Degree Bounds for Generators of Cohomology Modules and Castelnuovo-Mumford Regularity

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1. INTRODUCTION

Let $\mathcal{F}$ denote a coherent sheaf on the projective space $\mathbb{P}^n = \mathbb{P}^n_K$, K denotes an algebraically closed field. In [10], Lecture 14, $\mathcal{F}$ is called m-regular, $m \in \mathbb{Z}$, provided $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ for all $i > 0$. Then it turns out, see loc. cit., that $\mathcal{F}(k)$ is generated as $\mathcal{O}_{\mathbb{P}^n}$-module by its global sections if $k \geq m$. By more recent results, see e. g. [4], this is generalized to the generation of $\mathcal{S}_j$, the j-th sheaf of syzygies of $\mathcal{F}$. Here we want to show another generalization of Mumford's result. In order to formulate our approach we fix a few notation. For $s > 0$ let

$$r_s(\mathcal{F}) := \min\{m \in \mathbb{Z} \mid H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0 \text{ for all } i \geq s\}.$$

Note that $\operatorname{reg} \mathcal{F} = r_1(\mathcal{F})$ is called the Castelnuovo-Mumford regularity of $\mathcal{F}$. Hence $\mathcal{F}$ is m-regular for all $m \geq \operatorname{reg} \mathcal{F}$. Furthermore, define $e_i^+(\mathcal{F})$ the smallest integer $m \in \mathbb{Z}$ such that $H^i(\mathbb{P}^n, \mathcal{F}(k))$ is spanned by $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^i(\mathbb{P}^n, \mathcal{F}(k-1))$ for all $k > m$. By Serre's vanishing result this is true for all $m \gg 0$. More precisely, Mumford's result, see loc. cit., says $e_0^+(\mathcal{F}) \leq \operatorname{reg} \mathcal{F}$. Its extension is our first main result.

Theorem 1.1. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then there is the following bound*

$$e_i^+(\mathcal{F}) \leq r_{i+1}(\mathcal{F}) - i$$

for all $i \geq 0$.

This result is shown in Section 2 where we prove more general degree bounds for the minimal generators of local cohomology modules. That is, we prove 1.1 by considering local cohomology modules of graded modules.

Another point of our considerations are estimates of $\operatorname{reg} \mathcal{F}$ under additional assumptions on the local behaviour of $\mathcal{F}$, in particular when $\mathcal{F}$ is a Cohen-Macaulay $\mathcal{O}_{\mathbb{P}^n}$-module. More precisely, let $S = K[x_0, \ldots x_n]$, denote the polynomial ring in $n+1$ variables over K. Then a Cohen-Macaulay $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{F}$ is called k-Buchsbaum whenever the S-module $\oplus_{j \in \mathbb{Z}} H^i(\mathbb{P}^n_K, \mathcal{F}(j))$ is annihilated by $(x_0, \ldots, x_n)^k$ for all i with $1 \leq i < \dim \mathcal{F}$. Note that every Cohen-Macaulay sheaf is k-Buchsbaum for some k. Using our results on the generators of cohomology modules we explore some of the restrictions on the vanishing of the cohomology of k-Buchsbaum sheaves as demonstrated by:

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Theorem 1.2. *Let $\mathcal{F}$ denote a k-Buchsbaum $\mathcal{O}_{\mathbb{P}^n}$-module. Then*

$$\operatorname{reg}\mathcal{F} \leq e(\mathcal{F}) + (d-1)(k+1) + 2,$$

where $d = \dim \mathcal{F}$ and $e(\mathcal{F}) = \max\{m \in \mathbb{Z} \mid H^d(\mathbb{P}^n, \mathcal{F}(m)) \neq 0\}$.

The previous result shows that (in the case of a "nice" local behaviour of $\mathcal{F}$) the number $e(\mathcal{F})$ is dominating for $\operatorname{reg}\mathcal{F}$. A bound of this type has first been shown in [6] by completely different means. Theorem 1.2 is a considerable improvement of the corresponding estimate in [6]. It will be proved in Section 3. By some examples we show that certain of the finer bounds obtained in that section are best possible.

In the case of $\mathcal{F} = \mathcal{J}_X$, the ideal sheaf of a projective scheme $X \subset \mathbb{P}^n$, there are estimates of $e(\mathcal{J}_X)$ by simple invariants. Here X is called k-Buchsbaum scheme whenever $\mathcal{J}_X$ is a k-Buchsbaum sheaf. For an integral nondegenerate k-Buchsbaum scheme X this leads to bounds of the following type

$$\operatorname{reg} X \leq \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil + C(k),$$

where $\operatorname{reg} X = \operatorname{reg} \mathcal{J}_X$. In [7] it was shown that $C(k) \leq \binom{d+1}{2}k - d + 1, d = \dim X$. In [12] resp. in [6] (in a slightly weaker form) this was improved to $C(k) \leq (2d-1)k - d + 1$. Our applications to Castelnuovo bounds presented in Section 4 provide a further improvement.

Theorem 1.3. *Let $X \subset \mathbb{P}^n$ denote an integral nondegenerate k-Buchsbaum scheme, $k \geq 1$, of dimension d. Then there is the bound*

$$\operatorname{reg} X \leq \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil + dk.$$

So it turns out that $C(k) \leq dk$.

As mentioned above we translate the statements into the context of graded modules and their local cohomology. In our terminology we follow [12].

2. Degree Bounds for the Generators of Local Cohomology Modules

Let $R = \oplus_{n \geq 0} R_n$ denote a graded Noetherian ring such that $R = R_0[R_1]$ and $K := R_0$ is a field. Put $\mathfrak{m} = \oplus_{n>0} R_n$ the irrelevant maximal ideal of R. Let M denote a finitely generated graded R-module. We fix the basic notation of [12]. In particular, a homogeneous element $x \in R$ is called M-filter regular provided $0 :_M x$ is an R-module of finite length. A system of (homogeneous) elements $\underline{x} = \{x_1, \ldots, x_r\}$ is called an M-filter regular sequence whenever

$$(x_1, \ldots, x_{i-1})M : x_i/(x_1, \ldots, x_{i-1})M, \quad i = 1, \ldots, r,$$

is an R-module of finite length. For an arbitrary graded R-module N let $e(N)$ denote

$$e(N) := \sup\{j \in \mathbb{Z} \mid N_j \neq 0\}.$$

Here N_j denotes the j-th graded piece of the graded R-module N. Thus $e(\{0\}) = -\infty$. Furthermore put

$$e^+(N) := e(N/\mathfrak{m}N).$$

Hence, in the case of a finitely generated module N it denotes the maximal degree of an element in a minimal generating set of N.

The following technical result does not look impressive but it will be proven useful with respect to the estimates announced in the introduction.

Lemma 2.1. *Let $\underline{y} = \{y_1, \dots, y_r\} \subset R$ denote a set of homogeneous elements of degree $\leq s$. Let $x \in R_t$ be an M-filter regular element. Then we have*

$$e(H^i_{\mathfrak{m}}(M)/(x,\underline{y})H^i_{\mathfrak{m}}(M)) \leq \max\{e(H^{i+1}_{\mathfrak{m}}(M)) + t + s, e(H^i_{\mathfrak{m}}(M/xM)/\underline{y}H^i_{\mathfrak{m}}(M/xM))\}$$

for all $i \geq 0$.

Proof. Since x is an M-filter regular element the short exact sequence

$$0 \to M/0 :_M x(-t) \xrightarrow{x} M \to M/xM \to 0$$

induced by multiplication by x provides a long exact sequence

$$(*) \qquad H^i_{\mathfrak{m}}(M)(-t) \xrightarrow{x} H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(M/xM) \to H^{i+1}_{\mathfrak{m}}(M)(-t) \xrightarrow{x} H^{i+1}_{\mathfrak{m}}(M)$$

for all $i \geq 0$. Hence, it induces a short exact sequence

$$0 \to H_0(x; H^i_{\mathfrak{m}}(M)) \to H^i_{\mathfrak{m}}(M/xM) \to H_1(x; H^{i+1}_{\mathfrak{m}}(M)) \to 0.$$

By applying the Koszul homology functor $H_\bullet(\underline{y}; \cdot)$ it provides an exact sequence

$$H_1(\underline{y}; H_1(x; H^{i+1}_{\mathfrak{m}}(M)) \to H^i_{\mathfrak{m}}(M)/(x,\underline{y})H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(M/xM)/\underline{y}H^i_{\mathfrak{m}}(M/xM).$$

Call the module on the left hand side N. Note that it is a subquotient of

$$\oplus_j H_1(x; H^{i+1}_{\mathfrak{m}}(M))(-\deg y_j).$$

Whence it turns out that

$$e(N) \leq e(H^{i+1}_{\mathfrak{m}}(M)) + t + s.$$

So the claim follows by the previous exact sequence. □

As a consequence there is the following bound of $e^+(H^i_{\mathfrak{m}}(M))$.

Corollary 2.2. *Let $x \in R_t$ denote an M-filter regular element. Then*

$$e^+(H^i_{\mathfrak{m}}(M)) \leq \max\{e(H^{i+1}_{\mathfrak{m}}(M)) + t + 1, e^+(H^i_{\mathfrak{m}}(M/xM))\}$$

for all $i < \dim M$.

Proof. Choose $\underline{y}$ as a set of generators for the maximal ideal $\mathfrak{m}$. Note that all the generators have degree 1. So the claim is an immediate consequence of 2.1. □

For a system of elements $\underline{x} = \{x_1, \dots, x_r\}$ of R and an integer $0 \leq i \leq r$ let $\underline{x}_i = \{x_1, \dots, x_i\}$. Note that $\underline{x}_0$ is the empty set.

Theorem 2.3. *Let $\underline{x} = \{x_1, \dots, x_r\}$ be an M-filter regular sequence consisting of homogeneous elements of degree $\leq t$. Let i denote an integer with $0 \leq i \leq \dim M =: d$. Then there exist the following bounds:*

(a) $e(H^i_{\mathfrak{m}}(M)/\underline{x}H^i_{\mathfrak{m}}(M)) \leq \max\{e(H^i_{\mathfrak{m}}(M/\underline{x}M), e(H^{i+1}_{\mathfrak{m}}(M/\underline{x}_jM) + 2t \mid 0 \leq j \leq r-2\}$ *for all i with $1 \leq r \leq d - i$.*

(b) $e(H^i_{\mathfrak{m}}(M)/\underline{x}H^i_{\mathfrak{m}}(M)) \leq \max\{e(H^{i+1}_{\mathfrak{m}}(M/\underline{x}_jM)) + 2t \mid 0 \leq j \leq d-i-1\}$ *for all i with $i > d - r$.*

Proof. First consider i with $1 \leq r \leq d-i$. Then a repeated application of 2.1 provides

$$\begin{array}{l} e(H^i_{\mathfrak{m}}(M)/\underline{x}H^i_{\mathfrak{m}}(M)) \leq \\ \max\{e(H^{i+1}_{\mathfrak{m}}(M)+2t, e(H^i_{\mathfrak{m}}(M/x_1M)/(x_2,\dots,x_r)H^i_{\mathfrak{m}}(M/x_1M))\} \leq \\ \max\{e(H^{i+1}_{\mathfrak{m}}(M)+2t, e(H^{i+1}_{\mathfrak{m}}(M/x_1M))+2t, e(H^i_{\mathfrak{m}}(M/\underline{x}_2M)/(x_3,\dots,x_r)H^i_{\mathfrak{m}}(M/\underline{x}_2M))\} \\ \dots \\ \max\{e(H^{i+1}_{\mathfrak{m}}(M/\underline{x}_jM))+2t, e(H^i_{\mathfrak{m}}(M/\underline{x}_{r-1}M)/x_rH^i_{\mathfrak{m}}(M/\underline{x}_{r-1}M)) \mid 0 \leq j \leq r-2\}. \end{array}$$

But now by an exact sequence as in the proof of 2.1 it is easy to see that

$$e(H^i_{\mathfrak{m}}(M/\underline{x}_{r-1}M)/x_rH^i_{\mathfrak{m}}(M/\underline{x}_{r-1}M)) \leq e(H^i_{\mathfrak{m}}(M/\underline{x}_rM)).$$

Thus the statement in (a) follows. Now let $r > d-i$. Then first note that

$$e(H^i_{\mathfrak{m}}(M)/\underline{x}H^i_{\mathfrak{m}}(M)) \leq e(H^i_{\mathfrak{m}}(M)/\underline{x}_{d-i+1}H^i_{\mathfrak{m}}(M))$$

as easily seen. Similarly as above we obtain

$$\begin{array}{l} e(H^i_{\mathfrak{m}}(M)/\underline{x}_{d-i+1}H^i_{\mathfrak{m}}(M)) \leq \\ \max\{e(H^i_{\mathfrak{m}}(M/\underline{x}_{d-i}M)/x_{d-i+1}H^i_{\mathfrak{m}}(M/\underline{x}_{d-i}M), e(H^{i+1}_{\mathfrak{m}}(M/\underline{x}_jM)) \mid 0 \leq j \leq d-i-1\}. \end{array}$$

But now it turns out that

$$e(H^i_{\mathfrak{m}}(M/\underline{x}_{d-i}M)/x_{d-i+1}H^i_{\mathfrak{m}}(M/\underline{x}_{d-i}M)) = -\infty$$

because $H^i_{\mathfrak{m}}(M/\underline{x}_{d-i+1}M) = 0$. Observe that $\dim M/\underline{x}_{d-i+1}M < i$. Therefore (b) is shown to be true. □

Note that the previous result for $r=2$, $t=1$ was proved in [3], Lemma 4.1. In the special case of linear elements there is the following application.

Corollary 2.4. *Let $\underline{l} = \{l_1,\dots,l_d\} \subseteq R_1$ be an M-filter regular system of parameters, $d = \dim M$. Then*

(a) $e^+(H^i_{\mathfrak{m}}(M)) \leq \max\{e(H^{i+1}_{\mathfrak{m}}(M/\underline{l}_jM))+2 \mid 0 \leq j \leq d-i-1\}$ *for all i with $1 \leq i < d$.*
(b) $e^+(H^0_{\mathfrak{m}}(M)) \leq \max\{e^+(M), e(H^1_{\mathfrak{m}}(M/\underline{l}_jM))+2 \mid 0 \leq j < d\}$.

Proof. Because of $e^+(H^i_{\mathfrak{m}}(M)) \leq e(H^i_{\mathfrak{m}}(M)/\underline{l}H^i_{\mathfrak{m}}(M))$ the statement in (a) follows immediately by 2.3 (b). In order to prove (b) choose a system of elements $\underline{y} = \{y_1,\dots,y_s\}$ consisting of linear forms such that $(\underline{l},\underline{y})R = \mathfrak{m}$. By 2.1 it follows that

$$\begin{array}{rcl} e^+(H^0_{\mathfrak{m}}(M)) & = & e(H^0_{\mathfrak{m}}(M)/(\underline{l},\underline{y})H^0_{\mathfrak{m}}(M)) \\ & \leq & \max\{e(H^0_{\mathfrak{m}}(M/\underline{l}M)/\underline{y}H^0_{\mathfrak{m}}(M/\underline{l}M)), e(H^1_{\mathfrak{m}}(M/\underline{l}_jM)+2 \mid 0 \leq j < d\}. \end{array}$$

Now $\dim(M/\underline{l}M) = 0$ and therefore $H^0_{\mathfrak{m}}(M/\underline{l}M) \simeq M/\underline{l}M$, i.e.,

$$e(H^0_{\mathfrak{m}}(M/\underline{l}M)/\underline{y}H^0_{\mathfrak{m}}(M/\underline{l}M)) = e(M/(\underline{l},\underline{y})M) = e^+(M)$$

which proves the claim. □

In order to continue we recall a definition, see [12], Definition 6.1. For an integer $s \geq 0$ put

$$r_s(M) := \max\{i + e(H^i_{\mathfrak{m}}(M)) \mid i \geq s\}.$$

Then $\operatorname{reg} M := r_0(M) = r_{\operatorname{depth} M}(M)$ is called the Castelnuovo-Mumford regularity of M. It is known,see e.g. [4], that $e^+(M) \leq \operatorname{reg} M$.

Corollary 2.5. *There are the following estimates:*

(a) $e^+(H^i_{\mathfrak{m}}(M)) \leq r_{i+1}(M) - i + 1$ *for all $i > 0$.*
(b) $e^+(H^0_{\mathfrak{m}}(M)) \leq \max\{e^+(M), r_1(M)+1\}$ *provided $d > 0$.*

Proof. For $l \in R_1$ an M-filter regular element the short exact sequence $(*)$ in the proof of 2.1 provides

$$r_i(M/lM) \le r_i(M) \quad \text{for all } i,$$

see [11] for more details. Now let $\underline{l} = \{l_1, \dots, l_r\} \subset R_1$ be an M-filter regular sequence. Then by induction on r it turns out that $r_i(M/\underline{l}M) \le r_i(M)$ for all i. Thus the statements of this corollary follow by 2.4. □

Moreover $e^+(H^d_{\mathfrak{m}}(M)) = -\infty$ for $d = \dim M > 0$, since $H^d_{\mathfrak{m}}(M) = \mathfrak{m}H^d_{\mathfrak{m}}(M)$. It is also noteworthy to say that there is no bound for $e^+(H^0_{\mathfrak{m}}(M))$ which does not depend on $e^+(M)$. To this end note that

$$r_1(M) = r_1(M \oplus R_0(t))$$

for all $t \in \mathbb{Z}$.

For the following result let $H(\cdot) = \varinjlim \operatorname{Hom}(\mathfrak{m}^t, \cdot)$ denote the functor of global transform. Let $R^iH, i \ge 1$, its right derived functors. For an R-module M there are a natural exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to H(M) \to H^1_{\mathfrak{m}}(M) \to 0$$

and natural isomorphisms $H^{i+1}_{\mathfrak{m}}(M) \simeq R^iH(M)$ for $i \ge 1$.

Lemma 2.6. *Let M denote a finitely generated graded R-module. Then*

$$e^+(H(M)) \le r_2(M),$$

in particular $e^+(H(M))$ is a finite number.

Proof. If $d = \dim M \le 1$, then $e^+(H(M)) = -\infty$, so the claim is true. Let $d \ge 2$. Let $l \in R_1$ denote an M-filter regular element. The multiplication by l induces a short exact sequence

$$0 \to H(M)/lH(M) \to H(M/lM) \to H_1(l; H^2_{\mathfrak{m}}(M))(-1) \to 0.$$

Now a Koszul homology argument as in the proof of 2.2 provides that

$$e^+(H(M)) \le \max\{e^+(H(M/lM)), e(H^2_{\mathfrak{m}}(M) + 2)\}.$$

Furthermore, by induction hypothesis

$$e^+(H(M/lM)) \le r_2(M/lM).$$

Because of $r_2(M/lM) \le r_2(M)$ and $e(H^2_{\mathfrak{m}}(M)) + 2 \le r_2(M)$ the inductive step is complete. □

Now we prove Theorem 1.1 of the introduction.

Proof. We use the notation of the introduction. Choose M a finitely generated graded S-module such that $\tilde{M}$, the sheafification of M, satisfies $\tilde{M} = \mathcal{F}$. Then there are graded isomorphisms

$$H(M) \simeq \oplus_{j\in\mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(j)) \quad \text{and} \quad H^{i+1}_{\mathfrak{m}}(M) \simeq \oplus_{j\in\mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(j))$$

for $i \ge 1$, see, e.g., [5]. That is, $e_i^+(\mathcal{F}) = e^+(H^{i+1}_{\mathfrak{m}}(M))$ and $r_i(\mathcal{F}) = r_{i+1}(M)$ for $i \ge 1$. So the claim of 1.1 is a consequence of 2.5 and 2.6. □

3. Restrictions on the Cohomology Imposed by Large Cohomological Annihilators

For a graded R-module M let $\mathfrak{a}_i(M) = \operatorname{Ann}_R H^i_\mathfrak{m}(M), i \in \mathbb{Z}$, denote the i-th cohomological annihilator of M. See [12] for basic results and applications. For an M-filter regular element $x \in \mathfrak{a}_i(M) \cap \mathfrak{a}_{i+1}(M)$ the long exact cohomology sequence induced by multiplication by x provides a short exact sequence

$$0 \to H^i_\mathfrak{m}(M) \to H^i_\mathfrak{m}(M/xM) \to H^{i+1}_\mathfrak{m}(M)(-t) \to 0,$$

$t = \deg x$, see (*) in the proof of 2.1. So there is a good comparison of $r_i(M)$ and $r_i(M/xM)$. Pursuing this point of view further we show estimates of $e(H^i_\mathfrak{m}(M))$ by $e(H^d_\mathfrak{m}(M))$ and the "size" of $\mathfrak{a}_j(M), i \leq j < d$.

Theorem 3.1. *Let $\underline{l} = \{l_1, \ldots, l_{d-i+1}\} \subset R_1, 1 \leq i \leq d$, denote an M-filter regular sequence with $d = \dim M$. Suppose that*

$$\underline{l}^{\mu_j}_{d-j+1} H^j_\mathfrak{m}(M) = 0 \quad \text{for all} \quad i \leq j < d$$

and certain integers $\mu_j \geq 0$. Then

$$e(H^i_\mathfrak{m}(M)) \leq e(H^d_\mathfrak{m}(M)) + \sum_{j=i}^{d-1} (\mu_j + 1).$$

Before we shall prove 3.1 let us mention an interesting consequence. In fact, it is helpful in order to streamline the proof of 3.1. It gives bounds of $r_i(M)$ in terms of $e(H^d_\mathfrak{m}(M))$ and the "size" of $\mathfrak{a}_j(M)$. If in addition $H^i_\mathfrak{m}(M)$ is a finitely generated R-module, one can measure the "size" of $\mathfrak{a}_i(M)$ by the integer

$$\lambda_i(M) = \min\{\lambda \in \mathbb{N} \mid \mathfrak{m}^\lambda \subset \mathfrak{a}_i(M)\}.$$

Corollary 3.2. *With the assumptions of 3.1 there are the following estimates:*

(a) $r_i(M) \leq e(H^d_\mathfrak{m}(M)) + d + \sum_{j=i}^{d-1} \mu_j$, *provided* $i > 0$.

(b) $\operatorname{reg}(M) \leq \lambda_0(M) + \max\{e^+(M) - 1, e(H^d_\mathfrak{m}(M)) + d + \sum_{j=1}^{d-1} \mu_j\}$.

Proof. By the definition of $r_i(M)$, the claim in (a) follows by 3.1. If $\lambda_0(M) = 0$, i.e., equivalently $H^0_\mathfrak{m}(M) = 0$, then $\operatorname{reg}(M) = r_1(M)$ and the statement in (b) follows by (a). If $\lambda_0(M) > 0$, then $\operatorname{reg}(M) = \max\{e(H^0_\mathfrak{m}(M)), r_1(M)\}$. On the other hand by Lemma 3.3 below it follows that

$$e(H^0_\mathfrak{m}(M) \leq e^+(H^0_\mathfrak{m}(M)) + \lambda_0(M) - 1.$$

Therefore, by 2.5 we get

$$\operatorname{reg}(M) \leq \lambda_0(M) + \max\{e^+(M) - 1, r_1(M)\}.$$

So the statement in (b) follows by virtue of (a). □

In the proof of the previuos corollary we have already used the following observation.

Lemma 3.3. *Let $I \subset R$ be an ideal generated by elements of R_1 and let M be a finite graded R-module. Suppose there is an integer $\mu > 0$ such that $I^\mu M = 0$. Then*

$$e(M) \leq e(M/IM) + \mu - 1.$$

Proof. Let r denote the number of generators of I. For an integer $t \geq 1$ there is the natural epimorphism

$$(M/IM(-t))^{\binom{r+t-1}{r-1}} \twoheadrightarrow I^tM/I^{t+1}M.$$

Thus $e(I^tM/I^{t+1}M) \leq e(M/IM) + t$. Because of

$$e(M/I^tM) = \max\{e(I^{t-1}M/I^tM), e(M/I^{t-1}M)\}$$

the conclusion follows now. □

Now let us continue with the proof of 3.1

Proof. In order to prove the desired bound we make induction on $d - i \geq 0$. In the case $d - i = 0$ the statement is empty. Let $0 < i < d$. If $\mu_i = 0$, i.e., $H^i_{\mathfrak{m}}(M) = 0$, then $e(H^i_{\mathfrak{m}}(M)) = -\infty$ and the statement is true. Let $\mu_i > 0$. Then by 3.3

$$e(H^i_{\mathfrak{m}}(M)) \leq e(H^i_{\mathfrak{m}}(M)/\underline{l}H^i_{\mathfrak{m}}(M)) + \mu_i - 1.$$

By combining 2.3 (b) with the fact that

$$r_i(M/\underline{l}_jM) \leq r_i(M)$$

for all i, j with $0 \leq j \leq d - i - 1$ it turns out that

$$e(H^i_{\mathfrak{m}}(M)) \leq r_{i+1} - i + 1 + \mu_i - 1.$$

By the induction hypothesis the claim is true for $d-(i+1)$. Whence the above Corollary 3.2 provides

$$r_{i+1}(M) \leq e(H^d_{\mathfrak{m}}(M)) + d + \sum_{j=i+1}^{d-1} \mu_j.$$

Putting this together it completes the inductive step. □

Note that in 3.1 there is no assumption on the finiteness of $H^i_{\mathfrak{m}}(M), i = 1, \ldots, d-1$. Under the additional assumption of finiteness it follows:

Corollary 3.4. *In addition to the assumptions of* 3.1 *suppose that* $H^i_{\mathfrak{m}}(M), i = 0, \ldots, d-1$, *are finitely generated. Then there are the bounds:*

(a) $r_i(M) \leq e(H^d_{\mathfrak{m}}(M)) + d + \sum_{j=i}^{d-1} \lambda_j(M)$ *provided* $i > 0$.
(b) $\operatorname{reg}(M) \leq \max\{e^+(M) + \lambda_0(M) - 1, e(H^d_{\mathfrak{m}}(M)) + d + \sum_{j=0}^{d-1} \lambda_j(M)\}$.

There is no generalization of 3.1 relating $e(H^i_{\mathfrak{m}}(M))$ and $e(H^j_{\mathfrak{m}}(M))$ with $i < j < d$. This follows because for any integers m, n one may construct Buchsbaum modules with $e(H^i_{\mathfrak{m}}(M)) = m$ and $e(H^j_{\mathfrak{m}}(M)) = n$.

Let R be the coordinate ring of a projective curve $C \subset \mathbb{P}^n$. Then 3.1 specializes to [9], Proposition 2.8. If C is a rational curve, then $e(H^1_{\mathfrak{m}}(R)) \leq \lambda_1(R)$ by 3.1. Since $a(H^1_{\mathfrak{m}}(R)) \geq 1$ and $\lambda_1(R) \leq e(H^1_{\mathfrak{m}}(R)) - a(H^1_{\mathfrak{m}}(R)) + 1$ it follows (cf. also [9], 2.10) that

$$e(H^1_{\mathfrak{m}}(R)) = \lambda_1(R) \quad \text{and} \quad a(H^1_{\mathfrak{m}}(R)) = 1.$$

In particular, it turns out that 3.1 is optimal in this case. Note also that the previous equalities are generalizations of the main results in [2], proved there in the case of monomial curves in $\mathbb{P}^3$. Theorem 3.1 is also optimal in higher dimensions as seen by the following:

Example 3.5. Let $S := K[x_1, \dots, x_d]$ denote the polynomial ring in $x_1, \dots, x_d$ over the field K. For a positive integer μ let $M = \underline{x}^\mu S$, where $\underline{x} = \{x_1, \dots, x_d\}$. So there are the following isomorphisms

$$H^j_{\mathfrak{m}}(M) \simeq \begin{cases} S/\underline{x}^\mu S & \text{if } j = 1, \\ H^d_{\mathfrak{m}}(S) & \text{if } j = d, \\ 0 & \text{otherwise.} \end{cases}$$

Now $\underline{x}$ is an M-filter regular sequence. Thus 3.1 is applicable. It yields the following estimate.

$$e(H^1_{\mathfrak{m}}(M)) \le e(H^d_{\mathfrak{m}}(S)) + \mu + d - 1 = \mu - 1.$$

On the other hand $e(S/\underline{x}^\mu S) = \mu - 1$, as easily seen. So it follows that

$$e(H^1_{\mathfrak{m}}(M)) = e(S/\underline{x}^\mu S) = \mu - 1.$$

Hence the bound in 3.1 is optimal.

The bound in Theorem 3.1 is also optimal in the case when M has more than two non-vanishing cohomology modules. In order to illustrate this situation consider:

Example 3.6. For $r \ge 2$ put $S = K[x_1, \dots, x_{2r}]$ and $R = S/\mathfrak{a}$ with $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$, where $\mathfrak{b} = (x_1, \dots, x_r) \cap (x_{r+1}, \dots, x_{2r})$ and

$$\mathfrak{c} = \begin{cases} (x_1, \dots, x_{\frac{r+1}{2}}, (x_{\frac{r+1}{2}+1}, \dots, x_{2r})^3) & \text{if } r \text{ is odd,} \\ (x_1, \dots, x_{\frac{r}{2}}, x_{\frac{r}{2}+1}x_{2r}, \dots, x_r x_{2r}, (x_{\frac{r}{2}+1}, \dots, x_{2r})^3) & \text{if } r \text{ is even.} \end{cases}$$

Then $H^i_{\mathfrak{m}}(R) \simeq H^i_{\mathfrak{m}}(S/\mathfrak{b})$ for $i > 0$ and thus

$$H^i_{\mathfrak{m}}(R) = \begin{cases} K & \text{for } \quad i = 1, \\ 0 & \text{for } \quad 1 < i < r. \end{cases}$$

Finally $H^0_{\mathfrak{m}}(R) \simeq \mathfrak{b}/\mathfrak{a} \simeq K(-2)^{\binom{r-1}{2}}$. Therefore we have $e(H^r_{\mathfrak{m}}(R)) = -r, \lambda_0(R) = \lambda_1(R) = 1, \lambda_2(R) = \dots = \lambda_{r-1}(R) = 0$, and $\operatorname{reg} R = 2$. Thus in 3.4 (b) equality holds.

Now recall that M is called a k-Buchsbaum R-module if $\lambda_i(M) \le k$ for all i with $0 \le i < \dim M$. Note that 0-Buchsbaum means Cohen-Macaulay. Observe that 3.6 shows that the bound in [6], 2.8, is not true for 1-Buchsbaum rings which are not arithmetically Buchsbaum. Instead, we have the following estimations in case of k-Buchsbaum modules.

Corollary 3.7. *Let M be k-Buchsbaum R-module. Then there are the bounds:*

(a) $r_i(M) \le e(H^d_{\mathfrak{m}}(M)) + d + (d - i)k$ *for all* $i > 0$.
(b) $\operatorname{reg}(M) \le \max\{e^+(M) + k - 1, e(H^d_{\mathfrak{m}}(M)) + d(k + 1)\}$.

Proof. By the definitions this is an immediate consequence of 3.4. □

Remark 3.8. (1) First note that Theorem 1.2 of the introduction is a consequence of 3.7 by the same translation procedure as in the proof of Theorem 1.1.
(2) Put $M = R$. Then $e^+(R) = 0$. Moreover, it is known that $e(H^d_{\mathfrak{m}}(R)) + d \ge 0$, see e. g., [8]. Let R denote a k-Buchsbaum ring. Then 3.7 yields the following estimate

$$\operatorname{reg}(R) \le e(H^d_{\mathfrak{m}}(M)) + d + (d - t)k,$$

where $t = \operatorname{depth} R$.
(3) Note that 3.7 improves the main results of [6] for k-Buchsbaum modules. It is often much easier to check if a module M is k-Buchsbaum than to decide if $\mathfrak{m}^k$ is an M-standard ideal. Note that the main results of [6] stated under this latter assumption are also improved by 3.7 in case $i + k > d$.

4. Applications to Castelnuovo Bounds

First let us recall the definition of an (r,i)-standard sequence introduced in [12]. To this end let $\underline{x} = \{x_1, \ldots, x_r\}, 1 \leq r \leq \dim M =: d$, denote an M-filter regular sequence. For $i \leq d-r$ it is called an (r,i)-standard sequence with respect to M provided

$$x_{n+1} H_{\mathfrak{m}}^{i+j}(M/(x_1, \ldots, x_n)M) = 0$$

for all non-negative j, n with $0 \leq j+n < r$. This notion generalizes the notion of a standard system of parameters. In [12] it is shown to be useful in order to control the vanishing of graded local cohomology. This point of view is pursued further in this section.

Lemma 4.1. *Let $\underline{x} = \{x_1, \ldots, x_r\} \subset R_k$ be an (r,i)-standard sequence with respect to M. Then*

$$e(H_{\mathfrak{m}}^i(M/\underline{x}M)) = \max\{e(H_{\mathfrak{m}}^{i+j}(M)) + jk \mid 0 \leq j \leq r\}.$$

Proof. In [12], 6.3, it is shown that

$$e(H_{\mathfrak{m}}^{i+j}(M)) \leq e(H_{\mathfrak{m}}^i(M/\underline{x}M)) - jk$$

for $j = 0, 1, \ldots, r$. This proves that the left-hand side is bounded by the maximum on the right. Since $\underline{x}$ is an (r,i)-standard sequence there are short exact sequences of local cohomology modules

$$H_{\mathfrak{m}}^{i+j}(M/(x_1, \ldots, x_n)M) \to H_{\mathfrak{m}}^{i+j}(M/(x_1, \ldots, x_{n+1})M) \to H_{\mathfrak{m}}^{i+j+1}(M/(x_1, \ldots, x_n)M)(-k).$$

Thus an easy induction on r proves the claim. □

As an application 4.1 implies a bound of $r_i(M)$. Thereby we use the notation $\underline{l}^{(k)} := \{l_1^k, \ldots, l_r^k\}$ for $\underline{l} = \{l_1, \ldots, l_r\}$ a sequence of elements of R.

Proposition 4.2. *Let $\underline{l} = \{l_1, \ldots, l_{d-i}\} \subset R_1$ denote an M-filter regular sequence. Suppose that $\underline{l}^{(k)}$ is an $(d-i,i)$-standard sequence. Then*

$$r_i(M) \leq e(H_{\mathfrak{m}}^i(M/\underline{l}M)) + i + (d-i)(k-1).$$

Proof. By virtue of [12], 6.5, it follows that

$$e(H_{\mathfrak{m}}^j(M/\underline{l}^{(k)}M)) \leq e(H_{\mathfrak{m}}^j(M/\underline{l}M)) + (d-i)(k-1).$$

Therefore, by 4.1 it implies for $i \leq j \leq d$ that

$$\begin{aligned} e(H_{\mathfrak{m}}^j(M)) + (j-i)k &\leq e(H_{\mathfrak{m}}^j(M/\underline{l}^{(k)}M)) \\ &\leq e(H_{\mathfrak{m}}^j(M/\underline{l}M)) + (d-i)(k-1), \end{aligned}$$

which by definition proves the claim. □

In the case of M a Buchsbaum module and $l = \{l_1, \ldots, l_r\} \subset R_1$ a subsystem of a system of parameters 4.1 yields that

$$r_i(M) = r_i(M/\underline{l}M) \quad \text{for all } i \leq d-r.$$

This is the crucial observation in [13] in order to derive Castelnuovo bounds for Buchsbaum schemes. In contrast the basic result for our Castelnuovo bounds for k-Buchsbaum schemes is the following:

Proposition 4.3. *Let $\underline{l} = \{l_1, \dots, l_{d-i}\} \subset R_1$, $0 \le i < d$, be an M-filter regular sequence. Suppose there are integers $\mu_j \ge 0$ such that $\underline{l}_{d-j}^{\mu_j} H_{\mathfrak{m}}^j(M) = 0$ for all $i \le j < d$. Then*

$$r_i(M) \le e(H_{\mathfrak{m}}^i(M/\underline{l}M)) + i + c_i,$$

where

$$c_i = \begin{cases} \mu_i + \dots + \mu_{d-1} & \text{if } \mu_i + \dots + \mu_{d-1} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\mu_i + \ldots + \mu_{d-1} = 0$, i. e., $H_{\mathfrak{m}}^j(M) = 0$ for $j = i, \dots, d-1$. Then the claim is a consequence of 4.1. Otherwise we make an induction on $d - i \ge 1$. Let $i = d-1$. If $\mu_{d-1} = 0$, then the claim follows by the previous argument. Let $\mu_{d-1} > 0$. By 3.3 we have

$$e(H_{\mathfrak{m}}^{d-1}(M)) \le e(H_{\mathfrak{m}}^{d-1}(M/\underline{l}H_{\mathfrak{m}}^{d-1}(M)) + \mu_{d-1} - 1.$$

Whence by 2.3 (a)

$$e(H_{\mathfrak{m}}^{d-1}(M)) \le e(H_{\mathfrak{m}}^{d-1}(M/\underline{l}M)) + \mu_{d-1} - 1.$$

Moreover, by [12], 6.2, we know that

$$e(H_{\mathfrak{m}}^{d}(M)) + 1 \le e(H_{\mathfrak{m}}^{d-1}(M/\underline{l}M))$$

which proves the claim for $i = d - 1$.

Suppose $0 < i < d-1$. If $\mu_i = 0$, then $r_i(M) = r_{i+1}(M)$. So the statement follows by the induction hypothesis. Now suppose that $\mu_i > 0$. By 2.3 (a) and observing that $r_i(M/\underline{l}M) \le r_i(M)$ it turns out

$$(*) \qquad \begin{aligned} e(H_{\mathfrak{m}}^i(M)) &\le e(H_{\mathfrak{m}}^i(M)/\underline{l}H_{\mathfrak{m}}^i(M)) + \mu_i - 1 \\ &\le \max\{e(H_{\mathfrak{m}}^i(M/\underline{l}M), r_{i+1}(M) - i + 1\} + \mu_i - 1. \end{aligned}$$

Assume that $c_{i+1} > 0$. Then by the induction hypothesis and [12], 6.2, we get

$$\begin{aligned} r_{i+1}(M) &\le e(H_{\mathfrak{m}}^{i+1}(M/\underline{l}_{d-i-1}M)) + i + 1 + c_{i+1} \\ &\le e(H_{\mathfrak{m}}^i(M/\underline{l}M)) + i + c_{i+1}. \end{aligned}$$

So $(*)$ implies $e(H_{\mathfrak{m}}^i(M)) \le e(H_{\mathfrak{m}}^i(M/\underline{l}M)) + i + c_i$, i. e., the claim is true. In the remaining case of $c_{i+1} = 0$ we have $H_{\mathfrak{m}}^{i+1}(M/\underline{l}_jM) = 0$ for all $0 \le j \le d-i-2$, which follows by an easy induction. Thus 2.3 (a) reads as

$$e(H_{\mathfrak{m}}^i(M)/\underline{l}H_{\mathfrak{m}}^i(M)) \le e(H_{\mathfrak{m}}^i(M/\underline{l}M)).$$

Therefore $(*)$ and the induction hypothesis complete the inductive step. □

It is noteworthy to say that in 4.3 there is no finiteness condition for the cohomology modules in the case $i > 0$. Under additional finiteness conditions 4.3 yields the following:

Corollary 4.4. *Suppose that $H_{\mathfrak{m}}^j(M), j = i, \dots, d-1$, are finitely generated R-modules. Let $\underline{l} = \{l_1, \dots, l_{d-i}\} \subset R_1$, $0 \le i < d$, be an M-filter regular sequence. Then*

$$r_i(M) \le r_i(M/\underline{l}M) + d_i,$$

where

$$d_i = \begin{cases} \lambda_i(M) + \dots + \lambda_{d-1}(M) - 1 & \text{if } \lambda_1(M) + \dots + \lambda_{d-1}(M) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.5. Consider the ring R of Example 3.6. R is 1-Buchsbaum and $\operatorname{reg} R = 2$. Since $\operatorname{rank}_K[\mathfrak{a}]_2 = \binom{r+1}{2}$ we obtain for general linear forms $\underline{l} = \{l_1, \ldots, l_r\}$ that

$$\begin{aligned} \operatorname{rank}_K[(a, \underline{l})]_2 &= \operatorname{rank}_K[\mathfrak{a}]_2 + \operatorname{rank}_K[(\underline{l})]_2 \\ &= \binom{r+1}{2} + \binom{2r+1}{2} - \binom{r+1}{2} \\ &= \operatorname{rank}_K[K[x_1, \ldots, x_{2r}]]_2. \end{aligned}$$

Therefore $e(H^0_{\mathfrak{m}}(R/\underline{l}R)) = e(R/\underline{l}R) = 1$. Whence

$$2 = \operatorname{reg} R = e(H^0_{\mathfrak{m}}(R/\underline{l}R)) + \lambda_0(R) + \ldots + \lambda_{r-1}(R) - 1.$$

That is, the bound in 3.4 is the best possible.

We need some more notation. The unique polynomial $h_M(t)$ determined by $h_M(t) = \operatorname{rank}_{R_0} M_t$ for $t \gg 0$ is called the Hilbert polynomial of M. Let $d = \dim M > 0$. Then it may be written as

$$h_M(t) = \operatorname{mult}(M) \frac{t^{d-1}}{(d-1)!} + \text{terms of lower degree}$$

where $\operatorname{mult}(M) \neq 0$. Then the multiplicity of M is defined to be $\operatorname{mult}(M)$. If M is zero-dimensional its multiplicity is by definition $\operatorname{mult}(M) = \operatorname{length}(M)$. The codimension of R is $\operatorname{codim} R := \operatorname{rank}_{R_0} R_1 - \dim R$. Finally, recall that $\lceil a \rceil$ denotes the least integer $\geq a$ for $a \in \mathbb{R}$.

The following lemma concerns the most technical part of estimates of the Castelnuovo-Mumford regularity.

Lemma 4.6. *Let M denote a finitely generated graded R-module.*

(a) *Let $\underline{l} = \{l_1, \ldots, l_{d-1}\} \subset R_1$ be an M-filter regular sequence where $d = \dim M$. Then we have for all $i > 0$*

$$d + e(H^d_{\mathfrak{m}}(M)) \leq i + e(H^i_{\mathfrak{m}}(M/\underline{l}_{d-i}M)) \leq \operatorname{mult}(M) + e^+(M) - 1.$$

(b) *Suppose that R is integral and $R_0 = K$ is an algebraically closed field. Let $l_1, \ldots, l_{d-1}$ be general linear forms where $d = \dim R$. Then we get for all $i > 0$*

$$d + e(H^d_{\mathfrak{m}}(R)) \leq i + e(H^i_{\mathfrak{m}}(R/\underline{l}_{d-i}R)) \leq \left\lceil \frac{\operatorname{mult}(R) - 1}{\operatorname{codim} R} \right\rceil.$$

Proof. In both statements the bounds on the left-hand side follow by [12], 6.2. In order to show (a) put $M' := (M/\underline{l}M)/H^0_{\mathfrak{m}}(M/\underline{l}M)$. Note that M' is an one-dimensional Cohen-Macaulay R-module. Since $\underline{l}$ is an M-filter regular sequence it is well-known that

$$\operatorname{mult}(M) = \operatorname{mult}(M').$$

Furthermore,

$$e^+(M) \geq e^+(M/\underline{l}M) \geq e^+(M') \quad \text{and} \quad e(H^1_{\mathfrak{m}}(M/\underline{l}M)) = e(H^1_{\mathfrak{m}}(M'))$$

as easily seen. Now let us prove that

$$(**) \qquad 1 + e(H^1_{\mathfrak{m}}(M')) \leq e_0(M') + e^+(M') - 1.$$

To this end choose a general $l \in [R/\underline{l}R]_1$. Then we have $e^+(M') = e^+(M'/lM')$. Therefore $[M'/lM']_t = 0$ for a certain integer $t > e^+(M')$ implies $[M'/lM']_{t+1} = 0$, too. Now the multiplication by l on M' induces a short exact sequence

$$0 \to [M'/lM']_t \to [H^1_{\mathfrak{m}}(M')]_{t-1} \to [H^1_{\mathfrak{m}}(M')]_t \to 0$$

for any integer t. It provides

$$\operatorname{rank}[H^1_{\mathfrak{m}}(M')]_t \leq \max\{0, \operatorname{rank}[H^1_{\mathfrak{m}}(M')]_{t-1} - 1\}$$

for all $t > e^+(M')$. But

$$\operatorname{rank}[H^1_{\mathfrak{m}}(M')]_{e^+(M')} = \operatorname{mult}(M) - \operatorname{rank}[M']_{e^+(M')}.$$

Because of $\operatorname{rank}[M']_{e^+(M')} > 0$ the inequality in (**) follows. But now

$$i + e(H^i_{\mathfrak{m}}(M/\underline{l}_{d-i}M)) \leq 1 + e(H^1_{\mathfrak{m}}(M/\underline{l}M))$$

for all $i > 0$. This proves part (a) of the claim.

In order to prove (b) we use the same notation as above. Then R' is the coordinate ring of a set of $\operatorname{mult}(R)$ points in linear semi-uniform position, see [1]. Moreover, by [1] and Riemann-Roch it follows that

$$1 + e(H^1_{\mathfrak{m}}(R')) \leq \left\lceil \frac{\operatorname{mult}(R') - 1}{\operatorname{codim} R'} \right\rceil.$$

Then the same arguments as above show (b). □

Remark 4.7. (1) Because of $e^+(R) = 0$ part (a) of 3.6 is a generalization of [6], 3.1. Furthermore, part (b) of 3.6 is an extension of [11], Corollary 2, to the case of a ground field of arbitrary characteristic.
(2) The result in 4.4 is an improvement by one of the bound which follows by a direct combination of 3.4 and 4.6.

Now there are several bounds of Castelnuovo type by combining 4.2 resp. 4.3 with 4.7. Here we state only one which seems most interesting to us. Consider a Cohen-Macaulay scheme $X \subset \mathbb{P}^n$. Let R denote its homogeneous coordinate ring. In accordance with the introduction put $\operatorname{reg}(X) = \operatorname{reg}(R) + 1$. Moreover, define $\lambda_i(X) = \lambda_i(R)$.

Theorem 4.8. *Let $X \subset \mathbb{P}^n_K$ be a projective Cohen-Macaulay scheme of positive dimension d, where K is an algebraically closed field. Let*

$$c = \begin{cases} \lambda_1(X) + \cdots + \lambda_{d-1}(X) - 1 & \text{if } X \text{ is not arithmetically Buchsbaum,} \\ 0 & \text{if } X \text{ is arithmetically Buchsbaum.} \end{cases}$$

(a) *Then there is the following bound*

$$\operatorname{reg}(X) \leq \deg(X) + c.$$

(b) *Suppose in addition that X is integral and nondegenerate. Then*

$$\operatorname{reg}(X) \leq \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil + c + 1.$$

Proof. Let R be the homogeneous coordinate ring of X and let $\underline{l} = \{l_1, \ldots, l_{d-1}\} \subset R$ be general linear forms. Suppose that X is aritmetically Buchsbaum. Then $\underline{l}$ is an $(r,1)$-standard sequence and 4.1 provides that

$$\operatorname{reg}(R) = 1 + e(H^1_{\mathfrak{m}}(M/\underline{l}M)).$$

Furthermore $\deg(X) = \operatorname{mult}(R)$. Thus the asserted bounds are a consequence of 4.6. If X is not arithmetically Buchsbaum the claims follow by 4.3 and 4.6. □

Remark 4.9. (1) The statements in 4.8 are an improvement of [12], 6.9, and - as noted there - also of [6], 3.2 (ii) and 3.3 (ii). Moreover, Theorem 1.3 of the introduction is a particular case of 4.8.
(2) Let R denote a k-Buchsbaum ring with $k > 0$. Let $\underline{l}$ denote a system of linear parameters. Then $\operatorname{reg}(R/\underline{l}R) = e(H^0_{\mathfrak{m}}(R/\underline{l}R))$. Hence, it yields an improved bound in [12], 6.8. This follows by replacing the corresponding argument in [12], 6.7, by 4.4.

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