at both sides of (2.6), one sees that $A=c\tau+d$. Comparing coefficients of $dy\wedge d\tau$, one then finds $\tilde{y}=y(c\tau+d)-cz$, which gives $B=-c, C=cy$. Finally, one checks the vanishing of the relevant part of the coefficient of $dz\wedge d\tau$. This proves (2.4); (2.5) is checked similarly.

We will now construct a function of z,τ behaving in the same way as y in (2.4), (2.5).

Namely, consider the theta–function

$$\theta(z,\tau)=\sum_{n\in\mathbf{Z}}\exp(\pi i n^2\tau+2\pi i n z).$$

It has zeroes of the first order at $z\equiv\dfrac{1+\tau}{2}\bmod(1,\tau)$ and satisfies the following functional equations under the action of $\Gamma(2)$ and $\mathbf{Z}^2$:

$$\theta\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=\zeta(c\tau+d)^{1/2}\exp\left(\pi i c\frac{z^2}{c\tau+d}\right)\theta(z,\tau), \tag{2.7}$$

$$\theta(z+m\tau+n)=\exp(-\pi i m^2\tau-2\pi i m z)\,\theta(z,\tau), \tag{2.8}$$

where ζ is a root of unity of degree eight. Therefore the function $v(z,\tau):=-\dfrac{1}{2\pi i}\dfrac{\theta_z}{\theta}(z,\tau)$ has poles of the first order with residue $-\dfrac{1}{2\pi i}$ at $z\equiv\dfrac{1+\tau}{2}\bmod(1,\tau)$ and satisfies

$$v\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=v(z,\tau)(c\tau+d)-cz, \tag{2.9}$$

$$v(z+m\tau+n)=v(z,\tau)+m. \tag{2.10}$$

Comparing this to (2.4), we find finally:

2.3.2. Proposition. *The vertical (over H) differential*

$$\nu:=\left(2\pi i y+\frac{\theta_z}{\theta}\right)d_{\downarrow}z \tag{2.11}$$

on the phase space $\mathbf{C}\times\mathbf{C}\times H$ is $\Gamma(2)\ltimes\mathbf{Z}^2$–invariant, has residue one at its poles $z\equiv\dfrac{T_3}{2}\bmod(1,\tau)$, and therefore can be pushed down to the three–dimensional space $F:=\Gamma(2)\ltimes\mathbf{Z}^2\setminus(\mathbf{C}\times\mathbf{C}\times H)$ fibered over the total space E of the elliptic pencil $\{E_t\}$.

We will use (2.11) first of all in order to identify F with the phase space described in the Introduction, 0.2. Here is the formal construction.

2.4. Lemma–Definition. *Let $(\pi : E \to B, D_i)$ be an elliptic pencil with $\Gamma(2)$–rigidity, as in 0.2. Then there exists an affine line bundle $\lambda : F \to E$, and a relative 1–form $\nu_F \in \Omega^1_{F/B}(\lambda^{-1}(D_3))$ such that the map of sheaves of affine lines over $\mathcal{O}_E$ $\{$sections of F over $E\} \to \Omega^1_{E/B}(D_3)$: $s \mapsto s^*(\nu_F)$ identifies the sheaf of sections of F/E with that of forms with residue one $\mathcal{F} \subset \Omega^1_{E/B}(D_3)$. Moreover,*

a). $(F = F(E, \pi, \{D_i\}), \lambda, \nu_F)$ is unique up to a unique isomorphism over E.

b). $(\lambda : \mathbf{Z}^2 \setminus (\mathbf{C} \times \mathbf{C} \times H) \to \mathbf{Z}^2 \setminus (\mathbf{C} \times H), \nu = \left(2\pi i y + \frac{\theta_z}{\theta}\right) d_{\downarrow} z)$ is the F–space for the pencil $\{E_\tau\}$ over H, with $D_i \equiv \frac{T_i}{2} \bmod (1, \tau)$.

Proof. Uniqueness follows from general nonsense. For existence, we give a standard Čech–type construction which will be useful later.

Put $U_i = E \setminus D_i$ for $i = 0, 1, 2$.

Localizing on B, we may and will assume that $\Omega^1_{E/B}$ is $\mathcal{O}_E$–free.

Choose $\nu_i \in \Gamma(U_i, \mathcal{F}), i = 0, 1, 2$ (recall that $\mathcal{F}$ consists of relative 1–forms with residue 1 at D_3) and take for ν_3 a generator of $\Omega^1_{E/B}$ over $\mathcal{O}_E$.

Define the alternating Čech 1–cocycle in $Z^1((U_i), \mathcal{O}_E)$ by

$$f_{ij} = \frac{\nu_j - \nu_i}{\nu_3} \text{ on } U_i \cap U_j.$$

Use it to glue together $U_i \times \mathbf{A}^1$:

$$(x \in U_i \cap U_j,\ p_i \in \mathbf{A}^1) \mapsto (x \in U_j \cap U_i,\ p_j = p_i + f_{ij}(x)).$$

Denote by F the resulting space, with projection $\lambda : (x, p) \mapsto x$ on E. Denote by ν_F the form whose restriction to $U_j \times \mathbf{A}^1$ is $\nu_j - p_j \nu_3$. One easily checks the compatibility, so that ν_F is a section of $\lambda^*(\mathcal{F}) \subset \Omega^1_{F/B}(\lambda^{-1}(D_3))$.

Clearly, (F, λ, ν_F) satisfies the defining universal property. In fact, any section ν of $\mathcal{F}$ on U_i can be uniquely represented as a sum of ν_i and a unique regular differential, i. e. $\nu = \nu_i + f_i \nu_3$, $f_i \in \Gamma(U_i, \mathcal{O}_E)$. Therefore ν is induced by ν_F on the section locally given by $U_i \to F : x \mapsto (x, f_i(x))$. We leave the last statement to the reader.

This finishes the proof.

Notice that $\lambda : F \to E$ has no global sections, even over a single fibre of E, because there are no differentials of the third kind with a single pole. However, F can be trivialized over $E \setminus D_i$ for any i so that any Painlevé equation with one nontrivial α_i effectively lives on $E \times \mathbf{A}^1$.

Having thus described (F, ν_F), we can characterize the whole space of Painlevé forms along the lines of 0.2.

2.5. Theorem. *a). The form $\omega^{(0)}$ which in the (y, z, τ)–coordinates is defined by*

$$\omega^{(0)} := 2\pi i \,(dy \wedge dz - y dy \wedge d\tau)$$

is the unique closed holomorphic 2–form on F satisfying two conditions:

i). The restriction of $\omega^{(0)}$ to $\mathcal{T}_{F/B}$ coincides with $d_\downarrow \nu_F$.

ii). The canonical lifts to F of the multisections of finite order of E/B defined by

$$z = e\tau + f,\ y = e;\ e, f \in \mathbf{Q} \tag{2.12}$$

are leaves of of the null–foliation of $\omega^{(0)}$.

b). The form ω_j on E which in the (z,τ)–coordinates is defined by

$$\omega_j := \frac{1}{2\pi i}\wp_z(z + \frac{T_j}{2}, \tau) dz \wedge d\tau \tag{2.13}$$

is the unique closed meromorphic form on E satisfying two conditions:

iii). The divisor of ω_j is $\dfrac{D_k D_l D_m}{D_j^3}$, $\{j,k,l,m\} = \{0,1,2,3\}$.

iv). If we identify Ω_E^2 with $\pi^*(\Omega^1_{E/B})^{\otimes 3}$ with the help of the Kodaira–Spencer isomorphism $d\tau \mapsto 4\pi i (d_\downarrow z)^2$, then in a formal neighbourhood of D_j, ω_j becomes the cube of a differential with constant residue r, where $r^3 = -4$.

Proof. a). From (2.11) one sees that

$$d_\downarrow \nu = 2\pi i\, d_\downarrow y \wedge d_\downarrow z = \omega^{(0)}\Big|_{\mathcal{T}_{F/B}}.$$

From (2.1) and (2.2) for $\alpha_i = 0$ for $i = 0, \dots, 3$ it follows that (2.12) are solutions to this PVI.

Conversely, consider a holomorphic closed 2–form $\tilde{\omega}^{(0)}$ enjoying properties i), ii). Then

$$\frac{1}{2\pi i}\tilde{\omega}^{(0)} = dy \wedge dz + E dy \wedge d\tau + G dz \wedge d\tau,$$

where E, G are entire functions of y, z, τ with $E_z = -G_y$. The respective equations of motion are

$$\frac{dz}{d\tau} = -E(y,z,\tau),\ \frac{dy}{d\tau} = G(y,z,\tau). \tag{2.14}$$

If (2.12) satisfy (2.14) for all $e, f \in \mathbf{Q}$, we get that $E(e, e\tau + f, \tau) = -e$ for all real e, f by continuity, so that $E(y,z,\tau) \equiv -y$ by analyticity. Similarly, $G(e, e\tau + f, \tau) = 0$ for all $e, f \in \mathbf{R}$ so that $G \equiv 0$, and $\tilde{\omega}^{(0)} = \omega^{(0)}$.

b). The divisor of (2.13) is well known. If the Kodaira–Spencer isomorphism is normalized as above, ω_j becomes represented by the cubic differential

$$2\wp_z(z + \frac{T_j}{2}, \tau)(d_\downarrow z)^3 = \left(-\frac{4}{(z + \frac{T_j}{2})^3} + O(z + \frac{T_j}{2}) \right) (d_\downarrow z)^3$$

so that its formal cubic root near $z = -\frac{T_j}{2}$ exists and has residue $-\sqrt[3]{4}$. Any other cubic differential with the same divisor can be obtained from ours by multiplication by a function of τ. Fixing the residue, we lose this freedom.

2.6. Theorem. *The Painlevé forms are exact. More precisely,*

$$\omega(\alpha_0,\dots,\alpha_3) = d\Omega(\alpha_0,\dots,\alpha_3) \tag{2.15}$$

where $\omega(\alpha_0,\dots,\alpha_3)$ *is defined by (2.3), and*

$$\Omega(\alpha_0,\dots,\alpha_3) = 2\pi i\,(ydz - \frac{1}{2}y^2 d\tau) + d\log\theta(z,\tau) + 2\pi i\, G_2(\tau)d\tau +$$

$$+\frac{1}{2\pi i}\sum_{j=0}^{3}\alpha_j \wp(z+\frac{T_j}{2},\tau)d\tau \tag{2.16}$$

is a $\Gamma(2)\ltimes \mathbf{Z}^2$*-invariant meromorphic 1-form with poles of the second order at* D_j*.*
Here

$$G_2(\tau) := -\frac{1}{24} + \sum_{n=1}^{\infty}(\sum_{d/n} d)e^{2\pi i n\tau}. \tag{2.17}$$

Proof. Only $\Gamma(2)\ltimes \mathbf{Z}^2$–invariance needs to be checked. This is a straightforward calculation using (2.4), (2.5), (2.7), (2.8), and the pseudo–modular property of $G_2(\tau)$:

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \frac{c(c\tau+d)}{4\pi i}.$$

We leave it to the reader.

2.7. Theorem. *On a PVI phase space* (F,λ,ν_F)*, denote by* D *the divisor of the zeroes of* ν_F *considered as a section of the invertible sheaf* $\lambda^*(\Omega^1_{E/B}(D_3))$*. Then:*

a). D is a section of $\lambda : F\setminus\lambda^{-1}(D_3)\to E\setminus D_3$*.*

b). In the space of Painlevé 2-forms, there exists a unique form identically vanishing on D. It corresponds to the point $(\alpha_0,\dots,\alpha_3) = (0,0,0,\frac{1}{2})$ *(which is the* $\mathbf{P}^2$*-point up to a renumbering and a Landin transform, cf. 0.1 and 1.6.)*

c). D is generically transversal to the null-leaves of any PVI except for $(0,0,0,\frac{1}{2})$*, and so can serve as a common space of initial conditions for these equations (cf. [O3] for a framework for the more precise analysis.)*

Proof. a). From the construction given in the proof of Lemma 2.4 one sees that the equation of D in $U_j\times \mathbf{A}^1$ is $p_j = \nu_j/\nu_3$. Since ν_3 is everywhere invertible, and the only pole of ν_j is D_3, D is a section of λ outside D_3.

b). Since the difference of any two forms in the Painlevé space on F is lifted from E, its restriction to D can vanish identically only if these forms coincide. Hence at most one form can vanish on D identically. To exhibit the one corresponding to $(\alpha_0,\dots,\alpha_3) = (0,0,0,\frac{1}{2})$ we will prove a slightly stronger statement, that it is a differential of a form vanishing on D. Put

$$\Omega_0 := 2\pi i\,[ydz - \frac{1}{2}y^2 d\tau] + d\log\theta(z,\tau) + \frac{i}{4\pi}\frac{\partial^2}{\partial z^2}\log\theta(z,\tau)d\tau. \tag{2.18}$$

Then we can consecutively check that it is $\Gamma(2) \ltimes \mathbf{Z}^2$–invariant and that $d\Omega_0 = \omega(0,0,0,\frac{1}{2})$. For the latter, use the identity

$$\frac{\partial^2}{\partial z^2}\log\theta(z,\tau) = -\wp(z+\frac{1+\tau}{2},\tau)+\varphi(\tau)$$

where the precise form of $\varphi(\tau)$ is inessential here.

On the other hand, from the heat equation

$$\theta_\tau(z,\tau) = \frac{1}{4\pi i}\theta_{zz}(z,\tau)$$

it follows that

$$\Omega_0 = \left[2\pi i\, y + \frac{\theta_z}{\theta}\right]dz - \frac{1}{2}\left[2\pi i\, y^2 + \frac{i}{2\pi}\left(\frac{\theta_z}{\theta}\right)^2\right]d\tau =$$

$$= \left(2\pi i\, y + \frac{\theta_z}{\theta}\right)\left[dz - \frac{1}{4\pi i}\left(2\pi i\, y - \frac{\theta_z}{\theta}\right)d\tau\right]. \tag{2.19}$$

Comparing this with (2.11), one sees that Ω_0 vanishes on D.

c). From the previous discussion, it follows that if $\omega \neq d\Omega_0$, then the restriction of ω to D is generically of rank 2, so that its null–foliation is generically transversal to D.

2.8. The structure of the phase space in algebraic coordinates. In this subsection, we will work out the basic formulas on the algebraic model (0.3).

2.8.1. The vertical coordinate. According to the proof of Lemma 2.4, the natural vertical (over E) coordinate on the algebraic model of $F \setminus \lambda^{-1}(D_3)$ is

$$U := \frac{\nu}{d_\downarrow X/Y}. \tag{2.20}$$

From (2.11) and (1.14) one finds its expression through elliptic functions:

$$U = (\text{push down of})\ \frac{2\pi i\, y + \frac{\theta_z}{\theta}}{2(e_2(\tau)-e_1(\tau))^{1/2}}. \tag{2.21}$$

In particular, the equation of D is simply $U=0$.

We will now identify the Painlevé forms, using the classical parameters $(\alpha,\beta,\gamma,\delta)$ rather than α_i.

2.8.2. Theorem. *We have*

$$\Omega(\alpha,\beta,\gamma,\delta) = U\frac{dX}{Y} - U^2\frac{dt}{t(t-1)} +$$

$$+\frac{1}{2t(t-1)}\left(\alpha X - \beta\frac{t}{X} - \gamma\frac{t-1}{X-1} - \delta\frac{t(t-1)}{X-t}\right)dt, \tag{2.22}$$

$$\omega(\alpha,\beta,\gamma,\delta) = dU \wedge \frac{dX}{Y} - \frac{U}{2(X-t)Y}\, dX \wedge dt - 2UdU \wedge \frac{dt}{t(t-1)} +$$

$$+\frac{1}{2t(t-1)}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right) dX \wedge dt. \tag{2.23}$$

Proof. The main task is to show that (2.22) holds for $\alpha = \beta = \gamma = \delta = 0$. In fact, the part involving $\alpha, \beta, \gamma, \delta$ can be treated in the same way as at the end of the proof of Theorem 1.4, and (2.23) is then obtained by derivation.

Now, the form Ω corresponding to $\alpha = \beta = \gamma = \delta = 0$ is precisely Ω_0 from (2.19). Thus, we have to prove that

$$\Omega_0 = U\frac{dX}{Y} - U^2\frac{dt}{t(t-1)}. \tag{2.24}$$

Using (2.19) and (2.21), we find

$$\Omega_0 = \frac{2\pi i\, y + \frac{\theta_z}{\theta}}{2(e_2-e_1)^{1/2}} \cdot 2(e_2-e_1)^{1/2}\left[\frac{i}{4\pi}\frac{2\pi i\, y + \frac{\theta_z}{\theta}}{2(e_2-e_1)^{1/2}} \cdot 2(e_2-e_1)^{1/2} d\tau + dz + \frac{1}{2\pi i}\frac{\theta_z}{\theta} d\tau\right] =$$

$$= U^2\,\frac{i}{\pi}(e_2-e_1)d\tau + U \cdot 2(e_2-e_1)^{1/2}\left[dz + \frac{1}{2\pi i}\frac{\theta_z}{\theta}d\tau\right]. \tag{2.25}$$

From (1.13) one can deduce that

$$\frac{i}{\pi}(e_2-e_1)d\tau = -\frac{dt}{t(t-1)}.$$

Comparing (2.25) and (2.24), one sees that it remains to prove that

$$\frac{dX}{Y} = 2(e_2-e_1)^{1/2}\left[dz + \frac{1}{2\pi i}\frac{\theta_z}{\theta}d\tau\right]. \tag{2.26}$$

Now, from (1.13) we obtain

$$\frac{dX}{Y} = d\left(\frac{\wp - e_1}{e_2 - e_1}\right) \cdot \frac{2(e_2-e_1)^{3/2}}{\wp_z} =$$

$$= 2(e_2-e_1)^{1/2}dz - 2\frac{\wp_\tau - e_{1\tau}}{(e_2-e_1)^{1/2}\wp_z}d\tau. \tag{2.27}$$

Taking the difference of the right hand sides of (2.26) and (2.27), we first check that it cannot depend on z, because a calculation shows that $d\left(\frac{dX}{Y} - \mu\right) = 0$, where we temporarily denoted by μ the right hand side of (2.26). Put now $\frac{dX}{Y} - \mu = \varphi(\tau)d\tau$.

Then we can calculate $\varphi(\tau)$ by restricting this identity to the divisor $D_1 : X = 0$ or equivalently, $z = 1/2$. We get

$$\varphi(\tau) = \frac{1}{2\pi i} \frac{\theta_z(1/2,\tau)}{\theta(1/2,\tau)} = 0$$

finishing the proof.

2.8.3. PVI in the (U, X, Y, t)–space and the canonical lifts of the multisections of the finite order. From (2.23) one deduces the following equations of motion:

$$\frac{dX}{dt} = \frac{2UY}{t(t-1)},$$

$$\frac{dU}{dt} = -\frac{U}{2(X-t)} + \frac{Y}{2t(t-1)}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right). \qquad (2.28)$$

In particular, if $(X(t), Y(t))$ is a multisection of finite order, hence a solution of the $(\alpha = \beta = \gamma = 0,\ \delta = 1/2)$–equation, then from the first equation (2.28) we see that its lift to F is given by

$$U(t) = \frac{t(t-1)}{2}\frac{X'(t)}{Y(t)}. \qquad (2.29)$$

§3. Symmetries and special solutions

3.1. Reduced phase space and enhanced moduli space. The discrete symmetries of PVI of infinite order were discovered in the context of isomonodromic deformations by Schlessinger and rediscovered many times afterwards (cf. [JM].) We review here the Okamoto's treatment [O2] which nicely fits in our framework.

Our phase space F has an obvious $\mathbf{Z}_2$-symmetry induced by the inversion map on the fibers of E:

$$(y, z, \tau) \mapsto (-y, -z, \tau),\ \ (U, Y, X, t) \mapsto (-U, -Y, X, t).$$

Each Painlevé form and the respective equations of motion are invariant w.r.t. this symmetry. We delete eventual poles and consider the reduced phase space $F_0 := (F \setminus \cup_{i=0}^3 D_i)/\mathbf{Z}_2$. In this section we will work with the algebraic (U, Y, X, t)–model. Then F_0 has an obvious structure of affine algebraic variety.

We also replace the moduli space $P_0 := \operatorname{Spec}\mathbf{C}[\alpha_0, \dots, \alpha_3]$ by its cover

$$P := \operatorname{Spec}\mathbf{C}[a_0, \dots, a_3],\ a_i^2 = 2\alpha_i.$$

Finally, we introduce the pair $(\Phi := F_0 \times P, \omega)$, where ω is the (relative over P) closed regular algebraic 2–form on Φ denoted $\omega(\alpha, \beta, \gamma, \delta)$ in (2.23). This pair is an algebraic model of the space of all PVI equations.

3.2. Symmetries. Denote by W the group of symmetries of P generated by the following transformations:

a). $(a_i) \mapsto (\varepsilon_i a_i)$, where $\varepsilon_i = \pm 1$.

b). Permutaions of (a_i).

c). $(a_i) \mapsto (a_i + n_i)$, where $n_i \in \mathbf{Z}$ and $\sum_{i=0}^{3} n_i \equiv 0\,(2)$.

3.2.1. Theorem (Okamoto [O2]). *All transformations in W can be lifted to the birational transformations of $\Phi = F_0 \times P$ preserving the equations of motion defined by ω.*

Sign changes of a_i can be extended by identity on F_0. The action of S_4 on E was described in 1.6. To lift it to F, it suffices to remark that the four affine sheaves of differentials with a single pole and residue 1 at D_i can be pairwise identified by adding $\frac{1}{2} d\log f_{ij}$, $\operatorname{div} f_{ij} = D_j^2/D_i^2$.

The whole group W is generated by these elements and one shift $(a_i) \mapsto (a_i + \delta_{i0} + \delta_{i3})$, hence it suffices to construct its lifting. I will briefly sketch Okamoto's ingenious argument for doing this.

I start with comparing notation. Okamoto's q, t are ours X, t. The vertical coordinate in the phase space which Okamoto denotes p can be identified as

$$p = \frac{U}{Y} + \frac{1}{2}\left(\frac{a_1}{X} + \frac{a_2}{X-1} + \frac{a_3 - 1}{X - t}\right). \tag{3.1}$$

The verification reduces to a somewhat tedious calculation, showing that (3.1) transforms Okamoto's equations of motion ([O2], (1.5), p. 349) into ours (2.28). It is useful to remember that Okamoto's parameters $(\kappa_\infty, \kappa_0, \kappa_1, \theta)$ are ours (a_0, a_1, a_2, a_3).

Okamoto introduces an auxiliary function h ((1.6), p. 349), which in our coordinates is

$$h = U^2 + \frac{1}{4}\left[-a_0^2 X - a_1^2 \frac{t}{X} + a_2^2 \frac{t-1}{X-1} - (a_3 - 1)^2 \frac{t(t-1)}{X-t}\right] -$$

$$-\frac{1}{4}(a_3 - 1)^2 t + \frac{1}{8}[a_0^2 + a_1^2 - a_2^2 + (a_3 - 1)^2]. \tag{3.2}$$

We need also the Painlevé flow on Φ given by the total time derivative

$$\mathcal{D} := \partial_t + \frac{2UY}{t(t-1)}\partial_X -$$

$$-\left[\frac{U}{2(X-t)} - \frac{Y}{4t(t-1)}\left(a_0^2 - a_1^2 \frac{t}{X^2} + a_2^2 \frac{t-1}{(X-1)^2} - (a_3^2 - 1)\frac{t(t-1)}{(X-t)^2}\right)\right]\partial_U. \tag{3.3}$$

This is a restatement of (2.28).

Now Okamoto's description of the shift can be summarized as follows.

The action of the shift upon h given explicitly by

$$h \mapsto h - X(X-1)\left(\frac{U}{Y} + \frac{a_1}{2X} + \frac{a_2}{2(X-1)} + \frac{a_3 - 1}{2(X-t)}\right) +$$

$$+\frac{1}{2}(-a_0 + a_1 + a_2 + a_3 - 1)X + \frac{1}{4}(a_0 - 2a_1 - a_3 + 1)$$

has a unique birational extension to the whole affine ring of Φ compatible with $\mathcal{D}$.

The proof given by Okamoto is a calculation. He shows that the ring homomorphism

$$\mathbf{C}[a_0, \ldots, a_3; h, k, l] \to \text{affine ring of } \Phi$$

defined by $h \mapsto h$, $k \mapsto \mathcal{D}h$, $l \mapsto \mathcal{D}^2 h$, after a localization becomes surjective. Its kernel is generated by an explicit polynomial relation (see [O2], p. 349, Prop. 1.1.) The symmetries of this polynomial relation are slightly more visible than those of the initial setting.

The geometric meaning of this proof in the context of elliptic pencils remains unclear to me. On the level of the complete phase space $F \times P$, Okamoto's map becomes a correspondence.

3.3. Special solutions. The points of the Painlevé moduli space can be roughly divided into four groups, according to the dimension of the space of solutions reducible to "classical" functions. This is a purely experimental classification, since to the author's knowledge, no precise definition of this notion led to a precise classification picture. Nevertheless, it seems worthwhile to summarize a part of what is known.

Generally, some classical solutions at a point of P are constructed directly. Afterwards new solutions can be generated in principle by applying transformations from W and Landin's transform (which in the (a_i) coordinates is $(a_0, a_1, a_0, a_1) \leftrightarrow (2a_0, 2a_1, 0, 0)$.) Especially interesting are *algebraic* solutions: those for which $X(t)$ is an algebraic function of t. Symmetries (including Landin) preserve the algebricity.

a). *Equations with classical general solutions.* The basic point for them is the null–point $a_i = 0$ for all i. In the (z, τ)–space we get simply $z = e\tau + f$, $e, f \in \mathbf{C}$. Algebraic solutions are obtained precisely for $e, f \in \mathbf{Q}$. They are rigid (not deformable), but in a certain sense dense in the set of all solutions.

Applying shifts from W, we get infinitely many classically completely solvable equations: $(a_i) = (n_i)$, $(\alpha_i) = (\frac{n_i^2}{2})$, where n_i are integral and the sum of n_i is even. Let L be the lattice of such vectors. Inverse Landin transform applied to $(1, 1, 0, 0)$ then shows that the points (a_i) in $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) + L$ are also classically solvable; this includes Hitchin's equation.

I do not know of other PVI equations with this property.

b). *Equations with one–dimensional families of classical solutions.* The basic point here is the $\mathbf{P}^2$–point $(a_i) = (0, 0, 0, 1)$. One family of solutions is obvious in the algebraic coordinates (2.28): $X = \text{const}, U = 0$. These solutions have a clear geometric meaning in our phase space F: they form the foliation of the divisor D formed by the null–leaves entirely contained in D: cf. Theorem 2.7.

It is interesting that this time algebraic solutions are not rigid. If they look somewhat plain, they become more sophisticated on other elements of the orbit $(0,0,0,1)+L$ and on the Landin transforms $(0,0,\frac{1}{2},\frac{1}{2})+L$ and $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})+L$.

More interesting one–dimensional family of solutions can be constructed for any (a_i) belonging to the hyperplane $a_0+a_1+a_2+a_3=1$. They are expressed through Gauss hypergeometric equations: see [O2], p. 373–374. Again, W and Landin generate infinitely many new families.

c). Hitchin [H1] and Dubrovin [D] constructed isolated algebraic solutions using respectively twistor geometry and Frobenius manifolds.

Our last remark concerns some similarity between the (generalized) Lamé potentials in the theory of KdV–type equations and our classically integrable potentials of the non–linear equation (2.2). According to [TV], the former are of the form

$$\sum_{j=0}^{3}\frac{n_j(n_j+1)}{2}\wp(z+\frac{T_j}{2},\tau),$$

whereas according to our discussion the latter have coefficients (proportional to) $(n_j^2)/2$ or $(n_j+\frac{1}{2})^2/2$. Is there a direct connection between the two phenomena?

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